Chapter 5
Subgame Consistency Under Asynchronous Players' Horizons

In many game situations, the players' time horizons differ. This may arise from different life spans, different entry and exit times, and the different duration for leases and contracts. Asynchronous horizon game situations occur frequently in economic and social activities. In this Chapter, subgame consistent cooperative solutions are derived for differential games with asynchronous players' horizons and uncertain types of future players. Analytically tractable payoff distribution mechanisms which lead to the realization of these solutions are derived. This analysis extends the application of cooperative differential game theory to problems where the players' game horizons are asynchronous and the types of future players are uncertain. In particular, the Chapter is an integrated disquisition of the analysis in Yeung (2011) with an extension to incorporate stochastic state dynamics.

The organization of the chapter is as follows. Section 5.1 presents the game formulation and characterizes noncooperative outcomes. Dynamic cooperation among players coexisting in the same duration is examined in Sect. 5.2. Section 5.3 provides an analysis on payoff distribution procedures leading to dynamically consistent solutions in this asynchronous horizons scenario. An illustration in cooperative resource extraction is given in Sect. 5.4. An extension to stochastic dynamics is provided in Sect. 5.5. Chapter notes are given in Sect. 5.6 and problems in Sect. 5.7.

5.1 Game Formulation and Noncooperative Outcome

In this section we present an analytical framework of differential games with asynchronous players' horizons and characterize the noncooperative outcome.

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5.1.1 Game Formulation

For clarity in exposition and without loss of generality, we consider a general class of differential games, in which there are $\upsilon+1$ overlapping cohorts or generations of players. The game begins at time t_1 and terminates at time $t_{\upsilon+1}$. In the time interval $\left[t_1, t_2\right)$, there coexist a generation 0 player whose game horizon is $\left[t_1, t_2\right)$ and a generation 1 player whose game horizon is $\left[t_1, t_3\right)$. In the time interval $\left[t_k, t_{k+1}\right)$ for $k \in \{2, 3, \cdots, \upsilon-1\}$, there coexist a generation $k-1$ player whose game horizon is $\left[t_{k-1}, t_{k+1}\right)$ and a generation k player whose game horizon is $\left[t_k, t_{k+2}\right)$. In the last time interval $[t_\upsilon, t_{\upsilon+1}]$, there coexist a generation $\upsilon-1$ player and a generation υ player whose game horizon is just $[t_\upsilon, t_{\upsilon+1}]$.

When the game starts at initial time t_1, it is known that in the time interval $[t_1, t_2)$, there coexist a type ω_0^1 generation 0 player and a type ω_1^1 generation 1 player. At time t_1, it is also known that the probability of the generation k player being type $\omega_k^{a_k} \in \left\{\omega_k^1, \omega_k^2, \cdots, \omega_k^{\varsigma_k}\right\}$ is $\lambda_k^{a_k} \in \left\{\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^{\varsigma_k}\right\}$, for $k \in \{2, 3, \cdots, \upsilon\}$. The type of generation k player will become known with certainty at time t_{k}.

The instantaneous payoffs and terminal rewards of the type $\omega_k^{a_k}$ generation k player and the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player coexisting in the time interval $\left[t_k, t_{k+1}\right)$ are respectively:

$$g^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}\left[s, x(s), u_k^{\left(\omega_{k-1}^{a_{k-1}}, O\right)\omega_k^{a_k}}(s), u_k^{\left(\omega_k^{a_k}, Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right] \text{ and } q^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}[t_{k+1}, x(t_{k+1})] \text{ and}$$
$$g^{k\left(\omega_k^{a_k}\right)}\left[s, x(s), u_k^{\left(\omega_{k-1}^{a_{k-1}}, O\right)\omega_k^{a_k}}(s), u_k^{\left(\omega_k^{a_k}, Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right] \text{ and } q^{k\left(\omega_k^{a_k}\right)}[t_{k+2}, x(t_{k+2})], \tag{1.1}$$

for $k \in \{1, 2, 3, \cdots, \upsilon\}$,

where $u_k^{\left(\omega_{k-1}^{a_{k-1}}, O\right)\omega_k^{a_k}}(s)$ is the vector of controls of the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player when he is in his second (old) life stage while the type $\omega_k^{a_k}$ generation k player is coexisting;

and $u_k^{\left(\omega_k^{a_k}, Y\right)\omega_{k-1}^{a_{k-1}}}(s)$ is that of the type $\omega_k^{a_k}$ generation k player when he is in his first (young) life stage while the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player is coexisting.

Note that the superindex "O" in $u_k^{\left(\omega_{k-1}^{a_{k-1}}, O\right)\omega_k^{a_k}}(s)$ denote "Old" and the superindex "Y" in $u_k^{\left(\omega_k^{a_k}, Y\right)\omega_{k-1}^{a_{k-1}}}(s)$ denote "Young". The state dynamics of the game is characterized by the vector-valued differential equations:

$$\dot{x}(s) = f\left[s, x(s), u_k^{\left(\omega_{k-1}^{a_{k-1}}, O\right)\omega_k^{a_k}}(s), u_k^{\left(\omega_k^{a_k}, Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right], \text{ for } s \in \left[t_k, t_{k+1}\right), \tag{1.2}$$

if type $\omega_k^{a_k}$ generation k player and type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player co-exist in the time interval $\left[t_k, t_{k+1}\right)$ for $k \in \{1, 2, 3, \cdots, \upsilon\}$, and $x(t_1) = x_0 \in X$.

In the game interval $[t_k, t_{k+1})$ for $k \in \{1, 2, 3, \cdots, \upsilon - 1\}$ with type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player and type $\omega_k^{a_k}$ generation k player, the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player seeks to maximize:

$$\int_{t_k}^{t_{k+1}} g^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}\left[s, x(s), u_k^{\left(\omega_{k-1}^{a_{k-1}}, O\right)\omega_k^{a_k}}(s), u_k^{\left(\omega_k^{a_k}, Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right] e^{-r(s-t_k)} ds \\ + e^{-r(t_{k+1}-t_k)} q^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}[t_{k+1}, x(t_{k+1})], \tag{1.3}$$

and the type ω_k generation k player seeks to maximize:

$$\int_{t_k}^{t_{k+1}} g^{k\left(\omega_k^{a_k}\right)}\left[s, x(s), u_k^{\left(\omega_{k-1}^{a_{k-1}}, O\right)\omega_k^{a_k}}(s), u_k^{\left(\omega_k^{a_k}, Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right] e^{-r(s-t_k)} ds \\ + \sum_{\ell=1}^{\varsigma_{k+1}} \lambda_{k+1}^{\ell} \int_{t_{k+1}}^{t_{k+2}} g^{k\left(\omega_k^{a_k}\right)}\left[s, x(s), u_{k+1}^{\left(\omega_k^{a_k}, O\right)\omega_{k+1}^{\ell}}(s), u_{k+1}^{\left(\omega_{k+1}^{\ell}, Y\right)\omega_k^{a_k}}(s)\right] e^{-r(s-t_k)} ds \\ + e^{-r(t_{k+2}-t_k)} q^{k\left(\omega_k^{a_k}\right)}[t_{k+2}, x(t_{k+2})] \tag{1.4}$$

subject to dynamics (1.2), where r is the discount rate.

In the last time interval $[t_\upsilon, t_{\upsilon+1}]$ where the generation $\upsilon - 1$ player is of type $\omega_{\upsilon-1}^{a_{\upsilon-1}}$ and the generation υ player is of type $\omega_\upsilon^{a_\upsilon}$, the type $\omega_{\upsilon-1}^{a_{\upsilon-1}}$ generation $\upsilon - 1$ player seeks to maximize:

$$\int_{t_\upsilon}^{t_{\upsilon+1}} g^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}\left[s, x(s), u_\upsilon^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}, O\right)\omega_\upsilon^{a_\upsilon}}(s), u_\upsilon^{(\omega_\upsilon^{a_\upsilon}, Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(s)\right] e^{-r(s-t_\upsilon)} ds \\ + e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}[t_{\upsilon+1}, x(t_{\upsilon+1})], \tag{1.5}$$

and the type $\omega_\upsilon^{a_\upsilon}$ generation υ player seeks to maximize:

$$\int_{t_\upsilon}^{t_{\upsilon+1}} g^{\upsilon(\omega_\upsilon^{a_\upsilon})}\left[s, x(s), u_\upsilon^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}, O\right)\omega_\upsilon^{a_\upsilon}}(s), u_\upsilon^{(\omega_\upsilon^{a_\upsilon}, Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(s)\right] e^{-r(s-t_\upsilon)} ds \\ + e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon(\omega_\upsilon^{a_\upsilon})}[t_{\upsilon+1}, x(t_{\upsilon+1})], \tag{1.6}$$

subject to dynamics (1.2).

The game formulated is a finite overlapping generations version of Jørgensen and Yeung's (2005) infinite generations game.

5.1.2 Noncooperative Outcomes

To obtain a characterization of a noncooperative solution to the asynchronous horizons game mentioned above we first consider the solutions of the game in the last time interval $[t_\upsilon, t_{\upsilon+1}]$, that is the game (1.5 and 1.6). One way to characterize and derive a feedback solution to the game in $[t_\upsilon, t_{\upsilon+1}]$ is provided in the lemma below.

Lemma 1.1 If the generation $\upsilon - 1$ player is of type $\omega_{\upsilon-1}^{a_{\upsilon-1}} \in \{\omega_{\upsilon-1}^1, \omega_{\upsilon-1}^2, \cdots, \omega_{\upsilon-1}^{\varsigma_{\upsilon-1}}\}$ and the generation υ player is of type $\omega_\upsilon^{a_\upsilon} \in \{\omega_\upsilon^1, \omega_\upsilon^2, \cdots, \omega_\upsilon^{\varsigma_\upsilon}\}$ in the time interval $[t_\upsilon, t_{\upsilon+1}]$, a set of feedback strategies $\left\{\phi_\upsilon^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}, O\right)\omega_\upsilon^{a_\upsilon}}(t,x);\ \phi_\upsilon^{(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x)\right\}$ constitutes a Nash equilibrium solution for the game (1.5 and 1.6), if there exist continuously differentiable functions $V^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(t,x) : [t_\upsilon, t_{\upsilon+1}] \times R^m \to R$ and $V^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x) : [t_\upsilon, t_{\upsilon+1}] \times R^m \to R$ satisfying the following partial differential equations:

$$
\begin{aligned}
-V_t^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(t,x) &= \max_{u_\upsilon^O}\left\{g^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}\left[t,x,u_\upsilon^O,\phi_\upsilon^{(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x)\right]e^{-r(t-t_\upsilon)}\right.\\
&\quad\left.+V_x^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(t,x)f\left[t,x,u_\upsilon^O,\phi_\upsilon^{(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x)\right]\right\},\\
V^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(t_{\upsilon+1},x) &= e^{-r(t_{\upsilon+1}-t_\upsilon)}q^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}(t_{\upsilon+1},x),\ \text{and}\\
-V_t^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x) &= \max_{u_\upsilon^Y}\left\{g^{\upsilon(\omega_\upsilon^{a_\upsilon})}\left[t,x,\phi_{\upsilon-1}^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(t,x),u_\upsilon^Y\right]e^{-r(t-t_\upsilon)}\right.\\
&\quad\left.+V_x^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x)f\left[t,x,\phi_{\upsilon-1}^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(t,x),u_\upsilon^Y\right]\right\},\\
V^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t_{\upsilon+1},x) &= e^{-r(t_{\upsilon+1}-t_\upsilon)}q^{\upsilon(\omega_\upsilon^{a_\upsilon})}[t_{\upsilon+1},x(t_{\upsilon+1})]
\end{aligned}
\tag{1.7}
$$

Proof Follow the proof of Theorem 1.1 in Chap. 2. ■

For ease of exposition and sidestepping the issue of multiple equilibria, the analysis focuses on solvable games in which a particular noncooperative Nash equilibrium is chosen by the players in the entire subgame.

We proceed to examine the game in the second last interval $[t_{\upsilon-1}, t_\upsilon)$. If the generation $\upsilon - 2$ player is of type $\omega_{\upsilon-2}^{a_{\upsilon-2}} \in \{\omega_{\upsilon-2}^1, \omega_{\upsilon-2}^2, \cdots, \omega_{\upsilon-2}^{\varsigma_{\upsilon-2}}\}$ and the generation $\upsilon - 1$ player is of type $\omega_{\upsilon-1}^{a_{\upsilon-1}} \in \{\omega_{\upsilon-1}^1, \omega_{\upsilon-1}^2, \cdots, \omega_{\upsilon-1}^{\varsigma_{\upsilon-1}}\}$. The type $\omega_{\upsilon-2}^{a_{\upsilon-2}}$ generation $\upsilon - 2$ player seeks to maximize:

$$
\begin{aligned}
&\int_{t_{\upsilon-1}}^{t_\upsilon} g^{\upsilon-2\left(\omega_{\upsilon-2}^{a_{\upsilon-2}}\right)}\left[s,x(s),u_{\upsilon-2}^{\left(\omega_{\upsilon-2}^{a_{\upsilon-2}},O\right)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(s),u_{\upsilon-1}^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},Y\right)\omega_{\upsilon-2}^{a_{\upsilon-2}}}(s)\right]e^{-r(s-t_{\upsilon-1})}ds\\
&\quad+e^{-r(t_\upsilon-t_{\upsilon-1})}q^{\upsilon-2\left(\omega_{\upsilon-2}^{a_{\upsilon-2}}\right)}[t_\upsilon,x(t_\upsilon)].
\end{aligned}
\tag{1.8}
$$

In the subgame in the time interval $[t_{\upsilon-1}, t_\upsilon)$ the expected payoff of the type $\omega_{\upsilon-1}^{a_{\upsilon-1}}$ generation $\upsilon - 1$ player at time t_υ can be expressed as:

$$
\sum_{\ell=1}^{\varsigma_\upsilon}\lambda_\upsilon^\ell V^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^\ell}(t_\upsilon,x). \tag{1.9}
$$

Therefore the type $\omega_{\upsilon-1}^{a_{\upsilon-1}}$ generation $\upsilon-1$ player then seeks to maximize:

$$\int_{t_{\upsilon-1}}^{t_\upsilon} g^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}\left[s, x(s), u_{\upsilon-2}^{\left(\omega_{\upsilon-2}^{a_{\upsilon-2}},O\right)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(s), u_{\upsilon-1}^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},Y\right)\omega_{\upsilon-2}^{a_{\upsilon-1}}}(s)\right] e^{-r(s-t_{\upsilon-1})} ds$$
$$+ e^{-r(t_\upsilon - t_{\upsilon-1})} \sum_{\ell=1}^{\varsigma_\upsilon} \lambda_\upsilon^\ell V^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^\ell}(t_\upsilon, x(t_\upsilon)).$$

Similarly, in the subgame in the interval $[t_k, t_{k+1})$ the expected payoff of the type ω_k generation k player at time t_{k+1} can be expressed as:

$$\sum_{\ell=1}^{\varsigma_{k+1}} \lambda_{k+1}^\ell V^{k\left(\omega_k^{a_k},O\right)\omega_{k+1}^\ell}(t_{k+1}, x), \qquad \text{for } k \in \{1, 2, \cdots, \upsilon-3\}. \tag{1.10}$$

Consider the game in the time interval $[t_k, t_{k+1})$ involving the type $\omega_k^{a_k}$ generation k player and the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player, for $k \in \{1, 2, \cdots, \upsilon-3\}$. The type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player will maximize the payoff

$$\int_{t_k}^{t_{k+1}} g^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}\left[s, x(s), u_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(s), u_k{}^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right] e^{-r(s-t_k)} ds$$
$$+ e^{-r(t_{k+1}-t_k)} q^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}[t_{k+1}, x(t_{k+1})], \tag{1.11}$$

and the type $\omega_k^{a_k}$ generation k player will maximize the payoff:

$$\int_{t_k}^{t_{k+1}} g^{k\left(\omega_k^{a_k}\right)}\left[s, x(s), u_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(s), u_k{}^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right] e^{-r(s-t_k)} ds$$
$$+ e^{-r(t_{k+1}-t_k)} \sum_{\ell=1}^{\varsigma_{k+1}} \lambda_{k+1}^\ell V^{k\left(\omega_k^{a_k},O\right)\omega_{k+1}^\ell}(t_{k+1}, x(t_{k+1})), \tag{1.12}$$

subject to (1.2).

A feedback solution to the game (1.5, 1.6) and (1.11, 1.12) can be characterized by the lemma below.

Lemma 1.2 A set of feedback strategies $\left\{\phi_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t,x);\ \phi_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x)\right\}$ constitutes a Nash equilibrium solution for the game (1.5, 1.6) and (1.11, 1.12), if there exist continuously differentiable functions $V^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t,x) : [t_k, t_{k+1}] \times R^m \to R$ and $V^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x) : [t_k, t_{k+1}] \times R^m \to R$ satisfying the following partial differential equations:

$$-V_t^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x)=\max_{u_\upsilon^Y}\left\{g^{\upsilon(\omega_\upsilon^{a_\upsilon})}\left[t,x,\phi_{\upsilon-1}^{(\omega_{\upsilon-1}^{a_{\upsilon-1}},O)\omega_\upsilon^{a_\upsilon}}(t,x),u_\upsilon^Y\right]e^{-r(t-t_\upsilon)}\right.$$
$$\left.+V_x^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x)f\left[t,x,\phi_{\upsilon-1}^{(\omega_{\upsilon-1}^{a_{\upsilon-1}},O)\omega_\upsilon^{a_\upsilon}}(t,x),u_\upsilon^Y\right]\right\}$$
$$V^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t_{\upsilon+1},x)=e^{-r(t_{\upsilon+1}-t_\upsilon)}q^{\upsilon(\omega_\upsilon^{a_\upsilon})}[t_{\upsilon+1},x(t_{\upsilon+1})];\text{ and}$$
$$-V_t^{k-1(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}}(t,x)=\max_{u_k^O}\left\{g^{k-1(\omega_{k-1})}\left[t,x,u_k^O,\phi_k^{(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(t,x)\right]e^{-r(t-t_k)}\right.$$
$$\left.+V_x^{k-1(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}}(t,x)f\left[t,x,u_k^O,\phi_k^{(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(t,x)\right]\right\},$$
$$V^{k-1(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}}(t_{k+1},x)=e^{-r(t_{k+1}-t_k)}q^{k-1(\omega_{k-1}^{a_{k-1}})}(t_{k+1},x),\text{ and}$$
$$-V_t^{k(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(t,x)=\max_{u_k^Y}\left\{g^{(k,\omega_k)}\left[t,x,\phi_{k-1}^{(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}},u_k^Y\right]e^{-r(t-t_k)}\right.$$
$$\left.+V_x^{k(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(t,x)f\left[t,x,\phi_{k-1}^{(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}},u_k^Y\right]\right\}$$
$$V^{k(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(t_{k+1},x)=e^{-r(t_{k+1}-t_k)}\sum_{\ell=1}^{\varsigma_{k+1}}\lambda_{k+1}^{\ell}V^{k(\omega_k^{a_k},O)\omega_{k+1}^{\ell}}(t_{k+1},x),\tag{1.13}$$

for $k\in\{1,2,\cdots,\upsilon-1\}$.

Proof Again follow the proof of Theorem 1.1 in Chap. 2. ∎

5.2 Dynamic Cooperation Among Coexisting Players

Now consider the case when coexisting players want to cooperate and agree to act and allocate the cooperative payoff according to a set of agreed upon optimality principles. The agreement on how to act cooperatively and allocate cooperative payoff constitutes the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle for the cooperative game includes (i) an agreement on a set of cooperative strategies/controls, and (ii) an imputation of their payoffs.

Consider the game in the time interval $[t_k,t_{k+1})$ involving the type $\omega_k^{a_k}$ generation k player and the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player. Let $\varpi_h^{(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k})}$ denote the probability that the type $\omega_k^{a_k}$ generation k player and the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player would agree to the solution imputation $\left[\xi^{k-1(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}[h]}(t,x),\ \xi^{k(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}[h]}(t,x)\right]$, over the time interval $[t_k,t_{k+1})$,

$$\text{where}\quad\sum_{h=1}^{\varsigma(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k})}\varpi_h^{(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k})}=1.$$

At time t_1, the agreed-upon imputation for the type ω_0^1 generation 0 player and the type ω_1^1 generation 1 player are known to be $\left[\xi^{0\left(\omega_0^1,O\right)\omega_1^1[1]}(t,x),\ \xi^{1\left(\omega_1^1,Y\right)\omega_0^1[1]}(t,x)\right]$, over the time interval $\left[t_1,t_2\right)$.

The solution imputation may be governed by many specific principles. For instance, the players may agree to maximize the sum of their expected payoffs and equally divide the excess of the cooperative payoff over the noncooperative payoff. As another example, the solution imputation may be an allocation principle in which the players allocate the total joint payoff according to the relative sizes of the players' noncooperative payoffs. Finally, it is also possible that the players refuse to cooperate. In that case, the imputation vector becomes $\left[V^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t,x),\ V^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x)\right]$.

Both group optimality and individual rationality are required in a cooperative plan. Group optimality requires the players to seek a set of cooperative strategies/controls that yields a Pareto optimal solution. The allocation principle has to satisfy individual rationality in the sense that neither player would be no worse off than before under cooperation.

5.2.1 Group Optimality

Since payoffs are transferable, group optimality requires the players coexisting in the same time interval to maximize their expected joint payoff. Consider the last time interval $[t_\upsilon,t_{\upsilon+1}]$, in which the generation $\upsilon-1$ player is of type $\omega_{\upsilon-1}^{a_{\upsilon-1}}\in\left\{\omega_{\upsilon-1}^1,\omega_{\upsilon-1}^2,\cdots,\omega_{\upsilon-1}^{\varsigma_{\upsilon-1}}\right\}$ and the generation υ player is of type $\omega_\upsilon^{a_\upsilon}\in\left\{\omega_\upsilon^1,\omega_\upsilon^2,\cdots,\omega_\upsilon^{\varsigma_\upsilon}\right\}$. The players maximize their joint payoff

$$\begin{aligned}&\int_{t_\upsilon}^{t_{\upsilon+1}}\left(g^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}\left[s,x(s),u_\upsilon^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(s),u_\upsilon^{\left(\omega_\upsilon^{a_\upsilon},Y\right)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(s)\right]\right.\\&\quad\left.+\,g^{\upsilon\left(\omega_\upsilon^{a_\upsilon}\right)}\left[s,x(s),u_\upsilon^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(s),u_\upsilon^{\left(\omega_\upsilon^{a_\upsilon},Y\right)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(s)\right]\right)e^{-r(s-t_\upsilon)}ds\\&\qquad+e^{-r(t_{\upsilon+1}-t_\upsilon)}\left(q^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}[t_{\upsilon+1},x(t_{\upsilon+1})]+q^{\upsilon\left(\omega_\upsilon^{a_\upsilon}\right)}[t_{\upsilon+1},x(t_{\upsilon+1})]\right),\end{aligned}\tag{2.1}$$

subject to (1.2).

An optimal solution of the problem (2.1 and 1.2) can be characterized by the following lemma.

Lemma 2.1 A set of Controls $\left\{\psi_{\upsilon-1}^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(t,x);\ \psi_\upsilon^{\left(\omega_\upsilon^{a_\upsilon},Y\right)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x)\right\}$ constitutes an optimal solution for the control problem (2.1 and 1.2), if there exist continuously differentiable functions $W^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t,x):\ [t_\upsilon,t_{\upsilon+1}]\times R^m\rightarrow R$ satisfying the following partial differential equations:

$$-W_t^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t,x) = \max_{u_\upsilon^O,u_\upsilon^O}\left\{g^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}\left[t,x,u_\upsilon^O,u_\upsilon^Y\right]e^{-r(t-t_\upsilon)}\right.$$
$$\left.+\,g^{\upsilon\left(\omega_\upsilon^{a_\upsilon}\right)}\left[t,x,u_\upsilon^O,u_\upsilon^Y\right]e^{-r(t-t_\upsilon)} + W_x^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t,x)f\left[t,x,u_\upsilon^O,u_\upsilon^Y\right]\right\},$$
$$W^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)}\left[q^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}(t_{\upsilon+1},x) + q^{\upsilon\left(\omega_\upsilon^{a_\upsilon}\right]}(t_{\upsilon+1},x)\right]. \tag{2.2}$$

Proof Invoking Bellman's techniques of dynamic programming stated in Theorem A.1 of the Technical Appendices an optimal solution of the problem (2.1 and 1.2) can be characterized as (2.2). ■

We proceed to examine joint payoff maximization problem in the time interval $\left[t_{\upsilon-1},t_\upsilon\right)$ involving the type $\omega_{\upsilon-1}^{a_{\upsilon-1}}$ generation $\upsilon-1$ player and type $\omega_{\upsilon-2}^{a_{\upsilon-2}}$ generation $\upsilon-2$ player. A critical problem is to determent type $\omega_{\upsilon-1}^{a_{\upsilon-1}}$ generation $\upsilon-1$ player's expected valuation of his optimization problem in the time interval $\left[t_{\upsilon-1},t_\upsilon\right)$ at time t_υ. At time t_υ, the $\omega_{\upsilon-1}^{a_{\upsilon-1}}$ generation $\upsilon-1$ player may co-exist with the type $\omega_\upsilon^{a_\upsilon} \in \left\{\omega_\upsilon^1,\omega_\upsilon^2,\cdots,\omega_\upsilon^{\varsigma_\upsilon}\right\}$ generation υ player with probabilities $\left\{\lambda_\upsilon^1,\lambda_\upsilon^2,\cdots,\lambda_\upsilon^{\varsigma_\upsilon}\right\}$. Consider the case in the time interval $\left[t_\upsilon,t_{\upsilon+1}\right)$ in which the type $\omega_{\upsilon-1}^{a_{\upsilon-1}}$ generation $\upsilon-1$ player and the type $\omega_\upsilon^{a_\upsilon}$ generation υ player co-exist. The probability that the type $\omega_{\upsilon-1}^{a_{\upsilon-1}}$ generation player and the type $\omega_\upsilon^{a_\upsilon}$ generation player would agree to the solution imputation

$$\left[\xi^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}[h]}(t,x),\ \xi^{\upsilon\left(\omega_\upsilon^{a_\upsilon},Y\right)\omega_{\upsilon-1}^{a_{\upsilon-1}}[h]}(t,x)\right] \text{ is } \varpi_h^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)},$$
$$\text{where } \sum_{h=1}^{\zeta\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)} \varpi_h^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)} = 1. \tag{2.3}$$

In the optimization problem within the time interval $\left[t_{\upsilon-1},t_\upsilon\right)$, the expected reward to the $\omega_{\upsilon-1}$ generation $\upsilon-1$ player at time t_υ can be expressed as:

$$\sum_{\ell=1}^{\varsigma_\upsilon}\lambda_\upsilon^\ell \sum_{h=1}^{\zeta\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^\ell\right)} \varpi_h^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^\ell\right)} \xi^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^\ell[h]}(t_\upsilon,x). \tag{2.4}$$

Similarly for the optimization problem within the time interval $\left[t_k,t_{k+1}\right)$, the expected reward to the type $\omega_k^{a_k}$ generation k player at time t_{k+1} can be expressed as:

$$\sum_{\ell=1}^{\varsigma_{k+1}}\lambda_{k+1}^{\ell}\sum_{h=1}^{\zeta\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\varpi_h^{\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\xi^{k\left(\omega_k^{a_k},O\right)\omega_{k+1}^{\ell}[h]}(t_{k+1},x),\qquad \text{for}\, k\in\{1,2,\cdots,\upsilon-2\}. \tag{2.5}$$

The joint maximization problem in the time interval $\left[t_k,t_{k+1}\right)$, for $k\in\{1,2,\cdots,\upsilon-2\}$, involving the type $\omega_k^{a_k}$ generation k player and type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player can be expressed as the maximization of joint payoff

$$\begin{aligned}&\left\{\int_{t_k}^{t_{k+1}}\left(g^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}\left[s,x(s),u_k^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(s),u_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right]\right.\right.\\&\quad+g^{k\left(\omega_k^{a_k}\right)}\left[s,x(s),u_k^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(s),u_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right]\right)e^{-r(s-t_k)}ds\\&\quad+e^{-r(t_{k+1}-t_k)}\left(q^{k-1\left(\omega_{k-1}^{a_{k+1}}\right)}[t_{k+1},x(t_{k+1})]\right.\\&\quad\left.\left.+\sum_{\ell=1}^{\varsigma_{k+1}}\lambda_{k+1}^{\ell}\sum_{h=1}^{\zeta\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\varpi_h^{\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\xi^{k\left(\omega_k^{a_k},O\right)\omega_{k+1}^{\ell}[h]}(t_{k+1},x(t_{k+1}))\right)\right\},\end{aligned} \tag{2.6}$$

subject to (1.2).

The conditions characterizing an optimal solution of the problem of maximizing (2.6) subject to (1.2) are given in the following theorem.

Theorem 2.1 A set of controls $\left\{\psi_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t,x);\ \psi_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x)\right\}$ constitutes an optimal solution for the control problem (1.2 and 2.6), if there exist continuously differentiable functions $W^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}(t,x):\ \left[t_k,t_{k+1}\right)\times R^m\rightarrow R$ satisfying the following partial differential equations:

$$\begin{aligned}&-W_t^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t,x)=\max_{u_\upsilon^O,u_\upsilon^Y}\left\{g^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}\left[t,x,u_\upsilon^O,u_\upsilon^Y\right]e^{-r(t-t_\upsilon)}\right.\\&\left.+g^{\upsilon\left(\omega_\upsilon^{a_\upsilon}\right)}\left[t,x,u_\upsilon^O,u_\upsilon^Y\right]e^{-r(t-t_\upsilon)}+W_x^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t,x)f\left[t,x,u_\upsilon^O,u_\upsilon^Y\right]\right\}\\&W^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t_{\upsilon+1},x)=e^{-r(t_{\upsilon+1}-t_\upsilon)}\\&\left[q^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}(t_{\upsilon+1},x)+q^{\upsilon\left(\omega_\upsilon^{a_\upsilon}\right]}(t_{\upsilon+1},x)\right]\text{and}\\&-W_t^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}(t,x)=\max_{u_k^O,u_\upsilon^Y}\left\{g^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}\left[t,x,u_k^O,u_k^Y\right]e^{-r(t-t_k)}\right.\\&\left.+g^{k\left(\omega_k^{a_k}\right)}\left[t,x,u_k^O,u_k^Y\right]e^{-r(t-t_k)}+W_x^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}(t,x)f\left[t,x,u_k^O,u_k^Y\right]\right\}\\&W^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}(t_{k+1},x)=e^{-r(t_{k+1}-t_k)}\left(q^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}(t_{k+1},x)\right.\\&\left.+\sum_{\ell=1}^{\varsigma_{k+1}}\lambda_{k+1}^{\ell}\sum_{h=1}^{\zeta\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\varpi_h^{\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\xi^{k\left(\omega_k^{a_k},O\right)\omega_{k+1}^{\ell}[h]}(t_{k+1},x(t_{k+1}))\right),\text{for } k\in\{1,2,\cdots,\upsilon-1\}.\end{aligned} \tag{2.7}$$

Proof Invoking Bellman's (1957) technique of dynamic programming stated in Theorem A.1 of the Technical Appendices we obtain the conditions characterizing an optimal solution of the problem (1.2) and (2.6) as in (2.7). ■

In particular, $W^{[t_k,t_{k+1}](\omega_{k-1}^{a_{k-1}},\omega_k^{a_k})}(t,x)$ gives the maximized joint payoff of the type $\omega_k^{a_k}$ generation k player and type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player at time $t\in[t_k,t_{k+1}]$ with the state x in the control problem

$$\max_{u_k^{(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}},\,u_k^{(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}} \left\{ \int_t^{t_{k+1}} \left(g^{k-1(\omega_{k-1}^{a_{k-1}})}\left[s,x(s),u_k^{(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}}(s),u_k^{(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(s)\right] \right.\right.$$
$$+ g^{k(\omega_k^{a_k})}\left[s,x(s),u_k^{(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}}(s),u_k^{(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(s)\right] \Bigg) e^{-r(s-t_k)}ds$$
$$+ e^{-r(t_{k+1}-t_k)}\Bigg(q^{k-1(\omega_{k-1}^{a_{k+1}})}[t_{k+1},x(t_{k+1})]$$
$$\left.\left. + \sum_{\ell=1}^{\varsigma_{k+1}}\lambda_{k+1}^{\ell} \sum_{h=1}^{\zeta_{(\omega_k^{a_k},\omega_{k+1}^{\ell})}} \varpi_h^{(\omega_k^{a_k},\omega_{k+1}^{\ell})} \xi^{k(\omega_k^{a_k},O)\omega_{k+1}^{\ell}[h]}(t_{k+1},x(t_{k+1})) \right) \right\}$$

subject to

$$\dot{x}(s) = f\left[s,x(s),u_k^{(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}}(s),u_k^{(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(s)\right], \quad \text{for } s\in[t_k,t_{k+1})$$
$$x(t) = x.$$

Substituting the set of cooperative strategies into (1.2) yields the dynamics of the cooperative state trajectory in the time interval $[t_k,t_{k+1})$

$$\dot{x}(s) = f\left[s,x(s),\psi_{k-1}^{(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}}(s,x(s)),\psi_k^{(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(s,x(s))\right], \tag{2.8}$$

if type $\omega_k^{a_k}$ generation k player and type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player coexist in $[t_k,t_{k+1})$, for $s\in[t_k,t_{k+1})$, $k\in\{1,2,\cdots,\upsilon\}$ and $x(t_k)=x_{t_k}\in X$.

Let $\left\{x^{(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k})*}(t)\right\}_{t=t_k}^{t_{k+1}}$ denote the cooperative solution path governed by (2.8). For simplicity in exposition we denote $x^{(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k k})*}(t)$ by $x_t^{(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k})*}$.

To fulfill group optimality, the imputation vectors have to satisfy:

$$\xi^{k-1(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}[h]}(t,x^*) + \xi^{k(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}[h]}(t,x^*) = W^{[t_k,t_{k+1}](\omega_{k-1}^{a_{k-1}},\omega_k^{a_k})}(t,x^*), \tag{2.9}$$

for $t\in[t_k,t_{k+1})$, $\omega_k^{a_k}\in\{\omega_k^1,\omega_k^2,\cdots,\omega_k^{\varsigma_k}\}$, $\omega_{k-1}^{a_{k-1}}\in\{\omega_{k-1}^1,\omega_{k-1}^2,\cdots,\omega_{k-1}^{\varsigma_{k-1}}\}$, $h\in\left\{1,2,\cdots,\zeta_{(\omega_k^{a_k},\omega_{k+1}^{a_{k-1}})}\right\}$ and $k\in\{0,1,2,\cdots,\upsilon\}$.

5.2.2 Individual Rationality

In a dynamic framework, individual rationality requires that the imputation received by a player has to be no less than his noncooperative payoff throughout the time interval in concern. Hence for individual rationality to hold along the cooperative trajectory $\left\{x^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)*}(t)\right\}_{t=t_k}^{t_{k+1}}$,

$$\begin{aligned}&\xi^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}\left(t,x_t^*\right) \geq V^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}\left(t,x_t^*\right) \text{ and}\\&\xi^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}\left(t,x_t^*\right) \geq V^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}\left(t,x_t^*\right),\end{aligned} \tag{2.10}$$

for $t\in\left[t_k,t_{k+1}\right)$, $\omega_k^{a_k}\in\left\{\omega_k^1,\omega_k^2,\cdots,\omega_k^{\varsigma_k}\right\}$, $\omega_{k-1}^{a_{k-1}}\in\left\{\omega_{k-1}^1,\omega_{k-1}^2,\cdots,\omega_{k-1}^{\varsigma_{k-1}}\right\}$, $h\in\left\{1,2,\cdots,\zeta_{\left(\omega_k^{a_k},\omega_{k+1}^{a_{k-1}}\right)}\right\}$ and $k\in\{0,1,2,\cdots,\upsilon\}$,

where x_t^* is the short form for $x_t^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)*}$.

For instance, using the results derived, an imputation vector equally dividing the excess of the cooperative payoff over the noncooperative payoff can be expressed as:

$$\begin{aligned}\xi^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}\left(t,x_t^*\right) &= V^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}\left(t,x_t^*\right)+0.5\left[W^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}\left(t,x_t^*\right)\right.\\&\quad\left.-V^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}\left(t,x_t^*\right)-V^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}\left(t,x_t^*\right)\right], \text{ and}\\\xi^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}\left(t,x_t^*\right) &= V^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}\left(t,x_t^*\right)+0.5\left[W^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}\left(t,x_t^*\right)\right.\\&\quad\left.-V^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}\left(t,x_t^*\right)-V^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}\left(t,x_t^*\right)\right].\end{aligned} \tag{2.11}$$

One can readily see that the imputations in (2.11) satisfy individual rationality and group optimality.

5.3 Subgame Consistent Solutions and Payoff Distribution

A stringent requirement for solutions of cooperative differential games to be dynamically stable is the property of subgame consistency. Under subgame consistency, an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal. In particular, when the game proceeds, at each instant of time the players are guided by the same optimality principles. According to the solution optimality principle the players agree to share their cooperative payoff according to the imputations

$$\left[\xi^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}\left(t,x_t^*\right),\ \xi^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}\left(t,x_t^*\right)\right] \tag{3.1}$$

over the time interval $[t_k, t_{k+1})$.

To achieve dynamic consistency, a payment scheme has to be derived so that imputation (3.1) will be maintained throughout the time interval $[t_k, t_{k+1})$. Following the analysis in Chap. 3, we formulate a payoff distribution procedure (PDP) over time so that the agreed imputations (3.1) can be realized. Let $B_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}(s)$ and $B_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}(s)$ denote the instantaneous payments at time $s\in[t_k, t_{k+1})$ allocated to the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ (old) player and type $\omega_k^{a_k}$ generation k (young) player.

In particular, the imputation vector can be expressed as:

$$\begin{aligned}
&\xi^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}\left(t,x_t^*\right)\\
&\quad=\int_{t_k}^{t_{k+1}} B_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}(s)\,e^{-r(s-t_k)}ds+e^{-r(t_{k+1}-t_k)}q^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}\left[t_{k+1},x^*(t_{k+1})\right]\\
&\xi^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}\left(t,x_t^*\right)=\int_{t_k}^{t_{k+1}} B_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}(s)\,e^{-r(s-t_k)}ds\\
&\qquad+e^{-r(t_{k+1}-t_k)}\sum_{\ell=1}^{\varsigma_{k+1}}\lambda_{k+1}^{\ell}\sum_{h=1}^{\varsigma\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\varpi_h^{\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\xi^{k\left(\omega_k^{a_k},O\right)\omega_{k+1}^{\ell}[h]}\left(t_{k+1},x^*(t_{k+1})\right),
\end{aligned} \tag{3.2}$$

for $k\in\{1,2,\cdots,\upsilon-1\}$, and

$$\begin{aligned}
&\xi^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}[h]}\left(t,x_t^*\right)=\int_{t_\upsilon}^{t_{\upsilon+1}} B_{\upsilon-1}^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}[h]}(s)\,e^{-r(s-t_\upsilon)}ds\\
&\quad+e^{-r(t_{\upsilon+1}-t_\upsilon)}q^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}\left[t_{\upsilon+1},x^*(t_{\upsilon+1})\right]\\
&\xi^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}[h]}\left(t,x_t^*\right)=\int_{t_\upsilon}^{t_{\upsilon+1}} B_\upsilon^{(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}[h]}(s)\,e^{-r(s-t_\upsilon)}ds\\
&+e^{-r(t_{\upsilon+1}-t_\upsilon)}q^{\upsilon(\omega_\upsilon^{a_\upsilon})}\left[t_{\upsilon+1},x^*(t_{\upsilon+1})\right].
\end{aligned} \tag{3.3}$$

Using the analysis in Chap. 2 we obtain a PDP leading to the realization of the imputation vectors in (3.2 and 3.3) in the following theorem.

Theorem 3.1 If the imputation vector $\left[\xi^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}\left(t,x_t^*\right),\ \xi^{k\left(\omega_k^{a_k},O\right)\omega_{k-1}^{a_{k-1}}[h]}\left(t,x_t^*\right)\right]$ are functions that are continuously differentiable in t and x_t^*, a PDP with an instantaneous payment at time $t\in[t_k, t_{k+1})$:

$$B_{k-1}^{(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}[h]}(t) = -\xi_t^{k-1(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}[h]}(t,x_t^*)$$
$$-\xi_x^{k-1(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}[h]}(t,x_t^*)f\left[t,x_t^*,\psi_{k-1}^{(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}}(t,x_t^*),\psi_k^{(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(t,x_t^*)\right] \tag{3.4}$$

allocated to the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player;

and an instantaneous payment at time $t \in [t_k, t_{k+1})$:

$$B_k^{(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}[h]}(t) = -\xi_t^{k(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}[h]}(t,x_t^*)$$
$$-\xi_x^{k(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}[h]}(t,x_t^*)f\left[t,x_t^*,\psi_{k-1}^{(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}}(t,x_t^*),\psi_k^{(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}}(t,x_t^*)\right]$$

allocated to the type $\omega_k^{a_k}$ generation k player,

yields a mechanism leading to the realization of the imputation vector

$$\left[\xi^{k-1(\omega_{k-1}^{a_{k-1}},O)\omega_k^{a_k}[h]}(t,x_t^*),\ \xi^{k(\omega_k^{a_k},Y)\omega_{k-1}^{a_{k-1}}[h]}(t,x_t^*)\right],\ \text{for}\, k \in \{1,2,\cdots,v\}.$$

Proof Follow the proof leading to Theorem 3.1 in Chap. 2 with the imputation vector in present value (rather than in current value). ■

5.4 An Illustration in Resource Extraction

Consider the game in which there are 4 overlapping generations of players with generation 0 and generation 1 players in $[t_1, t_2)$, generation 1 and generation 2 players in $[t_2, t_3)$, generation 2 and generation 3 players in $[t_3, t_4]$. Players are of either type 1 or type 2. The instantaneous payoffs and terminal rewards of the type 1 generation k player and the type 2 generation k player are respectively:

$$\left[(u_k)^{1/2} - \frac{c_1}{x^{1/2}}u_k\right] \text{ and}\, q_1 x^{1/2}, \quad \text{and } \left[(u_k)^{1/2} - \frac{c_2}{x^{1/2}}u_k\right] \text{ and}\, q_2 x^{1/2}, \tag{4.1}$$

where the state variable $x(s)$ is the biomass of a renewable resource. $u_k(s)$ is the harvest of the generation k extraction firm. The type $i \in \{1,2\}$ generation k extraction firm's extraction cost is $c_i u_k(s) x(s)^{-1/2}$.

At initial time t_1, it is known that the generation 0 player is of type 1 and the generation 1 player is also of type 1. It is also known that the generation 2 and generation 3 players may be of type 1 with probability $\lambda_k^1 = 0.4$ and of type 2 with probability $\lambda_k^2 = 0.6$ in time interval $[t_k, t_{k+1})$ for $k \in \{2,3\}$.

The state dynamics of the game is characterized by:

$$\dot{x}(s) = ax(s)^{1/2} - bx(s) - u_k^{(i,O)j}(s) - u_k^{(j,Y)i}(s), \tag{4.2}$$

if the old generation $k-1$ extractor is of type i and the young generation k extractor is of type j, for $s \in [t_k, t_{k+1})$ and $k \in \{1,2,3\}$;

$$x(t_1) = x_0 \in X \subset R,$$

where $u_k^{(i,O)j}(s)$ denote the harvest of the type i generation $k-1$ old extractor and $u_k^{(j,Y)i}(s)$ denote the harvest of the type j generation k young extractor.

The death rate of the resource is b. The rate of growth is $a/x^{1/2}$ which reflects the decline in the growth rate as the biomass increases. The game is an asynchronous horizons version of the synchronous-horizon resource extraction game in Yeung and Petrosyan (2006b).

This asynchronous horizon game can be expressed as follows. In the time interval $[t_3, t_4]$, consider the case with a type $i \in \{1,2\}$ generation 2 firm and a type $j \in \{1,2\}$ generation 3 firm, the game becomes

$$\begin{aligned}
&\max_{u_3^{(i,O)j}} \left\{ \int_{t_3}^{t_4} \left[\left[u_3^{(i,O)j}(s) \right]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_3^{(i,O)j}(s) \right] \exp[-r(s-t_3)]\, ds \right. \\
&\qquad \left. + \exp[-r(t_4-t_3)] q_i x(t_4)^{\frac{1}{2}} \right\}, \\
&\max_{u_3^{(j,Y)i}} \left\{ \int_{t_3}^{t_4} \left[\left[u_3^{(j,Y)i}(s) \right]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_2^{(j,Y)i}(s) \right] \exp[-r(s-t_3)]\, ds \right. \\
&\qquad \left. + \exp[-r(t_4-t_3)] q_j x(t_4)^{\frac{1}{2}} \right\},
\end{aligned} \tag{4.3}$$

subject to (4.2).

In the time interval $[t_k, t_{k+1})$, for $k \in \{1,2\}$, consider the case with a type $i \in \{1,2\}$ generation $k-1$ firm and a type $j \in \{1,2\}$ generation k firm, the game becomes

$$\begin{aligned}
&\max_{u_k^{(i,O)j}} \left\{ \int_{t_k}^{t_{k+1}} \left[\left[u_k^{(i,O)j}(s) \right]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_k^{(i,O)j}(s) \right] \exp[-r(s-t_k)]\, ds \right. \\
&\quad \left. + \exp[-r(t_{k+1}-t_k)] q_i x(t_{k+1})^{\frac{1}{2}} \right\}, \\
&\max_{u_k^{(j,Y)i},\, u_{k+1}^{(j,O)\ell}} \left\{ \int_{t_k}^{t_{k+1}} \left[\left[u_k^{(j,Y)i}(s) \right]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_k^{(j,Y)i}(s) \right] \exp[-r(s-t_k)]\, ds \right. \\
&\quad + \sum_{\ell=1}^{2} \lambda_{k+1}^{\ell} \int_{t_{k+1}}^{t_{k+2}} \left[\left[u_{k+1}^{(j,O)\ell}(s) \right]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_{k+1}^{(j,O)\ell}(s) \right] \exp[-r(s-t_k)]\, ds \\
&\quad \left. + \exp[-r(t_{k+2}-t_k)] q_j x(t_{k+2})^{\frac{1}{2}} \right\},
\end{aligned} \tag{4.4}$$

subject to (4.2).

5.4.1 Noncooperative Outcomes

In this section we first characterize the noncooperative outcome of the asynchronous horizons game (4.2, 4.3 and 4.4) as follows.

Proposition 4.1 The feedback Nash equilibrium payoffs for the type $i\in\{1,2\}$ generation $k-1$ firm and the type $j\in\{1,2\}$ generation k firm coexisting in the game interval $\left[t_k, t_{k+1}\right)$ can be obtained as:

$$V^{k-1(i,O)j}(t,x) = \exp[-r(t-t_k)]\left[A_{k-1}^{(i,O)j}(t)x^{1/2} + C_{k-1}^{(i,O)j}(t)\right], \text{ and}$$
$$V^{k(j,Y)i}(t,x) = \exp[-r(t-t_k)]\left[A_{k}^{(j,Y)i}(t)x^{1/2} + C_{k}^{(j,Y)i}(t)\right], \tag{4.5}$$

for $k\in\{1,2,3\}$ and $i,j\in\{1,2\}$,

where

$A_{k-1}^{(i,O)j}(t)$, $C_{k-1}^{(i,O)j}(t)$, $A_{k}^{(j,Y)i}(t)$ and $C_{k}^{(j,Y)i}(t)$ satisfy:

$$\dot{A}_{k-1}^{(i,O)j}(t) = \left[r+\frac{b}{2}\right] A_{k-1}^{(i,O)j}(t) - \frac{1}{2\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]}$$
$$+\frac{c_i}{4\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2} + \frac{A_{k-1}^{(i,O)j}(t)}{8\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2} + \frac{A_{k-1}^{(i,O)j}(t)}{8\left[c_j + A_{k}^{(j,Y)i}(t)/2\right]^2},$$
$$\dot{C}_{k-1}^{(i,O)j}(t) = rC_{k-1}^{(i,O)j}(t) - \frac{a}{2}A_{k-1}^{(i,O)j}(t);$$
$$A_{k-1}^{(i,O)j}(t_{k+1}) = q_i \text{ and } C_{k-1}^{(i,O)j}(t_{k+1}) = 0, \quad \text{for } k\in\{1,2,3\}; \tag{4.6}$$
$$\dot{A}_{k}^{(j,Y)i}(t) = \left[r+\frac{b}{2}\right]A_{k}^{(j,Y)i}(t) - \frac{1}{2\left[c_j + A_{k}^{(j,Y)i}(t)/2\right]} + \frac{c_j}{4\left[c_j + A_{k}^{(j,Y)i}(t)/2\right]^2}$$
$$+\frac{A_{k}^{(j,Y)i}(t)}{8\left[c_j + A_{k}^{(j,Y)i}(t)/2\right]^2} + \frac{A_{k}^{(j,Y)i}(t)}{8\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2}$$
$$\dot{C}_{k}^{(j,Y)i}(t) = rC_{k}^{(j,Y)i}(t) - \frac{a}{2}A_{k}^{(j,Y)i}(t), \text{ for } k\in\{1,2,3\};$$
$$A_{k}^{(j,Y)i}(t_{k+1}) = e^{-r(t_{k+1}-t_k)}\sum_{\ell=1}^{2}\lambda_{k+1}^{\ell}A_{k+1}^{(j,O)\ell}(t_{k+1}) \text{ and}$$
$$C_{k}^{(j,Y)i}(t_{k+1}) = e^{-r(t_{k+1}-t_k)}\sum_{\ell=1}^{2}\lambda_{k+1}^{\ell}C_{k+1}^{(j,O)\ell}(t_{k+1}),$$
$$\text{for } k\in\{1,2\}, \text{ and } A_{3}^{(j,Y)i}(t_4) = q_j \text{ and } C_{3}^{(j,Y)i}(t_4) = 0. \tag{4.7}$$

Proof Performing the indicated maximization in (4.4) and solving the system yield (4.5). Hence Proposition 4.1 follows. ■

The solution time paths $A_{k-1}^{(i,O)j}(t)$, $C_{k-1}^{(i,O)j}(t)$, $A_k^{(j,Y)i}(t)$ and $C_k^{(j,Y)i}(t)$ for the system of first order differential equations in (4.6) and (4.7) can be computed numerically for given values of the model parameters $r, q_1, q_2, c_1, c_2, a, b, \lambda_k^1$ and λ_k^2.

The game equilibrium strategies then can be expressed as:

$$\phi_k^{(i,O)j}(t,x) = \frac{x}{4\left[c_i + A_k^{(i,O)j}(t)/2\right]^2} \text{ and } \phi_k^{(j,Y)i}(t,x) = \frac{x}{4\left[c_j + A_k^{(j,Y)i}(t)/2\right]^2}.$$

5.4.2 *Dynamic Cooperation*

Now consider the case when coexisting firms want to cooperate and agree to act and allocate the cooperative payoff according to a set of agreed upon optimality principles. Let there be three acceptable imputations for the extractor firms.

Imputation I: the firms would share the excess gain from cooperation equally with weights $w_k^{o(1)} = 0.5$ for the generation $k-1$ firm and $w_k^{Y(1)} = 0.5$ for the generation k firm.

Imputation II: the generation $k-1$ firm acquires $w_k^{o(2)} = 0.6$ of the excess gain from cooperation and the generation k firm acquires $w_k^{Y(2)} = 0.4$ of the excess gain.

Imputation III: the generation $k-1$ firm acquires $w_k^{o(3)} = 0.4$ of the excess gain from cooperation and the generation k firm acquires $w_k^{Y(3)} = 0.6$ of the excess gain.

In time interval $[t_k, t_{k+1})$, if both the generation $k-1$ firm and the generation k firm are of type 1, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_k^{(1,1)1} = 0.8$, $\varpi_k^{(1,1)2} = 0.1$ and $\varpi_k^{(1,1)3} = 0.1$, for $k \in \{2,3\}$.

If both the generation $k-1$ firm and the generation k firm are of type 2, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_k^{(2,2)1} = 0.7$, $\varpi_k^{(2,2)2} = 0.15$ and $\varpi_k^{(2,2)3} = 0.15$, for $k \in \{2,3\}$.

If the generation $k-1$ firm is of type 1 and the generation k firm are of type 2, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_k^{(1,2)1} = 0.15$, $\varpi_k^{(1,2)2} = 0.75$ and $\varpi_k^{(1,2)3} = 0.1$, for $k \in \{2,3\}$.

If the generation $k-1$ firm is of type 2 and the generation k firm are of type 1, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_k^{(2,1)1} = 0.15$, $\varpi_k^{(2,1)2} = 0.1$ and $\varpi_k^{(2,1)3} = 0.75$, for $k \in \{2,3\}$.

At initial time t_1, the type 1 generation 0 firm and the type 1 generation 1 firm are assumed to have agreed to Imputation II.

Since payoffs are transferable, group optimality requires the firms coexisting in the same time interval to maximize their joint payoff. Consider the last time interval $[t_3, t_4]$, in which the generation 2 firm is of type $i \in \{1, 2\}$ and the generation 3 firm is of type $j \in \{1, 2\}$. The firms maximize their joint profit

$$\left\{ \int_{t_3}^{t_4} \left[\left[u_3^{(i,O)j}(s) \right]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_3^{(i,O)j}(s) \right] \exp[-r(s-t_3)]\, ds \right.$$
$$+ \int_{t_3}^{t_4} \left[\left[u_3^{(j,O)i}(s) \right]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_3^{(j,O)i}(s) \right] \exp[-r(s-t_3)]\, ds$$
$$\left. + \exp[-r(t_4 - t_3)] q_i x(t_4)^{\frac{1}{2}} + \exp[-r(t_4 - t_3)] q_j x(t_4)^{\frac{1}{2}} \right\}, \quad (4.8)$$

subject to (4.2).

The maximized joint payoffs of the players in the last subgame interval can be characterized by the proposition below.

Proposition 4.2 The maximized joint payoff with type $i \in \{1, 2\}$ generation 2 firm and the type $j \in \{1, 2\}$ generation 3 firm coexisting in the game interval $[t_3, t_4)$ can be obtained as:

$$W^{[t_3,t_4](i,j)}(t, x) = \exp[-r(t - t_3)] \left[A^{[t_3,t_4](i,j)}(t) x^{1/2} + C^{[t_3,t_4](i,j)}(t) \right], \quad (4.9)$$

where $A^{[t_3,t_4](i,j)}(t)$ and $C^{[t_3,t_4](i,j)}(t)$ satisfy:

$$\dot{A}^{[t_3,t_4](i,j)}(t) = \left[r + \frac{b}{2} \right] A^{[t_3,t_4](i,j)}(t) - \frac{1}{2\left[c_i + A^{[t_3,t_4](i,j)}(t)/2\right]}$$
$$- \frac{1}{2\left[c_j + A^{[t_3,t_4](i,j)}(t)/2\right]} + \frac{c_i}{4\left[c_i + A^{[t_3,t_4](i,j)}(t)/2\right]^2}$$
$$+ \frac{c_j}{4\left[c_j + A^{[t_3,t_4](i,j)}(t)/2\right]^2}$$
$$+ \frac{A^{[t_3,t_4](i,j)}(t)}{8\left[c_i + A^{[t_3,t_4](i,j)}(t)/2\right]^2} + \frac{A^{[t_3,t_4](i,j)}(t)}{8\left[c_j + A^{[t_3,t_4](i,j)}(t)/2\right]^2}$$
$$\dot{C}^{[t_3,t_4](i,j)}(t) = r C^{[t_3,t_4](i,j)}(t) - \frac{a}{2} A^{[t_3,t_4](i,j)}(t),$$
$$A^{[t_3,t_4](i,j)}(t_4) = q_i + q_j \text{ and } C^{[t_3,t_4](i,j)}(t_4) = 0. \quad (4.10)$$

Proof Invoking the dynamic programming techniques in Theorem A.1 in the Technical Appendices one can obtain (4.9 and 4.10). ■

The solution time paths $A^{[t_3,t_4](i,j)}(t)$ and $C^{[t_3,t_4](i,j)}(t)$ for the system of first order differential equations in (4.9 and 4.10) can be computed numerically for given values of the model parameters r, q_1, q_2, c_1, c_2, a and b.

In the game interval $[t_3, t_4)$ with type $i \in \{1,2\}$ generation 2 firm and the type $j \in \{1,2\}$ generation 3 firm coexisting, if imputation $h \in \{1,2,3\}$ is chosen the imputations of the firms under cooperation can be expressed as:

$$\begin{aligned}
&\xi^{2(i,O)j[h]}(t,x) = V^{2(i,O)j}(t,x) + w_3^{o(h)}\left[W^{[t_3,t_4](i,j)}(t,x) - V^{2(i,O)j}(t,x)\right.\\
&\left.- V^{3(j,Y)i}(t,x)\right],\\
&\xi^{3(j,Y)i[h]}(t,x) = V^{3(j,Y)i}(t,x) + w_3^{Y(h)}\left[W^{[t_3,t_4](i,j)}(t,x) - V^{2(i,O)j}(t,x)\right.\\
&\left.-V^{3(j,Y)i}(t,x)\right].
\end{aligned} \tag{4.11}$$

Now we proceed to the second last interval $[t_k, t_{k+1})$ for $k = 2$. Consider the case in which the generation k firm is of type $j \in \{1,2\}$ and the generation $k-1$ firm is known to be of type $i = 2$. Following the analysis in (2.4) and (2.5), the expected terminal reward to the type j generation k firm at time t_{k+1} can be expressed as:

$$\sum_{\ell=1}^{2} \lambda_k^{\ell} \sum_{h=1}^{3} \varpi_h^{(j,\ell)} \xi^{k(j,O)\ell[h]}(t_{k+1}, x), \quad \text{for } k = 2. \tag{4.12}$$

A review of Proposition 4.1, Proposition 4.2 and (4.11) shows the term in (4.12) can be written as:

$$A_k^{\zeta(j,O)} x^{1/2} + C_k^{\zeta(j,O)}, \tag{4.13}$$

where $A_k^{\zeta(j,O)}$ and $C_k^{\zeta(j,O)}$ are constant terms.

The joint maximization problem in the time interval $[t_k, t_{k+1})$, for $k \in \{1,2\}$, involving the type j generation k player and type i generation $k-1$ player can be expressed as:

$$\begin{aligned}
&\max_{u_k^{(i,O)j},\, u_k^{(j,Y)i}} \left\{ \int_{t_k}^{t_{k+1}} \left[\left[u_k^{(i,O)j}(s)\right]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_k^{(i,O)j}(s) \right] \exp[-r(s-t_k)]\, ds \right.\\
&\quad + \int_{t_k}^{t_{k+1}} \left[\left[u_k^{(j,Y)i}(s)\right]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_k^{(j,Y)i}(s) \right] \exp[-r(s-t_k)]\, ds\\
&\quad \left. + \exp[-r(t_{k+1}-t_k)] \left[q_i x(t_{k+1})^{\frac{1}{2}} + \sum_{\ell=1}^{2} \lambda_k^{\ell} \sum_{h=1}^{3} \varpi_h^{(j,\ell)} \xi^{k(j,O)\ell[h]}(t_{k+1}, x) \right] \right\},
\end{aligned} \tag{4.14}$$

subject to (4.2).

The maximized joint payoff of the players in the first two subgame intervals can be characterized by the proposition below.

Proposition 4.3 The maximized joint payoff with type $i \in \{1,2\}$ generation $k-1$ firm and the type $j \in \{1,2\}$ generation k firm coexisting in the game interval $\left[t_k, t_{k+1}\right)$, for $k \in \{1,2\}$, can be obtained as:

$$W^{[t_k,t_{k+1}](i,j)}(t,x) = \exp[-r(t-t_k)]\left[A^{[t_k,t_{k+1}](i,j)}(t)x^{1/2} + C^{[t_k,t_{k+1}](i,j)}(t)\right], \quad (4.15)$$

where $A^{[t_k,t_{k+1}](i,j)}(t)$ and $C^{[t_k,t_{k+1}](i,j)}(t)$ satisfy:

$$\begin{aligned}
\dot{A}^{[t_k,t_{k+1}](i,j)}(t) &= \left[r + \frac{b}{2}\right]A^{[t_k,t_{k+1}](i,j)}(t) - \frac{1}{2\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]} \\
&\quad - \frac{1}{2\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]} + \frac{c_i}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} \\
&\quad + \frac{c_j}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} \\
&\quad + \frac{A^{[t_k,t_{k+1}](i,j)}(t)}{8\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} + \frac{A^{[t_k,t_{k+1}](i,j)}(t)}{8\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} \\
\dot{C}^{[t_k,t_{k+1}](i,j)}(t) &= rC^{[t_k,t_{k+1}](i,j)}(t) - \frac{a}{2}A^{[t_k,t_{k+1}](i,j)}(t), \\
A^{[t_k,t_{k+1}](i,j)}(t_{k+1}) &= q_i + A_k^{\zeta(j,O)} \text{ and } C^{[t_k,t_{k+1}](i,j)}(t_{k+1}) = C_k^{\zeta(j,O)}.
\end{aligned} \quad (4.16)$$

Proof Performing the maximization operator in (4.14) and invoking (4.13) one can obtain the results in (4.15) and (4.16). ■

The solution time paths $A^{[t_k,t_{k+1}](i,j)}(t)$ and $C^{[t_k,t_{k+1}](i,j)}(t)$ for the system of first order differential equations in (4.16) can be computed numerically for given values of the model parameters r, q_1, q_2, c_1, c_2, a, b, λ_k^1, λ_k^2, and $\varpi_h^{(j,\ell)}$ for $h \in \{1,2,3\}$ and $j,\ell \in \{1,2\}$.

The optimal cooperative controls can then be obtained as:

$$\begin{aligned}
\psi_{k-1}^{(i,O)j}(t,x) &= \frac{x}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}, \text{ and} \\
\psi_k^{(j,Y)i}(t,x) &= \frac{x}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}.
\end{aligned} \quad (4.17)$$

Substituting these control strategies into (4.2) yields the dynamics of the state trajectory under cooperation. The optimal cooperative state trajectory in the time interval $[t_k, t_{k+1})$ can be obtained as:

$$x^{(i,j)*}(t) = \left[\Omega_{(i,j)}(t_k,t)\right]^2 \left[(x_{t_k})^{1/2} + \int_{t_k}^{t} \Omega_{(i,j)}^{-1}(t_k,s)\frac{a}{2}ds \right]^2, \tag{4.18}$$

where $\Omega_{(i,j)}(t_k,t) = \exp\left[\int_{t_k}^{t} H_{(i,j)}(\upsilon)d\upsilon\right]$ and

$$H_{(i,j)}(s) = -\left[\frac{b}{2} + \frac{1}{8\left[c_i + A^{[t_k,t_{k+1}](i,j)}(s)/2\right]^2} + \frac{1}{8\left[c_j + A^{[t_k,t_{k+1}](i,j)}(s)/2\right]^2} \right]$$

The term x_t^* is used to denote $x^{(i,j)*}(t)$ whenever there is no ambiguity.

5.4.3 *Dynamically Consistent Payoff Distribution*

According to the solution optimality principle the players agree to share their cooperative payoff according to the solution imputations:

$$\xi^{k-1(i,O)j[h]}(t,x) = V^{k-1(i,O)j}(t,x) + w_{k-1}^{h}\left[W^{[t_k,t_{k+1}](i,j)}(t,x) - V^{k-1(i,O)j}(t,x) - V^{k(j,Y)i}(t,x)\right],$$
$$\xi^{k(j,Y)i[h]}(t,x) = V^{k(j,Y)i}(t,x) + w_{k}^{h}\left[W^{[t_k,t_{k+1}](i,j)}(t,x) - V^{k-1(i,O)j}(t,x) - V^{k(j,Y)i}(t,x)\right],$$

for $h \in \{1,2,3\}$, $i,j \in \{1,2\}$ and $k \in \{1,2,3\}$.

These imputations are continuous differentiable in x and t. If an imputation vector $\left[\xi^{k-1(i,O)j[h]}(t,x),\ \xi^{k(j,Y)i[h]}(t,x)\right]$ is chosen, a crucial process is to derive a payoff distribution procedure (PDP) so that this imputation could be realized for $t \in [t_k, t_{k+1})$ along the cooperative trajectory $\{x_t^*\}_{t=t_k}^{t_{k+1}}$.

Following Theorem 3.1, a PDP leading to the realization of the imputation vector $\left[\xi^{k-1(i,O)j[h]}(t,x),\ \xi^{k(j,Y)i[h]}(t,x)\right]$ can be obtained as:

Corollary 4.1 A PDP with an instantaneous payment at time $t \in [t_k, t_{k+1})$:

$$B_k^{(i,O)j[h]}(t) = -\xi_t^{k-1(i,O)j[h]}(t,x_t^*) - \xi_x^{k-1(i,O)j[h]}(t,x_t^*)\left[a(x_t^*)^{1/2} - bx_t^* - \frac{x_t^*}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} - \frac{x_t^*}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} \right], \tag{4.19}$$

allocated to the type i generation $k-1$ player;

and an instantaneous payment at time $t \in [t_k, t_{k+1})$:

$$B_k^{(j,Y)i[h]}(t) = -\xi_t^{k(j,Y)i[h]}(t, x_t^*) - \xi_x^{k(j,Y)i[h]}(t, x_t^*)\Bigg[a(x_t^*)^{1/2} - bx_t^* \\ - \frac{x_t^*}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} - \frac{x_t^*}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}\Bigg] \tag{4.20}$$

allocated to the type j generation k player,
yields a mechanism leading to the realization of the imputation vector

$\left[\xi^{k-1(i,O)j[h]}(t,x),\ \xi^{k(j,Y)i[h]}(t,x)\right]$, for $k \in \{1,2,3\}$, $h \in \{1,2,3\}$ and $i,j \in \{1,2\}$. ■

Since the imputations $\xi^{k-1(i,O)j[h]}(t,x)$ and $\xi^{k(j,Y)i[h]}(t,x)$ are in terms of explicit differentiable functions, the relevant derivatives can be derived using the results in Propositions 4.1, 4.2 and 4.3. Hence, the PDP $B_k^{(i,O)j[h]}(t)$ and $B_k^{(j,Y)i[h]}(t)$ in (4.19) and (4.20) can be obtained explicitly.

5.5 Extension to Stochastic Dynamics

In this Section we extend the analysis to the case where the state dynamics is stochastic and governed by the stochastic differential equations:

$$dx(s) = f\left[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)\right]ds + \sigma[s,x(s)]dz(s), \\ x(t_1) = x_0 \in X, \tag{5.1}$$

for $s \in [t_k, t_{k+1})$, if the type ω_{a_k} generation k player and the type $\omega_{a_{k-1}}$ generation a_{k-1} player coexist in the time interval $[t_k, t_{k+1})$ for $k \in \{1,2,3,\cdots,\upsilon\}$, and where $\sigma[s, x(s)]$ is a $n \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process. Let $\Omega[s,x(s)] = \sigma[s,x(s)]\,\sigma[s,x(s)]'$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s,x(s)]$.

5.5.1 *Noncooperative Outcomes and Joint Maximization*

Following the analysis in Sect. 5.1 of this Chapter and Sect. 3.1 of Chap. 3 a counterpart of Lemma 1.2 characterizing the noncooperative outcomes of the game the stochastic dynamic problem (1.3, 1.4, 1.5, 1.6 and 5.1) can be obtained as Lemma 5.1 below.

Lemma 5.1 A set of feedback strategies $\left\{\phi_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t,x);\ \phi_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x)\right\}$ constitutes a Nash equilibrium solution for the game (1.3, 1.4, 1.5, 1.6 and 5.1), if there exist continuously differentiable functions $V^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t,x):[t_k,t_{k+1}]\times R^m\to R$ and $V^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x):[t_k,t_{k+1}]\times R^m\to R$ satisfying the following partial differential equations:

$$
\begin{aligned}
&-V_t^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)\,V_{x^hx^\zeta}^{\upsilon(\omega_\upsilon^{a_\upsilon},O)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x)\\
&\quad=\max_{u_\upsilon^Y}\left\{g^{\upsilon(\omega_\upsilon^{a_\upsilon})}\left[t,x,\phi_{\upsilon-1}^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(t,x),u_\upsilon^Y\right]e^{-r(t-t_\upsilon)}\right.\\
&\qquad\left.+V_x^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t,x)f\left[t,x,\phi_{\upsilon-1}^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(t,x),u_\upsilon^Y\right]\right\},\\
&V^{\upsilon(\omega_\upsilon^{a_\upsilon},Y)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(t_{\upsilon+1},x)=e^{-r(t_{\upsilon+1}-t_\upsilon)}q^{\upsilon(\omega_\upsilon^{a_\upsilon})}[t_{\upsilon+1},x(t_{\upsilon+1})];\ \text{and}\\
&-V_t^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)\,V_{x^hx^\zeta}^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t,x)\\
&\quad=\max_{u_k^O}\left\{g^{k-1(\omega_{k-1})}\left[t,x,u_k^O,\phi_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x)\right]e^{-r(t-t_k)}\right.\\
&\qquad\left.+V_x^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t,x)f\left[t,x,u_k^O,\phi_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x)\right]\right\}\\
&V^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t_{k+1},x)=e^{-r(t_{k+1}-t_k)}q^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}(t_{k+1},x),\ \text{and}\\
&-V_t^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)\,V_{x^hx^\zeta}^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x)\\
&\quad=\max_{u_k^Y}\left\{g^{(k,\omega_k)}\left[t,x,\phi_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}},u_k^Y\right]e^{-r(t-t_k)}\right.\\
&\qquad\left.+V_x^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x)\,f\left[t,x,\phi_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}},u_k^Y\right]\right\}\\
&V^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t_{k+1},x)=e^{-r(t_{k+1}-t_k)}\sum_{\ell=1}^{\varsigma_{k+1}}\lambda_{k+1}^{\ell}\,V^{k\left(\omega_k^{a_k},O\right)\omega_{k+1}^{\ell}}(t_{k+1},x),
\end{aligned}
\tag{5.2}
$$

for $k\in\{1,2,\cdots,\upsilon-1\}$.

Proof Follow the proof of Theorem 1.1 in Chap. 3. ■

Now consider the case when coexisting players want to cooperate and maximize their joint expected payoff. Following the analysis in Sect. 5.2, the joint maximization problem in the time interval $\left[t_\upsilon, t_{\upsilon+1}\right)$ involving type $\omega_\upsilon^{a_\upsilon}$ generation υ player and type $\omega_{\upsilon-1}^{a_{\upsilon-1}}$ generation $\upsilon-1$ player can be expressed as the expected joint payoff

$$E_{t_\upsilon}\left\{\int_{t_\upsilon}^{t_{\upsilon+1}}\left(g^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}\left[s, x(s), u_{\upsilon-1}^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(s), u_\upsilon^{\left(\omega_\upsilon^{a_\upsilon},Y\right)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(s)\right]\right.\right.$$
$$+\left.g^{\upsilon\left(\omega_\upsilon^{a_\upsilon}\right)}\left[s, x(s), u_{\upsilon-1}^{\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},O\right)\omega_\upsilon^{a_\upsilon}}(s), u_\upsilon^{\left(\omega_\upsilon^{a_\upsilon},Y\right)\omega_{\upsilon-1}^{a_{\upsilon-1}}}(s)\right]\right)e^{-r(s-t_\upsilon)}ds$$
$$\left.+e^{-r(t_{\upsilon+1}-t_\upsilon)}\left(q^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}[t_{\upsilon+1}, x(t_{\upsilon+1})]+q^{\upsilon\left(\omega_\upsilon^{a_\upsilon}\right)}[t_{\upsilon+1}, x(t_{\upsilon+1})]\right)\right\}, \tag{5.3}$$

subject to (5.1).

The joint maximization problem in the time interval $\left[t_k, t_{k+1}\right)$, for $k\in\{1, 2, \cdots, \upsilon-1\}$, involving the type $\omega_k^{a_k}$ generation k player and type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player can be expressed as the maximization of the expected joint payoff:

$$E_{t_k}\left\{\int_{t_k}^{t_{k+1}}\left(g^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}\left[s, x(s), u_k^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(s), u_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right]\right.\right.$$
$$+\left.g^{k\left(\omega_k^{a_k}\right)}\left[s, x(s), u_k^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(s), u_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right]\right)e^{-r(s-t_k)}ds$$
$$+e^{-r(t_{k+1}-t_k)}\left(q^{k-1\left(\omega_{k-1}^{a_{k+1}}\right)}[t_{k+1}, x(t_{k+1})]\right.$$
$$\left.\left.+\sum_{\ell=1}^{\varsigma_{k+1}}\lambda_{k+1}^{\ell}\sum_{h=1}^{\varsigma\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\varpi_h^{\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\xi^{k\left(\omega_k^{a_k},O\right)\omega_{k+1}^{\ell}[h]}(t_{k+1}, x(t_{k+1}))\right)\right\}, \tag{5.4}$$

subject to (5.1).

Following the analysis in Sect. 5.2 a counterpart of Theorem 2.1 characterizing an optimal solution of the problem of maximizing (5.3) and (5.4) subject to (5.1) can be obtained as Theorem 5.1 below.

Theorem 5.1 A set of controls $\left\{\psi_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(t,x);\ \psi_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(t,x)\right\}$ constitutes an optimal solution for the control problem (5.1, 5.3 and 5.4), if there exist continuously differentiable function $W^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}(t,x):\left[t_k, t_{k+1}\right)\times R^m\to R$ satisfying the following partial differential equations:

$$-W_t^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)\,W_{x^h x^\zeta}^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t,x)$$
$$=\max_{u_\upsilon^O,\,u_\upsilon^Y}\left\{g^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}\left[t,x,u_\upsilon^O,u_\upsilon^Y\right]e^{-r(t-t_\upsilon)}\right.$$
$$\left.+\,g^{\upsilon(\omega_\upsilon^{a_\upsilon})}\left[t,x,u_\upsilon^O,u_\upsilon^Y\right]e^{-r(t-t_\upsilon)}+W_x^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t,x)\,f\left[t,x,u_\upsilon^O,u_\upsilon^Y\right]\right\},$$
$$W^{[t_\upsilon,t_{\upsilon+1}]\left(\omega_{\upsilon-1}^{a_{\upsilon-1}},\omega_\upsilon^{a_\upsilon}\right)}(t_{\upsilon+1},x)=e^{-r(t_{\upsilon+1}-t_\upsilon)}\left[q^{\upsilon-1\left(\omega_{\upsilon-1}^{a_{\upsilon-1}}\right)}(t_{\upsilon+1},x)+q^{\upsilon\left(\omega_\upsilon^{a_\upsilon}\right]}(t_{\upsilon+1},x)\right];\ \text{and}$$
$$-W_t^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)\,W_{x^h x^\zeta}^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}(t,x)$$
$$=\max_{u_k^O,\,u_\upsilon^Y}\left\{g^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}\left[t,x,u_k^O,u_k^Y\right]e^{-r(t-t_k)}\right.$$
$$\left.+\,g^{k\left(\omega_k^{a_k}\right)}\left[t,x,u_k^O,u_k^Y\right]e^{-r(t-t_k)}+W_x^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}(t,x)\,f\left[t,x,u_k^O,u_k^Y\right]\right\},$$
$$W^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}(t_{k+1},x)=e^{-r(t_{k+1}-t_k)}\Bigg(q^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}(t_{k+1},x),$$
$$+\sum_{\ell=1}^{\varsigma_{k+1}}\lambda_{k+1}^{\ell}\sum_{h=1}^{\varsigma\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\varpi_h^{\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\xi^{k\left(\omega_k^{a_k},O\right)\omega_{k+1}^{\ell}[h]}(t_{k+1},x(t_{k+1}))\Bigg),$$
$$\text{for } k\in\{1,2,\cdots,\upsilon-1\}. \tag{5.5}$$

Proof Follow the proof of Theorem A.3 in the Technical Appendices we obtain the conditions characterizing an optimal solution of the problem (5.1), (5.3) and (5.4) as in (5.5). ∎

In particular, $W^{[t_k,t_{k+1}]\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}(t,x)$ gives the maximized expected joint payoff of the type $\omega_k^{a_k}$ generation k player and type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player at time $t\in[t_k,t_{k+1}]$ with the state x in the stochastic control problem

$$\max_{u_k^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}},\,u_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}}$$
$$E_{t_k}\Bigg\{\int_t^{t_{k+1}}\Bigg(g^{k-1\left(\omega_{k-1}^{a_{k-1}}\right)}\left[s,x(s),u_k^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(s),u_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right]$$
$$+\,g^{k\left(\omega_k^{a_k}\right)}\left[s,x(s),u_k^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(s),u_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(s)\right]\Bigg)e^{-r(s-t_k)}ds$$
$$+\,e^{-r(t_{k+1}-t_k)}\Bigg(q^{k-1\left(\omega_{k-1}^{a_{k+1}}\right)}[t_{k+1},x(t_{k+1})]$$
$$+\sum_{\ell=1}^{\varsigma_{k+1}}\lambda_{k+1}^{\ell}\sum_{h=1}^{\varsigma\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\varpi_h^{\left(\omega_k^{a_k},\omega_{k+1}^{\ell}\right)}\xi^{k\left(\omega_k^{a_k},O\right)\omega_{k+1}^{\ell}[h]}(t_{k+1},x(t_{k+1}))\Bigg)\Bigg\},$$

subject to

$$dx(s) = f\left[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)\right]ds + \sigma[s,x(s)]dz(s),\ x(t) = x.$$

Substituting the set of cooperative strategies into (5.1) yields the dynamics of the cooperative state trajectory in the time interval $\left[t_k, t_{k+1}\right)$

$$\begin{aligned} dx(s) = f&\left[s, x(s), \psi_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}(s,x(s)), \psi_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}(s,x(s))\right]ds \\ &+ \sigma[s,x(s)]dz(s) \end{aligned} \qquad (5.6)$$

if type $\omega_k^{a_k}$ generation k player and type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player coexist in $\left[t_k, t_{k+1}\right)$, for $s \in \left[t_k, t_{k+1}\right)$, $k \in \{1, 2, \cdots, \upsilon\}$ and $x(t_k) = x_{t_k} \in X$.

We denote the set of realizable states at time t from (5.6) under the scenarios of different players by $X_t^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)*}$, for $t \in \left[t_k, t_{k+1}\right)$ and $k \in \{1, 2, \cdots, \upsilon\}$. We use the term $x_t^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)*}$ by $x_t^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)*} \in X_t^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)*}$ to denote an element in $X_t^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)*}$. The term x_t^* is used to denote $x_t^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)*}$ whenever there is no ambiguity. For simplicity in exposition we also use $x^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)*}(t)$ and $x_t^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)*}$ inter-changeably.

5.5.2 Subgame Consistent Solutions and Payoff Distribution

Now consider the case when coexisting players want to cooperate and agree to act and allocate the cooperative payoff according to a set of agreed upon optimality principles. Again in the time interval $\left[t_k, t_{k+1}\right)$ the probability that the type $\omega_k^{a_k}$ generation k player and the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player would agree to the solution imputation

$\left[\xi^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}(t,x),\ \xi^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}(t,x)\right]$ over the time interval $\left[t_k, t_{k+1}\right)$, is $\varpi_h^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)}$, where $\sum_{h=1}^{\zeta\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)} \varpi_h^{\left(\omega_{k-1}^{a_{k-1}},\omega_k^{a_k}\right)} = 1$. At time t_1, the agreed-upon imputation for the type ω_0^1 generation 0 player and the type ω_1^1 player are known.

Following the analysis in Sect. 5.3 a counter-part of Theorem 3.1 which derives the PDP that yields a subgame consistent solution for the cooperative game (5.1) and (5.3, 5.4) can be obtained in the theorem below.

Theorem 5.2 If the imputation vector $\left[\xi^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}\left(t,x_t^*\right),\ \xi^{k\left(\omega_k^{a_k},O\right)\omega_{k-1}^{a_{k-1}}[h]}\left(t,x_t^*\right)\right]$, are functions that are continuously differentiable in t and x_t^*, a PDP with an instantaneous payment at time $t\in\left[t_k,t_{k+1}\right)$:

$$\begin{aligned}B_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}(t) &= -\xi_t^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}\left(t,x_t^*\right)\\&\quad-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(t,x_t^*\right)\,\xi_{x^h x^\zeta}^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}\left(t,x_t^*\right)\\&\quad-\xi_x^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}\left(t,x_t^*\right)f\left[t,x_t^*,\psi_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}\left(t,x_t^*\right),\psi_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}\left(t,x_t^*\right)\right]\end{aligned}\tag{5.7}$$

allocated to the type $\omega_{k-1}^{a_{k-1}}$ generation $k-1$ player;

and an instantaneous payment at time $t\in\left[t_k,t_{k+1}\right)$:

$$\begin{aligned}B_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}(t) &= -\xi_t^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}\left(t,x_t^*\right)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(t,x_t^*\right)\,\xi_{x^h x^\zeta}^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}\left(t,x_t^*\right)\\&\quad-\xi_x^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}\left(t,x_t^*\right)f\left[t,x_t^*,\psi_{k-1}^{\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}}\left(t,x_t^*\right),\psi_k^{\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}}\left(t,x_t^*\right)\right]\end{aligned}$$

allocated to the type $\omega_k^{a_k}$ generation k player,

yields a mechanism leading to the realization of the imputation vector

$$\left[\xi^{k-1\left(\omega_{k-1}^{a_{k-1}},O\right)\omega_k^{a_k}[h]}\left(t,x_t^*\right),\ \xi^{k\left(\omega_k^{a_k},Y\right)\omega_{k-1}^{a_{k-1}}[h]}\left(t,x_t^*\right)\right],\ \text{for } k\in\{1,2,\cdots,\upsilon\}.$$

Proof Follow the proof leading to Theorem 3.1 in Chap. 3 with the imputation vector in present value (rather than in current value). ■

5.6 Chapter Notes

This Chapter considers cooperative differential games in which players enter the game at different times and have diverse horizons. Moreover, the types of future players are not known with certainty. Subgame consistent cooperative solutions and analytically tractable payoff distribution mechanisms leading to the realization of these solutions are derived. Finally, the overlapping generations of players can be extended to more complex structures. The game horizon of the players can include more than two time intervals and be different across players. The number of players in each time interval can also be more than two and be different across intervals. Hence, the analysis can be formulated as a general class of stochastic differential games with asynchronous horizons structures. An analysis on subgame consistent cooperative solutions in stochastic differential games with asynchronous horizons and uncertain types of players can be found in Yeung (2012).

5.7 Problems

1. Consider the game in which there are 4 overlapping generations of players with generation 0 and generation 1 players in $[0,2)$, generation 1 and generation 2 players in $[2,4)$, generation 2 and generation 3 players in $[4,6]$. Players are of either type 1 or type 2. The instantaneous payoffs and terminal rewards of the type 1 generation k player and the type 2 generation k player are respectively:

$$\left[2(u_k)^{1/2} - \frac{1}{x^{1/2}}u_k\right] \text{ and } q_1 x^{1/2}; \quad \text{and } \left[(u_k)^{1/2} - \frac{2}{x^{1/2}}u_k\right] \text{ and } q_2 x^{1/2},$$

   where the state variable $x(s)$ is the biomass of a renewable resource. $u_k(s)$ is the harvest of the generation k extraction firm. The type $i \in \{1,2\}$ generation k extraction firm's extraction cost is $c_i u_k(s) x(s)^{-1/2}$.

   At initial time 0, it is known that the generation 0 player is of type 1 and the generation 1 player is also of type 1. It is also known that the generation 2 and generation 3 players may be of type 1 with probability $\lambda^1 = 0.4$ and of type 2 with probability $\lambda^2 = 0.6$.

   The state dynamics of the game is characterized by:

$$\dot{x}(s) = 10x(s)^{1/2} - 2x(s) - u_k^{(i,O)j}(s) - u_k^{(j,Y)i}(s),$$

   if the old generation $k-1$ extractor is of type i and the young generation k extractor is of type j, for $s \in [t_k, t_{k+1})$ and $k \in \{1,2,3\}$ with $t_1 = 0$, $t_2 = 2$ and $t_3 = 4$; and $x(0) = 30$,

   where $u_k^{(i,O)j}(s)$ denote the harvest of the type i generation $k-1$ old extractor and $u_k^{(j,Y)i}(s)$ denote the harvest of the type j generation k young extractor. The discount rate is 0.05.

   Characterize the non-cooperative feedback Nash equilibrium for the generation 0 player and generation 1 player game.
2. Construct a subgame consistent cooperative solution in which all types of players would accept the imputation which shares the excess cooperative gains (over the individual payoffs) equally among themselves.