Biorder Ideals and Regular Rings

P.G. Romeo and R. Akhila

Abstract In [4] (Structure of regular semigroups, 1979) K.S.S. Nambooripad introduced biordered sets as a partial algebra (E, ω^r, ω^l) where ω^r and ω^l are two quasiorders on the set E satisfying biorder axioms; to study the structure of a regular semigroup. John von Neumann (Continuous Geometry, 1960 in [5]) described the complemented modular lattice of principle ideals of a regular ring. In this paper, we introduced the biorder ideals of a regular ring and showed that these ideals form a complemented modular lattice.

Keywords Biordered set · Sandwitch set

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In many algebraic systems like semigroups, rings, algebras, the idempotent elements are important structural objects and can be used effectively in analyzing the structure of the algebraic system under consideration. The concept of biordered set was originally introduced by Nambooripad [1972, 1979] to describe the structure of the set of idempotents of a semigroup. He identified a partial binary operation on the set of idempotents $E(S)$ of a semigroup S arising from the binary operation in S. The resulting structure on $E(S)$ involving two quasiorders is abstracted as a biordered set. In this paper, we propose to extend biordered set approach to rings to study the structure of regular rings.

1 Preliminaries

First, we recall some basic definitions regarding semigroups, biordered sets, and rings needed in the sequel. A set S in which for every pair of elements $a, b \in S$ there is an element $a \cdot b \in S$ which is called the product of a by b is called a groupoid.

P.G. Romeo (✉) · R. Akhila
Department of Mathematics, Cochin University of Science and Technology, Kochi, Kerala, India
e-mail: romeo_parackal@yahoo.com

R. Akhila
e-mail: akhila.ravikumar@gmail.com

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A groupoid S is a semigroup if the binary operation on S is associative. An element $a \in S$ is called regular if there exists an element $a' \in S$ such that $aa'a = a$, if every element of S is regular then S is a regular semigroup. An element $e \in S$ such that $e \cdot e = e$ is called an idempotent and the set of all idempotents in S will be denoted by $E(S)$.

1.1 Biordered Sets

By a partial algebra E, we mean a set together with a partial binary operation on E. Then $(e, f) \in D_E$ if and only if the product ef exists in the partial algebra E. If E is a partial algebra, we shall often denote the underlying set by E itself; and the domain of the partial binary operation on E will then be denoted by D_E. Also, for brevity, we write $ef = g$, to mean $(e, f) \in D_E$ and $ef = g$. The dual of a statement T about a partial algebra E is the statement T^* obtained by replacing all products ef by its left–right dual fe. When D_E is symmetric, T^* is meaningful whenever T is. On E we define

$$\omega^r = \{(e, f) : fe = e\} \; \omega^l = \{(e, f) : ef = e\}$$

and $\mathcal{R} = \omega^r \cap (\omega^r)^{-1}$, $\mathcal{L} = \omega^l \cap (\omega^l)^{-1}$, *and* $\omega = \omega^r \cap \omega^l$. We will refer ω^r and ω^l as the right and the left quasiorder of E.

Definition 1 Let E be a partial algebra. Then E is a biordered set if the following axioms and their duals hold

(1) ω^r and ω^l are quasi orders on E and

$$D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$$

(2) $f \in \omega^r(e) \Rightarrow f \mathcal{R} fe \omega e$
(3) $g \omega^l f \; and \quad f, g \in \omega^r(e) \Rightarrow ge \omega^l fe.$
(4) $g \omega^r f \omega^r e \Rightarrow gf = (ge)f$
(5) $g \omega^l f$ and $f, g \in \omega^r(e) \Rightarrow (fg)e = (fe)(ge)$.

We shall often write $E =< E, \omega^l, \omega^r >$ to mean that E is a biordered set with quasiorders ω^l, ω^r. The relation ω defined is a partial order and

$$\omega \cap (\omega)^{-1} \subset \omega^r \cap (\omega^l)^{-1} = 1_E.$$

Definition 2 Let $\mathcal{M}(e, f)$ denote the quasiordered set $(\omega^l(e) \cap \omega^r(f), <)$ where $<$ is defined by $g < h \Leftrightarrow eg \omega^r eh$, and $gf \omega^l hf$. Then the set

$$S(e, f) = \{h \in M(e, f) : g < h \text{ for all } g \in M(e, f)\}$$

is called the sandwich set of e and f.

(1) $f, g \in \omega^r(e) \Rightarrow S(f, g)e = S(fe, ge)$

The biordered set E is said to be regular if $S(e, f) \neq \emptyset\ \forall e, f \in E$. The following theorem shows that if S is a regular semigroup, then $E(S)$ is a regular biordered set.

Theorem 1 ([4], Theorem 1.1) *Let S be a semigroup such that $E(S) \neq \phi$.*

(1) The partial algebra $E(S)$ is a biordered set.
(2) For $e, f \in E(S)$ define

$$S_1(e, f) = \{h \in M(e, f) : ehf = ef\}$$

Then $S_1(e, f) \subset S(e, f)$.
(3) If $e, f \in E(S)$ then ef is a regular element of S if and only if $S_1(e, f) = S(e, f) \neq \phi$.
(4) If S is regular, then $E(S)$ is a regular biordered set.

Remark 1 For $e \in E$, $\omega^r(e)$ $\left[\omega^l(e)\right]$ are principal right [left] ideals and $\omega(e)$ is a principal two sided ideal and are called biorder ideals generated by e.

1.2 Lattices

A lattice is a partially ordered set in which each pair of elements has a least upper bound and a greatest lower bound. If a and b are elements of a lattice, we denote their greatest lower bound (meet) and least upper bound (join) by $a \wedge b$ and $a \vee b$, respectively. It is easy to see that $a \vee b$ and $a \wedge b$ are unique. The notations $a \wedge b$ and $a \vee b$ are analogous to the notations for the intersection and union of two sets. However some properties of union and intersection of sets do not carry over to lattices, for instance, the distributive laws are false in some lattices. But many of the well-known lattices posses the modularity property which is a weak form of distributive property.

Definition 3 A lattice is called modular (or a Dedekind lattice) if

$$(a \vee b) \wedge c = a \vee (b \wedge c) \text{ for all } a \leq c.$$

A lattice is bounded if it has both a maximum element and a minimum element, we use the symbols 0 and 1 to denote the minimum element and maximum element of a lattice. A bounded lattice L is said to be complemented if for each element a of L, there exists at least one element b such that $a \vee b = 1$ and $a \wedge b = 0$. The element b is referred to as a complement of a. It is quite possible for an element of a complemented lattice to have many different complements. An element x is called a complement of a in b if $a \vee x = b$ and $a \wedge x = 0$.

Definition 4 Two elements a and b of a lattice L are said to be perspective (in symbols $a \sim b$) if there exists x in L such that $a \vee x = b \vee x, a \wedge x = b \wedge x = 0$ Such an element x is called an axis of perspective.

1.3 Principal Ideals of Regular Ring

A ring is a set R together with two binary operations '+', '·' with the following properties.

(1) The set $(R, +)$ is an abelian group.
(2) The set $(R, \cdot)$ is a semigroup.
(3) The operation $\cdot$ is distributive over $+$.

A ring $(R, +, \cdot)$ is regular if for every $a \in R$ there exists an element a' such that $aa'a = a$, i.e., the ring is regular if the multiplicative semigroup is a regular semigroup.

Definition 5 A subset a of a ring $\mathcal{R}$ is called right ideal in case

$$x + y \in a, \;\; xz \in a$$

for each $x, y \in a$ and $z \in \mathcal{R}$.

Similarly, we a can define the left ideal and a is called an ideal if it is both a right and a left ideal. The set of all right (left) ideals of $\mathcal{R}$ is denoted by $R_{\mathcal{R}}(L_{\mathcal{R}})$. The intersection of any class of right(left) ideals is again a right (left) ideal and also for any $a \subset \mathcal{R}$ there is a unique least extension a_r, (a_l) which is a right (left) ideal.

Proposition 1 *If $R \subset R_{\mathcal{R}}$ is any class of right ideals, there exists both a smallest right ideal (least upper bound of R) containing every element of R and a greatest right ideal (greatest lower bound of R) contained in every element of R. Thus $R_{\mathcal{R}}$ is a continuous lattice with $\subset$ and the operations thus defined. The zero element of $R_{\mathcal{R}}$ is $(0)_r = 0$ and the unit element is $(1)_r = \mathcal{R}$.*

Definition 6 A principal right [left] ideal is one of the from $(a)_r[(a)_l]$. The class of all principal right [left] ideals will be denoted by $\bar{R}_{\mathcal{R}}$ $[\bar{L}_{\mathcal{R}}]$.

In [5] John von Neumann describes the structure of principal ideals of a regular ring. Here we recall some of those results.

Lemma 1 *Let $\mathcal{R}$ be a ring, $e \in \mathcal{R}$, then*

- *e is idempotent if and only if $(1 - e)$ is idempotent.*
- *$\langle e\rangle_r$ if the set of all x such that $x = ex$ is a principal right ideal.*
- *$\langle e\rangle_r$ and $\langle 1 - e\rangle_r$ are mutual inverses.*
- *If $\langle e\rangle_r = \langle f\rangle_r$ and If $\langle 1 - e\rangle_r = \langle 1 - f\rangle_r$ where e and f are idempotents, then $e = f$.*

Theorem 2 *Two right ideals a and b are inverses if and only if there exists an idempotent e such that* $a = \langle e\rangle_r$ *and* $b = \langle 1 - e\rangle_r$.

Proof Let a and b be inverse right ideals, then there exists elements x, y with $x + y = 1, x \in a, y \in b$. If $z \in a$ then $xz + yz = x$. Since $z, xz \in a, yz \in a$. But $yz \in b$, hence $yz = 0$. Thus $z = xz \in (x)_r$ for every $z \in a$ and $a \subset (x)_r$. Bust $x \in a$, hence $a = (x)_r$. Similarly $b = (y)_r = (1 - x)_r$, since $x + y = 1$. Finally, since $z = xz$ for every $z \in a$ this holds for $z = x$ and x is idempotent. □

Theorem 3 *The following statements are equivalent*

(1) Every principal right ideal $\langle a\rangle_r$ *has an inverse right ideal.*
(2) For every a there exists an idempotent e such that $\langle a\rangle_r = \langle e\rangle_r$.
(3) For every a there exists an idempotent x such that $axa = a$.
(4) For every a there exists an idempotent f such that $\langle a\rangle_l = \langle f\rangle_l$.
(5) Every principal ideal $\langle a\rangle_l$ *has an inverse left ideal.*

Definition 7 A ring $\mathcal{R}$ is said to be regular if $\mathcal{R}$ possesses anyone of the equivalent properties of the above Theorem.

Theorem 4 *The set* $\bar{R}_{\mathcal{R}}$ *is a complemented modular lattice partially ordered by* $\subset$, *the meet being* $\cap$ *and join* $\cup$, *its zero is* $(0)_r$ *and its unit is* $(1)_r$.

2 Biorder Ideals of Regular Rings

Analogous to von Neumann's construction of the principal ideals of a regular ring, we proceed to describe the structure of the biorder ideals of regular rings.

Proposition 2 *Let e and f be idempotents in a regular ring R. Then the following are equivalent:*

(1) $ef = 0$
(2) $e\omega^l(1 - f)$
(3) $f\omega^r(1 - e)$

Proof Suppose $ef = 0$. Then

$$e(1 - f) = e - ef = e.$$

Conversely, $e\omega^l(1 - f)$ then $e(1 - f) = e - ef = e$ and hence $ef = 0$. Proof (3) is similar. □

Proposition 3 *Let e and f be idempotents in a regular ring R. Then the following holds.*

(1) $e\omega^l f$ *if and only if* $(1 - f)\omega^r(1 - e)$
(2) $e\omega^r f$ *if and only if* $(1 - f)\omega^l(1 - e)$

Proof Let $e\omega^l f$. Then,

$$(1-e)(1-f) = 1-e-f+ef = 1-e-f+e = 1-f$$

Conversely, suppose $(1-e)(1-f) = (1-f)$, then

$$1-e-f+ef = 1-f$$

hence $e\omega^l f$. Proof (2) is similar. □

Corollary 1 *Let e and f be idempotents in the ring R. Then the following old.*

(1) $\omega^l(e) = \omega^l(f)$ if and only if $\omega^r(1-e) = \omega^r(1-f)$
(2) $\omega^r(e) = \omega^r(f)$ if and only if $\omega^l(1-e) = \omega^l(1-f)$

Proposition 4 *Let e and f be idempotents in the ring R, if $\omega^r(e) = \omega^r(f), \omega^r(1-e) = \omega^r(1-f)$, where e, f are idempotents, then $e = f$*

Proof Since $\omega^r(e) = \omega^r(f)$, $ef = f$. Therefore, $(1-e)f = 0$. Since $\omega^r(1-e) = \omega^r(1-f)$, by replacing e and f by $(1-e)$ and $(1-f)$ respectively, we get $e(1-f) = 0$. That is, $ef = e$ and so $e = f$. □

Lemma 2 *Let $e, f, g \in E_R$ with $ef = fe = 0$. Then $e+f$ is an idempotent and the following hold.*

(1) $e\omega(e+f)$ and $f\omega(e+f)$
(2) If $e\omega^l g$ and $f\omega^l g$, then $(e+f)\omega^l g$
(3) If $e\omega^r g$ and $f\omega^r g$, then $(e+f)\omega^r g$

Proof Given $e, f \in E_R$ with $ef = fe = 0$, then

$$(e+f)^2 = e^2 + ef + fe + f^2 = e+f.$$

(1) $e(e+f) = e^2 + ef = e + ef = e$, and $(e+f)e = e^2 + fe = e + fe = e$. Thus $e\omega(e+f)$. Similarly, we can prove $f\omega(e+f)$.
(2) Given $e\omega^l g$ and $f\omega^l g$. Therefore, $(e+f)g = eg + fg = e+f$, i.e., $(e+f)\omega^l g$. Proof (3) is similar. □

Lemma 3 *Let $\omega^r(e) \cup \omega^r(f) = \omega^r(e+f'')$ where $f''\mathcal{R}f'$ and $f' = (1-e)f$.*

Proof Define

$$\begin{aligned}\omega^r(e) \cup \omega^r(f) &= \{eg + fh : g, h \in E_R; gh = hg = 0\}\\ &= \{eg = efh + (1-e)fh : g, h \in E_R; gh = hg = 0\}\\ &= \{e(g+fh) + (1-e)fh : g, h \in E_R; gh = hg = 0\}\end{aligned}$$

Let $f' = (1-e)f$. Then $f' \in S(f, 1-e)$ so that $f' \in E_R$, $ff' = f'$ and $(1-e)f' = f'$. So $ef' = 0$ and $\omega^r(e) \cup \omega^r(f) = \omega^r(e) \cup \omega^r(f')$. Define

$f'' = f'(1-e)$., then $f'f'' = f'f'(1-e) = f'(1-e) = f''$ and $f''f' = f'(1-e)f' = f'f' = f'$. Further f'' is an idempotent, $\omega^r(f') = \omega^r(f'')$ and $\omega^r(e) \cup \omega^r(f) = \omega^r(e) \cup \omega^r(f'')$. Now, $ef'' = ef'(1-e) = 0$ and $f''e = f'(1-e)e = 0$ so, by Lemma above $(e + f'')$ is an idempotent.

Next we proceed to prove that $\omega^r(e) \cup \omega^r(f) = \omega^r(e + f'')$. For, consider $e + f''$, then

$$e + f'' = e^2 + (f'')^2 = e \cdot e + f'' \cdot f'' \in \omega^r(e) \cup \omega^r(f'') \text{ where } ef'' = 0.$$

So, $\omega^r(e + f'') \subseteq \omega^r(e) \cup \omega^r(f'')$ and $e\omega^r(e + f'')$ and $f''\omega^r(e + f'')$. That is

$$\omega^r(e) \subseteq \omega^r(e + f'') \text{ and } \omega^r(f'') \subseteq \omega^r(e + f'')$$

thus $\omega^r(e) \cup \omega^r(f'') \subseteq \omega^r(e + f'')$, hence $\omega^r(e) \cup \omega^r(f) = \omega^r(e + f'')$. □

Denote by Ω_R the class of all principal ω^r−ideals and by Ω_L the class of all principal ω^l-ideals. In the light of the above lemma we have the following theorem:

Theorem 5 *Ω_R is closed with respect to the operation $\cup$ defined in Ω_R.*

Next we introduce the notion of annihilators in principal ω^r and ω^l-ideals.

Definition 8 For every ω^r-ideal we define

$$(\omega^r(e))^L = \left\{y : yz = 0 \text{ for every } z \in \omega^r(e)\right\}$$

and for every ω^l-ideal,

$$(\omega^l(e))^R = \left\{y : zy = 0 \text{ for every } z \in \omega^l(e)\right\}$$

then $(\omega^r(e))^L$ is a left ideal and $(\omega^l(e))^R$ is a right ideal.

Proposition 5 *For $e \in E_R$, $(\omega^l(e))^R$ is a principal ω^r-ideal and $(\omega^r(e))^L$ is a principal ω^l-ideal. In fact, $(\omega^l(e))^R = \omega^r(1-e)$ and $(\omega^r(e))^L = \omega^l(1-e)$.*

Proof

$$\begin{aligned}
\omega^r(e) &= \{g : e.g. = g\} \\
&= \{g : (1-e)g = 0\} \\
&= \{g : u(1-e) = 0; \text{ for every } u \in E_R\} \\
&= \left\{g : \text{ for every } h \in \omega^l(e), hg = 0\right\}
\end{aligned}$$

where $h = u(1-e)$. Since $h(1-e) = u(1-e)(1-e) = u(1-e) = h$ we have $h \in \omega^l(1-e)$. Thus $\omega^r(e) = (\omega^l(1-e))^R$. □

Lemma 4 *Let $e, f \in E_R$ and $\omega^r(e)$ and $\omega^r(f)$ are ideals generated by e and f, then*

(1) $\omega^r(e) \subset \omega^r(f) \Rightarrow (\omega^r(e))^L \supset (\omega^r(f))^L$
(2) $\omega^r(e) = (\omega^r(e))^{LR}$ *and* $(\omega^r(e))^L = (\omega^r(e))^{LRL}$

Proof (1) Let $g \in (\omega^r(f))^L$, then $gh = 0$ for every $h \in \omega^r(f)$. If $\omega^r(e) \subset \omega^r(f)$ then for $h \in \omega^r(e)$, $gh = 0$ for every $h \in \omega^r(e)$. Thus $g \in (\omega^r(e))^L$ and so

$$(\omega^r(f))^L \subset (\omega^r(e))^L.$$

(2) Let $g \in \omega^r(e)$. Consider $h \in (\omega^r(e))^L$, for $z \in \omega^r(e)$, $hz = 0$. Hence $hg = 0$ so $g \in (\omega^r(e))^{LR}$ and $\omega^r(e) \subset (\omega^r(e)^{LR}$. Now by (1) we have

$$\omega^r(e) \subseteq (\omega^r(e))^{LR};\ (\omega^r(e))^L \supseteq (\omega^r(e))^{LRL}$$

Replace $\omega^r(e)$ by $(\omega^r(e))^L$ we get $(\omega^r(e))^L \subseteq (\omega^r(e))^{LR}$. Hence $(\omega^r(e))^L = (\omega^r(e))^{LRL}$. But $\omega^r(e) = (\omega^l(1-e))^{RLR} = (\omega^l(1-e))^R = \omega^r(e)$, thus $\omega^r(e) = (\omega^r(e))^L$. □

In the following proposition, we establish the relation between Ω_L and Ω_R by using the relation between principal ω-ideals and annihilators.

Proposition 6 *Let R be a regular ring and E_R the set of idempotents on R. Let Ω_L and Ω_R denote the lattice of principal ω^l-ideals and principal ω^r-ideals of E_R. Define ϕ and ψ on Ω_L and Ω_R by*

$$\phi(\omega^l(e)) = (\omega^l(e))^R \text{ and } \psi(\omega^r(e)) = (\omega^r(e))^L$$

then ϕ and ψ are mutually inverse anti-isomorphisms.

Proof Let $I \in \Omega_L$. Therefore, there exists an idempotent, e such that $I = \omega^l(e)$ and

$$\phi(I) = \phi(\omega^l(e)) = (\omega^l(e))^R = \omega^r(1-e)$$

Thus ϕ maps Ω_L to Ω_R. Also ϕ reverses the order, for let $I, J \in \Omega_L$ with $I \subseteq J$, then there exists idempotents $e, f \in E_R$ such that $\omega^l(e) \subseteq \omega^l(f)$. But if $\omega^l(e) \subseteq \omega^l(f)$ then $(\omega^l(f))^R \subseteq (\omega^l(e))^R$ and $\phi(J) \subseteq \phi(I)$. Similarly ψ is an order reserving map from Ω_R to Ω_L. Moreover for $I \in \Omega_L$ and $I = \omega^l(e)$ then

$$(\psi\phi(I)) = \psi(\phi(\omega^l(e))) = \psi(\omega^r(1-e)) = (\omega^r(1-e))^L = \omega^l(1-(1-e)) = \omega^l(e) = I.$$

For I in Ω_R, $(\phi\psi)(I) = I$. Hence ϕ and ψ are mutually inverse anti-isomorphisms between Ω_L and Ω_R. □

Lemma 5 *Let $\omega^r(e)$ and $\omega^r(f)$ be principal right ω ideals generated by e and f. Then $(\omega^r(e) \cup \omega^r(f))^L = (\omega^r(e))^L \cap (\omega^r(f))^L$.*

Proof

$$\begin{aligned}(\omega^r(e))^L \cap (\omega^r(f))^L &= \{g : gh = 0 \ \forall\ h \in \omega^r(e) \text{ and } g : gh = 0 \ \forall\ h \in \omega^r(f)\} \\ &= \{g : gh = 0 \ \forall\ h \in \omega^r(e) \cup \omega^r(f)\} \\ &= \{g : gh = 0 \ \forall\ h \in (\omega^r(e) \cup \omega^r(f)\}\end{aligned}$$

Hence $(\omega^r(e))^L \cap (\omega^r(f))^L = (\omega^r(e) \cup \omega^r(f))^L$. □

Lemma 6 *For two principal ω^r-ideals, $\omega^r(e)$ and $\omega^r(f)$ their intersection is also a principal ω^r-ideal.*

Proof By the above Lemma

$$\begin{aligned}\omega^r(e) \cap \omega^r(f) &= (\omega^r(e))^{LR} \cap (\omega^r(f))^{LR} \\ &= ((\omega^r(e))^L \cup (\omega^r(f))^L)^R\end{aligned}$$

But $(\omega^r(e))^L$ and $(\omega^r(f))^L$ are principal ω^l-ideals, and so $(\omega^r(e))^L \cup (\omega^r(f))^L$ is also a principal ω^l-ideal. Hence $\omega^r(e) \cap \omega^r(f)$ is a principal ω^r-ideal. □

For any idempotent $e \in E_R$, $(1-e) \in E_R$ and $\omega^r(e) \cup \omega^r(1-e) = \omega^r(e+1-e) = \omega^r(1) = E_R$ and $\omega^r(e) \cap \omega^r(1-e) = \{0\}$. Thus $\omega^r(e)$ and $\omega^r(1-e)$ are complements of each other in the lattice of principal right ω-ideals of E_R. Similarly, $\omega^l(e)$ and $\omega^l(1-e)$ are complements of each other in the lattice of all principal left ω-ideals of E_R.

Thus we have the following theorem:

Theorem 6 *Let R be a ring then the set of all principal ω^l-ideals Ω_L and the set of all principal ω^r-ideals Ω_R of E_R are complemented modular lattices ordered by the relation $\subset$, the meet being $\cap$ and the join $\cup$; its zero is 0, and its unit is $\omega^l(1)[\omega^r(1)]$.*

References

1. Clifford, A.H., Preston, G.B.: The Algebraic Theory of Semigroups, vol. 1, p. 7. Math. Surveys of the American. Math. Soc., Providence (1964)
2. Easdown, D.: Biordered sets comes from Semigroups. J. Algebra, **96**, 581–591, 87d:06020 (1985)
3. Howie, J.M.: An Introduction To Semigroup Theory, Academic Press Inc. (London) (1976). ISBN: 75-46333
4. Nambooripad, K.S.S.: Structure of Regular Semigroups (MEMOIRS, No.224), American Mathematical Society (1979). ISBN-13: 978-0821 82224
5. von Neumann, J.: Continuous Geometry. Princeton University Press, London (1960)