On Some Classes of Module Hulls

Jae Keol Park and S. Tariq Rizvi

Abstract The study of various types of hulls of a module has been of interest for a long time. Our focus in this paper is to present results on some classes of these hulls of modules, their examples, counter examples, constructions and their applications. Since the notion of hulls and its study were motivated by that of an injective hull, we begin with a detailed discussion on classes of module hulls which satisfy certain properties generalizing the notion of injectivity. Closely linked to these generalizations of injectivity, are the notions of a Baer ring and a Baer module. The study of Baer ring hulls or Baer module hulls has remained elusive in view of the underlying difficulties involved. Our main focus is to exhibit the latest results on existence, constructions, examples and applications of Baer module hulls obtained by Park and Rizvi. In particular, we show the existence and explicit description of the Baer module hull of a module N over a Dedekind domain R such that $N/t(N)$ is finitely generated and $\mathrm{Ann}_R(t(N)) \neq 0$, where $t(N)$ is the torsion submodule of N. When $N/t(N)$ is not finitely generated, it is shown that N may not have a Baer module hull. Among applications, our results yield that a finitely generated module N over a Dedekind domain is Baer if and only if N is semisimple or torsion-free. We explicitly describe the Baer module hull of the direct sum of $\mathbb{Z}$ with $\mathbb{Z}_p$ (p a prime integer) and extend this to a more general construction of Baer module hulls over any commutative PID. We show that the Baer hull of a direct sum of two modules is not necessarily isomorphic to the direct sum of the Baer hulls of the modules, even if each relevant Baer module hull exists. A number of examples and applications of various classes of hulls are included.

Dedication: Dedicated to the memory of Professor Bruno J. Müller

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1 Introduction

Since the discovery of the existence of the injective hull of an arbitrary module independently in 1952 by Shoda [49] and in 1953 by Eckmann and Schopf [14], there have been numerous papers dedicated to the study and description of various types of hulls. These hulls are basically smallest extensions of rings and modules satisfying some generalizations of injectivity (for example, quasi-injective, continuous, quasi-continuous hulls, etc.) or satisfying properties related to such generalizations of injectivity. For a given module M (or a given ring R), the investigations include in general, to construct the smallest essential extension of M (or of R) which belongs to a particular class of modules (or of rings) within a fixed injective hull of M (or a fixed maximal quotient ring of R). We call this a hull of M (or of R) belonging to that particular class. One benefit of these hulls is that such hulls generally lie closer to the module M (or to the ring R) than its injective hull. This closeness may allow for a better transfer of information between M (or R) and that particular hull of M (or of R) from these classes than between M (or R) and its injective hull. These hulls have also proved to be useful tools for the study of the structure of M (or of R). So an important focus of investigations has been to obtain results on the existence and explicit descriptions of various types of module hulls. This is the topic of this survey paper.

We recall that a module M is said to be *quasi-injective* if, for each $N \leq M$, any $f \in \text{Hom}(N, M)$ can be extended to an endomorphism of M. Among other well-known generalizations of injectivity, the study of the continuous, quasi-continuous, extending, and the FI-extending properties has been extensive in the literature (see for example [4, 8, 13, 34–36, 43]). A module M is said to be *extending* if, for each $V \leq M$, there exists a direct summand $W \leq^{\oplus} M$ such that $V \leq^{\text{ess}} W$. And an extending module M is called *quasi-continuous* if for all direct summands M_1 and M_2 of M with $M_1 \cap M_2 = 0$, $M_1 \oplus M_2$ is also a direct summand of M. Furthermore, an extending module M is said to be *continuous* if every submodule N of M which is isomorphic to a direct summand is also a direct summand of M. A module M is called *FI-extending* if every fully invariant submodule is essential in a direct summand of M. For more details on FI-extending modules, see [4, 8], and [10, Sect. 2.3]. The following implications hold true for modules:

$$\text{injective} \Rightarrow \text{quasi-injective} \Rightarrow \text{continuous}$$
$$\Rightarrow \text{quasi-continuous} \Rightarrow \text{extending} \Rightarrow \text{FI-extending}$$

while each of reverse implications does not hold true, in general.

Since the injective module hull of a module always exists [14, 49], the study of module hulls with certain properties inside the injective hull of the module is more natural in contrast to the study of ring hulls of a ring (the injective hull of a ring may not even be a ring in general–and even if it is, for it to have a compatible ring structure with the ring is another hurdle).

Section 1 of the paper is devoted to results and examples (of either existence or non-existence) of various hulls which generalize injective hulls. This includes the consideration of quasi-injective, continuous, quasi-continuous and (FI-)extending module hulls. For a given module M, let $H = \mathrm{End}_R(E(M))$ denote the endomorphism ring of its injective hull $E(M)$. By Johnson and Wong [23], the unique quasi-injective hull of the module M is precisely given by HM. Goel and Jain [16] showed that there always exists a unique quasi-continuous hull of every module. The quasi-continuous hull of M is given by ΩM, where Ω is the subring generated by all idempotents of $H = \mathrm{End}(E(M))$. In contrast to this, it was shown by Müller and Rizvi in [35] that continuous module hulls do not always exist. However, they did show the existence of continuous hulls of certain classes of modules over a commutative ring (such as nonsingular cyclic ones) and provided a description of these continuous hulls (see [35, Theorem 8]). Similar to the case of continuous module hulls, it is also known that extending module hulls do not always exist (for example, see [10, Example 8.4.13, p. 319]). For the case of FI-extending module hulls, it was proved in [8, Theorem 6] that every finitely generated projective module over a semiprime ring has an FI-extending hull.

Closely linked to these notions, are the notions of a Baer ring and a Baer module. A ring R in which the left (right) annihilator of every nonempty subset of R is generated by an idempotent is called a Baer ring. It is well-known that this is a left-right symmetric notion for rings. Kaplansky introduced the notion of Baer rings in [26] (also see [27]). Having their roots in Functional Analysis, the class of Baer rings and the more general class of quasi-Baer rings (discussed ahead) were studied extensively by Kaplansky and many others who obtained a number of interesting results on these classes of rings (see [1, 3, 6–12, 18, 19, 21, 22, 31–33, 37, 38, 41]).

More recently, the notion of a Baer ring was extended to an analogous module theoretic notion using the endomorphism ring of the module by Rizvi and Roman in [44]. According to [44], a module M is called a *Baer module* if, for any $N_R \leq M_R$, there exists $e^2 = e \in S$ such that $\ell_S(N) = Se$, where $\ell_S(N) = \{f \in S \mid f(N) = 0\}$ and $S = \mathrm{End}(M_R)$. Equivalently, a module M is Baer if and only if for any left ideal I of S, $r_M(I) = fM$ with $f^2 = f \in S$, where $r_M(I) = \{m \in M \mid Im = 0\}$. Examples of Baer modules include any nonsingular injective module. In particular, it is known that every ($\mathcal{K}$-)nonsingular extending module is a Baer module while the converse holds under a certain dual condition. To study Baer module hulls, we provide relevant results and properties of Baer modules and related notions in Sect. 3 of the paper. These results will also be used in Sect. 4 of the paper.

In the main section, Sect. 4 of this expository paper, we introduce and discuss Baer module hulls of certain classes modules over a Dedekind domain from our recent work in [40]. We exhibit explicit constructions and examples of Baer module hulls and provide their applications in this section. Properties of Baer module hulls will also be discussed.

Extending the notion of a Baer ring, a quasi-Baer ring was introduced by Clark in [12]. A ring for which the left annihilator of every ideal is generated by an idempotent, as a left ideal is called a quasi-Baer ring. It was initially defined by Clark to help characterize a finite dimensional algebra over an algebraically closed field F to be a *twisted semigroup algebra* of a matrix units semigroup over F. Historically, it is of interest to note that the Hamilton quaternion division algebra over the real numbers field $\mathbb{R}$ is a *twisted group algebra* of the Klein four group V_4 over $\mathbb{R}$. It was also shown in [12] that any finite distributive lattice is isomorphic to a certain sublattice of the lattice of all ideals of an artinian quasi-Baer ring. It is clear that every Baer ring is quasi-Baer while the converse is not true in general. It is also obvious that the two notions coincide for a commutative ring and for a reduced ring. In [41], a number of interesting properties of quasi-Baer rings are obtained. See [10] for more details on quasi-Baer rings.

Quasi-Baer modules were defined and investigated by Rizvi and Roman [44] in the module theoretic setting. Recall from [44] that a module M_R is called a *quasi-Baer module* if for each $N \trianglelefteq M$, $\ell_S(N) = Se$ for some $e^2 = e \in S$, where $S = \mathrm{End}(M_R)$. Thus M_R is quasi-Baer if and only if for any ideal J of S, $r_M(J) = fM$ for some $f^2 = f \in S$. In [44] and [47], it is shown that the endomorphism ring of a (quasi-)Baer module is a (quasi-)Baer ring. It is proved that there exist close connections between quasi-Baer modules and FI-extending modules. A number of interesting properties of quasi-Baer modules and applications have also been presented.

As mentioned earlier, the notion of a "hull" with a certain property allows us to work with an overmodule or overring which has better properties than the original module or ring. It is worth mentioning that very little is known even about Baer ring hulls. Recall from [10, Chap. 8] that the *Baer* (resp., *quasi-Baer*) *ring hull* of a ring R is the smallest Baer (resp., quasi-Baer) right essential overring of R in $E(R_R)$. To the best of our knowledge, the only explicit results about *Baer ring hulls* in earlier existing literature have been due to Mewborn [33] for commutative semiprime rings, Oshiro [37] and [38] for commutative von Neumann regular rings, and Hirano, Hongan and Ohori [19] for reduced right Utumi rings. All these results were recently extended and a unified result was obtained for the case of an arbitrary semiprime ring using *quasi-Baer ring hulls* by Birkenemier, Park, and Rizvi [7, Theorem 3.3]. The focus of the present paper is on module hulls, more specifically on results and study of Baer module hulls.

For a given module M, the smallest Baer overmodule of M in $E(M)$ is called the *Baer module hull* of M. In short, we will often call it the *Baer hull* of M and denote it by $\mathfrak{B}(M)$.

Park and Rizvi in [40] recently initiated the study of the Baer module hulls. We introduce and discuss the results obtained in [40] on the Baer module hulls in Sect. 4. We show that the Baer module hull exists for a module N over a Dedekind domain R such that $N/t(N)$ is finitely generated and $\text{Ann}_R(t(N)) \neq 0$, where $t(N)$ is the torsion submodule of N. An explicit description of this Baer module hull has been provided. In contrast, an example exhibits a module N for which $N/t(N)$ is not finitely generated and which does not have a Baer module hull.

Among applications presented, we show that a finitely generated module N over a Dedekind domain is Baer if and only if N is semisimple or torsion-free. We explicitly describe the Baer module hull of $N = \mathbb{Z}_p \oplus \mathbb{Z}$, where p is a prime integer, as $V = \mathbb{Z}_p \oplus \mathbb{Z}[1/p]$ and extend this to a more general construction of Baer module hulls over any commutative PID. It is shown that unlike the case of (quasi-)injective hulls, the Baer hull of the direct sum of two modules is not necessarily isomorphic to the direct sum of the Baer hulls of the modules, even if all relevant Baer module hulls exist. Several interesting examples and applications of various types of module hulls are included throughout the paper.

All rings are assumed to have identity and all modules are assumed to be unitary. For right R-modules M_R and N_R, we use $\text{Hom}(M_R, N_R)$, $\text{Hom}_R(M, N)$, or $\text{Hom}(M, N)$ to denote the set of all R-module homomorphisms from M_R to N_R. Likewise, $\text{End}(M_R)$, $\text{End}_R(M)$, or $\text{End}(M)$ denote the endomorphism ring of an R-module M. For a given R-homomorphism (or R-module homomorphism) $f \in \text{Hom}_R(M, N)$, $\text{Ker}(f)$ denotes the kernel of f. A submodule U of a module V is said to be *fully invariant* in V if $f(U) \subseteq U$ for all $f \in \text{End}(V)$.

We use $E(M_R)$ or $E(M)$ for an injective hull of a module M_R. For a module M, we use $K \leq M$, $L \trianglelefteq M$, $N \leq^{\text{ess}} M$, and $U \leq^{\oplus} M$ to denote that K is a submodule of M, L is a fully invariant submodule of M, N is an essential submodule of M, and U is a direct summand of M, respectively.

If M is an R-module, $\text{Ann}_R(M)$ stands for the annihilator of M in R. For a module M and a set Λ, let $M^{(\Lambda)}$ be the direct sum of $|\Lambda|$ copies of M, where $|\Lambda|$ is the cardinality of Λ. When Λ is finite with $|\Lambda| = n$, then $M^{(n)}$ is used for $M^{(\Lambda)}$. For a ring R and a positive integer n, $\text{Mat}_n(R)$ and $T_n(R)$ denote the $n \times n$ matrix ring and the $n \times n$ upper triangular matrix ring over R, respectively.

For a ring R, $Q(R)$ denotes the maximal right ring of quotients of R. The symbols $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{Z}_n (n > 1)$ stand for the field of rational numbers, the ring of integers, and the ring of integers modulo n, respectively. Ideals of a ring without the adjective "left" or "right" mean two-sided ideals.

As mentioned, we will use the term *Baer hull* for Baer module hull in this paper.

2 Quasi-Injective, Continuous, Quasi-Continuous, Extending, and FI-Extending Hulls

We begin this section with a discussion on some useful generalizations of injectivity which are related to the topics of study in this paper. In particular, we discuss the notions of quasi-injective, continuous, quasi-continuous, extending, and FI-extending modules. Relationships between these notions, their examples, characterizations, and other relevant properties are presented.

For a given module M, its injective hull $E(M)$ is the minimal injective overmodule of M (equivalently, its maximal essential extension) and is unique up to isomorphism over M (see [14] and [49]). We discuss module hulls satisfying some generalizations of injectivity. One may expect that such minimal overmodules H of a module M will allow for a rich transfer of information between M and H. This, because each of these hulls, with more general properties than injectivity, sits in between a module M and a fixed injective hull $E(M)$ of M. Therefore, that specific hull of the module M usually lies closer to the module M that $E(M)$.

A module M is said to be *quasi-injective* if for every submodule N of M, each $\varphi \in \text{Hom}(N, M)$ extends to an R-endomorphism of M. The following is a well-known result.

Theorem 2.1 *A module M is quasi-injective if and only if M is fully invariant in $E(M)$.*

Quasi-injectivity is an important generalization of injectivity. All quasi-injective modules satisfy the (C_1), (C_2), (C_3), and (FI) conditions given next.

Proposition 2.2 *Let M be a quasi-injective module. Then it satisfies the following conditions.*

(C_1) *Every submodule of M is essential in a direct summand of M.*
(C_2) *If $V \leq M$ and $V \cong N \leq^{\oplus} M$, then $V \leq^{\oplus} M$.*
(C_3) *If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.*
(FI) *Any fully invariant submodule of M is essential in a direct summand of M.*

It is easy to see the relationship between the condition (C_2) and the condition (C_3) as follows.

Proposition 2.3 *If a module M satisfies (C_2), then it satisfies (C_3).*

Conditions (C_1), (C_2), (C_3), and (FI) help define the following notions.

Definition 2.4 Let M be a module.

(i) M is called *continuous* if it satisfies the (C_1) and (C_2) conditions.
(ii) M is said to be *quasi-continuous* if it has the (C_1) and (C_3) conditions.
(iii) M is called *extending* (or *CS*) if it satisfies the (C_1) condition.

(iv) M is called *FI-extending* if it satisfies the (FI) condition.

From the preceding, the following implications hold true for modules. However, the reverse implications do not hold as illustrated in Example 2.5.

$$\text{injective} \Rightarrow \text{quasi-injective} \Rightarrow \text{continuous}$$
$$\Rightarrow \text{quasi-continuous} \Rightarrow \text{extending} \Rightarrow \text{FI-extending}.$$

Example 2.5 (i) Every injective module and every semisimple module are quasi-injective. There exist simple modules which are not injective (e.g., $\mathbb{Z}_p$ for any prime integer p as a $\mathbb{Z}$-module). Further, there is a quasi-injective module which is neither injective nor semisimple. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^n}$, with p a prime integer and n an integer such that $n > 1$. Then $E(M) = \mathbb{Z}_{p^\infty}$, the Prüfer p-group, and thus M is neither injective nor semisimple. But $f(M) \subseteq M$ for any $f \in \text{End}(E(M))$. So M is quasi-injective by Theorem 2.1 (see [15, Example, p. 22]).

(ii) Let K be a field and F be a proper subfield of K. Set $K_n = K$ for all $n = 1, 2 \ldots$. We take.

$$R = \left\{ (a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} K_n \mid a_n \in F \text{ eventually} \right\},$$

which is a subring of $\prod_{n=1}^{\infty} K_n$. Say $I \trianglelefteq R$. Then we can verify that $r_R(I) = eR$ with $e^2 = e \in R$. Therefore $I_R \leq^{\text{ess}} r_R(\ell_R(I)) = (1-e)R_R$ as R is semiprime. So R_R is extending. Further, since R is von Nuemann regular, R_R also satisfies (C_2) condition. Thus R_R is continuous. As $E(R_R) = \prod_{n=1}^{\infty} K_n$, R_R is not injective, so R_R is not quasi-injective.

(iii) Let R be a right Ore domain which is not a division ring (e.g., the ring $\mathbb{Z}$ of integers). Then R_R is quasi-continuous. Take $0 \neq x \in R$ such that $xR \neq R$. Then $xR_R \cong R_R$, but xR_R is not a direct summand of R_R. Thus R_R is not continuous.

(iv) Let F be a field and $R = T_2(F)$, the 2×2 upper triangular matrix ring over F. Then we see that R_R is extending. Let $e_{ij} \in R$ be the matrix with 1 in the (i, j)-position and 0 elsewhere. Put $e = e_{12} + e_{22}$ and $f = e_{22}$. Then $e^2 = e$ and $f^2 = f$. Note that $eR \cap fR = 0$. But $eR_R \oplus fR_R$ is not a direct summand of R_R. Thus R_R is not quasi-continuous.

(v) Let $R = \text{Mat}_n(\mathbb{Z}[x])$ (n is an integer such that $n > 1$). Then R_R is FI-extending, but R_R is not extending. Further, the module $M = \oplus_{n=1}^{\infty}\mathbb{Z}$ is an FI-extending $\mathbb{Z}$-module which is not extending.

The next theorem allows us to transfer any given decomposition of the injective hull $E(M)$ of a quasi-continuous module M to a similar decomposition for M (the converse always holds). This fact is also helpful in transference of properties between between a quasi-continuous module M and its injective hull $E(M)$ or a module in between.

Theorem 2.6 ([16], [20], and [39]) *The following are equivalent for a module M.*

(i) M *is quasi-continuous.*
(ii) $M = X \oplus Y$ *for any two submodules* X *and* Y *which are complements of each other.*
(iii) $fM \subseteq M$ *for every* $f^2 = f \in End_R(E(M))$.
(iv) $E(M) = \oplus_{i\in\Lambda} E_i$ *implies* $M = \oplus_{i\in\Lambda}(M \cap E_i)$.
(v) Any essential extension V *of* M *with a decomposition* $V = \oplus_{\alpha\in\Gamma} V_\alpha$ *implies that* $M = \oplus_{\alpha\in\Gamma}(M \cap V_\alpha)$.

Remark 2.7 The equivalence of the conditions (i), (ii), (iii), and (iv) of Theorem 2.6 are comprised by results obtained in [16] and [20], while the condition (v) of Theorem 2.6 is obtained in [39].

Definition 2.8 Let $\mathfrak{M}$ be a class of modules and M be any module. We call, when it exists, a module H the $\mathfrak{M}$ *hull* of M if H is the smallest essential extension of M in a fixed injective hull $E(M)$ that belongs to $\mathfrak{M}$.

It is clear from the preceding definition that an $\mathfrak{M}$ hull of a module is unique within a fixed injective hull $E(M)$ of M. It may be worth to note that in [42, Definitions 4.7, 4.8, and 4.9, pp. 36–37], three types of continuous hulls of a module, Type I, Type II, and Type III are introduced (see also [35, Definitions]). The authors of [42] and [35] chose the Type III continuous hull of a module to be called as the continuous hull of an arbitrary module for several reasons provided in [42] and [35]. Our Definition 2.8 follows the definition of continuous hull of Type III.

The next result due to Johnson and Wong [23] describes precisely how the quasi-injective hull of a module can be constructed and that the quasi-injective hull of any module always exists.

Theorem 2.9 *Assume that* M *is a right* R*-module and let* $S = End(E(M))$. *Then* $SM = \{\sum f_i(m_i) \mid f_i \in S \text{ and } m_i \in M\}$ *is the quasi-injective hull of* M.

The following result for the existence of the quasi-continuous hull of a module is obtained by Goel and Jain [16].

Theorem 2.10 *Assume that* M *is a right* R*-module and* $S = End(E(M))$. *Let* Ω *be the subring of* S *generated by the set of all idempotents of* S. *Then* $\Omega M = \{\sum f_i(m_i) \mid f_i \in \Omega \text{ and } m_i \in M\}$ *is the quasi-continuous hull of* M.

Recall that a module is called *uniform* if the intersection of any two nonzero submodule is nonzero (i.e., the module $\mathbb{Z}_{\mathbb{Z}}$). If M is a uniform module, then $E(M)$ is also uniform. Thus $S = \text{End}(E(M))$ has only trivial idempotents, so $\Omega M = M$. Therefore the quasi-continuous module hull of M is M itself.

A module is said to be *directly finite* if it is not isomorphic to a proper direct summand of itself. A module is called *purely infinite* if it is isomorphic to the direct sum of two copies of itself. Recall that a ring R is called directly finite if $xy = 1$

implies $yx = 1$ for $x, y \in R$. We remark that a module M is directly finite if and only if $\mathrm{End}(M)$ is directly finite.

The following result was obtained by Goodearl [17] in a categorical way. In [36], Müller and Rizvi gave an algebraic proof of the result and extended it. They also proved a strong "uniqueness" of the decomposition. The result was further extended by them to a similar decomposition of a quasi-continuous module as provided in Theorem 2.13 ahead.

Theorem 2.11 ([36, Theorem 1]) *Every injective module E has a direct sum decomposition, $E = U \oplus V$, where U is directly finite, V is purely infinite, and U and V have no nonzero isomorphic direct summands (or submodules). If $E = U_1 \oplus V_1 = U_2 \oplus V_2$ are two such decompositions, then $E = U_1 \oplus V_2$ holds too, and consequently $U_1 \cong U_2$ and $V_1 \cong V_2$.*

Given a quasi-continuous module M and a submodule A of M, it is easy to find the direct summand of M in which A is essential (just consider $M \cap E(A)$). This summand was called an *internal quasi-continuous hull* of A in M by Müller and Rizvi [36].

Another interesting property of a quasi-continuous module M obtained is that if A and B are two isomorphic submodules of M then the direct summands of M which are essential over A and B respectively, are unique up to isomorphism as follows.

Theorem 2.12 ([36, Theorem 4]) *Assume that M is a quasi-continuous module and $A_i \leq^{ess} P_i \leq^{\oplus} M$ $(i = 1, 2)$. If $A_1 \cong A_2$, then $P_1 \cong P_2$.*

By using Theorem 2.12, the decomposition theorem of injective modules (Theorem 2.11) can be extended to the case of quasi-continuous modules as follows.

Theorem 2.13 ([36, Proposition 6]) *Every quasi-continuous module M has a direct sum decomposition, $M = U \oplus V$, where U is directly finite, V is purely infinite, and U and V have no nonzero isomorphic direct summands (or submodules). If $M = U_1 \oplus V_1 = U_2 \oplus V_2$ are two such decompositions, then $M = U_1 \oplus V_2$ holds too, and consequently $U_1 \cong U_2$ and $V_1 \cong V_2$.*

The existence and description of continuous hulls of certain modules have been investigated in [42] (and [35]). In contrast to Theorems 2.9 and 2.10, Müller and Rizvi [35, Example 3] construct the example of a nonsingular uniform cyclic module over a *noncommutative ring* which cannot not have a continuous hull as follows.

Example 2.14 Let V be a vector space over a field F with basis elements v_m, w_k $(m, k = 0, 1, 2, \dots)$. We denote by V_n the subspace of V generated by the v_m $(m \geq n)$ and all the w_k. Also we denote by W_n the subspace generated by the w_k $(k \geq n)$. We write $\mathcal{S}$ for the shifting operator such that $\mathcal{S}(w_k) = w_{k+1}$ and $\mathcal{S}(v_i) = 0$ for all k, i.

Let R be the set of all $\rho \in \mathrm{End}_F(V)$ such that $\rho(v_m) \in V_m$, $\rho(w_0) \in W_0$ and $\rho(w_k) = \mathcal{S}^k \rho(w_0)$, for $m, k = 0, 1, 2, \dots$. Note that $\tau\rho(w_k) = \mathcal{S}^k \tau\rho(w_0)$, for $\rho, \tau \in R$, and so $\tau\rho \in R$. Thus it is routine to check that R is a subring of $\mathrm{End}_F(V)$. Further, we see that $V_n = Rv_n$, $W_n = Rw_n$, and $V_{n+1} \subseteq V_n$ for all n. (When $f \in R$ and $v \in V$, we also use fv for the image $f(v)$ of v under f.)

Consider the left R-module $M = W_0$. First, we show that $M = Rw_0$ is uniform. For this, take $fw_0 \neq 0$, $gw_0 \neq 0$ in M, where $f, g \in R$. We need to find $h_1, h_2 \in R$ such that $h_1 f w_0 = h_2 g w_0 \neq 0$. Let

$$fw_0 = b_0 w_0 + b_1 w_1 + \cdots + b_m w_m \in Rw_0$$

and

$$gw_0 = c_0 w_0 + c_1 w_1 + \cdots + c_m w_m \in Rw_0,$$

where $b_i, c_j \in F$, $i, j = 0, 1, \ldots, m$, and some terms of b_i and c_j may be zero.

Put $h_1 w_0 = x_0 w_0 + x_1 w_1 + \cdots + x_\ell w_\ell$ and $h_2 w_0 = y_0 w_0 + y_1 w_1 + \cdots + y_\ell w_\ell$, where $x_i, y_i \in F$, $i = 0, 1, \ldots, \ell$ (also some terms of x_i and y_j may be zero). Since $h_1(w_k) = \mathcal{S}^k h_1(w_0)$ and $h_2(w_k) = \mathcal{S}^k h_2(w_0)$ for $k = 0, 1, 2 \ldots,$ we need to find such $x_i, y_i \in F$, $0 \leq i \leq \ell$ so that $h_1 f w_0 = h_2 g w_0 \neq 0$ from the following equations:

$$b_0 x_0 = c_0 y_0, \;\; b_0 x_1 + b_1 x_0 = c_0 y_1 + c_1 y_0,$$

$$b_0 x_2 + b_1 x_1 + b_2 x_0 = c_0 y_2 + c_1 y_1 + c_2 y_0,$$

$$b_0 x_3 + b_1 x_2 + b_2 x_1 + b_3 x_0 = c_0 y_3 + c_2 y_1 + c_2 y_1 + c_3 y_0,$$

and so on.

Say $\alpha(t) = b_0 + \cdots + b_m t^m \neq 0$ and $\beta(t) = c_0 + \cdots + c_m t^m \neq 0$ in the polynomial ring $F[t]$. Then $\alpha(t)F[t] \cap \beta(t)F[t] \neq 0$.

We may note that finding such $x_0, x_1 \ldots, x_\ell, y_0, y_1 \ldots, y_\ell$ in F above is the same as the job of finding $x_0, x_1 \ldots, x_\ell, y_0, y_1, \ldots, y_\ell$ such that

$$\alpha(t)(x_0 + x_1 t + \cdots + x_\ell t^\ell) = \beta(t)(y_0 + y_1 t + \cdots + y_\ell t^\ell) \neq 0$$

in the polynomial ring $F[t]$. Observing that $0 \neq \alpha(t)\beta(t) \in \alpha(t)F[t] \cap \beta(t)F[t]$, take $h_1 w_0 = c_0 w_0 + c_1 w_1 + \cdots + c_m w_m$ by putting $\ell = m$, $x_i = c_i$ for $0 \leq i \leq m$, and $h_2 w_0 = b_0 w_0 + b_1 w_1 + \cdots + b_m w_m$ by putting $\ell = m$, $y_i = b_i$ for $0 \leq i \leq m$. Since $\alpha(t)\beta(t) \neq 0$, we see that $0 \neq h_1 f w_0 = h_2 g w_0 \in Rfw_0 \cap Rgw_0$. So M is uniform.

Next, we show that each V_n is an essential extension of M (hence each V_i is uniform). Indeed, let $0 \neq \mu v_n \in Rv_n = V_n$, where $\mu \in R$. Say

$$\mu v_n = a_{n+k} v_{n+k} + \cdots + a_{n+k+\ell} v_{n+k+\ell} + b_s w_s + \cdots + b_{s+m} w_{k+m}.$$

If $a_{n+k} = \cdots = a_{n+k+\ell} = 0$, then $\mu v_n \in W_0$. Otherwise, we may assume that $a_{n+k} \neq 0$. Let $\omega \in R$ such that $\omega(v_{n+k}) = w_0$ and $\omega(v_i) = 0$ for $i \neq n+k$ and $\omega(w_j) = 0$ for all j. Then $0 \neq \omega\mu v_n = a_{n+k} w_0 \in W_0$. Thus $M = W_0$ is essential in V_n. Since M is uniform, V_n is also uniform for all n.

We prove that ${}_RM$ is nonsingular. For this, assume that $u \in Z({}_RM)$ (where $Z({}_RM)$ is the singular submodule of ${}_RM$) and let $K = \{\alpha \in R \mid \alpha u = 0\}$. Then K is an essential left ideal of R. So $K \cap R\mathcal{S}^2 \neq 0$. Thus there is $\rho \in R$ with $\rho\mathcal{S}^2 \neq 0$ and $\rho\mathcal{S}^2(u) = 0$. Say

$$u = a_k w_k + a_{k+1} w_{k+1} + \cdots + a_n w_n$$

with $a_k, a_{k+1}, \ldots, a_n \in F$. Assume on the contrary that $u \neq 0$. Then we may suppose that $a_k \neq 0$. Because $\rho(w_n) = \mathcal{S}^n \rho(w_0)$ for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} 0 = \rho\mathcal{S}^2(u) &= a_k \rho\mathcal{S}^2(w_k) + a_{k+1}\rho\mathcal{S}^2(w_{k+1}) + \cdots + a_n \rho\mathcal{S}^2(w_n) \\ &= a_k \mathcal{S}^{k+2}\rho(w_0) + a_{k+1}\mathcal{S}^{k+3}\rho(w_0) + \cdots + a_n \mathcal{S}^{n+2}\rho(w_0).f \end{aligned}$$

Here we put $\rho(w_0) = b_\ell w_\ell + b_{\ell+1} w_{\ell+1} + \cdots + b_t w_t$.

If $\rho(w_0) = 0$, then $\rho\mathcal{S}^2(w_0) = \rho(w_2) = \mathcal{S}^2\rho(w_0) = 0$. Also, $\rho\mathcal{S}^2(w_m) = 0$ for all $m = 1, 2, \ldots$, and $\rho\mathcal{S}^2(v_i) = 0$ for all $i = 0, 1, \ldots$. So $\rho\mathcal{S}^2 = 0$, a contradiction. Hence $\rho(w_0) \neq 0$, and thus we may assume that $b_\ell \neq 0$. We note that

$$\mathcal{S}^{k+2}\rho(w_0) = b_\ell w_{\ell+k+2} + b_{\ell+1} w_{\ell+k+3} + \cdots + b_t w_{t+k+2},$$

$$\mathcal{S}^{k+3}\rho(w_0) = b_\ell w_{\ell+k+3} + b_{\ell+1} w_{\ell+k+4} + \cdots + b_t w_{t+k+3},$$

and so on. Thus

$$0 = \rho\mathcal{S}^2(u) = a_k b_\ell w_{\ell+k+2} + (a_k b_{\ell+1} + a_{k+1} b_\ell) w_{\ell+k+3} + \cdots,$$

and hence $a_k b_\ell = 0$, which is a contradiction because $a_k \neq 0$ and $b_\ell \neq 0$. Therefore $u = 0$, and so M is nonsingular.

We show now that V_n is continuous. Note that V_n is uniform. So clearly, V_n has the (C_1) condition. Thus, to show that V_n is continuous, it suffices to prove that every R-monomorphism of V_n is onto for V_n to satisfy the (C_2) condition.

Let $\varphi : V_n \to V_n$ be an R-monomorphism. We put

$$\varphi(v_n) = \rho v_n \in R v_n = V_n, \quad \text{where} \quad \rho \in R.$$

We claim that $\rho v_n \notin V_{n+1}$. For this, assume on the contrary that $\rho v_n \in V_{n+1}$. Let $\lambda \in R$ such that $\lambda v_n = v_n$, $\lambda v_k = 0$ for $k \neq n$, and $\lambda w_m = 0$ for all m. Then $\varphi(\lambda v_n) = \lambda(\rho v_n) = 0$ since $\rho(v_n) \in V_{n+1}$. But $\lambda v_n = v_n \neq 0$. Thus φ is not one-to-one, a contradiction. Therefore $\rho v_n \notin V_{n+1}$.

As $\rho v_n \in V_n$, write

$$\rho v_n = a_n v_n + a_{n+1} v_{n+1} + \cdots + a_{n+\ell} v_{n+\ell} + b_0 w_0 + \cdots + b_h w_h,$$

where $a_n, a_{n+1}, \ldots, a_{n+\ell}, b_0, b_1, \ldots, b_h \in F$, and $a_n \neq 0$.

Take $\nu \in R$ such that $\nu v_n = a_n^{-1} v_n$, $\nu v_k = 0$ for $k \neq n$ and $\nu w_m = 0$ for all m. Then we see that $v_n = \nu\rho v_n \in R\rho v_n$. Therefore $R v_n \subseteq R\rho v_n$, and hence $V_n = R v_n =$

$R\rho v_n$. Thus $\varphi(Rv_n) = R\varphi(v_n) = R\rho v_n = V_n$, so φ is onto. Therefore V_n is continuous for all n.

Finally, note that the uniform nonsingular module $M = Rw_0$ is not continuous, since the shifting operator $\mathcal{S}$ provides an R-monomorphism which is not onto. Hence, M does not have a continuous hull (in $E(M) = E(V)$), because such a hull would have to be contained in each V_n, and hence in $M = \bigcap_n V_n$.

Despite Example 2.14, continuous hulls do exist for certain classes of modules over a *commutative ring*. For the class of cyclic modules, the next result and Theorem 2.17 due to Müller and Rizvi [35] show the existence of continuous hulls over commutative rings.

Theorem 2.15 ([35, Theorem 8]) *Every cyclic module over a commutative ring whose singular submodule is uniform, has a continuous hull.*

The next example, due to Müller and Rizvi [35], shows that in general, the quasi-continuous hull of a module is distinct from the continuous hull, which in turn is distinct from the (quasi-)injective hull of the module.

Example 2.16 ([35, Example 2]) Let $F_n = \mathbb{R}$ for $n = 1, 2, \ldots$, and put $A = \prod_{n=1}^{\infty} F_n$, where $\mathbb{R}$ is the field of real numbers. Let R be the subring of A generated by $\oplus_{n=1}^{\infty} F_n$ and 1_A. Then $E(R_R) = Q(R) = A$. In this case, we see that

$$V = \{(a_n)_{n=1}^{\infty} \in A \mid a_n \in \mathbb{Z} \text{ eventually}\}$$

is the quasi-continuous hull of R_R, while

$$W = \{(a_n)_{n=1}^{\infty} \in A \mid a_n \in \mathbb{Q} \text{ eventually}\}$$

is the continuous hull of R_R because W is the smallest continuous von Neumann regular ring between R and $Q(R)$ (so W is the intersection of all intermediate continuous von Neumann regular rings between R and $Q(R)$). We note that A_W is an injective hull of W_W, and also A_W is a quasi-injective hull of W_W.

When M is a uniform cyclic module over a commutative ring, the following theorem shows that M has a continuous hull (see [42]). Furthermore, it explicitly describes the continuous hull of M. Recall that when M_R is a right R-module, an element $c \in R$ is said to *act regularly on* M if $mc = 0$ with $m \in M$ implies that $m = 0$. Let C be the multiplicative set of elements of R which act regularly on M, and let $MC^{-1} = \{mc^{-1} \mid m \in M, c \in C\}$.

Theorem 2.17 ([42, Theorem 4.15] and [10, Theorem 8.4.11, p. 319]) *Let R be a commutative ring, and M a uniform cyclic R-module. Then MC^{-1} is a continuous hull of M.*

In view of the existence of quasi-injective and quasi-continuous hulls for all modules and from the existence of continuous hulls for some classes of modules in

Theorems 2.15 and 2.17, it is natural to consider the existence of extending hulls of modules. However, the following example exhibits that there exists a free module of finite rank over a commutative domain which has no extending hull.

Example 2.18 (cf. [40, Example 2.19] and [10, Example 8.4.13, p. 319]) We let $R = \mathbb{Z}[x]$, the polynomial ring over $\mathbb{Z}$. Then $(R \oplus R)_R$ has no extending hull.

We recall that a module M satisfying the (FI) condition is called *FI-extending*. Thus a module M is FI-extending if and only if every fully invariant submodule of M is essential in a direct summand of M. A ring R is called *right FI-extending* if R_R is FI-extending. Similarly left FI-extending ring is defined. For more details on FI-extending modules and rings, see [4, 8, 10].

The notion of an FI-extending module generalizes that of an extending module by requiring that only *every fully invariant* submodule is essential in a direct summand rather than *every* submodule. Many well-known submodule of a given module are fully invariant. For example, the socle of a module, and the Jacobson radical of a module, and the singular submodule of a module, are fully invariant. For a ring R, all its fully invariant submodules are precisely the ideals of R. It was shown in [4, Theorem 1.3] that any direct sum of FI-extending modules is FI-extending without any additional requirements. Thus while a direct sum of extending modules may not be extending, it does satisfy the extending property for all its fully invariant submodules.

There are close connections between the FI-extending property and the quasi-Baer property. For example, assume that R is a semiprime ring. Then R is right FI-extending if and only if R is left extending if and only if R is a quasi-Baer ring from [4, Theorem 4.7]. Further, every nonsingular FI-extending module is a quasi-Baer module (in fact, this also holds true under much weaker nonsingularity conditions).

A commutative domain R is called *Prüfer* if R is semihereditary. Thus a commutative domain is Prüfer if and only if every finitely generated ideal is projective. Note that every extending module is FI-extending. If R is a commutative domain which is not Prüfer (e.g., $R = \mathbb{Z}[x]$) and n is an integer such that $n > 1$, then $R_R^{(n)}$ is FI-extending, but $R_R^{(n)}$ is not extending (cf. Example 2.5(v)).

For a ring R, recall that $Q(R)$ denotes the maximal right ring of quotients of R. Let $\mathcal{B}(Q(R))$ be the set of all central idempotents of $Q(R)$. By [2], the subring $R\mathcal{B}(Q(R))$ of $Q(R)$ generated by R and $\mathcal{B}(Q(R))$ is called *the idempotent closure* of R.

Between R and $R\mathcal{B}(Q(R))$, LO (Lying Over), GU (Going Up), and INC (Incomparable) hold. Thereby, $\text{kdim}(R) = \text{kdim}(R\mathcal{B}(Q(R))$, where $\text{kdim}(-)$ is the classical Krull dimension of a ring, i.e., the supremum of all length of chains of prime ideals. For prime radicals and Jacobson radicals of R and $R\mathcal{B}(Q(R))$, we have that $P(R\mathcal{B}(Q(R)) \cap R = P(R)$ and $J(R\mathcal{B}(Q(R)) \cap R = J(R)$, where $P(-)$ and $J(-)$ denote the prime radical and the Jacobson radical of a ring, respectively. Also, R is strongly π-regular if and only if $R\mathcal{B}(Q(R))$ is strongly π-regular (recall that a ring A is called *strongly π-regular* if for each $a \in A$ there exist $x \in A$ and a positive integer n, depending on a, such that $a^n = a^{n+1}x$. (See [10, Lemma 8.3.26 and Theorem 8.3.28, pp. 296–297].)

Further, by [10, Corollary 8.3.30, p. 298], R is von Neumann regular if and only if $R\mathcal{B}(Q(R))$ is von Neumann regular. When R is a semiprime ring with exactly n (n a positive integer) minimal prime ideals $P_1, P_2, \ldots, P_n$, we have the following structure theorem

$$R\mathcal{B}(Q(R)) \cong R/P_1 \oplus R/P_2 \oplus \cdots \oplus R/P_n$$

as rings from [10, Theorem 10.1.20, p. 370].

By using the above structure theorem for $R\mathcal{B}(Q(R))$, it was shown in [7, Corollary 4.17] that if A is a unital C^*-algebra and n is a positive integer, then A has exactly n minimal prime ideals if and only if $A\mathcal{B}(Q(A))$ is a direct sum of n prime C^*-algebras if and only if the extended centroid $\mathrm{Cen}(Q(A))$ of A is $\mathbb{C}^n$, where $\mathbb{C}$ is the field of complex numbers.

An overring T of a ring R is called a *right ring of quotients* of R if R_R is a dense submodule of T_R. Assume that R is a semiprime ring. Then from [7, Theorem 3.3], the ring $R\mathcal{B}(Q(R))$ is the smallest right FI-extending right ring of quotients of R. For more details on $R\mathcal{B}(Q(R))$, see [10, Sects. 8.3 and 10.1].

In the following definition, for a ring R, we fix a maximal right ring of quotients $Q(R)$ of R. Thus a right ring of quotients T of R is a subring of $Q(R)$.

Definition 2.19 (*see* [6, Definition 2.1]) The smallest right FI-extending right ring of quotients of a ring R is called the right FI-extending ring hull of R (when it exists). Such hull is denoted by $\widehat{Q}_{\mathbf{FI}}(R)$.

The existence of the right FI-extending ring hull $\widehat{Q}_{\mathbf{FI}}(R)$ of a semiprime ring R was obtained and explicitly described by Birkenmeier, Park, and Rizvi in the following interesting result.

Theorem 2.20 ([7, Theorem 3.3]) *Assume that R is a semiprime ring. Then $\widehat{Q}_{\mathbf{FI}}(R)$ exists and $\widehat{Q}_{\mathbf{FI}}(R) = R\mathcal{B}(Q(R))$.*

Let R be a commutative semiprime ring. Then $R\mathcal{B}(Q(R))$ is the smallest extending ring of quotients of R by Theorem 2.20.

In contrast to Theorem 2.20, there exists a semiprime ring for which the right extending ring hull does not exist. For this, we need the the next result.

Theorem 2.21 ([10, Theorem 6.1.4, p. 191]) *Let R be a commutative domain. Then the following are equivalent.*

(i) R is a Prüfer domain.
(ii) $Mat_n(R)$ is a (right) extending ring for every positive integer n.
(iii) $Mat_k(R)$ is a (right) extending ring for some integer $k > 1$.
(iv) $Mat_2(R)$ is a (right) extending ring.

The smallest right extending right ring of quotients of a ring R is called the *right extending ring hull* of R (when it exists). Such hull is denoted by $\widehat{Q}_{\mathbf{E}}(R)$. By using Theorem 2.21, we obtain the following example which exhibits that the right extending ring hull of a semiprime ring does not exist, in general.

Example 2.22 (see [10, Example 8.3.34, p. 300]) Let $R = \text{Mat}_k(F[x, y])$, where F is a field and k is an integer such that $k \geq 2$. Then the right extending ring hull $\widehat{Q}_{\mathbf{E}}(R)$ of R does not exist.

Assume on the contrary that $\widehat{Q}_{\mathbf{E}}(R)$ exists. Note that $F(x)[y]$ and $F(y)[x]$ are Prüfer domains, where $F(x)$ (resp., $F(y)$) is the field of fractions of $F[x]$ (resp., $F[y]$). So $\text{Mat}_k(F(x)[y])$ and $\text{Mat}_k(F(y)[x])$ are right extending rings by Theorem 2.21. Note $Q(R) = \text{Mat}_k(F(x, y))$, where $F(x, y)$ is the field of fractions of $F[x, y]$. Hence

$$\widehat{Q}_{\mathbf{E}}(R) \subseteq \text{Mat}_k(F(x)[y]) \cap \text{Mat}_k(F(y)[x]) = \text{Mat}_k(F(x)[y] \cap F(y)[x]).$$

To see that $F(x)[y] \cap F(y)[x] = F[x, y]$, let

$$\begin{aligned}\gamma(x, y) &= f_0(x)/g_0(x) + (f_1(x)/g_1(x))y + \cdots + (f_m(x)/g_m(x))y^m \\ &= h_0(y)/k_0(y) + (h_1(y)/k_1(y))x + \cdots + (h_n(y)/k_n(y))x^n\end{aligned}$$

be in $F(x)[y] \cap F(y)[x]$, where $f_i(x),\ g_i(x) \in F[x],\ h_j(y),\ k_j(y) \in F[y]$, and $g_i(x) \neq 0,\ k_j(y) \neq 0$ for $i = 0, 1, \ldots, m, j = 0, 1, \ldots, n$. Let $\overline{F}$ be the algebraic closure of F. If $\deg(g_0(x)) \geq 1$, then there exists $\alpha \in \overline{F}$ such that $g_0(\alpha) = 0$. Thus $\gamma(\alpha, y)$ cannot be defined. On the other hand, we note that

$$\gamma(\alpha, y) = h_0(y)/k_0(y) + (h_1(y)/k_1(y))\alpha + \cdots + (h_n(y)/k_n(y))\alpha^n,$$

which is a contradiction. Thus $g_0(x) \in F$. Similarly, $g_1(x), \ldots, g_m(x) \in F$. Hence $\gamma(x, y) \in F[x, y]$. Therefore $F(x)[y] \cap F(y)[x] = F[x, y]$, and so

$$\widehat{Q}_{\mathbf{E}}(R) = \text{Mat}_k(F(x)[y] \cap F(y)[x]) = \text{Mat}_k(F[x, y]).$$

Thus $\text{Mat}_k(F[x, y])$ is a right extending ring, a contradiction from Theorem 2.21 because the commutative domain $F[x, y]$ is not Prüfer. Therefore $R = \text{Mat}_k(F[x, y])$ has no right extending ring hull.

In contrast to Theorem 2.20, the existence of the right FI-extending ring hull of a ring is not always guaranteed, even in the presence of nonsingularity, as the next example shows.

Example 2.23 (see [5, Example 2.10(ii)], [6, Example 3.16], and [10, Example 8.2.9, p. 278]) Let F be a field and put

$$R = \left\{ \begin{bmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & c \end{bmatrix} \mid a, c, x, y \in F \right\} \cong \begin{bmatrix} F & F \oplus F \\ 0 & F \end{bmatrix}.$$

Then R is right nonsingular and $Q(R) = \text{Mat}_3(F)$.

Let

$$H_1 = \left\{ \begin{bmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix} \mid a, b, c, x, y \in F \right\} \cong \begin{bmatrix} F \oplus F & F \oplus F \\ 0 & F \end{bmatrix},$$

and let

$$H_2 = \left\{ \begin{bmatrix} a+b & a & x \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix} \mid a, b, c, x, y \in F \right\}.$$

Note that R, H_1, and H_2 are subrings of $\mathrm{Mat}_3(F)$. Define $\phi : H_1 \to H_2$ by

$$\phi \begin{bmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a & a-b & x-y \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix}.$$

Then ϕ is a ring isomorphism. It is routine to check that the ring R is not right FI-extending. But, we can verify that H_1 is a right FI-extending ring. Therefore H_2 is also right FI-extending because $H_1 \cong H_2$ (ring isomorphic).

Let $F = \mathbb{Z}_2$. Then there is no proper intermediate ring between R and H_1, also between R and H_2. If $\widehat{Q}_{\mathbf{FI}}(R)$ exists, then $\widehat{Q}_{\mathbf{FI}}(R) \subseteq H_1 \cap H_2 = R$, so $\widehat{Q}_{\mathbf{FI}}(R) = R$. Hence R is a right FI-extending ring, which is a contradiction.

In contrast to Example 2.18 where the extending hull of a finitely generated free module of rank 2 does not exist, it was shown that the FI-extending hulls of every finitely generated projective module over a semiprime ring does exist in [8]. Also such an FI-extending hulls is described explicitly using Theorem 2.20 as in the next theorem. For a module M, let $\mathfrak{F}\mathfrak{I}(M)$ denote the FI-extending hull of M, when it exists.

Theorem 2.24 ([8, Theorem 6]) *Any finitely generated projective module P_R over a semiprime ring R has the FI-extending hull $\mathfrak{F}\mathfrak{I}(P_R)$. Indeed, $\mathfrak{F}\mathfrak{I}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)$ where $P \cong e(\oplus^n R_R)$, for some $e^2 = e \in End(\oplus^n R_R)$ and some positive integer n.*

From Theorems 2.20 and 2.24, the following result is obtained.

Corollary 2.25 ([8, Corollary 7]) *Assume that R is a semiprime ring and P_R is a finitely generated projective module. Then $\widehat{Q}_{\mathbf{FI}}(End(P_R)) \cong End(\mathfrak{F}\mathfrak{I}(P_R))$.*

An application of Theorem 2.24 yields the following consequences.

Corollary 2.26 ([8, Corollary 13]) *Let R be a semiprime ring. Then:*

(i) If P_R is a progenerator of the category Mod-R of right R-modules, then $\mathfrak{F}\mathfrak{I}(P_R)_{\widehat{Q}_{\mathbf{FI}}(R)}$ is a progenerator of the category Mod-$\widehat{Q}_{\mathbf{FI}}(R)$ of right $\widehat{Q}_{\mathbf{FI}}(R)$-modules.

(ii) If R and a ring S are Morita equivalent, then $\widehat{Q}_{\mathbf{FI}}(R)$ and $\widehat{Q}_{\mathbf{FI}}(S)$ are Morita equivalent.

3 Baer Modules

We introduce the definition of a Baer module M_R via its endomorphism ring $S = \text{End}(M_R)$ in contrast to defining this notion in terms of the base ring R. The use of the endomorphism ring instead of the base ring R appears to offer a more natural generalization of a Baer ring in the general module theoretic setting (see Definition 3.1 and the comments after Example 3.2).

Properties of Baer modules are included and examples are provided. Similar to the ring theoretic concepts of nonsingularity and cononsingularity, $\mathcal{K}$-nonsingularity and $\mathcal{K}$-cononsingularity, respectively are discussed for modules. Using these concepts, strong connections between extending modules and Baer modules are provided, which generalizes the Chatters-Khuri theorem to the module theoretic setting. We include a characterization of rings R for which every projective right R-module is Baer. Properties of Baer modules from this section will also be used in Sect. 4. For more details on Baer modules and their properties, see [44–47], and [10, Chap. 4].

We start with the following definition.

Definition 3.1 ([44, Definition 2.2]) A right R-module M is called a *Baer module* if, for any $N_R \leq M_R$, there exists $e^2 = e \in S$ such that $\ell_S(N) = Se$, where $S = \text{End}(M_R)$ and $\ell_S(N) = \{f \in S \mid f(N) = 0\}$. A right R-module M is Baer if and only if for any left ideal I of S, $r_M(I) = fM$ with $f^2 = f \in S$, where $r_M(I) = \{m \in M \mid Im = 0\}$.

A ring R is said to be a *Baer ring* if the right annihilator of any nonempty subset of R is generated, as a right ideal, by an idempotent of R. Thus a ring R is a Baer ring if and only if R_R is a Baer module. Further, we can verify that a ring R is Baer if and only if the left annihilator of any nonempty subset of R is generated, as a left ideal, by an idempotent of R (see [27, Theorem 3, p. 2]).

Example 3.2 (i) Every semisimple module is a Baer module.
(ii) If R is a Baer ring and $e^2 = e \in R$, then eR_R is a Baer module (see Theorem 3.12).
(iii) ([44, Proposition 2.19]) A finitely generated Baer abelian group M is a Baer $\mathbb{Z}$-module if and only if M is semisimple or torsion-free.
(iv) ([10, Corollary 4.3.6, p. 112]) Any finitely generated right Hilbert A-module over an AW^*-algebra A is a Baer module.
(v) ([44, Theorem 2.23]) A module M is an indecomposable Baer module if and only if any nonzero endomorphism of M is a monomorphism.
(vi) Any nonsingular extending module is a Baer module (see [44, Theorem 2.14]).
(vii) For a commutative domain R and an integer $n > 1$, $R_R^{(n)}$ is a Baer module if and only if $R_R^{(n)}$ is an extending module if and only if R is a Prüfer domain.
(viii) ([47, Theorem 3.16]) Let R be an n-fir (n a positive integer). Then $R_R^{(n)}$ is a Baer module (recall that a ring R is said to be an n-fir if any right ideal of R generated by at most n elements is free of unique rank).

In [30, Definition 3.1], Lee and Zhou also called a module M_R *Baer* if, for any nonempty subset X of M, $r_R(X) = eR$ with $e^2 = e \in R$. But Definition 3.1 is distinct from their definition. In fact, any semisimple module is a Baer module by Definition 3.1 (see Example 3.2(i)), but it may not be a Baer module in the sense of Lee and Zhou [30] (for example $\mathbb{Z}_p$ as a $\mathbb{Z}$-module, where p is a prime integer, is a Baer module in our sense).

Definition 3.3 ([44, Theorem 2.5]) Let M be a module. Then M is called $\mathcal{K}$-*nonsingular* if, for $\phi \in \text{End}_R(M)$, $\text{Ker}(\phi) \leq^{\text{ess}} M$ implies $\phi = 0$.

Example 3.4 (i) Any semisimple module is $\mathcal{K}$-nonsingular.
(ii) ([44, Proposition 2.10]) Every nonsingular module is $\mathcal{K}$-nonsingular.
(iii) ([44, Example 2.11]) The $\mathbb{Z}$-module $\mathbb{Z}_p$, where p is a prime integer, is $\mathcal{K}$-nonsingular, but it is not nonsingular.
(iv) Any polyform module is $\mathcal{K}$-nonsingular. Recall that a module M is said to be *polyform* if every essential submodule of M is a dense submodule. A polyform module M is also called *non-M-singular*.
(v) For a ring R, R_R is $\mathcal{K}$-nonsingular if and only if R_R is nonsingular if and only if R_R is polyform.
(vi) [46, Example 2.5]) Let $M = \mathbb{Q} \oplus \mathbb{Z}_2$ as a $\mathbb{Z}$-module. Then M is $\mathcal{K}$-nonsingular. But M is neither nonsingular nor polyform.
(vii) ([44, Lemma 2.15]) Every Baer module is $\mathcal{K}$-nonsingular.
(viii) ([44, Lemma 2.6]) A module M is $\mathcal{K}$-nonsingular if and only if, for any left ideal I of S, $r_M(I) \leq^{\text{ess}} M$ implies $I = 0$, where $S = \text{End}(M)$.

While the nonsingularity of a module M provides the uniqueness of essential closures in M (i.e., M is a UC-module), the $\mathcal{K}$-nonsingularity provides the uniqueness of closures which happen to be direct summands of M.

Theorem 3.5 ([46, Proposition 2.8]) *Assume that M is a $\mathcal{K}$-nonsingular module, and let $N \leq M$. If $N \leq^{ess} N_i \leq^{\oplus} M$, for $i = 1, 2$, then $N_1 = N_2$.*

We recall that a ring R is said to be *right cononsingular* if for $I_R \leq R_R$, $\ell_R(I) = 0$ implies $I_R \leq^{\text{ess}} R_R$. Dual to the notion in Definition 3.3, the following is a module theoretic version of cononsingularity introduced in [44].

Definition 3.6 ([44, Definition 2.7]) A module M_R is called $\mathcal{K}$-*cononsingular* if for all $N_R \leq M_R$, $\ell_S(N) = 0$ implies $N_R \leq^{\text{ess}} M_R$, where $S = \text{End}(M_R)$.

Example 3.7 (i) For a ring R, R_R is $\mathcal{K}$-cononsingular if and only if R is right cononsingular.
(ii) ([44, Lemma 2.13]) Every extending module is $\mathcal{K}$-cononsingular.
(iii) For a commutative semiprime ring R, $R_R^{(n)}$ is $\mathcal{K}$-cononsingular for every positive integer n.
(iv) Let $R = \mathbb{Z}[x]$. Then $(R \oplus R)_R$ is $\mathcal{K}$-cononsingular by part (iii). But $(R \oplus R)_R$ is not extending by Theorem 2.21. Hence the converse of part (ii) is not true.

Proposition 3.8 ([44, Proposition 2.8(ii)]) *Assume that M is a right R-module. Then M is $\mathcal{K}$-cononsingular if and only if, for $N \leq M$, $r_M(\ell_S(N)) \leq^{\oplus} M$ implies $N \leq^{ess} r_M\ell_S(N)$, where $S = End_R(M)$.*

It is shown by Chatters and Khuri [11, Theorem 2.1] that a ring R is right extending right nonsingular if and only if R is a Baer ring and right cononsingular. This result is extended to an arbitrary module in the next theorem which exhibits strong connections between a Baer module and an extending module.

Theorem 3.9 ([44, Theorem 2.12]) *A module M is extending and $\mathcal{K}$-nonsingular if and only if M is Baer and $\mathcal{K}$-cononsigular.*

Definition 3.10 ([47, Definition 2.3]) Let M_R be an R-module and $S = \text{End}_R(M)$. Then M_R is called *quasi-retractable* if $\text{Hom}_R(M, r_M(I)) \neq 0$ for every left ideal I of S with $r_M(I) \neq 0$ (or, equivalently, if $r_S(I) \neq 0$ for every left ideal I with $r_M(I) \neq 0$).

Recall from [29] that a module M is said to be *retractable* if any $0 \neq N \leq M$, $\text{Hom}(M, N) \neq 0$. Examples of retractable modules include free modules, generators, and semisimple modules. Obviously retractable modules are quasi-retractable. But there exists a quasi-retractable module which is not retractable. For example, let F be a field. Put

$$R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix} \quad \text{and} \quad e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R.$$

Consider the module $M = eR$. Note that $S := \text{End}(M_R) \cong eRe \cong F$, which is a field. Let I be a left ideal of S such that $r_M(I) \neq 0$. Then $I = 0$ and so $r_M(I) = M$. Hence, $\text{Hom}(M_R, r_M(I)) = \text{End}(M_R) \cong F \neq 0$. Thus, M_R is quasi-retractable. But M_R is not retractable, since the endomorphism ring S of M_R, which is isomorphic to F, consists of isomorphisms and the zero endomorphism. On the other hand, as M_R is not simple, retractability of M_R implies that there exist nonzero endomorphisms of M_R which are not onto (see [10, Example 4.2.4, p. 101]).

By [44, Theorem 4.1], the endomorphism ring of a Baer module is a Baer ring. But the converse does not hold by [44, Example 4.3]. Indeed, let $M = \mathbb{Z}_{p^\infty}$, the Prüfer p-group (p a prime integer), as $\mathbb{Z}$-module. Then $S := \text{End}_{\mathbb{Z}}(M)$ is the ring of p-adic integers, so S is a commutative domain. Hence S is a Baer ring. But M is not a Baer $\mathbb{Z}$-module.

In spite of the above example, the following result shows a connection between the Baer property of a module and its endomorphism ring via its quasi-retractability.

Theorem 3.11 ([47, Theorem 2.5]) *A module M_R is Baer if and only if $\text{End}_R(M)$ is a Baer ring and M_R is quasi-retractable.*

Theorem 3.12 ([44, Theorem 2.17]) *Any direct summand of a Baer module is a Baer module.*

We noted before, $\mathbb{Z}_p \oplus \mathbb{Z}$ (p a prime integer) is not Baer as a $\mathbb{Z}$-module, while both $\mathbb{Z}_p$ and $\mathbb{Z}$ are Baer $\mathbb{Z}$-modules. For the Baer property of a finite direct sum of Baer modules, we need the following. Let M and N be R-modules. Then M is said to be N-*injective* if, for any $W \leq N$ and $f \in \text{Hom}(W, M)$, there exists $\varphi \in \text{Hom}(N, M)$ such that $\varphi|_W = f$. Recall from [47, Definition 1.3] that two modules M and N are said to be *relatively Rickart* if, for every $f \in \text{Hom}(M, N)$, $\text{Ker}(f) \leq^{\oplus} M$ and for every $g \in \text{Hom}(N, M)$, $\text{Ker}(g) \leq^{\oplus} N$.

Theorem 3.13 ([47, Theorem 3.19] also see [10, Theorem 4.2.17, p. 105]) *Assume that* $\{M_i \mid 1 \leq i \leq n\}$ *be a finite set of Baer modules. Let* M_i *and* M_j *be relatively Rickart for* $i \neq j$*, and* M_i *be* M_j*-injective for* $i < j$*. Then* $\oplus_{i=1}^{n} M_i$ *is a Baer module.*

The study of rings R for which a certain class of R-modules is Baer is of natural interest. In the following, R is semisimple artinian if and only if every injective R-module is Baer.

Theorem 3.14 ([46, Theorem 2.20]) *The following are equivalent for a ring R.*

(i) Every injective (right) R-module is Baer.
(ii) Every (right) R-module is Baer.
(iii) R is semisimple artinian.

A ring R is said to be *semiprimary* if $R/J(R)$ is artinian and $J(R)$ is nilpotent. Recall that a ring R is *right* (resp., *left*) *hereditary* if every right (resp., left) ideal of R is projective. It is well-known that if a ring R is semisprimary, then R is right hereditary if and only if R is left hereditary.

The following result provides a characterization of rings R for which every projective right R-module is Baer. Also see Theorem 4.11.

Theorem 3.15 ([47, Theorem 3.3]) *The following are equivalent for a ring R.*

(i) Every projective right R-module is a Baer module.
(ii) Every free right R-module is a Baer module.
(iii) R is a semiprimary, hereditary (Baer) ring.

Since condition (iii) is left-right symmetric, the left-handed versions of (i) and (ii) also hold.

A module M_R is called *torsionless* if it can be embedded in a direct product of copies of R_R. The following result characterizes a ring R for which every finitely generated right R-module is a Baer module.

Recall that an R-module M is said to be *finitely presented* if there exists a short exact sequence of R-modules $0 \to K \to R^{(n)} \to M \to 0$, where n is a positive integer and K is a finitely generated R-module.

A ring R is called *right* Π*-coherent* if every finitely generated torsionless right R-module is finitely presented. Left Π-coherent ring is defined similarly. Recall that a ring R is said to be *right semiheditary* if every finitely generated right ideal of R is projective. A left semihereditary ring is denied similarly.

Theorem 3.16 ([47, Theorem 3.5]) *The following are equivalent for a ring R.*

(i) Every finitely generated free right R-module is a Baer module.
(ii) Every finitely generated projective right R-module is a Baer module.
(iii) Every finitely generated torsionless right R-module is projective.
(iv) Every finitely generated torsionless left R-module is projective.
(v) R is left semihereditary and right Π*-coherent.*

(vi) *R is right semihereditary and left* Π*-coherent.*
(vii) $Mat_n(R)$ *is a Baer ring for every positive integer n.*

For a positive integer n, we recall that an *n-generated module* means a module which is generated by n elements. A ring R is said to be *right n-hereditary* if every n-generated right ideal of R is projective. Thus, a ring R is right semihereditary if and only if it is right n-hereditary for all positive integers n. Given a fixed positive integer n, we introduce the following characterization for every n-generated free R-module to be Baer.

Theorem 3.17 ([47, Theorem 3.12]) *Let R be a ring and n a positive integer. Then the following are equivalent.*

(i) *Every n-generated free right R-module is a Baer module.*
(ii) *Every n-generated projective right R-module is a Baer module.*
(iii) *Every n-generated torsionless right R-module is projective (therefore R is right n-hereditary).*
(iv) $Mat_n(R)$ *is a Baer ring.*

Corollary 3.18 *Let R be a ring. Then R is a Baer ring if and only if every cyclic torsionless right R-module is projective.*

4 Baer Module Hulls

We present recent results and examples on Baer hulls in this section. As mentioned in the introduction, the study of even Baer ring hulls has been rather limited. And the only results on Baer ring hulls that exist in earlier literature are from [19, 33, 37, 38], respectively for the classes of commutative semiprime rings, commutative von Neumann regular rings, and reduced right Utumi rings. Some newer developments on ring hulls were presented in [5–7, 9, 10]. The question about the existence of Baer module hulls and their existence has not been addressed till now and is quite challenging. The results presented here are the latest developments on Baer module hulls of finitely generated modules over a commutative domain.

From [44] it is known that $N = \mathbb{Z}_p \oplus \mathbb{Z}$ (p a prime integer) is not a Baer $\mathbb{Z}$-module, while $\mathbb{Z}_p$ and $\mathbb{Z}$ are. We construct the Baer hull of the module N in a more general setting. Let R be a commutative noetherian domain. We first introduce a result from [40] for intermediate modules between an analogous direct sum as an R-module N and its injective hull *E(N)* to be Baer (Theorem 4.1). Then we use this result to construct and characterize the Baer hull of a module N over a Dedekind domain R, when $\text{Ann}_R(t(N)) \neq 0$ and $N/t(N)$ is finitely generated, where $t(N)$ denotes the torsion submodule of N (Theorems 4.4, 4.5, and 4.8). As a consequence, every finitely generated module over a Dedekind domain, has a unique Baer hull precisely when its torsion submodule is semisimple. For a module N such that $N/t(N)$ is not finitely generated, an example shows that N does not have a Baer hull (Example 4.12).

Among applications presented, we show that a finitely generated module N over a Dedekind domain is Baer if and only if N is semisimple or torsion-free (Corollary 4.6). This extends a result on finitely generated abelian groups. The isomorphism problem between modules and their Baer hulls is discussed (Proposition 4.13 and Example 4.14). It is also shown that the Baer hull of a direct sum of two modules is not necessarily isomorphic to the direct sum of the Baer hulls of the modules, even if all Baer hulls exist (Example 4.16). The Baer hull of $N = \mathbb{Z}_p \oplus \mathbb{Z}$ (p a prime integer) as a $\mathbb{Z}$-module, is shown to be precisely $V = \mathbb{Z}_p \oplus \mathbb{Z}[1/p]$. The disparity of the Baer hull and the extending hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ is discussed (Example 4.17). A number of other examples which illustrate the results are provided.

Let R be a commutative noetherian domain and F be its field of fractions. Assume that $N = M_R \oplus (\oplus_{i\in\Lambda} K_i)$, where M is semisimple with a finite number of homogeneous components, and $\{K_i\}_{i\in\Lambda}$ is a set of nonzero submodules of F_R.

By using the preceding results, we obtain the following which identifies intermediate modules between N and $E(N)$ which happen to be Baer modules.

Theorem 4.1 ([40, Theorem 2.6]) *Let R be a commutative noetherian domain, which is not a field. Assume that M is a nonzero semisimple R-module with only a finite number of homogeneous components, and $\{K_i \mid i \in \Lambda\}$ is a nonempty set of nonzero submodules of F_R, where F is the field of fractions of R. Let V_R be an essential extension of $M_R \oplus (\oplus_{i\in\Lambda} K_i)_R$. Then the following are equivalent.*

(i) V is a Baer module.
(ii) (1) $V = M \oplus W$ for some Baer essential extension W of $(\oplus_{i\in\Lambda} K_i)_R$.
(2) $Hom_R(W, M) = 0$.

Let R be a commutative domain with the field of fractions F. A submodule K of F_R is called a *fractional ideal* of R if $rK \subseteq R$ for some $0 \neq r \in R$. Thus $K_R \cong (rK)_R$ and rK is an ideal of R. We note that any ideal of R is a fractional ideal.

For a fractional ideal K of R, we put $K^{-1} = \{q \in F \mid qK \subseteq R\}$, which is called the *inverse* of K. We say that a fractional ideal K is *invertible* if $KK^{-1} = R$. It is well-known that for a nonzero ideal I of a commutative domain R, I_R is projective if and only if $II^{-1} = R$. In this case, I_R is finitely generated and I^{-1} is a fractional ideal of R.

Recall that a commutative domain R is a *Dedekind domain* if and only if R is hereditary. Thus for each nonzero ideal I of a Dedekind domain R, it follows that $II^{-1} = R$ because I_R is projective. Furthermore, every nonzero fractional ideal of a Dedekind domain is invertible. We note that a Dedekind domain is noetherian because every ideal is projective (hence every ideal is finitely generated). See [28, p. 37]and [48, Chap. 6] for more details on Dedekind domains.

Assume that I is an invertible ideal of a commutative domain R. Then we let

$$I^{-2} = I^{-1}I^{-1}, \quad I^{-3} = I^{-1}I^{-1}I^{-1}, \quad \text{and so on.}$$

For convenience, we put $I^0 = R$.

Assume that R is a Dedekind domain. Then for nonzero ideals $I_1, I_2, \ldots, I_n$ of R, it can be checked that $(I_1 I_2 \cdots I_n)^{-1} = I_n^{-1} \cdots I_2^{-1} I_1^{-1}$ (see [40, Lemma 2.8]).

Proposition 4.2 ([40, Lemma 2.9]) *Assume that R is a Dedekind domain and I is a nonzero ideal of R. We let $A = \sum_{\ell \geq 0} I^{-\ell}$. Then:*

(i) $A = R[q_1, q_2, \ldots, q_n]$, where $1 = \sum_{u=1}^{n} r_u q_u$ for some $r_u \in I$ and $q_u \in I^{-1}$ with $1 \leq u \leq n$.
(ii) A is a Dedekind domain.

In Proposition 4.2, since R is a Dedekind domain, R is a Prüfer domain. Because $A = R[q_1, q_2, \ldots, q_n]$ is an intermediate domain between R and its field of fractions, A is a Prüfer domain (note that any intermediate domain between a Prüfer domain and its field of fractions is a Prüfer domain). Since A is a noetherian domain, A is a Dedekind domain.

Theorem 4.3 ([48, Theorem 6.11, p. 171]) *Let R be a Dedekind domain and M an R-module with nonzero annihilator in R. Then there exists a unique family $\{P_i,\ n_i\}_{i \in \Gamma}$ such that:*

(i) The P_i are maximal ideals of R and there are only finitely many distinct ones.
(ii) $\{n_i \mid i \in \Gamma\}$ is a bounded family of positive integers.
(iii) $M \cong \oplus_{i \in \Gamma} (R/P_i^{n_i})$ as R-modules.

Let R be a Dedekind domain and N an R-module. Say $t(N)$ is the torsion submodule of N. Suppose that $N/t(N)$ is finitely generated as an R-module. Since $N/t(N)$ is torsion-free, $N/t(N) \cong (\oplus_{j=1}^{m} K_j)$ (as R-modules) for some fractional ideals K_j, $1 \leq j \leq m$, of R from [48, Theorem 6.16, p. 177] (see also Corollary 4.6). So $N/t(N)$ is projective, and hence

$$N \cong t(N) \oplus N/t(N) \cong t(N) \oplus (\oplus_{j=1}^{m} K_j)$$

as R-modules.

Our next result is a complete characterization for the existence of the Baer hull of a module N when $N/t(N)$ is finitely generated and $\text{Ann}_R(t(N)) \neq 0$ (also see Theorem 4.5). Furthermore, we describe the Baer hull of N explicitly in this case.

We denote the Baer hull of a module M by $\mathfrak{B}(M)$ when it exists.

Theorem 4.4 ([40, Theorem 2.13]) *Let R be a Dedekind domain. Assume that M is an R-module with nonzero annihilator in R, and $\{K_1, K_2, \ldots, K_m\}$ is a finite set of nonzero fractional ideals of R. Then the following are equivalent.*

(i) $M_R \oplus (\oplus_{j=1}^{m} K_j)_R$ has a Baer hull.
(ii) M_R is semisimple.
(iii) $M_R \oplus (\oplus_{j=1}^{m} K_j)_R$ has a Baer essential extension.

In this case, $\mathfrak{B}(M_R \oplus (\oplus_{j=1}^m K_j)_R) = M_R \oplus (\oplus_{j=1}^m K_j A)_R$, *where* $A = \sum_{\ell \geq 0} I^{-\ell}$ *with* $I = \mathrm{Ann}_R(M)$. *Furthermore,* $A = R[q_1, q_2, \ldots, q_n]$, *where* $1 = \sum_{u=1}^n r_u q_u$ *with* $r_u \in I$ *and* $q_u \in I^{-1}$, $1 \leq u \leq n$.

The following is a restatement of Theorem 4.4 for characterization of the Baer hull of a module N over a Dedekind domain for the case when $N/t(N)$ is finitely generated and $\mathrm{Ann}_R(t(N)) \neq 0$.

Theorem 4.5 ([40, Theorem 2.15]) *Let R be a Dedekind domain. Assume that N is an R-module with $N/t(N)$ finitely generated and $\mathrm{Ann}_R(t(N)) \neq 0$. Then the following are equivalent.*

(i) N has a Baer hull.
(ii) $t(N)$ is semisimple.
(iii) N has a Baer essential extension.

By [44, Proposition 2.19 and Remark 2.20], a finitely generated module N over a commutative PID is a Baer module if and only if N is semisimple or torsion-free. This result is extended to the case when the base ring is a Dedekind domain as follows by applying Theorems 4.4 and 4.5.

Corollary 4.6 ([40, Corollary 2.17]) *Let R be a Dedekind domain and N be a finitely generated R-module. Then the following are equivalent.*

(i) N is a Baer module.
(ii) N is semisimple or torsion-free.

The next theorem details the structure of finitely generated modules over a Dedekind domain.

Theorem 4.7 ([48, Theorem 6.16, p. 177]) *Let R be a Dedekind domain and N a finitely generated R-module. Then there exist positive integers $n_1, n_2, \ldots, n_k$ (k is a nonnegative integer), nonzero maximal ideals $P_1, P_2, \ldots, P_k$, and nonzero fractional ideals $K_1, K_2, \ldots, K_m$ (m is a nonnegative integer) of R such that $N \cong (\oplus_{i=1}^k R/P_i^{n_i}) \oplus (\oplus_{j=1}^m K_j)$ as R-modules.*

Assume that N is a finitely generated module over a Dedekind domain. From Theorem 4.7, $N \cong (\oplus_{i=1}^k R/P_i^{n_i}) \oplus (\oplus_{j=1}^m K_j)$, where P_i are nonzero maximal ideals of R and K_j are nonzero fractional ideals of R (k and m are nonnegative integers). In the following theorem, we characterize the existence of the Baer hull of such N and describe the Baer hull of N explicitly.

Theorem 4.8 ([40, Theorem 2.18]) *Let R be a Dedekind domain, and let N be a finitely generated R-module. Then the following are equivalent.*

(i) N has a Baer hull.
(ii) $t(N)$ is semisimple.
(iii) N has a Baer essential extension.

In this case, $\mathfrak{B}(N_R) \cong (\oplus_{i=1}^{k}(R/P_i))_R \oplus (\oplus_{j=1}^{m} K_j A)_R$*, where* $A = \sum_{\ell \geq 0} I^{-\ell}$ *with* $I = Ann_R(t(N))$*. Further,* $A = R[q_1, q_2, \ldots, q_n]$*, where* $1 = \sum_{u=1}^{n} r_u q_u$ *with* $r_u \in I$ *and* $q_u \in I^{-1}$, $1 \leq u \leq n$.

The following remark exhibits an explicit description of $A = \sum_{\ell \geq 0} I^{-\ell}$ in Theorem 4.4.

Remark 4.9 We have the following (see [40, Remark 3.1]).

(i) In Theorem 4.3, we put $A = \sum_{\ell \geq 0} I^{-\ell}$, where $I = \mathrm{Ann}_R(M)$. By Theorems 4.3 and 4.4, $M \cong \oplus_{i \in \Gamma} R/P_i$ and $\{P_i \mid i \in \Gamma\}$ is a finite set of maximal ideals P_i. Let $P_1, P_2, \ldots, P_s$ is all the distinct maximal ideals of $\{P_i \mid i \in \Gamma\}$.
We can verify that $A = \sum P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s}$, where $\ell_1, \ell_2, \ldots, \ell_s$ run through all nonnegative integers. In fact, $I \subseteq P_i$ for all i since $I = P_1 P_2 \cdots P_s$. For i, $1 \leq i \leq s$, $P_i^{-1} \subseteq I^{-1}$ and therefore $P_i^{-\ell} \subseteq I^{-\ell}$ for every nonnegative integer ℓ. Hence,

$$P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s} \subseteq I^{-\ell_1} I^{-\ell_2} \cdots I^{-\ell_s} = I^{-(\ell_1 + \ell_2 + \cdots + \ell_s)} \subseteq A.$$

Thus $\sum P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s} \subseteq A$, where $\ell_1, \ell_2, \ldots, \ell_s$ run through all nonnegative integers. Conversely, $I^{-1} = (P_1 P_2 \cdots P_s)^{-1} = P_1^{-1} P_2^{-1} \cdots P_s^{-1}$. Therefore it follows that $I^{-\ell} = P_1^{-\ell} P_2^{-\ell} \cdots P_s^{-\ell}$ for any nonnegative integer ℓ. Hence we obtain that $A \subseteq \sum P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s}$, where $\ell_1, \ell_2, \ldots, \ell_s$ run through all nonnegative integers.
Consequently, $A = \sum P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s}$, where $\ell_1, \ell_2, \ldots, \ell_s$ run through all nonnegative integers.

(ii) Let R be a commutative PID. Assume that M is a nonzero semisimple R-module with nonzero annihilator in R. Then from Theorem 4.3, M has only a finite number of homogeneous components. Let $\{H_k \mid 1 \leq k \leq s\}$ be the set of all homogeneous components of M. For k, $1 \leq k \leq s$, we put $H_k = \oplus_\alpha M_{(k,\alpha)}$ with each $M_{(k,\alpha)}$ simple. So $M_{(k,\alpha)} \cong R/p_k R$ for k, $1 \leq k \leq s$, with p_k a nonzero prime.
We put $P_k = \mathrm{Ann}_R(H_k)$ for k, $1 \leq k \leq s$. Then $P_k = p_k R$. For a nonnegative integer ℓ, we can routinely verify that $P_k^{-\ell} = (1/p_k^{\ell})R$ for k, $1 \leq k \leq s$. Therefore,

$$P_1^{-\ell_1} P_2^{-\ell_2} \cdots P_s^{-\ell_s} = (1/p_1^{\ell_1})(1/p_2^{\ell_2}) \cdots (1/p_s^{\ell_s})R$$

for nonnegative integers $\ell_1, \ell_2, \ldots, \ell_s$.

Let $A = \sum_{\ell \geq 0} I^{-\ell}$, where $I = \mathrm{Ann}_R(M) = P_1 P_2 \cdots P_s = p_1 p_2 \cdots p_s R$. By the preceding argument, $A = R[1/p_1, 1/p_2, \ldots, 1/p_s]$. Put $a = p_1 p_2 \cdots p_s$. Then it follows that $A = R[1/a]$ because $I^{-\ell} = (1/a^{\ell})R$. Also note that $\mathrm{Ann}_R(M) = aR$.

Example 4.10 ([40, Example 3.2]) Let Γ_i, $i = 1, 2, 3$ be nonempty sets and m be a positive integer. Then by Remark 4.9(ii), we have

$$\mathfrak{B}(\mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)} \oplus \mathbb{Z}^{(m)}) = \mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)} \oplus \mathbb{Z}[1/30]^{(m)}$$

as $\mathbb{Z}$-modules because $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)}) = 30\mathbb{Z}$.

For a ring R and a nonempty set Λ, we use $\mathrm{CFM}_\Lambda(R)$ to denote the $\Lambda \times \Lambda$ column finite matrix ring over the ring R.

Theorem 4.11 ([50, Theorem 2] and [47, Theorem 3.3]) *Let R be a ring. Then the following are equivalent.*

(i) R is a semiprimary right (and left) hereditary ring.
(ii) $CFM_\Lambda(R)$ is a Baer ring for any nonempty set Λ.

Example 4.12 in the following shows that the hypothesis "$\{K_1, K_2, \ldots, K_m\}$ is a finite set" in Theorem 4.4 and the hypothesis "$N/t(N)$ is finitely generated" in Theorem 4.5 are not superfluous conditions for the existence of the Baer hull of N.

Example 4.12 (see [40, Example 3.6]) Let Γ_i, $i = 1, 2, 3$ be nonempty sets as in Example 4.10. Since $\mathbb{Z}[1/30]$ is not a field, $\mathbb{Z}[1/30]$ is not semiprimary because $\mathbb{Z}[1/30]$ is a domain. By Theorem 4.11, there exists a nonempty set Λ such that $\mathrm{CFM}_\Lambda(\mathbb{Z}[1/30])$ is not a Baer ring. Note that the set Λ is necessarily infinite. In fact, if Λ is finite with the cardinality n, then $\mathrm{CFM}_\Lambda(\mathbb{Z}[1/30]) = \mathrm{Mat}_n(\mathbb{Z}[1/30])$ is a Baer ring as $\mathbb{Z}[1/30]$ is a Prüfer domain (see [10, Theorem 6.1.4, p. 191]), a contradiction. Let

$$N = \mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)} \oplus \mathbb{Z}^{(\Lambda)}.$$

Then we have the following.

(i) $V := \mathbb{Z}_2^{(\Gamma_1)} \oplus \mathbb{Z}_3^{(\Gamma_2)} \oplus \mathbb{Z}_5^{(\Gamma_3)} \oplus \mathbb{Z}[1/30]^{(\Lambda)}$ is not a Baer $\mathbb{Z}$-module. In fact, if V is a Baer module, then $\mathbb{Z}[1/30]^{(\Lambda)}$ is Baer as a $\mathbb{Z}$-module by Theorem 3.12. We show that

$$\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/30]^{(\Lambda)}) = \mathrm{End}_{\mathbb{Z}[1/30]}(\mathbb{Z}[1/30]^{(\Lambda)}).$$

For this, first we note that $\mathrm{End}_{\mathbb{Z}[1/30]}(\mathbb{Z}[1/30]^{(\Lambda)}) \subseteq \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/30]^{(\Lambda)})$. Next, let $f \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/30]^{(\Lambda)})$. Assume on the contrary that $f \notin \mathrm{End}_{\mathbb{Z}[1/30]}(\mathbb{Z}[1/30]^{(\Lambda)})$. Then there exist $y \in \mathbb{Z}[1/30]^{(\Lambda)}$ and $q \in \mathbb{Z}[1/30]$ such that $f(yq) - f(y)q \neq 0$. Put $q = ab^{-1}$, where $a,\ b \in R$ and $b \neq 0$. So

$$0 \neq (f(yq) - f(y)q)b = f(yq)b - f(y)a = f(yqb) - f(ya) = f(ya) - f(ya) = 0,$$

which is a contradiction. Therefore, $f \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/30]^{(\Lambda)})$. Consequently, we have that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/30]^{(\Lambda)}) = \mathrm{End}_{\mathbb{Z}[1/30]}(\mathbb{Z}[1/30]^{(\Lambda)})$. From Theorem 3.11,

$$\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[1/30]^{(\Lambda)}) = \mathrm{End}_{\mathbb{Z}[1/30]}(\mathbb{Z}[1/30]^{(\Lambda)}) = \mathrm{CFM}_\Lambda(\mathbb{Z}[1/30])$$

is a Baer ring. So we get a contradiction.

(ii) $N/t(N)(\cong \mathbb{Z}^{(\Lambda)})$ is not finitely generated as a $\mathbb{Z}$-module because Λ is infinite.

(iii) *N has no Baer module hull* as a $\mathbb{Z}$-module.

In Proposition 4.13 and Example 4.14, we consider the isomorphism problem for Baer hulls as follows: Let N_1 and N_2 be modules with Baer hulls $\mathfrak{B}(N_1)$ and $\mathfrak{B}(N_2)$, respectively. Is it true that $N_1 \cong N_2$ if and only if $\mathfrak{B}(N_1) \cong \mathfrak{B}(N_2)$ in this case?

Proposition 4.13 ([40, Proposition 3.8]) *Let N_1 and N_2 are isomorphic modules. If N_1 has a Baer hull $\mathfrak{B}(N_1)$, then N_2 has a Baer hull $\mathfrak{B}(N_2)$, and $\mathfrak{B}(N_1) \cong \mathfrak{B}(N_2)$ as modules.*

The next example shows that the converse of Proposition 4.13 does not hold true. In other words, there exist modules N_1 and N_2 such that $\mathfrak{B}(N_1) = \mathfrak{B}(N_2)$ (hence $\mathfrak{B}(N_1) \cong \mathfrak{B}(N_2)$ as modules), but $N_1 \ncong N_2$. Thus the isomorphism problem does not hold for the case of Baer hulls.

Example 4.14 ([40, Example 3.9]) Let $N_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$. Then by Theorem 4.4 or Example 4.10, $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6]$ is the Baer hull of N_1 as $\mathbb{Z}$-modules.

Next, let $N_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/3]$. Say V is a Baer module with $N_2 \le V \le E(N_2)$. From Theorem 4.1, $V = \mathbb{Z}_2 \oplus Z_3 \oplus W$ for some Baer module W such that

$$\mathbb{Z}[1/3] \le W \le \mathbb{Q} \quad \text{and} \quad \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}_2 \oplus \mathbb{Z}_3) = 0.$$

Thus $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}_2) = 0$, and so $2^k W = W$ for any nonnegative integer k (see the proof of [40, Theorem 2.13]). Therefore $1/2^k \in W$ for any positive integer k, and thus $\mathbb{Z}[1/2, 1/3] \le W$. Hence we have

$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/2, 1/3] = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6] \le V.$$

Because $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6]$ is Baer as a $\mathbb{Z}$-module, $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}[1/6]$ is the Baer hull of N_2. However, $N_1 \ncong N_2$ because $\mathbb{Z} \ncong \mathbb{Z}[1/3]$ as $\mathbb{Z}$-modules.

In the next examples, we compare the direct sum of Baer hulls with the Baer hull of a direct sum of modules.

Example 4.15 ([40, Example 2.19]) There exist two modules W_1 and W_2 such that both W_1 and W_2 have Baer module hulls, but $W_1 \oplus W_2$ has no Baer hull.

Let $R = \mathbb{Z}[x]$, the polynomial ring over $\mathbb{Z}$. Put $N = (R \oplus R)_R$. Then $t(N) = 0$, so $t(N)$ is semisimple. However, N has no Baer hull. For this, note that if N is a Baer module, then $\mathrm{End}_R(N) = \mathrm{Mat}_2(R)$ is a Baer ring from Theorem 3.12. So [10, Theorem 6.1.4, p. 191] yields that the ring $R = \mathbb{Z}[x]$ must be Prüfer, which is a contradiction.

Say B is the Baer hull of N. Put $F = \mathbb{Q}(x)$, the field of fractions of R. Note that $E(N) = F \oplus F$. Put $U = F \oplus R$. Then by [10, Theorem 4.2.18, p. 107], U_R is a Baer module. Similarly, $V_R := (R \oplus F)_R$ is a Baer module. Thus $B \subseteq U \cap V = N$, so $B = N$. Hence N is Baer, a contradiction. Therefore N has no Baer hull.

Example 4.16 ([40, Example 3.10]) There exist two modules M and N such that M, N, and $M \oplus N$ have Baer hulls $\mathfrak{B}(M)$, $\mathfrak{B}(N)$, and $\mathfrak{B}(M \oplus N)$, respectively. But

$$\mathfrak{B}(M \oplus N) \ncong \mathfrak{B}(M) \oplus \mathfrak{B}(N).$$

Let $M = \mathbb{Z}_p$ (p a prime integer) and $N = \mathbb{Z}$ as $\mathbb{Z}$-modules. Then $\mathfrak{B}(M) = \mathbb{Z}_p$ and $\mathfrak{B}(N) = \mathbb{Z}$ since $\mathbb{Z}_p$ is a semisimple $\mathbb{Z}$-module and $\mathbb{Z}$ is a Baer ring. Therefore we have that $\mathfrak{B}(M) \oplus \mathfrak{B}(N) = \mathbb{Z}_p \oplus \mathbb{Z}$.

On the other hand, $\mathfrak{B}(M \oplus N) = \mathfrak{B}(\mathbb{Z}_p \oplus \mathbb{Z}) = \mathbb{Z}_p \oplus \mathbb{Z}[1/p]$ (see Theorem 4.8 and Remark 4.9(ii)). Hence $\mathfrak{B}(M \oplus N) \ncong \mathfrak{B}(M) \oplus \mathfrak{B}(N)$ because $\mathbb{Z} \ncong \mathbb{Z}[1/p]$ as $\mathbb{Z}$-modules.

The following example exhibits the disparity of the Baer hull and the extending hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ (p a prime integer).

Example 4.17 [40, Example 3.7]) (i) Let $V = \mathbb{Z}_p \oplus \mathbb{Z}[1/p]$, where p is a prime integer. Then by Remark 4.9(ii), V is the Baer hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module. Hence in view of Theorem 3.9, one might expect that V is also the extending hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module. But this is not true. Further, V is not even extending from [25, Corollary 2]. In fact, the extending hull of $\mathbb{Z}_p \oplus \mathbb{Z}$ is $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$, where $\mathbb{Z}_{p^\infty}$ is the Prüfer p-group.

(ii) In the chain of $\mathbb{Z}$-submodules $\mathbb{Z}_p \leq \mathbb{Z}_{p^2} \leq \cdots \leq \mathbb{Z}_{p^\infty}$ of $\mathbb{Z}_{p^\infty}$ (p a prime integer), $\mathbb{Z}_p$ is the Baer hull (also quasi-injective hull) of itself, and $\mathbb{Z}_{p^\infty}$ is the injective hull of each of the modules in the chain. However, $\mathbb{Z}_{p^n}$ ($n > 1$) has no Baer hull by Theorem 4.8. Also note that $\mathbb{Z}_{p^\infty}$ has no Baer hull.

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