Chapter 6
Subgame Consistent Cooperative Solution in NTU Differential Games

Subgame consistency is a fundamental element in the solution of cooperative stochastic differential games which ensures that the extension of the solution policy to a later starting time and any possible state brought about by prior optimal behavior of the players would remain optimal. In many game situations payoff (or utility) of players may not be transferable. It is well known that utility in economic study is assumed to be non-transferrable or comparable among economic agents. The Nash (1950, 1953) bargaining solution is a solution for non-transferable payoff cooperative games. Strategic interactions involving national security, social issues and political gains fall into the category of non-transferrable utility/payoff (NTU) games. In the case when payoffs are nontransferable, transfer payments cannot be made and subgame consistent solution mechanism becomes extremely complicated. In this Chapter, the issue of subgame consistency in cooperative stochastic differential games with nontransferable payoffs or utility is presented. In particular, the Chapter is an integrated exposition of the works in Yeung and Petrosyan (2005) and Yeung et al. (2007). The Chapter is organized as follows. The formulation of non-transferrable utility cooperative stochastic differential games, the corresponding Pareto optimal state trajectories and individual player's payoffs under cooperation are provided in Sect. 6.1. The notion of subgame consistency in NTU cooperative stochastic differential games under time invariant payoff weights is examined in Sect. 6.2. In Section 6.3, a class of cooperative stochastic differential games with nontransferable payoffs is developed to illustrate the derivation of subgame consistent solutions. Subgame consistent cooperative solutions of the class of NTU games developed in Sect. 6.3 are investigated in Sect. 6.4. Numerical delineations of the solutions presented in Sect. 6.4 are given in Sect. 6.5. An analysis on infinite horizon NTU cooperative stochastic differential games is provided in Sect. 6.6. A chapter appendices containing proofs are given in Sect. 6.7. Chapter notes are given Sect. 6.8 and problems in Sect. 6.9.

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6.1 NTU Cooperative Stochastic Differential Games

Consider the two-person cooperative stochastic differential game with initial state x_0 and duration $T - t_0$. The state space of the game is $X \in R^n$, with permissible state trajectories $\{x(s),\ t_0 \le s \le T\}$. The state dynamics of the game is characterized by the vector-valued stochastic differential equations:

$$dx(s) = f[s, x(s), u_1(s), u_2(s)]ds + \sigma[s, x(s)]dz(s), \qquad x(t_0) = x_0, \tag{1.1}$$

where $\sigma[s, x(s)]$ is a $n \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\sigma[s, x(s)]'$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)], u_i \in U_i \subset compR^{\ell}$ is the control vector of player i, for $i \in \{1, 2\}$.

At time instant $s \in [t_0, T]$, the instantaneous payoff of player i, for $i \in \{1, 2\}$, is denoted by $g^i[s, x(s), u_1(s), u_2(s)]$, and when the game terminates at time T, player i receives a terminal payment of $q^i(x(T))$. Payoffs are nontransferable across players. Given a time-varying instantaneous discount rate $r(s)$, for $s \in [t_0, T]$, values received t time after t_0 have to be discounted by the factor $\exp\left[-\int_{t_0}^{t} r(y)dy\right]$. Hence at time t_0, the expected payoff of player i, for $i \in \{1, 2\}$, is given as:

$$J^i(t_0, x_0) = E_{t_0}\left\{ \int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s)] \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds \right. \\ \left. + \exp\left[-\int_{t_0}^{T} r(y)dy\right] q^i(x(T)) \,\middle|\, x(t_0) = x_0 \right\}, \tag{1.2}$$

where E_{t_0} denotes the expectation operator performed at time t_0,

We use $\Gamma(x_0, T - t_0)$ to denote the game (1.1 and 1.2) and $\Gamma(x_\tau, T - \tau)$ to denote an alternative game with state dynamics (1.1) and payoff structure (1.2) which starts at time $\tau \in [t_0, T]$ with initial state $x_\tau \in X$. The benchmark noncooperative feedback Nash equilibrium solution can be characterized by Theorem 1.1 in Chap. 3.

6.1.1 *Pareto Optimal Trajectories*

Consider the situation when the players agree to cooperate. We use $\Gamma_c(x_0, T - t_0)$ to denote a cooperative game with dynamics (1.1) and payoffs (1.2). To achieve group optimality, the players have to consider cooperative outcomes belonging to the Pareto optimal set. Pareto optimal trajectories for $\Gamma_c(x_0, T - t_0)$ can be identified by choosing a specific weight $\alpha_1 \in (0, \infty)$ that solves the following stochastic control problem (See Leitmann (1974), Dockner and Jørgensen (1984) and Jørgensen and Zaccour (2001)):

$$\max_{u_1,u_2}\left\{J^1(t_0,x_0)+\alpha_1 J^2(t_0,x_0)\right\}\equiv$$
$$\max_{u_1,u_2} E_{t_0}\left\{\int_{t_0}^{T}\left(g^1[s,x(s),u_1(s),u_2(s)]+\alpha_1 g^2[s,x(s),u_1(s),u_2(s)]\right)\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds\right.$$
$$\left.+\left[q^1(x(T))+\alpha_1 q^2(x(T))\right]\exp\left[-\int_{t_0}^{T} r(y)dy\right]\middle|\, x(t_0)=x_0\right\}, \tag{1.3}$$

subject to dynamics (1.1). Note that the problem $\max_{u_1,u_2}\left\{J^1(t_0,x_0)+\alpha_i J^2(t_0,x_0)\right\}$ is identical to the problem $\max_{u_1,u_2}\left\{J^2(t_0,x_0)+\alpha_2 J^1(t_0,x_0)\right\}$ when $\alpha_1 = 1/\alpha_2$.

Invoking the technique developed by Fleming (1969) in Theorem A.3 of the Technical Appendices, we have

Corollary 1.1 A set of controls $\left\{u_i^{\alpha_1(t_0)}(t)=\psi_i^{\alpha_1(t_0)}(t,x), \text{ for } i\in\{1,2\}\right\}$ provides an optimal solution to the stochastic control problem (1.3), if there exists twice continuously differentiable function $W^{\alpha_1(t_0)}(t,x):[t_0,T]\times R^n\to R$ satisfying the partial differential equation:

$$-W_t^{\alpha_1(t_0)}(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{n}\Omega^{h\zeta}(t,x)W_{x^h x^\zeta}^{\alpha_1(t_0)}(t,x)=$$
$$\max_{u_1,u_2}\left\{\left(g^1[t,x,u_1,u_2]+\alpha_1 g^2[t,x,u_1,u_2]\right)\exp\left[-\int_{t_0}^{t} r(y)dy\right]\right.$$
$$\left.+W_x^{\alpha_1(t_0)}(t,x)f[t,x,u^1,u^2]\right\},$$
$$W^{\alpha_1(t_0)}(T,x)=\exp[-r(T-t_0)][q^1(x)+\alpha_1 q^2(x)].$$ ■

Substituting $\psi_1^{\alpha_1(t_0)}(t,x)$ and $\psi_2^{\alpha_1(t_0)}(t,x)$ into (1.1) yields the dynamics of the Pareto optimal trajectory associated with weight α_1:

$$dx(s)=f\left[s,x(s),\psi_1^{\alpha_1(t_0)}(s,x(s)),\psi_2^{\alpha_1(t_0)}(s,x(s))\right]ds+\sigma[s,x(s)]dz(s),\ x(t_0)=x_0. \tag{1.4}$$

We denote the set containing realizable values of $x^{\alpha_1 *}(t)$ by $X_t^{\alpha_1(t_0)}$, for $t\in(t_0,T]$.

The solution to (1.4) yields a Pareto optimal trajectory, which can be expressed as:

$$x(t)=x_0+\int_{t_0}^{t} f\left[s,x(s),\psi_1^{\alpha_1(t_0)}(s,x(s)),\psi_2^{\alpha_1(t_0)}(s,x(s))\right]ds+\int_{t_0}^{t}\sigma[s,x(s)]\,dz(s).$$

We denote the set containing realizable values of $x(t)$ along the optimal trajectory by $X_t^{\alpha_1(t_0)}$, for $t\in(t_0,T]$.

Now, consider the cooperative game $\Gamma_c(x_\tau,T-\tau)$ with state dynamics (1.1) and payoff structure (1.2), which starts at time $\tau\in[t_0,T]$ with initial state $x_\tau\in X_\tau^{\alpha_1(t_0)}$.

We use $\psi_i^{\alpha_1(\tau)}(t,x)$ to denote the optimal control in $\Gamma_c(x_\tau, T-\tau)$, for $\tau \in [t_0, T]$ and $t \in [\tau, T]$. Using Definition 1.1 we can characterize the solution of the control problem $\max_{u_1,u_2} \{J^1(\tau, x_\tau) + \alpha_1 J^2(\tau, x_\tau)\}$ in $\Gamma_c(x_\tau, T-\tau)$, for $\tau \in [t_0, T]$ and $t \in [\tau, T]$. In particular, we use $\left[\psi_1^{\alpha_1(\tau)}(t,x), \psi_2^{\alpha_1(\tau)}(t,x)\right]$ to denote the optimal control and $W^{\alpha_1(\tau)}(t,x) : [\tau, T] \times R^n \to R$ the corresponding maximized value function.

Remark 1.1 Invoking Definition 1.1, one can readily show that $\psi_i^{\alpha_1(\tau)}(t,x) = \psi_i^{\alpha_1(s)}(t,x)$ at the point (t,x), for $i \in \{1,2\}$, $t_0 \le \tau \le s \le t \le T$ and $x \in X_t^{\alpha_1(t_0)}$. ■

Remark 1.2 Invoking Definition 1.1, one can readily show that $W^{\alpha_1(\tau)}(t,x) = W^{\alpha_1(s)}(t,x)\exp[-r(\tau - s)]$, for $t_0 \le \tau \le s \le t \le T$ and $x \in X_t^{\alpha_1(t_0)}$. ■

6.1.2 Individual Player's Payoffs Under Cooperation

In this section, we present a methodology for the derivation of individual player's payoff under cooperation. To do this, we first substitute the optimal controls $\psi_1^{\alpha_1(t_0)}(t,x)$ and $\psi_2^{\alpha_1(t_0)}(t,x)$ into the objective functions (1.2) to derive the players' expected payoff under cooperation with α_1 being chosen as the cooperative weight.

Given that $x(t) = x \in X_t^{\alpha_1 *}$, for $t \in [\tau, T]$, we define player 1's expected cooperative payoff over the interval $[t, T]$ as:

$$\begin{aligned}
&\hat{W}^{\alpha_1(t_0)i}(t,x) = \\
&E_{t_0}\left\{ \int_t^T g^i\left[s, x(s), \psi_1^{\alpha_1(t_0)}(s,x(s)), \psi_2^{\alpha_1(t_0)}(s,x(s))\right] \exp\left[-\int_{t_0}^s r(y)dy\right] ds \right. \\
&\left. + \exp\left[-\int_{t_0}^T r(y)dy\right] q^i(x(T)) \middle| \, x(t) = x \right\}, \quad \text{for } i \in \{1,2\}, \qquad (1.5)
\end{aligned}$$

where

$$dx(s) = f\left[s, x(s), \psi_1^{\alpha_1(t_0)}(s,x(s)), \psi_2^{\alpha_1(t_0)}(s,x(s))\right] ds + \sigma[s, x(s)]\, dz(s), \qquad x(t) = x.$$

To facilitate the derivation individual players' cooperative payoffs a mechanism characterizing player i's cooperative payoff under payoff weights α_1 is given in the theorem below.

Theorem 1.1 If there exist continuously functions $\hat{W}^{\alpha_1(t_0)i}(t,x) : [t_0,T] \times R^n \to R,\ \ i \in \{1,2\}$, satisfying

$$-\hat{W}_t^{\alpha_1(t_0)i}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{n} \Omega^{h\zeta}(t,x)\hat{W}_{x^h x^\zeta}^{\alpha_1(t_0)i}(t,x) = $$
$$g^i\left[t,x,\psi_1^{\alpha_1(t_0)}(t,x),\psi_2^{\alpha_1(t_0)}(t,x)\right]\exp\left[-\int_{t_0}^{t} r(y)dy\right]$$
$$+\hat{W}_x^{\alpha_1(t_0)i}(t,x)f\left[t,x,\psi_1^{\alpha_1(t_0)}(t,x),\psi_2^{\alpha_1(t_0)}(t,x)\right] \text{ and}$$
$$\hat{W}^{\alpha_1(t_0)i}(T,x) = \exp\left[-\int_{t_0}^{T} r(y)dy\right]q^i(x)$$

then $\hat{W}^{\alpha_1(t_0)i}(t,x)$ gives player i's expected cooperative payoff over the interval $[t,T]$ with α_1 being chosen as the weight.

Proof Note that for $\Delta t \to 0$, we can express $\hat{W}^{\alpha_1(t_0)i}(t,x)$ in (1.5) as:

$$\begin{aligned}
&\hat{W}^{\alpha_1(t_0)i}(t,x) = \\
&E_{t_0}\left\{ \int_t^{t+\Delta t} g^i\left[s,x(s),\psi_1^{\alpha_1(t_0)}(s,x(s)),\psi_2^{\alpha_1(t_0)}(s,x(s))\right]\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds \right.\\
&\qquad \left. + \hat{W}^{\alpha_1(t_0)i}(t+\Delta t, x+\Delta x) \,\middle|\, x(t) = x \right\}\\
&= E_{t_0}\left\{ g^i\left[t,x,\psi_1^{\alpha_1(t_0)}(t,x),\psi_2^{\alpha_1(t_0)}(t,x)\right]\exp\left[-\int_{t_0}^{t} r(y)dy\right]\Delta t \right.\\
&\qquad + \hat{W}^{\alpha_1(t_0)i}(t,x) + \hat{W}_t^{\alpha_1(t_0)i}(t,x)\Delta t\\
&\qquad + \hat{W}_x^{\alpha_1(t_0)i}(t,x)f\left[t,x,\psi_1^{\alpha_1(t_0)}(t,x),\psi_2^{\alpha_1(t_0)}(t,x)\right]\Delta t\\
&\qquad \left. + \hat{W}_x^{\alpha_1(t_0)i}(t,x)\sigma(t,x)\Delta z + \frac{1}{2}\sum_{h,\zeta=1}^{n}\Omega^{h\zeta}(t,x)\hat{W}_{x^h x^\zeta}^{\alpha_1(t_0)i}(t,x) + o(\Delta t) \right\}\\
&\text{for } i \in \{1,2\},
\end{aligned} \tag{1.6}$$

where

$$\Delta x = f\left[t,x,\psi_1^{\alpha_1(t_0)}(t,x),\psi_2^{\alpha_1(t_0)}(t,x)\right]\Delta t + \sigma(t,x)\Delta z + o(\Delta t),$$
$$\Delta z = z(t+\Delta t) - z(t), \text{ and } E_{t_0}[o(\Delta t)]/\Delta t \to 0 \text{ as } \Delta t \to 0$$

Canceling terms, performing the expectation operator, dividing throughout by Δt and taking $\Delta t \to 0$, we obtain:

$$-\hat{W}_t^{\alpha_1(t_0)i}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{n} \Omega^{h\zeta}(t,x)\hat{W}_{x^h x^\zeta}^{\alpha_1(t_0)i}(t,x) =$$
$$g^i\left[t,x,\psi_1^{\alpha_1(t_0)}(t,x),\psi_2^{\alpha_1(t_0)}(t,x)\right]\exp\left[-\int_{t_0}^{t} r(y)dy\right]$$
$$+\hat{W}_x^{\alpha_1(t_0)i}(t,x)f\left[t,x,\psi_1^{\alpha_1(t_0)}(t,x),\psi_2^{\alpha_1(t_0)}(t,x)\right], \text{ for } i \in \{1,2\}. \tag{1.7}$$

Boundary conditions require:

$$\hat{W}^{\alpha_1(t_0)i}(T,x) = \exp\left[-\int_{t_0}^{T} r(y)dy\right]q^i(x(T)), \quad \text{for } i \in \{1,2\}. \tag{1.8}$$

Hence Theorem 1.1 follows. ■

6.2 Notion of Subgame Consistency

Under cooperation with nontransferable payoffs, the players negotiate to establish an agreement (optimality principle) on how to play the cooperative game and how to distribute the resulting payoff. In particular, the chosen optimality principle has to satisfy group optimality and individual rationality. Subgame consistency requires that the extension of the solution policy to a later starting time and any possible state brought about by prior optimal behavior of the players would remain optimal.

Consider the cooperative game $\Gamma_c(x_0, T-t_0)$ in which the players agree to an optimality principle. In particular, given x_0 at time t_0, according to the solution optimality principle the players will adopt

(i) a weight α_1^0 leading to a set of cooperative controls $\{\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right]$, for $t \in [t_0, T]\}$, and

(ii) an imputation $\left[\xi^{(t_0)1}\left(x_0, T-t_0; \alpha_1^0\right), \xi^{(t_0)2}\left(x_0, T-t_0; \alpha_1^0\right)\right] = \left[\hat{W}^{t_0\left(\alpha_1^0\right)1}(t_0, x_0), \hat{W}^{t_0\left(\alpha_1^0\right)2}(t_0, x_0)\right]$.

Now consider the game $\Gamma_c(x_\tau, T-\tau)$ where $x_\tau \in X_\tau^{\alpha_1(t_0)}$ and $\tau \in [t_0, T]$, under the same solution optimality principle the players will adopt

(i) a weight α_1^τ leading to a set of cooperative controls $\{\left[\psi_1^{\alpha_1^\tau(\tau)}(t,x), \psi_2^{\alpha_1^\tau(\tau)}(t,x)\right]$, for $t \in [\tau, T]\}$, and

(ii) an imputation $\left[\xi^{(\tau)1}\left(\tau, T-\tau; \alpha_1^\tau\right), \xi^{(\tau)2}\left(\tau, T-\tau; \alpha_1^\tau\right)\right] = \left[\hat{W}^{\tau\left(\alpha_1^\tau\right)1}(\tau, x_\tau), \hat{W}^{\tau\left(\alpha_1^\tau\right)2}(\tau, x_\tau)\right]$.

A formal definition of subgame consistency can be stated as:

Definition 2.1 An optimality principle yielding imputations $\xi^{(\tau)}\left(x_\tau, T-\tau; \alpha_1^\tau\right)$, for $\tau \in [t_0, T]$ and $x_\tau \in X_\tau^{\alpha_1^0(t_0)}$, constitutes a subgame consistent solution to the game $\Gamma_c\left(x_0, T-t_0; \alpha_1^0\right)$ if the following conditions are satisfied:

(i) $\xi^{(\tau)}\left(x_\tau, T-\tau; \alpha_1^\tau\right) = \left[\xi^{(\tau)1}\left(x_\tau, T-\tau; \alpha_1^\tau\right), \xi^{(\tau)2}\left(x_\tau, T-\tau; \alpha_1^\tau\right)\right]$, for $t_0 \le \tau \le t \le T$, is Pareto optimal;

(ii) $\xi^{(\tau)i}\left(x_\tau, T-\tau; \alpha_1^\tau\right) \ge V^{(\tau)i}(\tau, x_\tau)$, for $i \in \{1,2\}$, $\tau \in [t_0, T]$ and $x_\tau \in X_\tau^{\alpha_1^0(t_0)}$; and

(iii)
$$\xi^{(\tau)i}\left(x_t, T-t; \alpha_1^\tau\right) \exp[r(\tau - t)] = \xi^{(t)i}\left(x_t, T-t; \alpha_1^t\right),$$
for $i \in \{1,2\}$, $t_0 \le \tau \le t \le T$ and $x_t \in X_t^{\alpha_1^0(t_0)}$. ■

Part (i) of Definition 4.1 requires that according to the agreed upon optimality principle Pareto optimality is maintained at every instant of time. Hence group rationality is satisfied throughout the game interval. Part (ii) demands individual rationality to be met throughout the entire game interval. Part (iii) guarantees the consistency of the solution imputations throughout the game interval in the sense that the extension of the solution policy to a situation with a later starting time and any possible state brought about by prior optimal behavior of the players remains optimal.

6.3 A NTU Game for Illustration

Consider a two-person nonzero-sum stochastic differential game with initial state x_0 and duration $T - t_0$. The state space of the game is $X \subset R$, with permissible state trajectories $\{x(s), t_0 \le s \le T\}$. The state dynamics of the game is characterized by the stochastic differential equations:

$$dx(s) = [a - bx(s) - u_1(s) - u_2(s)]ds + \sigma x(s) dz(s), \ \ x(t_0) = x_0 \in X, \tag{3.1}$$

where $u_i \in U_i$ is the control vector of player i, for $i \in [1,2]$, a, b and σ are positive constants, and $z(s)$ is a Wiener process. Equation (3.1) could be interpreted as the stock dynamics of a biomass of renewable resource like forest or fresh water. The state $x(s)$ represents the resource size and $u_i(s)$ the (nonnegative) amount of resource extracted by player i.

At time t_0, the expected payoff of player $i \in \{1,2\}$ is:

$$J^i(t_0, x_0) = E_{t_0}\left\{ \int_{t_0}^T \left[h_i u_i(s) - c_i u_i(s)^2 x(s)^{-1} + k_i x(s)\right] \exp[-r(s - t_0)]\, ds + \exp[-r(T - t_0)]\, q_i x(T) \,\middle|\, x(t_0) = x_0 \right\},$$
$$\text{for } i \in \{1,2\}, \tag{3.2}$$

where h_i, c_i, k_i and q_i are positive parameters.

The term $h_i u_i(s)$ reflects player i's satisfaction level obtained from the consumption of the resource extracted, and $c_i u_i(s)^2 x(s)^{-1}$ measures the cost created in the extraction process. $k_i x(s)$ is the benefit to player i related to the existing level of the resource. Total utility of player i is the aggregate level of satisfaction. Payoffs in the form of utility are not transferable between players. There exists a time discount rate r, and utility received at time t has to be discounted by the factor $\exp[-r(t-t_0)]$. At time T, player i will receive a terminal benefit $q_i x(T)^{1/2}$, where q_i is nonnegative.

6.3.1 Noncooperative Outcome and Pareto Optimal Trajectories

We use $\Gamma(x_0, T-t_0)$ to denote the game (3.1 and 3.2) and $\Gamma(x_\tau, T-\tau)$ to denote an alternative game with state dynamics (3.1) and payoff structure (3.2), which starts at time $\tau \in [t_0, T]$ with initial state $x_\tau \in X$. Invoking the techniques of Isaacs (1965), Bellman (1957) and Fleming (1969) as stated in Theorem 1.1 of Chap. 3 a non-cooperative Nash equilibrium solution of the game $\Gamma(x_\tau, T-\tau)$ can be characterized as follows.

Corollary 3.1 A set of feedback strategies $\left\{ u_i^{(\tau)*}(t) = \phi_i^{(\tau)*}(t,x), \text{ for } i \in \{1,2\} \right\}$ provides a Nash equilibrium solution to the game $\Gamma(x_\tau, T-\tau)$, if there exist twice continuously differentiable functions $V^{(\tau)i}(t,x) : [\tau, T] \times R \to R$, $i \in \{1,2\}$, satisfying the following partial differential equations:

$$-V_t^{(\tau)i}(t,x) - \frac{1}{2}\sigma^2 x^2 V_{xx}^{(\tau)i}(t,x)$$
$$= \max_{u_i} \left\{ \left[h_i u_i - c_i u_i^2 x^{-1} + k_i x\right] \exp\left[-r(t-\tau)\right] + V_x^{(\tau)i}(t,x)\left[a - bx - u_i - u_j\right] \right\}, \text{ and}$$
$$V^{(\tau)i}(T,x) = \exp[-r(T-\tau)]\, q_i x, \quad \text{for } i \in \{1,2\}, \ \ j \in \{1,2\} \text{ and } j \neq i. \tag{3.3}$$

■

Performing the indicated maximization in Corollary 3.2 yields:

$$\phi_i^{(\tau)*}(t,x) = \frac{\left[h_i - V_x^{(\tau)i} \exp(r(t-\tau))\right] x}{2c_i}, \text{ for } i \in \{1,2\} \text{ and } x \in X. \tag{3.4}$$

The feedback Nash equilibrium payoffs of the players in the game $\Gamma(x_\tau, T-\tau)$ can be obtained as:

Proposition 3.1 The value function representing the feedback Nash equilibrium payoff of player i in the game $\Gamma(x_\tau, T-\tau)$ is:

$$V^{(\tau)i}(t,x) = \exp[-r(t-\tau)][A_i(t)x + B_i(t)], \quad \text{for } i \in \{1,2\} \text{ and } t \in [\tau, T], \quad (3.5)$$

where $A_i(t), B_i(t), A_j(t)$ and $B_j(t)$, for $i \in \{1,2\}$ and $j \in \{1,2\}$ and $i \neq j$, satisfy:

$$\dot{A}_i(t) = (r+b)A_i(t) - k_i - \frac{[h_i - A_i(t)]^2}{4c_i} + \frac{A_i(t)\left[h_j - A_j(t)\right]}{2c_j},$$
$$\dot{B}_i(t) = rB_i(t) - aA_i(t),$$
$$A_i(T) = q_i, \; B_i(T) = 0.$$

Proof Upon substitution of $\phi_i^{(\tau)*}(t,x)$ from (3.4) into (3.3) yields a set of partial differential equations. One can readily verify that (3.5) is a solution to this set of equations. ■

Consider the case where the players agree to cooperate in order to enhance their payoffs. Let $\Gamma_c(x_0, T-t_0)$ denote a cooperative game with payoff structure (3.1) and dynamics (3.2) starting at time t_0 with initial state x_0. If the players agree to adopt a weight $\alpha_1 > 0$, Pareto optimal trajectories for $\Gamma_c(x_0, T-t_0)$ can be identified by solving the following stochastic control problem:

$$\begin{aligned}
&\max_{u_1,u_2} \left\{J^1(t_0,x_0) + \alpha_1 J^2(t_0,x_0)\right\} \\
&\equiv \max_{u_1,u_2} E_{t_0}\Bigg\{ \int_{t_0}^{T} \Bigg(\left[h_1u_1(s) - c_1u_1(s)^2x(s)^{-1} + k_1x(s)\right] \\
&\quad + \alpha_1\left[h_2u_2(s) - c_2u_2(s)^2x(s)^{-1} + k_2x(s)\right] \Bigg)\exp[-r(s-t_0)]\,ds \\
&\quad \exp[-r(T-t_0)]\,[q_1x(T) + q_2x(T)]\Bigg|\, x(t_0) = x_0 \Bigg\},
\end{aligned} \quad (3.6)$$

subject to dynamics (3.1). Note that when $\alpha_1 = 1/\alpha_2$, the problem $\max_{u_1,u_2}\left\{J^1(t_0,x_0) + \alpha_1 J^2(t_0,x_0)\right\}$ is identical to the problem $\max_{u_1,u_2}\left\{J^2(t_0,x_0) + \alpha_2 J^1 (t_0,x_0)\right\}$ in the sense that $\max_{u_1,u_2}\left\{J^2(t_0,x_0) + \alpha_2 J^1(t_0,x_0)\right\} \equiv \max_{u_1,u_2}\left\{\alpha_2\left[J^1(t_0,x_0) + \alpha_1 J^2(t_0,x_0)\right]\right\}$ yields the same optimal controls as those from $\max_{u_1,u_2}\left\{J^1(t_0,x_0) + \alpha_1 J^2(t_0,x_0)\right\}$.

In $\Gamma_c(x_0, T-t_0)$, let α_1^0 be the selected weight according the agreed upon optimality principle. Invoking Corollary 1.1 in Sect. 6.1 the optimal solution of the stochastic control problem (3.1) and (3.6) can be characterized as:

Corollary 3.2 A set of controls $\left\{\left[\psi_1^{\alpha_1^0(t_0)}(t,x), \psi_2^{\alpha_1^0(t_0)}(t,x)\right], \text{for } t \in [t_0, T]\right\}$ provides an optimal solution to the stochastic control problem $\max_{u_1,u_2}\left\{J^1(t_0,x_0) + \alpha_1^0 J^2(t_0,x_0)\right\}$, if there exists twice continuously differentiable function $W^{\alpha_1^0(t_0)}(t,x) : [t_0, T] \times R \to R$ satisfying the partial differential equation:

$$-W_t^{\alpha_1^0(t_0)}(t,x)-\frac{1}{2}\sigma^2x^2W_{xx}^{\alpha_1^0(t_0)}(t,x)=$$
$$\max_{u_1,u_2}\Big\{\left(\left[h_1u_1-c_1u_1^2x^{-1}+k_1x\right]+\alpha_1^0\left[h_2u_2-c_2u_2^2x^{-1}+k_2x\right]\right)\exp[-r(t-t_0)]$$
$$+W_x^{\alpha_1^0(t_0)}(t,x)\left[a-bx-u_i-u_j\right]\Big\},$$
$$W^{\alpha_1^0(t_0)}(T,x)=\exp[-r(T-t_0)]\left[q_1x(T)+\alpha_1^0q_2x(T)\right] \qquad (3.7)\blacksquare$$

Performing the indicated maximization in Corollary 3.2 yields:

$$\psi_1^{\alpha_1^0(t_0)}(t,x)=\frac{\left[h_1-W_x^{\alpha_1^0(t_0)}(t,x)\exp(r(t-t_0))\right]x}{2c_1}, \text{ and}$$
$$\psi_2^{\alpha_1^0(t_0)}(t,x)=\frac{\left[\alpha_1^0h_2-W_x^{\alpha_1^0(t_0)}(t,x)\exp(r(t-t_0))\right]x}{2\alpha_1^0c_2}, \quad \text{for } t\in[t_0,T]. \qquad (3.8)$$

The maximized value function $W^{\alpha_1^0(t_0)}(t,x)$ of the control problem $\max_{u_1,u_2}\left\{J^1(t_0,x_0)+\alpha_1^0J^2(t_0,x_0)\right\}$ can be obtained as:

Proposition 3.2

$$W^{\alpha_1^0(t_0)}(t,x)=\exp[-r(t-t_0)]\left[A^{\alpha_1^0}(t)x+B^{\alpha_1^0}(t)\right], \quad \text{for } t\in[t_0,T], \qquad (3.9)$$

where $A^{\alpha_1^0}(t)$ and $B^{\alpha_1^0}(t)$ satisfy:

$$\dot{A}^{\alpha_1^0}(t)=(r+b)A^{\alpha_1^0}(t)-\frac{\left[h_1-A^{\alpha_1^0}(t)\right]^2}{4c_1}-\frac{\left[\alpha_1^0h_2-A^{\alpha_1^0}(t)\right]^2}{4\alpha_1^0c_2}-k_1-k_2,$$
$$\dot{B}^{\alpha_1^0}(t)=rB^{\alpha_1^0}(t)-A^{\alpha_1^0}(t)\,a,$$
$$A^{\alpha_1^0}(T)=q_1+\alpha_1^0q_2 \text{ and } B^{\alpha_1^0}(T)=0. \qquad (3.10)$$

Proof Upon substitution of $\psi_1^{\alpha_1^0(t_0)}(t,x)$ and $\psi_2^{\alpha_1^0(t_0)}(t,x)$ from (3.10) into (3.7) yields a partial differential equation. One can readily verify that (3.9) is a solution to this set of equations. ■

Substituting the partial derivatives $W_x^{\alpha_1^0(t_0)}(t,x)$ into $\psi_1^{\alpha_1^0(t_0)}(t,x)$ and $\psi_2^{\alpha_1^0(t_0)}(t,x)$ in (3.9) yields the optimal controls of the problem $\max_{u_1,u_2}\left\{J^1(t_0,x_0)+\alpha_1^0J^2(t_0,x_0)\right\}$ as:

$$\psi_1^{\alpha_1^0(t_0)}(t,x)=\frac{\left[h_1-A^{\alpha_1^0}(t)\right]x}{2c_1}, \text{ and}$$
$$\psi_2^{\alpha_1^0(t_0)}(t,x)=\frac{\left[\alpha_1^0h_2-A^{\alpha_1^0}(t)\right]x}{2\alpha_1^0c_2}, \quad \text{for } t\in[t_0,T]. \qquad (3.11)$$

Substituting these controls into (3.1) yields the dynamics of the Pareto optimal trajectory associated with a weight α_1^0. The Pareto optimal trajectory then can be solved as:

$$x^{\alpha_1^0(t_0)}(t) = \left\{\Phi\left(\alpha_1^0; t, t_0\right)\left[x_0 + \int_{t_0}^{t} \Phi^{-1}\left(\alpha_1^0; s, t_0\right) a ds\right]\right\}^2, \tag{3.12}$$

where

$$\Phi\left(\alpha_1^0; t, t_0\right) = \exp\left[\int_{t_0}^{t}\left(-b - \frac{h_1 - A^{\alpha_1^0}(s)}{2c_1} - \frac{\alpha_1 h_2 - A^{\alpha_1^0}(s)}{2\alpha_1^0 c_2} - \frac{\sigma^2}{2}\right) ds + \int_{t_0}^{t} \sigma \; dz(s)\right].$$

We use $X_t^{\alpha_1^0(t_0)}$ to denote the set of realizable values of $x^{\alpha_1^0(t_0)}(t)$ generated by (3.12) at $t \in \left(t_0, T\right]$.

Now, consider the cooperative game $\Gamma_c(x_\tau, T-\tau)$ with state dynamics (3.1) and payoff structure (3.2), which starts at time $\tau \in [t_0, T]$ with initial state $x_\tau \in X_\tau^{\alpha_1(t_0)}$. Let α_1^τ be the selected weight according the agreed upon optimality principle.

Following previous analysis, we can obtain the maximized value function, optimal controls and optimal trajectory of the control problem $\max_{u_1, u_2}\left\{J^1(\tau, x_\tau) + \alpha_1^\tau J^2(\tau, x_\tau)\right\}$.

Remark 3.1 One can readily show that when $\alpha_1^0 = \alpha_1^\tau = \alpha_1^*$, then $\psi_i^{\alpha_1^*(t_0)}(t, x_t) = \psi_i^{\alpha_1^*(\tau)}(t, x_t)$ at the point (t, x_t), for $i \in [1, 2]$, $t_0 \leq \tau \leq t \leq T$ and $x_t \in X_t^{\alpha_1^*(t_0)}$. ■

6.3.2 Individual Player's Payoff Under Cooperation

In order to verify individual rationality, we have to derive the players' expected payoffs in the cooperative game $\Gamma_c\left(x_\tau^*, T-\tau\right)$. Let α_1^τ be the weight dictated by the solution optimality principle. We substitute

$$\psi_1^{\alpha_1^\tau(\tau)}(t, x) = \frac{\left[h_1 - A^{\alpha_1^\tau}(t)\right]x}{2c_1} \text{ and } \psi_2^{\alpha_1^\tau(\tau)}(t, x) = \frac{\left[\alpha_1^\tau h_2 - A^{\alpha_1^\tau}(t)\right]x}{2\alpha_1^\tau c_2}$$

into the players' payoffs and define the following functions.

Definition 3.1 Given that $x(t) = x_t^{\alpha_1^\tau(\tau)} \in X_t^{\alpha_1^\tau(\tau)}$, for $t \in [\tau, T]$, player 1's expected payoff over the interval $[t, T]$ under the control problem $\max_{u_1, u_2}\left\{J^1(\tau, x_\tau) + \alpha_1^\tau J^2(\tau, x_\tau)\right\}$ as:

$$\hat{W}^{\tau(\alpha_1^\tau)1}(t,x) =$$
$$E_\tau\left\{ \int_t^T \left[\frac{h_1\left[h_1 - A^{\alpha_1^\tau}(s)\right]x(s)}{2c_1} - \frac{\left[h_1 - A^{\alpha_1^\tau}(s)\right]^2 x(s)}{4c_1} + k_1 x(s)\right] \exp[-r(s-\tau)]\, ds\right.$$
$$\left. + \exp[-r(T-t_0)]\, q_1 x(T) \right| x(t) = x \Bigg\},$$

and the corresponding expected payoff of player 2 over the interval $[t,T]$ as:

$$\hat{W}^{\tau(\alpha_1^\tau)2}(t,x) = E_\tau\Bigg\{$$
$$\int_t^T \left[\frac{h_2\left[\alpha_1^\tau h_2 - A^{\alpha_1^\tau}(s)\right]x(s)}{2\alpha_1^\tau c_2} - \frac{\left[\alpha_1^\tau h_2 - A^{\alpha_1^\tau}(s)\right]^2 x(s)}{4\left(\alpha_1^\tau\right)^2 c_2} + k_i x(s)\right] \exp[-r(s-\tau)]\, ds$$
$$\left. + \exp[-r(T-\tau)]\, q_2 x(T) \right| x(t) = x \Bigg\},$$

where

$$dx(s) = \left[a - bx(s) - \frac{\left[h_1 - A^{\alpha_1^\tau}(s)\right]x(s)}{2c_1} - \frac{\left[\alpha_1^\tau h_2 - A^{\alpha_1^\tau}(s)\right]x(s)}{2\alpha_1^\tau c_2} \right] ds$$
$$+ \sigma x(s) dz(s), x(t) = x. \qquad \blacksquare$$

Invoking Theorem 1.1 in Sect. 6.1, player 1's expected payoff $\hat{W}^{\tau(\alpha_1^\tau)1}(t,x_\tau)$ can be characterized as:

$$-\hat{W}_t^{\tau(\alpha_1^\tau)1}(t,x_t) - \frac{1}{2}\hat{W}_{x_t x_t}^{\tau(\alpha_1^\tau)1}(t,x_t)\sigma^2 x_t^2 =$$
$$\left[\frac{h_1\left[h_1 - A^{\alpha_1^\tau}(t)\right]x_t}{2c_1} - \frac{c_1\left[h_1 - A^{\alpha_1^\tau}(t)\right]^2 x_t}{4c_1^2} + k_1 x_t\right] \exp[-r(t-\tau)]$$
$$+ \hat{W}_{x_t}^{\tau(\alpha_1^\tau)1}(t,x_t)\left[a - bx_t - \frac{\left[h_1 - A^{\alpha_1^\tau}(t)\right]x_t}{2c_1} - \frac{\left[\alpha_1^\tau h_2 - A^{\alpha_1^\tau}(t)\right]x_t}{2\alpha_1^\tau c_2} \right]. \tag{3.13}$$

Boundary conditions require:

$$\hat{W}^{\tau(\alpha_1^\tau)1}(T,x) = \exp[-r(T-\tau)]q_1 x. \tag{3.14}$$

If there exist continuously differentiable functions $\hat{W}^{\tau(\alpha_1^\tau)1}(t,x) : [\tau,T] \times R \to R$ satisfying (3.13) and (3.14), then player 1's expected payoff in the cooperative game $\Gamma(x_\tau, T-\tau)$ under the cooperation scheme with weight α_1^τ is indeed $\hat{W}^{\tau(\alpha_1^\tau)1}(t,x)$. The value function $\hat{W}^{\tau(\alpha_1^\tau)1}(t,x)$ indicating the expected payoff of player 1 under cooperation can be obtained as:

Proposition 3.3 The function $\hat{W}^{\tau(\alpha_1^\tau)1}(t,x) : [t_0,T] \times R \to R$ satisfying (3.13) and (3.14) can be solved as:

$$\hat{W}^{\tau(\alpha_1^\tau)1}(t,x) = \exp[-r(t-\tau)]\left[\hat{A}_1^{\alpha_1^\tau}(t)x + \hat{B}_1^{\alpha_1^\tau}(t)\right], \tag{3.15}$$

where $\hat{A}_1^{\alpha_1^\tau}(t)$ and $\hat{B}_1^{\alpha_1^\tau}(t)$ satisfy:

$$\dot{\hat{A}}_1^{\alpha_1^\tau}(t) = \left[r + b + \frac{\left[h_1 - A^{\alpha_1^\tau}(t)\right]}{2c_1} + \frac{\left[\alpha_1 h_2 - A^{\alpha_1^\tau}(t)\right]}{2\alpha_1 c_2}\right]\hat{A}_1^{\alpha_1}(t)$$
$$- \frac{\left[h_1 - A^{\alpha_1^\tau}(t)\right]\left[h_1 + A^{\alpha_1^\tau}(t)\right]}{4c_1} - k_1,$$

$$\dot{\hat{B}}_1^{\alpha_1^\tau}(t) = r\hat{B}_1^{\alpha_1^\tau}(t) - a\hat{A}_1^{\alpha_1^\tau}(t),\ \hat{A}_1^{\alpha_1^\tau}(T) = q_1 \text{ and } \hat{B}_1^{\alpha_1^\tau}(T) = 0.$$

Proof Upon calculating the derivatives $\hat{W}_t^{\tau(\alpha_1^\tau)1}(t,x)$, $\hat{W}_{xx}^{\tau(\alpha_1^\tau)1}(t,x)$, and $\hat{W}_x^{\tau(\alpha_1^\tau)1}(t,x)$ from (3.15) and then substituting them into (3.13) yield Proposition 3.3. ■

Following the above analysis, a continuously differentiable function $\hat{W}^{\tau(\alpha_1^\tau)2}(t,x) : [\tau,T] \times R \to R$ giving the player 2's expected payoff under cooperation can be obtained as:

Proposition 3.4

$$\hat{W}^{\alpha_1^\tau(\tau)2}(t,x) = \exp[-r(t-\tau)]\left[\hat{A}_2^{\alpha_1^\tau}(t)x + \hat{B}_2^{\alpha_1^\tau}(t)\right], \tag{3.16}$$

where $\hat{A}_2^{\alpha_1^\tau}(t)$ and $\hat{B}_2^{\alpha_1^\tau}(t)$ has to satisfy:

$$\dot{\hat{A}}_2^{\alpha_1^\tau}(t) = \left[r + b + \frac{\left[h_1 - A^{\alpha_1^\tau}(t)\right]}{2c_1} + \frac{\left[\alpha_1 h_2 - A^{\alpha_1^\tau}(t)\right]}{2\alpha_1 c_2}\right]\hat{A}_2^{\alpha_1^\tau}(t)$$
$$- \frac{\left[\alpha_1 h_2 - A^{\alpha_1^\tau}(t)\right]\left[\alpha_1 h_2 + A^{\alpha_1^\tau}(t)\right]}{4\alpha_1^2 c_2} - k_2,$$
$$\dot{\hat{B}}_2^{\alpha_1^\tau}(t) = r\hat{B}_2^{\alpha_1^\tau}(t) - a\hat{A}_2^{\alpha_1^\tau}(t),\ \hat{A}_2^{\alpha_1^\tau}(T) = q_2 \text{ and } \hat{B}_2^{\alpha_1^\tau}(T) = 0.$$

Proof Follow the proof of Proposition 3.3. ■

6.4 Subgame Consistent Cooperative Solutions of the Game

In this section, we present subgame consistent solutions to the cooperative game $\Gamma_c(x_0, T - t_0)$. First note that group optimality will be maintained only if the solution optimality principle selects the same weight α_1 for all games

$\Gamma_c(x_\tau, T-\tau)$, $\tau \in [t_0, T]$ and $x_\tau \in X_\tau^{\alpha_1(t_0)}$. For any chosen α_1 to maintain individual rationality throughout the game interval, the following condition must be satisfied.

$$\xi^{(\tau)i}(x_\tau, T-\tau; \alpha_1) = \hat{W}^{\tau(\alpha_1)i}(\tau, x_\tau) \geq V^{(\tau)i}(\tau, x_\tau),$$
$$\text{for } i \in \{1,2\},\ \tau \in [t_0, T] \text{ and } x_\tau \in X_\tau^{\alpha_1(t_0)}. \tag{4.1}$$

Definition 4.1 We define the set $S_\tau^T = \bigcap_{\tau \leq t < T} S_t$, for $\tau \in [t_0, T)$. ■

S_t represents the set of α_1 satisfying individual rationality at time $t \in [t_0, T)$ and S_τ^T represents the set of α_1 satisfying individual rationality throughout the interval $[\tau, T)$. In general $S_\tau^T \neq S_t^T$ for $\tau, t \in [t_0, T)$ where $\tau \neq t$.

6.4.1 Typical Configurations of $\mathbf{S_t}$

To find out typical configurations of the set S_t for $t \in [t_0, T)$ of the game $\Gamma_c(x_0, T-t_0)$, we perform extensive numerical simulations with a wide range of parameter specifications for $a, b, \sigma, h_1, h_2, k_1, k_2, c_1, c_2, q_1, q_2, T, r, x_0$. We calculate the time paths of $A_1(t), B_1(t), A_2(t)$ and $B_2(t)$ in Proposition 3.1 for $t \in [t_0, T]$. Then we select weights α_1 and calculate the time paths of $\hat{A}_1^{\alpha_1}(t), \hat{A}_2^{\alpha_1}(t), \hat{B}_1^{\alpha_1}(t)$ and $\hat{B}_2^{\alpha_1}(t)$ in Propositions 3.3 and 3.4, for $t \in [t_0, T]$. At each time instant $t \in [t_0, T]$, we derive the set of α_1 that yields $\hat{A}_i^{\alpha_1}(t) \geq A_i(t)$ and $\hat{B}_i^{\alpha_1}(t) \geq B_i(t)$, for $i \in [1, 2]$, to derive the set S_t, for $t \in [t_0, T)$.

We denote the locus of the values of $\underline{\alpha}_1^t$ along $t \in [t_0, T)$ as curve $\underline{\alpha}_1$ and the locus of the values of $\overline{\alpha}_1^t$ as curve $\overline{\alpha}_1$. In particular, typical patterns include:

(i) The curves $\underline{\alpha}_1$ and $\overline{\alpha}_1$ are continuous and move in the same direction over the entire game duration: either both increase monotonically or both decrease monotonically (see Fig. 6.1).
(ii) The curves $\underline{\alpha}_1$ and $\overline{\alpha}_1$ are continuous. $\underline{\alpha}_1$ declines and $\overline{\alpha}_1$ rises over the entire game duration (see Fig. 6.2).
(iii) The curves $\underline{\alpha}_1$ and $\overline{\alpha}_1$ are continuous. One of these curves would rise/fall to a peak/trough and then fall/rise (see Fig. 6.3).
(iv) The set $S_{t_0}^T$ can be nonempty or empty.

6.4.2 Examples of Subgame Consistent Solutions

In this subsection, we present some subgame consistent solutions to $\Gamma_c(x_0, T-t_0)$.

Solution 4.1 Consider the cooperative differential game $\Gamma_c(x_0, T-t_0)$ with parameters leading to a set of payoff weights as in Panel (b) of Fig. 6.1. In

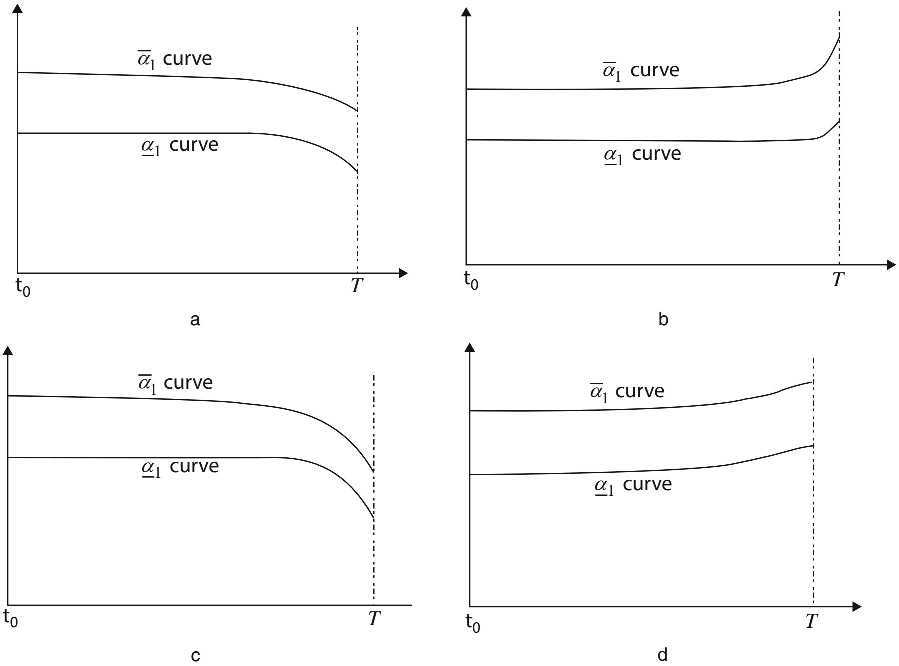

Fig. 6.1 Both upward $\underline{\alpha}_1$ and $\overline{\alpha}_1$ curves and both downward $\underline{\alpha}_1$ and $\overline{\alpha}_1$ curves

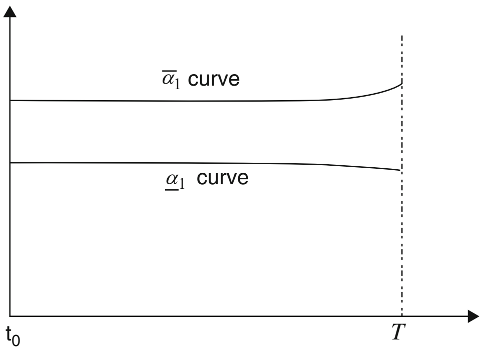

Fig. 6.2 Declining $\underline{\alpha}_1$ curve and rising $\overline{\alpha}_1$ curve

particular, there exist a set of weights $S_{t_0}^T \neq \varnothing$ under which individual rationality is satisfied throughout the game horizon $[0, T]$ and $\underline{\alpha}_1^{T-} \in S_{t_0}^T$. At initial time 0, in the cooperative $\Gamma_c(x_0, T - t_0)$, an optimality principle under which the players to choose the weight

$$\alpha_1^* = \underline{\alpha}_1^{T-}, \text{ in } \Gamma_c(x_\tau, T - \tau) \text{ for } \tau \in \left[t_0, T\right)$$

yields a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$.

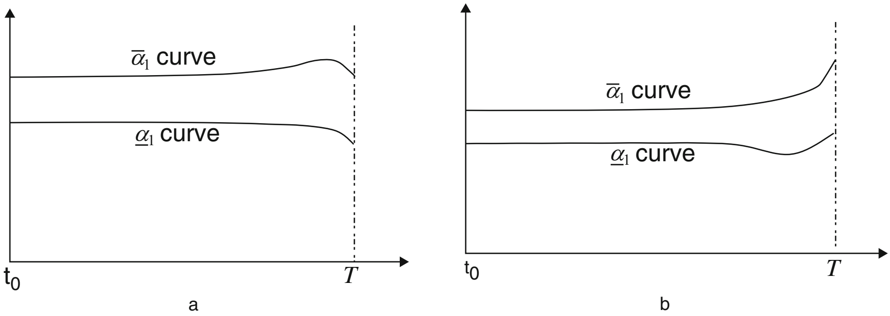

Fig. 6.3 Rising to a peak and then fall curve and falling to a trough and then rise curve

Proof According to the optimality principle in Solution 4.1, a unique $\alpha_1^* = \underline{\alpha}_1^{T-}$ will be chosen for all the subgames $\Gamma_c(x_\tau, T-\tau)$, for $t_0 \le \tau \le t < T$ and $x_\tau \in X_\tau^{\alpha_1^*(t_0)}$. The vector $\xi^{(\tau)}\left(x_\tau, T-\tau; \alpha_1^*\right) = \left[\hat{W}^{\tau(\alpha_1^*)1}(\tau, x_\tau), \hat{W}^{\tau(\alpha_1^*)2}(\tau, x_\tau)\right]$, for $\tau \in [t_0, T]$, yields a Pareto optimal pair of imputations. Hence part (i) of Definition 2.1 is proved.

One can readily verify that $\hat{W}^{\tau(\alpha_1^*)i}(t,x)\exp[r(\tau-t)] = \hat{W}^{t(\alpha_1^*)i}(t,x)$, for $i \in \{1,2\}$, $t_0 \le \tau \le t \le T$ and $x_t \in X_t^{\alpha_1^0(t_0)}$. Hence part (ii) of Definition 2.1 is satisfied.

Finally, from Definitions 4.1, one can verify that $\hat{W}^{\tau(\alpha_1^*)i}(\tau, x_\tau) = \exp[-r(t-\tau)]$ $\left[\hat{A}_i^{\alpha_1^*}(t)x^{1/2} + \hat{B}_i^{\alpha_1^*}(t)\right] \ge V^{(\tau)i}(\tau, x_\tau) = \exp[-r\ (t-\tau)]\left[A_i(t)x^{1/2} + B_i(t)\right]$, for $i \in \{1,2\}$, $\tau \in [t_0, T]$ and $x_\tau \in X_\tau^{\alpha_1^*(t_0)}$. Hence part (iii) of Definition 2.1 is fulfilled. ■

Solution 4.2 Consider the cooperative differential game $\Gamma_c(x_0, T-t_0)$ with parameters leading to a set of payoff weights as in Panel (a) of Fig. 6.1. In particular, there exist a set of weights $S_{t_0}^T \ne \varnothing$ under which individual rationality is satisfied throughout the game horizon $[0,T]$ and $\overline{\alpha}_1^{T-} \in S_{t_0}^T$. At initial time 0, in the cooperative $\Gamma_c(x_0, T-t_0)$, an optimality principle under which the players to choose the weight

$$\alpha_1^* = \overline{\alpha}_1^{T-},\ \text{in}\ \Gamma_c(x_\tau, T-\tau)\ \text{for}\ \tau \in \left[t_0, T\right)$$

yields a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T-t_0)$.

Proof Follow the proof of Solution 4.1. ■

Solution 4.3 Consider the cooperative differential game $\Gamma_c(x_0, T-t_0)$ with parameters leading to a set of payoff weights as in Fig. 6.2. In particular, there exist a set of weights $S_{t_0}^T \ne \varnothing$ under which individual rationality is satisfied throughout the game horizon $[0,T]$ and $\left(\underline{\alpha}_1^{T-}\right)^{0.5}\left(\overline{\alpha}_1^{T-}\right)^{0.5} \in S_{t_0}^T$, At initial time 0, in the cooperative $\Gamma_c(x_0, T - t_0)$, an optimality principle under which the players to choose the weight

$$\alpha_1^* = \left(\underline{\alpha}_1^{T-}\right)^{0.5} \left(\overline{\alpha}_1^{T-}\right)^{0.5} \text{ in } \Gamma_c(x_\tau, T - \tau) \text{ for } \tau \in \left[t_0, T\right)$$

yields a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$.

Proof Follow the proof of Solution 4.1. ■

6.5 Numerical Delineation

Numerical delineations of the 4 solutions presented in Sect. 6.4 are given in the following 4 cases.

Case 5.1 Consider the cooperative game $\Gamma_c(x_0, T - t)$ with the following parameter specifications: $a = 10, b = 1, \sigma = 0.05, h_1 = 8, h_2 = 7, k_1 = 1, k_2 = 0.5, c_1 = 1, c_2 = 1.2,\ q_1 = 0.8, q_2 = 0.4, T = 6, r = 0.02$.

The numerical results are displayed in Fig. 6.4. The curve $\underline{\alpha}_1$ is the locus of the values of $\underline{\alpha}_1^t$ along $t \in \left[t_0, T\right)$. The curve $\overline{\alpha}_1$ is the locus of the values of $\overline{\alpha}_1^t$ along $t \in \left[t_0, T\right)$. In particular, the set $S_{t_0}^T = \underset{t_0 \le t < T}{\cap} S_t = \left[\underline{\alpha}_1^{t_0}, \overline{\alpha}_1^{T-}\right] = [1.182686, 1.450783]$. Note that $\underline{\alpha}_1^{T-} \in S_{t_0}^T$ and $\overline{\alpha}_1^{T-} \notin S_{t_0}^T$, for $\tau \in \left[t_0, T\right)$. According to Solution 4.1, the players would agree to the optimality principle of choosing a weight $\alpha_1^* = \underline{\alpha}_1^{T-} = 1.182686$ throughout the game interval, and a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$ would result.

Case 5.2 Consider the cooperative game $\Gamma_c(x_0, T - t)$ with the following parameter specifications: $a = 6, b = 0.8, \sigma = 0.04, h_1 = 8, h_2 = 6, k_1 = 1, k_2 = 0.5, c_1 = 1, c_2 = 1.5, q_1 = 3, q_2 = 2, T = 3, r = 0.02$.

The numerical results are displayed in Fig. 6.5. In particular, the set $S_{t_0}^T = \underset{t_0 \le t < T}{\cap} S_t = \left[\underline{\alpha}_1^{t_0}, \overline{\alpha}_1^{T-}\right] = [1.246704,\ \ 1.443176]$. Note that $\overline{\alpha}_1^{T-} \in S_{t_0}^T$ and $\underline{\alpha}_1^{T-} \notin S_{t_0}^T$,

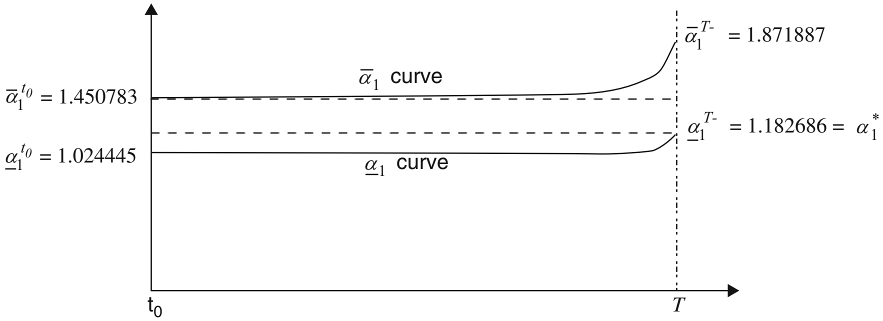

Fig. 6.4 A subgame consistent solution with optimality principle of a weight $\alpha_1{}^* = \underline{\alpha}_1{}^{T-} = 1.182686$

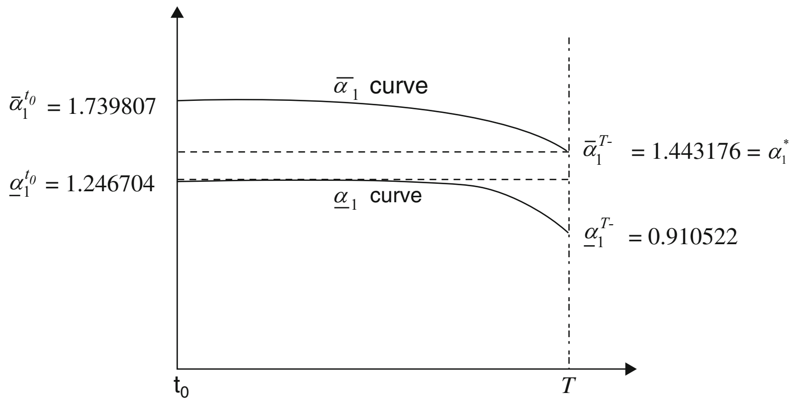

Fig. 6.5 A subgame consistent solution with optimality principle of a weight $\alpha_1{}^* = \bar{\alpha}_1{}^{T-} = 1.443176$

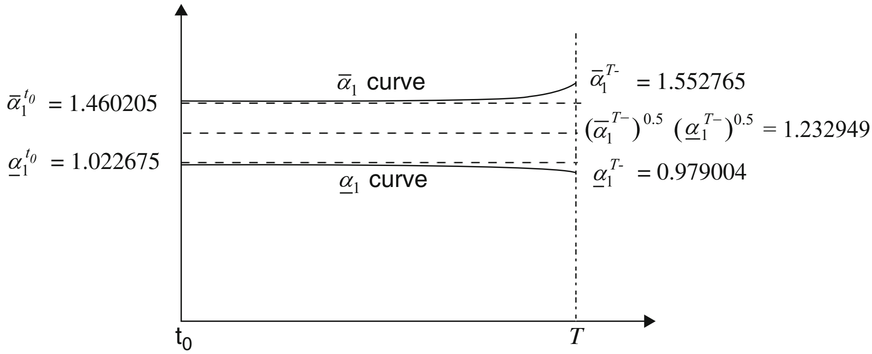

Fig. 6.6 A subgame consistent solution with optimality principle of a weight $\alpha_1{}^* = \left(\underline{\alpha}_1^{T-}\right)^{0.5}\left(\bar{\alpha}_1^{T-}\right)^{0.5} = 1.232949$

for $\tau \in [t_0, T)$. According to Solution 4.2, the players would agree to the optimality principle of choosing a weight $\alpha_1^* = \bar{\alpha}_1^{T-} = 1.443176$ throughout the game interval, and a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$ would result.

Case 5.3 Consider the cooperative game $\Gamma_c(x_0, T - t)$ with the following parameter specifications: $a = 10, b = 1.1, \sigma = 0.04, h_1 = 8, h_2 = 7, k_1 = 1, k_2 = 0.5, c_1 = 1, c_2 = 1.2, q_1 = 3, q_2 = 2, T = 3, r = 0.02$.

The numerical results are displayed in Fig. 6.6. In particular, the set $S_{t_0}^T = \bigcap_{t_0 \leq t < T} S_t = \left[\underline{\alpha}_1^{t_0}, \bar{\alpha}_1^{T-}\right] = [1.022675, \;\; 1.460205]$. Note that $\underline{\alpha}_1^{T-} \notin S_{t_0}^T$ and $\bar{\alpha}_1^{T-} \notin S_{t_0}^T$, for $\tau \in [t_0, T)$. According to Solution 4.3, the players would agree to the optimality principle of choosing a weight $\alpha_1^* = \left(\underline{\alpha}_1^{T-}\right)^{0.5}\left(\bar{\alpha}_1^{T-}\right)^{0.5} = 1.232949$ throughout the game interval, and a subgame consistent solution to the cooperative game $\Gamma_c(x_0, T - t_0)$ would result.

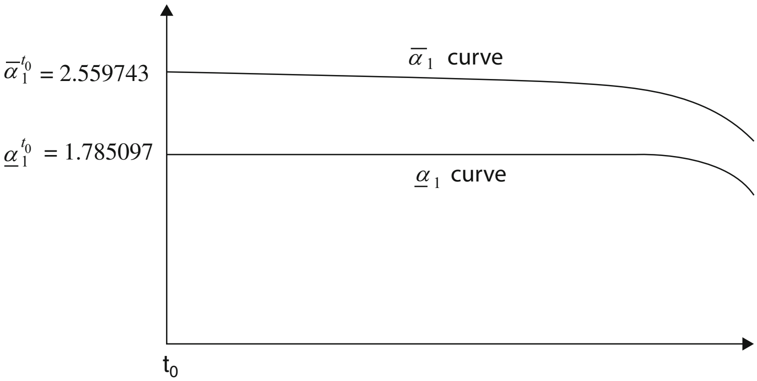

Fig. 6.7 The set $S_{to}^T = \underset{to \le t < T}{\cap} S_t = \emptyset$ and no candidate for a subgame consistent solution

Case 5.4 Consider the cooperative game $\Gamma_c(x_0, T-t)$ with parameters: $a = 6, b = 1, \sigma = 0.03, h_1 = 11, h_2 = 6, k_1 = 1, k_2 = 0.5, c_1 = 1, c_2 = 1.5, q_1 = 3, q_2 = 2, T = 6, r = 0.02$.

The numerical results are displayed in Fig. 6.7. In particular, the set $S_{t_0}^T = \underset{t_0 \le t < T}{\cap} S_t = \emptyset$. Hence there does not exist any candidate for a subgame consistent solution for the game $\Gamma_c(x_0, T - t_0)$.

6.6 Infinite Horizon Analysis

In this Section we examine the situation when the game horizon approaches infinity. Consider an infinite-horizon cooperative stochastic differential game in which player i's payoff to be maximized is

$$J^i(x_0) = E_{t_0}\left\{ \int_{t_0}^{\infty} \left[[k_i u_i(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_i(s) \right] \exp[-r(s-t_0)]\, ds \middle| x(t_0) = x_0 \right\}, \quad (6.1)$$

for $i \in \{1, 2\}$.

The state dynamics of the game is characterized by the stochastic differential equations:

$$dx(s) = \left[a x(s)^{1/2} - b x(s) - u_1(s) - u_2(s) \right] ds + \sigma x(s) dz(s), \ x(t_0) = x_0 \in X \quad (6.2)$$

where $u_i \in U_i$ is the control vector of player i, for $i \in \{1, 2\}$,

a, b, and σ are positive constants, and $z(s)$ is a Wiener process. Equation (6.2) could be interpreted as the stock dynamics of a biomass of renewable resource (see Jørgensen and Yeung (1996, 1999)).

Note that the infinite-horizon autonomous problem (6.1 and 6.2) is independent of the choice of t_0 and dependent only upon the state at the starting time, that is x_0. Hence, we use $\Gamma(x, \infty)$ and $\Gamma_c(x, \infty)$ to denote respectively a noncooperative and a cooperative game with payoffs (6.1) and dynamics (6.2) with starting state x. Following the previous analysis modified for an infinite horizon problem, we can obtain the value function reflecting the expected payoff (in current value) of player $i \in \{1, 2\}$ in the noncooperative game $\Gamma(x, \infty)$ as

Proposition 6.1

$$V^i(x) = \left[\overline{A}_i x^{1/2} + \overline{B}_i\right],$$

where $\overline{A}_i$, $\overline{B}_i, \overline{A}_j$ and $\overline{B}_j$, for $i \in \{1, 2\}$ and $j \in \{1, 2\}$ and $i \neq j$, satisfy:

$$\left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right]\overline{A}_i - \frac{k_i}{4\left[c_i + \overline{A}_i/2\right]} + \frac{\overline{A}_i k_j}{8\left[c_j + \overline{A}_j/2\right]^2} = 0, \text{ and } \overline{B}_i = \frac{a}{2r}\overline{A}_i.$$

Proof Applying Theorem 5.1 of Chap. 3 to the game (6.1 and 6.2) yields Proposition 6.1. ■

In the case of cooperation where α_1 is the chosen weight under the agreed optimality principle, the maximized value function reflecting the maximized expected weighted joint payoff of the stochastic control problem $\max_{u_1, u_2}\left\{J^1(x) + \alpha_1 J^2(x)\right\}$ subject to dynamics (6.1) can be obtained as:

Proposition 6.2 $W^{\alpha_1}(x) = \left[\overline{A}^{\alpha_1} x^{1/2} + \overline{B}^{\alpha_1}\right]$, where $\overline{A}^{\alpha_1}$ and $\overline{B}^{\alpha_1}$ satisfy:

$$\left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right]\overline{A}^{\alpha_1} - \frac{k_1}{4\left[c_1 + \overline{A}^{\alpha_1}/2\right]} - \frac{\alpha_1 k_2}{4\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]} = 0, \text{ and } \overline{B}^{\alpha_1} = \frac{a}{2r}\overline{A}^{\alpha_1}.$$

Proof Applying Theorem A.4 in the Technical Appendices to the stochastic control problem $\max_{u_1, u_2}\left\{J^1(x) + \alpha_1 J^2(x)\right\}$ subject to dynamics (6.1) yields Proposition 6.2. ■

The corresponding optimal controls are:

$$\psi_1^{\alpha_1(\infty)}(x) = \frac{k_1 x}{4\left[c_1 + \overline{A}^{\alpha_1}/2\right]^2} \text{ and } \psi_2^{\alpha_1(\infty)}(x) = \frac{k_2 x}{4\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]^2}, \text{ for } x \in X.$$

We define player 1's expected payoff over the interval $[0,\infty)$ under the control problem $\max_{u_1,u_2}\left\{J^1(x) + \alpha_1 J^2(x)\right\}$ as:

$$\hat{W}^{\alpha_1(1)}(x) = E_0\left\{\int_0^\infty \left[\frac{k_1 x(s)^{1/2}}{2\left[c_1 + \overline{A}^{\alpha_1}/2\right]} - \frac{c_1 k_1 x(s)^{1/2}}{4\left[c_1 + \overline{A}^{\alpha_1}/2\right]^2}\right] \exp(-rs)\, ds\right\};$$

and the corresponding expected payoff of player 2 over the interval $[0,\infty)$ as:

$$\hat{W}^{\alpha_1(2)}(x) = E_0\left\{\int_0^\infty \left[\frac{k_2 x(s)^{1/2}}{2\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]} - \frac{c_2 k_2 x(s)^{1/2}}{4\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]^2}\right] \exp(-rs)\, ds\right\};$$

where

$$dx(s) = \left[a x(s)^{1/2} - \left(b + \frac{k_1}{4\left[c_1 + \overline{A}^{\alpha_1}/2\right]^2} + \frac{k_2}{4\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]^2}\right) x(s)\right] ds$$
$$+ \sigma x(s) dz(s),\ x(t) = x.$$

An infinite-horizon counterpart of Theorem 1.1 characterizing player i's cooperative payoff under payoff weights α_1 is given in the theorem below.

Theorem 6.1 If there exist continuously functions $\hat{W}^{\alpha_1(i)}(x) : R^n \to R,\ i \in \{1,2\}$, satisfying

$$r\hat{W}_t^{\alpha_1(i)}(x) - \frac{1}{2}\sum_{h,\zeta=1}^{n} \Omega^{h\zeta}(x)\hat{W}_{x^h x^\zeta}^{\alpha_1(i)}(x) = g^i\left[x, \psi_1^{\alpha_1}(x), \psi_2^{\alpha_1}(x)\right]$$
$$+ \hat{W}_x^{\alpha_1(i)}(t,x) f\left[x, \psi_1^{\alpha_1}(x), \psi_2^{\alpha_1}(x)\right],$$

then $\hat{W}^{\alpha_1(i)}(t,x)$ gives player i's expected cooperative payoff when the state is x and α_1 is chosen as the weight.

Proof Following the analysis of developing an infinite horizon counter of the stochastic control leading to Theorem A.4 in the Technical appendices one can obtain an infinite-horizon counterpart of Theorem 1.1 in Section as Theorem 6.1. ∎

Using Theorem 6.1 the expected payoffs of Player 1 and Players 2 under cooperation can be obtained as follows.

Proposition 6.3 The expected payoffs of Player 1 and Player 2 (in current-value) under cooperation with bargaining weight α_1 are respectively:

$$\hat{W}^{\alpha_1(1)}(x) = \left[\hat{A}_1^{\alpha_1} x^{1/2} + \hat{B}_1^{\alpha_1}\right] \text{ and } \hat{W}^{\alpha_1(2)}(x) = \left[\hat{A}_2^{\alpha_1} x^{1/2} + \hat{B}_2^{\alpha_1}\right],$$

where

$$\left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right]\hat{A}_1^{\alpha_1} - \frac{k_1}{2\left[c_1 + \overline{A}^{\alpha_1}/2\right]} + \frac{c_1 k_1}{4\left[c_1 + \overline{A}^{\alpha_1}/2\right]^2} + \frac{\hat{A}_1^{\alpha_1} k_1}{8\left[c_1 + \overline{A}^{\alpha_1}/2\right]^2}$$
$$+ \frac{\hat{A}_1^{\alpha_1} k_2}{8\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]^2} = 0,\ \hat{B}_1^{\alpha_1}(t) = \frac{a}{2r}\hat{A}_1^{\alpha_1},$$
$$\left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right]\hat{A}_2^{\alpha_1} - \frac{k_2}{2\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]} + \frac{c_2 k_2}{4\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]^2}$$
$$+ \frac{\hat{A}_2^{\alpha_1} k_1}{8\left[c_1 + \overline{A}^{\alpha_1}/2\right]^2} + \frac{\hat{A}_2^{\alpha_1} k_2}{8\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]^2} = 0, \text{ and } \hat{B}_2^{\alpha_1} = \frac{a}{2r}\hat{A}_2^{\alpha_1}.$$

Proof Follow the Proof of Proposition 3.3 yields Proposition 6.3. ■

Since the solution to the control problem $\max_{u_1, u_2}\left\{J^1(x) + \alpha_1 J^2(x)\right\}$ yields a Pareto optimal outcome there exist (i) an $\underline{\alpha}_1^\infty$ such that $\hat{W}^{\underline{\alpha}_1^\infty(2)}(x) = V^2(x)$ and $\hat{W}^{\underline{\alpha}_1^\infty(1)}(x) \geq V^1(x)$, and (ii) an $\overline{\alpha}_1^\infty$ such that $\hat{W}^{\overline{\alpha}_1^\infty(1)}(x) = V^1(x)$ and $\hat{W}^{\overline{\alpha}_1^\infty(2)}(x) \geq V^2(x)$.

Comparing $\hat{W}^{\alpha_1(i)}(x)$ in Proposition 6.3 with $V^i(x)$ in Proposition 6.1 shows that $\hat{W}^{\alpha_1(i)}(x) \geq V^i(x)$ if and only if $\hat{A}_i^{\alpha_1} \geq \overline{A}_i$, for $i \in \{1, 2\}$.

A condition that would be used in subsequent analysis is:

Condition 6.1 $d\hat{A}_1^{\alpha_1}/d\alpha_1 < 0$ and $d\hat{A}_2^{\alpha_1}/d\alpha_1 > 0$.

Proof See Appendix A. ■

Therefore there exists a nonempty set S^∞ of α_1 such that $\hat{A}_i^{\alpha_1} \geq \overline{A}_i$, for $i \in \{1, 2\}$.

Using Condition 6.1, we can readily show that

Corollary 6.1 $S^\infty = \left[\underline{\alpha}_1^\infty, \overline{\alpha}_1^\infty\right]$, where $\underline{\alpha}_1^\infty$ is the lowest value of α_1 in S^∞, and $\overline{\alpha}_1^\infty$ the highest. Moreover, $\hat{A}_1^{\overline{\alpha}_1^\infty} = \overline{A}_1$ and $\hat{A}_2^{\underline{\alpha}_1^\infty} = \overline{A}_2$. ■

Now consider the case where the players agree to an optimality principle which chooses the payoff weight $\alpha_1^* = \left(\underline{\alpha}_1^\infty\right)^{0.5}\left(\overline{\alpha}_1^\infty\right)^{0.5}$. We then show that such an optimality principle yields a subgame consistent solution in the following Proposition.

Proposition 6.4 An optimality principle under which the players agree to choose the weight

$$\alpha_1^* = \left(\underline{\alpha}_1^\infty\right)^{0.5}\left(\overline{\alpha}_1^\infty\right)^{0.5} \tag{6.3}$$

yields a subgame consistent solution to the cooperative game $\Gamma_c(x, \infty)$.

Proof According to the optimality principle in Proposition 6.4 a unique weight $\alpha_1^* = \left(\underline{\alpha}_1^\infty\right)^{1/2}\left(\overline{\alpha}_1^\infty\right)^{1/2}$ is chosen for any game $\Gamma_c(x, \infty)$. Since $\left(\underline{\alpha}_1^\infty\right)^{1/2}\left(\overline{\alpha}_1^\infty\right)^{1/2} \in S^\infty$, the imputation vector $\xi^{(\tau)}(x, \infty) = \left[\hat{W}^{(\alpha_1^*)1}(x), \hat{W}^{(\alpha_1^*)2}(x)\right]$ yields a Pareto optimal pair. Hence part (i) of Definition 3.2 is proved.

The present-value (at time $\tau < t$) counterpart of the current-value payoff $\hat{W}^{\alpha_1^*(i)}(x),\ i \in \{1, 2\}$, can be expressed as

$$E_\tau\left\{ \exp[-r(t-\tau)]\int_t^\infty \left[\left\{k_i\psi_i^{\alpha_1^*(\infty)}[\mathrm{x(s)}]\right\}^{1/2} - \frac{c_i}{x(s)^{1/2}}\psi_i^{\alpha_1^*(\infty)}[\mathrm{x(s)}]\right]\right.$$
$$\left.\exp[-r(s-t)]\,ds \middle| \, x(t) = x \right\} = \exp[-r(t-\tau)]\hat{W}^{\alpha_1^*(i)}(x).$$

Hence, part (ii) of Definition 3.2 holds.

Since $\left(\underline{\alpha}_1^\infty\right)^{1/2}\left(\overline{\alpha}_1^\infty\right)^{1/2} \in S^\infty$, $\hat{W}^{\alpha_1^*(i)}(x) \geq V^i(x)$, for $i \in \{1, 2\}$ and $x \in X$. Hence, part (iii) of Definition 3.2 is satisfied. ■

In addition, the cooperative solution in Proposition 6.4 also satisfies the axioms of symmetry in the following remark.

Remark 6.1 The Pareto optimal cooperative solution proposed in Proposition 6.4 also satisfies the axioms of symmetry. See Appendix B for proof details. ■

6.7 Chapter Appendices

Appendix A: Proof of Condition 6.1 Note that $W^{\alpha_1}(x) = \hat{W}^{\alpha_1(1)}(x) + \alpha_1\hat{W}^{\alpha_1(2)}(x)$, therefore we have $\overline{A}^{\alpha_1} = \hat{A}_1^{\alpha_1} + \alpha_1\hat{A}_2^{\alpha_1}$. Since u_1 and u_2 are nonnegative, $\hat{W}^{\alpha_1(1)}(x) \geq 0$ and $\hat{W}^{\alpha_1(2)}(x) \geq 0$. Hence $\overline{A}^{\alpha_1}$, $\hat{A}_1^{\alpha_1}$ and $\hat{A}_2^{\alpha_1}$ are nonnegative.

Define the equation $\left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right]\overline{A}^{\alpha_1} - \frac{k_1}{4\left[c_1 + \overline{A}^{\alpha_1}/2\right]} - \frac{\alpha_1 k_2}{4\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]} = 0$ in Proposition 6.2 as $\Psi\left(\overline{A}^{\alpha_1}, \alpha_1\right) = 0$. Implicitly differentiating $\Psi\left(\overline{A}^{\alpha_1}, \alpha_1\right) = 0$ yields:

$$\frac{d\overline{A}^{\alpha_1}}{d\alpha_1} = \frac{k_2\left[c_2+\overline{A}^{\alpha_1}/\alpha_1\right]}{4\left[c_2+\overline{A}^{\alpha_1}/2\alpha_1\right]^2} \Big/ \left\{\left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]+\frac{k_1}{8\left[c_1+\overline{A}^{\alpha_1}/2\right]^2}+\frac{k_2}{8\left[c_2+\overline{A}^{\alpha_1}/2\alpha_1\right]^2}\right\} > 0 \tag{7.1}$$

Then we define the equation $\left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]\hat{A}_1^{\alpha_1} - \frac{k_1}{2\left[c_1+\overline{A}^{\alpha_1}/2\right]} + \frac{c_1k_1}{4\left[c_1+\overline{A}^{\alpha_1}/2\right]^2}$ $+\frac{\hat{A}_1^{\alpha_1}k_1}{8\left[c_1+\overline{A}^{\alpha_1}/2\right]^2} + \frac{\hat{A}_1^{\alpha_1}k_2}{8\left[c_2+\overline{A}^{\alpha_1}/2\alpha_1\right]^2} = 0$ in Proposition 6.3 as $\Psi^1\left(\hat{A}_1^{\alpha_1},\alpha_1\right)=0.$ The effect of a change in α_1 on $\hat{A}_1^{\alpha_1}$ can be obtained as:

$$\frac{d\hat{A}_1^{\alpha_1}}{d\alpha_1} = -\frac{\partial\Psi^1\left(\hat{A}_1^{\alpha_1},\alpha_1\right)/\partial\alpha_1}{\partial\Psi^1\left(\hat{A}_1^{\alpha_1},\alpha_1\right)/\partial\hat{A}_1^{\alpha_1}}, \tag{7.2}$$

Where

$$\frac{\partial\Psi^1}{\partial\hat{A}_1^{\alpha_1}} = \left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]+\frac{k_1}{8\left[c_1+\overline{A}^{\alpha_1}/2\right]^2}+\frac{k_2}{8\left[c_2+\overline{A}^{\alpha_1}/2\alpha_1\right]^2} > 0, \text{ and} \tag{7.3}$$

$$\frac{\partial\Psi^1}{\partial\alpha_1} = \frac{k_1\left[\overline{A}^{\alpha_1}-\hat{A}_1^{\alpha_1}\right]}{8\left[c_1+\overline{A}^{\alpha_1}/2\right]^3}\frac{d\overline{A}^{\alpha_1}}{d\alpha_1} + \frac{\hat{A}_1^{\alpha_1}k_2/\alpha_1{}^2}{8\left[c_2+\overline{A}^{\alpha_1}/2\alpha_1\right]^3}\left[\overline{A}^{\alpha_1}-\alpha_1\frac{d\overline{A}^{\alpha_1}}{d\alpha_1}\right] \tag{7.4}$$

From Proposition 6.3, we obtain:

$$\hat{A}_2^{\alpha_1} = \frac{k_2\left[c_2+\overline{A}^{\alpha_1}/\alpha_1\right]}{4\left[c_2+\overline{A}^{\alpha_1}/2\alpha_1\right]^2} \Big/ \left\{\left[r+\frac{1}{8}\sigma^2+\frac{b}{2}\right]+\frac{k_1}{8\left[c_1+\overline{A}^{\alpha_1}/2\right]^2}+\frac{k_2}{8\left[c_2+\overline{A}^{\alpha_1}/2\alpha_1\right]^2}\right\}. \tag{7.5}$$

Comparing (7.5) with (7.1) shows that $d\overline{A}^{\alpha_1}/d\alpha_1 = \hat{A}_2^{\alpha_1}$. Upon substituting $d\overline{A}^{\alpha_1}/d\alpha_1$ by $\hat{A}_2^{\alpha_1}$ and invoking the relation $\overline{A}^{\alpha_1} = \hat{A}_1^{\alpha_1} + \alpha_1\hat{A}_2^{\alpha_1}$, we have

$$\frac{\partial\Psi^1}{\partial\alpha_1} = \frac{k_1\alpha_1\left(\hat{A}_2^{\alpha_1}\right)^2}{8\left[c_1+\overline{A}^{\alpha_1}/2\right]^3} + \frac{\left(\hat{A}_1^{\alpha_1}\right)^2k_2/\alpha_1{}^2}{8\left[c_2+\overline{A}^{\alpha_1}/2\alpha_1\right]^3} > 0. \tag{7.6}$$

Therefore, $d\hat{A}_1^{\alpha_1}/d\alpha_1 < 0$. Following the above analysis, we have:

$$\frac{d\hat{A}_2^{\alpha_1}}{d\alpha_1} = \left\{ \frac{k_2\left(\hat{A}_1^{\alpha_1}\right)^2/\alpha_1{}^3}{8\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]^3} + \frac{\left(\hat{A}_2^{\alpha_1}\right)^2 k_1}{8\left[c_1 + \overline{A}^{\alpha_1}/2\right]^3} \right\}$$
$$\div \left\{ \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right] + \frac{k_1}{8\left[c_1 + \overline{A}^{\alpha_1}/2\right]^2} + \frac{k_2}{8\left[c_2 + \overline{A}^{\alpha_1}/2\alpha_1\right]^2} \right\} > 0. \tag{7.7}$$

Hence Condition 6.1 follows. ■

Appendix B: Proof of Remark 6.1 Let $[V^{(\max)1}(x), V^2(x)]$ denote a payoff pair along the Pareto optimal trajectory. From Condition 6.1 and Corollary 6.1, in the problem $\max_{u_1, u_2} \left\{J^1(x) + \alpha_1 J^2(x)\right\}$ if $\underline{\alpha}_1^\infty$ is chosen, $\hat{W}^{\underline{\alpha}_1^\infty(2)}(x) = V^2(x)$ and $\hat{W}^{\underline{\alpha}_1^\infty(1)}(x) = V^{(\max)1}(x)$. On the other hand, in the problem $\max_{u_1, u_2} \left\{J^2(x) + \alpha_2 J^1(x)\right\}$, in order to have player 2's expected payoff being $V^2(x)$ and player 1's payoff being $V^{(\max)1}(x)$ the weight $\overline{\alpha}_2^\infty$ has to be chosen. Recall that when $\alpha_1 = 1/\alpha_2$, the problem $\max_{u_1, u_2} \left\{J^1(x) + \alpha_1 J^2(x)\right\}$ is identical to the problem $\max_{u_1, u_2} \left\{J^2(x) + \alpha_2 J^1(x)\right\}$. Since $\max_{u_1, u_2} \left\{J^1(x) + \underline{\alpha}_1^\infty J^2(x)\right\}$ and $\max_{u_1, u_2} \left\{J^2(x) + \overline{\alpha}_2^\infty J^1(x)\right\}$ both yield $V^2(x)$ and $V^{1(\max)}(x)$, it is necessary that $\underline{\alpha}_1^\infty = 1/\overline{\alpha}_2^\infty$. With similar argument, $\overline{\alpha}_1^\infty = 1/\underline{\alpha}_2^\infty$ is verified.

According to Proposition 6.4, in the problem $\max_{u_1, u_2} \left\{J^1(x) + \alpha_1 J^2(x)\right\}$ an optimality principle under which the players agree to choose the weight $\alpha_1^* = \left(\underline{\alpha}_1^\infty\right)^{0.5} \left(\overline{\alpha}_1^\infty\right)^{0.5}$ yields a subgame consistent solution to the cooperative game $\Gamma_c(x, \infty)$.

Following the same optimality principle in the problem $\max_{u_1, u_2} \left\{J^2(x) + \alpha_2 J^1(x)\right\}$ under which the players agree to choose the weight $\alpha_2^* = \left(\underline{\alpha}_2^\infty\right)^{0.5} \left(\overline{\alpha}_2^\infty\right)^{0.5}$, which is equivalent to having $1/\alpha_1^* = 1/\left[\left(\underline{\alpha}_1^\infty\right)^{0.5} \left(\overline{\alpha}_1^\infty\right)^{0.5}\right]$.

Since $\alpha_2^* = 1/\alpha_1^*$, the controls in the problems $\max_{u_1, u_2} \left\{J^1(x) + \alpha_1^* J^2(x)\right\}$ and $\max_{u_1, u_2} \left\{J^2(x) + \alpha_2^* J^1(x)\right\}$ are identical. Hence the axiom of symmetry prevails. ■

6.8 Chapter Notes

The number of studies in cooperative dynamic games with non-transferrable utility/ payoff (NTU) is much less than that of cooperative dynamic games with transferrable payoffs. Leitmann (1974), Dockner and Jørgensen (1984), Hamalainen et al. (1986), Yeung and Petrosyan (2005), Yeung et al. (2007), de-Paz et al. (2013), and Marin-Solano (2014) studied continuous-time cooperative differential games with non-transferable payoffs. The stringent requirement of subgame consistency imposes additional hurdles to the derivation of solutions for cooperative stochastic differential games. In the case when players' payoffs are nontransferable, the derivation of solution candidates becomes even more complicated and intractable. In this Chapter, subgame consistent solutions of cooperative stochastic differential games with nontransferable payoffs are examined and a class of cooperative stochastic differential games with nontransferable payoffs is used to illustrate some possible solutions. Theorem 1.1 characterizing the players' expected payoff under cooperation was developed by Yeung (2004). Finally, the analysis can be applied to NTU cooperative differential games with the removal of the stochastic term $\sigma[s, x(s)]$. Finally, the notion of cooperative subgame consistency under variable payoff weights is examined in the discrete-time case in Chap. 11.

6.9 Problems

1. Consider a two-person stochastic differential game with initial state $x(0) = x_0 = 14$ and duration $[0, 4]$. The state dynamics of the game is characterized by the stochastic differential equations:

$$dx(s) = [15 - x(s) - u_1(s) - u_2(s)]ds + 0.01\,x(s)dz(s),$$

where $u_i \in U_i$ is the control vector of player i, for $i \in [1, 2]$, and $z(s)$ is a Wiener process. The state dynamics is the stock dynamics of a biomass of renewable resource like forest or fresh water. The state $x(s)$ represents the resource size and $u_i(s)$ the (nonnegative) amount of resource extracted by player i.

At time 0, the expected payoff of player 1 is:

$$J^1(0, x_0) = E_0\left\{ \int_0^4 \left[4u_1(s) - u_1(s)^2 x(s)^{-1} + 0.5x(s)\right] \exp[-0.05]\,ds + 2\exp[-0.2]x(T) \right\}, \text{ and,}$$

the expected payoff of player 2 is:

$$J^2(0, x_0) = E_0\left\{ \int_0^4 \left[3u_1(s) - 2u_1(s)^2 x(s)^{-1} + x(s)\right] \exp[-0.05]\,ds + 3\exp[-0.2]x(T) \right\}.$$

If the payoff weight $\alpha_1 = 0.4$ is chosen to maximize the expected weighted payoff

$\max\limits_{u_1, u_2} E_0 \left\{ J^1(0, x_0) + \alpha_1 J^2(0, x_0) \right\}$, derive the individual payoffs of the players under cooperation.

2. Consider an infinite horizon stochastic differential game with initial state $x(0) = x_0 = 10$. The state dynamics of the game is characterized by the stochastic differential equations:

$$dx(s) = [9 - 2x(s) - u_1(s) - u_2(s)]ds + 0.02x(s)dz(s),$$

where $u_i \in U_i$ is the control vector of player i, for $i \in [1, 2]$, and $z(s)$ is a Wiener process.

At time 0, the expected payoff of player 1 is:

$J^1(0, x_0) = E_0 \left\{ \int_0^\infty \left[4u_1(s) - u_1(s)^2 x(s)^{-1} + 0.2x(s) \right] \exp[-0.05]\, ds \right\}$, and the expected payoff of player 2 is:

$$J^2(0, x_0) = E_0 \left\{ \int_0^\infty \left[4u_1(s) - 2u_1(s)^2 x(s)^{-1} + 1.5x(s) \right] \exp[-0.05]\, ds \right\}.$$

If the payoff weight $\alpha_1 = 0.35$ is chosen to maximize the expected weighted payoff

$\max\limits_{u_1, u_2} E_0 \left\{ J^1(0, x_0) + \alpha_1 J^2(0, x_0) \right\}$, derive the individual payoffs of the players under cooperation.