The Finite Volume Diffusion Method on Nonmatched Polygonal Meshes Suited for the Lagrangian Slide Lines Calculation

Xuezhe Liu, Zhong Lin and Ruili Wang

Abstract An efficient diffusion finite volume method on nonmatched polygonal meshes suited for fluid slide line calculation is proposed. The method manages the sliding meshes and the internal meshes unifying as arbitrary polygonal meshes, takes the hanging-nodes on slip-lines naturally as the nodes of the polygon, and constructs unified diffusion schemes with high accuracy and highly efficient solving algorithms. Furthermore, the radiation diffusion code on unstructured polygonal meshes has been developed and coupled with the hydrocode. Numerical results show the validity of the radiation diffusion computational method for Lagrangian slide lines calculation on nonmatched polygonal meshes.

Keywords Finite volume diffusion scheme · Nonmatched polygonal meshes · Lagrangian slide lines calculation

1 Introduction

In realistic physical simulations, people often face the problem of shear flow at material interfaces. If the materials move along each other but are tied to a single computational mesh without any sliding treatment, severe mesh distortions appear which can eventually caused the failure of the simulation. This problem is usually treated by introducing a slide line framework into the Lagrangian code. The introduction of slide lines is an old but fruitful idea that dates back to Wilkins [1] as a chapter in a book (reproduced in Chap. 5 of Wilkins [2]).

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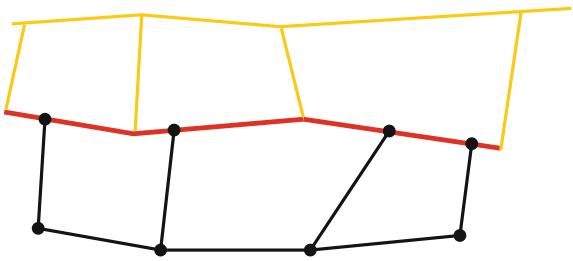

Fig. 1 A sample of nonmatched meshes with *slide line*

Suppose that there exist two different meshes interacting with each other through a common sliding line, one of them is specified as the master side defining the slide line shape, while the other -slave- side follows the slide line. Because of the different tangential-velocity component, the sliding treatment will cause non-matched meshes with arbitrary shapes (with hanging nodes), see Fig. 1.

If the diffusion solver is to be incorporated into a two-dimensional (2D) hydrodynamics code, the diffusion equation is solved on the hydrodynamic meshes, so it becomes a key issue whether the diffusion solver is suited with the nonmatched sliding meshes. Typically, slide line calculation requires special treatment due to the different mesh topology for the sliding meshes and the internal meshes, and this causes considerable difficulties for the design of diffusion schemes and iterative algorithms on such complex meshes, see reference [3, 4] and the reference cited there.

In this paper, we propose an efficient discretization method for solving diffusion equations on nonmatched meshes which has been caused by hydrodynamics slide lines calculation. The method manages the sliding meshes and the internal meshes unifying as arbitrary polygonal meshes, takes the hanging-nodes on slide lines naturally as the nodes of the polygon, and constructs unified diffusion schemes with high accuracy and highly efficient solving algorithms. The diffusion scheme on unstructured polygonal meshes has the advantages of easy coding and is quite promising for application in multi-dimensional radiation hydrodynamics codes.

2 Basic Equations

Let Ω be an open bounded subset of R^2 with $\partial\Omega$ being its boundary. We consider the following nonlinear diffusion equations with Robin boundary condition:

$$\begin{aligned} \rho\frac{d\varepsilon}{dt} - \nabla\cdot(\kappa\nabla T) &= f, \\ T(x,r,0) &= T^0, \\ \alpha\kappa\nabla T\cdot\boldsymbol{n} + \beta T &= g, \end{aligned} \tag{1}$$

where ρ is the mass density, $\varepsilon = \varepsilon(\rho, T)$ is the mass-specific internal energy, t is the time, $\kappa = \kappa(\rho, T)$ is the heat conduction coefficient, T is the temperature and f is the

volume-specific external energy sources. $\boldsymbol{n}$ is the unit outward normal vector. α, β, g are the given values as the function of the space coordinates (x, r).

3 Construction of Diffusion Scheme on Nonmatched Polygonal Meshes

Let Ω_h be a non-overlapping conformal partition of Ω onto polygonal elements Ω_i, we denote the cell and cell center by i, the node by α, and the cell edge by σ, where $i = 1, \cdots, I$, and I is the total number of the mesh cell. Denote A_i as the area of cell i, L_{σ_j} as the length of the j-th edge σ_j of cell i, $d_{i,\sigma_j}, d_{i_j,\sigma_j}$ as the distance of σ_j and cell center i, i_j respectively, θ as the angle between the line from i to i_j and edge $\sigma_j, \boldsymbol{n}_{i,\sigma_j}$ as the unit outward normal vector which points to cell i_j (see Fig. 2).

By integrating (1) over Ω_i and using the Green formula, we obtain the following discretized finite volume diffusion scheme (more details in [5]):

$$m_i \frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} + \sum_{j=1}^{l_i} F_{\sigma_j}^{n+1} S_{\sigma_j} = \nabla_i f_i^{n+1}, \tag{2}$$

Where m_i is the mass of the cell i, Δt is the time step, $t^{n+1} = t^n + \Delta t$, l_i is the total number of the edges for cell i, S_{σ_j} is the rotational area and ∇_i is the rotational volume. F_{σ_j} is the discretized normal flux on edge σ_j in terms of cell-centered unknowns and node-unknowns:

$$F_{\sigma_j}^{n+1} = \begin{cases} -\kappa_{\sigma_j}^{n+1} \left[\frac{T_{i_j}^{n+1} - T_i^{n+1}}{d_{i_j,\sigma_j} + d_{i,\sigma_j}} - D_{\sigma_j} \left(\frac{T_{\alpha_{j+1}}^{n+1} - T_{\alpha_j}^{n+1}}{L_{\sigma_j}} \right) \right], & \sigma_j - \text{internal edge} \\ -\kappa_i^{n+1} \frac{g_{\sigma_j} - \beta_{\sigma_j} T_i^{n+1}}{\alpha_{\sigma_j} \kappa_i^{n+1} + \beta_{\sigma_j} d_{i,\sigma_j}}, & \sigma_j - \text{boundary edge} \end{cases} \tag{3}$$

Where

$$\kappa_{\sigma_j}^{n+1} = \frac{d_{i_j,\sigma_j} + d_{i,\sigma_j}}{\frac{d_{i_j,\sigma_j}}{\kappa_{i_j}^{n+1}} + \frac{d_{i,\sigma_j}}{\kappa_i^{n+1}}}, \quad D_{\sigma_j} = \frac{(\boldsymbol{R}_{i_j} - \boldsymbol{R}_i)}{|\boldsymbol{R}_{i_j} - \boldsymbol{R}_i|} \cdot \frac{(\boldsymbol{R}_{\alpha_{j+1}} - \boldsymbol{R}_{\alpha_j})}{L_{\sigma_j}} \frac{|\boldsymbol{R}_{i_j} - \boldsymbol{R}_i|}{d_{i_j,\sigma_j} + d_{i,\sigma_j}}.$$

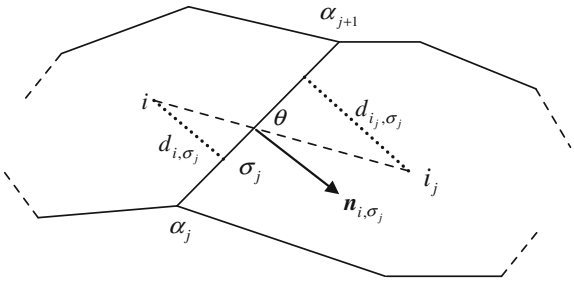

Fig. 2 Stencil of the diffusion scheme on polygonal meshes

κ_{σ_j} is the harmonic mean of the κ on the adjacent meshes i and i_j of edge σ_j, D_{σ_j} is the geometric quantity characterizing the mesh distortion, when the mesh is orthogonal, D_{σ_j} will be zero.

In order to obtain the normal flux approximation expressed in terms of cell-centered unknowns only, we should eliminate the terms associated with node unknowns, i.e. $T_{\alpha_{j+1}}$ and T_{α_j} in expression (3). It is usually approximated with the expression

$$T_\alpha = \frac{1}{\sum_{k=1}^{m_\alpha} \omega_{k,\alpha}} \sum_{k=1}^{m_\alpha} \omega_{k,\alpha} T_{i_k},$$

where m_α is the total number of the adjacent cells for node α,T_{ij} is the temperature of the k-th adjacent cell $i_k, \omega_{k,\alpha}$ is the weight, see Fig. 3.

In [6], a highly accurate method of eliminating the cell node unknowns has been proposed on quadrilateral meshes based on Taylor expansion. We extend this approach to unstructured polygonal meshes, which is well suited for distorted nonmatched meshes.

The diffusion scheme on nonmatched polygonal meshes is fully implicit and should be solved by a nonlinear iteration method. ε is the nonlinear function of temperature T, it can be represented by the temperature:

$$\varepsilon_i^{n+1(s+1)} = \varepsilon_i^{n+1(s)} + \left(\frac{\partial \varepsilon}{\partial T}\right)^{n+1(s)} \left(T_i^{n+1(s+1)} - T_i^{n+1(s)}\right),$$

and κ is taken as the value of the last iterative step:

$$\kappa(\rho^{n+1}, T^{n+1}) \approx \kappa^{(s)} = \kappa(\rho^{n+1}, T^{(s)}).$$

Thus, we can obtain the linear systems which can be rewritten as the matrix form:

$$A\left(T^{n+1(s)}\right) T^{n+1(s+1)} = b\left(T^{n+1(s)}\right).$$

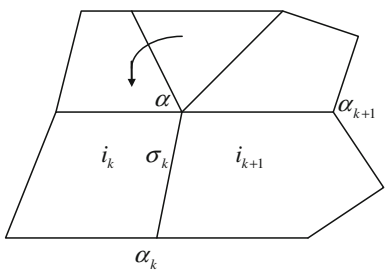

Fig. 3 Adjacent cell of node α

The symmetric linear systems are solved by the conjugate gradient (CG) method, and the nonsymmetric linear systems are solved by the biconjugate gradient stabilized algorithm (BICGSTAB) (see [7]).

4 Numerical Tests

Now, we will present some numerical results to illustrate the behavior of the proposed finite volume scheme.

Let Ω be the unit square, and let $\partial\Omega_S$, $\partial\Omega_E$, $\partial\Omega_N$, $\partial\Omega_W$ be the boundaries of Ω. There is a nonmatched interface located around $x = 0.5$ the mesh is independently generated by different methods on either side of the interface.

In our nonmatched grids we measure the mesh size approximately with the average size which is defined as:

$$h = \sqrt{\frac{S_\Omega}{I}},$$

where S_Ω is the area of the whole computation domain and I is the sum of the cell number in all patches.

Note that an exact solution can be found for all our tests. Let $\hat{T}_i^h$ be that exact solution at the centroid of the cell *i.* T_i^h is the numerical solution in the same cell. Then, the convergence rates can be estimated with the asymptotic errors using both the maximum norm

$$E_{\max}^h = \max_{i=1,\cdots,I} \left| T_i^h - \hat{T}_i^h \right|,$$

and the mean-square norm

$$E_{L^2}^h = \sqrt{\sum_{i=1}^{I} \left(T_i^h - \hat{T}_i^h \right)^2 A_i}.$$

The asymptotic error is estimated by

$$E^h = Ch^q + O\left(h^{q+1}\right),$$

where q is the order of truncation error and the constant C is independent of h.

Calculations are performed on a sequence of grids with different values of h. For two grids with mesh size h_1 and h_2, we denote the asymptotic error on these two

meshes with E_{h_1} and E_{h_2}, respectively. Then the order of convergence is evaluated using

$$q = \frac{\log\left(\frac{E^{h_1}}{E^{h_2}}\right)}{\log\left(\frac{h_1}{h_2}\right)}.$$

4.1 Linear Diffusion Problem

We consider the following linear diffusion equation

$$\begin{array}{lll} T_t - \nabla \cdot \nabla T = -2\pi^2 e^{-\pi^2 t} & \text{in} & \Omega, \\ T = e^{-\pi^2 t}(2 + \cos(\pi x) + \sin(\pi y)) & \text{on} & \partial\Omega_S \cup \partial\Omega_N, \\ \nabla T \cdot \boldsymbol{n} = 0 & \text{on} & \partial\Omega_E \cup \partial\Omega_W, \\ T(x,0) = 2 + \cos(\pi x) + \sin(\pi y) & \text{on} & \Omega. \end{array}$$

The exact solution for this problem is given by $T = e^{-\pi^2 t}(2 + \cos(\pi x) + \sin(\pi y))$.

We perform the calculation on a sequence of nonmatched polygonal meshes shown in Fig. 4, the numbers of cells are 158, 570, 2180 respectively. Figure 5 shows that the numerical solution is convergent to the exact solution as the mesh is further refined.

In addition the asymptotic error in the maximum norm and in the mean-square norm are displayed in the Fig. 6. The line labeled with 2nd order is also plotted to give the theoretical second convergence error. The results for our method show that it has almost a second order convergence rate in both the max and L_2 norms on nonmatched polygonal meshes.

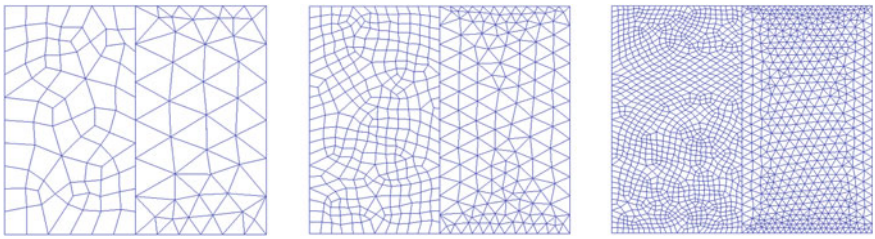

Fig. 4 A sequence of nonmatched polygonal meshes

Fig. 5 Contours of the numerical and exact solution for the linear problem

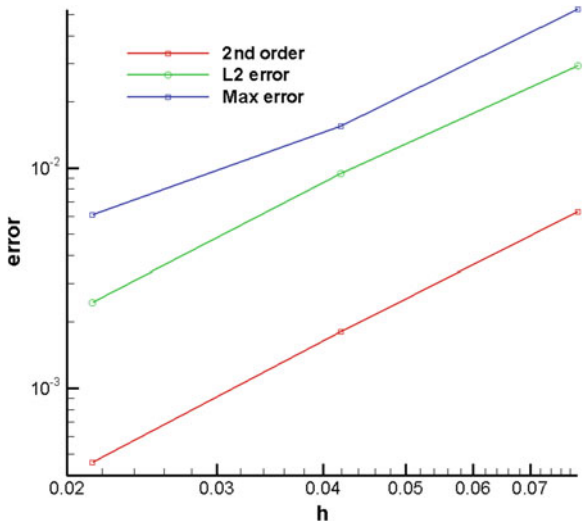

Fig. 6 Asymptotic error in logarithmic scale for the linear problem

4.2 Nonlinear Diffusion Problem

Consider the following nonlinear problem

$$\begin{aligned} T_t - \nabla \cdot (\kappa \nabla T) &= f \quad \text{in} \quad \Omega \\ T(x, y, 0) &= T_0 \quad \text{on} \quad \Omega \\ T(x, y, t) &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

The heat conduction coefficient is $\kappa(T) = T^{7/5} + 1$, and the nonlinear exact solution is given as $T = e^t(2x^3 - 3x^2 + 2)(2y^3 - 3y^2 + 3)$. The source term, Dirichlet boundary condition and the initial value of the temperature are given by the exact solution. We compute the solution until $t = 0.1$.

The asymptotic errors are evaluated on a sequence of nonmatched random meshes with three different resolutions. Specifically, there is a grid of 10×10

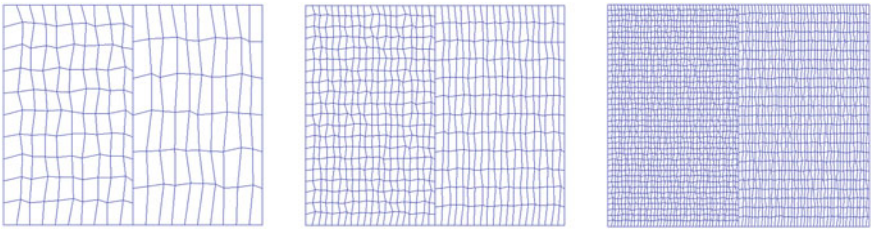

Fig. 7 A sequence of nonmatched random meshes

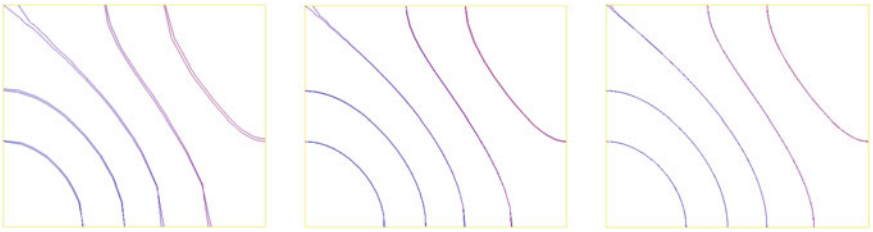

Fig. 8 Contours of the numerical and exact solution for nonlinear problem

(resp. $20 \times 20, 40 \times 40$) cells on the left hand side of the sliding interface. And on the right-hand side is a grid of 10×6 (resp. $20 \times 12, 40 \times 24$) cells, see Fig. 7. It is shown in Fig. 8 It is shown in Fig. 8 that the numerical solution is convergent to the exact solution as the mesh is further refined.

In addition the asymptotic error in the maximum norm and in the mean-square norm are displayed in the Fig. 9. The line labeled with 2nd order is also plotted to

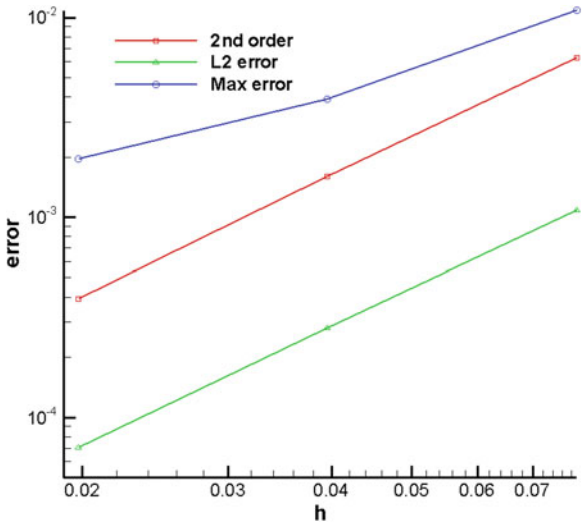

Fig. 9 Asymptotic error in logarithmic scale for the nonlinear problem

give the theoretical second convergence error. The results for our method show that it has almost a second order convergence rate in both the max and L_2 norms on nonmatched polygonal meshes.

5 Conclusion

In this paper, we propose a finite volume diffusion method on nonmatched polygonal meshes caused by the fluid sliding computation. The diffusion scheme has some benefits:

- Cell center unknowns only
- Numerical flux has explicit physical meaning
- Numerical tests show the method has almost second order convergence for linear and nonlinear diffusion problems on distorted nonmatched meshes
- Suited for arbitrary polygonal meshes and easy to incorporated into a Lagrangian hydrodynamics code.

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