Notes on Commutativity of Prime Rings

Shuliang Huang

Abstract Let R be a prime ring with center $Z(R)$, J a nonzero left ideal, α an automorphism of R and R admits a generalized (α,α)-derivation F associated with a nonzero (α,α)-derivation d such that $d(Z(R)) \neq (0)$. In the present paper, we prove that if any one of the following holds: *(i)* $F([x, y]) - \alpha([x, y]) \in Z(R)$ (ii) $F([x, y]) + \alpha([x, y]) \in Z(R)$ (iii) $F(x \circ y) - \alpha(x \circ y) \in Z(R)$ (iv) $F(x \circ y) - \alpha(x \circ y) \in Z(R)$ for all $x, y \in J$, then R is commutative. Also some related results have been obtained.

Keywords Commutativity · Prime and semiprime rings · Generalized (α,α)-derivations

2000 Mathematics Subject Classification: 16N60 · 16W25 · 16U80 · 16D90

1 Introduction

In all that follows, unless stated otherwise, R will be an associative ring with the center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for the Lie commutator $xy - yx$ and Jordan commutator $xy + yx$, respectively. A ring R is called 2-torsion free, if whenever $2x = 0$, with $x \in R$, then $x = 0$. If $S \subseteq R$, then we can define the left (resp. right) annihilator of S as $l(S) = \{x \in R \mid xs = 0 \ \textit{for all} \ \ s \in S\}$ (resp. $r(S) = \{x \in R \mid sx = 0 \ \textit{for all} \ \ s \in S\}$).

Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$,

The paper is supported by the Anhui Provincial Natural Science Foundation (1408085QA08) and the Key University Science Research Project of Anhui Province (KJ2014A183) and also the Training Program of Chuzhou University (2014PY06) of China.

S. Huang (✉)
School of Mathematics and Finance, Chuzhou University,
Chuzhou 239000, People's Republic of China
e-mail: shulianghuang@163.com

S.T. Rizvi et al. (eds.), *Algebra and its Applications*, Springer Proceedings in Mathematics & Statistics 174, DOI 10.1007/978-981-10-1651-6_5

and a Lie ideal U is called square-closed if $u^2 \in U$ for all $u \in U$. By a derivation, we mean an additive mapping $d : R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let α and β be endomorphisms of R, an additive mapping $d : R \longrightarrow R$ is said to be an (α, β)-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. Following [1], an additive mapping $F : R \longrightarrow R$ is called a generalized (α, β)-derivation on R if there exists an (α, β)-derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. Note that for I_R the identity map on R, this notion includes those of (α, β)-derivation when $F = d$, of derivation when $F = d$ and $\alpha = \beta = I_R$, and of generalized derivation, which is the case when $\alpha = \beta = I_R$.

Many results indicate that the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R. A well known result of Posner [10] states that if R is a prime ring and d a nonzero derivation of R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R must be commutative. Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R (see [3–5, 7] where further references can be found).

Daif and Bell [6] showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x, y]) - [x, y] = 0$ or $d([x, y]) + [x, y] = 0$ for all $x, y \in I$, then $I \subseteq Z(R)$. In particular, if $I = R$ then R is commutative. At this point, the natural question is what happens in case the derivation is replaced by a generalized derivation. In [11], Quadri et al., proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that any one of the following holds: (i) $F([x, y]) - [x, y] = 0$ (ii) $F([x, y]) + [x, y] = 0$ (iii) $F(x \circ y) - x \circ y = 0$ (iv) $F(x \circ y) + x \circ y = 0$ for all $x, y \in I$, then R is commutative. Following this line of investigation, Ali, Kumar and Miyan [2], explored the commutativity of a prime ring R admitting a generalized derivation F satisfying any one of the following conditions: (i) $F([x, y]) - [x, y] \in Z(R)$ (ii) $F([x, y]) + [x, y] \in Z(R)$ (iii) $F(x \circ y) - x \circ y \in Z(R)$ (iv) $F(x \circ y) + x \circ y \in Z(R)$ for all $x, y \in I$, a nonzero right ideal of R. On the other hand, Marubayashi et al. [8], established that if a 2-torsion free prime ring R admits a nonzero generalized (α, β)-derivation F associated with an (α, β)-derivation d such that either $F([u, v]) = 0$ or $F(u \circ v) = 0$ for all $u, v \in U$, where U is a nonzero square-closed Lie ideal of R, then $U \subseteq Z(R)$. In the present paper, our purpose is to prove the cited results for the case when the generalized (α, α)-derivation F acts on one sided ideal of R.

2 Main Results

In the remaining part of this paper, α and β will denote automorphisms of R. And we shall do a great deal of calculation with commutators and anti-commutators, routinely using the following basic identities: For all $x, y, z \in R$;

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z$$

$$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$$

$$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z].$$

Theorem 2.1 *Let R be a prime ring with center $Z(R)$ and J a nonzero left ideal of R. Suppose that R admits a generalized (α, α)-derivation F associated with a nonzero (α, α)-derivation d such that $d(Z(R)) \neq (0)$. If $F([x, y]) - \alpha([x, y]) \in Z(R)$ for all $x, y \in J$, then R is commutative.*

Proof It is easy to check that $d(Z(R)) \subseteq Z(R)$. Since $d(Z(R)) \neq (0)$, there exists $0 \neq c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. By assumption, we have

$$F([x, y]) - \alpha([x, y]) \in Z(R) \text{ for all } x, y \in J. \tag{1}$$

Replacing y by cy in (1), we get

$$(F([x, y]) - \alpha([x, y]))\alpha(c) + \alpha([x, y])d(c) \in Z(R) \text{ for all } x, y \in J. \tag{2}$$

Combining (1) and (2) and noting that the fact $\alpha(c) \in Z(R)$, we find that $\alpha([x, y])$ $d(c) \in Z(R)$, which implies that $[\alpha([x, y])d(c), r] = 0 = [\alpha([x, y]), r]d(c)$ for all $x, y \in J$ and $r \in R$. Since R is prime and $0 \neq d(c) \in Z(R)$, we have

$$[\alpha([x, y]), r] = 0 \text{ for all } x, y \in J; r \in R. \tag{3}$$

Replacing y by yx in (3) and using (3), we get

$$\alpha([x, y])[\alpha(x), r] = 0 \text{ for all } x, y \in J; r \in R. \tag{4}$$

Replacing r by $r\alpha(s)$ in (4) and using (4), we arrive at $\alpha([x, y])r[\alpha(x), \alpha(s)] = 0$ for all $x, y \in J$ and $r, s \in R$. The primeness of R yields that for each $x \in J$, either $\alpha([x, y]) = 0$ or $[\alpha(x), \alpha(s)] = 0$. Equivalently, either $[x, J] = 0$ or $[x, R] = 0$. Set $J_1 = \{x \in J \mid [x, J] = 0\}$ and $J_2 = \{x \in J \mid [x, R] = 0\}$. Then, J_1 and J_2 are both additive subgroups of I such that $J = J_1 \cup J_2$. Thus, by Brauer's trick, we have either $J = J_1$ or $J = J_2$. If $J = J_1$, then $[J, J] = 0$, and if $J = J_2$, then $[J, R] = 0$. In both cases, we conclude that J is commutative and so, by a result of [9], R is commutative.

Corollary 2.2 *Let R be a prime ring with center $Z(R)$ and J a nonzero left ideal of R. Suppose that R admits a generalized (α, α)-derivation F associated with a nonzero (α, α)-derivation d such that $d(Z(R)) \neq (0)$. If $F(xy) - \alpha(xy) \in Z(R)$ for all $x, y \in J$, then R is commutative.*

Proof For any $x, y \in J$, we have $F([x, y]) - \alpha([x, y]) = (F(xy) - \alpha(xy)) - (F(yx) - \alpha(yx)) \in Z(R)$, and hence the result follows.

Theorem 2.3 *Let R be a prime ring with center $Z(R)$ and J a nonzero left ideal of R. Suppose that R admits a generalized (α, α)-derivation F associated with a nonzero (α, α)-derivation d such that $d(Z(R)) \neq (0)$. If $F([x, y]) + \alpha([x, y]) \in Z(R)$ for all $x, y \in J$, then R is commutative.*

Proof If $F([x, y]) + \alpha([x, y]) \in Z(R)$ for all $x, y \in J$, then the generalized (α, α)-derivation $-F$ satisfies the condition $(-F)([x, y]) - \alpha([x, y]) \in Z(R)$ for all $x, y \in J$. It follows from Theorem 2.1 that R is commutative.

Theorem 2.4 *Let R be a prime ring with center $Z(R)$ and J a nonzero left ideal of R. Suppose that R admits a generalized (α, α)-derivation F associated with a nonzero (α, α)-derivation d such that $d(Z(R)) \neq (0)$. If $F(x \circ y) - \alpha(x \circ y) \in Z(R)$ for all $x, y \in J$, then R is commutative.*

Proof We are given that

$$F(x \circ y) - \alpha(x \circ y) \in Z(R) \text{ for all } x, y \in J. \tag{5}$$

Since $d(Z(R)) \neq (0)$, there exists $0 \neq c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Replacing y by cy in (5), we get

$$(F(x \circ y) - \alpha(x \circ y))\alpha(c) + \alpha(x \circ y)d(c) \in Z(R) \text{ for all } x, y \in J. \tag{6}$$

Combining (5) and (6), we find that $\alpha(x \circ y)d(c) \in Z(R)$ and hence $\alpha(x \circ y) \in Z(R)$. This implies that

$$[\alpha(x \circ y), r] = 0 \text{ for all } x, y \in J; r \in R. \tag{7}$$

Replacing yx for y in (7) and using (7), we have

$$\alpha(x \circ y)[\alpha(x), r] = 0 \text{ for all } x, y \in J; r \in R. \tag{8}$$

Replacing r by $r\alpha(s)$ in (8) and using (8), we have $\alpha(x \circ y)r[\alpha(x), \alpha(s)] = 0$ for all $x, y \in J$ and $r, s \in R$. The primeness of R yields that for each $x \in J$, either $\alpha(x \circ y) = 0$ or $[\alpha(x), \alpha(s)] = 0$. Now applying similar arguments as used in the proof of Theorem 2.1, we have either $x \circ y = 0$ for all $x, y \in J$; or $[J, R] = 0$. In the former case, replacing x by xz and using the fact $x \circ y = 0$ we find $[x, y]z = 0$ for all $x, y, z \in J$. This implies that $[x, y]J = 0$ and hence $[x, y]RJ = 0$. Since J is nonzero and R is prime, we get $[J, J] = 0$. Thus, J is commutative and so R. In the latter case, we have $[J, R] = 0$, in particular $[J, J] = 0$ and hence we get the required result.

Theorem 2.5 *Let R be a prime ring with center $Z(R)$ and J a nonzero left ideal of R. Suppose that R admits a generalized (α, α)-derivation F associated with a nonzero*

(α, α)-derivation d such that $d(Z(R)) \neq (0)$. If $F(x \circ y) + \alpha(x \circ y) \in Z(R)$ for all $x, y \in J$, then R is commutative.

Proof If $F(x \circ y) + \alpha(x \circ y) \in Z(R)$ for all $x, y \in J$, then the generalized (α, α)-derivation $-F$ satisfies the condition $(-F)(x \circ y) - \alpha(x \circ y) \in Z(R)$ for all $x, y \in J$. It follows from Theorem 2.4 that R is commutative.

Corollary 2.6 *Let R be a prime ring with center $Z(R)$ and J a nonzero left ideal of R. Suppose that R admits a generalized (α, α)-derivation F associated with a nonzero (α, α)-derivation d such that $d(Z(R)) \neq (0)$. If $F(xy) + \alpha(xy) \in Z(R)$ for all $x, y \in J$, then R is commutative.*

Proof For any $x, y \in I$, we have $F(x \circ y) + \alpha(x \circ y) = (F(xy) + \alpha(xy)) + (F(yx) + \alpha(yx)) \in Z(R)$, and hence our result follows.

Theorem 2.7 *Let R be a prime ring and J a nonzero left ideal of R such that $r(J) = 0$. If R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $F([x, y]) = 0$ for all $x, y \in J$, then R is commutative.*

Proof By assumption, we have

$$F([x, y]) = 0 \text{ for all } x, y \in J. \tag{9}$$

Replacing y by yx in (9) and using (9), we get $\beta([x, y])d(x) = 0$, which implies

$$[x, y]\beta^{-1}(d(x)) = 0 \text{ for all } x, y \in J. \tag{10}$$

Now substituting ry for y in (10) and using (10), we obtain $[x, r]y\beta^{-1}(d(x)) = 0$ for all $x, y \in J$ and $r \in R$. In particular, $[x, R]RJ\beta^{-1}(d(x)) = 0$ for all $x \in J$. The primeness of R yields that for each $x \in J$, either $[x, R] = 0$ or $J\beta^{-1}(d(x)) = 0$, in this case $d(x) = 0$. In view of similar arguments as used in the proof of Theorem 2.1, we have either $[J, R] = 0$ or $d(J) = 0$. If $[J, R] = 0$, then J is commutative and we are done. If $d(J) = 0$, then $0 = d(RJ) = d(R)\alpha(J) + \beta(R)d(J)$, which reduces to $d(R)\alpha(J) = 0$. And hence $d(R)\alpha(RJ) = 0 = d(R)\alpha(R)\alpha(J) = d(R)R\alpha(I)$. Since J is nonzero and the last relation forces that $d = 0$, contradiction.

Using the same techniques with necessary variations, we can prove the following:

Theorem 2.8 *Let R be a prime ring and J a nonzero left ideal of R such that $r(J) = 0$. If R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $F(x \circ y) = 0$ for all $x, y \in J$, then R is commutative.*

The following example demonstrates that R to be prime is essential in the hypothesis of Theorems 2.1, 2.3, 2.4 and 2.5.

Example 2.9 Let S be any ring. Next, let $R=\left\{\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix} | a, b, c \in S\right\}$ and $J=\left\{\begin{pmatrix}0 & a & b\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} | a, b \in S\right\}$, a nonzero left ideal of R. Define maps $F,\ d\ , \alpha : R \longrightarrow R$ as follows: $F\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & -c\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $d\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & c\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $\alpha\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & -a & b\\ 0 & 0 & -c\\ 0 & 0 & 0\end{pmatrix}$, Then, it is straightforward to check that F is a generalized (α, α)-derivation associated with a nonzero (α, α)-derivation d such that $d(Z(R)) \neq (0)$. It is easy to see that (i) $F([x, y]) - \alpha([x, y]) \in Z(R)$ (ii) $F([x, y]) + \alpha([x, y]) \in Z(R)$ (iii) $F(x \circ y) - \alpha(x \circ y) \in Z(R)$ (iv) $F(x \circ y) - \alpha(x \circ y) \in Z(R)$ for all $x, y \in J$, however R is not commutative.

Acknowledgments The author would like to thank the referee for giving helpful comments and suggestions.

References

1. Ali, A., Kumar, D., Miyan, P.: On generalized derivations and commutativity of prime ang semiprime rings. Hacet. J. Math. Stat. **40**(3), 367–374 (2011)
2. Ashraf, M., Ali, A., Ali, S.: On Lie ideals and generalized (θ, ϕ)-derivations in prime rings. Commun. Algebr. **32**(8), 2977–2985 (2004)
3. Ashraf, M., Rehman, N.: On commutativity of rings with derivations. Results Math. **42**, 3–8 (2002)
4. Bell, H.E., Daif, M.N.: On commutativity and strong commutativity-preserving maps. Can. Math. Bull. **37**, 443–447 (1994)
5. Bell, H.E., Daif, M.N.: On derivations and commutativity in prime rings. Acta Math. Hung. **66**(4), 337–343 (1995)
6. Daif, M.N., Bell, H.E.: Remarks on derivations on semiprime rings. Int. J. Math. Math. Sci. **15**, 205–206 (1992)
7. Golbasi, O.: Commutativity of semiprime rings with genearlized derivations. Indian J. Pure Appl. Math. **40**, 191–199 (2009)
8. Marubayashi, H., Ashraf, M., Rehman, N., Ali, S.: On generalized (α, β)-derivations in prime rings. Algebr. Colloq. **17**(1), 865–874 (2010)
9. Mayne, J.H.: Centralizing mappings of prime rings. Can. Math. Bull. **27**, 122–126 (1984)
10. Posner, E.C.: Derivations in prime rings. Proc. Am. Math. Soc. **8**, 1093–1100 (1957)
11. Quadri, M.A., Khan, M.S., Rehman, N.: Generalized derivations and commutativity of prime rings. Indian J. Pure Appl. Math. **34**, 1393–1396 (2003)