Chapter 9
Subgame Consistency in Randomly-Furcating Cooperative Stochastic Dynamic Games

This Chapter considers subgame consistent cooperative solutions in randomly furcating stochastic discrete-time dynamic games. In particular, in this type of games the evolution of the state is stochastic and future payoff structures are not known with certainty. The presence of random elements in future payoff structures and stock dynamics are prevalent in many practical game situations like regional economic cooperation, corporate joint ventures and transboundary environmental management. The analysis is based on Yeung and Petrosyan (2013a). It first develops a class of randomly furcating stochastic dynamic games in which future payoff structures of the game furcates or branches out randomly and the discrete-time game dynamics evolves stochastically. Nash equilibria of this class of games are characterized for non-cooperative outcomes and subgame-consistent solutions are derived for cooperative paradigms. A discrete-time analytically tractable payoff distribution procedure contingent upon specific random realizations of the state and payoff structure is derived. Worth mentioning is that in computer modeling and operations research discrete-time analysis often proved to be more applicable and compatible with actual data than continuous-time analysis. The Chapter is organized as follows. The game formulation and non-cooperative equilibria are given in Sect. 9.1. Group optimality and individual rationality under dynamic cooperation are discussed in Sect. 9.2. Subgame consistent solutions and payment mechanism leading to the realization of these solutions are obtained in Sect. 9.3. Section 9.4 presents an illustration in cooperative resource extraction. Extensions of the model are provided in Sect. 9.5. Chapter appendices, chapter notes and problems are presented in Sect. 9.6, Sect. 9.7, and Sect. 9.8 respectively.

9.1 Game Formulation and Non-cooperative Outcome

In this Section, we first consider the formulation of a general class of randomly-furcating stochastic dynamic games and then derive the non-cooperative outcome.

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9.1.1 Randomly-Furcating Stochastic Dynamic Games

Consider the $T-$ stage $n-$ person nonzero-sum dynamic game with initial state x^0. The state space of the game is $X \in R^m$ and the state dynamics of the game is characterized by the stochastic difference equation:

$$x_{k+1} = f_k\left(x_k, u_k^1, u_k^2, \cdots, u_k^n\right) + \vartheta_k, \tag{1.1}$$

for $k \in \{1, 2, \cdots, T\}$ and $x_1 = x^0$,

where $u_k^i \in U^i \subset R^{m_i}$ is the control vector of player i at stage $k, x_k \in X$ is the state, and ϑ_k is a sequence of statistically independent random variables.

The payoff of player i at stage k is $g_k^i\left(x_k, u_k^1, u_k^2, \cdots, u_k^n; \theta_k\right)$ which is affected by a random variable θ_k. In particular, θ_k for $k \in \{1, 2, \cdots, T\}$ are independent discrete random variables with range $\left\{\theta_k^1, \theta_k^2, \cdots, \theta_k^{\eta_k}\right\}$ and corresponding probabilities $\left\{\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^{\eta_k}\right\}$, where η_k is a positive integer for $k \in \{1, 2, \cdots, T\}$. In stage 1, it is known that θ_1 equals θ_1^1 with probability $\lambda_1^1 = 1$.

The objective that player i seeks to maximize is

$$E_{\theta_1, \theta_2, \cdots, \theta_T; \vartheta_1, \vartheta_2, \cdots, \vartheta_T}\left\{\sum_{k=1}^{T} g_k^i\left(x_k, u_k^1, u_k^2, \cdots, u_k^n; \theta_k\right) + q^i(x_{T+1})\right\},$$
$$\text{for } i \in \{1, 2, \cdots, n\} \equiv N, \tag{1.2}$$

where $E_{\theta_1, \theta_2, \cdots, \theta_T; \vartheta_1, \vartheta_2, \cdots, \vartheta_T}$ is the expectation operation with respect to the random variables $\theta_1, \theta_2, \cdots, \theta_T$ and $\vartheta_1, \vartheta_2, \cdots, \vartheta_T$, and $q^i(x_{T+1})$ is a terminal payment given at stage $T + 1$. The payoffs of the players are transferable.

9.1.2 Noncooperative Equilibria

Let $u_t^{(\sigma_t)i}$ denote the strategy of player i at stage t given that the realized random variable affecting the players' payoffs is $\theta_t^{\sigma_t}$. In a stochastic dynamic game framework, a strategy space with state-dependent property has to be considered. In particular, a pre-specified class Γ^i of mapping $\phi_t^{(\sigma_t)i}(\cdot) : X \to U^i$ with the property $u_t^{(\sigma_t)i} = \phi_t^{(\sigma_t)i}(x) \in \Gamma^i$ is the strategy space of player i and each of its elements is a permissible strategy.

To solve the game, we invoke the principle of optimality in Bellman's (1957) technique of dynamic programming and begin with the subgame starting at the last operating stage, that is stage T. If $\theta_T^{\sigma_T} \in \left\{\theta_T^1, \theta_T^2, \cdots, \theta_T^{\eta_T}\right\}$ has occurred at stage T and the state $x_T = x$, the subgame becomes:

$$\max_{u_T^i} E_{\vartheta_T}\left\{g_T^i\left(x, u_T^1, u_T^2, \cdots, u_T^n; \theta_T^{\sigma_T}\right) + q^i(x_{T+1})\right\}, \text{ for } i \in N,$$

subject to

$$x_{T+1} = f_T\left(x, u_T^1, u_T^2, \cdots, u_T^n\right) + \vartheta_T. \tag{1.3}$$

A set of state-dependent strategies $\phi_T^{(\sigma_T)*}(x) = \left\{\phi_T^{(\sigma_T)1*}(x), \phi_T^{(\sigma_T)2*}(x), \cdots, \phi_T^{(\sigma_T)n*}(x)\right\}$ constitutes a Nash equilibrium solution to the subgame (1.3) if the following conditions are satisfied:

$$\begin{aligned} V^{(\sigma_T)i}(T, x) &= E_{\vartheta_T}\left\{g_T^i\left[x, \phi_T^{(\sigma_T)*}(x); \theta_T^{\sigma_T}\right] + q^i(x_{T+1})\right\} \\ &\geq E_{\vartheta_T}\left\{g_T^i\left[x, \phi_T^{(\sigma_T)\neq i*}(x); \theta_T^{\sigma_T}\right] + q^i(\widetilde{x}_{T+1})\right\}, \text{ for } i \in N, \end{aligned}$$

where $x_{T+1} = f_T\left[x, \phi_T^{(\sigma_T)*}(x)\right] + \vartheta_T$,

$$\begin{aligned} &\phi_T^{(\sigma_T)\neq i*}(x) \\ &\quad = \left[\phi_T^{(\sigma_T)1*}(x), \phi_T^{(\sigma_T)2*}(x), \cdots, \phi_T^{(\sigma_T)i-1*}(x), u_T^{(\sigma_T)i}, \phi_T^{(\sigma_T)i+1*}(x), \cdots, \phi_T^{(\sigma_T)n*}(x)\right], \end{aligned}$$

and $\widetilde{x}_{T+1} = f_T\left[x, \phi_T^{(\sigma_T)\neq i*}(x)\right] + \vartheta_T$.

A characterization of the Nash equilibrium of the subgame (1.3) is provided in the following lemma.

Lemma 1.1 A set of strategies $\phi_T^{(\sigma_T)*}(x) = \left\{\phi_T^{(\sigma_T)1*}(x), \phi_T^{(\sigma_T)2*}(x), \cdots, \phi_T^{(\sigma_T)n*}(x)\right\}$ provides a Nash equilibrium solution to the subgame (1.3) if there exist functions $V^{(\sigma_T)i}(T, x)$, for $i \in N$, such that the following conditions are satisfied:

$$\begin{aligned} V^{(\sigma_T)i}(T, x) &= \max_{u_T^{(\sigma_T)i}} E_{\vartheta_T}\left\{g_T^i\left[x, \phi_T^{(\sigma_T)\neq i*}(x); \theta_T^{\sigma_T}\right]\right. \\ &\qquad \left. + V^{(\sigma_{T+1})i}\left[T+1, f_T\left(x, \phi_T^{(\sigma_T)\neq i*}(x)\right) + \vartheta_T\right]\right\}, \\ V^{(\sigma_T)i}(T+1, x) &= q^i(x); \quad \text{for } i \in N\,. \end{aligned} \tag{1.4}$$

Proof The system of equations in (1.4) satisfies the standard stochastic dynamic programming property and the Nash property for each player $i \in N$. Hence a Nash

(1951) equilibrium of the subgame (1.3) is characterized. Details of the proof of the results can be found in Theorem 6.10 in Basar and Olsder (1999). ■

For the sake of exposition, we sidestep the issue of multiple equilibria and focus on games in which there is a unique noncooperative Nash equilibrium in each subgame. Using Lemma 1.1, one can characterize the value functions $V^{(\sigma_T)i}(T,x)$ for all $\sigma_T \in \{1,2,\cdots,\eta_T\}$ if they exist. In particular, $V^{(\sigma_T)i}(T,x)$ yields player i's expected game equilibrium payoff in the subgame starting at stage T given that $\theta_T^{\sigma_T}$ occurs and $x_T = x$.

Then we proceed to the subgame starting at stage $T-1$ when $\theta_{T-1}^{\sigma_{T-1}} \in \left\{\theta_{T-1}^1, \theta_{T-1}^2, \cdots, \theta_{T-1}^{\eta_{T-1}}\right\}$ occurs and $x_{T-1} = x$. In this subgame player $i \in N$ seeks to maximize his expected payoff

$$E_{\theta_T;\vartheta_{T-1},\vartheta_T}\left\{g_{T-1}^i\left(x, u_{T-1}^1, u_{T-1}^2, \cdots, u_{T-1}^n; \theta_{T-1}^{\sigma_{T-1}}\right) + g_T^i\left(x_T, u_T^1, u_T^2, \cdots, u_T^n; \theta_T\right) + q^i(x_{T+1})\right\}$$
$$= E_{\vartheta_{T-1},\vartheta_T}\left\{g_{T-1}^i\left(x, u_{T-1}^1, u_{T-1}^2, \cdots, u_{T-1}^n; \theta_{T-1}^{\sigma_{T-1}}\right) + \sum_{\sigma_T=1}^{\eta_T}\lambda_T^{\sigma_T} g_T^i\left(x_T, u_T^1, u_T^2, \cdots, u_T^n; \theta_T^{\sigma_T}\right) + q^i(x_{T+1})\right\}, \text{ for } i \in N, \quad (1.5)$$

subject to

$$x_{k+1} = f_k\left(x_k, u_k^1, u_k^2, \cdots, u_k^n\right) + \vartheta_k, \text{ for } k \in \{T-1, T\} \text{ and } x_{T-1} = x. \quad (1.6)$$

If the functions $V^{(\sigma_T)i}(T,x)$ for all $\sigma_T \in \{1,2,\cdots,\eta_T\}$ characterized in Lemma 1.1 exist, the subgame (1.5 and 1.6) can be expressed as a game in which player i seeks to maximize the expected payoff

$$E_{\vartheta_{T-1}}\left\{g_{T-1}^i\left(x, u_{T-1}^1, u_{T-1}^2, \cdots, u_{T-1}^n; \theta_{T-1}^{\sigma_{T-1}}\right) + \sum_{\sigma_T=1}^{\eta_T}\lambda_T^{\sigma_T} V^{(\sigma_T)i}\left[T, f_{T-1}\left(x, u_{T-1}^1, u_{T-1}^2, \cdots, u_{T-1}^n\right) + \vartheta_{T-1}\right]\right\}, \text{ for } i \in N, \quad (1.7)$$

using his control u_{T-1}^i.

A set of strategies $\phi_{T-1}^{(\sigma_{T-1})*}(x) = \left\{\phi_{T-1}^{(\sigma_{T-1})1*}(x), \phi_{T-1}^{(\sigma_{T-1})2*}(x), \cdots, \phi_{T-1}^{(\sigma_{T-1})n*}(x)\right\}$ constitutes a Nash equilibrium solution to the subgame (1.7) if the following conditions are satisfied:

$$V^{(\sigma_{T-1})i}(T-1,x) = E_{\vartheta_{T-1}}\left\{ g^i_{T-1}\left[x,\phi^{(\sigma_{T-1})*}_{T-1}(x);\theta^{\sigma_{T-1}}_{T-1}\right]\right.$$
$$\left. + \sum_{\sigma_T=1}^{\eta_T}\lambda^{\sigma_T}_T V^{(\sigma_T)i}\left[T, f_{T-1}\left[x,\phi^{(\sigma_{T-1})*}_{T-1}(x)\right] + \vartheta_{T-1}\right]\right\}$$
$$\geq E_{\vartheta_{T-1}}\left\{ g^i_{T-1}\left[x,\phi^{(\sigma_{T-1})\neq i*}_{T-1}(x);\theta^{\sigma_{T-1}}_{T-1}\right]\right.$$
$$\left. + \sum_{\sigma_T=1}^{\eta_T}\lambda^{\sigma_T}_T V^{(\sigma_T)i}\left[T, f_{T-1}\left(x,\phi^{(\sigma_{T-1})\neq i*}_{T-1}(x)\right) + \vartheta_{T-1}\right]\right\}, \text{ for } i\in N, \tag{1.8}$$

where

$$\phi^{(\sigma_{T-1})\neq i*}_{T-1}(x) =$$
$$\left[\phi^{(\sigma_{T-1})1*}_{T-1}(x), \phi^{(\sigma_{T-1})2*}_{T-1}(x), \cdots, \phi^{(\sigma_{T-1})i-1*}_{T-1}(x), u^{(\sigma_{T-1})i}_{T-1}, \phi^{(\sigma_{T-1})i+1*}_{T-1}(x), \cdots, \phi^{(\sigma_{T-1})n*}_{T-1}(x)\right].$$

A characterization of the Nash equilibrium of the subgame (1.7) is provided in the following lemma.

Lemma 1.2 A set of strategies $\phi^{(\sigma_{T-1})*}_{T-1}(x) = \left\{\phi^{(\sigma_{T-1})1*}_{T-1}(x), \phi^{(\sigma_{T-1})2*}_{T-1}(x), \cdots,\right.$ $\left.\phi^{(\sigma_{T-1})n*}_{T-1}(x)\right\}$ provides a Nash equilibrium solution to the subgame (1.7) if there exist functions $V^{(\sigma_T)i}(T, x_T)$ for $i\in N$ and $\sigma_T = \{1, 2, \cdots, \eta_T\}$ characterized in Lemma 1.1, and functions $V^{(\sigma_{T-1})i}(T-1,x)$, for $i\in N$, such that the following conditions are satisfied:

$$V^{(\sigma_{T-1})i}(T-1,x) = \max_{u^{(\sigma_{T-1})i}_{T-1}} E_{\vartheta_{T-1}}\left\{ g^i_{T-1}\left[x,\phi^{(\sigma_{T-1})\neq i*}_{T-1}(x);\theta^{\sigma_{T-1}}_{T-1}\right]\right.$$
$$\left. + \sum_{\sigma_T=1}^{\eta_T}\lambda^{\sigma_T}_T V^{(\sigma_T)i}\left[T, f_{T-1}\left(x,\phi^{(\sigma_{T-1})\neq i*}_{T-1}\right) + \vartheta_{T-1}\right]\right\}, \text{ for } i\in N. \tag{1.9}$$

Proof The conditions in Lemma 1.1 and the system of equations in (1.9) satisfies the standard discrete-time stochastic dynamic programming property and the Nash property for each player $i\in N$. Hence a Nash equilibrium of the subgame (1.7) is characterized. ∎

In particular, $V^{(\sigma_{T-1})i}(T-1,x)$, if it exists, yields player i's expected game equilibrium payoff in the subgame starting at stage $T-1$ given that $\theta^{\sigma_{T-1}}_{T-1}$ occurs and $x_{T-1} = x$.

Consider the subgame starting at stage $t\in\{T-2, T-3, \cdots, 1\}$ when $\theta^{\sigma_t}_t \in \left\{\theta^1_t, \theta^2_t, \cdots, \theta^{\eta_t}_t\right\}$ occurs and $x_t = x$, in which player $i\in N$ maximizes his expected payoff

$$E_{\theta_{t+1};\vartheta_t,\vartheta_{t+1},\cdots,\vartheta_T}\Bigg\{g_t^i\left(x,u_t^1,u_t^2,\cdots,u_t^n;\theta_t^{\sigma_t}\right)$$
$$+\sum_{\zeta=t+1}^{T}g_\zeta^i\Big(x_\zeta,u_\zeta^1,u_\zeta^2,\cdots,u_\zeta^n;\theta_\zeta\Big)+q^i(x_{T+1})\Bigg\},\ \text{for } i\in N, \tag{1.10}$$

subject to

$$x_{k+1}=f_k\left(x_k,u_k^1,u_k^2,\cdots,u_k^n\right)+\vartheta_k,\ \text{for } k\in\{t,t+1,\cdots,T\} \text{ and } x_t=x. \tag{1.11}$$

Following the above analysis, the subgame (1.10 and 1.11) can be expressed as a game in which player $i\in N$ maximizes his expected payoff

$$E_{\vartheta_t}\Bigg\{g_t^i\left(x,u_t^1,u_t^2,\cdots,u_t^n;\theta_t^{\sigma_t}\right)$$
$$+\sum_{\sigma_{t+1}=1}^{\eta_{t+1}}\lambda_{t+1}^{\sigma_{t+1}}V^{(\sigma_{t+1})i}\left[t+1,f_t\left(x,u_t^1,u_t^2,\cdots,u_t^n\right)+\vartheta_t\right]\Bigg\},\ \text{for } i\in N, \tag{1.12}$$

with his control u_t^i,

where $V^{(\sigma_{t+1})i}\left[t+1,f_t\left(x,u_t^1,u_t^2,\cdots,u_t^n\right)+\vartheta_t\right]$ is player i's expected game equilibrium payoff in the subgame starting at stage $t+1$ given that $\theta_{t+1}^{\sigma_{t+1}}$ occurs and $x_{t+1}=f_t\left(x,u_t^1,u_t^2,\cdots,u_t^n\right)+\vartheta_t$.

A set of strategies $\phi_t^{(\sigma_t)*}(x)=\left\{\phi_t^{(\sigma_t)1*}(x),\phi_t^{(\sigma_t)2*}(x),\cdots,\phi_t^{(\sigma_t)n*}(x)\right\}$, constitutes a Nash equilibrium solution to the subgame (1.12) if the following conditions are satisfied:

$$V^{(\sigma_t)i}(t,x)=E_{\vartheta_t}\Bigg\{g_t^i\left[x,\phi_t^{(\sigma_t)*}(x);\theta_t^{\sigma_t}\right]$$
$$+\sum_{\sigma_{t+1}=1}^{\eta_{t+1}}\lambda_{t+1}^{\sigma_{t+1}}V^{(\sigma_{t+1})i}\left[t+1,f_t\left[x,\phi_t^{(\sigma_t)*}(x)\right]+\vartheta_t\right]\Bigg\}$$
$$\geq E_{\vartheta_t}\Bigg\{g_t^i\left[x,\phi_t^{(\sigma_t)\neq i*}(x);\theta_t^{\sigma_t}\right]+\sum_{\sigma_{t+1}=1}^{\eta_{t+1}}\lambda_{t+1}^{\sigma_{t+1}}V^{(\sigma_{t+1})i}\left[t+1,f_t\left(x,\phi_t^{(\sigma_t)\neq i*}(x)\right)+\vartheta_t\right]\Bigg\}$$

where

$$\phi_t^{(\sigma_t)\neq i*}(x)=\left\{\phi_t^{(\sigma_t)1*}(x),\phi_t^{(\sigma_t)2*}(x),\cdots,\phi_t^{(\sigma_t)i-1*}(x),u_t^{(\sigma_t)i},\phi_t^{(\sigma_t)i+1*}(x),\cdots,\phi_t^{(\sigma_t)n*}(x)\right\}.$$

A Nash equilibrium solution for the game (1.1 and 1.2) can be characterized by the following theorem.

Theorem 1.1 A set of strategies $\phi_t^{(\sigma_t)*}(x) = \left\{\phi_t^{(\sigma_t)1*}(x), \phi_t^{(\sigma_t)2*}(x), \cdots, \phi_t^{(\sigma_t)n*}(x)\right\}$, for $\sigma_t \in \{1, 2, \cdots, \eta_t\}$ and $t \in \{1, 2, \cdots, T\}$, constitutes a Nash equilibrium solution to the game (1.1 and 1.2) if there exist functions $V^{(\sigma_t)i}(t, x)$, for $\sigma_t \in \{1, 2, \cdots, \eta_t\}$, $t \in \{1, 2, \cdots, T\}$ and $i \in N$, such that the following recursive relations are satisfied:

$$
\begin{aligned}
V^{(\sigma_T)i}(T+1, x) &= q^i(x), \\
V^{(\sigma_T)i}(T, x) &= \max_{u_T^{(\sigma_T)i}} E_{\vartheta_T}\Bigg\{ g_T^i\left[x, \phi_T^{(\sigma_T)\neq i*}(x); \theta_T^{\sigma_T}\right] \\
&\quad + V^{(\sigma_{T+1})i}\left[T+1, f_T\left(x, \phi_T^{(\sigma_T)\neq i*}(x)\right) + \vartheta_T\right]\Bigg\}, \\
V^{(\sigma_t)i}(t, x) &= \max_{u_t^{(\sigma_t)i}} E_{\vartheta_t}\Bigg\{ g_t^i\left[x, \phi_t^{(\sigma_t)\neq i*}(x); \theta_t^{\sigma_t}\right] \\
&\quad + \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} V^{(\sigma_{t+1})i}\left[t+1, f_t\left(x, \phi_t^{(\sigma_t)\neq i*}(x)\right) + \vartheta_t\right]\Bigg\}; \\
&\text{for } \sigma_t \in \{1, 2, \cdots, \eta_t\},\ t \in \{1, 2, \cdots, T-1\} \text{ and } i \in N.
\end{aligned}
\tag{1.13}
$$

Proof The results in (1.13) characterizing the game equilibrium in stage T and stage $T-1$ are proved in Lemma 1.1 and 1.2. Invoking the subgame in stage $t \in \{1, 2, \cdots, T-2\}$ as expressed in (1.12), the results in (1.13) satisfy the optimality conditions in stochastic dynamic programming and the Nash equilibrium property for each player in each of these subgames. Therefore, a feedback Nash equilibrium of the game (1.1 and 1.2) is characterized. ■

Theorem 1.1 is the discrete-time analog of the Nash equilibrium in the continuous-time randomly furcating stochastic differential games in Chap. 4.

9.2 Dynamic Cooperation

Now consider the case when the players agree to cooperate and distribute the joint payoff among themselves according to an optimality principle. As pointed out before two essential properties that a cooperative scheme has to satisfy are group optimality and individual rationality.

9.2.1 Group Optimality

In this subsection, we consider the issue of ensuring group optimality in a cooperative scheme. To achieve group optimality by maximizing their expected joint payoff the players have to solve the discrete-time stochastic dynamic programming problem of maximizing

$$E_{\theta_1,\theta_2,\cdots,\theta_T;\vartheta_1,\vartheta_2,\cdots,\vartheta_T}\left\{\sum_{j=1}^{n}\sum_{k=1}^{T}\left[g_k^j\left(x_k,u_k^1,u_k^2,\cdots,u_k^n;\theta_k\right)\right]+\sum_{j=1}^{n}q^j(x_{T+1})\right\} \tag{2.1}$$

subject to (1.1).

The stochastic dynamic programming problem (1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13 and 2.1) can be regarded as a single-player case of the game problem (1.1 and 1.2) with $n = 1$ and the payoff being the sum of the all the players' payoffs. In a stochastic dynamic framework, again strategy space with state-dependent property has to be considered. In particular, a pre-specified class $\hat{\Gamma}^i$ of mapping $\psi_i^{(\sigma_t)^i}(\cdot): X \to U^i$ with the property $u_t^{(\sigma_t)i} = \psi_t^{(\sigma_t)i}(x) \in \hat{\Gamma}^i$, for $\sigma_t \in \{1, 2, \cdots, \eta_t\}$ and $t \in \{1, 2, \cdots, T\}$, is the strategy space of player i and each of its elements is a permissible strategy.

To solve the dynamic programming problem (1.1) and (2.1), we first consider the problem starting at stage T. If $\theta_T^{\sigma_T} \in \left\{\theta_T^1, \theta_T^2, \cdots, \theta_T^{\eta_T}\right\}$ has occurred at stage T and the state $x_T = x$, the problem becomes:

$$\max_{u_T^1,u_T^2,\cdots,u_T^n} E_{\vartheta_T}\left\{\sum_{j=1}^{n} g_T^j\left(x,u_T^1,u_T^2,\cdots,u_T^n;\theta_T^{\sigma_T}\right)+\sum_{j=1}^{n}q^j(x_{T+1})\right\} \tag{2.2}$$

subject to

$$x_{T+1} = f_T\left(x,u_T^1,u_T^2,\cdots,u_T^n\right)+\vartheta_T. \tag{2.3}$$

An optimal solution to the stochastic control problem (2.2 and 2.3) is characterized by the following lemma.

Lemma 2.1 A set of controls $u_T^{(\sigma_T)*} = \psi_T^{(\sigma_T)*}(x) = \left\{\psi_T^{(\sigma_T)1*}(x), \psi_T^{(\sigma_T)2*}(x), \cdots\cdots, \psi_T^{(\sigma_T)n*}(x)\right\}$ provides an optimal solution to the stochastic control problem (2.2 and 2.3) if there exist functions $W^{(\sigma_{T+1})}(T,x)$, for $i \in N$, such that the following conditions are satisfied:

$$\begin{aligned}W^{(\sigma_T)}(T,x) = &\max_{u_T^{(\sigma_T)1},u_T^{(\sigma_T)2},\cdots,u_T^{(\sigma_T)n}} E_{\vartheta_T}\Bigg\{\sum_{j=1}^{n} g_T^j\left[x,u_T^{(\sigma_T)1},u_T^{(\sigma_T)2},\cdots,u_T^{(\sigma_T)n};\theta_T^{\sigma_T}\right]\\&+W^{(\sigma_{T+1})}\left[T+1,f_T\left(x,u_T^{(\sigma_T)1},u_T^{(\sigma_T)2},\cdots,u_T^{(\sigma_T)n}\right)+\vartheta_T\right]\Bigg\},\\W^{(\sigma_T)}(T+1,x) + &\sum_{j=1}^{n}q^j(x).\end{aligned} \tag{2.4}$$

Proof The system of equations in (2.4) satisfies the standard discrete-time stochastic dynamic programming property. Details of the proof of the results can be found in Basar and Olsder (1999). ■

Note that $W^{(\sigma_T)}(T,x)$ yields the expected cooperative payoff starting at stage T given that $\theta_T^{\sigma_T}$ occurs and $x_T = x$ according to the dynamic programming problem (2.2 and 2.3) if $\theta_T^{\sigma_T}$. Using Lemma 2.1, one can characterize the functions $W^{(\sigma_T)}(T,x)$ for all $\theta_T^{\sigma_T} \in \{\theta_T^1, \theta_T^2, \cdots, \theta_T^{\eta_T}\}$, if they exist. Following the analysis in Sect. 9.2, the control problem starting at stage t when $\theta_t^{\sigma_t} \in \{\theta_t^1, \theta_t^2, \cdots, \theta_t^{\eta_t}\}$ occurs and $x_t = x$ can be expressed as:

$$\max_{u_t} E_{\vartheta_t}\Bigg\{\sum_{j=1}^{n} g_t^j\left(x, u_t^1, u_t^2, \cdots, u_t^n; \theta_t^{\sigma_t}\right) + \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} W^{(\sigma_{t+1})}\left[t+1, f_t\left(x, u_t^1, u_t^2, \cdots, u_t^n\right) + \vartheta_t\right]\Bigg\}, \tag{2.5}$$

where $W^{(\sigma_{t+1})}\left[t+1, f_t\left(x, u_t^1, u_t^2, \cdots, u_t^n\right) + \vartheta_t\right]$ is the expected optimal cooperative payoff in the control problem starting at stage $t+1$ when $\theta_{t+1}^{\sigma_{t+1}} \in \{\theta_{t+1}^1, \theta_{t+1}^2, \cdots, \theta_{t+1}^{\eta_{t+1}}\}$ occurs and $x_{t+1} = f_t\left(x, u_t^1, u_t^2, \cdots, u_t^n\right) + \vartheta_t$.

An optimal solution for the stochastic control problem (1.1) and (2.1) can be characterized by the following theorem.

Theorem 2.1 A set of controls $u_t^{(\sigma_t)i*} = \psi_t^{(\sigma_t)*}(x) = \{\psi_t^{(\sigma_t)1*}(x), \psi_t^{(\sigma_t)2*}(x), \cdots\cdots, \psi_t^{(\sigma_t)n*}(x)\}$, for $\sigma_t \in \{1, 2, \cdots, \eta_t\}$ and $t \in \{1, 2, \cdots, T\}$ provides an optimal solution to the stochastic control problem (1.1) and (2.1) if there exist functions $W^{(\sigma_t)}(t,x)$, for $\sigma_t \in \{1, 2, \cdots, \eta_t\}$ and $t \in \{1, 2, \cdots, T\}$, such that the following recursive relations are satisfied:

$$W^{(\sigma_T)}(T+1, x) = \sum_{j=1}^{n} q^j(x),$$

$$W^{(\sigma_T)}(T, x) = \max_{u_T^{(\sigma_T)1}, u_T^{(\sigma_T)2}, \cdots, u_T^{(\sigma_T)n}} E_{\vartheta_T}\Bigg\{\sum_{j=1}^{n} g_T^j\left[x, u_T^{(\sigma_T)1}, u_T^{(\sigma_T)2}, \cdots, u_T^{(\sigma_T)n}; \theta_T^{\sigma_T}\right] + W^{(\sigma_{T+1})}\left[T+1, f_T\left(x, u_T^{(\sigma_T)1}, u_T^{(\sigma_T)2}, \cdots, u_T^{(\sigma_T)n}\right) + \vartheta_T\right]\Bigg\},$$

$$W^{(\sigma_t)}(t, x) = \max_{u_t^{(\sigma_t)1}, u_t^{(\sigma_t)2}, \cdots, u_t^{(\sigma_t)n}} E_{\vartheta_t}\Bigg\{\sum_{j=1}^{n} g_t^j\left[x, u_t^{(\sigma_t)1}, u_t^{(\sigma_t)2}, \cdots, u_t^{(\sigma_t)n}; \theta_t^{\sigma_t}\right] + \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} W^{(\sigma_{t+1})}\left[t+1, f_t\left(x, u_t^{(\sigma_t)1}, u_t^{(\sigma_t)2}, \cdots, u_t^{(\sigma_t)n}\right) + \vartheta_t\right]\Bigg\},$$

for $\sigma_t \in \{1, 2, \cdots, \eta_t\}$ and $t \in \{1, 2, \cdots, T-1\}$. (2.6)

Proof The results in (2.6) characterizing the optimal solution in stage T is proved in Lemma 2.1. Invoking the specification of the control problem starting in stage $t \in \{1, 2, \cdots, T-1\}$ as expressed in (2.5), the results in (2.6) satisfy the optimality conditions in stochastic dynamic programming. Therefore, an optimal solution of the stochastic control problem (1.1) and (2.1) is characterized. ■

Theorem 2.1 is the discrete-time analog of the optimal cooperative scheme in randomly furcating stochastic differential games in Petrosyan and Yeung (2007).

Substituting the optimal control $\{\psi_k^{(\sigma_k)i*}(x)$, for $k \in \{1, 2, \cdots T\}$ and $i \in N\}$ into the state dynamics (1.1), one can obtain the dynamics of the cooperative trajectory as:

$$x_{k+1} = f_k\left(x_k, \psi_k^{(\sigma_k)1*}(x_k), \psi_k^{(\sigma_k)2*}(x_k), \cdots, \psi_k^{(\sigma_k)n*}(x_k)\right) + \vartheta_k, \text{ if } \theta_k^{\sigma_k} \text{ occurs }, \quad (2.7)$$

for $k \in \{1, 2, \cdots, T\}, \sigma_k \in \{1, 2, \cdots, \eta_k\}$ and $x_1 = x^0$.

We use X_k^* to denote the set of realizable values of x_k^* at stage k generated by (2.7). The term $x_k^* \in X_k^*$ is used to denote an element in X_k^*.

The term $W^{(\sigma_k)}(k, x_k^*)$ gives the expected total cooperative payoff over the stages from k to T if $\theta_k^{\sigma_k}$ occurs and $x_k^* \in X_k^*$ is realized at stage k.

9.2.2 Individual Rationality

The players then have to agree to an optimality principle in distributing the total cooperative payoff among themselves. For individual rationality to be upheld the expected payoffs a player receives under cooperation have to be no less than his expected noncooperative payoff along the cooperative state trajectory $\{x_k^*\}_{k=1}^{T+1}$. The players may (i) share the excess of the total expected cooperative payoff over the expected sum of individual noncooperative payoffs equally, or (ii) share the total expected cooperative payoff proportional to their expected noncooperative payoffs.

Let $\xi^{(\sigma_k)}(k, x_k^*) = [\xi^{(\sigma_k)1}(k, x_k^*), \xi^{(\sigma_k)2}(k, x_k^*), \cdots, \xi^{(\sigma_k)n}(k, x_k^*)]$ denote the imputation vector guiding the distribution of the total expected cooperative payoff under the agreed-upon optimality principle along the cooperative trajectory given that $\theta_k^{\sigma_k}$ has occurred in stage k, for $\sigma_k \in \{1, 2, \cdots, \eta_k\}$ and $k \in \{1, 2, \cdots, T\}$. In particular, the imputation $\xi^{(\sigma_k)i}(k, x_k^*)$ gives the expected cumulative payments that player i will receive from stage k to stage $T+1$ under cooperation.

If for example, the optimality principle specifies that the players share the excess of the total cooperative payoff over the sum of individual noncooperative payoffs equally, then the imputation to player i becomes:

$$\xi^{(\sigma_k)i}(k, x_k^*) = V^{(\sigma_k)i}(k, x_k^*) + \frac{1}{n}\left[W^{(\sigma_k)}(k, x_k^*) - \sum_{j=1}^{n} V^{(\sigma_k)j}(k, x_k^*)\right], \quad (2.8)$$

for $i \in N$ and $k \in \{1, 2, \cdots, T\}$.

For individual rationality to be maintained throughout all the stages $k \in \{1, 2, \cdots, T\}$, it is required that the imputation satisfies:

$$\xi^{(\sigma_k)i}(k, x_k^*) \geq V^{(\sigma_k)i}(k, x_k^*),$$
$$\text{for } i \in N,\ \sigma_k \in \{1, 2, \cdots, \eta_k\} \text{ and } k \in \{1, 2, \cdots, T\}. \tag{2.9}$$

To guarantee group optimality, the imputation vector has to satisfy

$$W^{(\sigma_k)}(k, x_k^*) = \sum_{j=1}^{n} \xi^{(\sigma_k)j}(k, x_k^*),$$
$$\text{for } \sigma_k \in \{1, 2, \cdots, \eta_k\} \text{ and } k \in \{1, 2, \cdots, T\}. \tag{2.10}$$

Hence, a valid imputation $\xi^{(\sigma_k)i}(k, x_k^*)$, for $i \in N$, $\sigma_k \in \{1, 2, \cdots, \eta_k\}$ and $k \in \{1, 2, \cdots, T\}$, has to satisfy conditions (2.9) and (2.10).

9.3 Subgame Consistent Solutions and Payment Mechanism

As demonstrated in Chap. 7, to guarantee dynamical stability in a stochastic dynamic cooperation scheme, the solution has to satisfy the property of subgame consistency in addition to group optimality and individual rationality. In particular, an extension of a subgame-consistent cooperative solution policy to a subgame starting at a later time with a feasible state brought about by prior optimal behavior would remain optimal. Thus subgame consistency ensures that as the game proceeds players are guided by the same optimality principle at each stage of the game, and hence do not possess incentives to deviate from the previously adopted optimal behavior. For subgame consistency to be satisfied, the imputation according to the original optimality principle has to be maintained at all the T stages along the cooperative trajectory $\{x_k^*\}_{k=1}^{T}$. In other words, the imputation

$$\xi^{(\sigma_k)}(k, x_k^*) = \left[\xi^{(\sigma_k)1}(k, x_k^*), \xi^{(\sigma_k)2}(k, x_k^*), \cdots, \xi^{(\sigma_k)n}(k, x_k^*)\right], \tag{3.1}$$

for $\sigma_k \in \{1, 2, \cdots, \eta_k\}$, $x_k^* \in X_k^*$ and $k \in \{1, 2, \cdots, T\}$, has to be upheld.

9.3.1 Payoff Distribution Procedure

Following the analysis of Yeung and Petrosyan (2010 and 2011), we formulate a Payoff Distribution Procedure (PDP) so that the agreed-upon imputation (3.1) can be realized. Let $B_k^{(\sigma_k)i}(x_k^*)$ denote the payment that player i will received at stage

k under the cooperative agreement if $\theta_k^{\sigma_k} \in \{\theta_k^1, \theta_k^2, \cdots, \theta_k^{\eta_k}\}$ occurs and $x_k^* \in X_k^*$ is realized at stage $k \in \{1, 2, \cdots, T\}$. The payment scheme $\{B_k^{(\sigma_k)i}(x_k^*)$ contingent upon the event $\theta_k^{\sigma_k}$ and state x_k^*, for $k \in \{1, 2, \cdots, T\}\}$ constitutes a PDP in the sense that the imputation to player i over the stages 1 to $T+1$ can be expressed as:

$$\xi^{(\sigma_1)i}(1, x_{1(0)}) = B_1^{(\sigma_1)i}(x_{1(0)}) + E_{\theta_2, \cdots, \theta_T; \vartheta_1, \vartheta_2, \cdots, \vartheta_T}\left(\sum_{\zeta=2}^{T} B_\zeta^{(\sigma_\zeta)i}\left(x_\zeta^*\right) + q^i\left(x_{T+1}^*\right)\right), \tag{3.2}$$

for $i \in N$.

Moreover, according to the agreed-upon optimality principle in (3.1), if $\theta_k^{\sigma_k}$ occurs and $x_k^* \in X_k^*$ is realized at stage k the imputation to player i is $\xi^{(\sigma_k)i}(k, x_k^*)$. For subgame consistency to be satisfied, the imputation according to the agreed-upon optimality principle has to be maintained at all the T stages along the cooperative trajectory $\{x_k^*\}_{k=1}^{T}$. Therefore to guarantee subgame consistency, the payment scheme $\left\{B_k^{(\sigma_k)i}(x_k^*)\right\}$ has to satisfy the conditions

$$\xi^{(\sigma_k)i}(k, x_k^*) = B_k^{(\sigma_k)i}(x_k^*) + E_{\theta_{k+1}, \theta_{k+2}, \cdots, \theta_T; \vartheta_k, \vartheta_{k+1}, \cdots, \vartheta_T}\left(\sum_{\zeta=k+1}^{T} B_\zeta^{(\sigma_\zeta)i}\left(x_\zeta^*\right) + q^i\left(x_{T+1}^*\right)\right) \tag{3.3}$$

for $i \in N$ and $k \in \{1, 2, \cdots, T\}$.

Using (3.3) one can readily obtain $\xi^{(\sigma_{T+1})i}(T+1, x_{T+1}^*)$ equals $q^i(x_{T+1}^*)$ with probability 1. Crucial to the formulation of a subgame consistent solution is the derivation of a payment scheme $\{B_k^{(\sigma_k)i}(x_k^*)$, for $i \in N$, $\sigma_k \in \{1, 2, \cdots, \eta_k\}$, $x_k^* \in X_k^*$ and $k \in \{1, 2, \cdots, T\}$ so that the imputation in (3.3) can be realized. This will be done in the sequel.

A theorem for the derivation of a subgame consistent PDP can be established as follows.

Theorem 3.1 A payment equaling

$$B_k^{(\sigma_k)i}(x_k^*) = \xi^{(\sigma_k)i}(k, x_k^*) - E_{\vartheta_k}\left[\sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}}\left(\xi^{(\sigma_{k+1})i}\left[k+1, f_k\left(x_k^*, \psi_k^{(\sigma_k)*}(x_k^*)\right) + \vartheta_k\right]\right)\right], \tag{3.4}$$

for $i \in N$,

given to player i at stage $k \in \{1, 2, \cdots, T\}$, if $\theta_k^{\sigma_k}$ occurs and $x_k^* \in X_k^*$, leads to the realization of the imputation in (3.3).

Proof To construct the proof of Theorem 3.1, we first consider the imputation

$$
\begin{aligned}
&E_{\theta_{k+1},\theta_{k+2},\cdots,\theta_T;\vartheta_k,\vartheta_{k+1},\cdots,\vartheta_T}\left(\sum_{\zeta=k+1}^{T} B_\zeta^{(\sigma_\zeta)i}\left(x_\zeta^*\right) + q^i\left(x_{T+1}^*\right)\right)\\
&\quad = E_{\vartheta_k}\left\{\sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}}\left[B_{k+1}^{(\sigma_{k+1})i}\left(x_{k+1}^*\right)\right.\right.\\
&\quad \left.\left. + E_{\theta_{k+2},\theta_{k+3},\cdots,\theta_T;\vartheta_{k+2},\vartheta_{k+3},\cdots,\vartheta_T}\left(\sum_{\zeta=k+2}^{T} B_\zeta^{(\sigma_\zeta)i}\left(x_\zeta^*\right) + q^i\left(x_{T+1}^*\right)\right)\right]\right\}.
\end{aligned}
\tag{3.5}
$$

Then, using (3.3) we can derive the term $\xi^{(\sigma_{k+1})i}\left(k+1, x_{k+1}^*\right)$ as

$$
\begin{aligned}
&\xi^{(\sigma_{k+1})i}\left(k+1, x_{k+1}^*\right) = B_{k+1}^{(\sigma_{k+1})i}\left(x_{k+1}^*\right)\\
&\quad + E_{\theta_{k+2},\theta_{k+3},\cdots,\theta_T;\vartheta_{k+2},\vartheta_{k+3},\cdots,\vartheta_T}\left(\sum_{\zeta=k+2}^{T} B_\zeta^{(\sigma_\zeta)i}\left(x_\zeta^*\right) + q^i\left(x_{T+1}^*\right)\right)
\end{aligned}
\tag{3.6}
$$

The expression on the right-hand-side of equation (3.6) is the same as the expression inside the square brackets of (3.5). Invoking equation (3.6) we can replace the expression inside the square brackets of (3.5) by $\xi^{(\sigma_{k+1})i}\left(k+1, x_{k+1}^*\right)$ and obtain:

$$
\begin{aligned}
&E_{\theta_{k+1},\theta_{k+2},\cdots,\theta_T;\vartheta_k,\vartheta_{k+1},\cdots,\vartheta_T}\left(\sum_{\zeta=k+1}^{T} B_\zeta^{(\sigma_\zeta)i}\left(x_\zeta^*\right) + q^i\left(x_{T+1}^*\right)\right)\\
&\quad = E_{\vartheta_k}\left[\sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}}\left(\xi^{(\sigma_{k+1})i}\left[k+1, f_k\left(x_k^*, \psi_k^{(\sigma_k)*}\left(x_k^*\right)\right) + \vartheta_k\right]\right)\right].
\end{aligned}
$$

Substituting the term $E_{\theta_{k+1},\theta_{k+2},\cdots,\theta_T;\vartheta_k,\vartheta_{k+1},\cdots,\vartheta_T}\left(\sum_{\zeta=k+1}^{T} B_\zeta^{(\sigma_\zeta)i}\left(x_\zeta^*\right) + q^i\left(x_{T+1}^*\right)\right)$ by E_{ϑ_k} $\left[\sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}}\left(\xi^{(\sigma_{k+1})i}\left[k+1, f_k\left(x_k^*, \psi_k^{(\sigma_k)*}\left(x_k^*\right)\right) + \vartheta_k\right]\right)\right]$ in (3.3) we can express (3.3) as:

$$
\begin{aligned}
&\xi^{(\sigma_k)i}\left(k, x_k^*\right) = B_k^{(\sigma_k)i}\left(x_k^*\right)\\
&\quad + E_{\vartheta_k}\left[\sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}}\left(\xi^{(\sigma_{k+1})i}\left[k+1, f_k\left(x_k^*, \psi_k^{(\sigma_k)*}\left(x_k^*\right)\right) + \vartheta_k\right]\right)\right].
\end{aligned}
\tag{3.7}
$$

For condition (3.7), which is an alternative form of (3.3), to hold it is required that:

$$B_k^{(\sigma_k)i}\left(x_k^*\right) = \xi^{(\sigma_k)i}\left(k, x_k^*\right) \\ - E_{\vartheta_k}\left[\sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}}\left(\xi^{(\sigma_{k+1})i}\left[k+1, f_k\left(x_k^*, \psi_k^{(\sigma_k)*}\left(x_k^*\right)\right) + \vartheta_k\right]\right)\right], \quad (3.8)$$

for $i \in N$ and $k \in \{1, 2, \cdots, T\}$.

Therefore by paying $B_k^{(\sigma_k)i}\left(x_k^*\right)$ to player $i \in N$ at stage $k \in \{1, 2, \cdots, T\}$, if $\theta_k^{\sigma_k}$ occurs and $x_k^* \in X_k^*$, leads to the realization of the imputation in (3.3). Hence Theorem 3.1 follows. ■

For a given imputation vector

$$\xi^{(\sigma_k)}\left(k, x_k^*\right) = \left[\xi^{(\sigma_k)1}\left(k, x_k^*\right), \xi^{(\sigma_k)2}\left(k, x_k^*\right), \cdots, \xi^{(\sigma_k)n}\left(k, x_k^*\right)\right],$$

for $\sigma_k \in \{1, 2, \cdots, \eta_k\}$ and $k \in \{1, 2, \cdots, T\}$,

Theorem 3.1 can be used to derive the PDP that leads to the realization this vector.

9.3.2 Transfer Payments

When all players are using the cooperative strategies, the payoff that player i will directly received at stage k given that $x_k^* \in X_k^*$ and $\theta_k^{\sigma_k}$ occurs becomes

$$g_k^i\left[x_k^*, \psi_k^{(\sigma_k)1*}\left(x_k^*\right), \psi_k^{(\sigma_k)2*}\left(x_k^*\right), \cdots, \psi_k^{(\sigma_k)n*}\left(x_k^*\right); \theta_k^{\sigma_k}\right].$$

However, according to the agreed upon imputation, player i is supposed to received $B_k^{(\sigma_k)i}\left(x_k^*\right)$ at stage k as given in Theorem 3.1. Therefore a transfer payment (which can be positive or negative)

$$\varpi_k^{(\sigma_k)i}\left(x_k^*\right) = B_k^{(\sigma_k)i}\left(x_k^*\right) \\ - g_k^i\left[x_k^*, \psi_k^{(\sigma_k)1*}\left(x_k^*\right), \psi_k^{(\sigma_k)2*}\left(x_k^*\right), \cdots, \psi_k^{(\sigma_k)n*}\left(x_k^*\right); \theta_k^{\sigma_k}\right], \quad (3.9)$$

for $k \in \{1, 2, \cdots, T\}$ and $i \in N$,

will be assigned to player i to yield the cooperative imputation $\xi^i(k, x_k^*)$.

The transfer payments system in (3.9) constitutes an instrument to guide the execution of the agreed-upon payoff sharing mechanism. Coordination of payments is jointly performed by the participating players.

9.4 An Illustration in Cooperative Resource Extraction Under Uncertainty

Consider an economy endowed with a renewable resource and with 2 resource extractors (firms). The lease for resource extraction begins at stage 1 and ends at stage 3 for these two firms. Let u_k^i denote the resource extracted by firm i at stage k, for $i \in \{1,2\}$. Let U^i be the set of admissible amount of resource extracted by firm i, and $x_k \in X \subset R^+$ be the size of the resource stock at stage k.

It is known at each stage there is a random element, θ_k for $k \in \{1,2,3\}$, affecting the prices of the outputs produced by these firms and their costs of extraction. If $\theta_k^{\sigma_k} \in \{\theta_k^1, \theta_k^2\}$ happens at stage $k \in \{2,3\}$ the profits (in present-value) that firm 1 and firm 2 will obtain at stage k are respectively:

$$\left[P_k^{(\sigma_k)1} u_k^1 - \frac{c_k^{(\sigma_k)1}}{x_k} \left(u_k^1\right)^2\right] \left(\frac{1}{1+r}\right)^{k-1}$$
$$\text{and} \left[P_k^{(\sigma_k)2} u_k^2 - \frac{c_k^{(\sigma_k)2}}{x_k} \left(u_k^2\right)^2\right] \left(\frac{1}{1+r}\right)^{k-1}, \tag{4.1}$$

where $P_k^{(\sigma_k)i}$ is the price of the resource extracted and processed by firm i, and $c_k^{(\sigma_k)i} \left(u_k^i\right)^2 / x_k$ is the production cost of firm i in stage k if $\theta_k^{\sigma_k}$ occurs.

It is known in stage 1 that θ_1 is θ_1^1 with probability $\lambda_1^1 = 1$. The probability that $\theta_k^{\sigma_k} \in \{\theta_k^1, \theta_k^2\}$ will occur at stage $k \in \{2,3\}$ is $\lambda_k^{\sigma_k}$. In stage 4, a terminal payment (again in present-value) contingent upon the resource size equaling $q^i x_4 \left(\frac{1}{1+r}\right)^3$ will be paid to firm i.

The growth dynamics of the resource is governed by the stochastic difference equation:

$$x_{k+1} = x_k + a - b x_k - \sum_{j=1}^{2} u_k^j + \vartheta_k, \tag{4.2}$$

for $k \in \{1,2,3\}$ and $x_1 = x^0$,

where ϑ_k is a random variable with non-negative range $\{\vartheta_k^1, \vartheta_k^2, \vartheta_k^3\}$ and corresponding probabilities $\{\gamma_k^1, \gamma_k^2, \gamma_k^3\}$; moreover $\vartheta_1, \vartheta_2, \vartheta_3$ are independent. Moreover, we have the constraint $u_k^1 + u_k^2 \leq (1-b)x_k + a$.

The objective of extractor $i \in \{1,2\}$ is to maximize the present value of the expected stream of future profits:

$$E_{\theta_1,\theta_2,\theta_3;\vartheta_1,\vartheta_2,\vartheta_3}\left\{\sum_{k=1}^{3}\left[P_k^{(\sigma_k)i}u_k^i-\frac{c_k^{(\sigma_k)i}}{x_k}\left(u_k^i\right)^2\right]\left(\frac{1}{1+r}\right)^{k-1}+q^ix_4\left(\frac{1}{1+r}\right)^3\right\} \quad (4.3)$$

subject to the stochastic dynamics (4.2).

Invoking Lemma 1.2, one can characterize the noncooperative Nash equilibrium strategies for the game (4.2 and 4.3) as follows. In particular, a set of strategies $\{u_k^{(\sigma_k)i*}=\phi_k^{(\sigma_k)i*}(x)\in\Gamma^i$, for $\sigma_1\in\{1\},\sigma_2,\sigma_3\in\{1,2\}$, $k\in\{1,2,3\}$ and $i\in\{1,2\}\}$ provides a Nash equilibrium solution to the game (4.2 and 4.3) if there exist functions $V^{(\sigma_k)i}(k,x)$, for $i\in\{1,2\}$ and $k\in\{1,2,3\}$, such that the following recursive relations are satisfied:

$$\begin{aligned}
V^{(\sigma_k)i}(k,x) &= \max_{u_k^{(\sigma_k)i}} E_{\vartheta_k}\left\{\left[P_k^{(\sigma_k)i}u_k^{(\sigma_k)i}-\frac{c_k^{(\sigma_k)i}}{x}\left(u_k^{(\sigma_k)i}\right)^2\right]\left(\frac{1}{1+r}\right)^{k-1}\right.\\
&\quad\left.+\sum_{\sigma_{k+1}=1}^{2}\lambda_{k+1}^{\sigma_{k+1}}V^{(\sigma_{k+1})i}\left[k+1,x+a-bx-u_k^{(\sigma_k)i}-\phi_k^{(\sigma_k)j*}(x)+\vartheta_k\right]\right\}\\
&= \max_{u_k^{(\sigma_k)i}}\left\{\left[P_k^{(\sigma_k)i}u_k^{(\sigma_k)i}-\frac{c_k^{(\sigma_k)i}}{x}\left(u_k^{(\sigma_k)i}\right)^2\right]\left(\frac{1}{1+r}\right)^{k-1}\right.\\
&\quad\left.+\sum_{y=1}^{3}\gamma_k^y\sum_{\sigma_{k+1}=1}^{2}\lambda_{k+1}^{\sigma_{k+1}}V^{(\sigma_{k+1})i}\left[k+1,x+a-bx-u_k^{(\sigma_k)i}-\phi_k^{(\sigma_k)j*}(x)+\vartheta_k^y\right]\right\};\\
V^{(\sigma_3)i}(4,x) &= q^ix\left(\frac{1}{1+r}\right)^3.
\end{aligned} \quad (4.4)$$

Performing the indicated maximization in (4.4) yields:

$$\begin{aligned}
&\left[P_k^{(\sigma_k)i}-\frac{2c_k^{(\sigma_k)i}u_k^{(\sigma_k)i}}{x}\right]\left(\frac{1}{1+r}\right)^{k-1}\\
&-\sum_{y=1}^{3}\gamma_k^y\sum_{\sigma_{k+1}=1}^{2}\lambda_{k+1}^{\sigma_{k+1}}V_{x_{k+1}}^{(\sigma_{k+1})i}\left[k+1,x+a-bx-u_k^{(\sigma_k)i}-\phi_k^{(\sigma_k)j*}(x)+\vartheta_k^y\right]=0;
\end{aligned} \quad (4.5)$$

for $i\in\{1,2\}$ and $k\in\{1,2,3\}$.

From (4.5), the game equilibrium strategies can be expressed as:

$$\begin{aligned}
\phi_k^{(\sigma_k)i*}(x) &= \frac{x}{2c_k^{(\sigma_k)i}}\left(P_k^{(\sigma_k)i}-(1+r)^{k-1}\sum_{y=1}^{3}\gamma_k^y\right.\\
&\quad\left.\sum_{\sigma_{k+1}=1}^{2}\lambda_{k+1}^{\sigma_{k+1}}V_{x_{k+1}}^{(\sigma_{k+1})i}\left[k+1,x+a-bx-\phi_k^{(\sigma_k)1*}(x)-\phi_k^{(\sigma_k)2*}(x)+\vartheta_k^y\right]\right),
\end{aligned} \quad (4.6)$$

for $i\in\{1,2\}$ and $k\in\{1,2,3\}$.

The expected game equilibrium payoffs of the extractors can be obtained as:

Proposition 4.1 The value function indicating the expected game equilibrium payoff of player i is

$$V^{(\sigma_k)i}(k,x) = \left[A_k^{(\sigma_k)i} x + C_k^{(\sigma_k)i}\right], \text{ for } i \in \{1,2\} \text{ and } k \in \{1,2,3\}, \tag{4.7}$$

where $A_k^{(\sigma_k)i}$ and $C_k^{(\sigma_k)i}$, for $i \in \{1,2\}$ and $k \in \{1,2,3\}$, are constants in terms of the parameters of the game (4.2 and 4.3).

Proof See Appendix A. ■

Substituting the relevant derivatives of the value functions in Proposition 4.1 into the game equilibrium strategies (4.6) yields a noncooperative Nash equilibrium solution of the game (4.2 and 4.3).

Now consider the case when the extractors agree to maximize their expected joint profit and share the excess of cooperative gains over their expected noncooperative payoffs equally. To maximize their expected joint payoff, they solve the problem of maximizing

$$E_{\theta_1,\theta_2,\theta_3;\vartheta_1,\vartheta_2,\vartheta_3}\left\{\sum_{j=1}^{2}\left[\sum_{k=1}^{3}\left(P_k^{(\sigma_k)j} u_k^j - \frac{c_k^{(\sigma_k)j}}{x_k}\left(u_k^j\right)^2\right)\left(\frac{1}{1+r}\right)^{k-1} + q^j x_4\left(\frac{1}{1+r}\right)^3\right]\right\} \tag{4.8}$$

subject to (4.2).

Invoking Theorem 2.1, one can characterize the optimal controls in the stochastic dynamic programming problem (4.2) and (4.8). In particular, a set of control strategies $\left\{u_k^{(\sigma_k)i*} = \psi_k^{(\sigma_k)i*}(x) \in \hat{\Gamma}^i, \text{ for } \sigma_k \in \{1,2\}, k \in \{1,2,3\} \text{ and } i \in \{1,2\}\right\}$ provides an optimal solution to the problem (4.2) and (4.8) if there exist functions $W^{(\sigma_k)}(k,x)$, for $k \in \{1,2,3\}$, such that the following recursive relations are satisfied:

$$\begin{aligned}
W^{(\sigma_4)}(4,x) &= \sum_{j=1}^{2} q^j x\left(\frac{1}{1+r}\right)^3, \\
W^{(\sigma_k)}(k,x) &= \max_{u_k^1,u_k^2} E_{\vartheta_k}\left\{\sum_{j=1}^{2}\left(P_k^{(\sigma_k)j} u_k^j - \frac{c_k^{(\sigma_k)j}}{x}\left(u_k^j\right)^2\right)\left(\frac{1}{1+r}\right)^{k-1}\right. \\
&\quad \left. + \sum_{\sigma_{k+1}=1}^{2} \lambda_{k+1}^{\sigma_{k+1}} W^{(\sigma_{k+1})}\left[k+1, x+a-bx-u_k^1-u_k^2+\vartheta_k\right]\right\} \\
&= \max_{u_k^1,u_k^2}\left\{\sum_{j=1}^{2}\left(P_k^{(\sigma_k)j} u_k^j - \frac{c_k^{(\sigma_k)j}}{x}\left(u_k^j\right)^2\right)\left(\frac{1}{1+r}\right)^{k-1}\right. \\
&\quad \left. + \sum_{y=1}^{3}\gamma_k^y \sum_{\sigma_{k+1}=1}^{2} \lambda_{k+1}^{\sigma_{k+1}} W^{(\sigma_{k+1})}\left[k+1, x+a-bx-u_k^1-u_k^2+\vartheta_k^y\right]\right\},
\end{aligned}$$

$$\text{for } k \in \{1,2,3\} \text{ and } \sigma_k \in \{1,2\}. \tag{4.9}$$

Performing the indicated maximization in (4.9) yields:

$$\left(P_k^{(\sigma_k)j} - \frac{2c_k^{(\sigma_k)j}u_k^j}{x_k}\right)\left(\frac{1}{1+r}\right)^{k-1} \\ -\sum_{y=1}^{3}\gamma_k^y \sum_{\sigma_{k+1}=1}^{2} \lambda_{k+1}^{\sigma_{k+1}} W_{x_{k+1}}^{(\sigma_{k+1})}\left[k+1, x+a-bx-u_k^1-u_k^2+\vartheta_k^y\right] = 0, \tag{4.10}$$

for $k \in \{1,2,3\}$ and $\sigma_k \in \{1,2\}$.

In particular, the optimal cooperative strategies can be obtained from (4.10) as:

$$\psi_k^{(\sigma_k)i*}(x) = \frac{x}{2c_k^{(\sigma_k)i}}\Bigg(P_k^{(\sigma_k)i} - \sum_{y=1}^{3}\gamma_k^y \sum_{\sigma_{k+1}=1}^{2} \lambda_{k+1}^{\sigma_{k+1}} W_{x_{k+1}}^{(\sigma_{k+1})}\left[k+1, x+a-bx \right. \\ \left. -\psi_k^{(\sigma_k)1*}(x) - \psi_k^{(\sigma_k)2*}(x) + \vartheta_k^y\right](1+r)^{k-1}\Bigg), \tag{4.11}$$

for $k \in \{1,2,3\}$ and $\sigma_k \in \{1,2\}$.

The expected joint payoff under cooperation can be obtained as:

Proposition 4.2 The value function indicating the maximized expected joint payoff is

$$W^{(\sigma_k)}(k,x) = \left[\widetilde{A}_k^{(\sigma_k)}x + \widetilde{C}_k^{(\sigma_k)}\right], \text{ for } k \in \{1,2,3\} \text{ and } \sigma_k \in \{1,2\}, \tag{4.12}$$

where $\widetilde{A}_k^{(\sigma_k)}$ and $\widetilde{C}_k^{(\sigma_k)}$, for $k \in \{1,2,3\}$ and $\sigma_k \in \{1,2\}$, are constants in terms of the parameters of the problem (4.8) and (4.2).

Proof See Appendix B. ■

Using (4.11) and Proposition 4.2, the optimal cooperative strategies of the agents can be expressed as:

$$\psi_k^{(\sigma_k)i*}(x) = \frac{x}{2c_k^{(\sigma_k)i}}\left(P_k^{(\sigma_k)i} - \sum_{\sigma_{k+1}=1}^{2} \lambda_{k+1}^{\sigma_{k+1}} A_{k+1}^{(\sigma_{k+1})}(1+r)^{k-1}\right), \tag{4.13}$$

for $i \in \{1,2\}$, $k \in \{1,2,3\}$ and $\sigma_k \in \{1,2\}$.

Substituting $\psi_k^{(\sigma_k)i}(x)$ from (4.13) into (4.2) yields the optimal cooperative state trajectory:

$$x_{k+1} = x_k + a - bx_k \\ - \sum_{j=1}^{2}\frac{x}{2c_k^{(\sigma_k)j}}\left(P_k^{(\sigma_k)j} - \sum_{\sigma_{k+1}=1}^{2} \lambda_{k+1}^{\sigma_{k+1}} \widetilde{A}_{k+1}^{(\sigma_{k+1})}(1+r)^{k-1}\right) + \vartheta_k, \tag{4.14}$$

if $\theta_k^{\sigma_k}$ occurs at stage k for $k \in \{1,2,3\}$ and $x_1 = x^0$.

Dynamics (4.14) is a linear stochastic difference equation readily solvable by standard techniques. Let $\{x_k^*$, for $k \in \{1,2,3\}\}$ denote the solution to (4.14).

Since the extractors agree to share the excess of cooperative gains over their expected noncooperative payoffs equally, an imputation

$$\begin{aligned}\xi^{(\sigma_k)i}(k,x_k^*) &= V^{(\sigma_k)i}(k,x_k^*) + \frac{1}{2}\left[W^{(\sigma_k)}(k,x_k^*) - \sum_{j=1}^{2} V^{(\sigma_k)j}(k,x_k^*)\right] \\ &= \left(A_k^{(\sigma_k)i}x_k^* + C_k^{(\sigma_k)i}\right) + \frac{1}{2}\left[\left(\widetilde{A}_k^{(\sigma_k)}x_k^* + \widetilde{C}_k^{(\sigma_k)}\right) - \sum_{j=1}^{2}\left(A_k^{(\sigma_k)j}x_k^* + C_k^{(\sigma_k)j}\right)\right], \quad (4.15)\end{aligned}$$

if $\theta_k^{\sigma_k}$ occurs at stage k for $k \in \{1,2,3\}$, $\sigma_k \in \{1,2\}$ and $i \in \{1,2\}$ has to be maintained.

Invoking Theorem 3.1, if $\theta_k^{\sigma_k}$ occurs and $x_k^* \in X$ is realized at stage k a payment equaling

$$\begin{aligned}B_k^{(\sigma_k)i}(x_k^*) &= (1+r)^{k-1}\Big[\xi^i(k,x_k^*) \\ &\quad - E_{\theta_{k+1}}\left(\xi^i\left[k+1, f_k\left(x_k^*, \psi_k^{(\sigma_k)*}(x_k^*)\right) + \theta_k\right]\right)\Big] \\ &= (1+r)^{k-1}\Big\{\left(A_k^{(\sigma_k)i}x_k^* + C_k^{(\sigma_k)i}\right) \\ &\quad + \frac{1}{2}\left(\left(\widetilde{A}_k^{(\sigma_k)}x_k^* + \widetilde{C}_k^{(\sigma_k)}\right) - \sum_{j=1}^{2}\left(A_k^{(\sigma_k)j}x_k^* + C_k^{(\sigma_k)j}\right)\right) \\ &\quad - \sum_{y=1}^{3}\gamma_k^y \sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}}\Big[\left(A_{k+1}^{(\sigma_{k+1})i}x_{k+1}^{*(\vartheta_k^y)} + C_{k+1}^{(\sigma_{k=1})i}\right) \\ &\quad + \frac{1}{2}\left(\left(\widetilde{A}_{k+1}^{(\sigma_{k+1})}x_{k+1}^{*(\vartheta_k^y)} + \widetilde{C}_{k+1}^{(\sigma_{k+1})}\right) - \sum_{j=1}^{2}\left(A_{k+1}^{(\sigma_{k+1})j}x_{k+1}^{*(\vartheta_k^y)} + C_{k+1}^{(\sigma_{k+1})j}\right)\right)\Big]\Big\},\end{aligned} \quad (4.16)$$

where

$$x_{k+1}^{*(\vartheta_k^y)} = x_k^* + a - bx_k^* - \sum_{j=1}^{2}\frac{x_k^*}{2c_k^{(\sigma_k)j}}\left(P_k^{(\sigma_k)j} - \sum_{\sigma_{k+1}=1}^{2}\lambda_{k+1}^{\sigma_{k+1}}\widetilde{A}_{k+1}^{(\sigma_{k+1})}(1+r)^{k-1}\right) + \vartheta_k^*$$

for $y \in \{1,2,3\}$,

given to firm i at stage $k \in \{1,2,3\}$ would lead to the realization of the imputation (4.15).

A subgame consistent solution can be readily obtained using (4.13), (4.15) and (4.16).

9.5 Extensions

The analysis can be expanded in a few directions.

Case 1: Random Changes in the State Dynamics Structures
Following Yeung (2011) one allow the structure of the state dynamics in (1.1) be affected by the random variable θ_k for $k \in \{1, 2, \cdots, T\}$. In particular the state dynamics become:

$$x_{k+1} = f_k\left(x_k, u_k^1, u_k^2, \cdots, u_k^n; \theta_k\right) + \vartheta_k, \tag{5.1}$$

for $k \in \{1, 2, \cdots, T\}$ and $x_1 = x^0$,

where $u_k^i \in U^i \subset R^{m_i}$ is the control vector of player i at stage k, $x_k \in X$ is the state, ϑ_k is a sequence of statistically independent random variables, and θ_k is an independent discrete random variables with range $\left\{\theta_k^1, \theta_k^2, \cdots, \theta_k^{\eta_k}\right\}$ and corresponding probabilities $\left\{\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^{\eta_k}\right\}$.

Following the analyses in Sects. 9.1, 9.2 and 9.3, a theorem deriving a subgame consistent PDP can be established as follows.

Theorem 5.1 A payment equaling

$$B_k^{(\sigma_k)i}\left(x_k^*\right) = \xi^{(\sigma_k)i}\left(k, x_k^*\right) - E_{\vartheta_k}\left[\sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}} \left(\xi^{(\sigma_{k+1})i}\left[k+1, f_k\left(x_k^*, \psi_k^{(\sigma_k)*}\left(x_k^*\right); \theta_k^{\sigma_k}\right) + \vartheta_k\right]\right)\right], \tag{5.2}$$

for $i \in N$,

given to player i at stage $k \in \{1, 2, \cdots, T\}$, if $\theta_k^{\sigma_k}$ occurs and $x_k^* \in X_k^*$, leads to the realization of the imputation according to the agreed upon optimality principle. ∎

Case 2: More Complex Branching Processes
The random event θ_k affecting the payoff structures of the players in stage k may be more complex branching processes. For instance, the random variables may not be independent and may stem from a branching process in which the random variable θ_k for $k \in \{1, 2, \cdots, T\}$ is conditional upon the realization of the random variables in its preceding stages. An example of this type of processes is the one adopted in Yeung (2003) as a random variable stemming from the branching process as described below.

$\theta^1 = \left\{\theta_1^1, \theta_2^1, \ \ldots, \theta_{\eta_1}^1\right\}$ with corresponding probabilities $\left\{\lambda_1^1, \lambda_2^1, \ \ldots, \lambda_{\eta_1}^1\right\}$.

Given that $\theta_{a_1}^1$ is realized in time interval $[t_1, t_2)$, for $a_1 = 1, 2, \ldots, \eta_1$,

$\theta^2 = \left\{ \theta_1^{2[(1,a_1)]}, \theta_2^{2[(1,a_1)]}, \quad \ldots, \theta_{\eta_{2[(1,a_1)]}}^{2[(1,a_1)]} \right\}$ would be realized with the corresponding probabilities $\left\{ \lambda_1^{2[(1,a_1)]}, \lambda_2^{2[(1,a_1)]}, \quad \ldots, \lambda_{\eta_{2[(1,a_1)]}}^{2[(1,a_1)]} \right\}$.

Given that $\theta_{a_1}^1$ is realized in time interval $[t_1, t_2)$ and $\theta_{a_2}^{2[(1,a_1)]}$ is realized in time interval $[t_2, t_3)$, for $a_1 = 1, 2, \ldots, \eta_1$ and $a_2 = 1, 2, \ldots, \eta_{2[(1,a_1)]}$,

$\theta^3 = \left\{ \theta_1^{3[(1,a_1)\,(2,a_2)]}, \theta_2^{3[(1,a_1)\,(2,a_2)]}, \quad \ldots, \theta_{\eta_{3[(1,a_1)\,(2,a_2)]}}^{3[(1,a_1)\,(2,a_2)]} \right\}$ would be realized with the corresponding probabilities

$\left\{ \lambda_1^{3[(1,a_1)\,(2,a_2)]}, \lambda_2^{3[(1,a_1)\,(2,a_2)]}, \quad \ldots, \lambda_{\eta_{3[(1,a_1)\,(2,a_2)]}}^{3[(1,a_1)\,(2,a_2)]} \right\}$.

In general, given that $\theta_{a_1}^1$ is realized in time interval $[t_1, t_2)$, $\theta_{a_2}^{2[(1,a_1)]}$ is realized in time interval $[t_2, t_3)$, …, and $\theta_{a_{k-1}}^{k-1[(1,a_1)(2,a_2)\ldots(k-2,a_{k-2})]}$ is realized in time interval $[t_{k-1}, t_k)$, for $a_1 = 1, 2, \ldots, \eta_1$, $a_2 = 1, 2, \ldots, \eta_{2[(1,a_1)]}$, $\ldots, a_{k-1} = 1, 2, \ldots, \eta_{k-1[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}$,

$$\theta^k = \left\{ \theta_1^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \theta_2^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \quad \ldots, \theta_{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]} \right\}$$

would be realized with the corresponding probabilities

$$\left\{ \lambda_1^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \lambda_2^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \quad \ldots, \lambda_{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]} \right\}$$

for $k = 1, 2, \ldots, \tau$.

Applying the techniques derived in the analysis in this paper, subgame consistent solutions can be derived accordingly.

Case 3: Games with Deterministic Dynamics

The analysis can be readily applied to derive subgame consistent solutions in randomly-furcating dynamic games in which the random variables ϑ_k in the stock dynamics are not present. In particular, the objective that player i seeks to maximize becomes

$$E_{\theta_1,\theta_2,\cdots,\theta_T} \left\{ \sum_{k=1}^{T} g_k^i \left[x_k, u_k^1, u_k^2, \cdots, u_k^n; \theta_k \right] + q^i(x_{T+1}) \right\}, \text{ for } i \in N \tag{5.3}$$

subject to the deterministic dynamics:

$$x_{k+1} = f_k\left(x_k, u_k^1, u_k^2, \cdots, u_k^n \right). \tag{5.4}$$

Following the analysis in Sects 9.3 and 9.4 and the proof of Theorem 3.1, a theorem deriving a subgame consistent PDP for the randomly-furcating dynamic game (5.3 and 5.4) can be established as follows.

Theorem 5.2 A payment equaling

$$B_k^{(\sigma_k)i}\left(x_k^*\right) = \xi^{(\sigma_k)i}\left(k, x_k^*\right) - \left[\sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}}\left(\xi^{(\sigma_{k+1})i}\left[k+1, f_k\left(x_k^*, \psi_k^{(\sigma_k)*}\left(x_k^*\right)\right)\right]\right)\right], \tag{5.5}$$

for $i \in N$,

given to player i at stage $k \in \{1, 2, \cdots, T\}$, if $\theta_k^{\sigma_k}$ occurs and $\in X_k^*$, would yield the PDP leading to a subgame consistent solution of the game (5.3 and 5.4). ■

9.6 Chapter Appendices

Appendix A. Proof of Proposition 4.1 Consider first the last stage, that is stage 3, when $\theta_3^{\sigma_3}$ occurs. Invoking that $V^{(\sigma_3)i}(3, x) = \left[A_3^{(\sigma_3)i} x + C_3^{(\sigma_3)i}\right]$ from Proposition 4.1 and $V^{(\sigma_3)i}(4, x) = q^i x\left(\frac{1}{1+r}\right)^3$, the conditions in equation (4.4) become

$$\begin{aligned}\left[A_3^{(\sigma_3)i} x + C_3^{(\sigma_3)i}\right] &= \max_{u_3^{(\sigma_3)i}} \left\{\left[P_3^{(\sigma_3)i} u_3^{(\sigma_3)i} - \frac{c_3^{(\sigma_3)i}}{x}\left(u_3^{(\sigma_3)\mathrm{i}}\right)^2\right] \left(\frac{1}{1+r}\right)^{k-1}\right. \\ &\left.+ \sum_{y=1}^{3} \gamma_3^y q^i \left[x + a - bx - u_3^{(\sigma_3)i} - \phi_3^{(\sigma_3)j*}(x) + \vartheta_3^y\right]\right\}, \text{ for } i \in \{1, 2\}.\end{aligned} \tag{A.1}$$

Performing the indicated maximization in (A.1) yields:

$$\left[P_3^{(\sigma_3)i} - \frac{2c_3^{(\sigma_3)i} u_3^{(\sigma_3)\mathrm{i}}}{x}\right] \left(\frac{1}{1+r}\right)^{k-1} - \sum_{y=1}^{3} \gamma_3^y q^i = 0, \text{ for } i \in \{1, 2\}. \tag{A.2}$$

The game equilibrium strategies in stage 3 can then be expressed as:

$$\phi_3^{(\sigma_3)i*}(x) = \left[P_3^{(\sigma_3)i} - (1+r)^2 q^i\right] \frac{x}{2c_3^{(\sigma_3)i}}, \text{ for } i \in \{1, 2\}. \tag{A.3}$$

Substituting (A.3) into (A.1) yields:

$$\begin{aligned}\left[A_3^{(\sigma_3)i} x + C_3^{(\sigma_3)i}\right] &= \left[P_3^{(\sigma_3)i}\left[P_3^{(\sigma_3)i} - (1+r)^2 q^i\right] \frac{x}{2c_3^{(\sigma_3)i}}\right. \\ &\left.- \frac{1}{4c_3^{(\sigma_3)i}}\left[P_3^{(\sigma_3)i} - (1+r)^2 q^i\right]^2 x\right] \left(\frac{1}{1+r}\right)^{k-1} \\ &+ \sum_{y=1}^{3} \gamma_3^y q^i \left(x + a - bx - \sum_{j=1}^{2}\left[P_3^{(\sigma_3)j} - (1+r)^2 q^j\right] \frac{x}{2c_3^{(\sigma_3)j}} + \vartheta_3^y\right),\end{aligned} \tag{A.4}$$

for $i \in \{1, 2\}$.

Note that both sides of equation (A.4) are linear expression of x, the terms $A_3^{(\sigma_3)i}$ and $C_3^{(\sigma_3)i}$, for $i \in \{1,2\}$ and $\sigma_3 \in \{1,2\}$, are explicitly given in (A.4).

Now we proceed to stage 2, the conditions in equation (4.4) become

$$\left[A_2^{(\sigma_2)i}x + C_2^{(\sigma_2)i}\right] = \max_{u_2^{(\sigma_2)i}} \left\{ \left[P_2^{(\sigma_2)i}u_2^{(\sigma_2)i} - \frac{c_2^{(\sigma_2)i}}{x}\left(u_2^{(\sigma_2)i}\right)^2\right] \left(\frac{1}{1+r}\right) \right.$$
$$\left. + \sum_{y=1}^{3}\gamma_2^y \sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}\left[A_3^{(\sigma_3)i}\left[x + a - bx - u_2^{(\sigma_2)i} - \phi_2^{(\sigma_2)j*}(x) + \vartheta_2^y\right] + C_3^{(\sigma_3)i}\right]\right\}, \quad \text{(A.5)}$$

for $i \in \{1,2\}$.

Performing the indicated maximization in (A.5) yields:

$$\left[P_2^{(\sigma_2)i} - \frac{2c_2^{(\sigma_2)i}u_2^{(\sigma_2)i}}{x}\right]\left(\frac{1}{1+r}\right) - \sum_{y=1}^{3}\gamma_2^y \sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}A_3^{(\sigma_3)i} = 0, \text{ for } i \in \{1,2\}. \quad \text{(A.6)}$$

The game equilibrium strategies in stage 2 can then be expressed as:

$$\phi_2^{(\sigma_2)i*}(x) = \left[P_2^{(\sigma_2)i} - (1+r)\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}A_3^{(\sigma_3)i}\right]\frac{x}{2c_2^{(\sigma_2)i}}, \text{ for } i \in \{1,2\}. \quad \text{(A.7)}$$

Substituting (A.7) into (A.5) yields:

$$\left[A_2^{(\sigma_2)i}x + C_2^{(\sigma_2)i}\right] = \left[P_2^{(\sigma_2)i}\left[P_2^{(\sigma_2)i} - (1+r)\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}A_3^{(\sigma_3)i}\right]\frac{x}{2c_2^{(\sigma_2)i}}\right.$$
$$\left. - \frac{1}{4c_2^{(\sigma_2)i}}\left[P_2^{(\sigma_2)i} - (1+r)\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}A_3^{(\sigma_3)i}\right]^2 x\right]\left(\frac{1}{1+r}\right)$$
$$+ \sum_{y=1}^{3}\gamma_2^y \sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}\left[A_3^{(\sigma_3)i}\left(x + a - bx\right.\right.$$
$$\left.\left. - \sum_{j=1}^{2}\left[P_2^{(\sigma_2)j} - (1+r)\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}A_3^{(\sigma_3)i}\right]\frac{x}{2c_2^{(\sigma_2)j}} + \vartheta_2^y\right) + C_3^{(\sigma_3)i}\right], \quad \text{(A.8)}$$

for $i \in \{1,2\}$.

Once again, both sides of equation (A.8) are linear expression of x, the terms $A_2^{(\sigma_2)i}$ and $C_2^{(\sigma_2)i}$, for $i \in \{1,2\}$ and $\sigma_2 \in \{1,2\}$, can be obtained explicitly using (A.8).

Finally, we proceed to the first stage, the conditions in equation (4.4) become

$$\left[A_1^{(\sigma_1)i}x + C_1^{(\sigma_1)i}\right] = \max_{u_1^{(\sigma_1)i}}\left\{\left[P_1^{(\sigma_1)i}u_1^{(\sigma_1)i} - \frac{c_1^{(\sigma_1)i}}{x}\left(u_1^{(\sigma_1)\mathrm{i}}\right)^2\right]\right.$$
$$\left. + \sum_{y=1}^{3}\gamma_1^y\sum_{\sigma_2=1}^{2}\lambda_2^{\sigma_2}\left[A_2^{(\sigma_2)i}\left[x + a - bx - u_1^{(\sigma_1)i} - \phi_1^{(\sigma_1)j}(x) + \vartheta_1^y\right] + C_2^{(\sigma_2)i}\right]\right\} \quad \text{(A.9)}$$

for $i \in \{1,2\}$.

Following the analysis in (A.6 and A.7), the game equilibrium strategies in stage 1 can then be expressed as:

$$\phi_1^{(\sigma_1)i*}(x) = \left[P_1^{(\sigma_1)i} - \sum_{\sigma_2=1}^{2}\lambda_2^{\sigma_2}A_2^{(\sigma_2)i}\right]\frac{x}{2c_1^{(\sigma_1)i}}, \ \text{for}\, i \in \{1,2\}. \quad \text{(A.10)}$$

Substituting (A.10) into (A.9) yields:

$$\left[A_1^{(\sigma_1)i}x + C_1^{(\sigma_1)i}\right] = \left[P_1^{(\sigma_1)i}\left[P_1^{(\sigma_1)i} - \sum_{\sigma_2=1}^{2}\lambda_2^{\sigma_2}A_2^{(\sigma_2)i}\right]\frac{x}{2c_1^{(\sigma_1)i}}\right.$$
$$\left. - \frac{1}{4c_1^{(\sigma_1)i}}\left[P_1^{(\sigma_1)i} - \sum_{\sigma_2=1}^{2}\lambda_2^{\sigma_2}A_2^{(\sigma_2)i}\right]^2 x\right] + \sum_{y=1}^{3}\gamma_1^y\sum_{\sigma_3=1}^{2}\lambda_2^{\sigma_2}$$
$$\left[A_2^{(\sigma_2)i}\left(x + a - bx\right.\right.$$
$$\left.\left. - \sum_{j=1}^{2}\left[P_1^{(\sigma_1)j} - (1+r)\sum_{\sigma_2=1}^{2}\lambda_2^{\sigma_2}A_2^{(\sigma_2)i}\right]\frac{x}{2c_1^{(\sigma_1)j}} + \vartheta_1^y\right) + C_2^{(\sigma_2)i}\right], \quad \text{(A.11)}$$

for $i \in \{1,2\}$.

Once again, both sides of equation (A.11) are linear expression of x, the terms $A_1^{(\sigma_1)i}$ and $C_1^{(\sigma_1)i}$, for $i \in \{1,2\}$ and $\sigma_1 = 1$, can be obtained explicitly using (A.11).

Appendix B. Proof of Proposition 4.2 Consider first the last stage, that is stage 3, when $\theta_3^{\sigma_3}$ occurs. Invoking that $W^{(\sigma_3)}(3,x) = \left[\widetilde{A}_3^{(\sigma_3)}x + \widetilde{C}_3^{(\sigma_3)}\right]$ from Proposition 4.2 and $W^{(\sigma_3)}(4,x) = \sum_{j=1}^{2} q^j x \left(\frac{1}{1+r}\right)^3$, the condition in equation (4.9) becomes

$$\left[\widetilde{A}_3^{(\sigma_3)}x + \widetilde{C}_3^{(\sigma_3)}\right] = \max_{u_3^{(\sigma_3)1},\, u_3^{(\sigma_3)2}}\left\{\sum_{j=1}^{2}\left[P_3^{(\sigma_3)j}u_3^{(\sigma_3)j} - \frac{c_3^{(\sigma_3)j}}{x}\left(u_3^{(\sigma_3)\mathrm{j}}\right)^2\right]\left(\frac{1}{1+r}\right)^{k-1}\right.$$
$$\left. + \sum_{y=1}^{3}\gamma_3^y\sum_{j=1}^{2}q^j\left[x + a - bx - \sum_{\ell=1}^{2}u_3^{(\sigma_3)\ell} + \vartheta_3^y\right]\right\}. \quad \text{(B.1)}$$

Performing the indicated maximization in (B.1) yields:

$$\left[P_3^{(\sigma_3)i} - \frac{2c_3^{(\sigma_3)i}u_3^{(\sigma_3)\mathrm{i}}}{x}\right] \left(\frac{1}{1+r}\right)^{k-1} - \sum_{y=1}^{3}\gamma_3^y\sum_{j=1}^{2}q^j = 0, \text{ for } i\in\{1,2\}. \quad \text{(B.2)}$$

The optimal cooperative strategies in stage 3 can then be expressed as:

$$\psi_3^{(\sigma_3)i*}(x) = \left[P_3^{(\sigma_3)i} - (1+r)^2\sum_{j=1}^{2}q^j\right]\frac{x}{2c_3^{(\sigma_3)i}}, \text{ for } i\in\{1,2\}. \quad \text{(B.3)}$$

Substituting (B.3) into (B.1) yields:

$$\begin{aligned}\left[\widetilde{A}_3^{(\sigma_3)}x + \widetilde{C}_3^{(\sigma_3)}\right] = &\sum_{j=1}^{2}\left[P_3^{(\sigma_3)j}\left[P_3^{(\sigma_3)j} - (1+r)^2\sum_{\ell=1}^{2}q^\ell\right]\frac{x}{2c_3^{(\sigma_3)j}}\right.\\ &\left.-\frac{1}{4c_3^{(\sigma_3)j}}\left[P_3^{(\sigma_3)j} - (1+r)^2\sum_{\ell=1}^{2}q^\ell\right]^2 x\right]\left(\frac{1}{1+r}\right)^{k-1}\\ &+\sum_{y=1}^{3}\gamma_3^y\sum_{j=1}^{2}q^j\left(x + a - bx\right.\\ &\left.-\sum_{\ell=1}^{2}\left[P_3^{(\sigma_3)\ell} - (1+r)^2\sum_{\zeta=1}^{2}q^\zeta\right]\frac{x}{2c_3^{(\sigma_3)j}} + \vartheta_3^y\right), \end{aligned} \quad \text{(B.4)}$$

for $i\in\{1,2\}$.

Note that both sides of equation (B.4) are linear expression of x, the terms $\widetilde{A}_3^{(\sigma_3)}$ and $\widetilde{C}_3^{(\sigma_3)}$, for $\sigma_3\in\{1,2\}$, are explicitly given in (B.4).

Now we proceed to stage 2, the condition in equation (4.9) becomes

$$\begin{aligned}\left[\widetilde{A}_2^{(\sigma_2)}x + \widetilde{C}_2^{(\sigma_2)}\right] = &\max_{u_2^{(\sigma_2)1},u_2^{(\sigma_2)2}}\left\{\sum_{j=1}^{2}\left[P_2^{(\sigma_2)j}u_2^{(\sigma_2)j} - \frac{c_2^{(\sigma_2)j}}{x}\left(u_2^{(\sigma_2)\mathrm{j}}\right)^2\right]\left(\frac{1}{1+r}\right)\right.\\ &\left.+\sum_{y=1}^{3}\gamma_2^y\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}\left[\widetilde{A}_3^{(\sigma_3)}\left[x + a - bx - \sum_{j=1}^{2}u_2^{(\sigma_2)j} + \vartheta_2^y\right] + \widetilde{C}_3^{(\sigma_3)}\right]\right\}. \end{aligned} \quad \text{(B.5)}$$

Performing the indicated maximization in (B.5) yields:

$$\left[P_2^{(\sigma_2)i} - \frac{2c_2^{(\sigma_2)i}u_2^{(\sigma_2)\mathrm{i}}}{x}\right] \left(\frac{1}{1+r}\right) - \sum_{y=1}^{3}\gamma_2^y\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}\widetilde{A}_3^{(\sigma_3)} = 0, \text{ for } i\in\{1,2\}. \quad \text{(B.6)}$$

The optimal cooperative strategies in stage 2 can then be expressed as:

$$\psi_2^{(\sigma_2)i*}(x) = \left[P_2^{(\sigma_2)i} - (1+r)\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}\widetilde{A}_3^{(\sigma_3)}\right]\frac{x}{2c_2^{(\sigma_2)i}}, \text{ for } i\in\{1,2\}. \tag{B.7}$$

Substituting (B.7) into (B.5) yields:

$$\begin{aligned}\left[\widetilde{A}_2^{(\sigma_2)}x + \widetilde{C}_2^{(\sigma_2)}\right] = &\sum_{j=1}^{2}\left[P_2^{(\sigma_2)j}\left[P_2^{(\sigma_2)j} - (1+r)\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}\widetilde{A}_3^{(\sigma_3)}\right]\frac{x}{2c_2^{(\sigma_2)j}}\right.\\ &\left.-\frac{1}{4c_2^{(\sigma_2)j}}\left[P_2^{(\sigma_2)j} - (1+r)\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}\widetilde{A}_3^{(\sigma_3)}\right]^2 x\right]\left(\frac{1}{1+r}\right)\\ &+\sum_{y=1}^{3}\gamma_2^y\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}\left[\widetilde{A}_3^{(\sigma_3)}\left(x+a-bx\right.\right.\\ &\left.\left.-\sum_{j=1}^{2}\left[P_2^{(\sigma_2)j} - (1+r)\sum_{\sigma_3=1}^{2}\lambda_3^{\sigma_3}\widetilde{A}_3^{(\sigma_3)}\right]\frac{x}{2c_2^{(\sigma_2)j}} + \vartheta_2^y\right) + \widetilde{C}_3^{(\sigma_3)}\right].\end{aligned} \tag{B.8}$$

Once again, both sides of equation (B.8) are linear expression of x, the terms $\widetilde{A}_2^{(\sigma_2)}$ and $\widetilde{C}_2^{(\sigma_2)}$, for $\sigma_2\in\{1,2\}$, can be obtained explicitly using (B.8).

Finally, we proceed to the first stage, the conditions in equation (4.9) become

$$\begin{aligned}\left[\widetilde{A}_1^{(\sigma_1)}x + \widetilde{C}_1^{(\sigma_1)}\right] = &\max_{u_1^{(\sigma_1)1},\,u_1^{(\sigma_1)2}}\left\{\sum_{j=1}^{2}\left[P_1^{(\sigma_1)j}u_1^{(\sigma_1)j} - \frac{c_1^{(\sigma_1)j}}{x}\left(u_1^{(\sigma_1)j}\right)^2\right]\right.\\ &\left.+\sum_{y=1}^{3}\gamma_1^y\sum_{\sigma_2=1}^{2}\lambda_2^{\sigma_2}\left[\widetilde{A}_2^{(\sigma_2)}\left[x+a-bx-\sum_{j=1}^{2}u_1^{(\sigma_1)j}+\vartheta_1^y\right]+\widetilde{C}_2^{(\sigma_2)}\right]\right\}.\end{aligned} \tag{B.9}$$

Following the analysis in (B.6 and B.7), the optimal cooperative strategies in stage 1 can then be expressed as:

$$\psi_1^{(\sigma_1)i*}(x) = \left[P_1^{(\sigma_1)i} - \sum_{\sigma_1=1}^{2}\lambda_2^{\sigma_2}\widetilde{A}_2^{(\sigma_2)}\right]\frac{x}{2c_1^{(\sigma_1)i}}, \text{ for } i\in\{1,2\}. \tag{B.10}$$

Substituting (B.10) into (B.9) yields:

$$\begin{aligned}\left[\widetilde{A}_1^{(\sigma_1)}x + \widetilde{C}_1^{(\sigma_1)}\right] = &\sum_{j=1}^{2}\left[P_1^{(\sigma_1)j}\left[P_1^{(\sigma_1)j} - \sum_{\sigma_2=1}^{2}\lambda_2^{\sigma_2}\widetilde{A}_2^{(\sigma_2)}\right]\frac{x}{2c_1^{(\sigma_1)j}}\right.\\ &\left.-\frac{1}{4c_1^{(\sigma_1)j}}\left[P_1^{(\sigma_1)j} - \sum_{\sigma_2=1}^{2}\lambda_2^{\sigma_2}\widetilde{A}_2^{(\sigma_2)}\right]^2 x\right]\\ &+\sum_{y=1}^{3}\gamma_1^y\sum_{\sigma_2=1}^{2}\lambda_2^{\sigma_2}\left[\widetilde{A}_2^{(\sigma_2)}\left(x+a-bx\right.\right.\\ &\left.\left.-\sum_{j=1}^{2}\left[P_1^{(\sigma_1)j} - \sum_{\sigma_2=1}^{2}\lambda_2^{\sigma_2}\widetilde{A}_2^{(\sigma_2)}\right]\frac{x}{2c_1^{(\sigma_1)j}} + \vartheta_1^y\right) + \widetilde{C}_2^{(\sigma_2)}\right].\end{aligned} \tag{B.11}$$

Once again, both sides of equation (B.11) are linear expression of x, the terms $\widetilde{A}_1^{(\sigma_1)}$ and $\widetilde{C}_1^{(\sigma_1)}$, for $\sigma_1 = 1$, can be obtained explicitly using (B.11).

9.7 Chapter Notes

This Chapter considers subgame-consistent cooperative solutions in randomly furcating stochastic dynamic games developed by Yeung and Petrosyan (2013a). The extension of continuous-time randomly furcating stochastic differential games to an analysis in discrete time is not just of theoretical interest but also for practical reasons in applications in operations research. In the process of obtaining the main results for subgame consistent solution, Nash equilibrium for randomly furcating stochastic dynamic games and optimal control for randomly furcating stochastic control problems are also derived. Yeung and Petrosyan (2014b) considered subgame consistent cooperative provision of public goods under accumulation and payoff uncertainties. Yeung and Petrosyan (2014a) examined subgame consistent solution for a dynamic game of pollution management in which future environmental costs are not known with certainty.

9.8 Problems

1. Consider an economy endowed with a renewable resource and with 2 resource extractors (firms). The lease for resource extraction begins at stage 1 and ends at stage 3 for these two firms. Let u_k^i denote the resource extracted by firm i at stage k, for $i \in \{1, 2\}$. Let U^i be the set of admissible amount of resource extracted by firm i, and $x_k \in X \subset R^+$ be the size of the resource stock at stage k.

   It is known at each stage there is a random element, θ_k for $k \in \{1, 2, 3\}$, affecting the revenues of the outputs produced by these firms and their costs of extraction. If θ_k^1 happens at stage $k \in \{2, 3\}$ the profits (in present-value) that firm 1 and firm 2 will obtain at stage k are respectively:

$$\left[4u_k^1 - \frac{2}{x_k}\left(u_k^1\right)^2\right] \left(\frac{1}{1+r}\right)^{k-1} \text{ and } \left[2u_k^2 - \frac{1}{x_k}\left(u_k^2\right)^2\right] \left(\frac{1}{1+r}\right)^{k-1},$$

   where $r = 0.05$ is the discount rate.

   If θ_k^2 happens at stage $k \in \{2, 3\}$ the profits (in present-value) that firm 1 and firm 2 will obtain at stage k are respectively:

$$\left[2u_k^1 - \frac{2}{x_k}\left(u_k^1\right)^2\right] \left(\frac{1}{1+r}\right)^{k-1} \text{ and } \left[3u_k^2 - \frac{2}{x_k}\left(u_k^2\right)^2\right] \left(\frac{1}{1+r}\right)^{k-1}.$$

It is known in stage 1 that θ_1^1 has occurred. The probability that θ_k^1 will occur at stage $k \in \{2, 3\}$ is 0.6 and the probability that θ_k^2 will occur at stage $k \in \{2, 3\}$ is 0.4. In stage 4, a terminal payment (again in present-value) equaling $x_4 \left(\frac{1}{1+r}\right)^3$ will be paid to firm 1 and a terminal payment (again in present-value) equaling $0.5x_4 \left(\frac{1}{1+r}\right)^3$ will be paid to firm 2.

The growth dynamics of the resource is governed by the stochastic difference equation:

$$x_{k+1} = x_k + 15 - 0.1x_k - \sum_{j=1}^{2} u_k^j + \vartheta_k,$$

for $k \in \{1, 2, 3\}$ and $x_1 = 12$,

where ϑ_k is a random variable with non-negative range $\{0, 1, 2\}$ and corresponding probabilities $\{0.1, 0.7, 0.2\}$; moreover $\vartheta_1, \vartheta_2, \vartheta_3$ are independent. Moreover, we have the constraint $u_k^1 + u_k^2 \le 0.9x_k + 15$.

The objective of extractor $i \in \{1, 2\}$ is to maximize the present value of the expected stream of future profits:

Characterize the feedback Nash equilibrium.

2. Obtain a group optimal solution that maximizes the joint expected profit.
3. Consider the case when the extractors agree to share the excess of cooperative gains over their expected noncooperative profits equally. Derive a subgame consistent solution.