n-Strongly Gorenstein Projective and Injective Complexes

C. Selvaraj and R. Saravanan

Abstract In this paper, we introduce and study the notions of n-strongly Gorenstein projective and injective complexes, which are generalizations of n-strongly Gorenstein projective and injective modules, respectively. Further, we characterize the so-called notions and prove that the Gorenstein projective (resp., injective) complexes are direct summands of n-strongly Gorenstein projective (resp., injective) complexes. Also, we discuss the relationships between n-strongly Gorenstein injective and n-strongly Gorenstein flat complexes, and for any two positive integers n and m, we exhibit the relationships between n-strongly Gorenstein projective (resp., injective) and m-strongly Gorenstein projective (resp., injective) complexes.

Keywords n-SG-projective complex · n-SG-injective complex · n-SG-flat complex

AMS Subject classification (2010) 16E05 · 16E30

1 Introduction

Throughout this paper, let R be an associative ring with identity and $\mathscr{C}$ be the abelian category of complexes of R-modules. Unless stated otherwise, a complex and an R-module will be understood to be a complex of left R-modules and a left R-module respectively.

Bennis and Mahdou [2] introduced the notions of strongly Gorenstein projective, injective and flat modules which are further studied and characterized by Liu [8]. Later, Bennis and Mahdou [3] generalized the notion of strongly Gorenstein projective modules to n-strongly Gorenstein projective modules and [11] Zhao studied the homological behaviors of n-strongly Gorenstein projective, injective and flat modules. Zhang et al. [10] studied the notions of strongly Gorenstein projective

C. Selvaraj (✉) · R. Saravanan
Department of Mathematics, Periyar University, Salem 636 011, TN, India
e-mail: selvavlr@yahoo.com

S.T. Rizvi et al. (eds.), *Algebra and its Applications*, Springer Proceedings in Mathematics & Statistics 174, DOI 10.1007/978-981-10-1651-6_17

and injective complexes. Motivated by the above works in this article, we introduce and study the notions of n-strongly Gorenstein projective and injective complexes, which are generalizations of n-strongly Gorenstein projective and injective modules, respectively. In [2, Theorem 2.7], it is proved that a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module. Using [9, Theorem 2.3], we prove the following.

Theorem *Let G be a complex. Then the following holds:*

(1) G is Gorenstein projective if and only if it is a direct summand of an n-SG-projective complex.
(2) G is Gorenstein injective if and only if it is a direct summand of an n-SG-injective complex.

In [9, Theorem 3.1], the relationship between Gorenstein flat and Gorenstein injective complexes is given. In connection to [9, Theorems 3.1 and 3.3] and [7, Proposition 4.7], we have the following result.

Theorem *Let R be a left artinian ring and let the injective envelope of every simple left R-module be finitely generated. Then the following hold:*

(1) If a complex G of left R-modules is n-SG-injective, then G^+ is an n-SG-flat complex of right R-modules.
(2) If a complex G of right R-modules is n-SG-flat, then G^+ is an n-SG-injective complex of left R-modules.

In Sect. 2, we recall some known definitions and terminologies which will be needed in the sequel.

In Sect. 3, we introduce and study the notions of n-strongly Gorenstein projective and injective complexes. We show that a complex is Gorenstein projective (resp., injective) if and only if it is a direct summand of an n-SG-projective (resp., injective) complex and prove that the modules in an n-SG-projective (resp., injective) complex are precisely the n-SG-projective (resp., injective) modules. Further, over a left artinian ring R, we discuss the relationships between n-SG-injective and n-SG-flat complexes.

In the last section, we study the relationships between n-SG-projective (resp., injective) and m-SG-projective (resp., injective) complexes for any two positive integers n and m.

2 Preliminaries

In this section, we first recall some known definitions and terminologies which we need in the sequel.

In this paper, a complex

$$\cdots \to C^{-1} \stackrel{\delta^{-1}}{\to} C^0 \stackrel{\delta^0}{\to} C^1 \stackrel{\delta^1}{\to} \cdots$$

will be denoted by C or (C, δ). We will use subscripts to distinguish complexes. So if $\{C_i\}_{i \in I}$ is a family of complexes, C_i will be

$$\cdots \to C_i^{-1} \stackrel{\delta_i^{-1}}{\to} C_i^0 \stackrel{\delta_i^0}{\to} C_i^1 \stackrel{\delta_i^1}{\to} \cdots .$$

Given an R-module M, we will denote by $\overline{M}$ the complex

$$\cdots 0 \to 0 \to M \stackrel{id}{\to} M \to 0 \to 0 \cdots$$

with M in the 1st and 0th degrees. Similarly, we denote by $\underline{M}$ the complex with M in the 0th degree and 0 in the other places. Note that an R-module M is injective (resp., projective) if and only if the complex $\overline{M}$ is injective (resp., projective).

Given a complex C and an integer m, $C[m]$ denotes the complex such that $C[m]^n = C^{m+n}$ and whose boundary operators are $(-1)^m \delta^{m+n}$. The nth cycle of a complex C is defined as $\mathrm{Ker}\delta^n$ and is denoted by $Z^n C$. The nth boundary of C is defined as $\mathrm{Im}\delta^{n-1}$ and is denoted by $B^n C$.

Let C be a complex of left R-modules (resp., of right R-modules) and let D be a complex of left R-modules. We denote by $\mathrm{Hom}(C, D)$ (respectively, $C \otimes D$) the usual homomorphism complex (resp., tensor product) of the complexes C and D. The nth degree term of the complex $\mathrm{Hom}(C, D)$ is given by

$$\mathrm{Hom}(C, D)^n = \prod_{t \in \mathbb{Z}} \mathrm{Hom}(C^t, D^{n+t})$$

and whose boundary operators are

$$(\delta^n f)^m = \delta_D^{n+m} f^m - (-1)^n f^{m+1} \delta_C^m.$$

The nth degree term of $C \otimes D$ is given by

$$(C \otimes D)^n = \bigoplus_{t \in \mathbb{Z}} (C^t \otimes_R D^{n-t})$$

and

$$\delta(x \otimes y) = \delta_C^t(x) \otimes y + (-1)^t x \otimes \delta_D^{n-t}(y),$$

for $x \in C^t$ and $y \in D^{n-t}$.

For a complex C of left R-modules, we have a functor $- \otimes \mathscr{C} : \mathscr{C}_R \to \mathscr{C}_{\mathbb{Z}}$, where $\mathscr{C}_R$ denotes the category of right R-modules. The functor $- \otimes \mathscr{C} : \mathscr{C}_R \to \mathscr{C}_{\mathbb{Z}}$ being right exact, we can construct the left derived functors which we denote by

$Tor_i(-, C)$. Given two complexes C and D of $\mathscr{C}$, we use $Ext^i(C, D)$ for $i \geq 0$ to denote the groups we obtain from the right derived functors of Hom and we use C^+ to denote the complex $\underline{Hom}(C, \overline{\mathbb{Q}/\mathbb{Z}})$.

Recall that a complex C is projective (respectively, injective) if C is exact and $Z^n C$ is a projective (respectively, an injective) R-module for each $i \in \mathbb{Z}$. A complex C is flat if C is exact and $Z^n C$ is flat R-module for each $i \in \mathbb{Z}$. Equivalently, a complex C is projective (respectively, injective) if and only if Hom$(C, -)$ (respectively, Hom $(-, C)$) is exact. Also a complex C is flat if and only if $- \otimes C$ is exact. For unexplained terminologies and notations we refer to [1, 4–6].

Definition 2.1 ([10]) A complex G is called strongly Gorenstein projective (for short SG-projective) if there exists an exact sequence of complexes

$$\mathbb{P} : \cdots \to P \xrightarrow{\delta} P \xrightarrow{\delta} P \xrightarrow{\delta} \cdots$$

such that (i) P is a projective complex;
(ii) Ker $\delta_0 \cong G$;
(iii) Hom$(\mathbb{P}, Q)$ is exact for any projective complex Q.

Similarly, the SG-injective complexes are defined.

Definition 2.2 ([7]) A complex G of right R-modules is called strongly Gorenstein flat (for short SG-flat) if there exists an exact sequence of complexes of right R-modules

$$\mathbb{F} : \cdots \to F \xrightarrow{\delta} F \xrightarrow{\delta} F \xrightarrow{\delta} \cdots$$

such that (i) F is flat;
(ii) Ker $\delta_0 \cong G$;
(iii) $\mathbb{F} \otimes I$ is exact for any injective complex I.

Definition 2.3 ([7]) Let n be a positive integer. A complex G of right R-modules is said to be an n-SG-flat if there exists an exact sequence of complexes

$$0 \to G \xrightarrow{\delta_{n+1}} F_n \xrightarrow{\delta_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\delta_1} G \to 0$$

with F_i projective for any $1 \leq i \leq n$, such that $- \otimes I$ leaves the sequence exact whenever I is an injective complex.

Next, we present the characterizations of n-SG-flat complexes in order to use it further.

Proposition 2.4 ([7]) *Let R be a right coherent ring and G be any complex of right R-modules. Then the following are equivalent;*

(1) G is n-SG-flat;

(2) There exists an exact sequence of complexes of right R-modules

$$0 \to G \overset{\delta_{n+1}}{\to} F_n \overset{\delta_n}{\to} F_{n-1} \to \cdots \to F_1 \overset{\delta_1}{\to} G \to 0$$

with F_i flat for any $1 \leq i \leq n$, such that $\bigoplus_{i=2}^{n+1} Im\ \delta_i$ is SG-flat;

(3) There exists an exact sequence of complexes of right R-modules

$$0 \to G \overset{\delta_{n+1}}{\to} F_n \overset{\delta_n}{\to} F_{n-1} \to \cdots \to F_1 \overset{\delta_1}{\to} G \to 0$$

with F_i flat for any $1 \leq i \leq n$, such that $\bigoplus_{i=2}^{n+1} Im\ \delta_i$ is Gorenstein flat.

3 *n*-Strongly Gorenstein Projective and Injective Complexes

In this section, we introduce and study the n-SG-projective and injective complexes which are generalizations of SG-projective and injective modules, respectively. Also we extend the results in [3, 11] on n-strongly Gorenstein projective and injective modules to that of complexes.

Definition 3.1 Let n be a positive integer. A complex G is said to be an n-strongly Gorenstein projective (for short n-SG-projective) if there exists an exact sequence of complexes

$$0 \to G \overset{\delta_{n+1}}{\to} P_n \overset{\delta_n}{\to} P_{n-1} \to \cdots \to P_1 \overset{\delta_1}{\to} G \to 0$$

with P_i projective for any $1 \leq i \leq n$, such that $\mathrm{Hom}(-, Q)$ leaves the sequence exact whenever Q is a projective complex.

Definition 3.2 Let n be a positive integer. A complex G is said to be an n-strongly Gorenstein injective (for short n-SG-injective) if there exists an exact sequence of complexes

$$0 \to G \overset{\alpha_{n+1}}{\to} I_n \overset{\alpha_n}{\to} I_{n-1} \to \cdots \to I_1 \overset{\alpha_1}{\to} G \to 0$$

with I_i injective for any $1 \leq i \leq n$, such that $\mathrm{Hom}(E, -)$ leaves the sequence exact whenever E is an injective complex.

Note that 1-SG-projective (resp., injective) complexes are just SG-projective (resp., injective) complexes. It is also clear that for any i with $2 \leq i \leq n+1$, the complex $\mathrm{Im}\ \delta_i$ (resp., $\mathrm{Im}\ \alpha_i$) in the above exact sequence is n-SG-projective (resp., injective). The following proposition shows that the class of all n-SG-projective (resp., injective) complexes is between the class of all SG-projective (resp., injective) complexes and the class of all Gorenstein projective (resp., injective) complexes.

Proposition 3.3 *Let n be a positive integer. Then:*

(1) Every SG-projective (resp., injective) complex is an n-SG-projective (resp., injective) complex.
(2) Every n-SG-projective (resp., injective) complex is a Gorenstein projective (resp., injective) complex.

Proof Since the SG-injective complex is the dual notion of SG-projective, we prove the results for SG-projective case.

(1) Let G be an SG-projective complex. There exists an exact sequence of complexes

$$0 \to G \xrightarrow{f} P \xrightarrow{g} G \to 0,$$

where P is a projective complex, such that $\mathrm{Hom}(-, Q)$ leaves the sequence exact whenever Q is a projective complex. Then we get an exact sequence of complexes of the form

$$X : 0 \to G \xrightarrow{f} P \xrightarrow{fg} P \xrightarrow{fg} \cdots \to P \xrightarrow{g} G \to 0$$

such that $\mathrm{Hom}(X, Q)$ is exact for any projective complex Q. Therefore G is an n-SG-projective complex.

(2) Let G be an n-SG-projective complex. There exists an exact sequence of complexes

$$Y : 0 \to G \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\delta_1} G \to 0$$

with P_i projective for any $1 \le i \le n$, such that $\mathrm{Hom}(-, Q)$ leaves the sequence exact whenever Q is a projective complex. Thus, we get the following exact sequence of complexes

$$Y' : \cdots \to P_1 \xrightarrow{\delta_{n+1}\delta_1} P_n \xrightarrow{\delta_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\delta_{n+1}\delta_1} P_n \xrightarrow{\delta_n} \cdots .$$

such that $Im(\delta_{n+1}\delta_1) \cong G$. Let Q be any projective complex. Then the exactness of $\mathrm{Hom}(Y', Q)$ follows from the exactness of $\mathrm{Hom}(Y, Q)$ and hence G is a Gorenstein projective complex. □

Proposition 3.4 *Let $\{G_i\}_I$ be any family of complexes. Then*

(1) If G_i is n-SG-projective for every $i \in I$, then $\bigoplus_I G_i$ is an n-SG-projective complex.
(2) If G_i is n-SG-injective for every $i \in I$, then $\prod_I G_i$ is an n-SG-injective complex.

Proof (1) For each i in I there exists an exact sequence of complexes

$$\mathbb{X}_i : 0 \to G_i \to P_{in} \to P_{in-1} \to \cdots \to P_{i1} \to G_i \to 0$$

with P_{ij} projective for $1 \le j \le n$, such that $\mathrm{Hom}(\mathbb{X}_i, Q)$ is exact for any projective complex Q. Since the direct sum of projective complexes is projective, we obtain the following exact sequence of complexes

$$\bigoplus_{i\in I} \mathbb{X}_i : 0 \to \bigoplus_{i\in I} G_i \to \bigoplus_{i\in I} P_{in} \to \cdots \to \bigoplus_{i\in I} P_{i1} \to \bigoplus_{i\in I} G_i \to 0$$

with $\bigoplus_{i\in I} P_{ij}$ projective for $1 \le j \le n$. Let Q be any projective complex. Then $\mathrm{Hom}(\bigoplus \mathbb{X}_i, Q) \cong \prod \mathrm{Hom}(\mathbb{X}_i, Q)$ is exact, and hence $\bigoplus G_i$ is an n-SG-projective complex.

(2) The proof is similar to (1). □

In [2, Theorem 2.7], it is proved that a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module. Using [9, Theorem 2.3] and Proposition 3.3, we have the following.

Theorem 3.5 *Let G be a complex. Then the following hold:*

(1) G is Gorenstein projective if and only if it is a direct summand of an n-SG-projective complex.
(2) G is Gorenstein injective if and only if it is a direct summand of an n-SG-injective complex.

Proof (1) Let G be a Gorenstein projective complex. Then it is a direct summand of an SG-projective complex by [10, Theorem 1]. Hence G is a direct summand of an n-SG-projective complex by Proposition 3.3. Conversely, let G be a direct summand of an n-SG-projective complex C. Then C is Gorenstein projective by Proposition 3.3 (2). Since the class of all Gorenstein projective complexes is closed under direct summands by [9, Theorem 2.3], it follows that G is Gorenstein projective.

(2) The proof is similar to (1). □

In [11, Theorem 3.9], Zhao and Huang have given some characterizations of n-SG-projective modules. Now, we have the similar characterization for n-SG-projective complexes in the following.

Proposition 3.6 *Let G be any complex. Then the following are equivalent;*

(1) G is n-SG-projective;
(2) There exists an exact sequence of complexes

$$0 \to G \stackrel{\delta_{n+1}}{\to} P_n \stackrel{\delta_n}{\to} P_{n-1} \to \cdots \to P_1 \stackrel{\delta_1}{\to} G \to 0$$

with P_i projective for any $1 \le i \le n$, such that $\bigoplus_{i=2}^{n+1} \mathrm{Im}\, \delta_i$ is SG-projective;

(3) There exists an exact sequence of complexes

$$0 \to G \stackrel{\delta_{n+1}}{\to} P_n \stackrel{\delta_n}{\to} P_{n-1} \to \cdots \to P_1 \stackrel{\delta_1}{\to} G \to 0$$

with F_i projective for any $1 \leq i \leq n$, such that $\bigoplus\limits_{i=2}^{n+1} Im\, \delta_i$ is Gorenstein projective.

Proof (1) $\Rightarrow$ (2). Let G be an SG-projective complex. Then there exists an exact sequence of complexes

$$0 \to G \stackrel{\delta_{n+1}}{\to} P_n \stackrel{\delta_n}{\to} P_{n-1} \to \cdots \to P_1 \stackrel{\delta_1}{\to} G \to 0$$

with P_i projective for any $1 \leq i \leq n$, such that $\mathrm{Hom}(-, Q)$ leaves the sequence exact whenever Q is a projective complex. Now for each i with $2 \leq i \leq n+1$, we have an exact sequence of complexes

$$0 \to \mathrm{Im}\, \delta_i \stackrel{\alpha_i}{\to} P_{i-1} \stackrel{\delta_{i-1}}{\to} \cdots \to P_1 \stackrel{\delta_{n+1}\delta_1}{\longrightarrow} P_n \stackrel{\delta_n}{\to} \cdots \to P_i \stackrel{\delta_i}{\to} \mathrm{Im}\, \delta_i \to 0.$$

By adding these exact sequences, we obtain the following exact sequence

$$0 \to \bigoplus_{i=2}^{n+1} \mathrm{Im}\, \delta_i \stackrel{\alpha}{\to} \bigoplus_{i=1}^{n} P_i \stackrel{\delta}{\to} \cdots \to P_n \oplus P_0 \oplus \cdots \oplus P_{n-1} \to \cdots$$

where $\alpha = \mathrm{diag}\{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\delta = \mathrm{diag}\, \{\delta_{n+1}\delta_1, \delta_2, ..., \delta_n\}$. Hence it is clear that $\mathrm{Im}\, \delta \cong \bigoplus\limits_{i=2}^{n+1} \delta_i$ and $Ext_1(\bigoplus\limits_{i=2}^{n+1} \mathrm{Im}\, \delta_i, Q) \cong \prod\limits_{i=2}^{n+1} Ext_1(\mathrm{Im}\, \delta_i, Q) = 0$ for any projective complex Q. Therefore $\bigoplus\limits_{i=2}^{n+1} \mathrm{Im}\, \delta_i$ is SG-flat.

(2) $\Rightarrow$ (3) It follows from the Proposition 3.3.

(3) $\Rightarrow$ (1) It is obvious. □

Similarly, we can characterize the n-SG-injective complexes.

In [9, Theorem 3.1], the relationship between Gorenstein flat and Gorenstein injective complexes is given. In connection to [9, Theorems 3.1 and 3.3] and Proposition 2.4, we have the following.

Theorem 3.7 *Let R be a left artinian ring and let the injective envelope of every simple left R-module be finitely generated. Then the following hold:*

(1) If a complex G of left R-modules is n-SG-injective, then G^+ is an n-SG-flat complex of right R-modules.
(2) If a complex G of right R-modules is n-SG-flat, then G^+ is an n-SG-injective complex of left R-modules.

Proof (1) Let G be an n-SG-injective complex. Then using the characterization of n-SG-injective complexes similar to Proposition 3.6, we get an exact sequence of complexes

$$\mathbf{I} : 0 \to G \stackrel{\delta_{n+1}}{\to} I_n \stackrel{\delta_n}{\to} I_{n-1} \to \cdots \to I_1 \stackrel{\delta_1}{\to} G \to 0$$

where I_j is an injective complex for $1 \leq j \leq n$ and $\bigoplus_{j=2}^{n+1} \mathrm{Im}\, \delta_j$ is Gorenstein injective. Thus we have the following exact sequence of right R-modules

$$\mathbf{I}^+ : 0 \to G^+ \xrightarrow{\delta_1^+} I_1^+ \xrightarrow{\delta_2^+} I_2^+ \to \cdots \to I_n^+ \xrightarrow{\delta_{n+1}^+} G^+ \to 0$$

where I_j^+ is a flat complex for $1 \leq j \leq n$. Since G is Gorenstein injective by Proposition 3.3, we have that $\mathrm{Im}\, \delta_1^+ \cong G^+$ is Gorenstein flat by [9, Theorem 3.5]. Since $\bigoplus_{j=2}^{n+1} \mathrm{Im}$ δ_j is Gorenstein injective, we get that $\mathrm{Im}\, \delta_j$ is Gorenstein injective for $2 \leq j \leq n+1$ by [9, Theorem 2.10]. Thus for every j with $1 \leq j \leq n$, $\mathrm{Im}\, \delta_j^+$ is Gorenstein flat by [9, Theorem 3.5]. Hence $\bigoplus_{j=1}^{n} \mathrm{Im}\, \delta_j^+$ is Gorenstein flat since Gorenstein flat complexes are closed under direct sums. Therefore G^+ is n-SG-flat by Proposition 2.4.

(2) Let G be an n-SG-flat complex. Then by Proposition 2.4, we get an exact sequence of complexes of right R-modules

$$\mathbf{F} : 0 \to G \stackrel{\delta_{n+1}}{\to} F_n \stackrel{\delta_n}{\to} F_{n-1} \to \cdots \to F_1 \stackrel{\delta_1}{\to} G \to 0$$

where F_j is a flat complex for $1 \leq j \leq n$ and $\bigoplus_{j=2}^{n+1} \mathrm{Im}\, \delta_j$ is Gorenstein flat. Thus we have the following exact sequence of complexes of R-modules

$$\mathbf{F}^+ : 0 \to G^+ \xrightarrow{\delta_1^+} F_1^+ \xrightarrow{\delta_2^+} F_2^+ \to \cdots \to F_n^+ \xrightarrow{\delta_{n+1}^+} G^+ \to 0$$

where F_j^+ is an injective complex for $1 \leq j \leq n$. Since G is Gorenstein flat by [7, Proposition 4.2], we have that $\mathrm{Im}\, \delta_1^+ \cong G^+$ is Gorenstein injective by [9, Theorem 3.1]. Since $\bigoplus_{j=2}^{n+1} \mathrm{Im}\, \delta_j$ is Gorenstein flat, we get that $\mathrm{Im}\, \delta_j$ is Gorenstein flat for $2 \leq j \leq n+1$ by [9, Theorem 3.3]. Thus for every j with $1 \leq j \leq n$, $\mathrm{Im}\, \delta_j^+$ is Gorenstein injective by [9, Theorem 3.1]. Hence $\bigoplus_{j=1}^{n} \mathrm{Im}\, \delta_j^+$ is Gorenstein injective since Gorenstein injective complexes are closed under finite direct sums. Therefore G^+ is n-SG-injective by Proposition 3.3. □

Corollary 3.8 *Let R be a left artinian ring and let the injective envelope of every simple left R-module be finitely generated. Then the following hold:*

(1) If a complex G of R-modules is n-SG-injective, then G^{++} is an n-SG-injective complex.
(2) If a complex G of right R-modules is n-SG-flat, then G^{++} is an n-SG-flat complex.

Proof The proof follows from Theorem 3.7. □

The following result shows the relationship between n-SG-projective complexes and n-SG-projective modules.

Proposition 3.9 *Let G be a complex. If G is n-SG-projective, then G^i is an n-SG-projective R-module for all $i \in \mathbb{Z}$.*

Proof Suppose G is an n-SG-projective complex. By Proposition 3.6, there exists an exact sequence of complexes

$$0 \to G \overset{\delta_{n+1}}{\to} P_n \overset{\delta_n}{\to} P_{n-1} \to \cdots \to P_1 \overset{\delta_1}{\to} G \to 0$$

where P_j is a projective complex for $1 \leq j \leq n$ and $\bigoplus_{j=2}^{n+1} \operatorname{Im} \delta_j$ is Gorenstein projective. Then for each $i \in \mathbb{Z}$, we get an exact sequence of modules

$$0 \to G^i \xrightarrow{\delta^i_{n+1}} P^i_n \xrightarrow{\delta^i_n} P^i_{n-1} \to \cdots \to P^i_1 \xrightarrow{\delta^i_1} G^i \to 0$$

such that P^i_j is a projective R-module for $1 \leq j \leq n$. Since $\bigoplus_{j=2}^{n+1} \operatorname{Im} \delta_j$ is a Gorenstein projective complex if and only if $\operatorname{Im} \delta_j$ is a Gorenstein projective complex for $2 \leq j \leq n+1$ by [9, Theorem 2.3]. Then by [9, Theorem 2.2], we have $\operatorname{Im} \delta_j$ is a Gorenstein projective complex if and only if $\operatorname{Im} \delta^i_j$ is a Gorenstein projective R-module for every $i \in \mathbb{Z}$ and $2 \leq j \leq n+1$. Thus we get that $\bigoplus_{j=2}^{n+1} \operatorname{Im} \delta^i_j$ is a Gorenstein projective R-module since the class of all Gorenstein projective modules is closed under direct sums. Therefore the result follows from [11, Theorem 3.9]. □

Corollary 3.10 *Let M be an R-module. Then M is n-SG-projective if and only if the complex $\overline{M}$ is n-SG-projective.*

Proof Suppose M is an n-SG-projective module. Then there exists an exact sequence of R-modules

$$X : 0 \to M \to P_n \to P_{n-1} \to \cdots \to P_1 \to M \to 0,$$

where P_i is a projective R-module for $1 \leq i \leq n$, such that $\operatorname{Hom}_R(-, Q)$ leaves the sequence exact for any projective module Q. Thus, we get an exact sequence of complexes

$$\overline{X} : 0 \to \overline{M} \to \overline{P}_n \to \overline{P}_{n-1} \to \cdots \to \overline{P}_1 \to \overline{M} \to 0$$

with $\overline{P}_i$ a projective complex for $1 \leq i \leq n$. Now let Q' be any projective complex. Then it is a direct product of complexes of the form $\overline{P}[n]$ for some projective module P and $n \in \mathbb{Z}$. Then

$$\begin{aligned} Hom(\overline{X}, Q') &\cong Hom(\overline{X}, \prod_{n\in\mathbb{Z}} \overline{P}[n]) \\ &\cong \prod_{n\in\mathbb{Z}} Hom(\overline{X}, \overline{P}[n]) \end{aligned}$$

is exact for all $n \in \mathbb{Z}$ and hence $\overline{M}$ is n-SG-projective. The converse follows from Proposition 3.9.

The following example describes that there are 2-SG-projective complexes which are not necessarily 1-SG-projective.

Example 3.11

(1) Let R be a local ring and consider the ring $S = R[X, Y]/(XY)$. Let $[X]$ and $[Y]$ be the residue classes in S of X and Y respectively. Then by [3, Example 2.6], we observe that the R-modules $[X]$ and $[Y]$ are 2-SG-projective but are not 1-SG-projective. Then by Corollary 3.10, the complexes $\overline{[X]}$ and $\overline{[Y]}$ are 2-SG-projective but are not SG-projective.
(2) In general, n-SG-projective complexes need not be m-SG-projective whenever $n \nmid m$. Based on the assumptions in [11, Example 3.2], we observe that the modules S_i $(1 \leq i \leq n)$ are n-strongly Gorenstein projective but are not m-strongly Gorenstein projective. Then by the Corollary 3.10, we see that the complexes $\overline{S_i}$ are n-SG-projective but are not m-SG-projective whenever $n \nmid m$.

4 n-SG-Projective and m-SG-Projective Complexes

In this section, we study the relationships between n-SG-projective (resp., injective) and m-SG-projective (resp., injective) complexes for any two positive integers n and m.

Lemma 4.1 *Let m, n and r be any positive integers such that $m = rn$. Then the class of all m-SG-projective (resp., injective) complexes contains the class of all n-SG-projective (resp., injective) complexes.*

Proof Let G be an n SG-projective complex. Then there exists an exact sequence of complexes

$$\mathbf{X} : 0 \to G \overset{\alpha_{n+1}}{\to} I_n \overset{\alpha_n}{\to} I_{n-1} \to \cdots \to I_1 \overset{\alpha_1}{\to} G \to 0$$

with I_j injective for any $1 \leq j \leq n$, such that $\bigoplus_{j=2}^{n+1} \operatorname{Im} \alpha_j$ is a Gorenstein projective complex. So $\operatorname{Im} \delta_j$ is Gorenstein projective for every $1 \leq j \leq n$ by [9, Theorem 2.3]. Using the exact sequence **X** for r times, we have the following exact sequence

$$\mathbf{Y} : 0 \to G \stackrel{\alpha_{n+1}}{\to} I_n \stackrel{\alpha_n}{\to} I_{n-1} \to \cdots \to I_1 \stackrel{\delta}{\to} I_n \to \cdots I_1 \stackrel{\alpha_1}{\to} G \to 0$$

with I_j injective for any $1 \leq j \leq n$ and $\delta = \alpha_{n+1}\alpha_1$. Then $\bigoplus_{j=2}^{n+1} \operatorname{Im} \alpha_j$ is Gorenstein projective since $\operatorname{Im} \alpha_j$ and G are Gorenstein projective. □

For any positive integer n, we use n-SG-Proj($\mathscr{C}$) (resp., n-SG-Inj($\mathscr{C}$)) to denote the subcategory of ${}_R\mathscr{C}$ consisting of n-SG-projective (resp., injective) complexes of left R-modules. The following results extend [11, Proposition 3.4 (2) and Theorem 3.5] to that of complexes.

Proposition 4.2 *Let n and m be positive integers. Then the following hold:*

(1) If $n|m$, then n-SG-Proj($\mathscr{C}$) $\bigcap$ m-SG-Proj($\mathscr{C}$) = n-SG-Proj($\mathscr{C}$).
(2) If $n \nmid m$ and $m = kn + j$, where k is a positive integer and $0 < j < n$, then n-SG-Proj($\mathscr{C}$) $\bigcap$ m-SG-Proj($\mathscr{C}$) $\subseteq$ j-SG-Proj($\mathscr{C}$).

Proof (1) It follows from Lemma 4.1.

(2) By Lemma 4.1, we have that m-SG-Proj($\mathscr{C}$) $\bigcap$ n-SG-Proj($\mathscr{C}$) $\subseteq$ m-SG-Proj($\mathscr{C}$) $\bigcap$ kn-SG-Proj($\mathscr{C}$). Suppose that a complex G is in m-SG-Proj($\mathscr{C}$) $\bigcap$ kn-SG-Proj($\mathscr{C}$). Then there exists an exact sequence of complexes

$$\mathbb{P} : 0 \to G \to P_m \to \cdots \to P_2 \to P_1 \to 0$$

with P_i projective for any $1 \leq i \leq m$. Put $L_i = \operatorname{Ker}(P_i \to P_{i-1})$ for any $2 \leq i \leq m$. Since G is kn-SG-projective, we see that G and L_{kn} are projectively equivalent, i.e., there exist projective complexes P and Q in $\mathscr{C}$ such that $G \oplus P \cong Q \oplus L_{kn}$.

Now consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Q & = & Q & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & L_{kn+1} & \to & X & \to & G \oplus P & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & L_{kn+1} & \to & P_{kn} & \to & L_{kn} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Then X is a projective complex. Next, consider the following pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{kn+1} & \longrightarrow & Y & \longrightarrow & G & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{kn+1} & \longrightarrow & X & \longrightarrow & G \oplus P & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & P & = & P & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Hence Y is also projective. Combining the exact sequence $\mathbb{P}$ and the first row in the above diagram, we get the following exact sequence of complexes

$$0 \to G \to P_m \to \cdots \to P_{kn+1} \to Y \to G \to 0$$

such that $\mathrm{Hom}(-, Q')$ leaves the sequence exact for any projective complex Q'. Thus G is j-SG-projective and hence n-SG-Proj$(\mathscr{C}) \bigcap m$-SG-Proj$(\mathscr{C}) \subseteq j$-$SG$-Proj$(\mathscr{C})$. □

Dually, we have the following result for n-SG-injective complexes.

Proposition 4.3 *Let n and m be positive integers. Then the following hold:*

(1) If $n|m$, then n-SG-Inj$(\mathscr{C}) \bigcap m$-SG-Inj$(\mathscr{C}) = n$-SG-Inj$(\mathscr{C})$.
(2) If $n \nmid m$ and $m = kn + j$, where k is a positive integer and $0 < j < n$, then n-SG-Inj$(\mathscr{C}) \bigcap m$-SG-Inj$(\mathscr{C}) \subseteq j$-SG-Inj$(\mathscr{C})$.

For any two positive integers m and n, we use (m, n) (resp., $[m, n]$) to denote the greatest common divisor (resp., least common multiple) of m and n.

Proposition 4.4 *For any two positive integers m and n, we have the following:*

(1) m-SG-Proj$(\mathscr{C}) \bigcap n$-SG-Proj$(\mathscr{C}) = (m, n)$-SG-Proj$(\mathscr{C})$.
(2) m-SG-Proj$(\mathscr{C}) \bigcap (m+1)$-SG-Proj$(\mathscr{C}) = 1$-SG-Proj$(\mathscr{C})$.

Proof (1) If $n|m$, then the result follows from Proposition 4.3 (1). Now suppose $n \nmid m$ and $m = k_0 n + j_0$, where k_0 is a positive integer and $0 < j_0 < n$. By Proposition 4.3 (2), we have that m-SG-Proj$(\mathscr{C}) \bigcap n$-SG-Proj$(\mathscr{C}) \subseteq j_0$-$SG$-Proj$(\mathscr{C})$. If $j_0 \nmid n$ and $n = k_1 j_0 + j_1$, with $0 < j_1 < j_0$, then by Proposition 4.3 (2) again, we have that m-SG-Proj$(\mathscr{C}) \bigcap n$-SG-Proj$(\mathscr{C}) \subseteq n$-$SG$-Proj$(\mathscr{C}) \bigcap j_0$-$SG$-Proj$(\mathscr{C})$ $\subseteq j_1$-SG-Proj$(\mathscr{C})$. Continuing the process, after finite steps, there exists a positive integer t such that $j_t = k_{t+2} j_{t+1}$ and $j_{t+1} = (m, n)$. Thus m-SG-Proj$(\mathscr{C}) \bigcap$

n-SG-Proj($\mathscr{C}$) $\subseteq$ j_t-SG-Proj($\mathscr{C}$) $\bigcap$ j_{t+1}-SG-Proj($\mathscr{C}$) $=$ j_{t+1}-SG-Proj($\mathscr{C}$)=(m, n)-SG-Proj($\mathscr{C}$). Then the result follows from the fact that (m, n)-SG-Proj($\mathscr{C}$) $\subseteq$ m-SG-Proj($\mathscr{C}$) $\bigcap$ n-SG-Proj($\mathscr{C}$).

(2) It follows from (1). □

Corollary 4.5 *For any two positive integers m and n, we have the following: m-SG-Proj($\mathscr{C}$) $\bigcup$ n-SG-Proj($\mathscr{C}$) $\subseteq$ $[m, n]$-SG-Proj($\mathscr{C}$).*

Proof It is clear from the fact that every n-SG-projective complex is m-SG-projective whenever $n|m$. □

For the case of n-SG-injective complexes, we have the following.

Proposition 4.6 *For any two positive integers m and n, we have the following:*

(1) m-SG-Inj($\mathscr{C}$) $\bigcap$ n-SG-Inj($\mathscr{C}$) $=$ (m, n)-SG-Inj($\mathscr{C}$).
(2) m-SG-Inj($\mathscr{C}$) $\bigcap$ $(m + 1)$-SG-Inj($\mathscr{C}$) $=$ 1-SG-Inj($\mathscr{C}$).

Proof The proof is similar to Proposition 4.4. □

Corollary 4.7 *For any two positive integers m and n, we have the following: m-SG-Inj($\mathscr{C}$) $\bigcup$ n-SG-Inj($\mathscr{C}$) $\subseteq$ $[m, n]$-SG-Inj($\mathscr{C}$).*

Proof It is clear from the fact that every n-SG-injective complex is m-SG-injective whenever $n|m$. □

References

1. Anderson, F.W., Fuller, K.R.: Rings and Categories of Modules. Graduate Texts in Mathematics, vol. 13, 2nd edn. Springer, New York (1992)
2. Bennis, D., Mahdou, N.: Strongly Gorenstein projective, injective and flat modules. J. Pure Appl. Algebra **210**, 437–445 (2007)
3. Bennis, D., Mahdou, N.: A generalization of strongly Gorenstein projective modules. J. Algebra Appl. **8**, 219–227 (2009)
4. Enochs, E., García Rozas, J.R.: Tensor products of complexes. Math. J. Okayama Univ. **39**, 17–39 (1997)
5. Enochs, E., Jenda, O.M.G.: Relative Homological Algebra. de Gruyter Expositions in Mathematics, vol. 30. Walter de Gruyter, Berlin (2000)
6. García Rozas, J.R.: Covers and Envelopes in the Category of Complexes of Modules. CRC Press, Boca Raton (1999)
7. Selvaraj, C., Saravanan, R.: SG-flat complexes and their generalizations, communicated to Vietnam J. Math
8. Yang, X., Liu, Z.: Strongly Gorenstein projective, injective and flat modules. J. Algebra **320**, 2659–2674 (2008)
9. Yang, X., Liu, Z.: Gorenstein projective, injective and flat complexes. J. Commut. Algebra **39**, 1705–1721 (2011)
10. Zhang, D., Ouyang, B.: Strongly Gorenstein projective and injective complexes. J. Nat. Sci. Hunan Norm. Univ. **35**, 1–5 (2012)
11. Zhao, G., Huang, Z.: n-strongly Gorenstein projective, injective and flat modules. Commun. Algebra **39**, 3044–3062 (2011)