Products of Generalized Semiderivations of Prime Near Rings

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Abstract Let N be a near ring. An additive mapping $F : N \longrightarrow N$ is said to be a generalized semiderivation on N if there exists a semiderivation $d : N \longrightarrow N$ associated with a function $g : N \longrightarrow N$ such that $F(xy) = F(x)y + g(x)d(y) = d(x)g(y) + xF(y)$ and $F(g(x)) = g(F(x))$ for all $x, y \in N$. The purpose of the present paper is to prove some theorems in the setting of semigroup ideal of a 3-prime near ring admitting a pair of suitably-constrained generalized semiderivations, thereby extending some known results on derivations and generalized derivations. We show that if N is 2-torsion free and F_1 and F_2 are generalized semiderivations such that $F_1F_2 = 0$, then $F_1 = 0$ or $F_2 = 0$; we prove other theorems asserting triviality of F_1 or F_2; and we also prove some commutativity theorems.

Keywords 3-prime near-rings · Semiderivations · Generalized semiderivations

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1 Introduction

Throughout the paper, N denotes a zero-symmetric left near ring with multiplicative centre Z; and for any pair of elements $x, y \in N$, $[x, y]$ denotes the commutator $xy - yx$. A near ring N is called zero-symmetric if $0x = 0$, for all $x \in N$ (recall that left distributivity yields that $x0 = 0$). The near ring N is said to be 3-prime if $xNy = \{0\}$ for $x, y \in N$ implies that $x = 0$ or $y = 0$. A near ring N is called 2-torsion free if $(N, +)$ has no element of order 2. A nonempty subset U of N is

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called a semigroup right (resp. semigroup left) ideal if $UN \subseteq U$ (resp. $NU \subseteq U$); and if U is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal. An additive mapping $f : N \longrightarrow N$ is said to be a right (resp. left) generalized derivation with associated derivation D if $f(xy) = f(x)y + xD(y)$ (resp. $f(xy) = D(x)y + xf(y)$), for all $x, y \in N$, and f is said to be a generalized derivation with associated derivation D on N if it is both a right generalized derivation and a left generalized derivation on N with associated derivation D. Motivated by a definition given by Bergen [5] for rings, we define an additive mapping $d : N \longrightarrow N$ is said to be a semiderivation on a near ring N if there exists a function $g : N \longrightarrow N$ such that $(i)\, d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $(ii)\, d(g(x)) = g(d(x))$, for all $x, y \in N$. In case g is the identity map on N, d is of course just a derivation on N, so the notion of semiderivation generalizes that of derivation. But the generalization is not trivial for example take $N = N_1 \oplus N_2$, where N_1 is a zero symmetric near ring and N_2 is a ring. Then the map $d : N \longrightarrow N$ defined by $d((x, y)) = (0, y)$ is a semiderivation associated with function $g : N \longrightarrow N$ such that $g(x, y) = (x, 0)$. However d is not a derivation on N. An additive mapping $F : N \to N$ is said to be a generalized semiderivation of N if there exists a semiderivation $d : N \longrightarrow N$ associated with a map $g : N \longrightarrow N$ such that $(i)\, F(xy) = F(x)y + g(x)d(y) = d(x)g(y) + xF(y)$ and $(ii)\, F(g(x)) = g(F(x))$ for all $x, y \in N$. All semiderivations are generalized semiderivations. If g is the identity map on N, then all generalized semiderivations are merely generalized derivations, again the notion of generalized semiderivation generalizes that of generalized derivation. Moreover, the generalization is not trivial as the following example shows:

Example 1.1 Let S be a 2-torsion free left near ring and let

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \in S \right\}.$$

Define maps $F, d, g : N \to N$ by

$$F\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & xy & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}$$

and

$$g\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be verified that N is a left near ring and F is a generalized semiderivation with associated semiderivation d and a map g associated with d. However F is not a generalized derivation on N.

2 Preliminary Results

We begin with several lemmas, most of which have been proved elsewhere.

Lemma 2.1 ([2, Lemmas 1.2 and 1.3]) *Let N be a 3-prime near ring.*

(i) If $z \in Z\backslash\{0\}$, then z is not a zero divisor.
(ii) If $Z\backslash\{0\}$ and x is an element of N for which $xz \in Z$, then $x \in Z$.
(iii) If x is an element of N which centralizes some nonzero semigroup right ideal, then $x \in Z$.
(iv) If $Z\backslash\{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.

Lemma 2.2 ([2, Lemmas 1.3 and 1.4]) *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N.*

(i) If $x \in N$ and $xU = \{0\}$, or $Ux = \{0\}$, then $x = 0$.
(ii) If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.

Lemma 2.3 ([2, Lemma 1.5]) *If N is a 3-prime near ring and Z contains a nonzero semigroup left ideal or a nonzero semigroup right ideal, then N is a commutative ring.*

Lemma 2.4 ([4, Lemma 2.4]) *Let N be an arbitrary near ring. Let S and T be non empty subsets of N such that $st = -ts$ for all $s \in S$ and $t \in T$. If $a, b \in S$ and c is an element of T for which $-c \in T$, then $(ab)c = c(ab)$.*

Lemma 2.5 *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. If N admits a nonzero semiderivation d of N associated with a map g, then $d \neq 0$ on U.*

Proof Let $d(u) = 0$, for all $u \in U$. Replacing u by xu, we get $d(xu) = 0$, for all $x \in N$ and $u \in U$. Thus $d(x)g(u) + xd(u) = 0$, for all $x \in N$ and $u \in U$, i.e., $d(x)g(u) = 0$. The result follows by Lemma 2.2(i).

Lemma 2.6 *Let N be a 3-prime near ring admitting a nonzero semiderivation d with a map g such that $g(xy) = g(x)g(y)$ for all $x, y \in N$. Then N satisfies the following partial distributive law:*

$$(d(x)y + g(x)d(y))z = d(x)yz + g(x)d(y)z \text{ for all } x, y, z \in N.$$

Proof Let $x, y, z \in N$, by defining d we have

$$\begin{aligned} d(xyz) &= d(xy)z + g(xy)d(z) \\ &= (d(x)y + g(x)d(y))z + g(x)g(y)d(z). \end{aligned} \tag{2.1}$$

On the other hand,

$$\begin{aligned} d(xyz) &= d(x)yz + g(x)d(yz) \\ &= d(x)yz + g(x)(d(y)z + g(y)d(z)) \\ &= d(x)yz + g(x)d(y)z + g(x)g(y)d(z). \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$\begin{aligned} &(d(x)y + g(x)d(y))z + g(x)g(y)d(z) \\ &\quad = d(x)yz + g(x)d(y)z + g(x)g(y)d(z) \text{ for all } x, y, z \in N. \end{aligned}$$

$$(d(x)y + g(x)d(y))z = d(x)yz + g(x)d(y)z \text{ for all } x, y, z \in N.$$

Lemma 2.7 *Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. If d is a nonzero semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$, for all $u, v \in U$. If $a \in N$ and $ad(U) = \{0\}$ (or $d(U)a = \{0\}$), then $a = 0$.*

Proof Let $ad(u) = 0$, for all $u \in U$. Replacing u by uv, $a(d(u)g(v) + ud(v)) = 0$, for all $u, v \in U$. Thus $ad(u)g(v) + aud(v) = 0$, for all $u, v \in U$ or $aud(v) = 0$, for all $u, v \in U$. Choosing v such that $d(v) \neq 0$ and applying Lemma 2.2(ii), we get $a = 0$.

Lemma 2.8 *Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that d is a semiderivation on N associated with a map g such that $g(U) = U$. If $d^2(U) = \{0\}$, then $d = 0$.*

Proof Suppose $d^2(U) = \{0\}$. Then for $u, v \in U$ exploit the definition of d in different ways to obtain

$$0 = d^2(uv) = d(d(uv)) = d(d(u)v + g(u)d(v)) \text{ for all } u, v \in U,$$

$$= d^2(u)v + g(d(u))d(v) + d(g(u))d(v) + g(u)d^2(v),$$

$$= d(g(u))d(v) + d(g(u))d(v).$$

Note that $g(d(u)) = d(g(u))$ and $g(U) = U$, we get

$$2d(u)d(v) = 0 \text{ for all } u, v \in U.$$

Since N is a 2-torsion free, we get

$$d(u)d(v) = 0 \text{ for all } u, v \in U.$$

Replacing v by wv in the above relation, we get

$$d(u)d(wv) = 0 \text{ for all } u, v, w \in U.$$

$$d(u)(d(w)v + g(w)d(v)) = 0 \text{ for all } u, v, w \in U.$$

$$d(u)d(w)v + d(u)g(w)d(v) = 0 \text{ for all } u, v, w \in U.$$

This implies that

$$d(u)g(w)d(v) = 0 \text{ for all } u, v, w \in U.$$

$$d(u)wd(v) = 0 \text{ for all } u, v, w \in U.$$

$$d(U)Ud(U) = \{0\}.$$

Thus we obtain that $d = 0$ on U by Lemma 2.2(ii).

Lemma 2.9 *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose d is a nonzero semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$, for all $u, v \in U$. If $d(U) \subseteq Z$, then N is a commutative ring.*

Proof We begin by showing that $(N, +)$ is abelian, which by Lemma 2.1(iv) is accomplished by producing $z \in Z\backslash\{0\}$ such that $z + z \in Z$. Let a be an element of U such that $d(a) \neq 0$. Then for all $x \in N$, $ax \in U$ and $ax + ax = a(x + x) \in U$, so that $d(ax) \in Z$ and $d(ax) + d(ax) \in Z$; hence we need only show that there exists $x \in N$ such that $d(ax) \neq 0$. Suppose this is not the case, so that $d((ax)a) = 0 = d(ax)g(a) + axd(a) = axd(a)$ for all $x \in N$. Since $d(a)$ is not zero divisor by Lemma 2.1(i), we get $aN = \{0\}$, so that $a = 0$—a contradiction. Therefore $(N, +)$ is abelian as required.

We are given that $[d(u), x] = 0$ for all $u \in U$ and $x \in N$. Replacing u by uv, we get $[d(uv), x] = 0$, which yields $[d(u)v + g(u)d(v), x] = 0$ for all $u, v \in U$ and $x \in N$. Since $(N, +)$ is abelian and $d(U) \subseteq Z$, we have

$$d(u)[v, x] + d(v)[x, g(u)] = 0 \text{ for all } u, v \in U \text{ and } x \in N. \tag{2.3}$$

Replacing x by $g(u)$, we obtain $d(u)[v, g(u)] = 0$ for all $u, v \in U$; and choosing $u \in U$ such that $d(u) \neq 0$ and applying Lemma 2.1(iii), we get $g(u) \in Z$. It then follows from (2.3) that $d(u)[v, x] = 0$ for all $v \in U$ and $x \in N$; therefore $U \subseteq Z$ and Lemma 2.3 completes the proof.

Lemma 2.10 *Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that d is a nonzero semiderivation of N associated with a map g such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $[d(U), d(U)] = \{0\}$, then N is a commutative ring.*

Proof By hypothesis $[d(U), d(U)] = \{0\}$. Thus $d(u)d(vd(w)) = d(vd(w))d(u)$, for all $u, v, w \in U$, i.e., $d(u)(d(v)g(d(w)) + vd^2(w)) = (d(v)g(d(w)) + vd^2(w))$ $d(u)$, for all $u, v, w \in U$. Then by Lemma 2.6, we get $d(u)d(v)g(d(w)) + d(u)$ $vd^2(w) = d(v)g(d(w))d(u) + vd^2(w)d(u)$. This implies that $d(u)d(v)d(g(w)) + d(u)vd^2(w) = d(v)d(g(w))d(u) + vd^2(w)d(u)$ i.e., $d(u)d(v)d(w) + d(u)vd^2$ $(w) = d(v)d(w)d(u) + vd^2(w)d(u)$ for all $u, v, w \in U$ and since $[d(U), d(U)] = \{0\}$, we obtain

$$d(u)vd^2(w) = vd^2(w)d(u) \text{ for all } u, v, w \in U. \tag{2.4}$$

Replace v by xv, to get

$$d(u)xvd^2(w) = xvd^2(w)d(u) \text{ for all } u, v, w \in U \text{ and } x \in N.$$

Using (2.4), the above relation yields that $d(u)xvd^2(w) = xd(u)vd^2(w)$, for all $u, v, w \in U$ and $x \in N$, i.e., $[d(u), x]vd^2(w) = 0$, for all $u, v, w \in U$ and $x \in N$ by Lemma 2.6. Thus $[d(u), x]Ud^2(w) = 0$, for all $u, w \in U$ and $x \in N$. Since $d^2(U) \neq 0$ by Lemma 2.8, Lemma 2.2(ii) gives $d(U) \subseteq Z$, and the result follows by Lemma 2.9.

Lemma 2.11 *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. If F is a nonzero generalized semiderivation of N with associated semiderivation d and a map g associated with d such that $g(U) = U$, then $F \neq 0$ on U.*

Proof Let $F(u) = 0$ for all $u \in U$. Replacing u by ux, we get $F(ux) = 0$ for all $u \in U$ and $x \in N$. Thus

$$F(u)x + g(u)d(x) = 0 = Ud(x) \text{ for all } x \in N$$

and it follows by Lemma 2.2(i) that $d = 0$. Therefore, we have

$$F(xu) = F(x)u = 0 \text{ for all } u \in U \text{ for all } x \in N$$

and another appeal to Lemma 2.2(i) gives $F = 0$, which is a contradiction.

Lemma 2.12 *Let N be a 3-prime near ring admitting a generalized semiderivation F associated with a semiderivation d. If g is an onto map associated with d such that $g(xy) = g(x)g(y)$ for all $x, y \in N$, then N satisfies the following partial distributive laws:*

(i) $(F(x)y + g(x)d(y))z = F(x)yz + g(x)d(y)z$ *for all* $x, y, z \in N$.
(ii) $(d(x)g(y) + xF(y))z = d(x)g(y)z + xF(y)z$ *for all* $x, y, z \in N$.

Proof (i) Let $x, y, z \in N$,

$$\begin{aligned} F(xyz) &= F(xy)z + g(xy)d(z) \\ &= (F(x)y + g(x)d(y))z + g(x)g(y)d(z). \end{aligned}$$

On the other hand,

$$\begin{aligned} F(xyz) &= F(x)yz + g(x)d(yz) \\ &= F(x)yz + g(x)(d(y)z + g(y)d(z)) \\ &= F(x)yz + g(x)d(y)z + g(x)g(y)d(z). \end{aligned}$$

Combining both expressions of $F(xyz)$, we obtain

$$(F(x)y + g(x)d(y))z = F(x)yz + g(x)d(y)z \text{ for all } x, y, z \in N.$$

(ii) For all $x, y, z \in N$ we have $F((xy)z) = F(xy)z + g(xy)d(z) = (d(x)g(y) + xF(y))z + g(x)g(y)d(z)$ and $F(x(yz)) = d(x)g(yz) + xF(yz) = d(x)g(y)g(z) + x(F(y)z + g(y)d(z)) = d(x)g(y)z + xF(y)z + g(x)g(y)d(z)$. Comparing the two expression, we get the required result.

Lemma 2.13 *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that F is a nonzero generalized semiderivation of N with associated semiderivation d and a map g associated with d such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $a \in N$ and $aF(U) = 0$ (or $F(U)a = 0$), then $a = 0$.*

Proof Suppose that $aF(U) = \{0\}$. Then for $u, v \in U$

$$aF(uv) = aF(u)v + ag(u)d(v) = aud(v) = 0 \text{ for all } u, v \in U \text{ and } a \in N.$$

So by Lemma 2.2(ii), $a = 0$ or $d(U) = \{0\}$. If $d(U) = \{0\}$, then

$$ad(u)g(v) + auF(v) = 0 = auF(v) \text{ for all } u, v \in U;$$

and since $F(U) \neq \{0\}$ by Lemma 2.11, $a = 0$.

Lemma 2.14 *Let N be a 3-prime near ring admitting a generalized semiderivation F associated with a semiderivation d and an additive map g associated with d. Then N satisfies the following laws:*

(i) $d(x)y + g(x)d(y) = g(x)d(y) + d(x)y$ *for all* $x, y \in N$.
(ii) $d(x)g(y) + xd(y) = xd(y) + d(x)g(y)$ *for all* $x, y \in N$.
(iii) $F(x)y + g(x)d(y) = g(x)d(y) + F(x)y$ *for all* $x, y \in N$.
(iv) $d(x)g(y) + xF(y) = xF(y) + d(x)g(y)$ *for all* $x, y \in N$.

Proof (i) $d(x(y + y)) = d(x)(y + y) + g(x)d(y + y) = d(x)y + d(x)y + g(x)d(y) + g(x)d(y)$, and $d(xy + xy) = d(xy) + d(xy) = d(x)y + g(x)d(y) + d(x)y + g(x)d(y)$. Comparing these two equations, we get the desired result.
(ii) Again, calculate $d((x + x)y)$ and $d(xy + xy)$ and compare.

(iii) $F(x(y + y)) = F(x)(y + y) + g(x)d(y + y) = F(x)y + F(x)y + g(x)d(y) + g(x)d(y)$, and $F(xy + xy) = F(x)y + g(x)d(y) + F(x)y + g(x)d(y)$. Comparing these two equations, we get the desired result.
(iv) Again, calculate $F((x + x)y)$ and $F(xy + xy)$ and compare.

Lemma 2.15 *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that N admits nonzero semiderivations d_1, d_2 associated with a map g such that $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $d_1(x)d_2(y) + d_2(y)d_1(x) \in Z$ for all $x, y \in U$ and at least one of $d_1(U) \cap Z$ and $d_2(U) \cap Z$ is nonzero, then N is a commutative ring.*

Proof Assume that $d_1(U) \cap Z \neq \{0\}$. Let $x \in U$ such that $d_1(x) \in Z\backslash\{0\}$, and $y \in U$. Then $d_1(x)d_2(y) + d_2(y)d_1(x) = d_1(x)(2d_2(y)) = d_1(x)(d_2(2y)) \in Z$. Therefore, $d_2(2U) \subseteq Z$. Since $2U$ is nonzero semigroup left ideal, our conclusion follows by Lemma 2.9, then N is commutative ring.

Lemma 2.16 *Let N be a 2-torsion free 3-prime near ring. If U is a nonzero semigroup ideal of N, then $2U \neq \{0\}$ and $d(2U) \neq \{0\}$ for any nonzero semiderivation d associated with a map g such that $g(U) = U$.*

Proof Let $x \in N$ with $x + x \neq 0$. Then for every $u \in U$, $u(x + x) = ux + ux \in 2U$; and by Lemma 2.2(i), we get $\{0\} \neq U(x + x) \subseteq 2U$. Since $2U$ is a semigroup left ideal, it follows by Lemma 2.5 that $d(2U) \neq \{0\}$.

Lemma 2.17 *Let N be a 3-prime near ring. If F is a generalized semiderivation with associated semiderivation d and a map g associated with d such that $g(U) = U$, then $F(Z) \subseteq Z$.*

Proof Let $z \in Z$ and $x \in N$. Then F(zx) = F(xz); that is $F(z)x + g(z)d(x) = d(x)g(z) + xF(z)$. Applying Lemma 2.14(iii), we get $g(z)d(x) + F(z)x = d(x)g(z) + xF(z)$; $zd(x) + F(z)x = d(x)z + xF(z)$. It follows that $F(z)x = xF(z)$ for all $x \in N$, so $F(Z) \subseteq Z$.

Lemma 2.18 *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that N admits a semiderivation d associated with a map g such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $d^2(U) \neq \{0\}$ and $a \in N$ such that $[a, d(U)] = \{0\}$, then $a \in Z$.*

Proof Let $C(a) = \{x \in N | ax = xa\}$. Note that $d(U) \subseteq C(a)$. Thus, if $y \in C(a)$ and $u \in U$, both $d(yu)$ and $d(u)$ are in $C(a)$; hence $(d(y)g(u) + yd(u))a = a(d(y)g(u) + yd(u))$ and $d(y)g(u)a + yd(u)a = ad(y)g(u) + ayd(u)$; $d(y)ua + yd(u)a = ad(y)u + ayd(u)$. Since $yd(u) \in C(a)$, we conclude that $d(y)ua = ad(y)u$. Thus

$$d(C(a))U \subseteq C(a). \tag{2.5}$$

Choosing $z \in U$ such that $d^2(z) \neq 0$, and let $y = d(z)$. Then $y \in C(a)$; and by (2.5), $d(y)u \in C(a)$ and $d(y)uv \in C(a)$ for all $u, v \in U$. Thus, $0 = [a, d(y)uv] = ad(y)uv - d(y)uva = d(y)uav - d(y)uva = d(y)u(av - va)$. Thus $d(y)U(av - va) = 0$ for all $v \in U$; and by Lemma 2.2(ii), a centralizes U. By Lemma 2.1(iii), $a \in Z$.

Lemma 2.19 *Let N be a 3-prime near ring and F be a generalized semiderivation of N with associated nonzero semiderivation d and a map g associated with d such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $d(F(N)) = \{0\}$, then $d^2(x)d(y) + d(x)d^2(y) = 0$ for all $x, y \in N$ and $F(d(N)) = \{0\}$.*

Proof Assume that $d(F(x)) = 0$ for all $x \in N$. It follows that $d(F(xy)) = d(F(x)y) + d(g(x)d(y)) = d(F(x)y) + d(xd(y)) = 0$ for all $x, y \in N$, that is,

$$d(F(x))g(y) + F(x)d(y) + d(x)g(d(y)) + xd^2(y) = 0 \text{ for all } x, y \in N.$$

This implies that

$$F(x)d(y) + d(x)d(g(y)) + xd^2(y) = 0.$$

$$F(x)d(y) + d(x)d(y) + xd^2(y) = 0 \text{ for all } x, y \in N. \tag{2.6}$$

Applying d again, we get

$$F(x)d^2(y) + d^2(x)d(y) + d(x)d^2(y) + d(x)d^2(y) + xd^3(y) = 0 \text{ for all } x, y \in N. \tag{2.7}$$

Taking $d(y)$ instead of y in (2.6) gives $F(x)d^2(y) + d(x)d^2(y) + xd^3(y) = 0$, hence (2.7) yields

$$d^2(x)d(y) + d(x)d^2(y) = 0 \text{ for all } x, y \in N. \tag{2.8}$$

Now, substitute $d(x)$ for x in (2.6), to obtain $F(d(x))d(y) + d^2(x)d(y) + d(x)d^2(y) = 0$; and use (2.8) to conclude that $F(d(x))d(y) = 0$ for all $x, y \in N$. Thus, by Lemma 2.7, $F(d(x)) = 0$ for all $x \in N$.

Lemma 2.20 *Let N be a 2-torsion free 3-prime near ring and F be a nonzero generalized semiderivation of N with associated semiderivation d and a map g associated with d such that $g(U) = U$; $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $F(V) \subseteq U$ for some nonzero semigroup ideal V contained in U. If $a \in N$ and $[a, F(U)] = \{0\}$, then $a \in Z$.*

Proof If $d = 0$, then for all $x \in U$ and $y \in N$, $aF(x)y = F(x)ya$; hence $F(U)[a, y] = \{0\}$ and $a \in Z$ by Lemma 2.13. Therefore, we may assume $d \neq 0$. Let $C(a)$ denotes the centralizer of a, and let $y \in C(a)$ for all $u \in U$, $F(yu) \in C(a)$ -i.e. $(d(y)g(u) + yF(u))a = a(d(y)g(u) + yF(u))$ and by Lemma 2.12(ii) $d(y)g(u)a + yF(u)a = ad(y)g(u) + ayF(u)$; $d(y)ua + yF(u)a = ad(y)u + ayF(u)$. Now $yF(u)a = ayF(u)$, and it follows that $d(y)u \in C(a)$; therefore $d(C(a))U$ is a semigroup right ideal which centralizes a, and if $d(C(a))U \neq \{0\}$. Lemma 2.1(iii) yields $a \in Z$. Assume now that $d(C(a))U = \{0\}$, in which case $d(C(a)) = \{0\}$ and hence $d(F(U)) = \{0\}$. It follows that for all $x \in N$ and $v \in V$, $d(F(xF(v))) = 0 = d(F(x)F(v) + g(x)d(F(v))) = d(F(x)F(v)) = d(F(x))g(F(v)) + F(x)d(F(v)) = d(F(x))F(v)$, so that $d(F(N))F(V) = \{0\}$ and by Lemma 2.13, $d(F(N)) = \{0\}$. By Lemma 2.19

$$d^2(x)d(y) + d(x)d^2(y) = 0 \text{ for all } x, y \in N \text{ and } F(d(N)) = \{0\}. \tag{2.9}$$

As in the proof of Theorem 4.1 of [3], we calculate $F(d(x)d(y))$ in two ways, obtaining $F(d(x)d(y)) = F(d(x))d(y) + g(d(x))d^2(y) = d(g(x))d^2(y) = d(x)d^2(y)$ and $F(d(x)d(y)) = d^2(x)g(d(y)) + d(x)F(d(y)) = d^2(x)d(g(y)) = d^2(x)d(y)$. Comparing the two results, we get $d(x)d^2(y) = d^2(x)d(y)$ for all $x, y \in N$, which together with (2.9) gives $d^2(x)d(y) = 0$ for all $x, y \in N$ and hence $d^2 = 0$. But by Lemma 2.8, this contradicts our assumption that $d \neq 0$; thus $d(C(a))U \neq \{0\}$ and our proof is complete.

3 Some Results Involving Two Generalized Semiderivations

The theorems that we prove in this section extend the results proved in [4].

Theorem 3.1 *Let N be a 3-prime near ring and U be a nonzero semigroup ideal of N. Suppose that N admits a nonzero generalized semiderivation F with associated semiderivation d and a map g associated with d such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $F(U) \subseteq Z$, then $(N, +)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.*

Proof We begin by showing that $(N, +)$ is abelian, which by Lemma 2.1(iv) is accomplished by producing $z \in Z\backslash\{0\}$ such that $z + z \in Z$. Let a be an element of U such that $F(a) \neq 0$. Then for all $x \in N$, $ax \in U$ and $ax + ax = a(x + x) \in U$, so that $F(ax) \in Z$ and $F(ax) + F(ax) \in Z$; hence we need only to show that there exists $x \in N$ such that $F(ax) \neq 0$. Suppose that this is not the case, so that $F((ax)a) = 0 = F(ax)a + g(ax)d(a) = g(a)g(x)d(a) = axd(a)$ for all $x \in N$. By Lemma 2.2(ii) either $a = 0$ or $d(a) = 0$.

If $d(a) = 0$, then $F(xa) = F(x)a + g(x)d(a)$; that is, $F(xa) = F(x)a \in Z$, for all $x \in N$. Thus, $[F(u)a, y] = 0$ for all $y \in N$ and $u \in U$. This implies that $F(u)[a, y] = 0$ for all $u \in U$ and $y \in N$ and Lemma 2.1(i) gives $a \in Z$. Thus, $0 = F(ax) = F(xa) = F(x)a$ for all $x \in N$. Replacing x by $u \in U$, we have $F(U)a = 0$, and by Lemmas 2.1(i) and 2.11, we get $a = 0$. Thus we have a contradiction.

To complete the proof, we show that if N is 2-torsion free, then N is commutative.

Consider first case $d = 0$. This implies that $F(ux) = F(u)x \in Z$ for all $u \in U$ and $x \in N$. By Lemma 2.11, we have $u \in U$ such that $F(u) \in Z\backslash\{0\}$, so N is commutative by Lemma 2.1(ii).

Now consider the case $d \neq 0$. Let $c \in Z\backslash\{0\}$. This implies that $x \in U$, $F(xc) = F(x)c + g(x)d(c) = F(x)c + xd(c) \in Z$. Thus $(F(x)c + xd(c))y = y(F(x)c + xd(c))$ for all $x, y \in U$ and $c \in Z$. Therefore, by Lemma 2.12(i), $F(x)cy + xd(c)y = yF(x)c + yxd(c)$ for all $x, y \in U$ and $c \in Z$. Since $d(c) \in Z$ and $F(x) \in Z$, we obtain $d(c)[x, y] = 0$ for all $x, y \in U$ and $c \in Z$. Let $d(Z) \neq \{0\}$. Choosing c such that $d(c) \neq 0$ and noting that $d(c)$ is not a zero divisor, we have $[x, y] = 0$ for all $x, y \in U$. By Lemma 2.1(iii), $U \subseteq Z$; hence N is commutative by Lemma 2.3.

The remaining case is $d \neq 0$ and $d(Z) = \{0\}$. Suppose we can show that $U \cap Z \neq \{0\}$. Taking $z \in (U \cap Z)\backslash\{0\}$ and $x \in N$, we have $F(xz) = F(x)z \in Z$; therefore $F(N) \subseteq Z$ by Lemma 2.1(ii). Let $F(x) \in Z$ for all $x \in N$.

Since $d(Z) = 0$, for all $x, y \in N$. We have

$$0 = d(F(xy)).$$

$$0 = d(F(x)y + g(x)d(y)).$$

$$0 = F(x)d(y) + g(x)d^2(y) + d(g(x))g(d(y)) \text{ for all } x, y \in Z.$$

Hence $F(xd(y)) = -d(g(x))g(d(y)) \in Z$ for all $x, y \in N$. By hypothesis, we have $d(x)d(y) \in Z$ for all $x, y \in N$. This implies that

$$d(x)(d(x)d(y) - d(y)d(x)) = 0 \text{ for all } x, y \in N.$$

Left multiplying by $d(y)$, we arrive at

$$d(y)d(x)N(d(x)d(y) - d(y)d(x)) = \{0\} \text{ for all } x, y \in N.$$

Since N is a 3-prime near ring, we get

$$[d(x), d(y)] = 0 \text{ for all } x, y \in N.$$

Using Lemma 2.10, N is a commutative ring.

Assume that $U \cap Z = \{0\}$. For each $u \in U$, $F(u^2) = F(u)u + g(u)d(u) = F(u)u + ud(u) = u(F(u) + d(u)) \in U \cap Z$. So $F(u^2) = 0$, thus for all $u \in U$ and $x \in N$, $F(u^2x) = F(u^2)x + g(u^2)d(x) = u^2d(x) \in U \cap Z$. So $u^2d(x) = 0$ and Lemma 2.7, $u^2 = 0$. Since $F(xu) = F(x)u + g(x)d(u) = F(x)u + xd(u) \in Z$ for all $u \in U$ and $x \in N$. We have $(F(x)u + xd(u))u = u(F(x)u + xd(u))$ and right multipling by u gives $uxd(u)u = 0$. Consequently, $d(u)uNd(u)u = \{0\}$. So that $d(u)u = 0$ for all $u \in U$,so $F(u)u = 0$ for all $u \in U$. But by Lemma 2.11, there exist $u_0 \in U$ for which $F(u_0) \neq 0$; and $F(u_0) \in Z$, we get $u_0 = 0$, contradiction. Therefore, $U \cap Z \neq \{0\}$ as required.

Theorem 3.2 *Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. Let F_1 and F_2 be generalized semiderivations on N with associated semiderivations d_1 and d_2 respectively with at least one of d_1, d_2 not zero and a map g associated with d_1 and d_2 such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If $F_1(x)d_2(y) + F_2(x)d_1(y) = 0$ for all $x, y \in U$, then $F_1 = 0$ or $F_2 = 0$.*

Proof By hypothesis

$$F_1(x)d_2(y) + F_2(x)d_1(y) = 0 \text{ for all } x, y \in U. \tag{3.1}$$

Replacing x by uv in (3.1), we get

$$(d_1(u)g(v) + uF_1(v))d_2(y) + (d_2(u)g(v) + uF_2(v))d_1(y) = 0 \text{ for all } u, v, y \in U.$$

Using Lemmas 2.12(ii) and 2.14(iv), we conclude that

$$(d_1(u)g(v) + uF_1(v))d_2(y) + (uF_2(v) + d_2(u)g(v))d_1(y) = 0.$$

$$d_1(u)g(v)d_2(y) + uF_1(v)d_2(y) + uF_2(v)d_1(y) + d_2(u)g(v)d_1(y) = 0.$$

$$d_1(u)vd_2(y) + u(F_1(v)d_2(y) + F_2(v)d_1(y)) + d_2(u)vd_1(y) = 0 \text{ for all } u, v, y \in U.$$

Since middle summand is 0 by (3.1), we conclude that

$$d_1(u)vd_2(y) + d_2(u)vd_1(y) = 0 \text{ for all } u, v, y \in U. \tag{3.2}$$

Substituting yt for y in (3.2), we get

$$d_1(u)vd_2(yt) + d_2(u)vd_1(yt) = 0 \text{ for all } u, v, y, t \in U.$$

$$d_1(u)v(d_2(y)g(t) + yd_2(t)) + d_2(u)v(d_1(y)g(t) + yd_1(t)) = 0.$$

Using Lemma 2.14(ii), we have

$$d_1(u)v(d_2(y)g(t) + yd_2(t)) + d_2(u)v(yd_1(t) + d_1(y)g(t)) = 0.$$

This implies that

$$d_1(u)vd_2(y)t + (d_1(u)vyd_2(t) + d_2(u)vyd_1(t)) + d_2(u)vd_1(y)t = 0.$$

Again the middle summand is 0, so

$$d_1(u)vd_2(y)t + d_2(u)vd_1(y)t = 0 \text{ for all } u, v, y, t \in U. \tag{3.3}$$

Replacing t by $td_1(w)$ in (3.3), where $w \in U$, we have

$$d_1(u)v(d_2(y)td_1(w)) + d_2(u)(vd_1(y)t)d_1(w) = 0 \text{ for all } u, v, y, t, w \in U.$$

Using (3.2), we get

$$d_1(u)v(-d_1(y)td_2(w)) - d_1(u)vd_1(y)td_2(w) = 0.$$

This implies that

$$2d_1(u)vd_1(y)td_2(w) = 0 \text{ for all } u, v, y, t, w \in U.$$

Since N is 2-torsion free, we get

$$d_1(u)vd_1(y)td_2(w) = 0 \text{ for all } u, v, y, t, w \in U.$$

Thus $d_1(U)Ud_1(U)Ud_2(U) = \{0\}$; and by Lemmas 2.2(ii) and 2.5, one of d_1, d_2 must be 0. Assuming without loss that $d_1 = 0$, in which case $d_2 \neq 0$, we get $F_1(U)d_2(U) = \{0\}$, so by Lemmas 2.7 and 2.11, we have $F_1 = 0$.

Theorem 3.3 *Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. Let F_1 and F_2 be generalized semiderivations on N with associated semiderivations d_1 and d_2 respectively and a map g associated with d_1 and d_2 such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If d_1 and d_2 are not both zero and F_1F_2 acts on U as a generalized semiderivation with associated semiderivation d_1d_2 and a map g associated with d_1d_2, then $F_1 = 0$ or $F_2 = 0$.*

Proof By the hypothesis, we have

$$F_1F_2(xy) = F_1F_2(x)y + g(x)d_1d_2(y) \text{ for all } x, y \in U.$$

$$F_1F_2(xy) = F_1F_2(x)y + xd_1d_2(y) \text{ for all } x, y \in U. \tag{3.4}$$

We also have

$$\begin{aligned} F_1F_2(xy) = F_1(F_2(xy)) &= F_1(F_2(x)y + g(x)d_2(y)) \\ &= F_1(F_2(x)y) + F_1(g(x)d_2(y)) \\ &= F_1(F_2(x)y) + F_1(xd_2(y)). \end{aligned}$$

i.e.

$$\begin{aligned} F_1F_2(xy) &= F_1F_2(x)y + g(F_2(x))d_1(y) + F_1(x)d_2(y) + g(x)d_1d_2(y) \\ &= F_1F_2(x)y + F_2(g(x))d_1(y) + F_1(x)d_2(y) + g(x)d_1d_2(y) \\ &= F_1F_2(x)y + F_2(x)d_1(y) + F_1(x)d_2(y) + xd_1d_2(y) \text{ for all } x, y \in U. \end{aligned} \tag{3.5}$$

Comparing (3.4) and (3.5), we get

$$F_2(x)d_1(y) + F_1(x)d_2(y) = 0 \text{ for all } x, y \in U.$$

Hence application of Theorem 3.2 yields that $F_1 = 0$ or $F_2 = 0$.

Theorem 3.4 *Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N. Let F_1 and F_2 be generalized semiderivations on N with associated semiderivations d_1 and d_2 respectively and a map g associated with d_1 and*

d_2 such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If $F_1F_2(U) = \{0\}$, then $F_1 = 0$ or $F_2 = 0$.

Proof By the hypothesis

$$F_1F_2(U) = \{0\}.$$

$$F_1F_2(xy) = F_1(F_2(xy)) = 0 = F_1(F_2(x)y + g(x)d_2(y))$$

$$= F_1(F_2(x)y) + F_1(xd_2(y))$$

$$= F_1F_2(x)y + g(F_2(x))d_1(y) + F_1(x)d_2(y) + g(x)d_1d_2(y)$$

$$= F_2(g(x))d_1(y) + F_1(x)d_2(y) + xd_1d_2(y) \text{ for all } x, y \in U.$$

This implies that

$$F_2(x)d_1(y) + xd_1d_2(y) + F_1(x)d_2(y) = 0 \text{ for all } x, y \in U. \tag{3.6}$$

Replacing x by zx in (3.6), we have

$$F_2(zx)d_1(y) + zxd_1d_2(y) + F_1(zx)d_2(y) = 0 \text{ for all } x, y, z \in U.$$

$$(d_2(z)g(x) + zF_2(x))d_1(y) + zxd_1d_2(y) + (d_1(z)g(x) + zF_1(x))d_2(y) = 0.$$

$$(d_2(z)g(x) + zF_2(x))d_1(y) + zxd_1d_2(y) + (zF_1(x) + d_1(z)g(x))d_2(y) = 0.$$

$$d_2(z)g(x)d_1(y) + zF_2(x)d_1(y) + zxd_1d_2(y) + zF_1(x)d_2(y) + d_1(z)g(x)d_2(y) = 0.$$

$$d_2(z)xd_1(y) + z(F_2(x)d_1(y) + xd_1d_2(y) + F_1(x)d_2(y)) + d_1(z)xd_2(y) = 0.$$

Since the middle summand is 0 by (3.6), we have

$$d_2(z)xd_1(y) + d_1(z)xd_2(y) = 0 \text{ for all } x, y, z \in U.$$

But this is just (3.2) of Theorem 3.2, so we argue as in the proof of Theorem 3.2 that $d_1 = 0$ or $d_2 = 0$. It now follows from (3.6) that

$$F_2(x)d_1(y) + F_1(x)d_2(y) = 0 \text{ for all } x, y \in U.$$

If one of d_1, d_2 is nonzero, then F_1 or F_2 is 0 by Theorem 3.2, so we assume that $d_1 = d_2 = 0$. Then $F_1F_2(xy) = 0 = F_1(F_2(x)y) = F_2(x)F_1(y)$ for all $x, y \in U$, so that $F_2(U)F_1(U) = \{0\}$. Applying Lemma 2.13, we conclude that $F_1 = 0$ or $F_2 = 0$.

We now consider a somewhat different condition that elements of $F_1(U)$ and $F_2(U)$ anti-commute.

Theorem 3.5 *Let N be a 2-torsion free 3-prime near ring with nonzero semigroup ideal U; and let F_1 and F_2 be generalized semiderivations on N with associated semiderivations d_1 and d_2 respectively such that $F_1(U^2) \subseteq U$ and $F_2(U^2) \subseteq U$ and a map g associated with d_1 and d_2 such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. If*

$$F_1(x)F_2(y) + F_2(y)F_1(x) = 0 \text{ for all } x, y \in U, \tag{3.7}$$

then $F_1 = 0$ or $F_2 = 0$.

Proof Assume that $F_1 \neq 0$ and $F_2 \neq 0$. Note that if $w \in F_2(U^2)$, $-w \in F_2(U)$; and apply Lemma 2.4 to get $(uv)w = w(uv)$ for all $u, v \in F_1(U)$ and $w \in F_2(U^2)$. It follows by Lemma 2.20 that $F_1(U)F_1(U) \subseteq Z$, and it is easy to see that

$$F_1(x)F_1(y)(F_1(x)F_1(y) - F_1(y)F_1(x)) = 0 \text{ for all } x, y \in U.$$

This implies that

$$F_1(y)F_1(x)(F_1(x)F_1(y) - F_1(y)F_1(x)) = 0 \text{ for all } x, y \in U.$$

Since $F_1(x)F_1(y)$ and $F_1(y)F_1(x)$ are central, Lemma 2.1(i) shows that either both are zero or one can be cancelled to yield

$$F_1(x)F_1(y) = F_1(y)F_1(x).$$

Thus $[F_1(U), F_1(U)] = \{0\}$ and by Lemma 2.20, $F_1(U) \subseteq Z$, hence N is a commutative ring by Theorem 3.1. This fact together with (3.7) gives $F_1(U)F_2(U) = \{0\}$. Contradicting our assumption that $F_1 \neq 0 \neq F_2$. Therefore $F_1 = 0$ or $F_2 = 0$ as required.

If U is closed under addition, then $F(U^2) \subseteq U$ for any generalized semiderivation F; hence we have

Corollary 3.6 *Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N which is closed under addition. If F_1 and F_2 are generalized semiderivations on N with associated semiderivations d_1 and d_2 respectively and a map g associated with d_1 and d_2 such that $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $g(U) = U$. if*

$$F_1(x)F_2(y) + F_2(y)F_1(x) = 0 \text{ for all } x, y \in U,$$

then $F_1 = 0$ or $F_2 = 0$.

We now replace the hypothesis that $F_1(U) \subseteq U$ and $F_2(U) \subseteq U$ in Theorem 3.5 by some commutativity hypothesis.

Theorem 3.7 *Let N be a 2-torsion free 3-prime near ring with nonzero semigroup ideal U; and let F_1 and F_2 be generalized semiderivations on N with associated semiderivations d_1 and d_2 respectively and a map g associated with d_1 and d_2 such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. If*

$$F_1(x)F_2(y) + F_2(y)F_1(x) = 0 \textit{ for all } x, y \in U,$$

then $F_1 = 0$ *or* $F_2 = 0$ *and one of the following is satisfied: (a)* $d_1(Z) \neq \{0\}$ *and* $d_2(Z) \neq \{0\}$*; (b)* $U \cap Z \neq \{0\}$.

Proof (a) Let $z_1 \in Z$ such that $d_1(z_1) \neq 0$. Then for all $x, y \in U$, we have

$$F_1(z_1x)F_2(y) + F_2(y)F_1(z_1x) = 0.$$

$$(d_1(z_1)g(x) + z_1F_1(x))F_2(y) + F_2(y)(F_1(x)z_1 + g(x)d_1(z_1)) = 0.$$

$$d_1(z_1)g(x)F_2(y) + z_1F_1(x)F_2(y) + F_2(y)F_1(x)z_1 + F_2(y)g(x)d_1(z_1) = 0.$$

$$d_1(z_1)xF_2(y) + z_1(F_1(x)F_2(y) + F_2(y)F_1(x)) + F_2(y)xd_1(z_1) = 0.$$

It follows that

$$d_1(z_1)xF_2(y) + F_2(y)xd_1(z_1) = 0 \text{ for all } x, y \in U.$$

Choosing $z_2 \in Z$ such that $d_2(z_2) \neq 0$ and using a similar argument, we now get

$$xy + yx = 0 \text{ for all } x, y \in U;$$

and applying Lemma 2.4 with $S = U$ and $T = U^2$ shows that U^2 centralizes U^2, so that $U^2 \subseteq Z$ by Lemma 2.1(iii) and hence N is commutative ring by Lemma 2.3. It now follows that $F_1(x)F_2(y) = F_2(y)F_1(x) = -F_2(y)F_1(x)$ for all $x, y \in U$. Hence $F_1(U)F_2(U) = \{0\}$. Therefore $F_1 = 0$ or $F_2 = 0$.
(b) We assume that $F_1 \neq 0$ and $F_2 \neq 0$. Let $z_0 \in (U \cap Z)\backslash\{0\}$. By Lemma 2.17, $F_1(z_0) \in Z$; hence if $F_1(z_0) \neq 0$ the condition

$$F_1(z_0)F_2(x) + F_2(x)F_1(z_0) = 0 \text{ for all } x \in U$$

gives $2F_2(x) = 0 = F_2(x)$ for all $x \in U$, so that $F_1 = 0$ by Lemma 2.11. Therefore, $F_1(z_0) = 0$ and similarly $F_2(z_0) = 0$. Now $z_0^2 \in (U \cap Z)\backslash\{0\}$ also, so $F_1(z_0^2) = 0 = F_2(z_0^2)$; and since $F_1(z_0^2) = F_1(z_0)z_0 + g(z_0)d_1(z_0) = z_0d_1(z_0)$ and $F_2(z_0^2) = F_2(z_0)z_0 + g(z_0)d_2(z_0) = z_0d_2(z_0)$. we have $d_1(z_0) = d_2(z_0) = 0$. Observing that $F_1(z_0x) = F_1(z_0)x + g(z_0)d_1(x) = F_1(z_0)x + z_0d_1(x)$ and $F_1(xz_0) = F_1(x)z_0 + g(x)d_1(z_0) = F_1(x)z_0 + xd_1(z_0)$ for all $x \in N$, we see that $F_1(x) = d_1(x)$ for all $x \in N$, So that F_1 is a semiderivation; and similarly F_2 is a semiderivation. We can now derive a contradiction as in the proof of Theorem 3.5, with Lemmas 2.8 and 2.18 used instead of Lemma 2.20.

4 Some Commutativity Conditions

The skew-commutativity hypothesis of Theorems 3.4 and 3.5 suggests investigating conditions of the form $F_1(x)F_2(y) + F_2(y)F_1(x) \in Z$ or $xF(y) + F(y)x \in Z$.

Theorem 4.1 *Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N which is closed under addition.*
(i) Suppose N has nonzero generalized semiderivations F_1, F_2 with associated semiderivations d_1 and d_2 respectively and a map g associated with d_1 and d_2 such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $F_1(x)F_2(y) + F_2(y)F_1(x) \in Z$, for all $x, y \in U$ and at least one of $F_1(U) \cap Z$ and $F_2(U) \cap Z$ is nonzero, then N is a commutative ring.
(ii) If N admits a nonzero generalized semiderivation F with associated semiderivation d and a map g associated with d such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$ and $U \cap Z \neq \{0\}$ and $xF(y) + F(y)x \in Z$, for all $x, y \in U$, then N is commutative ring.

Proof (i) Assume that $F_1(U) \cap Z \neq \{0\}$. Let $x \in U$ such that $F_1(x) \in Z\backslash\{0\}$. Then $F_1(x)F_2(y) + F_2(y)F_1(x) = 2F_1(x)F_2(y) = F_1(x)F_2(2y) \in Z$ for all $y \in U$. Since $F_1(x) \in Z\backslash\{0\}$, Lemma 2.1(ii) gives $F_2(2y) \in Z$ for all $y \in U$ -i.e. $F_2(2U) \subseteq Z$. Since $0 \in Z$, we get $F_2(2U) = \{0\}$ -i.e. $2F_2(U) = \{0\}$. But N is 2-torsion free, we get $F_2(U) = \{0\}$ would contradict our hypothesis that $F_2 \neq 0$; hence $F_2(2U) \neq \{0\}$ and we may choose $y \in U$ such that $F_2(2y) \in Z\backslash\{0\}$. Since $2U \subseteq U$, this shows that $F_2(2y)$ and $2F_2(2y) = F_2(4y)$ are in $F_2(U) \cap Z\backslash\{0\}$, so that for all $x \in U$, $F_1(x)(2F_2(2y)) \in Z$ and hence $F_1(x) \in Z$. Thus, $F_1(U) \subseteq Z$ and by Theorem 3.1, N is a commutative ring.
(ii) Essentially the same argument yields $U \subseteq Z$, and the result follows by Lemma 2.3.

Theorem 4.2 *Let N be a 2-torsion free 3-prime near ring and U be a nonzero semigroup ideal of N which is closed under addition. Suppose N admits nonzero generalized semiderivations F_1 and F_2 with associated semiderivations d_1 and d_2 respectively and a map g associated with d_1 and d_2 such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. Suppose that $F_1(x)F_2(y) + F_2(y)F_1(x) \in Z$, for all $x, y \in U$ and $F_1(U) \subseteq U$; $F_2(U) \subseteq U$. If $F_1(N) \cap Z \neq \{0\}$ or $F_2(N) \cap Z \neq \{0\}$, then N is a commutative ring.*

Proof By Corollary 3.6, we cannot have $F_1(x)F_2(y) + F_2(y)F_1(x) = 0$ for all $x, y \in U$, hence there exist $x_0, y_0 \in U$ such that $u_0 = F_1(x_0)F_2(y_0) + F_2(y_0) F_1(x_0) \in (Z\backslash\{0\}) \cap U$. Since $F_1(Z)$ and $F_2(Z)$ are central by Lemma 2.17, if $F_1(u_0) \neq 0$ or $F_2(u_0) \neq 0$ we have $F_1(U) \cap Z \neq \{0\}$ or $F_2(U) \cap Z \neq \{0\}$ and our conclusion follows by Theorem 4.1(i).

Assume, therefore, that $F_1(u_0) = F_2(u_0) = 0$. For all $x, y \in U$, $F_1(u_0x)F_2 (u_0y) + F_2(u_0y)F_1(u_0x) = u_0^2(d_1(x)d_2(y) + d_2(y)d_1(x)) \in Z$, hence $d_1(x)d_2(y) + d_2(y)d_1(x) \in Z$; and if $d_1(u_0) \neq 0$ or $d_2(u_0) \neq 0$ our desired conclusion follows by Lemma 2.15. Therefore we may assume $d_1(u_0) = d_2(u_0) = 0$. For all $x, y \in N$, $F_1(xu_0)F_2(yu_0) + F_2(yu_0)F_1(xu_0) \in Z$, so $u_0^2(F_1(x)F_2(y) + F_2(y)F_1(x)) \in Z$

and $F_1(x)F_2(y) + F_2(y)F_1(x) \in Z$. Since $F_1(N) \cap Z \neq \{0\}$ or $F_2(N) \cap Z \neq \{0\}$, our result follows by Theorem 4.1(i).

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