Properties of Semi-Projective Modules and their Endomorphism Rings

Manoj Kumar Patel

Abstract In this paper, we have studied the properties of semi-projective module and its endomorphism rings related with Hopfian, co-Hopfian, and directly finite modules. We have provide an example of module which are semi-projective but not quasi-projective. We also prove that for semi-projective module M with $dim M < \infty$ or $Codim M < \infty$, M^n is Hopfian for every integer $n \geq 1$. Apart from this we have studied the properties of pseudo-semi-injective module and observed that for pseudo-semi-injective module, co-Hopficity weakly co-Hopficity and directly finiteness are equivalent. Finally proved that for pseudo-semi-injective module M, N be fully invariant M-cyclic submodule of M with N is essential in M, then N is weakly co-Hopfian if and only if M is weakly co-Hopfian.

Keywords Semi-projective · Pseudo-semi-injective · Hopfian · Co-Hopfian

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1 Introduction

The notion of quasi-principally projective module was introduced by Wisbauer [14] under the terminology of semi-projective modules. Tansee and Wongwai [11] introduced the idea of M-principally projective module and defined a module M quasi-principally projective if it is M-principally projective. They also established several properties of the endomorphism ring of such modules and proved that quasi-principally projective modules are equivalent to semi-projective module. In this paper, we have established some properties of endomorphism ring of quasi-principally projective module in terms of Hopfian modules and proved that a quasi-principally projective module M is Hopfian if and only if M/N is Hopfian, where N is fully invariant small submodule of M.

M.K. Patel (✉)
Department of Mathematics, NIT Nagaland, Dimapur 797103, India
e-mail: mkpitb@gmail.com

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2 Preliminaries

Throughout this paper, by a ring R we always mean an associative ring with identity and every R-module M is an unitary right R-module. Let M be an R-module; a module N is called M-generated, if there is an epimorphism $M^{(I)} \longrightarrow N$ for some index set I. If I is finite then N is called finitely M-generated. In particular, a submodule N of M is called an M-cyclic submodule of M if $N = s(M)$ for some $s \in EndM_R$ or if there exist an epimorphism from M to N, equivalently it is isomorphic to M/L for some submodule L of M. A submodule K of an R-module M is said to be small in M, written $K \ll M$, if for every submodule $L \subseteq M$ with $K + L = M$ implies $L = M$. A nonzero R-module M is called hollow if every proper submodule of it is small in M. A submodule N of M is called fully invariant submodule of M, if $f(N) \subseteq N$ for any $f \in S = EndM_R$. A module M is called indecomposable, if $M \neq 0$ and cannot be written as a direct sum of nonzero submodules.

Consider the following conditions for an R-module M:

(D_1): For every submodule A of M there is a decomposition $M = M_1 \bigoplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 \ll M$.

(D_2): If $A \subseteq M$ such that M/A is isomorphic to a summand of M, then A is a summand of M.

(D_3): If M_1 and M_2 are summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a summand of M.

An R-module M is called a lifting module if M satisfies (D_1), M is called discrete module if it satisfies (D_1) and (D_2) and quasi-discrete if it satisfies (D_1) and (D_3).

We will freely make use of the standard notations, terminologies, and results of [1, 3, 14].

3 M-Principally Projective Module

Let M be a right R-module. A right R-module N is called M-principally projective

$$\begin{array}{ccc} & & N \\ & g \swarrow & \downarrow f \\ M & \underset{s}{\longrightarrow} & s(M) \longrightarrow 0 \end{array}$$

if every R-homomorphism f from N to an M-cyclic submodule $s(M)$ of M can be lifted to an R-homomorphism g from N to M, such that the above diagram is commutative, i.e., $s \cdot g = f$. A right R-module M is called quasi-principally projective (or semi-projective) if it is M-principally projective. Some examples of semi-projective modules are $\mathbb{Z}_4$, $\mathbb{Z}_6$ over $\mathbb{Z}$ (set of integers). Clearly, every projective module and quasi-projective module are semi-projective. But converse need not be true:

1. The $\mathbb{Z}$-module $\mathbb{Q}$ is semi-projective but not quasi-projective.
2. Let R be any integral domain with quotient field $F \neq R$. Then $M = F \oplus R$ is semi-projective (but in general not quasi-projective).
3. For any prime p in $\mathbb{Z}$, the Prufer p-group $\mathbb{Z}(p\infty)$ is not semi-projective.

Now, we provide an example of semi-projective module which is not M-principally projective module.

Example 3.1 Let $M_1 = \mathbb{Z}/p\mathbb{Z}$ and $M_2 = \mathbb{Z}/p^2\mathbb{Z}$ for any prime $p \in \mathbb{Z}$ be modules over $\mathbb{Z}$. Then we can easily check that both M_1 and M_2 are semi-projective modules. However M_1 is not M_2-principally projective.

Proposition 3.2 *If M is quasi-projective module and K is fully invariant submodule of M then M/K is semi-projective module.*

Proof The Proof is straightforward and hence we omit it.

An R-module M is called Hopfian (resp. co-Hopfian), if every surjective (resp. injective) R-homomorphism $f : M \longrightarrow M$ is an automorphism. For example, every Noetherian R-modules are Hopfian and every Artinian R-modules are co-Hopfian. A module M is called directly finite, if M is not isomorphic to a proper summand of itself.

Lemma 3.3 *(Proposition 3.25, Mohamed and Muller* (1990)[6]*) An R-module M is directly finite if and only if $f \cdot g = 1$ implies $g \cdot f = 1$ for any $f, g \in End M_R$.*

In the following propositions, we relate semi-projective module with Hopfian, co-Hopfian and directly finite modules.

Proposition 3.4 *Let M be semi-projective co-Hopfian, then it is Hopfian.*

Proof Let f be surjective endomorphism on M and $I_M : M \longrightarrow M$ be an identity map on M. By semi-projectivity of M there exists an R-homomorphism $g : M \longrightarrow M$ such that $f \cdot g = I_M$, implies that g is monomorphism. Since M is co-Hopfian, then it follows that $f = g^{-1}$ is an automorphism on M. Therefore M is Hopfian.

Proposition 3.5 *For the semi-projective modules M, the following statements are equivalent:*
(i) M is Hopfian;
(ii) M is co-Hopfian;
(iii) M is directly finite.

Proof Proof is trivial.

Proposition 3.6 *Let M be semi-projective and N is fully invariant small submodule of M. Then M is Hopfian if and only if M/N is Hopfian.*

Proof Assume that M/N is Hopfian. Let $f : M \longrightarrow M$ be any epimorphism, then semi-projectivity of M implies that there exist an homomorpshim $g : M \longrightarrow M$ such that $f \cdot g = I_M$. Hence $M \cong M \oplus (kerf)$ hence $K = (kerf)$ is direct summand of M. Since N is fully invariant implies $f(N) \subseteq N$, now we have induced a map $f^{'} : M/N \longrightarrow M/N$ which is clearly an epimorphism, the Hopficity of M/N implies that $f^{'} : M/N \longrightarrow M/N$ is an isomorphism. Now by $(f^{'}.\pi)(K) = (\pi \cdot f)(K) = 0$, where $\pi : M \longrightarrow M/N$ be natural epimorphism, we see that $\pi(K) = 0$, it means $K \subseteq N$, but $K \subseteq N \ll M$ implies that $K \ll M$. Since M is semi-projective there exist a splitting for f, i.e., $K = kerf$ is direct summand of M. Therefore $K = kerf = 0$, implies that M is Hopfian.

Conversely, assume that M is Hopfian and $N \ll M$ if $f : M/N \longrightarrow M/N$ is an epimorphism. We have $f \cdot \pi : M \longrightarrow M/N$, where π is natural epimorphism from $M \longrightarrow M/N$. Then by semi-projectivity of M, there exists $g \in EndM_R$, such that $\pi \cdot g = f \cdot \pi$ implies that g is an epimorphism by 19.2, Wisbauer (1991) [14] as π is a small epimorphism. Since M is Hopfian then g is an isomorphism.

Assume $kerf \neq 0$, then there exists $x \in M$ such that $f(x + N) = N$ implies $f.\pi(x) = \pi.g(x) = g(x) + N = N$ gives that $g(x) \in N \Rightarrow x \in g^{-1}(N) \subseteq N$. It follows that $kerf = N$, therefore M/N is Hopfian.

Corollary 3.7 *Let M be finitely generated semi-projective module. Then M is Hopfian if and only if $M/J(M)$ is Hopfian.*

Proof We know that $J(M)$ is fully invariant submodule of M. If M is finitely generated then we have $J(M) \ll M$. Thus by the above proposition proof is obvious.

Corollary 3.8 *Let M be semi-projective, N and L are submodules of M such that $N + L = M$ and $N \cap L \ll M$. Then M/N and M/L are Hopfian.*

Proof We have $M/(N \cap L) = N/(N \cap L) \oplus L/(N \cap L)$, by above Proposition 3.6, $M/(N \cap L)$ is Hopfian, hence so its direct summand, as $N/(N \cap L) \cong (N + L)/L = M/L$, similarly $L/(N \cap L) \cong (N + L)/N = M/N$ is Hopfian.

The next proposition is the generalization of Pandeya et.al. (Proposition 3.8) [7], whose proof is straightforward and hence we omit it.

Proposition 3.9 *Let M be finitely generated semi-projective hollow module then M is directly finite if and only if each homomorphic image is directly finite.*

For any module M, we denote the Goldie dimension of M by $dimM$ and the dual Goldie dimension of M by $CodimM$.

Proposition 3.10 *Let M be semi-projective modules with $dimM < \infty$ or $CodimM < \infty$. Then M^n is Hopfian for every integer $n \geq 1$.*

Proof We can easily seen that M^n satisfies the hypothesis of the statement, since $dimM^n = n(dimM)$, $CodimM^n = n(CodimM)$, and M is semi-projective module implies that M^n is semi-projective. Hence it remains to prove that M is Hopfian. Let $f : M \longrightarrow M$ be any epimorphism, then semi-projectivity of M implies that there exist an homomorpshim $g : M \longrightarrow M$ such that $f \cdot g = I_M$. Hence $M \cong M \oplus (kerf)$. This yields $dimM = dimM + dim(kerf)$ and $CodimM = CodimM + Codim(kerf)$. If $dimM < \infty$ then first of these equations will imply that $dim(kerf) = 0$, hence $kerf = 0$ that is f is an automorphism. If $CodimM < \infty$, then second of these equations will imply that $Codim(kerf) = 0$, hence $kerf = 0$ that is f is an automorphism. Thus in both cases, we get our assumed surjective endomorphism is an automorphism that is M is Hopfian implies that M^n is Hopfian.

Corollary 3.11 *Let M be semi-projective modules with $CodimM < \infty$. Then for any fully invariant submodule K of M and any integer $n \geq 1$, the module $(M/K)^n$ is Hopfian.*

Proof Immediate consequence of Propositions 3.2 and 3.10.

Corollary 3.12 *Let R be a ring with $dimR_R < \infty$. Then $M_n(R)$ is directly finite for every integer $n \geq 1$.*

Proof Since R_R is projective, assume that $dimR_R < \infty$ then by Proposition 3.9, we see that R^n is Hopfian for all integer $n \geq 1$. Then it is proved by the observation that M is Hopfian then $EndM_R$ is directly finite.

Lemma 3.13 *Let N be a submodule of a semi-projective module M. Then N is a summand if M/N is isomorphic to a summand of M.*

Proof The Proof is straightforward and hence we omit it.

Therefore, we say that a semi-projective module satisfies (D_2) condition. In general, we have the following implication:

Projective $\Rightarrow$ Quasi-projective $\Rightarrow$ semi-projective $\nRightarrow$ Discrete.

Corollary 3.14 *Let M be semi-projective module, then the following statements are equivalent:*
(1)M is discrete;
(2)M is quasi-discrete;
(3)M is lifting.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear from definitions and (3) $\Rightarrow$ (1) immediate from Lemma 3.13.

Corollary 3.15 *An indecomposable semi-projective module M is discrete if and only if M is hollow.*

Proof The Proof is straightforward and hence we omit it.

4 Pseudo-Semi-Injective Modules

Let M be a right R-module. M is called semi-injective if for any M-cyclic submodule N of M, monomorphism $g : N \longrightarrow M$ and corresponding to any homomorphism $f : N \longrightarrow M$ there exists a map $h \in EndM_R$, such that $h \cdot g = f$, i.e., diagram is commutative.

We wish to consider the situation where the map h in this definition is required to be a monomorphism. For this to happen, a map f must be a monomorphism. This leads to the following definition.

A right R-module M is called pseudo-M-principally injective (or pseudo-semi-injective) if for any M-cyclic submodule N of M and R-monomorphism $f, g : N \longrightarrow M$ there exists a monomorphism $h \in EndM_R$, such that $h \cdot g = f$.

It is easy to show that if M is pseudo-semi-injective module, then every monomorphism in $EndM_R$ is an automorphism, that is every pseudo-semi-injective module is co-Hopfian.

It is clear that every semi-injective module is pseudo-semi-injective, however, converse need not be true. In the following Proposition, we impose the uniformness on pseudo-semi-injective module that is desirable to make it semi-injective modules.

Proposition 4.1 *Every uniform pseudo-semi-injective module is semi-injective.*

Proof Let M be uniform pseudo-semi-injective module and N be M-cyclic submodule of M, let $f : N \longrightarrow M$ be any homomorphism implies that $kerf \subseteq N$. If $kerf = N$ case is trivial. If $kerf = 0$, then f is a monomorphism which extend to a homomorphism h from M to M. If $kerf \neq 0$, since N is uniform then it can be easily checked that $g = I_N - f : N \longrightarrow M$ is injective map that is $kerg = 0$, where $I_N : N \longrightarrow M$ be the inclusion map. By definition of pseudo-semi-injectivity of M, there exists an extension h of g from M to M such that $g = I_N - f = h \cdot i$ implies that $f = (1 - h) \cdot i$, which gives that $(1 - h)$ is an extension of f to M. Thus, we conclude that M is semi-injective module.

Corollary 4.2 *Every semi-simple pseudo-semi-injective module is semi-injective.*

Proposition 4.3 *Let M be a pseudo-semi-injective module and $f : M \longrightarrow M$ be a monomorphism. Then $f(M)$ is a direct summand of M.*

Proof The proof is straightforward and hence we omit it.

Proposition 4.4 *Let N be indecomposable pseudo M-principally injective modules, then every element $f \in EndN_R$ is invertible if and only if $kerf = 0$.*

Proof The invertible in $EndN_R$ is just the R-isomorphism from N to N. Thus it is clear that, if f is an invertible elements of $EndN_R$ then $kerf = 0$. Conversely suppose that $kerf = 0$ then f is a monomorphism and $f(N)$ is injective and so pseudo M-principally injective module. Then $f(N)$ is a direct summand of every extension of itself, thus $f(N)$ is a direct summand of N, and $f(N) \neq 0$ so $f(N) = N$, since N is indecomposable. Therefore f is a surjective homomorphism and so f is an invertible element of $EndN_R$.

A R-module M is called weakly co-Hopfian if any injective endomorphism f of M is essential, i.e., $f(M) \subseteq^e M$. The set of Integer $\mathbb{Z}$ is weakly co-Hopfian but not co-Hopfian.

Proposition 4.5 *Let M be pseudo-semi-injective module, then the following statements are equivalent:*
$(i)M$ is co-Hopfian;
$(ii)M$ is weakly co-Hopfian;
$(iii)M$ is directly finite.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. For (3) $\Rightarrow$ (1) Assume that $f : M \longrightarrow M$ be an injective endomorphism, then $f(M) \cong M$ and so $f(M)$ is pseudo-M-principally injective. Thus, $f(M)$ is direct summand of M that is there exist a submodule K of M such that $f(M) \oplus K = M$. Hence, $M \oplus K \cong M \Rightarrow K = 0$ since M is directly finite. Therefore, $f(M) = M$ implies that f is surjective and hence M is co-Hopfian.

Corollary 4.6 *If M is indecomposable pseudo-semi-injective module, then it is co-Hopfian.*

Proposition 4.7 *Let M be pseudo-semi-injective and nonsingular module. Then M Hopfian if and only if M co-Hopfian.*

Proof Let M is co-Hopfian and $f : M \longrightarrow M$ be surjective endomorphism of M. Then $M/kerf$ is nonsingular, and so $kerf$ is essentially closed in M. since M is pseudo-semi-injective modules, then $kerf$ is also pseudo-semi-injective. Thus, $M \cong M \oplus kerf$. As M is co-Hopfian, it is directly finite module by Proposition 4.5, so the above isomorphism implies that $kerf = 0$, i.e., f is an automorphism. Thus M is Hopfian. Conversely, It is well known that every Hopfian and co-Hopfian modules is directly finite so prove is done in the light of Proposition 4.5.

Proposition 4.8 *Let M be pseudo-semi-injective module and N be fully invariant M-cyclic submodule of M with N is essential in M. Then N is weakly co-Hopfian if and only if M is weakly co-Hopfian.*

Proof A sume that N is weakly co-Hopfian. Let $f : M \longrightarrow M$ be an injective endomorphism then by Proposition 2.3, $f(M)$ is direct summand of M. Since N is fully invariant $f(N) \subseteq N$. Thus $f|_N : N \longrightarrow N$ is an injective homomophism, the weakly co-Hopficity of N implies that $f(N) \subseteq^e N$, since $N \subseteq^e M$ we deduce that $f(N) \subseteq^e M$ and we have $f(N) \subseteq f(M) \subseteq M$, thus $f(M) \subseteq^e M$ therefore M is weakly co-Hopfian.

Conversely, let $f : N \longrightarrow N$ be an injective endomorphism and $i : N \longrightarrow M$ be an inclusion map. Since M is pseudo-semi-injective module, there exists a monomorphism $h : M \longrightarrow M$ such that $i \cdot f = h \cdot i$. Since M is weakly co-Hopfian by Proposition 4.5, M is co-Hopfian, so h is an isomorphism. N is fully invariant M-cyclic submodule of M so it is pseudo-semi-injective and $h(N) \subseteq N \Rightarrow h^{-1}(N) \subseteq N$ so $h(N) = N$. But $f = h|_N$ hence $f : N \longrightarrow N$ is surjective, so N is co-Hopfian then by Proposition 4.5, proof is complete.

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