Chapter 2
Subgame Consistent Cooperative Solution in Differential Games

In game theory, strategic behavior and decision making are modeled in terms of the characteristics of players, the objective or payoff function of each individual, the actions open to each player throughout the game, the order of such actions, and the information available at each stage of play. Optimal decisions are then determined under different assumptions regarding the availability and transmission of information, and the opportunities and possibilities for individuals to communicate, negotiate, collude, offer inducements, and enter into agreements which are binding or enforceable to varying degrees and at varying costs. Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic action. As discussed in Chap. 1, individual rationality, group optimality and subgame consistency are crucial elements of a cooperative game solution. This chapter presents an analysis on subgame consistent solutions which entail group optimality and individual rationality for cooperative differential games. It integrates the works of Chapter 2 of Yeung and Petrosyan (2006b), Chapter 4 of Yeung and Petrosyan (2012a) and the deterministic version of Yeung and Petrosyan (2004).

The organization of the Chapter is as follows. Section 2.1 presents the basic formulation of cooperative differential games. Section 2.2 presents an analysis on subgame consistent dynamic cooperation. Derivation of a subgame consistent payoff distribution procedure is provided in Sect. 2.3. An illustration of the solution mechanism is given in a cooperative fishery game in Sect. 2.4. Subgame consistency in infinite horizon cooperative differential games is examined in Sect. 2.5. In Sect. 2.6, a subgame consistent solution of an infinite horizon cooperative resource extraction scheme is derived. Chapter notes are given in Sect. 2.7 and problems in Sect. 2.8.

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2.1 Basic Formulation of Cooperative Differential Games

Consider the general form of n-person differential games in which player i seeks to maximize its objective:

$$\int_{t_0}^{T} g^i[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds + \exp\left[-\int_{t_0}^{T} r(y)dy\right] q^i(x(T)), \tag{1.1}$$

for $i \in N = \{1, 2, \cdots, n\}$, where $r(y)$ is the discount rate, $x(s) \in X \subset R^m$ denotes the state variables of game, $q^i(x(T))$ is player i's valuation of the state at terminal time T and $u_i \in U^i$ is the control of player i, for $i \in N$. The payoffs of the players are transferrable.

The state variable evolves according to the dynamics

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)],\ x(t_0) = x_0, \tag{1.2}$$

where $x(s) \in X \subset R^m$ denotes the state variables of game, and $u_i \in U^i$ is the control of player i, for $i \in N$. The functions $f[s, x, u_1, u_2, \cdots, u_n]$, $g^i[s, \cdot, u_1, u_2, \cdots, u_n]$ and $q^i(\cdot)$, for $i \in N$, and $s \in [t_0, T]$ are differentiable functions.

2.1.1 Non-cooperative Feedback Equilibria

To analyze the cooperative outcome we first characterize the non-cooperative equilibria as a benchmark for negotiation in a cooperative scheme. Since in a non-cooperative situation it is difficult to prevent the players from revising their strategies during the game duration, therefore they would consider adopting feedback strategies which are decision rules that are dependent upon the current state $x(t)$ and current time t, for $t_0 \le t \le s$.

For the n-person differential game (1.1 and 1.2), an n -tuple of feedback strategies $\{u_i^*(s) = \phi_i^*(s, x) \in U^i$, for $i \in N\}$ constitutes a *Nash equilibrium solution* if the following relations for each $i \in N$ are satisfied:

$$V^{(t_0)i}(t, x) = \int_{t}^{T} g^i\left[s, x^*(s), \phi_1^*(s, x^*(s)), \phi_2^*(s, x^*(s)), \ldots, \phi_n^*(s, x^*(s))\right] \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds + q^i(x^*(T)) \exp\left[-\int_{t_0}^{T} r(y)dy\right]$$

$$\geq \int_t^T g^i\left[s, x^i(s), \phi_1^*\left(s, x^i(s)\right), \phi_2^*\left(s, x^i(s)\right), \ldots, \phi_{i-1}^*\left(s, x^i(s)\right), \phi_i\left(s, x^i(s)\right),\right.$$
$$\left.\phi_{i+1}^*\left(s, x^i(s)\right), \ldots\ldots, \phi_n^*\left(s, x^i(s)\right)\right] \exp\left[-\int_{t_0}^s r(y)dy\right] ds$$
$$+ q^i\left(x^i(T)\right) \exp\left[-\int_{t_0}^T r(y)dy\right], \forall \phi_i^*(s,x) \in U^i, x \in R^m, \tag{1.3}$$

where on the interval $[t_0, T]$,

$$\dot{x}^*(s) = f\left[s, x^*(s), \phi_1^*\left(s, x^*(s)\right), \phi_2^*\left(s, x^*(s)\right), \ldots, \phi_n^*\left(s, x^*(s)\right)\right], \quad x^*(t) = x;$$

and

$$\dot{x}^i(s) = f\left[s, x^i(s), \phi_1^*(s, x^i(s)), \phi_2^*(s, x^i(s)), \ldots, \phi_{i-1}^*(s, x^i(s)), \phi_i(s, x^i(s)), \phi_{i+1}^*(s, x^i(s)), \ldots\right.$$
$$\left.\ldots, \phi_n^*(s, x^i(s))\right], \quad x^i(t) = x, \text{ for } i \in N.$$

A feedback Nash equilibrium solution of the game (1.1 and 1.2) satisfying (1.3) can be characterized by the following Theorem.

Theorem 1.1 An n-tuple of strategies $\left\{u_i^*(t) = \phi_i^*(t,x) \in U^i, \text{ for } i \in N\right\}$ provides a feedback Nash equilibrium solution to the game (1.1 and 1.2) if there exist continuously differentiable functions $V^{(t_0)i}(t,x) : [t_0, T] \times R^m \to R, i \in N$, satisfying the following set of partial differential equations:

$$-V_t^{(t_0)i}(t,x) = \max_{u_i}\left\{ g^i\left[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \Lambda, \phi_{i-1}^*(t,x), u_i(t,x), \phi_{i+1}^*(t,x), \Lambda\right.\right.$$
$$\left.\Lambda, \phi_n^*(t,x)\right] \exp\left[-\int_{t_0}^t r(y)dy\right]$$
$$\left.+V_x^{(t_0)i}(t,x) f\left[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \Lambda, \phi_{i-1}^*(t,x), u_i(t,x), \phi_{i+1}^*(t,x), \Lambda, \phi_n^*(t,x)\right]\right\}$$
$$= g^i\left[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \Lambda, \phi_n^*(t,x) \exp\left[-\int_{t_0}^t r(y)dy\right]\right.$$
$$+V_x^{(t_0)i}(t,x) f\left[t, x, \phi_1^*(t,x), \phi_2^*(t,x), \Lambda, \phi_n^*(t,x)\right],$$
$$V^{(t_0)i}(T,x) = q^i(x) \exp\left[-\int_{t_0}^T r(y)dy\right], \quad i \in N.$$

Proof Invoking the dynamic programming technique in Theorem A.1 of the Technical Appendices, $V^{(t_0)i}(t,x)$ is the maximized payoff of player i for given strategies $\left\{u_j^*(s) = \phi_j^*(t,x) \in U^j, \text{ for } j \in N \text{ and } j \neq i\right\}$ of the other $n-1$ players. Hence a Nash equilibrium appears. ■

A remark that will be utilized in subsequent analysis is given below.

Remark 1.1 Let $V^{(\tau)i}(t,x)$ denote the feedback Nash equilibrium payoff of player i at time t given the state x in a game with payoffs (1.1) and dynamics (1.2) which starts at time τ for $\tau \in [t_0, T)$. Note that the equilibrium feedback strategies depend on current time and current state. One can readily verify that

$$\exp\left[\int_{t_0}^{\tau} r(y)dy\right] V^{(t_0)i}(t,x) = \exp\left[\int_{t_0}^{\tau} r(y)dy\right]$$
$$\times \int_t^T g^i\left[s, x^*(s), \phi_1^*(s,x^*(s)), \phi_2^*(s,x^*(s)), \Lambda, \phi_n^*(s,x^*(s))\right] \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds$$
$$= \int_t^T g^i\left[s, x^*(s), \phi_1^*(s,x^*(s)), \phi_2^*(s,x^*(s)), \Lambda, \phi_n^*(s,x^*(s))\right] \exp\left[-\int_{\tau}^{s} r(y)dy\right] ds$$
$$= V^{(\tau)i}(t,x),$$

for $\tau \in [t_0, T)$. ■

While non-cooperative outcomes are (in general) not Pareto optimal the players would consider cooperation to enhance their payoffs. This will be analyzed in the following section.

2.1.2 Dynamic Cooperation

Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic action. Now consider the case when the players agree to cooperate and distribute the payoff among themselves according to an optimality principle. Two essential properties that a cooperative scheme has to satisfy are group optimality and individual rationality. Group optimality ensures that the joint payoff of all the players under cooperation is maximized. Failure to fulfill group optimality leads to the condition where the participants prefer to deviate from the agreed-upon solution plan in order to extract the unexploited gains. Individual rationality is required to hold so that the payoff allocated to any player under cooperation will be no less than his noncooperative payoff. Failure to guarantee individual rationality leads to the condition where the concerned participants would deviate from the agreed upon solution plan and play noncooperatively.

2.1.2.1 Group Optimality Under Cooperation

Since payoffs are transferable, group optimality requires the players to maximize their joint payoff. The players must then solve the following optimal control problem:

$$\max_{u_1, u_2, \cdots, u_n} \left\{ \int_{t_0}^{T} \sum_{j=1}^{n} g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp\left[-\int_{t_0}^{s} r(y)dy \right] ds \right.$$
$$\left. + \exp\left[-\int_{t_0}^{T} r(y)dy \right] \sum_{j=1}^{n} q^j(x(T)) \right\} \tag{1.4}$$

subject to (1.2).

An optimal solution to the control problem (1.2) and (1.4) characterizing the set of group optimal control strategies is provided by the theorem below.

Theorem 1.2 A set of controls $\{\psi_i^*(t, x)$, for $i \in N$ and $t \in [t_0, T]\}$ provides an optimal solution to the control problem (1.2) and (1.4) if there exists continuously differentiable function $W^{(t_0)}(t, x) : [t_0, T] \times R^m \to R$ satisfying the following Bellman equation:

$$-W_t^{(t_0)}(t, x)$$
$$= \max_{u_1, u_2, \cdots, u_n} \left\{ \sum_{j=1}^{n} g^j[t, x, u_1, u_2, \cdots, u_n] \exp\left[-\int_{t_0}^{t} r(y)dy \right] + W_x^{(t_0)} f[t, x, u_1, u_2, \cdots, u_n] \right\},$$
$$W^{(t_0)}(T, x) = \exp\left[-\int_{t_0}^{T} r(y)dy \right] \sum_{j=1}^{n} q^j(x).$$

Proof Follow the proof of Theorem A.1 in the Technical Appendices. ■

Hence the players will adopt the cooperative control $\{\psi_i^*(t, x)$, for $i \in N$ and $t \in [t_0, T]\}$ to obtain the maximized level of joint profit. Substituting this set of control into (1.2) yields the dynamics of the optimal (cooperative) trajectory as:

$$\dot{x}(s) = f\left[s, x(s), \psi_1^*(s, x(s)), \psi_2^*(s, x(s)), \cdots, \psi_n^*(s, x(s))\right], \quad x(t_0) = x_0. \tag{1.5}$$

Let $x^*(t)$ denote the solution to (1.5). The optimal trajectory $\{x^*(t)\}_{t=t_0}^{T}$ can be expressed as:

$$x^*(t) = x_0 + \int_{t_0}^{t} f\left[s, x^*(s), \psi_1^*\left(s, x^*(s)\right), \psi_2^*\left(s, x^*(s)\right), \cdots, \psi_n^*\left(s, x^*(s)\right)\right] ds. \tag{1.6}$$

For notational convenience, we use the terms $x^*(t)$ and x_t^* interchangeably.

Note that for group optimality to be achievable, the cooperative controls $\{\psi_i^*(t, x^*(t))$, for $i \in N$ and $t \in [t_0, T]\}$ must be exercised throughout time interval $[t_0, T]$.

The maximized cooperative payoff over the interval $[t, T]$, for $t \in \left[t_0, T\right)$, can be expressed as:

$$W^{(t_0)}\left(t, x_t^*\right) =$$
$$\int_t^T \sum_{j=1}^n g^j\left[s, x^*(s), \psi_1^*\left(s, x^*(s)\right), \psi_2^*\left(s, x^*(s)\right), \cdots, \psi_n^*\left(s, x^*(s)\right)\right] \exp\left[-\int_{t_0}^s r(y)dy\right] ds$$
$$+ \exp\left[-\int_{t_0}^T r(y)dy\right] \sum_{j=1}^n q^j\left(x^*(T)\right)$$

A remark that will be utilized in subsequent analysis is given below.

Remark 1.2 Let $W^{(\tau)}(t, x_t^*)$ denote the maximized cooperative payoff of the control problem

$$\max_{u_1, u_2, \cdots, u_n} \left\{ \int_t^T \sum_{j=1}^n g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp\left[-\int_\tau^s r(y)dy\right] ds \right.$$
$$\left. + \exp\left[-\int_\tau^T r(y)dy\right] \sum_{j=1}^n q^j(x(T)) \right\},$$

subject to

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)],\ x(t) = x_t^*.$$

One can readily verify that

$$\exp\left[\int_{t_0}^\tau r(y)dy\right] W^{(t_0)}\left(t, x_t^*\right) = \exp\left[\int_{t_0}^\tau r(y)dy\right] \times$$
$$\left\{ \int_t^T \sum_{j=1}^n g^j\left[s, x^*(s), \psi_1^*\left(s, x^*(s)\right), \psi_2^*\left(s, x^*(s)\right), \cdots, \psi_n^*\left(s, x^*(s)\right)\right] \exp\left[-\int_{t_0}^s r(y)dy\right] ds \right.$$
$$\left. + \exp\left[-\int_{t_0}^T r(y)dy\right] \sum_{j=1}^n q^j\left(x^*(T)\right) \right\} =$$
$$\int_t^T \sum_{j=1}^n g^j\left[s, x^*(s), \psi_1^*\left(s, x^*(s)\right), \psi_2^*\left(s, x^*(s)\right), \cdots, \psi_n^*\left(s, x^*(s)\right)\right] \exp\left[-\int_\tau^s r(y)dy\right] ds$$
$$+ \exp\left[-\int_\tau^T r(y)dy\right] \sum_{j=1}^n q^j\left(x^*(T)\right) = W^{(\tau)}\left(t, x_t^*\right),$$

for $\tau \in [t_0, T]$ and $t \in \left[\tau, T\right)$. ■

2.1.2.2 Individual Rationality

After the players agree to cooperate and maximize their joint payoff, they have to distribute the cooperative payoff among themselves. At time t_0, with the state being x_0, the term $\xi^{(t_0)i}(t_0, x_0)$ is used to denote the imputation of payoff (received over the time interval $[t_0, T]$) to player i. A necessary condition for group optimality and individual rationality to be upheld is:

$$\text{(i)} \quad \sum_{j=1}^{n} \xi^{(t_0)j}(t_0, x_0) = W^{(t_0)}(t_0, x_0), \text{ and}$$
$$\text{(ii)} \quad \xi^{(t_0)i}(t_0, x_0) \geq V^{(t_0)i}(t_0, x_0), \quad \text{for } i \in N \tag{1.7}$$

Condition (i) of (1.7) ensures group optimality and condition (ii) guarantees individual rationality at time t_0.

For the optimization scheme to be upheld throughout the game horizon both group rationality and individual rationality are required to be satisfied throughout the cooperation period $[t_0, T]$. At time $\tau \in [t_0, T]$, let $\xi^{(\tau)i}(\tau, x_\tau^*)$ denote the imputation of payoff to player i over the time interval $[\tau, T]$. Therefore the conditions

$$\text{(i)} \quad \sum_{j=1}^{n} \xi^{(\tau)j}\left(\tau, x_\tau^*\right) = W^{(\tau)}\left(\tau, x_\tau^*\right), \text{ and}$$
$$\text{(ii)} \quad \xi^{(\tau)i}\left(\tau, x_\tau^*\right) \geq V^{(\tau)i}\left(\tau, x_\tau^*\right); \text{ for } i \in N \text{ and } \tau \in [t_0, T], \tag{1.8}$$

have to be fulfilled.

In particular, condition (i) ensures Pareto optimality and condition (ii) guarantees individual rationality, throughout the cooperation period $[t_0, T]$. Failure to guarantee individual rationality leads to the condition where the concerned participants would reject the agreed upon solution plan and play noncooperatively.

Dockner and Jørgensen (1984); Dockner and Long (1993), Tahvonen (1994); Mäler and de Zeeuw (1998) and Rubio and Casino (2002) examines group optimal solutions in cooperative differential games. Haurie and Zaccour (1986, 1991); Kaitala and Pohjola (1988, 1990, 1995); Kaitala et al. (1995) and Jørgensen and Zaccour (2001) presented classes of transferable-payoff cooperative differential games with solutions which satisfy group optimality and individual rationality.

2.1.3 Distribution of Cooperative Payoffs

With the players using the cooperative strategies $\left\{\psi_i^*\left(s, x_s^*\right), \text{for } s \in [t_0, T] \text{ and } i \in N\right\}$, player i would derive a direct payoff :

$$W^{(t_0)i}(t_0,x_0)=$$
$$\int_{t_0}^{T} g^i\left[s,x^*(s),\psi_1^*\left(s,x^*(s)\right),\psi_2^*\left(s,x^*(s)\right),\cdots,\psi_n^*\left(s,x^*(s)\right)\right]\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds$$
$$+\exp\left[-\int_{t_0}^{T} r(y)dy\right]q^i\left(x^*(T)\right),\ \text{for}\, i\in N. \tag{1.9}$$

At initial time t_0, for cooperation to begin the cooperative payoff to player i $W^{(t_0)i}(t_0,x_0)$ must be no less than the non-cooperative $V^{(t_0)i}(t_0,x_0)$ for all player $i\in N$. However as time proceeds there is no guarantee that adopting the cooperative strategies would lead to $W^{(t)i}\left(t,x_t^*\right)\geq V^{(t)i}\left(t,x_t^*\right)$ for all player $i\in N$. In case there exists some player i such that $V^{(t)i}\left(t,x_t^*\right)>W^{(t)i}\left(t,x_t^*\right)$, then player i would have an incentive to deviate from the cooperation plan. Hence the cooperation scheme has to include transfer payments to overcome this problem. Let $\chi^i(s)$ denote the instantaneous transfer payment allocated to agent i at time $s\in[t_0,T]$. With players using the cooperative strategies $\left\{\psi_i^*\left(s,x_s^*\right),\text{ for } s\in[t_0,T] \text{ and } i\in N\right\}$, the payoff that player i's payoff under cooperation at time t_0 becomes:

$$\xi^{(t_0)i}(t_0,x_0)=$$
$$\int_{t_0}^{T}\left\{g^i\left[s,x^*(s),\psi_1^*\left(s,x^*(s)\right),\psi_2^*\left(s,x^*(s)\right),\cdots,\psi_n^*\left(s,x^*(s)\right)\right]+\chi^i(s)\right\}\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds$$
$$+\exp\left[-\int_{t_0}^{T} r(y)dy\right]q^i\left(x^*(T)\right),$$
$$\text{for } i\in N;$$
$$\text{and} \sum_{j=1}^{n}\int_{t_0}^{T}\chi^j(s)\exp\left[-\int_{t_0}^{s} r(y)dy\right]ds=0. \tag{1.10}$$

In order to uphold individual rationality one has to device a time path of instantaneous transfer payments $\chi^i(s)$ for $s\in[t_0,T]$ satisfying:

$$\int_{\tau}^{T}\left\{g^i\left[s,x^*(s),\psi_1^*\left(s,x^*(s)\right),\psi_2^*\left(s,x^*(s)\right),\cdots,\psi_n^*\left(s,x^*(s)\right)\right]+\chi^i(s)\right\}\exp\left[-\int_{\tau}^{s} r(y)dy\right]ds$$
$$+\exp\left[-\int_{\tau}^{T} r(y)dy\right]q^i\left(x^*(T)\right)\geq V^{(\tau)i}\left(\tau,x_\tau^*\right),\ \text{for } i\in N; \tag{1.11}$$

and

$$\sum_{j=1}^{n}\int_{\tau}^{T}\chi^j(s)\exp\left[-\int_{\tau}^{s} r(y)dy\right]ds=0,\ \text{for } \tau\in[t_0,T]. \tag{1.12}$$

There exist a large number of $\chi^i(s)$ for $s \in [t_0, T]$ paths which leads to the satisfaction of individual rationality for all players to be selected. Nevertheless, just satisfying individual rationality may not be acceptable. For instance, players with larger non-cooperative payoffs (or sizes) would demand a larger share proportionally. In the next Section we will consider the derivation of $\chi^i(s)$ for $s \in [t_0, T]$ paths which would keep the original agree-upon imputation throughout the cooperation duration.

2.2 Subgame Consistent Dynamic Cooperation

Though group optimality and individual rationality constitute two essential properties for cooperation, their fulfillment does not necessarily guarantee a dynamically stable solution in cooperation because there is no guarantee that the agreed-upon optimality principle is fulfilled throughout the cooperative duration. The question of dynamic stability in differential games has been explored rigorously in the past four decades. Haurie (1976) discussed the problem of instability in extending the Nash bargaining solution to differential games. Petrosyan (1977) formalized mathematically the notion of dynamic stability in solutions of differential games. Petrosyan and Danilov (1979, 1982) introduced the notion of "imputation distribution procedure" for cooperative solution.

To ensure stability in dynamic cooperation over time, a stringent condition is required: the specific agreed-upon optimality principle must be maintained at any instant of time throughout the game along the optimal state trajectory. This condition is the notion of *subgame consistency*.

Let $\Gamma_c(x_0, T - t_0)$ denote a cooperative game in which player i's payoff is (1.1) and the state dynamics is (1.2). The players agree to act according to an agreed-upon optimality principle. The agreement on how to act cooperatively and allocate cooperative payoff constitutes the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle includes

(i) an agreement on a set of cooperative strategies/controls,
(ii) an imputation vector stating the allocation of the cooperative payoff to individual players, and
(iii) a mechanism to distribute total payoff among players.

2.2.1 Optimality Principle

Let there be an optimality principle agreed upon by all players in the cooperative game $\Gamma_c(x_0, T - t_0)$. Based on the agreed upon optimality principle the solution of the game $\Gamma_c(x_0, T - t_0)$ at time t_0 includes

(i) a set of cooperative strategies $u^{(t_0)*}(s) = \left[u_1^{(t_0)*}(s), u_2^{(t_0)*}(s), \cdots, u_n^{(t_0)*}(s)\right]$, for $s \in [t_0, T]$;

(ii) an imputation vector $\xi^{(t_0)}(t_0, x_0) = \left[\xi^{(t_0)1}(t_0, x_0), \xi^{(t_0)2}(t_0, x_0), \cdots, \xi^{(t_0)n}(t_0, x_0)\right]$ to allot the cooperative payoff to the players; and

(iii) a payoff distribution procedure $B^{t_0}(s) = \left[B_1^{t_0}(s), B_2^{t_0}(s), \cdots, B_n^{t_0}(s)\right]$ for $s \in [t_0, T]$, where $B_i^{t_0}(s)$ is the instantaneous payments for player i at time s. In particular,

$$\xi^{(t_0)i}(t_0, x_0) = \int_{t_0}^{T} B_i^{t_0}(s) \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds + q^i(x_T) \exp\left[-\int_{t_0}^{T} r(y)dy\right],$$

for $i \in N$.

This means that the players agree at the outset on a set of cooperative strategies $u^{(t_0)*}(s)$, an imputation $\xi^{(t_0)i}(t_0, x_0)$ of the gains to the i th player covering the time interval $[t_0, T]$, and a payoff distribution procedure $\{B^{t_0}(s)\}_{s=t_0}^{T}$ to allocate payments to the players over the game interval.

Using the agreed-upon cooperative strategies the state evolves according to the state dynamics:

$$\dot{x}(s) = f\left[s, x(s), u_1^{(t_0)*}(s), u_2^{(t_0)*}(s), \cdots, u_n^{(t_0)*}(s)\right], \quad x(t_0) = x_0. \tag{2.1}$$

The solution to (2.1) yields the optimal cooperative trajectory which is denoted by $\{x^c(s)\}_{s=t_0}^{T}$. For notational convenience we use $x^c(s)$ and x_s^c interchangeably.

When time $t \in (t_0, T]$ has arrived, the situation becomes a cooperative game in which player i's payoff is:

$$\begin{aligned} &\int_t^T g^i[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp\left[-\int_t^s r(y)dy\right] ds \\ &\quad + \exp\left[-\int_t^T r(y)dy\right] q^i(x(T)), \quad \text{for } i \in N, \end{aligned} \tag{2.2}$$

and the evolutionary dynamics of the state is

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)], \; x(t) = x_t^c. \tag{2.3}$$

We use $\Gamma_c(x_t^c, T - t)$ to denote a cooperative game in which player i's objective is (2.2) with state dynamics (2.3). At time $t \in (t_0, T]$ when the state is x_t^c, according to the agreed-upon principle the solution of the game $\Gamma_c(x_t^c, T - t)$ includes:

(i) a set of cooperative strategies $u^{(t)*}(s) = \left[u_1^{(t)*}(s), u_2^{(t)*}(s), \cdots, u_n^{(t)*}(s)\right]$, for $s \in [t, T]$;

(ii) an imputation vector $\xi^{(t)}\left(t,x_t^c\right)=\left[\xi^{(t)1}\left(t,x_t^c\right),\xi^{(t)2}\left(t,x_t^c\right),\cdots,\xi^{(t)n}\left(t,x_t^c\right)\right]$ to allot the cooperative payoff to the players; and

(iii) a payoff distribution procedure $B^t(s)=\left[B_1^t(s),B_2^t(s),\cdots,B_n^t(s)\right]$ for $s\in[t,T]$, where $B_i^t(s)$ is the instantaneous payments for player i at time s. In particular,

$$\xi^{(t)i}\left(t,x_t^c\right)=\int_t^T B_i^t(s)\exp\left[-\int_t^s r(y)dy\right]ds+q^i\left(x_T^c\right)\exp\left[-\int_t^T r(y)dy\right],\quad(2.4)$$

for $i\in N$ and $t\in[t_0,T]$.

This means that under the agreed-upon optimality principle, the players agree on a set of cooperative strategies $u^{(t)*}(s)$, an imputation of the gains in such a way that the gain under cooperation of the i th player over the time interval $[t,T]$ is equal to $\xi^{(t)i}(t,x_t^c)$ and a payoff distribution procedure $\{B^t(s)\}_{s=t}^T$ to allocate payments to the players over the game interval $[t,T]$.

Examples of optimality principles include:

(i) joint payoff maximization and equal sharing of gains from cooperation,
(ii) joint payoff maximization and sharing gains proportional to non-cooperative payoffs,
(iii) joint payoff maximization and time varying sharing weights,
(iv) different combinations of (i), (ii) and (iii),
(v) joint payoff maximization and sharing gains according to the Shapley value,
(vi) joint payoff maximization and sharing gains according to the von Neumann-Morgenstern solution, or
(vii) joint payoff maximization and sharing gains according to the nucleolus.

2.2.2 Cooperative Subgame Consistency

To satisfy subgame consistency, the cooperative strategies, imputation and payoff distribution procedure $\left\{u^{(t_0)*}(s)\text{ and }B^{t_0}(s)\text{ for }s\in[t_0,T];\ \xi^{(t_0)}(t_0,x_0)\right\}$ generated by the agreed-upon optimality principle in the cooperative game $\Gamma_c(x_0,T-t_0)$ must be consistent with the cooperative strategies, imputation and payoff distribution procedure $\{u^{(t)*}(s)\text{ and }B^t(s)\text{ for }s\in[t,T];\ \xi^{(t)}\left(t,x_t^c\right)\}$ generated by the same optimality principle in the cooperative game $\Gamma_c\left(x_t^c,T-t\right)$ along the optimal cooperative trajectory $\left\{x_s^c\right\}_{s=t_0}^T$.

If this consistency does not appear, there is no guarantee that the players would not abandon the cooperative scheme and switch to other plans including the non-cooperative scheme. Dynamical instability would arise as participants found that their agreed upon optimality principle could not be maintained after cooperation has gone on for some time.

2.2.2.1 Subgame Consistent Cooperative Strategies

First we consider the cooperative strategies adopted under the agreed-upon optimality principle in the game $\Gamma_c(x_0, T - t_0)$. At time t_0 when the initial state is x_0, the set of cooperative strategies is

$$u^{(t_0)*}(s) = \left[u_1^{(t_0)*}(s), u_2^{(t_0)*}(s), \cdots, u_n^{(t_0)*}(s)\right], \text{ for } s \in [t_0, T].$$

Consider the case when the game has proceeded to time t and the state variable became x_t^c. Then one has a cooperative game $\Gamma_c\left(x_t^c, T - t\right)$ which starts at time t with initial state x_t^c. According to the agreed upon optimality principle a set of cooperative strategies

$$u^{(t)*}(s) = \left[u_1^{(t)*}(s), u_2^{(t)*}(s), \cdots, u_n^{(t)*}(s)\right], \text{ for } s \in [t, T],$$

will be adopted.

Definition 2.1 The set of cooperative strategies $u^{(t_0)*}(s) = \left[u_1^{(t_0)*}(s), u_2^{(t_0)*}(s), \cdots, u_n^{(t_0)*}(s)\right]$ in the game $\Gamma_c(x_0, T - t_0)$ is subgame consistent if $\left[u_1^{(t_0)*}(s), u_2^{(t_0)*}(s), \cdots, u_n^{(t_0)*}(s)\right] = \left[u_1^{(t)*}(s), u_2^{(t)*}(s), \cdots, u_n^{(t)*}(s)\right]$ in the game $\Gamma_c\left(x_t^c, T - t\right)$ under the agreed-upon optimality principle, for $s \in [t, T]$ and $t \in [t_0, T]$. ■

If the condition in Definition 2.1 is satisfied at each instant of time $t \in [t_0, T]$ along the optimal trajectory $\{x^c(t)\}_{t=t_0}^{T}$, the continuation of the original cooperative strategies $u^{(t_0)*}(s)$ coincides with the cooperative strategies $u^{(t)*}(s)$ in the cooperative game $\Gamma_c\left(x_t^c, T - t\right)$. Hence the set of cooperative strategies $u^{(t_0)*}(s)$ is subgame consistent. Recall that to ensure group optimality the players have to maximize the players' joint payoffs. An optimality principle which requires group optimality would yield a set of cooperative controls that solves the problem:

$$\max_{u_1, u_2, \cdots, u_n} \left\{ \int_{t_0}^{T} \sum_{j=1}^{n} g^j[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds \right.$$
$$\left. + \exp\left[-\int_{t_0}^{T} r(y)dy\right] \sum_{j=1}^{n} q^j(x(T)) \right\}, \tag{2.5}$$

$$\text{subject to } \dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \cdots, u_n(s)], \quad x(t_0) = x_0. \tag{2.6}$$

A set of group optimal cooperative strategies $\{\psi_i^*(s, x^*(s))$, for $i \in N$ and $s \in [t_0, T]\}$ which solves the problem (2.5 and 2.6) could be characterized by Theorem 1.2. In particular, $\left\{x^*(t)\right\}_{t=t_0}^{T}$ is the solution path of the optimal cooperative trajectory:

$$\dot{x}(s) = f\left[s, x(s), \psi_1^*(s, x(s)), \psi_2^*(s, x(s)), \cdots, \psi_n^*(s, x(s))\right], \ x(t_0) = x_0.$$

Invoking Remark 1.2 in Sect. 2.1, one can show that the joint payoff maximizing controls for the cooperative game $\Gamma_c\left(x_t^*, T - t\right)$ over the time interval $[t, T]$ is identical to the joint payoff maximizing controls for the cooperative game $\Gamma_c(x_0, T - t_0)$ over the same time interval.

Therefore the solution to an optimality principle which requires group optimality yields a system of subgame consistent cooperative strategies. Given that group optimality is an essential element in dynamic cooperation, a valid optimality principle would require the maximization of joint payoff and the cooperative strategies $u^{(t_0)*}(s) = u_1^{(t)*}(s) = \psi_i^*\left(s, x^*(s)\right)$, for $s \in [t, T]$ and $t \in [t_0, T]$.

2.2.2.2 Subgame Consistent Imputation

Now, we consider subgame consistency in imputation and payoff distribution procedure. In the cooperative game $\Gamma_c(x_0, T - t_0)$, according to the agreed-upon optimality principle the players would use the payoff distribution procedure $\{B^{t_0}(s)\}_{s=t_0}^{T}$ to bring about an imputation to player i as:

$$\xi^{(t_0)i}(t_0, x_0) = \int_{t_0}^{T} B_i^{t_0}(s) \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds + q^i(x_T) \exp\left[-\int_{t_0}^{T} r(y)dy\right], \quad (2.7)$$

for $i \in N$.

When the game proceeds to time $t \in \left(t_0, T\right]$, the current state is x_t^c. According to the same optimality principle player i will receive an imputation (in present value viewed at time t_0) equaling

$$\xi^{(t_0)i}\left(t, x_t^c\right) = \int_{t}^{T} B_i^{t_0}(s) \exp\left[-\int_{t_0}^{s} r(y)dy\right] ds + q^i\left(x_T^c\right) \exp\left[-\int_{t_0}^{T} r(y)dy\right], \quad (2.8)$$

over the time interval $[t, T]$.

At time $t \in \left(t_0, T\right]$ when the current state is x_t^c, we have a cooperative game $\Gamma_c\left(x_t^c, T - t\right)$. According to the agreed-upon optimality principle the players would use the payoff distribution procedure $\{B^t(s)\}_{s=t}^{T}$ to bring about an imputation to player i as:

$$\xi^{(t)i}\left(t, x_t^c\right) = \int_{t}^{T} B_i^{t}(s) \exp\left[-\int_{t}^{s} r(y)dy\right] ds + q^i\left(x_T^c\right) \exp\left[-\int_{t}^{T} r(y)dy\right], \quad (2.9)$$

for $i \in N$.

For the imputation and payoff distribution procedure in the game $\Gamma_c(x_0, T - t_0)$ to be consistent with those from $\Gamma_c\left(x_t^c, T - t\right)$, it is essential that

$$\exp\left[\int_{t_0}^{t} r(y)dy\right]\xi^{(t_0)}\left(t, x_t^c\right) = \xi^{(t)}\left(t, x_t^c\right), \text{ for } t \in [t_0, T].$$

In addition, in the game $\Gamma_c(x_0, T - t_0)$ according to the agreed-upon optimality principle the payoff distribution procedure is

$$B^{t_0}(s) = \left[B_1^{t_0}(s), B_2^{t_0}(s), \cdots, B_n^{t_0}(s)\right], \text{ for } s \in [t_0, T].$$

Consider the case when the game has proceeded to time t and the state variable became x_t^c. Then one has a cooperative game $\Gamma_c\left(x_t^c, T - t\right)$ which starts at time t with initial state x_t^c. According to the agreed-upon optimality principle the payoff distribution procedure

$$B^t(s) = \left[B_1^t(s), B_2^t(s), \cdots, B_n^t(s)\right], \text{ for } s \in [t, T]$$

will be adopted.

For the continuation of the payoff distribution procedure $B^{t_0}(s)$ for $s \in [t, T]$ to be consistent with $B^t(s)$ in the game $\Gamma_c\left(x_t^c, T - t\right)$, it is required that

$$B^{t_0}(s) = B^t(s), \text{ for } s \in [t, T] \text{ and } t \in [t_0, T].$$

Therefore a formal definition can be presented as below.

Definition 2.2 The imputation and payoff distribution procedure $\left\{\xi^{(t_0)}(t_0, x_0) \text{ and } B^{t_0}(s) \text{ for } s \in [t_0, T]\right\}$ are subgame consistent if

(i) $$\exp\left[\int_{t_0}^{t} r(y)dy\right]\xi^{(t_0)i}\left(t, x_t^c\right)$$
$$\equiv \exp\left[\int_{t_0}^{t} r(y)dy\right]\left\{\int_t^T B_i^{t_0}(s)\exp\left[-\int_t^s r(y)dy\right]ds + q^i\left(x_T^c\right)\exp\left[-\int_{t_0}^{T} r(y)dy\right]\right\}$$
$$= \xi^{(t)i}\left(t, x_t^c\right) \equiv \int_t^T B_i^t(s)\exp\left[-\int_t^s r(y)dy\right]ds + q^i\left(x_T^c\right)\exp\left[-\int_t^T r(y)dy\right],$$
for $i \in N$ and $t \in [t_0, T]$, and (2.10)

(ii) the payoff distribution procedure $B^{t_0}(s) = \left[B_1^{t_0}(s), B_2^{t_0}(s), \cdots, B_n^{t_0}(s)\right]$ for $s \in [t, T]$ is identical to

$$B^t(s) = \left[B_1^t(s), B_2^t(s), \cdots, B_n^t(s)\right], \text{ for } t \in [t_0, T] \tag{2.11}$$

■

Thus cooperative strategies, payoff distribution procedures and imputations satisfying the conditions in Definitions 2.1 and 2.2 are subgame consistent.

2.3 Subgame Consistent Payoff Distribution Procedure

Crucial to obtaining a subgame consistent solution is the derivation of a payoff distribution procedure satisfying Definition 2.2 in Sect. 2.2.

2.3.1 Derivation of Payoff Distribution Procedures

Invoking part (ii) of Definition 2.2, we have $B^{t_0}(s) = B^t(s)$ for $t \in [t_0, T]$ and $s \in [t, T]$. We use $B(s) = \{B_1(s), B_2(s), \cdots, B_n(s)\}$ to denote $B^t(s)$ for all $t \in [t_0, T]$. Along the optimal trajectory $\{x^c(s)\}_{s=t_0}^T$ we then have:

$$\xi^{(\tau)i}\left(\tau, x_\tau^c\right) = \int_\tau^T B_i(s) \exp\left[-\int_\tau^s r(y)dy\right] ds + q^i\left(x_T^c\right) \exp\left[-\int_\tau^T r(y)dy\right], \quad (3.1)$$

for $i \in N$ and $\tau \in [t_0, T]$; and

$$\sum_{j=1}^n B_j(s) = \sum_{j=1}^n g^j\left[s, x_s^c, u_1^{(\tau)*}(s), u_2^{(\tau)*}(s), \cdots, u_n^{(\tau)*}(s)\right].$$

Moreover, for $t \in [\tau, T]$, we use the term

$$\xi^{(\tau)i}\left(t, x_t^c\right) = \int_t^T B_i(s) \exp\left[-\int_\tau^s r(y)dy\right] ds + q^i\left(x_T^c\right) \exp\left[-\int_\tau^T r(y)dy\right], \quad (3.2)$$

to denote the present value (with initial time being τ) of player i's payoff under cooperation over the time interval $[t, T]$ according to the agreed-upon optimality principle along the cooperative state trajectory.

Invoking (3.1) and (3.2) we have

$$\xi^{(\tau)i}\left(t, x_t^c\right) = \exp\left[-\int_\tau^t r(y)dy\right] \xi^{(t)i}\left(t, x_t^c\right),$$
$$\text{for } i \in N \text{ and } \tau \in [t_0, T] \text{ and } t \in [\tau, T]. \quad (3.3)$$

One can readily verify that a payoff distribution procedure $\{B(s)\}_{s=t_0}^T$ which satisfies (3.3) would give rise to subgame consistent imputations satisfying part (ii) of Definition 2.2. The next task is the derivation of a payoff distribution procedure $\{B(s)\}_{s=t_0}^T$ that leads to the realization of (3.1, 3.2 and 3.3).

We first consider the following condition concerning the imputation $\xi^{(\tau)}(t, x_t^c)$, for $\tau \in [t_0, T]$ and $t \in [\tau, T]$.

Condition 3.1 For $i \in N$ and $t \in [\tau, T]$ and $\tau \in [t_0, T]$, the imputation $\xi^{(\tau)i}(t, x_t^c)$, for $i \in N$, is a function that is continuously differentiable in t and x_t^c. ■

A theorem characterizing a formula for $B_i(s)$, for $s \in [t_0, T]$ and $i \in N$, which yields (3.1, 3.2 and 3.3) is provided as follows.

Theorem 3.1 If Condition 3.1 is satisfied, a PDP with a terminal payment $q^i(x_T^c)$ at time T and an instantaneous payment at time $s \in [\tau, T]$:

$$B_i(s) = -\left[\xi_t^{(s)i}(t, x_t^c)\Big|_{t=s}\right] - \left[\xi_{x_s^c}^{(s)i}(s, x_s^c)\right] f\left[s, x_s^c, \psi_1^*(s, x_s^c), \psi_2^*(s, x_s^c), \cdots, \psi_n^*(s, x_s^c)\right], \text{for } i \in N, \quad (3.4)$$

yields imputation vector $\xi^{(\tau)}(\tau, x_\tau^c)$, for $\tau \in [t_0, T]$ which satisfy (3.1, 3.2 and3.3).

Proof Invoking (3.1, 3.2 and 3.3), one can obtain

$$\xi^{(\upsilon)i}(\upsilon, x_\upsilon^c) = \int_\upsilon^{\upsilon+\Delta t} B_i(s) \exp\left[-\int_\upsilon^s r(y)dy\right] ds + \exp\left[-\int_\upsilon^{\upsilon+\Delta t} r(y)dy\right] \xi^{(\upsilon+\Delta t)i}(\upsilon + \Delta t, x_\upsilon^c + \Delta x_\upsilon^c),$$
$$\text{for } \upsilon \in [\tau, T] \text{ and } i \in N; \quad (3.5)$$

where $\Delta x_\upsilon^c = f\left[\upsilon, x_\upsilon^c, \psi_1^*(\upsilon, x_\upsilon^c), \psi_2^*(\upsilon, x_\upsilon^c), \cdots, \psi_n^*(\upsilon, x_\upsilon^c)\right] \Delta t + o(\Delta t)$, and $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$.

From (3.2) and (3.5), one obtains

$$\int_\upsilon^{\upsilon+\Delta t} B_i(s) \exp\left[-\int_\upsilon^s r(y)dy\right] ds$$
$$= \xi^{(\upsilon)i}(\upsilon, x_\upsilon^c) - \exp\left[-\int_\upsilon^{\upsilon+\Delta t} r(y)dy\right] \xi^{(\upsilon+\Delta t)i}(\upsilon + \Delta t, x_\upsilon^c + \Delta x_\upsilon^c)$$
$$= \xi^{(\upsilon)i}(\upsilon, x_\upsilon^c) - \xi^{(\upsilon)i}(\upsilon + \Delta t, x_\upsilon^c + \Delta x_\upsilon^c),$$
$$\text{for all } \upsilon \in [t_0, T] \text{ and } i \in N \quad (3.6)$$

If the imputations $\xi^{(\upsilon)}(t, x_t^c)$, for $\upsilon \in [t_0, T]$, satisfy Condition 3.1, as $\Delta t \to 0$, one can express condition (3.6) as:

$$B_i(\upsilon)\Delta t = -\left[\xi_t^{(\upsilon)i}(t, x_t^c)\Big|_{t=\upsilon}\right] \Delta t - \left[\xi_{x_\upsilon^c}^{(\upsilon)i}(\upsilon, x_\upsilon^c)\right] f\left[\upsilon, x_\upsilon^c, \psi_1^*(\upsilon, x_\upsilon^c), \psi_2^*(\upsilon, x_\upsilon^c), \cdots, \psi_n^*(\upsilon, x_\upsilon^c)\right] \Delta t - o(\Delta t). \quad (3.7)$$

Dividing (3.7) throughout by Δt, with $\Delta t \to 0$, yields (3.4).

$$B_i(\upsilon) = -\left[\xi_t^{(\upsilon)i}\left(t, x_t^c\right)\Big|_{t=\upsilon}\right] - \left[\xi_{x_\upsilon^c}^{(\upsilon)i}\left(\upsilon, x_\upsilon^c\right)\right] f\left[\upsilon, x_\upsilon^c, \psi_1^*\left(\upsilon, x_\upsilon^c\right), \psi_2^*\left(\upsilon, x_\upsilon^c\right), \cdots, \psi_n^*\left(\upsilon, x_\upsilon^c\right)\right]$$

Thus the payoff distribution procedure in $B_i(s)$ in (3.4) would lead to the realization of $\xi^{(\tau)i}(\tau, x_\tau^c)$, for $\tau \in [t_0, T]$ which satisfy (3.1, 3.2 and 3.3). ■

Assigning the instantaneous payments according to the payoff distribution procedure in (3.4) leads to the realization of the imputation $\xi^{(\tau)}(\tau, x_\tau^c)$ governed by the agreed-upon optimality principle in the game $\Gamma_c\left(x_\tau^c, T-\tau\right)$ for $\tau \in [t_0, T]$. Therefore the payoff distribution procedure in $B_i(s)$ in (3.4) yields a subgame consistent solution.

With players using the cooperative strategies $\left\{\psi_i^*\left(\tau, x_\tau^*\right), \text{for } \tau \in [t_0, T] \text{ and } i \in N\right\}$, the instantaneous payment received by player i at time instant τ is:

$$\zeta_i(\tau) = g^i\left[\tau, x_\tau^*, \psi_1^*\left(\tau, x_\tau^*\right), \psi_2^*\left(\tau, x_\tau^*\right), \cdots, \psi_n^*\left(\tau, x_\tau^*\right)\right)\right],$$
$$\text{for } \tau \in [t_0, T] \text{ and } i \in N. \tag{3.8}$$

According to Theorem 3.1, the instantaneous payment that player i should receive under the agreed-upon optimality principle is $B_i(\tau)$ as stated in (3.2). Hence an instantaneous transfer payment

$$\chi^i(\tau) = B_i(\tau) - \zeta_i(\tau) \tag{3.9}$$

has to be given to player i at time τ, for $i \in N$ and $\tau \in [t_0, T]$.

2.3.2 Subgame Consistent Solution under Specific Optimality Principle

In this section we present examples of subgame consistent solutions under various optimality principles.

Case I Consider the cooperative differential game $\Gamma_c(x_0, T - t_0)$. In particular, the players agree with an optimality principle which entails

(i) group optimality and
(ii) the division of the excess of the total cooperative payoff over the sum of individual noncooperative payoffs equally.

According to the optimality principle the imputation to player j in $\Gamma_c(x_0, T-t_0)$ is:

$$\xi^{(\tau)i}\left(\tau, x_\tau^c\right) = V^{(\tau)i}\left(\tau, x_\tau^c\right) + \frac{1}{n}\left[W^{(\tau)}\left(\tau, x_\tau^c\right) - \sum_{j=1}^{n} V^{(\tau)j}\left(\tau, x_\tau^c\right)\right], \tag{3.10}$$

for $i \in N$ and $\tau \in [t_0, T]$.

The imputation in (3.10) yields

(i) $\xi^{(\tau)i}\left(\tau, x_\tau^c\right) \geq V^{(\tau)i}\left(\tau, x_\tau^c\right)$, for $i \in N$ and $\tau \in [t_0, T]$; and

(ii) $\sum_{j=1}^{n} \xi^{(\tau)j}\left(\tau, x_\tau^c\right) = W^{(\tau)}\left(\tau, x_\tau^c\right)$ for $\tau \in [t_0, T]$.

Hence the imputation vector $\xi^{(\tau)i}(\tau, x_\tau^*)$ satisfies individual rationality and group optimality.

Applying Theorem 3.1 a subgame consistent solution under the optimal principle can be characterized by $\left\{u(s) \text{ and } B(s) \text{ for } s \in [t_0, T] \text{ and } \xi^{(t_0)}(t_0, x_0)\right\}$ in which

(i) $u(s)$ for $s \in [t_0, T]$ is the set of group optimal strategies $\psi^*(s, x_s^*)$ in the game $\Gamma_c(x_0, T-t_0)$, and

(ii) the imputation distribution procedure

$$B(s) = \{B_1(s), B_2(s), \cdots, B_n(s)\} \text{ for } s \in [t_0, T] \text{ where}$$

$$B_i(s) = -\frac{\partial}{\partial t}\left[V^{(s)i}\left(t, x_t^*\right) + \frac{1}{n}\left(W^{(s)}\left(t, x_t^*\right) - \sum_{j=1}^{n} V^{(s)j}\left(t, x_t^*\right)\right)\Bigg|_{t=s}\right]$$

$$- \frac{\partial}{\partial x_s^*}\left[V^{(s)i}\left(s, x_s^*\right) + \frac{1}{n}\left(W^{(s)}\left(s, x_s^*\right) - \sum_{j=1}^{n} V^{(s)j}\left(s, x_s^*\right)\right)\right]$$

$$\times f\left[s, x_s^*, \psi_1^*\left(s, x_s^*\right), \psi_2^*\left(s, x_s^*\right), \cdots, \psi_n^*\left(s, x_s^*\right)\right], \tag{3.11}$$

for $i \in N$.

Case II Consider the cooperative differential game $\Gamma_c(x_0, T-t_0)$. In particular, the players agree with an optimality principle which entails

(i) group optimality and

(ii) the sharing of the excess of the total cooperative payoff over the sum of individual noncooperative payoffs proportional to the players' noncooperative payoffs.

$$\xi^{(\tau)i}\left(\tau, x_\tau^c\right) = \frac{V^{(\tau)i}\left(\tau, x_\tau^c\right)}{\sum_{j=1}^{n} V^{(\tau)j}\left(\tau, x_\tau^c\right)} + W^{(\tau)}\left(\tau, x_\tau^c\right), \tag{3.12}$$

for $i \in N$ and $\tau \in [t_0, T]$.

Applying Theorem 3.1 a subgame consistent solution under the optimal principle will yield the imputation distribution procedure
$B(s) = \{B_1(s), B_2(s), \cdots, B_n(s)\}$ for $s \in [t_0, T]$ where

$$B_i(s) = -\frac{\partial}{\partial t}\left[\frac{V^{(s)i}(t, x_t^*)}{\sum_{j=1}^{n} V^{(s)j}(t, x_t^*)} W^{(s)}(t, x_t^*)\Big|_{t=s}\right]$$
$$-\frac{\partial}{\partial x_s^*}\left[\frac{V^{(s)i}(s, x_s^*)}{\sum_{j=1}^{n} V^{(s)j}(s, x_s^*)} W^{(s)}(s, x_s^*)\right]$$
$$\times f\left[s, x_s^*, \psi_1^*(s, x_s^*), \psi_2^*(s, x_s^*), \cdots, \psi_n^*(s, x_s^*)\right] \text{ for } i \in N \qquad (3.13)$$

Case III Consider the cooperative differential game $\Gamma_c(x_0, T - t_0)$ with two players. In particular, the players agree with an optimality principle which entails

(i) group optimality and
(ii) the division of the excess of the total cooperative payoff over the sum of individual noncooperative payoffs by the time-varying weights $-\frac{\tau}{T+\alpha}$ for player 1 and $\frac{T+\alpha-\tau}{T+\alpha}$ for player 2 at time $\tau \in [t_0, T]$.

According to optimality principle the imputations to player 1 and player 2 in $\Gamma_c(x_0, T - t_0)$ are:

$$\xi^{(\tau)1}(\tau, x_\tau^c) = V^{(\tau)1}(\tau, x_\tau^c) + \frac{\tau}{T+\alpha}\left[W^{(\tau)}(\tau, x_\tau^c) - \sum_{j=1}^{2} V^{(\tau)j}(\tau, x_\tau^c)\right]$$

for player 1, and

$$\xi^{(\tau)2}(\tau, x_\tau^c) = V^{(\tau)2}(\tau, x_\tau^c) + \frac{T+\alpha-\tau}{T+\alpha}\left[W^{(\tau)}(\tau, x_\tau^c) - \sum_{j=1}^{2} V^{(\tau)j}(\tau, x_\tau^c)\right] \qquad (3.14)$$

for player 2; $\tau \in [t_0, T]$.

Applying Theorem 3.1 a subgame consistent solution under the optimal principle will yield the imputation distribution procedure
$B(s) = \{B_1(s), B_2(s), \cdots, B_n(s)\}$ for $s \in [t_0, T]$ where

$$B_1(s) = -\frac{\partial}{\partial t}\left[V^{(s)1}\left(t, x_t^*\right) + \frac{t}{T+\alpha}\left(W^{(s)}\left(t, x_t^*\right) - \sum_{j=1}^{2} V^{(s)j}\left(t, x_t^*\right) \right) \Bigg|_{t=s} \right]$$
$$- \frac{\partial}{\partial x_s^*}\left[V^{(s)1}\left(s, x_s^*\right) + \frac{s}{T+\alpha}\left(W^{(s)}\left(s, x_s^*\right) - \sum_{j=1}^{2} V^{(s)j}\left(s, x_s^*\right) \right) \right]$$
$$\times f\left[s, x_s^*, \psi_1^*\left(s, x_s^*\right), \psi_2^*\left(s, x_s^*\right)\right]$$
$$B_2(s) = -\frac{\partial}{\partial t}\left[V^{(s)2}\left(t, x_t^*\right) + \frac{T-t+\alpha}{T+\alpha}\left(W^{(s)}\left(t, x_t^*\right) - \sum_{j=1}^{2} V^{(s)j}\left(t, x_t^*\right) \right) \Bigg|_{t=s} \right]$$
$$- \frac{\partial}{\partial x_s^*}\left[V^{(s)1}\left(s, x_s^*\right) + \frac{T-s+\varepsilon}{T+\alpha}\left(W^{(s)}\left(s, x_s^*\right) - \sum_{j=1}^{2} V^{(s)j}\left(s, x_s^*\right) \right) \right]$$
$$\times f\left[s, x_s^*, \psi_1^*\left(s, x_s^*\right), \psi_2^*\left(s, x_s^*\right)\right]. \tag{3.15}$$

A variety of optimality principles with various imputation schemes can be constructed.

2.4 An Illustration in Cooperative Fishery

Consider a deterministic version of an example in Yeung and Petrosyan (2004) in which two nations are harvesting fish in common waters. The growth rate of the fish stock is characterized by the differential equation:

$$\dot{x}(s) = ax(s)^{1/2} - bx(s) - u_1(s) - u_2(s), \;\; x(t_0) = x_0 \in X, \tag{4.1}$$

where $u_i \in U_i$ is the (nonnegative) amount of fish harvested by nation i, for $i \in \{1, 2\}$, a and b are positive constants.

The harvesting cost for nation $i \in \{1, 2\}$ depends on the quantity of resource extracted $u_i(s)$, the resource stock size $x(s)$, and a parameter c_i. In particular, nation i's extraction cost can be specified as $c_i u_i(s) x(s)^{-1/2}$. The fish harvested by nation i at time s will generate a net benefit of the amount $[u_i(s)]^{1/2}$. The horizon in concern is $[t_0, T]$. At time T, nation i will receive a termination bonus $q_i x(T)^{1/2}$, where q_i is nonnegative. There exists a positive discount rate r.

At time t_0 the payoff of nation $i \in [1, 2]$ is:

$$\int_{t_0}^{T} \left[[u_i(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_i(s) \right] \exp[-r(s - t_0)]\, ds$$
$$+ \exp[-r(T - t_0)] q_i x(T)^{\frac{1}{2}}. \tag{4.2}$$

Following the above analysis a set of feedback strategies $\{u_i^*(t) = \phi_i^*(t,x)$, for $i \in \{1,2\}\}$ provides a feedback Nash equilibrium solution to the game (4.1 and 4.2), if there exist continuously differentiable functions $V^{(\tau)i}(t,x) : [\tau,T] \times R \to R$, $i \in \{1,2\}$, satisfying the following partial differential equations:

$$-V_t^{(\tau)i}(t,x) = \max_{u_i}\left\{ \left[u_i^{1/2} - \frac{c_i}{x^{1/2}}u_i\right]\exp[-r(t-\tau)] + V_x^{(\tau)i}(t,x)\left[ax^{1/2} - bx - u_i - \phi_j^*(t,x)\right] \right\}, \text{ and}$$
$$V^{(\tau)i}(T,x) = q_i x^{1/2}\exp[-r(T-\tau)] \text{ for } i \in \{1,2\},\ j \in \{1,2\} \text{ and } j \neq i. \tag{4.3}$$

Performing the indicated maximization yields:

$$\phi_i^*(t,x) = \frac{x}{4\left[c_i + V_x^{(\tau)i}\exp[r(t-\tau)]x^{1/2}\right]^2}, \text{ for } i \in \{1,2\} \tag{4.4}$$

Substituting $\phi_1^*(t,x)$ and $\phi_2^*(t,x)$ into (4.3) and upon solving (4.43 one obtains can obtain the feedback Nash equilibrium payoff of nation i in the game (4.1 and 4.2) as:

$$V^{(\tau)i}(t,x) = \exp[-r(t-\tau)]\left[A_i(t)x^{1/2} + C_i(t)\right],$$
$$\text{for } i \in \{1,2\} \text{ and } t \in [\tau,T] \text{ and } \tau \in [t_0,T], \tag{4.5}$$

where $A_i(t), C_i(t), A_j(t)$ and $C_j(t)$, for $i \in \{1,2\}$ and $j \in \{1,2\}$ and $i \neq j$, satisfy:

$$\dot{A}_i(t) = \left[r + \frac{b}{2}\right]A_i(t) - \frac{1}{2[c_i + A_i(t)/2]} + \frac{c_i}{4[c_i + A_i(t)/2]^2} + \frac{A_i(t)}{8[c_i + A_i(t)/2]^2} + \frac{A_i(t)}{8\left[c_j + A_j(t)/2\right]^2}$$
$$\dot{C}_i(t) = rC_i(t) - \frac{a}{2}A_i(t) \text{ and } A_i(T) = q, \text{ and } C_i(T) = 0. \tag{4.6}$$

Now consider the case when the nations agree to cooperate in harvesting the fishery. Let $\Gamma_c(x_0, T - t_0)$ denote a cooperative game with the game structure of $\Gamma(x_0, T - t_0)$ in which the players agree to act according to the optimality principle that they would

(i) maximize the sum of their payoffs and
(ii) divide the excess of the total cooperative payoff over the sum of individual noncooperative payoffs equally.

To maximize the joint payoffs, the nations would consider the optimal control problem:

$$\int_{t_0}^{T}\left(\left[u_1(s)^{1/2}-\frac{c_1}{x(s)^{1/2}}u_1(s)\right]+\left[u_2(s)^{1/2}-\frac{c_2}{x(s)^{1/2}}u_2(s)\right]\right)\exp[-r(t-t_0)]\,ds$$
$$+\,2\exp[-r(T-t_0)]qx(T)^{\frac{1}{2}}, \tag{4.7}$$

subject to (4.1).

Let $[\psi_1^*(t,x),\psi_2^*(t,x)]$ denote a set of controls that provides a solution to the optimal control problem (4.1) and (4.7) and $W^{(t_0)}(t,x):[t_0,T]\times R^n\to R$ denote the value function that satisfies the equations:

$$\begin{aligned}
&-W_t^{(t_0)}(t,x)\\
&=\max_{u_1,u_2}\left\{\left(\left[u_1{}^{1/2}-\frac{c_1}{x^{1/2}}u_1\right]+\left[u_2{}^{1/2}-\frac{c_2}{x^{1/2}}u_2\right]\right)\exp[-r(t-t_0)]\right.\\
&\left.+\,W_x^{(t_0)}(t,x)\left[ax^{1/2}-bx-u_1-u_2\right]\right\},\ \text{and}\\
&W^{(t_0)}(T,x)=2\exp[-r(T-t_0)]qx^{\frac{1}{2}}.
\end{aligned}\tag{4.8}$$

Performing the indicated maximization we obtain:

$$\psi_1^*(t,x)=\frac{x}{4\left[c_1+W_x^{(t_0)}\exp[r(t-t_0)]x^{1/2}\right]^2},\ \text{and}$$
$$\psi_2^*(t,x)=\frac{x}{4\left[c_2+W_x^{(t_0)}\exp[r(t-t_0)]x^{1/2}\right]^2}.$$

Substituting $\psi_1^*(t,x)$ and $\psi_2^*(t,x)$ above into (4.8) yields the value function

$$W^{(t_0)}(t,x)=\exp[-r(t-t_0)]\left[\hat{A}(t)x^{1/2}+\hat{C}(t)\right],$$

where $\dot{\hat{A}}(t)=\left[r+\frac{b}{2}\right]\hat{A}\,(t)-\frac{1}{2\left[c_1+\hat{A}\,(t)/2\right]}-\frac{1}{2\left[c_2+\hat{A}\,(t)/2\right]}$

$$+\frac{c_1}{4\left[c_1+\hat{A}\,(t)/2\right]^2}+\frac{c_2}{4\left[c_2+\hat{A}\,(t)/2\right]^2}+\frac{\hat{A}\,(t)}{8\left[c_1+\hat{A}\,(t)/2\right]^2}+\frac{\hat{A}\,(t)}{8\left[c_2+\hat{A}\,(t)/2\right]^2},$$

$$\dot{\hat{C}}(t)=r\hat{C}\,(t)-\frac{a}{2}\hat{A}\,(t),\ \ \hat{A}\,(T)=2q,\ \text{and}\ \hat{B}\,(T)=0. \tag{4.9}$$

The optimal cooperative controls can then be obtained as:

$$\psi_1^*(t,x) = \frac{x}{4\left[c_1 + \hat{A}(t)/2\right]^2} \text{ and } \psi_2^*(t,x) = \frac{x}{4\left[c_2 + \hat{A}(t)/2\right]^2}. \tag{4.10}$$

Substituting these control strategies into (4.1) yields the dynamics of the state trajectory under cooperation:

$$\dot{x}(s) = ax(s)^{1/2} - bx(s) - \frac{x(s)}{4\left[c_1 + \hat{A}(s)/2\right]^2} - \frac{x(s)}{4\left[c_2 + \hat{A}(s)/2\right]^2},\ x(t_0) = x_0. \tag{4.11}$$

Solving (4.11) yields the optimal cooperative state trajectory for $\Gamma_c(x_0, T-t_0)$ as:

$$x^*(s) = \varpi(t_0,s)^2\left[x_0^{1/2} + \int_{t_0}^{s}\varpi^{-1}(t_0,t)H_1 dt\right]^2, \text{ for } s\in[t_0,T], \tag{4.12}$$

where $\varpi(t_0,s) = \exp\left[\int_{t_0}^{s} H_2(\tau)d\tau\right]$, $H_1 = \frac{1}{2}a$, and $H_2(s) = -\left[\frac{1}{2}b + \frac{1}{8\left[c_1+\hat{A}(s)/2\right]^2} + \frac{1}{8\left[c_2+\hat{A}(s)/2\right]^2}\right]$.

The cooperative control for the game $\Gamma_c(x_0, T-t_0)$ over the time interval $[t_0,T]$ along the optimal trajectory $\left\{x^*(t)\right\}_{t=t_0}^{T}$ can be expressed precisely as:

$$\psi_1^*(t,x_t^*) = \frac{x_t^*}{4\left[c_1 + \hat{A}(t)/2\right]^2}, \text{ and } \psi_2^*(t,x_t^*) = \frac{x_t^*}{4\left[c_2 + \hat{A}(t)/2\right]^2}. \tag{4.13}$$

Following the above analysis, the value function of the optimal control problem with dynamics structure (4.1) and payoff structure (4.7) which starts at time τ with initial state x_τ^* can be obtained as $W^{(\tau)}(t,x) = \exp[-r(t-\tau)]\left[\hat{A}(t)x^{1/2} + \hat{B}(t)\right]$, and the corresponding optimal controls as

$$\psi_1^*(t,x_t^*) = \frac{x_t^*}{4\left[c_1 + \hat{A}(t)/2\right]^2}, \text{ and } \psi_2^*(t,x_t^*) = \frac{x_t^*}{4\left[c_2 + \hat{A}(t)/2\right]^2},$$

over the time interval $[\tau, T]$.

The agreed-upon optimality principle entails an imputation

$$\xi^{(\tau)i}(\tau,x_\tau^*) = V^{(\tau)i}(\tau,x_\tau^*) + \frac{1}{n}\left[W^{(\tau)}(\tau,x_\tau^*) - \sum_{j=1}^{n} V^{(\tau)j}(\tau,x_\tau^*)\right],\ i\in\{1,2\}, \tag{4.14}$$

in the cooperative game $\Gamma_c(x_\tau^*, T-\tau)$ for $\tau\in\left\{t_0,T\right]$.

Applying Theorem 3.1 a subgame consistent solution under the above optimal principle for the cooperative game $\Gamma_c(x_0, T-t_0)$ can be obtained as:

$\{u(s)$ and $B(s)$ for $s \in [t_0, T]$ and $\xi^{(t_0)}(t_0, x_0)\}$ in which

(i) $u(s)$ for $s \in [t_0, T]$ is the set of group optimal strategies $\psi_1^*(s, x_s^*) = \frac{x_s^*}{4[c_1 + \hat{A}(s)/2]^2}$, and $\psi_2^*(s, x_s^*) = \frac{x_s^*}{4[c_2 + \hat{A}(s)/2]^2}$; and

(ii) the imputation distribution procedure

$B(s) = \{B_1(s), B_2(s)\}$ for $s \in [t_0, T]$ where

$$
\begin{aligned}
B_i(s) = \frac{-1}{2}\Bigg\{ & \left(\left[\dot{A}_i(s)(x_s^*)^{1/2} + \dot{C}_i(s)\right] + r\left[A_i(s)(x_s^*)^{1/2} + C_i(s)\right]\right) \\
&+ \left[\frac{1}{2}A_i(s)(x_s^*)^{-1/2}\right]\left[a(x_s^*)^{1/2} - bx_s^* - \frac{x_s^*}{4[c_i + \hat{A}(s)/2]^2} - \frac{x_s^*}{4[c_j + \hat{A}(s)/2]^2}\right]\Bigg\} \\
&-\frac{1}{2}\Bigg\{ \left(\left[\dot{\hat{A}}(s)(x_s^*)^{1/2} + \dot{\hat{C}}(s)\right] + r\left[\hat{A}(s)(x_s^*)^{1/2} + \hat{C}(s)\right]\right) \\
&+ \left[\frac{1}{2}\hat{A}(s)(x_s^*)^{-1/2}\right] \\
&\left[a(x_s^*)^{1/2} - bx_s^* - \frac{x_s^*}{4[c_i + \hat{A}(s)/2]^2} - \frac{x_s^*}{4[c_j + \hat{A}(s)/2]^2}\right]\Bigg\} \\
&+\frac{1}{2}\Bigg\{ \left(\left[\dot{A}_j(s)(x_s^*)^{1/2} + \dot{C}_j(s)\right] + r\left[A_j(s)(x_s^*)^{1/2} + C_j(s)\right]\right) \\
&+ \left[\frac{1}{2}A_j(s)(x_s^*)^{-1/2}\right] \\
&\left[a(x_s^*)^{1/2} - bx_s^* - \frac{x_s^*}{4[c_i + \hat{A}(s)/2]^2} - \frac{x_s^*}{4[c_j + \hat{A}(s)/2]^2}\right]\Bigg\}, \\
&\text{for } i, j \in \{1, 2\} \text{ and } i \neq j,
\end{aligned}
\tag{4.15}
$$

where $\dot{A}_i(s)$ and $\dot{C}_i(s)$ are given in (4.6); and $\dot{\hat{A}}(s)$ and $\dot{\hat{C}}(s)$ are given in (4.9).

With players using the cooperative strategies, the instantaneous receipt of player i at time instant τ is:

$$
\zeta_i(\tau) = \frac{(x_\tau^*)^{1/2}}{2[c_i + A(\tau)/2]} - \frac{c_i(x_\tau^*)^{1/2}}{4[c_i + A(\tau)/2]^2}, \tag{4.16}
$$

Under cooperation the instantaneous payment that player i should receive is $B_i(\tau)$ as stated in (4.15). Hence an instantaneous transfer payment

$$\chi^i(\tau) = B_i(\tau) - \zeta_i(\tau) \quad (4.17)$$

has to be given to player i at time τ, for $i \in \{1, 2\}$ and $\tau \in [t_0, T]$.

2.5 Infinite Horizon Analysis

In this section we consider infinite horizon cooperative differential games in which player i's payoff is:

$$\int_{\tau}^{\infty} g^i[x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp[-r(s-\tau)]ds, \text{ for } i \in N. \quad (5.1)$$

The state dynamics is

$$\dot{x}(s) = f[x(s), u_1(s), u_2(s), \cdots, u_n(s)], x(\tau) = x_\tau. \quad (5.2)$$

Since s does not appear in $g^i[x(s), u_1(s), u_2(s)]$ and the state dynamics, the game (5.1 and 5.2) is an autonomous problem. Consider the alternative game $\Gamma(x)$ which starts at time $t \in \left[t_0, \infty\right)$ with initial state $x(t) = x$:

$$\max_{u_i} \int_{t}^{\infty} g^i[x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp[-r(s-t)]ds, \text{ for } i \in N, \quad (5.3)$$

subject to the state dynamics

$$\dot{x}(s) = f[x(s), u_1(s), u_2(s), \cdots, u_n(s)], \; x(t) = x. \quad (5.4)$$

The infinite-horizon autonomous game $\Gamma(x)$ is independent of the choice of t and dependent only upon the state at the starting time, that is x.

A feedback Nash equilibrium solution for the infinite-horizon autonomous game (5.3) and (5.4) can be characterized as follows:

Theorem 5.1 An n-tuple of strategies $\left\{u_i^* = \phi_i^*(\cdot) \in U^i, \text{ for } i \in N\right\}$ provides a feedback Nash equilibrium solution to the infinite-horizon game (5.3) and (5.4) if there exist continuously differentiable functions $\hat{V}^i(x) : R^m \to R, i \in N$, satisfying the following set of partial differential equations:

$$r\hat{V}^i(x) = \max_{u_i} \Big\{ g^i\left[x, \phi_1^*(x), \phi_2^*(x), \cdots, \phi_{i-1}^*(x), u_i, \phi_{i+1}^*(x), \cdots, \phi_n^*(x)\right]$$
$$+ \hat{V}_x^i(x) f\left[x, \phi_1^*(x), \phi_2^*(x), \cdots, \phi_{i-1}^*(x), u_i, \phi_{i+1}^*(x), \cdots, \phi_n^*(x)\right] \Big\}, \text{ for } i \in N.$$

Proof By Theorem A.2 in the Technical Appendices, $\hat{V}^i(x)$ is the value function associated with the optimal control problem of player i, $i \in N$. Hence the conditions in Theorem 5.1 imply a Nash equilibrium. ■

Now consider the case when the players agree to act cooperatively. Let $\Gamma_c(\tau, x_\tau)$ denote a cooperative game in which player i's payoff is (5.1) and the state dynamics is (5.2). The players agree to act according to an agreed upon optimality principle $P(\tau, x_\tau)$ which entails

(i) group optimality and
(ii) the distribution of the total cooperative payoff according to an imputation vector $\xi^{(\upsilon)}(\upsilon, x_\upsilon^*)$ for $\upsilon \in [\tau, \infty)$ over the game duration. Moreover, the function $\xi^{(\upsilon)i}(\upsilon, x_\upsilon^*) \in \xi^{(\upsilon)}(\upsilon, x_\upsilon^*)$, for $i \in N$, is continuously differentiable in υ and x_υ^*.

The solution of the cooperative game $\Gamma_c(\tau, x_\tau)$ includes

(i) a set of group optimal cooperative strategies $u^{(\tau)*}(s) = \left[u_1^{(\tau)*}(s), u_2^{(\tau)*}(s), \cdots, u_n^{(\tau)*}(s)\right]$, for $s \in [\tau, \infty)$;
(ii) an imputation vector $\xi^{(\tau)}(\tau, x_\tau) = \left[\xi^{(\tau)1}(\tau, x_\tau), \xi^{(\tau)2}(\tau, x_\tau), \cdots, \xi^{(\tau)n}(\tau, x_\tau)\right]$ to allot the cooperative payoff to the players; and
(iii) a payoff distribution procedure $B^\tau(s) = \left[B_1^\tau(s), B_2^\tau(s), \cdots, B_n^\tau(s)\right]$ for $s \in [\tau, \infty)$, where $B_i^\tau(s)$ is the instantaneous payments for player i at time s. In particular,

$$\xi^{(\tau)i}(\tau, x_\tau) = \int_\tau^\infty B_i^\tau(s) \exp[-r(s-\tau)] \, ds, \quad \text{for } i \in N \tag{5.5}$$

In the following sub-sections, we characterize the cooperative strategies and payoff distribution procedure of the cooperative game $\Gamma_c(\tau, x_\tau)$ under the agreed-upon optimality principle.

2.5.1 Group Optimal Cooperative Strategies

To ensure group rationality the players maximize the sum of their payoffs, the players solve the problem:

$$\max_{u_1, u_2, \cdots, u_n} \left\{ \int_\tau^\infty \sum_{j=1}^n g^j[x(s), u_1(s), u_2(s), \cdots, u_n(s)] \exp[-r(s-\tau)] ds \right\}, \tag{5.6}$$

subject to (5.2).

Following Theorem A.2 in the Technical Appendices, we note that a set of controls $\{\psi_1^*(x), \text{for } i \in N\}$ provides a solution to the optimal control problem (5.6) if there exists continuously differentiable function $W(x) : R^m \to R$ satisfying the infinite-horizon Bellman equation:

$$rW(x) = \max_{u_1, u_2, \cdots, u_n} \left\{ \sum_{j=1}^{2} g^j[x, u_1, u_2, \cdots, u_n] + W_x f[x, u_1, u_2, \cdots, u_n] \right\}. \tag{5.7}$$

The players will adopt the cooperative control $\{\psi_1^*(x)$, for $i \in N\}$ characterized in (5.7). Note that these controls are functions of the current state x only. Substituting this set of control into the state dynamics yields the optimal (cooperative) trajectory as;

$$\dot{x}(s) = f\left[x(s), \psi_1^*(x(s)), \psi_2^*(x(s)), \cdots, \psi_n^*(x(s))\right], \ x(\tau) = x_\tau. \tag{5.8}$$

Let $x^*(s)$ denote the solution to (5.8). The optimal trajectory $\{x^*(s)\}_{s=\tau}^{\infty}$ can be expressed as:

$$x^*(s) = x_\tau + \int_\tau^s f\left[x^*(\upsilon), \psi_1^*(x^*(\upsilon)), \psi_2^*(x^*(\upsilon)), \cdots, \psi_n^*(x^*(\upsilon))\right] d\upsilon.$$

For notational convenience, we use the terms $x^*(s)$ and x_s^* interchangeably.

The cooperative control for the game can be expressed more precisely as:

$$\{\psi_i^*(x_s^*), \text{ for } i \in N \text{ and } s \in [\tau, \infty)\},$$

which are functions of the current state x_s^* only. The term

$$W(x_\tau^*) = \int_\tau^\infty \sum_{j=1}^{n} g^j\left[x^*(s), \psi_1^*(x^*(s)), \psi_2^*(x^*(s)), \cdots, \psi_n^*(x^*(s))\right] \exp[-r(s-\tau)]\, ds$$

yields the maximized cooperative payoff at current time τ, given that the state is x_τ^*

2.5.2 Subgame Consistent Imputation and Payoff Distribution Procedure

According to the agreed-upon optimality principle $P(\tau, x_\tau)$ the players would use the Payoff Distribution Procedure $\{B^\tau(s)\}_{s=\tau}^{\infty}$ to bring about an imputation to player i as:

$$\xi^{(\tau)i}(\tau, x_\tau) = \int_\tau^\infty B_i^\tau(s) \exp[-r(s-\tau)] \ ds, \text{ for } i \in N. \tag{5.9}$$

At time τ, we define the present value of player i's payoff over the time interval $\left[t,\infty\right)$ as:

$$\xi^{(\tau)i}\left(t,x_t^*\right) = \int_t^\infty B_i^\tau(s)\exp[-r(s-\tau)]\ ds,\ \text{for}\, i\in N, \tag{5.10}$$

where $t>\tau$ and $x_t^* \in \left\{x^*(s)\right\}_{s=\tau}^\infty$.

Consider the case when the game has proceeded to time t and the state variable became x_t^*. Then one has a cooperative game $\Gamma_c(t,x_t^*)$ which starts at time t with initial state x_t^*. According to the agreed-upon optimality principle, an imputation

$$\xi^{(t)i}\left(t,x_t^*\right) = \int_t^\infty B_i^t(s)\exp[-r(s-t)]\ ds,$$

will be allotted to player i, for $i\in N$.

However, according to the optimality principle, the imputation (in present value viewed at time τ) to player i over the period $\left[t,\infty\right)$ is

$$\xi^{(\tau)i}\left(t,x_t^*\right) = \int_t^\infty B_i^\tau(s)\exp[-r(s-\tau)]\ ds,\ \text{for}\, i\in N, \tag{5.11}$$

For the imputations from the optimality principle to be consistent throughout the cooperation duration, it is essential that

$$\exp[r(t-\tau)]\xi^{(\tau)i}\left(t,x_t^*\right) = \xi^{(t)i}\left(t,x_t^*\right),\ \text{for}\, t\in(\tau,\infty).$$

In addition, at time τ when the initial state is x_τ, according to the optimality principle the payoff distribution procedure is

$$B^\tau(s) = \left[B_1^\tau(s), B_2^\tau(s), \cdots, B_n^\tau(s)\right],\ \text{for}\, s\in\left[\tau,\infty\right).$$

When the game has proceeded to time t and the state variable became x_t^*. According to the optimality principle the payoff distribution procedure

$$B^t(s) = \left[B_1^t(s), B_2^t(s), \cdots, B_n^t(s)\right],\ \text{for}\, s\in\left[t,\infty\right),$$

will be adopted.

For the continuation of the payoff distribution procedure to be consistent it is required that

$$B^{t_0}(s) = B^t(s),\ \text{for}\ s\in\left[t,\infty\right)\text{and}\, t\in\left[\tau,\infty\right).$$

Definition 5.1 The imputation and payoff distribution procedure $\left\{\xi^{(\tau)}(\tau, x_\tau) \text{ and } B^\tau(s) \text{ for } s \in [\tau, \infty)\right\}$ are subgame consistent if

$$\text{(i)}\quad \exp[r(t-\tau)]\xi^{(\tau)i}\left(t, x_t^*\right) \equiv \exp[r(t-\tau)]\int_t^\infty B_i^\tau(s)\exp[-r(s-\tau)]\ ds \\ = \xi^{(t)i}\left(t, x_t^*\right),\ \text{for}\, t \in (\tau, \infty) \text{ and}\, i \in N;\ \text{and} \tag{5.12}$$

(ii) the payoff distribution procedure $B^\tau(s) = \left[B_1^\tau(s), B_2^\tau(s), \cdots, B_n^\tau(s)\right]$ for $s \in [t, \infty)$ is identical to $B^t(s) = \left[B_1^t(s), B_2^t(s), \cdots, B_n^t(s)\right] \in \left(t, x_t^*\right)$. ■

Definition 5.1 yields the infinite horizon subgame consistent imputation and payoff distribution procedure.

2.5.3 Derivation of Subgame Consistent Payoff Distribution Procedure

A payoff distribution procedure leading to subgame consistent imputation has to satisfy Definition 5.1. Invoking Definition 5.1, we have $B_i^\tau(s) = B_i^t(s) = B_i(s)$, for $s \in [\tau, \infty)$ and $t \in [\tau, \infty)$ and $i \in N$.

Therefore along the cooperative trajectory $\left\{x^*(t)\right\}_{t \geq t_0}$,

$$\xi^{(\tau)i}\left(\tau, x_\tau^*\right) = \int_\tau^\infty B_i(s)\exp[-r(s-\tau)]ds,\ \text{for}\, i \in N,\ \text{and}$$
$$\xi^{(\upsilon)i}\left(\upsilon, x_\upsilon^*\right) = \int_\upsilon^\infty B_i(s)\exp[-r(s-\upsilon)]ds,\ \text{for}\, i \in N,\ \text{and}$$
$$\xi^{(t)i}\left(t, x_t^*\right) = \int_t^\infty B_i(s)\exp[-r(s-t)]ds,\ \text{for}\, i \in N \,\text{and}\, t \geq \upsilon \geq \tau \tag{5.13}$$

Moreover, for $i \in N$ and $t \in [\tau, \infty)$, we define the term

$$\xi^{(\upsilon)i}\left(t, x_t^*\right) = \left\{\left(\int_t^\infty B_i(s)\exp[-r(s-\upsilon)]ds\right) \middle|\ x(t) = x_t^*\right\}, \tag{5.14}$$

to denote the present value of player i's cooperative payoff over the time interval $[t, \infty)$, given that the state is x_t^* at time $t \in [\upsilon, \infty)$, under the solution $\mathrm{P}(\upsilon, x_\upsilon^*)$.

Invoking (5.13) and (5.14) one can readily verify that $\exp[r(t-\tau)]\xi^{(\tau)i}\left(t, x_t^*\right) = \xi^{(t)i}\left(t, x_t^*\right)$, for $i \in N$ and $\tau \in [t_0, T]$ and $t \in [\tau, T]$.

The next task is to derive $B_i(s)$, for $s \in [\tau, \infty)$ and $t \in [\tau, \infty)$ so that (5.13) can be realized. Consider again the following condition.

Condition 5.1 For $i \in N$ and $t \geq \upsilon$ and $\upsilon \in [\tau, T]$, the term $\xi^{(\upsilon)i}(t, x_t^*)$ is a function that is continuously differentiable in t and x_t^*.

A theorem characterizing a formula for $B_i(s)$, for $i \in N$ and $s \in [\upsilon, \infty)$, which yields (5.14) is provided as follows.

Theorem 5.2 If Condition 5.1 is satisfied, a PDP with instantaneous payments at time s equaling

$$B_i(s) = -\left[\xi_t^{(s)i}(t, x_t^*)\Big|_{t=s}\right] - \xi_{x_s^*}^{(s)i}(s, x_s^*) f\left[x_s^*, \psi_1^*(x_s^*), \psi_2^*(x_s^*), \cdots, \psi_n^*(x_s^*)\right], \quad (5.15)$$

for $i \in N$ and $s \in [\upsilon, \infty)$,

yields imputation $\xi^{(\upsilon)i}(\upsilon, x_\upsilon^c)$, for $\upsilon \in [\tau, \infty)$ which satisfy (5.13).

Proof Note that along the cooperative trajectory $\{x^*(t)\}_{t \geq \tau}$

$$\xi^{(\upsilon)i}(t, x_t^*) = \int_t^\infty B_i(s) \exp[-r(s-\upsilon)] ds = \exp[-r(t-\upsilon)]\, \xi^{(t)i}(t, x_t^*),$$

for $i \in N$ and $t \in [\upsilon, \infty)$. (5.16)

For $\Delta t \to 0$, Eq. (5.13) can be expressed as

$$\begin{aligned} \xi^{(\upsilon)i}(\tau, x_\tau^*) &= \int_\upsilon^\infty B_i(s) \exp[-r(s-\upsilon)] ds \\ &= \int_\upsilon^{\upsilon+\Delta t} B_i(s) \exp[-r(s-\upsilon)]\ ds + \xi^{(\upsilon)i}(\upsilon + \Delta t, x_\upsilon^* + \Delta x_\upsilon^*), \end{aligned} \quad (5.17)$$

where

$\Delta x_\upsilon^* = f\left[x_\upsilon^*, \psi_1^*(x_\upsilon^*), \psi_2^*(x_\upsilon^*), \cdots, \psi_n^*(x_\upsilon^*)\right] \Delta t + o(\Delta t)$, and $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$.

Replacing the term $x_\upsilon^* + \Delta x_\upsilon^*$ with $x_{\upsilon+\Delta t}^*$ and rearranging (5.17) yields:

$$\begin{aligned} &\int_\upsilon^{\upsilon+\Delta t} B_i(s) \exp[-r(s-\upsilon)]\ ds \\ &= \xi^{(\upsilon)i}(\upsilon, x_\upsilon^*) - \xi^{(\upsilon)i}(\upsilon + \Delta t, x_{\upsilon+\Delta t}^*), \text{ for all } \upsilon \in [\tau, \infty) \text{ and } i \in N. \end{aligned} \quad (5.18)$$

Consider the following condition concerning $\xi^{(\upsilon)i}(t, x_t^*)$, for $\upsilon \in [\tau, \infty)$ and $t \in [\upsilon, \infty)$:

With Condition 5.1 holding and $\Delta t \to 0$, (5.18) can be expressed as:

$$\begin{aligned} B_i(\upsilon)\Delta t &= -\left[\xi_t^{(\upsilon)i}(t, x_t^*)\Big|_{t=\tau}\right] \Delta t \\ &- \xi_{x_\upsilon^*}^{(\upsilon)i}(\upsilon, x_\upsilon^*) f\left[x_\upsilon^*, \psi_1^*(x_\upsilon^*), \psi_2^*(x_\upsilon^*), \cdots, \psi_n^*(x_\upsilon^*)\right] \Delta t - o(\Delta t). \end{aligned} \quad (5.19)$$

Dividing (5.19) throughout by Δt, with $\Delta t \to 0$, yields (5.15). Thus the payoff distribution procedure in $B_i(\upsilon)$ in (5.15) would lead to the realization of the imputations which satisfy (5.15). ■

Since the payoff distribution procedure in $B_i(\tau)$ in (5.15) leads to the realization of (5.13), it would yields subgame consistent imputations satisfying Definition 5.1.

A more succinct form of Theorem 5.2 can be derived as follows. Note that, a PDP with instantaneous payments at time s equaling

$$B_i(s) = r\,\xi^{(s)i}(s, x_s^*) - \xi_{x_s^*}^{(s)i}(s, x_s^*) f\left[x_s^*, \psi_1^*(x_s^*), \psi_2^*(x_s^*), \cdots, \psi_n^*(x_s^*)\right],$$
$$\text{for } i \in N \text{ and } s \in [\upsilon, \infty), \tag{5.20}$$

yields imputation $\xi^{(\upsilon)i}(\upsilon, x_\upsilon^c)$, for $\upsilon \in [\tau, \infty)$ which satisfy (5.13).

To demonstrate that (5.20) is an alternative form for (5.15) in Theorem 5.2, we define

$$\hat{\xi}^i(x_\upsilon^*) = \left\{ \int_\upsilon^\infty B_i(s)\exp[-r(s-\upsilon)]ds \,\middle|\, x(\upsilon) = x_\upsilon^* \right\} = \xi^{(\upsilon)i}(\tau, x_\upsilon^*), \text{ and}$$
$$\hat{\xi}^i(x_t^*) = \left\{ \int_t^\infty B_i(s)\exp[-r(s-t)]ds \,\middle|\, x(t) = x_t^* \right\} = \xi^{(t)i}(t, x_t^*),$$

for $i \in N$ and $\upsilon \in [\tau, \infty)$ and $t \in [\upsilon, \infty)$ along the optimal cooperative trajectory $\{x_s^*\}_{s=\tau}^\infty$.

We then have:

$$\xi^{(\upsilon)i}(t, x_t^*) = \exp[-r(t-\upsilon)]\,\hat{\xi}^i(x_t^*).$$

Differentiating $\xi^{(\upsilon)i}(t, x_t^*)$ with respect to t yields:

$$\left[\xi_t^{(\upsilon)i}(t, x_t^*)\Big|_{t=\upsilon}\right] = -r\exp[-r(t-\upsilon)]\,\hat{\xi}^i(x_t^*) = -r\,\xi^{(\upsilon)i}(t, x_t^*).$$

At $t = \upsilon$, $\xi^{(\upsilon)i}(t, x_t^*) = \xi^{(\upsilon)i}(\upsilon, x_\upsilon^*)$, therefore

$$\left[\xi_t^{(\upsilon)i}(t, x_t^*)\Big|_{t=\upsilon}\right] = r\xi^{(\upsilon)i}(t, x_t^*) = r\xi^{(\upsilon)i}(\upsilon, x_\upsilon^*). \tag{5.21}$$

Substituting (5.21) into (5.15) yields (5.20). Since the infinite-horizon autonomous game $\Gamma(x)$ is independent of the choice of time s and dependent only upon the state, Eq. (5.20) can be expressed as:

$$B_i(x_s^*) = r\,\hat{\xi}^i(x_s^*) - \hat{\xi}_{x_s^*}^i(x_s^*) f\left[x_s^*, \psi_1^*(x_s^*), \psi_2^*(x_s^*), \cdots, \psi_n^*(x_s^*)\right], \text{ for } i \in N. \tag{5.22}$$

Therefore a subgame consistent solution for the cooperative game $\Gamma_c(\tau, x_\tau)$ with optimality principle $P(\tau, x_\tau)$ includes the cooperative strategies and Payoff Distribution Procedure:

$\{u(s)$ and $B(x_s^*)$ for $s \in [\tau, \infty)\}$ in which

(i) $u(s)$ is the set of group optimal strategies $\psi^*(x_s^*)$ for the game $\Gamma_c(\tau, x_\tau)$, and
(ii) the payoff distribution procedure

$$B(x_s^*) = \{B_1(x_s^*), B_2(x_s^*), \cdots, B_n(x_s^*)\} \text{ where}$$
$$B_i(x_s^*) = r\hat{\xi}^i(x_s^*) - \hat{\xi}^i_{x_s^*}(x_s^*) f[x_s^*, \psi_1^*(x_s^*), \psi_2^*(x_s^*), \cdots, \psi_n^*(x_s^*)], \tag{5.23}$$

for $i \in N$.

With players using the cooperative strategies $\{\psi_i^*(x_\upsilon^*)$, for $i \in N$ and $\upsilon \in [\tau, \infty)\}$, the instantaneous receipt of player i at time instant υ is:

$$\zeta_i(x_\upsilon^*) = g^i[x_\upsilon^*, \psi_1^*(x_\upsilon^*), \psi_2^*(x_\upsilon^*), \cdots, \psi_n^*(x_\upsilon^*)],$$
$$\text{for } i \in N. \tag{5.24}$$

According to Theorem 5.2, the instantaneous payment that player i should receive under the agreed-upon optimality principle is $B_i(\upsilon)$ in (5.15) or equivalently $B_i(x_\upsilon^*)$ in (5.23). Hence an instantaneous transfer payment

$$\chi^i(x_\upsilon^*) = B_i(x_\upsilon^*) - \zeta_i(x_\upsilon^*) \tag{5.25}$$

has to be given to player i at time υ, for $i \in N$.

2.6 Infinite Horizon Resource Extraction

Consider an infinite horizon version of the cooperative fishery game in Sect. 2.5. At initial time τ, the payoff of nation 1 and that of nation 2 are respectively:

$$\int_\tau^\infty \left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}} u_1(s)\right] \exp[-r(t-\tau)]\, ds$$

and

$$\int_\tau^\infty \left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}} u_2(s)\right] \exp[-r(t-\tau)]\, ds. \tag{6.1}$$

The resource stock $x(s) \in X \subset R$ follows the dynamics

$$\dot{x}(s) = ax(s)^{1/2} - bx(s) - u_1(s) - u_2(s), \; x(\tau) = x_\tau \in X, \tag{6.2}$$

Using Theorem 5.1, the value function $\hat{V}^i(t,x)$ reflecting the payoff of nation i in a noncooperative feedback Nash equilibrium can be obtained as:

$$\hat{V}^i(t,x) = \left[A_i x^{1/2} + C_i\right], \tag{6.3}$$

where for $i,j \in \{1,2\}$ and $i \neq j, A_i, C_i, A_j$ and C_j satisfy:

$$\begin{aligned}
&\left[r + \frac{b}{2}\right]A_i - \frac{1}{2[c_i + A_i/2]} + \frac{c_i}{4[c_i + A_i/2]^2} \\
&+ \frac{A_i}{8[c_i + A_i/2]^2} + \frac{A_i}{8\left[c_j + A_j/2\right]^2} = 0, \text{ and} \\
&C_i = \frac{a}{2}A_i.
\end{aligned}$$

The game equilibrium strategies can be obtained as:

$$\phi_1^*(x) = \frac{x}{4[c_1 + A_1/2]^2}, \text{ and } \phi_2^*(x) = \frac{x}{4[c_2 + A_2/2]^2}. \tag{6.4}$$

Consider the case when these two nations agree to act according to an agreed upon optimality principle which entails

(i) group optimality, and
(ii) the distribution of the cooperative payoff according to the imputation that divides the excess of the total cooperative payoff over the sum of individual noncooperative payoffs equally.

To maximize their joint payoff for group optimality, the nations have to solve the control problem of maximizing

$$\int_\tau^\infty \left(\left[u_1(s)^{1/2} - \frac{c_1}{x(s)^{1/2}} u_1(s) \right] + \left[u_2(s)^{1/2} - \frac{c_2}{x(s)^{1/2}} u_2(s) \right] \right) \exp[-r(t-\tau)]\, ds \tag{6.5}$$

subject to (6.2).

Invoking Eq. (5.7), we obtain:

$$\begin{aligned}
rW(x) = \max_{u_1, u_2} \Bigg\{ &\left(\left[u_1{}^{1/2} - \frac{c_1}{x^{1/2}} u_1 \right] + \left[u_2{}^{1/2} - \frac{c_2}{x^{1/2}} u_2 \right] \right) \\
&+ W_x(x)\left[ax^{1/2} - bx - u_1 - u_2\right] \Bigg\}.
\end{aligned}$$

The value function $W(x)$ which reflects the maximized joint payoff can be obtained as:

$$W(x) = \left[Ax^{1/2} + C\right],$$

where $\left[r + \frac{b}{2}\right]A - \frac{1}{2[c_1+A/2]} - \frac{1}{2[c_2+A/2]}$

$$+\frac{c_1}{4[c_1 + A/2]^2} + \frac{c_2}{4[c_2 + A/2]^2} + \frac{A}{8[c_1 + A/2]^2} + \frac{A}{8[c_2 + A/2]^2} = 0, \text{ and}$$
$$C = \frac{a}{2r}A$$

The optimal cooperative controls can then be obtained as:

$$\psi_1^*(x) = \frac{x}{4[c_1 + A/2]^2} \text{ and } \psi_2^*(x) = \frac{x}{4[c_2 + A/2]^2}. \tag{6.6}$$

Substituting these control strategies into (6.2) yields the dynamics of the state trajectory under cooperation:

$$\dot{x}(s) = ax(s)^{1/2} - bx(s) - \frac{x(s)}{4[c_1 + A/2]^2} - \frac{x(s)}{4[c_2 + A/2]^2}, \; x(\tau) = x_\tau. \tag{6.7}$$

Solving (6.7) yields the optimal cooperative state trajectory $\{x^*(s)\}_{\tau=t_0}^{\infty}$ for the cooperative game (6.1 and 6.2) as:

$$x^*(s) = \left[\frac{a}{2H} + \left((x_\tau)^{\frac{1}{2}} - \frac{a}{2H}\right)\exp[-H(s-\tau)]\right]^2, \tag{6.8}$$

where $H = -\left[\frac{b}{2} + \frac{1}{8[c_1+A/2]^2} + \frac{1}{8[c_2+A/2]^2}\right]$.

According to the agreed-upon optimality principle these nations will distribute the cooperative payoff according to the imputation which divides the excess of the total cooperative payoff over the sum of individual noncooperative payoffs equally. Hence the imputation $\xi(\upsilon, x_\upsilon^*) = \left[\hat{\xi}^1(x_\upsilon^*), \hat{\xi}^2(x_\upsilon^*)\right]$ has to satisfy:

Condition 6.1

$$\hat{\xi}^i(x_\upsilon^*) = \hat{V}^i(x_\upsilon^*) + \frac{1}{2}\left[W(x_\upsilon^*) - \sum_{j=1}^{2}\hat{V}^j(x_\upsilon^*)\right], \tag{6.9}$$

for $i \in \{1,2\}$ and $\upsilon \in [\tau, \infty)$. ■

Applying Theorem 5.2 and Eq. (5.23) a subgame consistent solution payoff distribution procedure $B\left(x_s^*\right) = \left\{B_1\left(x_s^*\right), B_2\left(x_s^*\right)\right\}$ for $s \in \left[\tau, \infty\right)$ can be obtained as:

$$\begin{aligned} B_i\left(x_s^*\right) = &\frac{1}{2}\left\{ r\left[A_i\left(x_s^*\right)^{1/2} + C_i\right] + r\left[A\left(x_s^*\right)^{1/2} + C\right] - r\left[A_j\left(x_s^*\right)^{1/2} + C_j\right] \right\} \\ &- \frac{1}{4}\left\{ A_i\left(x_s^*\right)^{-1/2} + A\left(x_s^*\right)^{-1/2} - A_j\left(x_s^*\right)^{-1/2} \right\} \\ &\times \left[a\left(x_s^*\right)^{1/2} - bx_s^* - \frac{x_s^*}{4[c_1 + A/2]^2} - \frac{x_s^*}{4[c_2 + A/2]^2} \right], \end{aligned} \tag{6.10}$$

for $i, j \in \{1, 2\}$ and $i \neq j$.

With players using the cooperative strategies $\left\{\psi_i^*\left(x_\upsilon^*\right), i \in \{1, 2\}\right\}$ along the cooperative trajectory, the instantaneous receipt of player i at time instant υ becomes:

$$\zeta_i\left(x_\upsilon^*\right) = \frac{\left(x_\upsilon^*\right)^{1/2}}{2[c_i + A/2]} - \frac{c_i\left(x_\upsilon^*\right)^{1/2}}{4[c_i + A/2]^2}, \tag{6.11}$$

According to (6.10), the instantaneous payment that player i should receive under the agreed-upon optimality principle is $B_i(x_\upsilon^*)$. Hence an instantaneous transfer payment

$$\chi^i\left(x_\upsilon^*\right) = B_i\left(x_\upsilon^*\right) - \zeta_i\left(x_\upsilon^*\right) \tag{6.12}$$

has to be given to player i at time $\upsilon \in \left[\tau, \infty\right)$, for $i \in \{1, 2\}$.

2.7 Chapter Notes

Significant contributions to general game theory include von Neumann and Morgenstern (1944); Nash (1950, 1953); Vorob'ev (1972); Shapley (1953) and Shubik (1959a, b). Dynamic optimization techniques are essential in the derivation of solutions to differential games. The origin of differential games was established by Rufus Isaacs in the late 1940s (the complete work was published in Isaacs (1965)). In the meantime, control theory reached its maturity in the *Optimal Control Theory* of Pontryagin et al. (1962) and Bellman's *Dynamic Programming* (1957). Berkovitz (1964) developed a variational approach to differential games, and Leitmann and Mon (1967) investigated the geometry of differential games. Pontryagin (1966) solved differential games in open-loop solution in terms of the maximum principle. Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic action. As discussed above, Individual rationality and group optimality are essential element of a cooperative game solution. Dockner and Jørgensen (1984); Dockner and Long (1993); Tahvonen (1994); Mäler and de Zeeuw (1998) and Rubio and Casino (2002) presented cooperative solutions satisfying group optimality in differential games. The majority of cooperative differential games adopt solutions satisfying the essential criteria for dynamic stability – group optimality and individual rationality. Haurie and Zaccour (1986, 1991), Kaitala and Pohjola (1988, 1990, 1995), Kaitala et al. (1995) and Jørgensen and Zaccour (2001) presented classes of transferable-payoff cooperative differential games with solutions which satisfy group optimality and individual rationality. Miao et al. (2010) studied a cooperative differential game on transmission rate in wireless networks. Lin et al. (2014) presented a cooperative differential game for model energy-bandwidth efficiency tradeoff in the Internet. Huang et al. (2016) presented a cooperative differential game of transboundary industrial pollution with a Stackelberg game between firms and local governments while the governments cooperate in pollution reduction. Tolwinski et al. (1986) considered cooperative equilibria in differential games in which memory-dependent strategies and threats are introduced to maintain the agreed-upon control path. Petrosyan and Danilov (1982); Petrosyan and Zenkevich (1996) and Petrosyan (1997) provided a detailed analysis of subgame consistent (then referred to as time consistent solutions in the deterministic framework) imputation distribution schemes in cooperative differential games. Filar and Petrosyan (2000) considered dynamic cooperative games in characteristic functions which evolve over time in a dynamic equation that is influenced by the current (instantaneous) characteristic function and cooperative solution concept adopted.

Yeung and Petrosyan (2004) presented subgame consistent solution in stochastic differential games and Yeung and Petrosyan (2012a) gave a comprehensive account of the topic. Application of subgame consistent solutions in differential games in cost-saving joint venture, collaborative environmental management and dormant firm cartel can be found in Yeung and Petrosyan (2012a). Other examples of cooperative differential games with solutions satisfying subgame consistency can be found in Petrosyan (1997), Jørgensen and Zaccour (2001). A note concerning the notations used in Petrosyan (1997) and Yeung and Petrosyan (2004) is given in Yeung and Petrosyan (2012d). A non-cooperative-equivalent imputation formula in cooperative differential games is provided by Yeung (2007b) and an irrational-behaviour proof condition in cooperative differential games is given in Yeung (2006a). A study on the tragedy of the commons in a dynamic game framework can be found in Hartwick and Yeung (1997).

2.8 Problems

1. Consider the case of three nations harvesting fish in common waters. The growth rate of the fish biomass is characterized by the differential equation:

$$\dot{x}(s) = 4x(s)^{1/2} - 0.5x(s) - u_1(s) - u_2(s),\ x(0) = 50,$$

where $u_i \in U_i$ is the (nonnegative) amount of fish harvested by nation i, for $i \in \{1,2\}$. The horizon of the game is $[0,5]$.

The harvesting cost for nation $i \in \{1,2\}$ depends on the quantity of resource extracted $u_i(s)$ and the resource stock size $x(s)$. In particular, nation 1's extraction cost is $u_1(s)x(s)^{-1/2}$ and nation 2's is $2u_2(s)x(s)^{-1/2}$. The revenue of fish harvested by nation 1 at time s is $2[u_1(s)]^{1/2}$ and that by nation 2 is $[u_2(s)]^{1/2}$. The interest rate is 0.05.

Characterize a feedback Nash equilibrium solution for this fishery game.

2. If these nations agree to cooperate and maximize their joint payoff, obtain a group optimal cooperative solution.
3. Furthermore, if these nations agree to share the excess of their gain from cooperation equally along the optimal trajectory, derive a subgame consistent cooperative solution.
4. If the game horizon of the above problems is extended to infinity, what would be the answers to Problems 1, 2 and 3?