Chapter 3
Subgame Consistent Cooperation in Stochastic Differential Games

An essential characteristic of time – and hence decision making over time – is that though the individual may, through the expenditure of resources, gather past and present information, the future is inherently unknown and therefore (in the mathematical sense) uncertain. An empirically meaningful theory must therefore incorporate time-uncertainty in an appropriate manner. This Chapter considers subgame consistent cooperation in stochastic differential games. It provides an integrated exposition the works of Yeung and Petrosyan (2004), Chapter 4 of Yeung and Petrosyan (2006b), and Chapter 8 of Yeung and Petrosyan (2012a).

The organization of the Chapter is as follows. Section 3.1 presents the basic formulation of cooperative stochastic differential games. Section 3.2 presents an analysis on cooperative subgame consistency under uncertainty. Derivation of a subgame consistent payoff distribution procedure is provided in Sect. 3.3. An illustration in cooperative fishery under uncertainty is given in Sect. 3.4. Infinite horizon subgame consistency under uncertainty is examined in Sect. 3.5. In Sect. 3.6, a subgame consistent solution for infinite horizon cooperative fishery under uncertainty is presented. Chapter notes are provided in Sect. 3.7 and problems in Sect. 3.8.

3.1 Cooperative Stochastic Differential Games

Consider the general form of n-person stochastic differential games in which player i seeks to maximize its expected payoffs:

$$E_{t_0}\left\{ \int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds + \exp\left[-\int_{t_0}^{T} r(y)dy\right] q^i(x(T)) \right\}, \quad \text{for } i \in N, \tag{1.1}$$

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with $E_{t_0}\{\cdot\ \}$ denoting the expectation operation taken at time t_0, and the dynamics of the state is

$$dx(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, ds + \sigma[s, x(s)]dz(s), \qquad x(t_0) = x_0, \quad (1.2)$$

where $\sigma[s, x(s)]$ is a $m \times \Theta$ matrix and $z(s)$ is a Θ-dimensional Wiener process and the initial state x_0 is given. Let $\Omega[s, x(s)] = \sigma[s, x(s)]\ \sigma[s, x(s)]'$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s, x(s)]$. Moreover, $E[dz_\varpi] = 0$ and $E[dz_\varpi dt] = 0$ and $E\left[(dz_\varpi)^2\right] = dt$, for $\varpi \in [1, 2, \cdots, \Theta]$; and $E[dz_\varpi dz_\omega] = 0$, for $\varpi \in [1, 2, \cdots, \Theta]$, $\varpi \in [1, 2, \cdots, \Theta]$ and $\varpi \neq \omega$.

3.1.1 Non-cooperative Equilibria

Again, we first characterize the non-cooperative equilibria of the game as a benchmark for negotiation in the cooperative scheme. A feedback Nash equilibrium solution of the stochastic differential game (1.1) and (1.2) can be characterized by the following Theorem.

Theorem 1.1 An N-tuple of feedback strategies $\left\{\phi_i^*(t,x) \in U^i; i \in N\right\}$ provides a Nash equilibrium solution to the game (1.1) and (1.2) if there exist suitably smooth functions $V^{(t_0)i}(t,x) : [t_0, T] \times R^m \to R, i \in N$, satisfying the partial differential equations

$$-V_t^{(t_0)i}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{(t_0)i}(t,x) =$$
$$\max_{u_i}\left\{g^i\left[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \cdots, \phi_{i-1}^*(t,x), u_i(t), \phi_{i+1}^*(t,x), \cdots, \phi_n^*(t,x)\right]\right.$$
$$\exp\left[-\int_{t_0}^{s} r(y)dy\right]$$
$$\left.+ V_x^{(t_0)i}(t,x) f\left[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \cdots, \phi_{i-1}^*(t,x), u_i(t), \phi_{i+1}^*(t,x), \cdots, \phi_n^*(t,x)\right]\right\},$$
$$V^{(t_0)i}(T,x) = q^i(x)\exp\left[-\int_{t_0}^{T} r(y)dy\right], \; i \in N.$$

Proof This result follows readily from the definition of Nash equilibrium and from the stochastic control result in Theorem A.3 of the Technical Appendices. ■

In particular, $V^{(t_0)i}(t,x)$ represents the expected game equilibrium payoff of player i at time $t \in [t_0, T]$ with the state being x, that is

$$
\begin{aligned}
&V^{(t_0)i}(t,x) \\
&\quad = E_{t_0}\left\{ \int_t^T g^i\left[s, x^*(s), \phi_1^*\left(s, x^*(s)\right), \phi_2^*\left(s, x^*(s)\right), \cdots, \phi_n^*\left(s, x^*(s)\right)\right] \right. \\
&\exp\left[-\int_{t_0}^{s} r(y)dy\right] ds \\
&\quad\quad \left. + q^i\left(x^*(T)\right)\exp\left[-\int_{t_0}^{T} r(y)dy\right] \right\}.
\end{aligned}
$$

A remark that will be utilized in subsequent analysis is given below.

Remark 1.1 Let $V^{(\tau)i}(t,x)$ denote the feedback Nash equilibrium payoff of nation i in the game with stochastic dynamics (1.1) and expected payoffs (1.2) which starts at time τ for $\tau \in [t_0, T)$. Note that the equilibrium feedback strategies are Markovian in the sense that they depend on current time and current state. One can readily verify that

$$
\begin{aligned}
&\exp\left[\int_{t_0}^{\tau} r(y)dy\right] V^{(t_0)i}(t,x) = \exp\left[\int_{t_0}^{\tau} r(y)dy\right] \\
&\quad \times E_{t_0}\left\{ \int_t^T g^i\left[s, x^*(s), \phi_1^*\left(s, x^*(s)\right), \phi_2^*\left(s, x^*(s)\right), \cdots, \phi_n^*\left(s, x^*(s)\right)\right] \right. \\
&\left. \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds \right\} \\
&\quad \times E_{t_0}\left\{ \int_t^T g^i\left[s, x^*(s), \phi_1^*\left(s, x^*(s)\right), \phi_2^*\left(s, x^*(s)\right), \cdots, \phi_n^*\left(s, x^*(s)\right)\right] \right. \\
&\left. \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds \right\} \\
&\quad = E_t\left\{ \int_t^T g^i\left[s, x^*(s), \phi_1^*\left(s, x^*(s)\right), \phi_2^*\left(s, x^*(s)\right), \cdots, \phi_n^*\left(s, x^*(s)\right)\right] \right. \\
&\left. \exp\left[-\int_{\tau}^{s} r(y)dy\right] ds \right\} \\
&\quad = V^{(\tau)i}(t,x), \text{ for } \tau \in [t_0, T).
\end{aligned}
$$

■

3.1.2 *Dynamic Cooperation Under Uncertainty*

The participating players agree to act according to an agreed-upon optimality principle. Based on this optimality principle, the solution of the cooperative differential game $\Gamma_c(x_0, T - t_0)$ at time t_0 includes

(i) a set of cooperative strategies
$u^{(t_0)*}(s, x_s) = \left[u_1^{(t_0)*}(s, x_s), u_2^{(t_0)*}(s, x_s), \cdots, u_n^{(t_0)*}(s, x_s)\right]$, for $s \in [t_0, T]$ given that the state is x_s at time s;

(ii) an imputation vector $\xi^{(t_0)}(t_0,x_0) = \left[\xi^{(t_0)1}(t_0,x_0), \xi^{(t_0)2}(t_0,x_0), \cdots, \xi^{(t_0)n}(t_0,x_0)\right]$ to allot the cooperative payoff to the players; and

(iii) a payoff distribution procedure $B^{t_0}(s,x_s) = \left[B_1^{t_0}(s,x_s), B_2^{t_0}(s,x_s), \cdots, B_n^{t_0}(s,x_s)\right]$ for $s \in [t_0,T]$, where $B_i^{t_0}(s,x_s)$ is the instantaneous payments for player i at time s given that the state is x_s. In particular,

$$\xi^{(t_0)i}(t_0,x_0) = E_{t_0}\left\{ \int_{t_0}^{T} B_i^{t_0}(s,x_s)\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds + q^i(x_T)\ \exp\left[-\int_{t_0}^{T} r(y)dy\right] \right\},$$
for $i \in N$.
$$(1.3)$$

This means that the players agree at the outset on a set of cooperative strategies $u^{(t_0)*}(s,x_s)$, an imputation $\xi^{(t_0)i}(t_0,x_0)$ of the gains to the ith player covering the time interval $[t_0,T]$, and a payoff distribution procedure $\{B^{t_0}(s,x_s)\}\ _{s=t_0}^{T}$ to allocate payments to the players over the game interval.

Recall that group optimality is an essential element in dynamic cooperation, an optimality principle has to require the players have to maximize their expected joint payoff:

$$E_{t_0}\left\{ \sum_{j=1}^{n}\int_{t_0}^{T} g^j[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)]\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds \right.$$
$$\left. +\sum_{j=1}^{n}\exp\left[-\int_{t_0}^{T} r(y)dy\right]q^i(x(T)) \right\}, \tag{1.4}$$

subject to (1.2).

Let $W^{(t_0)}(t,x)$ denote maximized expected payoff of the stochastic control problem at time t given that the state is x, that is:

$$W^{(t_0)}(t,x)$$
$$= \max_{\substack{u_1(s),u_2(s),\cdots,u_n(s);\\ for\ \ s\in[t,T]}} E_{t_0}\Bigg\{$$
$$\sum_{j=1}^{n}\int_{t}^{T} g^j[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)]\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds + \sum_{j=1}^{n}\exp\left[-\int_{t_0}^{T} r(y)dy\right]q^i(x(T))\Bigg\}.$$

An optimal solution to the stochastic dynamic programming control problem (1.2) and (1.4) is provided by the theorem below.

Theorem 1.2 A set of controls $\left\{u_i^*(t) = \psi_i^*(t,x),\text{ for } i \in N\right\}$ constitutes an optimal solution to the stochastic control problem (1.2) and (1.4), if there exist continuously twice differentiable functions $W^{(t_0)}(t,x) : [t_0,T] \times R_m \to R$, satisfying the following partial differential equation:

$$-W_t^{(t_0)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)W_{x^h x^\zeta}^{(t_0)}(t,x) =$$
$$\max_{u_1,u_2,\cdots,u_n}\left\{\sum_{j=1}^{n} g^j[t,x,u_1,u_2,\cdots,u_n]\exp\left[-\int_{t_0}^{t} r(y)dy\right]\right.$$
$$\left. + W_x^{(t_0)}(t,x)f[t,x,u_1,u_2,\cdots,u_n]\right\}, \text{ and}$$
$$W^{(t_0)}(T,x) = \sum_{j=1}^{n} q^j(x)\exp\left[-\int_{t_0}^{T} r(y)dy\right]. \tag{1.5}$$

Proof Follow the proof of Theorem A.3 in the Technical Appendices. ■

Hence the players will adopt the cooperative control $\{\psi_i^*(t,x)$, for $i \in N$ and $t \in [t_0,T]\}$ to obtain the maximized level of expected joint profit. Substituting this set of control into (1.1) yields the dynamics of the optimal (cooperative) trajectory as:

$$dx(s) = f\left[s,x(s),\psi_1^*(s,x(s)),\psi_2^*(s,x(s)),\cdots,\psi_n^*(s,x(s))\right]ds$$
$$+\sigma[s,x(s)]dz(s),\ \ x(t_0) = x_0 \tag{1.6}$$

The solution to (1.6) can be expressed as:

$$x^*(t) = x_0 + \int_{t_0}^{t} f\left[s,x^*(s),\psi_1^*\left(s,x^*(s)\right),\psi_2^*\left(s,x^*(s)\right),\cdots,\psi_n^*\left(s,x^*(s)\right)\right]ds$$
$$+\int_{t_0}^{t}\sigma\left[s,x^*(s)\right]dz(s) \tag{1.7}$$

We use X_t^* to denote the set of realizable values of $x^*(t)$ at time t generated by (1.7). The term $x_t^* \in X_t^*$ is used to denote an element in X_t^*. We use the terms $x^*(t)$ and x_t^* interchangeably in case where there is no ambiguity.

The cooperative control for the game (1.2) and (1.4) over the time interval $[t_0,T]$ can be expressed more precisely as

$$\{\psi_i^*\left(t,x_t^*\right), for\, i \in N \,\text{and}\, t \in [t_0,T] \,\text{when}\, x_t^* \in X_t^* \,\text{is realized}\}. \tag{1.8}$$

The expected cooperative payoff over the interval $[t,T]$, for $t \in \left[t_0,T\right)$, can be expressed as:

$$W^{(t_0)}\left(t,x_t^*\right) = E_{t_0}\Bigg\{$$
$$\int_t^T \sum_{j=1}^{n} g^j\left[s,x^*(s),\psi_1^*\left(s,x^*(s)\right),\psi_2^*\left(s,x^*(s)\right),\cdots,\psi_n^*\left(s,x^*(s)\right)\right]\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds$$
$$+\exp\left[-\int_{t_0}^{T} r(y)dy\right]\sum_{j=1}^{n} q^j\left(x^*(T)\right)\Bigg|\ x^*(t) = x_t^* \in X_t^*\Bigg\}. \tag{1.9}$$

To verify whether the player would find it optimal to adopt the cooperative controls (1.8) throughout the cooperative duration, we consider a stochastic control problem with dynamics (1.2) and payoff (1.4) which begins at time $\tau \in [t_0, T]$ with initial state $x_\tau^* \in X_t^*$. At time τ, the optimality principle ensuring group optimality requires the players to solve the problem:

$$\max_{u_1, u_2, \cdots, u_n} E_\tau \left\{ \int_\tau^T \sum_{j=1}^n g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp\left[-\int_\tau^s r(y)dy\right] ds \right.$$
$$\left. + \exp\left[-\int_\tau^T r(y)dy\right] \sum_{j=1}^n q^j(x(T)) \right\}, \tag{1.10}$$

subject to

$$dx(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)]\, ds + \sigma[s, x(s)]dz(s),\ x(\tau) = x_\tau^* \in X_t^*. \tag{1.11}$$

Note that

$$\max_{u_1, u_2, \cdots, u_n} E_{t_0} \left\{ \int_\tau^T \sum_{j=1}^n g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp\left[-\int_{t_0}^s r(y)dy\right] ds \right.$$
$$\left. + \exp\left[-\int_{t_0}^T r(y)dy\right] \sum_{j=1}^n q^j(x(T)) \right| x(\tau) = x_\tau^* \in X_\tau^* \Bigg\}$$
$$= \max_{u_1, u_2, \cdots, u_n} E_{t_0} \left\{ \exp\left[-\int_{t_0}^\tau r(y)dy\right] \times \right.$$
$$\left(\int_\tau^T \sum_{j=1}^n g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp\left[-\int_\tau^s r(y)dy\right] ds \right.$$
$$\left. \left. + \exp\left[-\int_\tau^T r(y)dy\right] \sum_{j=1}^n q^j(x(T)) \right) \right| x(\tau) = x_\tau^* \in X_\tau^* \Bigg\}.$$
$$= \exp\left[-\int_{t_0}^\tau r(y)dy\right] \times$$
$$\max_{u_1, u_2, \cdots, u_n} E_\tau \Bigg\{$$
$$\left(\int_\tau^T \sum_{j=1}^n g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp\left[-\int_\tau^s r(y)dy\right] ds \right.$$
$$\left. \left. + \exp\left[-\int_\tau^T r(y)dy\right] \sum_{j=1}^n q^j(x(T)) \right) \right| x(\tau) = x_\tau^* \Bigg\}. \tag{1.12}$$

Hence the stochastic optimal controls strategies for problem (1.10) and (1.11) are analogous to the controls strategies for problem (1.2) and (1.4) in the time interval $[t, T]$.

A remark that will be utilized in subsequent analysis is given below.

Remark 1.2 Let $W^{(\tau)}(t, x_t^*)$ denote the expected cooperative payoff of control problem (1.10) and (1.11). One can readily verify that

$$\exp\left[\int_{t_0}^{\tau} r(y)dy\right] W^{(t_0)}\left(t, x_t^*\right) = \exp\left[\int_{t_0}^{\tau} r(y)dy\right] W^{(\tau)}\left(t, x_t^*\right),$$

for $\tau \in [t_0, T]$ and $t \in [\tau, T)$ and $x_\tau^* \in X_t^*$. ■

Again, we use $\Gamma_c\left(x_t^*, T-t\right)$ to denote the cooperative game with player payoffs (1.1) and dynamics (1.2) which starts at time $t \in [t_0, T)$ given the state $x(\tau) = x_\tau^* \in X_\tau^*$. Let there exist a solution under the agreed-upon optimality principle, $t_0 \le t \le T$ along the optimal trajectory $\left\{x^*(t)\right\}_{t=t_0}^{T}$. If this condition is not satisfied it is impossible for the players to adhere to the chosen principle of optimality.

For $\xi^{(t)}\left(t, x_t^*\right)$, $t \in [t_0, T]$, to be valid imputations, it is required that both group optimality and individual rationality have to be satisfied. Hence a valid optimality principle $P\left(x_t^*, T-t\right)$ would yield a solution which contains

(i)
$$\sum_{j=1}^{n} \xi^{(t)j}\left(t, x_t^*\right) = W^{(t)}\left(t, x_t^*\right), \text{ for } t \in [t_0, T]; \text{ and}$$

(ii)
$$\xi^{(t)i}\left(t, x_t^*\right) \ge V^{(t)i}\left(t, x_t^*\right), \text{ for } i \in N \text{ and } t \in [t_0, T].$$

3.2 Cooperative Subgame Consistency Under Uncertainty

In this Section we examine the properties of subgame consistency in cooperative stochastic differential games.

3.2.1 Principle of Subgame Consistency

In a stochastic environment, the condition of subgame consistency requires the optimality principle agreed upon at the outset to remain effective in a subgame with a later starting time and any realizable state brought about by prior optimal behavior. Assume that at the start of the game the players execute the solution under an agreed-upon optimality principle (which includes a set of cooperative strategies, an imputation to distribute the cooperative payoff and a payoff distribution procedure). When the game proceeds to time t and the state becomes $x_t^* \in X_t^*$, the continuation of the scheme for the game $\Gamma_c(x_0, T - t_0)$ has to be consistent with the solution to the game $\Gamma_c\left(x_t^*, T - t\right)$ under the same optimality principle. If this consistency condition is violated, some of the players will have an incentive to deviate from the initial agreement and instability arises.

To verify whether the solution is indeed subgame consistent, one has to verify whether the agreed upon cooperative strategies, payoff distribution procedures and imputations are all subgame consistent. Using Remark 1.2, one can show that joint expected payoff maximizing strategies are subgame consistent. In the next subsection, subgame consistent imputation and payoff distribution procedure are examined.

3.2.2 Subgame-Consistency in Imputation and Payoff Distribution Procedure

In this Section, we consider subgame consistency in imputation and payoff distribution procedure. In the cooperative game $\Gamma_c(x_0, T - t_0)$ according to the solution generated by the agreed-upon optimality principle, the players would use the payoff distribution procedure $\left\{B^{t_0}\left(s, x_s^*\right)\right\}_{s=t_0}^{T}$ to bring about an imputation to player i as:

$$\xi^{(t_0)i}(t_0, x_0) = E_{t_0}\left\{ \int_{t_0}^{T} B_i^{t_0}\left(s, x_s^*\right)\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds + q^i\left(x^*(T)\right) \exp\left[-\int_{t_0}^{T} r(y)dy\right] \right\}, \tag{2.1}$$

for $i \in N$.

When the game proceeds to time $t \in \left(t_0, T\right]$, the current state is $x_t^* \in X_t^*$. According to the solution of the game $\Gamma_c(x_0, T - t_0)$ generated by the agreed-upon optimality principle player i will receive an imputation (in present value viewed at time t_0) equaling

$$\xi^{(t_0)i}\left(t, x_t^*\right) = E_{t_0}\left\{ \int_{t}^{T} B_i^{t_0}\left(s, x_s^*\right)\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds + q^i\left(x^*(T)\right)\exp\left[-\int_{t_0}^{T} r(y)dy\right] \middle| x(t) = x_t^* \right\}, \tag{2.2}$$

over the time interval $[t, T]$.

Note that at time $t \in (t_0, T]$ when the current state is $x_t^* \in X_t^*$, we have a cooperative game $\Gamma_c(x_t^*, T-t)$. According to the solution generated by the same optimality principle, the players would use the payoff distribution procedure $\{B^t(s, x_s^*)\}_{s=t}^T$ to bring about an imputation to player i as:

$$\xi^{(t)i}(t, x_t^*) = E_t\left\{ \int_t^T B_i^t(s, x_s^*) \exp\left[-\int_t^s r(y)dy\right] ds \right.$$
$$\left. + q^i(x^*(T)) \exp\left[-\int_t^T r(y)dy\right] \right\}, \text{ for } i \in N. \quad (2.3)$$

For the imputation and payoff distribution procedure of the game $\Gamma_c(x_0, T-t_0)$ to be consistent with those of the game $\Gamma_c(x_t^c, T-t)$ under the same agreed-upon optimality principle, it is essential that

$$\exp\left[\int_{t_0}^t r(y)dy\right] \xi^{(t_0)}(t, x_t^*) = \xi^{(t)}(t, x_t^*), \text{ for } t \in [t_0, T].$$

In addition, the payoff distribution procedure of the game $\Gamma_c(x_0, T-t_0)$ generated by the agreed upon optimality principle is

$$B^{t_0}(s, x_s^*) = [B_1^{t_0}(s, x_s^*), B_2^{t_0}(s, x_s^*), \cdots, B_n^{t_0}(s, x_s^*)], \text{ for } s \in [t_0, T].$$

Consider the case when the game has proceeded to time t and the state variable became $x_t^* \in X_t^*$. Then one has a cooperative game $\Gamma_c(x_t^*, T-t)$ which starts at time t with initial state x_t^*. According to the same optimality principle, the payoff distribution procedure

$$B^t(s, x_s^*) = [B_1^t(s, x_s^*), B_2^t(s, x_s^*), \cdots, B_n^t(s, x_s^*)], \text{ for } s \in [t, T],$$

will be adopted.

For the continuation of the payoff distribution procedure $B^{t_0}(s, x_s^*)$ of the game $\Gamma_c(x_0, T-t_0)$ to be consistent with $B^t(s, x_s^*)$ of the game $\Gamma_c(x_t^*, T-t)$, it is required that

$$B^{t_0}(s, x_s^*) = B^t(s, x_s^*), \text{ for } s \in [t, T] \text{ and } t \in [t_0, T].$$

Therefore a formal definition can be presented as below.

Definition 2.1 The imputation and payoff distribution procedure

$\{\xi^{(t_0)}(t_0, x_0)$ and $B^{t_0}(s, x_s^*)$ for $s \in [t_0, T]\}$ under the agreed-upon optimality principle are subgame consistent if

(i)
$$\exp\left[\int_{t_0}^{t} r(y)dy\right]\xi^{(t_0)i}(t,x_t^*)$$
$$\equiv \exp\left[\int_{t_0}^{t} r(y)dy\right]E\left\{\int_t^T B_i^{t_0}(s,x_s^*)\exp\left[-\int_t^s r(y)dy\right]ds\right.$$
$$\left.+q^i(x^*(T))\exp\left[-\int_{t_0}^{T} r(y)dy\right]\middle| x(t)=x_t^*\right\} = \xi^{(t)i}(t,x_t^*)\equiv$$
$$E_t\left\{\int_t^T B_i^t(s,x_s^*)\exp\left[-\int_t^s r(y)dy\right]ds + q^i(x^*(T))\exp\left[-\int_t^T r(y)dy\right]\right\}$$

under $P(x_t^*, T-t)$, for $i \in N$ and $t \in [t_0, T]$; and

(ii) the payoff distribution procedure $B^{t_0}(s,x_s^*) = [B_1^{t_0}(s,x_s^*), B_2^{t_0}(s,x_s^*), \cdots, B_n^{t_0}(s,x_s^*)]$ for $s \in [t,T]$ is identical to $B^t(s,x_s^*) = [B_1^t(s,x_s^*), B_2^t(s,x_s^*), \cdots, B_n^t(s,x_s^*)]$ of the game $\Gamma_c(x_t^*, T-t)$. ■

3.3 Subgame Consistent Payoff Distribution Procedure

Crucial to obtaining a subgame consistent solution is the derivation of a payoff distribution procedure satisfying Definition 2.1 in Sect. 3.2. Invoking part (ii) of Definition 2.1, we have $B^{t_0}(s,x_s^*) = B^t(s,x_s^*)$ for $t \in [t_0,T]$ and $s \in [t,T]$. We use $B(s,x_s^*) = \{B_1(s,x_s^*), B_2(s,x_s^*), \cdots, B_n(s,x_s^*)\}$ to denote $B^t(s,x_s^*)$ for all $t \in [t_0,T]$. Along the optimal trajectory $\{x^*(s)\}_{s=t_0}^{T}$ we then have:

$$\xi^{(\tau)i}(\tau,x_\tau^*) = E_\tau\left\{\int_\tau^T B_i^\tau(s,x^*(s))\exp\left[-\int_\tau^s r(y)dy\right]ds\right.$$
$$\left.+q^i(x_T^*)\exp\left[-\int_\tau^T r(y)dy\right]\middle| x^*(\tau)=x_\tau^*\in X_\tau^*\right\}, \tag{3.1}$$

for $i \in N$ and $\tau \in [t_0,T]$.

Moreover, for $t \in [\tau,T]$, we use the term

$$\xi^{(\tau)i}(t,x_t^*) = E_\tau\left\{\int_t^T B_i^\tau(s,x^*(s))\exp\left[-\int_\tau^s r(y)dy\right]ds\right.$$
$$\left.+q^i(x_T^*)\exp\left[-\int_\tau^T r(y)dy\right]\middle| x^*(t)=x_t^*\in X_t^*\right\}, \tag{3.2}$$

to denote the expected present value (with initial time being τ) of player i's expected payoff under cooperation over the time interval $[t,T]$ according to the optimality principle $P(x_\tau^*, T-\tau)$ along the cooperative state trajectory.

Invoking (3.1) and (3.2) we have

$$\xi^{(\tau)i}\left(t, x_t^c\right) = \exp\left[-\int_\tau^t r(y)dy\right]\xi^{(t)i}\left(t, x_t^*\right),$$
$$\text{for } i \in N \text{ and } \tau \in [t_0, T] \text{ and } t \in [\tau, T] \tag{3.3}$$

One can readily verify that a payoff distribution procedure $\left\{B\left(s, x_s^*\right)\ \right\}_{s=t_0}^T$ which satisfies (3.3) would give rise to time-consistent imputations satisfying part (i) of Definition 2.1. The next task is the derivation of a payoff distribution procedure $\left\{B\left(s, x_s^*\right)\ \right\}_{s=t_0}^T$ that leads to the realization of (3.1), (3.2), and (3.3).

We first consider the following condition concerning the imputation $\xi^{(\tau)}(t, x_t^*)$, for $\tau \in [t_0, T]$ and $t \in [\tau, T]$.

Condition 3.1 For $i \in N$ and $t \in [\tau, T]$ and $\tau \in [t_0, T]$, the imputation $\xi^{(\tau)i}(t, x_t^*)$, for $i \in N$, is a function that is twice continuously differentiable in t and $x_t^* \in X_t^*$. ■

A theorem characterizing a formula for $B_i(s, x_s^*)$, for $s \in [t_0, T]$, $x_s^* \in X_s^*$ and $i \in N$, which yields (3.1), (3.2), and (3.3) can be provided as follows.

Theorem 3.1 If Condition 3.1 is satisfied, a PDP with a terminal payment $q^i(x_T^*)$ at time T and an instantaneous payment at time $s \in [\tau, T]$:

$$\begin{aligned} B_i\left(s, x_s^*\right) = & -\left[\xi_t^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right] \\ & -\left[\xi_{x_t^*}^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right] f\left[s, x_s^*, \psi_1^*\left(s, x_s^*\right), \psi_2^*\left(s, x_s^*\right), \cdots, \psi_n^*\left(s, x_s^*\right)\right] \\ & -\frac{1}{2}\sum_{h,\zeta=1}^m \Omega^{h\zeta}\left(s, x_s^*\right)\left[\xi_{x_t^h x_t^\zeta}^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right], \text{ for } i \in N \text{ and } x_s^* \in X_s^*, \end{aligned} \tag{3.4}$$

yields imputation vector $\xi^{(\tau)}(\tau, x_\tau^*)$, for $\tau \in [t_0, T]$ which satisfy (3.1), (3.2), and (3.3).

Proof Invoking (3.1), (3.2) and (3.3), one can obtain

$$\begin{aligned} \xi^{(\upsilon)i}\left(\upsilon, x_\upsilon^*\right) = E_\upsilon\Bigg\{ & \int_\upsilon^{\upsilon+\Delta t} B_i\left(s, x_s^*\right)\exp\left[-\int_\upsilon^s r(y)dy\right]ds + \\ & \exp\left[-\int_\upsilon^{\upsilon+\Delta t} r(y)dy\right]\xi^{(\upsilon+\Delta t)i}\left(\upsilon+\Delta t, x_\upsilon^* + \Delta x_\upsilon^*\right)\Bigg|\ x(\upsilon) = x_\upsilon^* \in X_\upsilon^*\Bigg\}, \\ & \text{for } \upsilon \in [\tau, T] \text{ and } i \in N; \end{aligned} \tag{3.5}$$

where

$\Delta x_\upsilon^* = f\left[\upsilon, x_\upsilon^c, \psi_1^*\left(\upsilon, x_\upsilon^*\right), \psi_2^*\left(\upsilon, x_\upsilon^*\right), \cdots, \psi_n^*\left(\upsilon, x_\upsilon^*\right)\right]\Delta t + \sigma\left[\upsilon, x_\upsilon^*\right]\Delta z_\upsilon + o(\Delta t)$,
$\Delta z_\upsilon = Z(\upsilon + \Delta t) - z(\upsilon)$, and $E_\upsilon[o(\Delta t)]/\Delta t \to 0$ as $\Delta t \to 0$.

From (3.2) and (3.5), one obtains

$$E_{\upsilon}\left\{ \int_{\upsilon}^{\upsilon+\Delta t} B_i\left(s,x_s^*\right)\exp\left[-\int_{\upsilon}^{s} r(y)dy\right]ds \middle| \, x(\upsilon)=x_{\upsilon}^* \right\}$$
$$= E_{\upsilon}\left\{ \xi^{(\upsilon)i}\left(\upsilon,x_{\upsilon}^*\right) - \exp\left[-\int_{\upsilon}^{\upsilon+\Delta t} r(y)dy\right]\xi^{(\upsilon+\Delta t)i}\left(\upsilon+\Delta t, x_{\upsilon}^*+\Delta x_{\upsilon}^*\right) \right\}$$
$$= E_{\upsilon}\left\{ \xi^{(\upsilon)i}\left(\upsilon,x_{\upsilon}^*\right) - \xi^{(\upsilon)i}\left(\upsilon+\Delta t, x_{\upsilon}^*+\Delta x_{\upsilon}^*\right) \right\},$$
$$\text{for all } \upsilon\in[t_0,T] \text{ and } i\in N. \tag{3.6}$$

If the imputations $\xi^{(\upsilon)}(t,x_t^*)$, for $\upsilon\in[t_0,T]$, satisfy Condition 3.1, as $\Delta t\to 0$, one can express condition (3.6) as:

$$E_{\upsilon}\left\{ B_i\left(\upsilon,x_{\upsilon}^*\right)\Delta t + o(\Delta t) \right\} = E_{\upsilon}\left\{ -\left[\xi_t^{(\upsilon)i}\left(t,x_t^c\right)\Big|_{t=\upsilon}\right]\Delta t \right.$$
$$-\left[\xi_{x_{\upsilon}^c}^{(\upsilon)i}\left(\upsilon,x_{\upsilon}^c\right)\right]f\left[\upsilon,x_{\upsilon}^c,\psi_1^*\left(\upsilon,x_{\upsilon}^c\right),\psi_2^*\left(\upsilon,x_{\upsilon}^c\right),\cdots,\psi_n^*\left(\upsilon,x_{\upsilon}^c\right)\right]\Delta t$$
$$-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(\upsilon,x_{\upsilon}^*\right)\left[\xi_{x_t^h x_t^\zeta}^{(\upsilon)i}\left(t,x_t^*\right)\Big|_{t=\upsilon}\right] - \left[\xi_{x_{\upsilon}^c}^{(\upsilon)i}\left(\upsilon,x_{\upsilon}^c\right)\right]\sigma\left[\upsilon,x_{\upsilon}^*\right]\Delta z_{\upsilon}$$
$$\left. - o(\Delta t). \right. \tag{3.7}$$

Dividing (3.7) throughout by Δt, with $\Delta t\to 0$, and taking expectation yield (3.4). Thus the payoff distribution procedure in $B_i(s,x_s^*)$ in (3.4) would lead to the realization of $\xi^{(\tau)i}(\tau,x_\tau^c)$, for $\tau\in[t_0,T]$ which satisfy (3.1)–(3.3). ■

Assigning the instantaneous payments according to the payoff distribution procedure in (3.4) leads to the realization of the imputation $\xi^{(\tau)}\left(\tau,x_\tau^*\right)\in P\left(x_\tau^*,T-\tau\right)$ for $\tau\in[t_0,T]$ and $x_\tau^*\in X_\tau^*$.

With players using the cooperative strategies $\left\{\psi_i^*\left(\tau,x_\tau^*\right),\text{ for } \tau\in[t_0,T] \text{ and } i\in N\right\}$, the instantaneous payment received by player i at time instant τ is:

$$\zeta_i\left(\tau,x_\tau^*\right) = g^i\left[\tau,x_\tau^*,\psi_1^*\left(\tau,x_\tau^*\right),\psi_2^*\left(\tau,x_\tau^*\right),\cdots,\psi_n^*\left(\tau,x_\tau^*\right)\right)\right],$$
$$\text{for } \tau\in[t_0,T],\ x_\tau^*\in X_\tau^* \text{ and } i\in N. \tag{3.8}$$

According to Theorem 3.1, the instantaneous payment that player i should receive under the agreed-upon optimality principle is $B_i(\tau,x_\tau^*)$ as stated in (3.4). Hence an instantaneous transfer payment

$$\chi^i\left(\tau,x_\tau^*\right) = B_i\left(\tau,x_\tau^*\right) - \zeta_i\left(\tau,x_\tau^*\right) \tag{3.9}$$

has to be given to player i at time τ, for $i\in N$ and $\tau\in[t_0,T]$ when the state is $x_\tau^*\in X_\tau^*$.

3.4 An Illustration in Cooperative Fishery Under Uncertainty

Consider the stochastic resource extraction game with two asymmetric extractors.

The resource stock $x(s) \in X \subset R$ follows the stochastic dynamics:

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - u_1(s) - u_2(s)\right]ds + \sigma x(s)dz(s), \ \ x(t_0) = x_0 \in X \tag{4.1}$$

where $u_i(s)$ is the harvest rate of extractor $i \in \{1, 2\}$. The instantaneous payoffs at time $s \in [t_0, T]$ for extractor 1 and extractor 2 are, respectively, $\left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}}u_1(s)\right]$ and $\left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}}u_2(s)\right]$, where c_1 and c_2 are constants and $c_1 \neq c_2$. At time T, each extractor will receive a termination bonus $qx(T)^{1/2}$. Payoffs are transferable between extractors and over time. Given the constant discount rate r, values received at time t are discounted by the factor $\exp[-r(t - t_0)]$.

At time t_0, the expected payoff of extractor i is:

$$E_{t_0}\left\{ \int_{t_0}^{T} \left[u_i(s)^{1/2} - \frac{c_i}{x(s)^{1/2}}u_i(s)\right] \exp[-r(t - t_0)]\,ds + \exp[-r(T - t_0)]qx(T)^{\frac{1}{2}} \right\}, \text{ for } i \in \{1, 2\}. \tag{4.2}$$

Let $[\phi_1^*(t, x), \phi_2^*(t, x)]$ for $t \in [t_0, T]$ denote a set of strategies that provides a feedback Nash equilibrium solution to the game (4.1) and (4.2), and $V^{(t_0)i}(t, x) : [t_0, T] \times R^n \to R$ denote the feedback Nash equilibrium payoff of extractor $i \in \{1, 2\}$ that satisfies the equations:

$$\begin{aligned} &-V_t^{(t_0)i}(t, x) - \frac{1}{2}\sigma^2 x^2 V_{xx}^{(t_0)i}(t, x) \\ &\quad = \max_{u_i}\left\{ \left[(u_i)^{1/2} - \frac{c_i}{x^{1/2}}u_i\right] \exp[-r(t - t_0)] \right. \\ &\quad\quad \left. + V_x^{(t_0)i}(t, x)\left[ax^{1/2} - bx - u_i - \phi_j^{(t_0)*}(t, x)\right] \right\}, \text{ and} \\ &V^{(t_0)i}(T, x) = \exp[-r(T - t_0)]qx(T)^{\frac{1}{2}}, \text{ for } i \in \{1, 2\} \text{ and } j \in \{1, 2\} \text{ and } j \neq i. \end{aligned} \tag{4.3}$$

Performing the indicated maximization in (4.3) yields:

$$\phi_i^*(t,x) = \frac{x}{4\left[c_i + V_x^{(t_0)i}\exp[r(t-t_0)]x^{1/2}\right]^2}, \text{ for } i \in \{1,2\}.$$

To completely characterize a feedback solution, we derive the feedback Nash equilibrium payoffs of the extractors in the game (4.1) and (4.2) as:

Proposition 4.1 The feedback Nash equilibrium payoff of extractor $i \in \{1,2\}$ in the game (4.1) and (4.2) is:

$$V^{(t_0)i}(t,x) = \exp[-r(t-t_0)]\left[A_i(t)x^{1/2} + C_i(t)\right], \quad (4.4)$$

where for $i,j \in \{1,2\}$ and $i \neq j, A_i(t), B_i(t), A_j(t)$ and $B_j(t)$ satisfy:

$$\begin{aligned}
\dot{A}_i(t) &= \left[r + \frac{1}{8}\sigma^2 + \frac{b}{2}\right]A_i(t) - \frac{1}{2[c_i + A_i(t)/2]} + \frac{c_i}{4[c_i + A_i(t)/2]^2} \\
&\quad + \frac{A_i(t)}{8[c_i + A_i(t)/2]^2} + \frac{A_i(t)}{8\left[c_j + A_j(t)/2\right]^2}. \\
\dot{C}_i(t) &= rC_i(t) - \frac{a}{2}A_i(t), \\
A_i(T) &= q \text{ and } C_i(T) = 0
\end{aligned}$$

Proof First substitute $\phi_1^*(t,x)$ and $\phi_2^*(t,x), V^{(t_0)i}(t,x)$ from (4.4) and the corresponding derivatives $V_t^{(t_0)i}(t,x), V_x^{(t_0)i}(t,x)$ and $V_{xx}^{(t_0)i}(t,x)$ into (4.3). Upon solving (4.3) one obtains Proposition 4.1. ■

Invoking Remark 4.1 in Chap. 2, we can obtain the feedback Nash equilibrium payoff of player i in the game with dynamics (4.1) and expected payoffs (4.2) which starts at time τ for $\tau \in [t_0, T)$ as:

$$V^{(\tau)i}(t,x) = \exp[-r(t-\tau)]\left[A_i(t)x^{1/2} + B_i(t)\right], \text{ for } i \in \{1,2\}.$$

3.4.1 Cooperative Extraction Under Uncertainty

Now consider the case when the resource extractors agree to act cooperatively and follow the optimality principle under which they would

(i) maximize their joint expected payoffs and
(ii) share the excess of the total expected cooperative payoff over the sum of expected individual noncooperative payoffs proportional to the extractors' expected noncooperative payoffs.

Hence the extractors maximize the sum of their expected profits:

$$E_{t_0}\Bigg\{ \int_{t_0}^{T}\left(\left[u_1(s)^{1/2}-\frac{c_1}{x(s)^{1/2}}u_1(s)\right]+\left[u_2(s)^{1/2}-\frac{c_2}{x(s)^{1/2}}u_2(s)\right]\right)\,\exp[-r(t-t_0)]\,ds + 2\exp[-r(T-t_0)]qx(T)^{\frac{1}{2}}\Bigg\}, \tag{4.5}$$

subject to the stochastic dynamics (4.1).

Invoking Theorem A.3 in the Technical Appendices yields the characterization of solution of the problem (4.1) and (4.5) as a set of controls $\{u_i^*(t)=\psi_i^*(t,x)$, for $i\in\{1,2\}\}$ which satisfies the following partial differential equation:

$$\begin{aligned}
&-W_t^{(t_0)}(t,x)-\frac{1}{2}\sigma^2x^2W_{xx}^{(t_0)}(t,x)\\
&\quad=\max_{u_1,u_2}\Bigg\{\left(\left[{u_1}^{1/2}-\frac{c_1}{x^{1/2}}u_1\right]+\left[{u_2}^{1/2}-\frac{c_2}{x^{1/2}}u_2\right]\right)\exp[-r(t-t_0)]\\
&\qquad+W_x^{(t_0)}(t,x)\left[ax^{1/2}-bx-u_1-u_2\right]\Bigg\},\text{ and}\\
&W^{(t_0)}(T,x)=2\exp[-r(T-t_0)]qx^{\frac{1}{2}}.
\end{aligned}\tag{4.6}$$

Performing the indicated maximization we obtain:

$$\begin{aligned}
\psi_1^{(t_0)*}(t,x)&=\frac{x}{4\left[c_1+W_x^{(t_0)}\exp[r(t-t_0)]x^{1/2}\right]^2},\text{ and}\\
\psi_2^{(t_0)*}(t,x)&=\frac{x}{4\left[c_2+W_x^{(t_0)}\exp[r(t-t_0)]x^{1/2}\right]^2}.
\end{aligned}\tag{4.7}$$

The maximized expected joint profit of the extractors can be obtained as:

Proposition 4.2

$$W^{(t_0)}(t,x)=\exp[-r(t-t_0)]\left[A(t)x^{1/2}+C(t)\right],\tag{4.8}$$

where

$$\begin{aligned}
\dot{A}(t)&=\left[r+\frac{\sigma^2}{8}+\frac{b}{2}\right]A(t)-\frac{1}{2[c_1+A(t)/2]}-\frac{1}{2[c_2+A(t)/2]}+\frac{c_1}{4[c_1+A(t)/2]^2}\\
&\quad+\frac{c_2}{4[c_2+A(t)/2]^2}+\frac{A(t)}{8[c_1+A(t)/2]^2}+\frac{A(t)}{8[c_2+A(t)/2]^2},\\
\dot{C}(t)&=rC(t)-\frac{a}{2}A(t),\ A(T)=2q,\ \text{and } C(T)=0.
\end{aligned}$$

Proof Upon substituting the optimal strategies in (4.7), $W^{(t_0)}(t,x)$ in (4.8), and the relevant derivatives $W_t^{(t_0)}(t,x), W_x^{(t_0)}(t,x)$ and $W_{xx}^{(t_0)}(t,x)$ into (4.6) yields the results in Proposition 4.2. ■

The optimal cooperative controls can then be obtained as:

$$\psi_1^*(t,x) = \frac{x}{4[c_1 + A(t)/2]^2} \text{ and } \psi_2^*(t,x) = \frac{x}{4[c_2 + A(t)/2]^2}. \tag{4.9}$$

Substituting these control strategies into (4.1) yields the dynamics of the state trajectory under cooperation:

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - \frac{x(s)}{4[c_1 + A(s)/2]^2} - \frac{x(s)}{4[c_2 + A(s)/2]^2} \right] ds \\ + \sigma x(s)dz(s),\ x(t_0) = x_0. \tag{4.10}$$

Solving (4.11) yields the optimal cooperative state trajectory as:

$$x^*(s) = \varpi(t_0,s)^2 \left[x_0^{1/2} + \int_{t_0}^{s} \varpi^{-1}(t_0,t) H_1 dt \right]^2, \text{ for } s \in [t_0,T] \tag{4.11}$$

Where $\varpi(t_0,s) = \exp\left[\int_{t_0}^{s} \left[H_2(\tau) - \frac{\sigma^2}{8}\right] dv + \int_{t_0}^{s} \frac{\sigma}{2} dz(v)\right], H_1 = \frac{1}{2}a,$

and $H_2(s) = -\left[\frac{1}{2}b + \frac{1}{8[c_1+A(s)/2]^2} + \frac{1}{8[c_2+A(s)/2]^2} + \frac{\sigma^2}{8} \right].$

The cooperative control for the game $\Gamma_c(x_0, T - t_0)$ over the time interval $[t_0,T]$ along the optimal trajectory can be expressed as:

$$\psi_1^*(t,x_t^*) = \frac{x_t^*}{4[c_1 + A(t)/2]^2}, \text{ and } \psi_2^*(t,x_t^*) = \frac{x_t^*}{4[c_2 + A(t)/2]^2}, \\ \text{for } t \in [t_0,T] \text{ and } x_t^* \in X_t^*. \tag{4.12}$$

3.4.2 Subgame Consistent Cooperative Extraction

The agreed-upon optimality principle requires the extractors to share the excess of the total expected cooperative payoff over the sum of individual noncooperative payoffs proportional to the extractors' expected noncooperative payoffs. Therefore the following imputation has to be satisfied.

Condition 4.1 An imputation

$$\xi^{(\tau)i}\left(\tau,x_\tau^*\right)=\frac{V^{(\tau)i}\left(\tau,x_\tau^*\right)}{\sum_{j=1}^{2}V^{(\tau)j}\left(\tau,x_\tau^*\right)}W^{(\tau)}\left(\tau,x_\tau^*\right)$$
$$=\frac{\left[A_i(\tau)\left(x_\tau^*\right)^{1/2}+C_i(\tau)\right]}{\sum_{j=1}^{2}\left[A_j(\tau)\left(x_\tau^*\right)^{1/2}+C_j(\tau)\right]}\left[A(\tau)\left(x_\tau^*\right)^{1/2}+C(\tau)\right] \tag{4.13}$$

is assigned to extractor i, for $i\in\{1,2\}$ if $x_\tau^*\in X_\tau^*$ occurs at time $\tau\in[t_0,T]$. ■

Applying Theorem 3.1 a subgame-consistent solution under the optimal principle $P(x_0,T-t_0)$ for the cooperative game $\Gamma_c(x_0,T-t_0)$ can be obtained as: $\left\{\ u\left(s,x_s^*\right)\text{ and }B(s,x_s^*)\text{ for }s\in[t_0,T]\text{ and }\xi^{(t_0)}(t_0,x_0)\right\}$ in which

(i) $u(s,x_s^*)$ for $s\in[t_0,T]$ is the set of group optimal strategies $\psi_1^*\left(s,x_s^*\right)=\frac{x_s^*}{4[c_1+A(s)/2]^2}$, and $\psi_2^*\left(s,x_s^*\right)=\frac{x_s^*}{4[c_2+A(s)/2]^2}$; and

(ii) $B\left(s,x_s^*\right)=\left\{B_1\left(s,x_s^*\right),B_2\left(s,x_2^*\right)\right\}$ for $s\in[t_0,T]$ where

$$B_i\left(s,x_s^*\right)=-\left[\xi_t^{(s)i}\left(t,x_t^*\right)\Big|_{t=s}\right]$$
$$-\left[\xi_{x_s^*}^{(s)i}\left(s,x_s^*\right)\right]\left[\ a\left(x_s^*\right)^{1/2}-bx_s^*-\frac{x_s^*}{4[c_1+A(s)/2]^2}-\frac{x_s^*}{4[c_2+A(s)/2]^2}\ \right]$$
$$-\frac{1}{2}\sigma^2\left(x_s^*\right)^2\left[\xi_{x_s^*x_s^*}^{(s)i}\left(s,x_s^*\right)\right],\ \text{for } i\in\{1,2\} \tag{4.14}$$

where

$$\left[\xi_t^{(s)i}\left(t,x_t^*\right)\Big|_{t=s}\right]$$
$$=\frac{\left[A_i(s)\left(x_s^*\right)^{1/2}+C_i(s)\right]}{\left(\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{1/2}+C_j(s)\right]\right)}\left\{\left[\dot{A}\left(s\right)\left(x_s^*\right)^{1/2}+\dot{C}\left(s\right)\right]-r\left[A(s)\left(x_s^*\right)^{1/2}+C(s)\right]\right\}$$
$$+\frac{\left[A(s)\left(x_s^*\right)^{1/2}+B(s)\right]}{\left(\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{1/2}+B_j(s)\right]\right)}$$
$$\left\{\left[\dot{A}_i(s)\left(x_s^*\right)^{1/2}+\dot{B}_i(s)\right]-r\left[A_i(s)\left(x_s^*\right)^{1/2}+B_i(s)\right]\right\}$$

$$-\frac{\left[A_i(s)\left(x_s^*\right)^{1/2}+B_i(s)\right]\left[A(s)\left(x_s^*\right)^{1/2}+B(s)\right]}{\left(\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{1/2}+B_j(s)\right]\right)^2}$$

$$\times\sum_{j=1}^{2}\left\{\left[\dot{A}_j(s)\left(x_s^*\right)^{1/2}+\dot{C}_j(s)\right]-r\left[A_j(s)\left(x_s^*\right)^{1/2}+C_j(s)\right]\right\}$$

$$\left[\xi_{x_s^*}^{(s)i}\left(s,x_s^*\right)\right]=$$
$$\frac{\left[A_i(s)\left(x_s^*\right)^{1/2}+C_i(s)\right]A(s)\left(x_s^*\right)^{-1/2}+\left[A(s)\left(x_s^*\right)^{1/2}+C(s)\right]A_i(s)\left(x_s^*\right)^{-1/2}}{2\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{1/2}+C_j(s)\right]}$$
$$-\frac{\left[A_i(s)\left(x_s^*\right)^{1/2}+C_i(s)\right]\left[A(s)\left(x_s^*\right)^{1/2}+C(s)\right]}{\left(\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{1/2}+C_j(s)\right]\right)^2}\left(\frac{1}{2}\sum_{j=1}^{2}A_j(s)\left(x_s^*\right)^{-1/2}\right);$$

and

$$\left[\xi_{x_s^*x_s^*}^{(s)i}\left(s,x_s^*\right)\right]=-\frac{C_i(s)A(s)\left(x_s^*\right)^{-3/2}+C(s)A_i(s)\left(x_s^*\right)^{-3/2}}{4\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{1/2}+C_j(s)\right]}$$
$$-\frac{\left[A_i(s)\left(x_s^*\right)^{1/2}+C_i(s)\right]A(s)\left(x_s^*\right)^{-1/2}+\left[A(s)\left(x_s^*\right)^{1/2}+C(s)\right]A_i(s)\left(x_s^*\right)^{-1/2}}{\left(2\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{1/2}+C_j(s)\right]\right)^2}$$
$$\times\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{-1/2}\right]$$
$$+\frac{\left[A_i(s)\left(x_s^*\right)^{1/2}+C_i(s)\right]\left[A(s)\left(x_s^*\right)^{1/2}+C(s)\right]}{\left(\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{1/2}+C_j(s)\right]\right)^2}\left(\frac{1}{4}\sum_{j=1}^{2}A_j(s)\left(x_s^*\right)^{-3/2}\right)$$
$$-\left(\frac{1}{2}\sum_{j=1}^{2}A_j(s)\left(x_s^*\right)^{-1/2}\right)$$
$$\times\left[\frac{A_i(s)A(s)+\frac{1}{2}\left[A_i(s)C(s)+A(s)C_i(\tau)\right]\left(x_s^*\right)^{-1/2}}{\left(\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{1/2}+C_j(s)\right]\right)^2}\right.$$
$$\left.-\frac{\left[A_i(s)\left(x_s^*\right)^{1/2}+C_i(s)\right]\left[A(s)\left(x_s^*\right)^{1/2}+C(s)\right]}{\left(\sum_{j=1}^{2}\left[A_j(s)\left(x_s^*\right)^{1/2}+C_j(s)\right]\right)^3}\sum_{j=1}^{2}A_j(s)\left(x_s^*\right)^{-1/2}\right].$$

With extractors using the cooperative strategies in (4.13), the instantaneous receipt of extractor i at time instant τ is:

$$\zeta_i(\tau, x_\tau^*) = \frac{(x_\tau^*)^{1/2}}{2[c_i + A(\tau)/2]} - \frac{c_i(x_\tau^*)^{1/2}}{4[c_i + A(\tau)/2]^2},$$
$$\text{for } \tau \in [t_0, T], x_\tau^* \in X_\tau^* \text{ and } i \in \{1, 2\}. \tag{4.15}$$

Under cooperation the instantaneous payment that extractor $i \in \{1, 2\}$ should receive $B_i(\tau, x_\tau^*)$ in (4.15). Hence an instantaneous transfer payment

$$\chi^i(\tau, x_\tau^*) = B_i(\tau, x_\tau^*) - \zeta_i(\tau, x_\tau^*) \tag{4.16}$$

has to be given to extractor i at time τ, for $i \in \{1, 2\}$ and $\tau \in [t_0, T]$ when the state is $x_\tau^* \in X_\tau^*$.

3.5 Infinite Horizon Subgame Consistency Under Uncertainty

Consider the infinite stochastic differential game in which player i seeks to

$$\max_{u_i} E_\tau \left\{ \int_\tau^\infty g^i[x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp[-r(s-\tau)]\, ds \right\},$$
$$\text{for } i \in N, \tag{5.1}$$

subject to the stochastic dynamics

$$dx(s) = f[x(s), u_1(s), u_2(s), \cdots, u_n(s)]ds + \sigma[x(s)]dz(s), \qquad x(\tau) = x_\tau. \tag{5.2}$$

Consider the alternative game which starts at time $t \in [t_0, \infty)$ with initial state $x(t) = x$:

$$\max_{u_i} E_t \left\{ \int_t^\infty g^i[x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp[-r(s-t)] ds \right\},$$
$$\text{for } i \in N, \tag{5.3}$$

subject to the stochastic dynamics

$$dx(s) = f[x(s), u_1(s), u_2(s), \cdots, u_n(s)]ds + \sigma[x(s)]dz(s), \qquad x(t) = x_t. \tag{5.4}$$

Let $\Omega[x(s)] = \sigma[x(s)]\sigma[x(s)]^T$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[x(s)]$.

The infinite horizon autonomous game (5.4) and (5.5) is independent of the choice of t and dependent only upon the state at the starting time, that is x. A Nash equilibrium solution for the infinite-horizon stochastic differential game (5.4) and (5.5) can be characterized by the following theorem.

Theorem 5.1 An n-tuple of strategies $\{u_i^* = \phi_i^*(\cdot) \in U^i$, for $i \in N\}$ provides a Nash equilibrium solution to the game (5.4) and (5.5) if there exist continuously twice differentiable functions $\hat{V}^i(x) : R^m \to R$, $i \in N$, satisfying the following set of partial differential equations:

$$r\hat{V}^i(x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(x)\hat{V}^i_{x^h x^\zeta}(x)$$
$$= \max_{u_i} \left\{ g^i\left[x, \phi_1^*(x), \phi_2^*(x), \cdots, \phi_{i-1}^*(x), u_i(x), \phi_{i+1}^*(x), \cdots, \phi_n^*(x)\right] \right.$$
$$\left. + \hat{V}^i_x(x) f\left[x, \phi_1^*(x), \phi_2^*(x), \cdots, \phi_{i-1}^*(x), u_i(x), \phi_{i+1}^*(x), \cdots, \phi_n^*(x)\right] \right\}$$
$$= \left\{ g^i\left[x, \phi_1^*(x), \phi_2^*(x), \cdots, \phi_n^*(x)\right] + \hat{V}^i_x(x) f\left[x, \phi_1^*(x), \phi_2^*(x), \cdots, \phi_n^*(x)\right] \right\},$$

for $i \in N$.

Proof This result follows readily from the definition of Nash equilibrium and from the infinite horizon stochastic control Theorem A.4 in the Technical Appendices. ■

Now consider the case when the players agree to act cooperatively. Let $\Gamma_c(\tau, x_\tau)$ denote a cooperative game in which player i's payoff is (5.2) and the state dynamics is (5.3). The players agree to act according to an agreed upon optimality principle which entails

(i) group optimality and
(ii) the distribution of the total cooperative payoff according to an imputation which equals $\xi^{(\upsilon)}(\upsilon, x_\upsilon^*)$ for $\upsilon \in [\tau, \infty)$ over the game duration. Moreover, the function $\xi^{(\upsilon)i}(\upsilon, x_\upsilon^*)$, for $i \in N$, is continuously differentiable in υ and x_υ^*.

The solution of the cooperative game $\Gamma_c(\tau, x_\tau)$ under the agreed-upon optimality principle includes

(i) a set of cooperative strategies

$$u^{(\tau)*}(s, x_s^*) = \left[u_1^{(\tau)*}(s, x_s^*), u_2^{(\tau)*}(s, x_s^*), \cdots, u_n^{(\tau)*}(s, x_s^*)\right], \text{ for } s \in [\tau, \infty);$$

(ii) an imputation vector $\xi^{(\tau)}(\tau, x_\tau) = \left[\xi^{(\tau)1}(\tau, x_\tau), \xi^{(\tau)2}(\tau, x_\tau), \cdots, \xi^{(\tau)n}(\tau, x_\tau)\right]$ to allot the cooperative payoff to the players; and
(iii) a payoff distribution procedure $B^\tau(s, x_s^*) = \left[B_1^\tau(s, x_s^*), B_2^\tau(s, x_s^*), \cdots, B_n^\tau(s, x_s^*)\right]$ for $s \in [\tau, \infty)$, where $B_i^\tau(s, x_s^*)$ is the i at time s when the state is $x_s^* \in X_s^*$. In particular,

$$\xi^{(\tau)i}(\tau, x_\tau) = E_\tau\left\{ \int_\tau^\infty B_i^\tau(s, x_s^*) \exp[-r(s-\tau)]\, ds \right\}, \text{ for } i \in N. \tag{5.5}$$

3.5.1 Group Optimal Cooperative Strategies

To ensure group rationality the players maximize the sum of their expected payoffs, the players solve the problem:

$$\max_{u_1, u_2, \cdots, u_n} E_\tau \left\{ \int_\tau^\infty \sum_{j=1}^n g^j[x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp[-r(s-\tau)] ds \right\}, \quad (5.6)$$

subject to (5.3).

Invoking Theorem A.4 in the Technical Appendices, a set of controls $\{\psi_i^*(x) \in U^i; i \in N\}$ constitutes an optimal solution to the infinite horizon stochastic control problem (5.3) and (5.7) if there exists continuously twice differentiable function $W(x)$ defined on $R^m \to R$ which satisfies the following equation:

$$rW(x) - \frac{1}{2} \sum_{h,\zeta=1}^m \Omega^{h\zeta}(x) W_{x^h x^\zeta}(x)$$
$$= \max_{u_1, u_2, \cdots, u_n} \left\{ \sum_{j=1}^n g^j[x, u_1, u_2, \cdots, u_n] + W_x(x) f[x, u_1, u_2, \cdots, u_n] \right\}. \quad (5.7)$$

Hence the players will adopt the cooperative control $\{\psi_i^*(x)$, for $i \in N\}$ to obtain the maximized level of expected joint profit. Substituting this set of control into (6.5) yields the dynamics of the optimal (cooperative) trajectory as:

$$dx(s) = f\left[x(s), \psi_1^*(x(s)), \psi_2^*(x(s)), \cdots, \psi_n^*(x(s))\right] ds + \sigma[x(s)] dz(s), \;\; x(\tau) = x_\tau. \quad (5.8)$$

The solution to (5.9) can be expressed as:

$$x^*(s) = x_\tau + \int_\tau^s f\left[x^*(\upsilon), \psi_1^*(x^*(\upsilon)), \psi_2^*(x^*(\upsilon)), \cdots, \psi_n^*(x^*(\upsilon))\right] d\upsilon$$
$$+ \int_\tau^s \sigma\left[x^*(\upsilon)\right] dz(\upsilon). \quad (5.9)$$

We use X_s^* to denote the set of realizable values of $x^*(s)$ at time s generated by (5.9). The term $x_s^* \in X_s^*$ is used to denote an element in X_s^*. The terms $x^*(s)$ and x_s^* will be used interchangeably in case where there is no ambiguity.

The expected cooperative payoff can be expressed as:

$$W\left(x_\tau^*\right) = E_\tau\Bigg\{ \int_\tau^\infty \sum_{j=1}^n g^j\left[x^*(s), \psi_1^*\left(x^*(s)\right), \psi_2^*\left(x^*(s)\right), \cdots, \psi_n^*\left(x^*(s)\right)\right] \exp[-r(s-\tau)] ds \,\Bigg|\, x^*(\tau) = x_\tau^* \Bigg\}.$$

Moreover, one can easily verify that the joint payoff maximizing controls for the cooperative game $\Gamma_c(\tau, x_\tau)$ over the time interval $[t, \infty)$ is identical to the joint payoff maximizing controls for the cooperative game $\Gamma_c(t, x_t^*)$ over the time interval $[t, \infty)$.

3.5.2 Subgame Consistent Imputation and Payoff Distribution Procedure

In the game $\Gamma_c(t, x_t^*)$, according to optimality principle the players would use the Payoff Distribution Procedure $\left\{B^\tau\left(s, x_s^*\right)\right\}_{s=\tau}^{\infty}$ to bring about an imputation to player i such that:

$$\xi^{(\tau)i}(\tau, x_\tau) = E_\tau\left\{ \int_\tau^\infty B_i^\tau\left(s, x_s^*\right) \exp[-r(s-\tau)] \; ds \right\}, \text{ for } i \in N.$$

We define

$$\xi^{(\tau)i}\left(t, x_t^*\right) = E_\tau\left\{ \int_t^\infty B_i^\tau\left(s, x_s^*\right) \exp[-r(s-\tau)] \; ds \,\middle|\, x(t) = x_t^* \in X_t^* \right\}, \quad \text{for } i \in N, \tag{5.10}$$

where $t > \tau$ and $x_t^* \in \left\{x^*(s)\right\}_{s=\tau}^{\infty}$.

At time τ, according to $P(\tau, x_\tau)$ player i is supposed to receive a payoff $\xi^{(\tau)i}(t, x_t^*)$ over the remaining time interval $[t, \infty)$ if the state is $x_t^* \in X_t^*$.

Consider the case when the game has proceeded to time t and the state variable becames $x_t^* \in X_t^*$. Then one has a cooperative game $\Gamma_c(t, x_t^*)$ which starts at time t with initial state x_t^*. According to the agreed-upon optimality principle, an imputation

$$\xi^{(t)i}\left(t, x_t^*\right) = E_t\left\{ \int_t^\infty B_i^t\left(s, x_s^*\right) \exp[-r(s-t)] \; ds \,\middle|\, x(t) = x_t^* \in X_t^* \right\},$$

will be allotted to player i, for $i \in N$.

However, according to the solution to the game $\Gamma_c(\tau, x_\tau)$, the imputation (in present value viewed at time τ) to player i over the period $[t, \infty)$ is $\xi^{(\tau)i}(t, x_t^*)$

in (5.11). For the imputation from $\Gamma_c(\tau, x_\tau)$ to be consistent with those from $\Gamma_c(t, x_t^*)$, it is required that

$$\begin{aligned}&\exp[r(t-\tau)]\xi^{(\tau)i}\left(t, x_t^*\right) = \xi^{(t)i}\left(t, x_t^*\right) \text{ from the game } \Gamma_c\left(t, x_t^*\right)\\&\text{under the same optimality principle,} \qquad \text{for } t \in (\tau, \infty).\end{aligned} \tag{5.11}$$

The payoff distribution procedure of the game $\Gamma_c(\tau, x_\tau)$ according to the agreed-upon optimality principle is

$$B^\tau\left(s, x_s^*\right) = \left[B_1^\tau\left(s, x_s^*\right), B_2^\tau\left(s, x_s^*\right), \cdots, B_n^\tau\left(s, x_s^*\right)\right], \text{ for } s \in \left[\tau, \infty\right) \text{ and } x_s^* \in X_s^*.$$

When the game has proceeded to time t and the state variable became $x_t^* \in X_t^*$, we have the game $\Gamma_c(t, x_t^*)$. According to the agreed-upon optimality principle the payoff distribution procedure of the game $\Gamma_c(t, x_t^*)$ is

$$B^t\left(s, x_s^*\right) = \left[B_1^t\left(s, x_s^*\right), B_2^t\left(s, x_s^*\right), \cdots, B_n^t\left(s, x_s^*\right)\right], \text{ for } s \in \left[t, \infty\right) \text{ and } x_s^* \in X_s^*.$$

For the continuation of the payoff distribution procedure $B^\tau(s, x_s^*)$ to be consistent with $B^t(s, x_s^*)$, it is required that

$$B^{t_0}\left(s, x_s^*\right) = B^t\left(s, x_s^*\right), \text{ for } s \in \left[t, \infty\right) \text{ and } t \in \left[\tau, \infty\right) \text{ and } x_s^* \in X_s^*.$$

Definition 5.1 The imputation and payoff distribution procedure $\left\{\xi^{(\tau)}(\tau, x_\tau) \text{ and } B^\tau(s, x_s^*) \text{ for } s \in \left[\tau, \infty\right)\right\}$ are subgame consistent if

(i)
$$\begin{aligned}&\exp[r(t-\tau)]\,\xi^{(\tau)i}\left(t, x_t^*\right)\\&\equiv \exp[r(t-\tau)]E_\tau\left\{\int_t^\infty B_i^\tau\left(s, x_s^*\right)\exp[-r(s-\tau)]\;ds \,\middle|\, x(t) = x_t^* \in X_t^*\right\}\\&= \xi^{(t)i}\left(t, x_t^*\right), \text{ for } t \in (\tau, \infty) \text{ and } i \in N; \text{ and}\end{aligned} \tag{5.12}$$

(ii) the payoff distribution procedure $B^\tau(s, x_s^*)$ for $s \in \left[t, \infty\right)$ is identical to $B^t(s, x_s^*)$. ■

3.5.3 *Payoff Distribution Procedure Leading to Subgame Consistency*

A payoff distribution procedure leading to subgame consistent imputation has to satisfy Definition 5.1. Invoking Definition 5.1, we have $B_i^\tau\left(s, x_s^*\right) = B_i^t\left(s, x_s^*\right) = B_i\left(s, x_s^*\right)$, for $s \in \left[\tau, \infty\right)$, $x_s^* \in X_s^*$ and $t \in \left[\tau, \infty\right)$ and $i \in N$.

Therefore along the cooperative trajectory,

$$\xi^{(\tau)i}(\tau, x_\tau) = E_\tau\left\{ \int_\tau^\infty B_i\left(s, x_s^*\right)\exp[-r(s-\tau)]ds \right\}, \text{ for } i \in N, \text{ and}$$
$$\xi^{(\upsilon)i}\left(\upsilon, x_\upsilon^*\right) = E_\upsilon\left\{ \int_\upsilon^\infty B_i\left(s, x_s^*\right)\exp[-r(s-\upsilon)]ds \middle| x(\upsilon) = x_\upsilon^* \in X_\upsilon^* \right\}, \text{ for } i \in N, \text{ and}$$
$$\xi^{(t)i}\left(t, x_t^*\right) = E_t\left\{ \int_t^\infty B_i\left(s, x_s^*\right)\exp[-r(s-t)]ds \middle| x(t) = x_t^* \in X_t^* \right\},$$
$$\text{for } i \in N \text{ and } t \geq \upsilon \geq \tau. \tag{5.13}$$

Moreover, for $i \in N$ and $t \in [\tau, \infty)$, we define the term

$$\xi^{(\upsilon)i}\left(t, x_t^*\right) = E_\upsilon\left\{ \left(\int_t^\infty B_i\left(s, x_s^*\right)\exp[-r(s-\upsilon)]ds \right) \middle| x(t) = x_t^* \right\}, \tag{5.14}$$

to denote the present value of player i's cooperative payoff over the time interval $[t, \infty)$, given that the state is x_t^* at time $t \in [\upsilon, \infty)$, under the optimality principle $P(\upsilon, x_\upsilon^*)$.

Invoking (5.14) and (5.15) one can readily verify that $\exp[r(t-\tau)]\xi^{(\tau)i}\left(t, x_t^*\right) = \xi^{(t)i}\left(t, x_t^*\right)$, for $i \in N$ and $\tau \in [t_0, T]$ and $t \in [\tau, T]$.

The next task is to derive $B_i(s, x_s^*)$, for $s \in [\tau, \infty)$ and $t \in [\tau, \infty)$ so that (5.14) can be realized. Consider again the following condition.

Condition 5.1 For $i \in N$ and $t \geq \upsilon$ and $\upsilon \in [\tau, T]$, the term $\xi^{(\upsilon)i}(t, x_t^*)$ is a function that is continuously differentiable in t and x_t^*.

A theorem characterizing a formula for $B_i(s, x_s^*)$, for $i \in N$ and $s \in [\upsilon, \infty)$, which yields (5.15) is provided as follows.

Theorem 5.2 If Condition 5.1 is satisfied, a PDP with instantaneous payments at time s with the state being $x_s^* \in X_s^*$ equaling

$$\begin{aligned} B_i\left(s, x_s^*\right) = &-\left[\xi_t^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right] \\ &-\left[\xi_{x_t^*}^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right] f\left[x_s^*, \psi_1^*\left(s, x_s^*\right), \psi_2^*\left(s, x_s^*\right), \cdots, \psi_n^*\left(s, x_s^*\right)\right] \\ &-\frac{1}{2}\sum_{h,\zeta=1}^m \Omega^{h\zeta}\left(x_s^*\right)\left[\xi_{x_t^h x_t^\zeta}^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right], \text{ for } i \in N \text{ and } s \in [\upsilon, \infty), \end{aligned} \tag{5.15}$$

yields imputation $\xi^{(\upsilon)i}(\upsilon, x_\upsilon^*)$ for $\upsilon \in [\tau, \infty)$ and $x_\upsilon^* \in X_\upsilon^*$ which satisfy (5.14).

Proof Note that along the cooperative trajectory

$$\begin{aligned} \xi^{(\upsilon)i}\left(t, x_t^*\right) &= E_\upsilon\left\{ \int_t^\infty B_i\left(s, x_s^*\right)\exp[-r(s-\upsilon)]ds \middle| x(t) = x_t^* \in X_t^* \right\} \\ &= \exp[-r(t-\upsilon)]\xi^{(t)i}\left(t, x_t^*\right), \text{ for } i \in N \text{ and } t \in [\upsilon, \infty). \end{aligned} \tag{5.16}$$

For $\Delta t \to 0$, equation (5.14) can be expressed as

$$\xi^{(\upsilon)i}\left(\upsilon, x_\upsilon^*\right) = E_\upsilon\left\{ \int_\upsilon^\infty B_i\left(s, x_s^*\right)\exp[-r(s-\upsilon)]ds \right\}$$
$$= E_\upsilon\left\{ \int_\upsilon^{\upsilon+\Delta t} B_i\left(s, x_s^*\right)\exp[-r(s-\upsilon)]\ ds + \xi^{(\upsilon)i}\left(\upsilon+\Delta t, x_\upsilon^* + \Delta x_\upsilon^*\right) \right\}, \quad (5.17)$$

where

$$\Delta x_\upsilon^* = f\left[x_\upsilon^*, \psi_1^*\left(x_\upsilon^*\right), \psi_2^*\left(x_\upsilon^*\right), \cdots, \psi_n^*\left(x_\upsilon^*\right)\right]\Delta t + \sigma\left(x_\upsilon^*\right)\Delta z_\upsilon + o(\Delta t),$$
$$\Delta z_\upsilon = Z(\upsilon+\Delta t) - z(\upsilon),\ \text{and}\ E_\upsilon[o(\Delta t)]/\Delta t \to 0 \text{ as } \Delta t \to 0.$$

Replacing the term $x_\upsilon^* + \Delta x_\upsilon^*$ with $x_{\upsilon+\Delta t}^*$ and rearranging (5.18) yields:

$$E_\upsilon\left\{ \int_\upsilon^{\upsilon+\Delta t} B_i(s)\exp[-r(s-\upsilon)]\ ds \right\} = E_\upsilon\left\{ \xi^{(\upsilon)i}\left(\upsilon, x_\upsilon^*\right) - \xi^{(\upsilon)i}\left(\upsilon+\Delta t, x_{\upsilon+\Delta t}^*\right) \right\},$$
$$\text{for all } \upsilon \in [\tau, \infty) \text{ and } i \in N. \quad (5.18)$$

With Condition 5.1 holding and $\Delta t \to 0$, (5.19) can be expressed as:

$$E_\upsilon\left\{ B_i\left(s, x_s^*\right)\Delta t + o(\Delta t) \right\} = E_\upsilon\left\{ -\left[\xi_t^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right]\Delta t \right.$$
$$-\left[\xi_{x_t^*}^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right] f\left[x_s^*, \psi_1^*\left(s, x_s^*\right), \psi_2^*\left(s, x_s^*\right), \cdots, \psi_n^*\left(s, x_s^*\right)\right]\Delta t$$
$$\left. -\frac{1}{2}\sum_{h,\zeta=1}^m \Omega^{h\zeta}\left(x_s^*\right)\left[\xi_{x_t^h x_t^\zeta}^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right]\Delta t - \left[\xi_{x_t^*}^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right]\sigma\left(x_\upsilon^*\right)\Delta z_\upsilon - o(\Delta t) \right\}. (5.19)$$

Dividing (5.20) throughout by Δt, with $\Delta t \to 0$ and taking expectation yields (5.16). Thus the payoff distribution procedure in $B_i(\upsilon, x_\upsilon^*)$ in (5.16) would lead to the realization of the imputations which satisfy (5.14). ■

Since the payoff distribution procedure in $B_i(\tau)$ in (5.16) leads to the realization of (5.14), it would yields subgame consistent imputations satisfying Definition 5.1.

A more succinct form of the PDP instantaneous payment in (5.14) can be derived as follows. First we define

$$\hat{\xi}^i\left(x_\upsilon^*\right) = E_\upsilon\left\{ \int_\upsilon^\infty B_i(s)\exp[-r(s-\upsilon)]ds \,\Big|\, x(\upsilon) = x_\upsilon^* \right\}\xi^{(\upsilon)i}\left(\tau, x_\upsilon^*\right), \text{ and}$$
$$\hat{\xi}^i\left(x_t^*\right) = E_t\left\{ \int_t^\infty B_i(s)\exp[-r(s-t)]ds \,\Big|\, x(t) = x_t^* \right\} = \xi^{(t)i}\left(t, x_t^*\right),$$

for $i \in N$ and $\upsilon \in [\tau, \infty)$ and $t \in [\upsilon, \infty)$ along the optimal cooperative trajectory $\{x_s^*\}_{s=\tau}^\infty$.

We then have:

$$\xi^{(v)i}\left(t, x_t^*\right) = \exp[-r(t-v)]\hat{\xi}^i\left(x_t^*\right).$$

Differentiating the above condition with respect to t yields:

$$\left[\xi_t^{(v)i}\left(t, x_t^*\right)\Big|_{t=v}\right] = -r\exp[-r(t-v)]\hat{\xi}^i\left(x_t^*\right) = -r\xi^{(v)i}\left(t, x_t^*\right).$$

At $t = v$, $\xi^{(v)i}\left(t, x_t^*\right) = \xi^{(v)i}\left(v, x_v^*\right)$, therefore

$$\left[\xi_t^{(v)i}\left(t, x_t^*\right)\Big|_{t=v}\right] = r\xi^{(v)i}\left(t, x_t^*\right) = r\xi^{(v)i}\left(v, x_v^*\right). \tag{5.20}$$

Substituting (5.21) into (5.16) yields,

$$B_i\left(s, x_s^*\right) = r\,\xi^{(s)i}\left(s, x_s^*\right) - \xi_{x_s^*}^{(s)i}\left(s, x_s^*\right) f\left[x_s^*, \psi_1^*\left(x_s^*\right), \psi_2^*\left(x_s^*\right), \cdots, \psi_n^*\left(x_s^*\right)\right]$$
$$-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(x_s^*\right)\left[\xi_{x_t^h x_t^\zeta}^{(s)i}\left(t, x_t^*\right)\Big|_{t=s}\right], \text{ for } i \in N,\ x_s^* \in X_s^* \text{ and } s \in [v, \infty). \tag{5.21}$$

An alternative form of Theorem 5.2 can be expressed as:

Theorem 5.3 A PDP with instantaneous payments with the state being x^* equaling

$$B_i\left(x^*\right) = r\,\hat{\xi}^i\left(x^*\right) - \xi_{x^*}^i\left(x^*\right) f\left[x^*, \psi_1^*\left(x^*\right), \psi_2^*\left(x^*\right), \cdots, \psi_n^*\left(x^*\right)\right]$$
$$-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(x^*\right)\hat{\xi}_{x^h x^\zeta}^i\left(x^*\right), \text{ for } i \in N. \tag{5.22}$$

yields imputation $\hat{\xi}^i\left(x^*\right)$.

Proof Multiplying (5.22) throughout by $\exp[r(t-v)]$ yields

$$B_i\left(x_s^*\right) = r\,\hat{\xi}^i\left(x_s^*\right) - \hat{\xi}_{x_s^*}^i\left(x_s^*\right) f\left[x_s^*, \psi_1^*\left(x_s^*\right), \psi_2^*\left(x_s^*\right), \cdots, \psi_n^*\left(x_s^*\right)\right]$$
$$-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}\left(x_s^*\right)\hat{\xi}_{x_s^h x_s^\zeta}^i\left(x_s^*\right), \text{ for } i \in N,\ x_s^* \in X_s^* \text{ and } s \in [v, \infty).$$

Recall that the infinite-horizon autonomous game $\Gamma(x)$ is independent of the choice of time s and dependent only upon the state, equation (5.22) can be expressed as (5.23). ■

With agents using the cooperative strategies, when the state is $x^* \in X^*$ the instantaneous receipt of agent i is:

$$\zeta_i\left(x^*\right) = g^i\left[x^*, \psi_1^*\left(x^*\right), \psi_2^*\left(x^*\right), \cdots, \psi_n^*\left(x^*\right)\right], \text{ for } i \in N. \tag{5.23}$$

According to Theorem 5.2 and (5.23), the instantaneous payment that player i should receive under the agreed-upon optimality principle is $B_i(x^*)$ as stated in (5.23). Hence an instantaneous transfer payment

$$\chi^i\left(x^*\right) = B_i\left(x^*\right) - \zeta_i\left(x^*\right), \text{ for } i \in N \tag{5.24}$$

has to be given to player i when the state is $x^* \in X^*$.

3.6 Infinite Horizon Cooperative Fishery Under Uncertainty

Consider an infinite horizon version of the cooperative fishery in Sect. 3.5. At time τ, the expected payoff of extractor 1 and that of extractor 2 are respectively:

$$E_\tau\left\{ \int_\tau^\infty \left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}} u_1(s) \right] \exp[-r(t-\tau)]\, ds \right\} \text{ and}$$
$$E_\tau\left\{ \int_\tau^\infty \left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}} u_2(s) \right] \exp[-r(t-\tau)]\, ds \right\}. \tag{6.1}$$

The fish resource stock $x(s) \in X \subset R$ follows the stochastic dynamics:

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - u_1(s) - u_2(s) \right] ds + \sigma x(s) dz(s), \; x(\tau) = x_\tau, \tag{6.2}$$

Invoking Theorem 5.1, the set of strategies $[\phi_1^*(x), \phi_2^*(x)]$ for $t \in [t_0, T]$ that provides a feedback Nash equilibrium solution to the game (6.2) and (6.3) can be characterized by:

$$r\hat{V}^i(x) - \frac{1}{2}\sigma^2 x^2 \hat{V}^i_{xx}(x) = \max_{u_i} \left\{ u_i^{1/2} - \frac{c_i}{x^{1/2}} u_i + \hat{V}^i_x(x) \left[ax^{1/2} - bx - u_i - \phi_j^*(x) \right] \right\}$$
$$\text{for } i, j \in \{1, 2\} \text{ and } i \neq j. \tag{6.3}$$

Performing the indicated maximization in (6.4) and using the derived game equilibrium strategies one obtains the value function of extractor $i \in \{1, 2\}$ as:

$$\hat{V}^i(t, x) = \left[A_i x^{1/2} + C_i \right], \tag{6.4}$$

where for $i, j \in \{1, 2\}$ and $i \neq j, A_i, C_i, A_j$ and C_j satisfy:

$$\left[r + \frac{\sigma^2}{8} + \frac{b}{2}\right] A_i - \frac{1}{2[c_i + A_i/2]} + \frac{c_i}{4[c_i + A_i/2]^2}$$
$$+ \frac{A_i}{8[c_i + A_i/2]^2} + \frac{A_i}{8\left[c_j + A_j/2\right]^2} = 0; \text{ and}$$
$$C_i = \frac{a}{2} A_i.$$

3.6.1 Cooperative Extraction

Consider the case when these two nations agree to act according to an agreed upon optimality principle which entails

(i) group optimality, and
(ii) the distribution of the excess of the total expected cooperative payoff over the sum of expected individual noncooperative payoffs proportional to the extractors' expected noncooperative payoffs.

To maximize their joint expected payoff for group optimality, the nations have to solve the stochastic control problem of maximizing

$$E_t \left\{ \int_t^\infty \left(\left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}} u_1(s) \right] + \left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}} u_2(s) \right] \right) \exp[-r(t-t)]\, ds \right\}. \tag{6.5}$$

subject to (6.3).

Invoking Theorem A.4 in the Technical Appendices yields the characterization of solution of the problem (6.3) and (6.6) as:

Corollary 6.1 A set of controls $\{\psi_i^*(x), \text{ for } i \in \{1, 2\}\}$ constitutes an optimal solution to the stochastic control problem (6.3) and (6.6), if there exist continuously twice differentiable functions $W(x) : R^m \to R$,, satisfying the following partial differential equation:

$$rW(x) - \frac{1}{2}\sigma^2 x^2 W_{xx}(x) = \max_{u_1, u_2} \left\{ \left(\left[u_1{}^{1/2} - \frac{c_1}{x^{1/2}} u_1 \right] + \left[u_2{}^{1/2} - \frac{c_2}{x^{1/2}} u_2 \right] \right) + W_x(x)\left[a x^{1/2} - bx - u_1 - u_2 \right] \right\}. \tag{6.6}$$ ■

Performing the indicated maximization and solving (6.7) one obtains the maximized expected joint profit can be derived as:

$$W(x) = \left[Ax^{1/2} + C\right], \tag{6.7}$$

where

$$\left[r + \frac{\sigma^2}{8} + \frac{b}{2}\right]A - \frac{1}{2[c_1 + A/2]} - \frac{1}{2[c_2 + A/2]}$$
$$+ \frac{c_1}{4[c_1 + A/2]^2} + \frac{c_2}{4[c_2 + A/2]^2} + \frac{A}{8[c_1 + A/2]^2} + \frac{A}{8[c_2 + A/2]^2} = 0, \text{ and}$$
$$C = \frac{a}{2r}A. \tag{6.8}$$

The optimal cooperative controls can then be obtained as:

$$\psi_1^*(x) = \frac{x}{4[c_1 + A/2]^2}, \text{ and } \psi_2^*(x) = \frac{x}{4[c_2 + A/2]^2}. \tag{6.9}$$

Substituting these control strategies into (6.3) yields the dynamics of the state trajectory under cooperation:

$$dx(s) = \left[ax(s)^{1/2} - bx(s) - \frac{x(s)}{4[c_1 + A/2]^2} - \frac{x(s)}{4[c_2 + A/2]^2} \right] ds$$
$$+ \sigma x(s) dz(s), \;\; x(t_0) = x_0. \tag{6.10}$$

Solving (6.11) yields the optimal cooperative state trajectory as:

$$x^*(s) = \varpi(t_0, s)^2 \left[x_0^{1/2} + \int_{t_0}^{s} \varpi^{-1}(t_0, t) H_1 dt \right]^2, \text{ for } s \in [t_0, T], \tag{6.11}$$

where

$$\varpi(t_0, s) = \exp\left[\int_{t_0}^{s} \left[H_2(\tau) - \frac{\sigma^2}{8}\right] dv + \int_{t_0}^{s} \frac{\sigma}{2} dz(v)\right], H_1 = \frac{1}{2}a,$$

$$\text{and } H_2(s) = -\left[\frac{1}{2}b + \frac{1}{8[c_1 + A(s)/2]^2} + \frac{1}{8[c_2 + A(s)/2]^2} + \frac{\sigma^2}{8} \right].$$

3.6.2 Subgame Consistent Payoff Distribution

With the extractors using the cooperative strategies (6.10) along the stochastic cooperative path, they agree to share the excess of the total expected cooperative payoff over the sum of individual noncooperative payoffs proportional to the extractors' expected noncooperative payoffs. Therefore the following imputation has to be satisfied.

Condition 6.1 An imputation

$$\xi^{(\upsilon)i}\left(\upsilon,x_\upsilon^*\right)=\frac{\hat{V}^i\left(x_\upsilon^*\right)}{\sum_{j=1}^{2}\hat{V}^j\left(x_\upsilon^*\right)}W\left(x_\upsilon^*\right)=\frac{\left[A_i\left(x_\upsilon^*\right)^{1/2}+C_i\right]}{\sum_{j=1}^{2}\left[A_j\left(x_\upsilon^*\right)^{1/2}+C_j\right]}\left[A\left(x_\upsilon^*\right)^{1/2}+C\right] \quad (6.12)$$

is assigned to extractor i, for $i\in\{1,2\}$ if $x_\upsilon^*\in X_\upsilon^*$ occurs at time $\upsilon\in[\tau,\infty)$. ■

Applying Theorem 5.3 a subgame-consistent solution for the cooperative game $\Gamma_c(\tau,x_\tau)$ includes:

(i) a set of group optimal strategies

$$\psi_1^*\left(x_s^*\right)=\frac{x_s^*}{4[c_1+A/2]^2}\text{ and }\psi_2^*\left(x_s^*\right)=\frac{x_s^*}{4[c_2+A/2]^2}\text{; and}$$

(ii) a Payoff Distribution Procedure
$B\left(s,x_s^*\right)=\left\{B_1\left(s,x_s^*\right),B_2\left(s,x_s^*\right),\cdots,B_n\left(s,x_s^*\right)\right\}$ for $s\in[\tau,\infty)$ with

$$\begin{aligned}B_i\left(s,x_s^*\right)&=r\,\xi^{(s)i}\left(s,x_s^*\right)\\&-\xi_{x_s^*}^{(s)i}\left(s,x_s^*\right)\left[a\left(x_s^*\right)^{1/2}-bx_s^*-\frac{x_s^*}{4[c_1+A/2]^2}-\frac{x_s^*}{4[c_2+A/2]^2}\right]\\&-\frac{1}{2}\sigma^2\left(x_s^*\right)^2\xi_{x_s^hx_s^\zeta}^{(\tau)i}\left(s,x_s^*\right),\ \text{for } i\in\{1,2\},\end{aligned}$$

where

$$\xi_{x_s^*\ xs*}^{(s)i\ si}\left(s,x_s^*\right)=\frac{\left[A_i\left(x_s^*\right)^{1/2}+C_i\right]A\left(x_s^*\right)^{-1/2}+\left[A\left(x_s^*\right)^{1/2}+C\right]A_i\left(x_s^*\right)^{-1/2}}{2\sum_{j=1}^{2}\left[A_j\left(x_s^*\right)^{1/2}+C_j\right]}$$

$$-\frac{\left[A_i\left(x_s^*\right)^{1/2}+C_i\right]\left[A\left(x_s^*\right)^{1/2}+C\right]}{\left(\sum_{j=1}^{2}\left[A_j\left(x_s^*\right)^{1/2}+C_j\right]\right)^2}\left(\frac{1}{2}\sum_{j=1}^{2}A_j\left(x_s^*\right)^{-1/2}\right);$$

$$\text{and }\xi_{x_s^*x_s^*}^{(\tau)i}\left(s,x_s^*\right)=-\frac{C_iA\left(x_s^*\right)^{-3/2}+CA_i\left(x_s^*\right)^{-3/2}}{4\sum_{j=1}^{2}\left[A_j\left(x_s^*\right)^{1/2}+C_j\right]}$$

$$-\frac{\left[A_i\left(x_s^*\right)^{1/2}+C_i\right]A\left(x_s^*\right)^{-1/2}+\left[A\left(x_s^*\right)^{1/2}+C\right]A_i\left(x_s^*\right)^{-1/2}}{\left(2\sum_{j=1}^{2}\left[A_j\left(x_s^*\right)^{1/2}+C_j\right]\right)^2}$$

$$\sum_{j=1}^{2}\left[A_j\left(x_s^*\right)^{-1/2}\right]$$
$$+\frac{\left[A_i\left(x_s^*\right)^{1/2}+C_i\right]\left[A\left(x_s^*\right)^{1/2}+C\right]}{\left(\sum_{j=1}^{2}\left[A_j\left(x_s^*\right)^{1/2}+C_j\right]\right)^2}\left(\frac{1}{4}\sum_{j=1}^{2}A_j\left(x_s^*\right)^{-3/2}\right)$$
$$-\left(\frac{1}{2}\sum_{j=1}^{2}A_j\left(x_s^*\right)^{-1/2}\right)\times\left[\frac{A_iA+\frac{1}{2}\left[A_iC+AC_i\right]\left(x_s^*\right)^{-1/2}}{\left(\sum_{j=1}^{2}\left[A_j\left(x_s^*\right)^{1/2}+C_j\right]\right)^2}\right.$$
$$\left.-\frac{\left[A_i\left(x_\tau^*\right)^{1/2}+C_i\right]\left[A\left(x_\tau^*\right)^{1/2}+C\right]}{\left(\sum_{j=1}^{2}\left[A_j\left(x_\tau^*\right)^{1/2}+C_j\right]\right)^3}\sum_{j=1}^{2}A_j\left(x_\tau^*\right)^{-1/2}\right]. \quad (6.13)$$

With extractors using the cooperative strategies in (6.13), the instantaneous receipt of extractor i at time instant $\upsilon \in [\tau, \infty)$ with the state being x_υ^* is:

$$\zeta_i\left(\upsilon, x_\upsilon^*\right)=\frac{\left(x_\upsilon^*\right)^{1/2}}{2[c_i+A/2]}-\frac{c_i\left(x_\upsilon^*\right)^{1/2}}{4[c_i+A/2]^2}, \text{ for } i \in \{1,2\}, \quad (6.14)$$

Under the cooperative agreement, the instantaneous payment that extractor $i \in \{1,2\}$ should receive under the agreed-upon optimality principle is $B_i(\upsilon, x_\upsilon^*)$ in (6.14). Hence an instantaneous transfer payment

$$\chi^i\left(\upsilon, x_\upsilon^*\right)=B_i\left(\upsilon, x_\upsilon^*\right)-\zeta_i\left(\upsilon, x_\upsilon^*\right) \quad (6.15)$$

has to be given to extractor i at time υ, for $i \in \{1,2\}$ and $x_\upsilon^* \in X_\upsilon^*$.

3.7 Chapter Notes

The analysis on subgame consistent solution in stochastic differential games was presented in Yeung and Petrosyan (2004). In particular, a generalized theorem for the derivation of an analytically tractable "payoff distribution procedure" which would lead to subgame-consistent solutions was developed. Examples of cooperative stochastic differential games with solutions satisfying subgame consistency can be found in Yeung (2005, 2007a, 2008, 2010) and Yeung and Petrosyan (2004, 2006a, b, 2007a, b, c, 2008, 2012c, 2014a). Theorem 3.1 could be applied to obtain subgame consistent cooperative solution for existing differential games in economic analysis. Solution mechanisms for cooperative stochastic differential games can be found in Yeung (2006b).

3.8 Problems

1. Consider the case of two nations harvesting fish in common waters. The growth rate of the fish biomass is subject to stochastic shocks and follows the differential equation:

$$dx(s) = \left[12x(s)^{1/2} - x(s) - u_1(s) - u_2(s)\right]ds + 0.1x(s)dz(s), \;\; x(0) = 100,$$

where $z(s)$ is a Wiener process, $x(s)$ is the fish stock and $u_i(s)$ is the amount of fish harvested by nation i, for $i \in \{1, 2\}$. The horizon of the game is $[0, 3]$.

The harvesting cost for nation $i \in \{1, 2\}$ depends on the quantity of resource extracted $u_i(s)$ and the resource stock size $x(s)$. In particular, nation 1's extraction cost is $2u_1(s)x(s)^{-1/2}$ and nation 2's is $u_2(s)x(s)^{-1/2}$. The fish harvested by nation 1 at time s will generate a net benefit of the amount $3[u_1(s)]^{1/2}$ and the fish harvested by nation 2 at time s will generate a net benefit of the amount $2[u_2(s)]^{1/2}$. At terminal time 5, nations 1 and 2 will receive termination bonuses $8x(3)^{1/2}$ and $6x(3)^{1/2}$ while the interest rate is 0.05.

Characterize a feedback Nash equilibrium solution for this stochastic fishery game.
2. If these nations agree to cooperate and maximize their expected joint payoff, obtain a group optimal cooperative solution.
3. Furthermore, if these nations agree to share the expected gain proportional to their non-cooperative payoffs, derive a subgame consistent solution.
4. Consider the case when the game horizon in exercise 1 is extended to infinity.

   (i) Characterize a feedback Nash equilibrium solution for this stochastic dynamic game.
   (ii) If these nations agree to cooperate and maximize their expected joint payoff and share the excess of their expected gain equally, derive a subgame consistent solution.