Chapter 9
Propagation of Light Pulse in Fiber and Optical Soliton

This chapter studies the phenomenon of nonlinear interaction between the laser pulse and the propagation fiber. At first, we will deduce the nonlinear Schrodinger equation for describing the propagation of light pulse in the fiber. Under this foundation, we will analyze the two effects of dispersion and self-phase modulation how to affect the propagation of light pulse in the fiber and how to form the time optical soliton in both combined action. And we will give the time optical soliton equation, the fundamental wave solution and the characteristic of the time optical soliton. Finally, we will briefly introduce the basic conception and characteristic of the space optical soliton.

9.1 Nonlinear Schrodinger Equation [1]

Most nonlinear effects in fiber are related with that the light pulses with pulsewidth in the range of $10\,\mathrm{ns} \sim 10\,\mathrm{fs}$ propagate in the fiber. In the propagation process, both the dispersion and the nonlinearity all affect the shape and spectrum of light pulse. For describing the propagation of light pulses in the nonlinear fiber we need use the time-domain nonlinear Schrodinger equation. We will deduce this equation taking following four steps:

① Reforming the general nonlinear time-domain wave equation to be the time-domain wave equation in the isotopic medium;
② Through Fourier transform to deduce the frequency-domain wave equation for describing the propagation of monochromic light in isotopic medium, namely Helmholtz equation;
③ In the single mode fiber condition solving Helmholtz equation, to deduce the frequency-domain wave equation for describing the propagation of monochromic light in single mode fiber;

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④ Through revers Fourier transform to deduce the time-domain nonlinear Schrodinger equation, which describes that when the light pulse propagate in the fiber to generate the dispersion, absorption, and Kerr effect.

9.1.1 Helmholtz Equation

If the medium is without absorption loss ($\sigma = 0$), according to Eq. (2.1.17), the general wave equation can be written to

$$\nabla \times \nabla \times \boldsymbol{E} + \mu_0 \frac{\partial^2 \boldsymbol{\varepsilon} \cdot \boldsymbol{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \boldsymbol{P}_{NL}}{\partial t^2}. \tag{9.1.1}$$

Assuming the fiber medium is isotopic, we have $\nabla \cdot \boldsymbol{E} = 0$, so that $\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\nabla^2 \boldsymbol{E}$. Because dielectric coefficient is $\varepsilon = \varepsilon_0(1 + \chi^{(1)})$, where $\boldsymbol{\chi}^{(1)}$ linear susceptibility, and linear polarization is $\mathrm{P}_L = \varepsilon_0 \boldsymbol{\chi}^{(1)} \cdot \mathrm{E}$, using $c = 1/\sqrt{\mu_0 \varepsilon_0}$, Eq. (9.1.1) can be rewritten to

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \boldsymbol{P}_L}{\partial t^2} + \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \boldsymbol{P}_{NL}}{\partial t^2}. \tag{9.1.2}$$

The total polarization of medium is

$$\boldsymbol{P}(\boldsymbol{r}, t) = \boldsymbol{P}_L(\boldsymbol{r}, t) + \boldsymbol{P}_{NL}(\boldsymbol{r}, t). \tag{9.1.3}$$

In the following discussion we assume: ① the medium is isotopic, non-absorption, far from resonance; ② $\boldsymbol{P}_{NL}$ can be regarded as the perturbation of $\boldsymbol{P}$, because in general the nonlinear variation of refractive index $< 10^{-6}$; ③ when the light propagates along the fiber length direction, its polarization keeps no change, so we can treatment by using scalar method; ④ the light wave is quasi-monochromic, namely $\Delta\omega/\omega_0 << 1$ is satisfied, because the center frequency of light wave is about $\omega_0 \approx 10^{15}$ Hz, and spectrum wide of light pulse is $\Delta\omega < 0.1\text{ps} = 10^{13}\text{Hz}$.

We suppose the light wave propagates in fiber along z-direction, the polarization is along x-direction, so we have

$$\boldsymbol{E}(\boldsymbol{r}, t) = \hat{\boldsymbol{x}}[E(\boldsymbol{r}, t)\exp(-i\omega_0 t) + c.c.], \tag{9.1.4}$$

$$\boldsymbol{P}_L(\boldsymbol{r}, t) = \hat{\boldsymbol{x}}[P_L(\boldsymbol{r}, t)\exp(-i\omega_0 t) + c.c.], \tag{9.1.5}$$

$$\boldsymbol{P}_{NL}(\boldsymbol{r},t) = \hat{\boldsymbol{x}}[P_{NL}(\boldsymbol{r},t)\exp(-i\omega_0 t) + c.c.], \tag{9.1.6}$$

where $\hat{\boldsymbol{x}}$ is the unit vector for polarization along x-direction. Substituting Eqs. (9.1.4)–(9.1.6) into Eq. (9.1.2), in the same time using the Fourier transform to $\boldsymbol{E}(\boldsymbol{r},t)$, the electrical field intensity is denoted as the frequency-domain form:

$$\tilde{\boldsymbol{E}}(\mathbf{r},\omega-\omega_0) = \int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},t)\exp[i(\omega-\omega_0)t]dt. \tag{9.1.7}$$

Therefore we deduce out the Helmholtz equation, which is satisfied by the monochromic wave in the isotopic medium:

$$\nabla^2\tilde{\boldsymbol{E}} + \varepsilon(\omega)k_0^2\tilde{\boldsymbol{E}} = 0, \tag{9.1.8}$$

where $k_0 = \omega/c$ is the wave vector in vacuum; $\varepsilon(\omega)$ is the complex dielectric coefficient. Now we investigate the physical mining of $\varepsilon(\omega)$.

If we only consider the linear effect and the three-order nonlinear effect, the polarization can be written as

$$\boldsymbol{P} = \boldsymbol{P}^{(1)} + \boldsymbol{P}^{(3)} = \varepsilon_0\chi^{(1)}\boldsymbol{E} + 3\varepsilon_0\chi^{(3)}|\boldsymbol{E}|^2\boldsymbol{E}. \tag{9.1.9}$$

According to the definition of electric induction intensity and using Eq. (9.1.9), we have

$$\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P} = \varepsilon_0\boldsymbol{E} + \varepsilon_0\chi^{(1)}\boldsymbol{E} + 3\varepsilon_0\chi^{(3)}|\boldsymbol{E}|^2\boldsymbol{E} = \varepsilon(\omega)\boldsymbol{E}. \tag{9.1.10}$$

So the dielectric coefficient $\varepsilon(\omega)$ is given by

$$\varepsilon(\omega) = \varepsilon_0\left(1 + \chi^{(1)}(\omega) + 3\chi^{(3)}|\boldsymbol{E}(\omega)|^2\right). \tag{9.1.11}$$

In general $\varepsilon(\omega)$ is a complex umber, the real part is corresponding to the refractive index $\tilde{n}$, and the imaginary part is corresponding to the absorption coefficient $\tilde{\alpha}$, so $\varepsilon(\omega)$ can be defined as

$$\varepsilon(\omega) = (\boldsymbol{n} + \boldsymbol{i\alpha}/\boldsymbol{2k_0})^{\boldsymbol{2}}. \tag{9.1.12}$$

And the refractive index and the absorption coefficient can be divided into linear and nonlinear two parts:

$$\tilde{n} = n + \Delta n, \tag{9.1.13}$$

$$\tilde{\alpha} = \alpha + \Delta\alpha. \tag{9.1.14}$$

where n and α are linear refractive index and linear absorption (as same as n_0 and α_0 in previous chapter); the relations of Δn and $\Delta\alpha$ with the light intensity I have been given in second chapter:

$$\Delta n = \frac{3}{\varepsilon_0 c n^2}\chi^{(3)\prime}(\omega)I,$$
$$\Delta\alpha = \frac{6\omega}{\varepsilon_0 c^2 n^2}\chi^{(3)\prime\prime}(\omega)I,$$

Using formula $I = \frac{1}{2}\varepsilon_0 cn|\boldsymbol{E}|^2$, Eqs. (9.1.13) and (9.1.14) become:

$$\tilde{n} = n + \bar{n}_2|\boldsymbol{E}|^2, \tag{9.1.15}$$

$$\tilde{\alpha} = \alpha + \bar{\alpha}_2|\boldsymbol{E}|^2, \tag{9.1.16}$$

where

$$\bar{n}_2 = \frac{3}{2n}\chi^{(3)\prime}, \tag{9.1.17}$$

$$\bar{\alpha}_2 = \frac{3\omega}{cn}\chi^{(3)\prime\prime}. \tag{9.1.18}$$

$\bar{n}_2$ is nonlinear refraction coefficient; $\bar{\alpha}_2$ can be called two-photon absorption coefficient (or write as β), for silica fiber the absorption coefficient $\tilde{\alpha}$ is very small, it can be neglected, so the dielectric coefficient can be approximately written as $\varepsilon \approx \tilde{n}^2$.

Using Eqs. (9.1.12) and (9.1.13), omitting the items containing $(\Delta n)^2$, $(i\tilde{\alpha}/2k_0)^2$, and $\Delta n(i\tilde{\alpha}/2k_0)$, and using Eq. (9.1.15), the dielectric coefficient $\varepsilon(\omega)$ can be approximately written as

$$\varepsilon(\omega) = \left(\tilde{n} + \frac{i\tilde{\alpha}}{2k_0}\right)^2 \approx \left[n^2 + 2n\left(\bar{n}_2|E|^2 + \frac{i\tilde{\alpha}}{2k_0}\right)\right]. \tag{9.1.19}$$

To define the complex number nonlinear refractive index $\Delta\boldsymbol{n}$ as

$$\Delta\boldsymbol{n} = \bar{n}_2|E|^2 + \frac{i\tilde{\alpha}}{2k_0}. \tag{9.1.20}$$

Then Eq. (9.1.19) can be approximately expressed as

$$\varepsilon(\omega) \approx n^2 + 2n\Delta\boldsymbol{n}. \tag{9.1.21}$$

Visible, $\Delta\boldsymbol{n}$ can be regarded the first perturbation of the dielectric coefficient $\varepsilon(\omega)$, which includes the nonlinearity and the absorption loss of fiber

If in Eq. (9.1.20) we replace $\bar{n}_2$ by n_2, replace $|E|^2$ by $I = P/\mathrm{S}_{\mathrm{eff}}$, and approximately replace $\tilde{\alpha}$ by α, Eq. (9.1.20) can be written as

$$\Delta \boldsymbol{n} \approx n_2 \frac{P}{S_{eff}} + \frac{i\alpha}{2k_0}, \tag{9.1.22}$$

where the unit of nonlinear refraction coefficient n_2 is m^2/W; the unit of power P is W, the unit of efficient cross-section of fiber core S_{eff} is m^2.

9.1.2 Derivation of Frequency-Domain Wave Equation in Fiber

Helmholtz Eq. (9.1.8) can be solved by using method of separation of variables. Assuming that the solution form is

$$\tilde{E}(r, \omega - \omega_0) = F(x, y)\tilde{A}(z, \omega - \omega_0)\exp(i\beta_0 z), \tag{9.1.23}$$

where $F(x, y)$ is transverse-mode distribution function of light pulse in the fiber; $\tilde{A}(z, \omega)$ is light electrical filed amplitude; $\exp(i\beta_0 z)$ is the phase factor, in which $\beta_0 = \beta(\omega_0)$ is the wave number for central wavelength.

Substituting the trying solution (9.1.23) into Eq. (9.1.8), which can be divided into two equations related to $F(x, y)$ and $\tilde{A}(z, \omega)$ respectively:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + [\varepsilon(\omega)k_0^2 - \tilde{\beta}^2]F = 0, \tag{9.1.24}$$

$$2i\beta_0 \frac{\partial \tilde{A}}{\partial z} + (\tilde{\beta}^2 - \beta_0^2)\tilde{A} = 0. \tag{9.1.25}$$

In the process of deduction of Eq. (9.1.25), because $\tilde{A}(z, \omega)$ is a slow-variation function of z, we have neglected the item containing $\partial^2\tilde{A}/\partial z^2$.

$\tilde{\beta}$ in Eqs. (9.1.24) and (9.1.25) can be expressed to be linear and nonlinear two parts:

$$\tilde{\beta} = \beta(\omega) + \Delta\beta(\omega). \tag{9.1.26}$$

$\tilde{\beta}^2$ in Eq. (9.1.24) can be written to $\tilde{\beta}^2 \approx \beta(\omega)^2 + 2\beta(\omega)\Delta\beta(\omega)$, using Eq. (9.1.21), then Eq. (9.1.24) can be rewritten to

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + 2[n(\omega)\Delta\boldsymbol{n}(\omega/c)^2 - \beta(\omega)\Delta\beta(\omega)]F = 0. \tag{9.1.27}$$

Taking double integral of x and y to function $F(x, y), \Delta\beta(\omega)$ can be regard a constant, moving it to outside of the integral sign, from Eq. (9.1.27) we obtain

$$\Delta\beta(\omega) = \frac{\omega^2 n(\omega)}{c^2\beta(\omega)} \frac{\int\int_{-\infty}^{\infty} \Delta\boldsymbol{n}(\omega)|F(x, y)|^2 dxdy}{\int\int_{-\infty}^{\infty} |F(x, y)|^2 dxdy}. \tag{9.1.28}$$

The physical meaning of this formula is: in the envelop of the light pulse propagated in the fiber, the wave number change of a monochromatic light $\Delta\beta(\omega)$ comes from light induced complex number refractive-index change $\Delta\boldsymbol{n}$, which is the first perturbation of $\varepsilon(\omega)$; and $\Delta\beta(\omega)$ can be regarded as the first perturbation of $\tilde{\beta}$.

Using Eq. (9.1.26) and approximate relation $\tilde{\beta}^2 - \beta_0^2 \approx 2\beta_0(\tilde{\beta} - \beta_0)$, the Eq. (9.1.25) becomes

$$\frac{\partial \tilde{A}}{\partial z} = i[\beta(\omega) + \Delta\beta(\omega) - \beta_0]\tilde{A}. \tag{9.1.29}$$

In order to obtain the exact expression and meaning of $\beta(\omega)$ in above equation, we expend $\beta(\omega)$ to be Taylor series nearby the central frequency ω_0:

$$\beta(\omega) = \beta_0 + (\omega - \omega_0)\beta_1 + \frac{1}{2}(\omega - \omega_0)^2\beta_2 + \frac{1}{6}(\omega - \omega_0)^3\beta_3 + \ldots. \tag{9.1.30}$$

In which β_n is

$$\beta_n = \left(\frac{d^n\beta}{d\omega^n}\right)_{\omega=\omega_0} \quad (n = 0, 1, 2, \ldots). \tag{9.1.31}$$

β_0 is the wave number of central frequency; β_1 is the wave number of wave packet, which is related with the group velocity v_g and the refractive index of group velocity n_g, β_2; denotes the dispersion of group velocity (GVD), which is related with frequency broaden of light pulse. β_1 and β_2 are respectively defined as

$$\beta_1 = \frac{d\beta}{d\omega} = \frac{1}{v_g} = \frac{n_g}{c} = \frac{1}{c}\left(n + \omega\frac{dn}{d\omega}\right), \tag{9.1.32}$$

$$\beta_2 = \frac{d^2\beta}{d\omega^2} = \frac{d\beta_1}{d\omega} = \frac{1}{c}\left(2\frac{dn}{d\omega} + \omega\frac{d^2n}{d\omega^2}\right). \tag{9.1.33}$$

We have assumed spectrum width $\Delta\omega < < \omega_0$, so in expansion Eq. (9.1.30), the high-order (higher than three-order) items can be neglected (only for the specific frequency, such as near zero dispersion waveguide of fiber, $\beta_2 \approx 0$, than we consider β_3 in the three-order item), therefore we only take preceding three items in Eq. (9.1.30), i.e.,

$$\Delta\beta(\omega) = \beta_0 + (\omega - \omega_0)\Delta\beta_1 + \frac{1}{2}(\omega - \omega_0)^2\Delta\beta_2. \tag{9.1.34}$$

Substituting Eq. (9.1.34) into Eq. (9.1.29), we obtain

$$\frac{\partial\tilde{A}}{\partial z} = i[(\omega - \omega_0)\Delta\beta_1 + \frac{1}{2}(\omega - \omega_0)^2\Delta\beta_2 + \Delta\beta(\omega)]\tilde{A}. \tag{9.1.35}$$

9.1.3 Derivation of Nonlinear Schrodinger Equation

At first, using Eqs. (9.1.4) and (9.1.23), the time-domain light electrical filed intensity expression can be written as:

$$\mathbf{E}(\boldsymbol{r}, t) = \hat{\boldsymbol{x}}[E(x, y)A(z, t)\exp\, i(\beta_0 z - \omega_0 t) + c.c.], \tag{9.1.36}$$

where $A(z, t)$ is time-domain slow-variation electrical filed amplitude, which can be obtained from the reverse Fourier transform of frequency electrical filed amplitude $\tilde{A}(z, \omega - \omega_0)$:

$$A(z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{A}(z, \omega - \omega_0)\exp[i(\omega - \omega_0)t]d\omega. \tag{9.1.37}$$

After than through reverse Fourier transform, we transfer the frequency-domain amplitude Eq. (9.1.35) to be time-domain amplitude equation. In the reverse Fourier transform, we can use the operator $i(\partial/\partial t)$ to instead of $\omega - \omega_0$ in Eq. (9.1.35), the time-domain form of that equation is given by

$$\frac{\partial A}{\partial z} + \beta_1\frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} = i\Delta\beta(\omega)A. \tag{9.1.38}$$

$\Delta\beta(\omega)$ in the right side of Eq. (9.1.38) includes the fiber loss (α) and the nonlinearity ($\Delta\boldsymbol{n}$).

Now we use Eq. (9.1.28) to find $\Delta\beta(\omega)$. Under first-class perturbation, we suppose $\Delta\boldsymbol{n}$ does not affect the model-filed distribution $F(x, y)$ in the pulsewidth range, so we can pull out $\Delta\boldsymbol{n}$ to outside the integral, further use expression (9.1.22) of $\Delta\boldsymbol{n}$, and setting $\beta(\omega) \approx n(\omega)\frac{\omega}{c}$, under condition of $\Delta\omega < < \omega_0$, we can approximately use $\Delta\beta_0$ instead of $\Delta\beta(\omega)$, therefore the $\Delta\beta(\omega)$ in Eq. (9.1.38) can be written as

$$\Delta\beta(\omega) = \Delta\beta_0 \approx \frac{n_2(\omega_0)\omega_0}{cS_{\mathrm{eff}}}|A|^2 + \frac{i\alpha}{2}, \tag{9.1.39}$$

here we used $P = |A|^2$, because the amplitude A is normalized. $|A|^2$ represents power (the unit is W). Substituting the Eq. (9.1.39) into Eq. (9.1.38), the time-domain amplitude Eq. (9.1.38) becomes the following form:

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + \frac{i\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2}A = i\gamma(\omega_0)|A|^2 A, \tag{9.1.40}$$

where γ is the nonlinear coefficient, its definition is

$$\gamma(\omega_0) = \frac{n_2(\omega_0)\omega_0}{cS_{eff}}. \tag{9.1.41}$$

The unit of $\gamma|A|^2$ is m^{-1}. S_{eff} is the effective cross-section of fiber core, which is defined as

$$S_{eff} = \frac{\left(\int\int_{-\infty}^{\infty} |F(x,y)|^2 dxdy\right)^2}{\int\int_{-\infty}^{\infty} |F(x,y)|^4 dxdy}, \tag{9.1.42}$$

where $F(x, y)$ is distribution function of fundamental mode light filed of fiber. For single-mode fiber, its fundamental mode can expressed as a Gaussian distribution, i.e.,

$$F(x,y) = \exp[-(x^2 + y^2)/w^2]. \tag{9.1.43}$$

The effective cross-section of fiber core is

$$S_{eff} = \pi w^2. \tag{9.1.44}$$

For the fiber at near the normalized cut-off frequency $V = 2$, the parameter w is equal to the radius of fiber core, i.e., $w \approx a$, and $S_{eff} = \pi a^2$. When wavelength at vicinity of 1.5 μm, the general cross-section of fiber core is $S_{eff} = 20 - 100\,\mu\text{m}^2$. If taking $n_2 \approx 2.6 \times 10^{-20}\text{m}^2/\text{W}$, the range of nonlinear coefficient is $\gamma = 1 \sim 10\,\text{W}^{-1}/\text{km}$.

Equation (9.1.40) describes the propagation law of picosecond light pulse in single-mode fiber. The equation is called Nonlinear Schrödinger (NLS) Equation. In which α denotes absorption loss, γ denotes nonlinearity effect, the moving group velocity of pulse waveform is $v_g \equiv 1/\beta_1$, the group velocity dispersion (GVD) effect is depended on the parameter β_2. The plus or minus of β_2 is depended on whether greater or smaller than zero-dispersion wavelength 1.31 μm, please see following appendix.

Appendix: Dispersion of Fiber

Dispersion of single-mode fiber is usually used the dispersion coefficient D_{m} to describe, the relationship between D_{m} and β_2 is

$$D_m = \frac{d\beta_1}{d\lambda} = -\frac{2\pi c}{\lambda^2}\beta_2 \approx -\frac{\lambda}{c}\frac{d^2 n}{d\lambda^2}. \tag{9.1.45}$$

This is the material dispersion. The wave number of wave packet $\beta_1 = 1/v_g$ is the time delay due to that the wave packet propagated with group velocity passes through a unit distance, its unit is ps/km; the dispersion coefficient D_{m} is the time delay induced by unit spectrum width, its unit is $\mathrm{ps}/(\mathrm{km}\cdot\mathrm{nm})$; and the unit of group-velocity dispersion parameter β_2 is $\mathrm{ps}^2/\mathrm{km}$.

Figure 9.1 gives the relation of variation of group-velocity dispersion parameter of single-mode silica fiber β_2 with the wavelength. We can see that β_2 tends to zero in the vicinity of wavelength 1.31 μm, for even longer wavelength it becomes negative value, for example, when $\lambda \approx 1.55\,\mu\mathrm{m}$, $\beta_2 \approx -25\mathrm{ps}^2/\mathrm{km}$. The wavelength at $\beta_2 = 0$ is called zero-dispersion wavelength λ_D.

The material dispersion D_{m} is related with the fiber doping situation, for different doping, the law of refractive-index variation with the wavelength is different, i.e., the characteristic of dispersion is different. There is also a dispersion, which relies on the waveguide structure of fiber, it is called waveguide dispersion D_w. For different fiber waveguide structure, in which the refractive index of efficient mode is different with the material refractive index, therefore dispersion characteristic is different. The main action is change the location of zero-dispersion wavelength λ_D, to lead it shifts along long wavelength direction.

The total dispersion of single mode fiber D is a sum of material dispersion and waveguide dispersion: $D = D_m + D_w$.

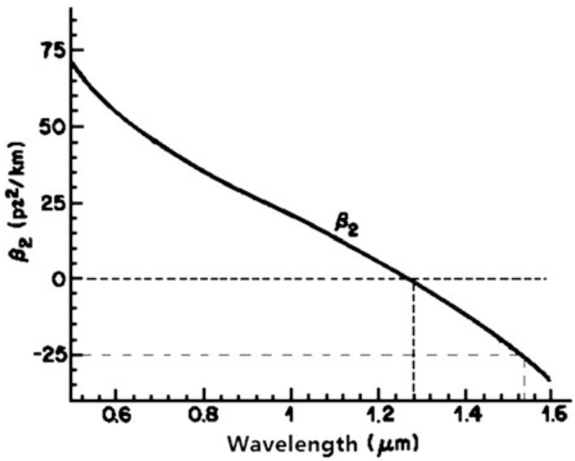

Fig. 9.1 Relation between the group velocity dispersion β_2 and the wavelength for single mode silica fiber

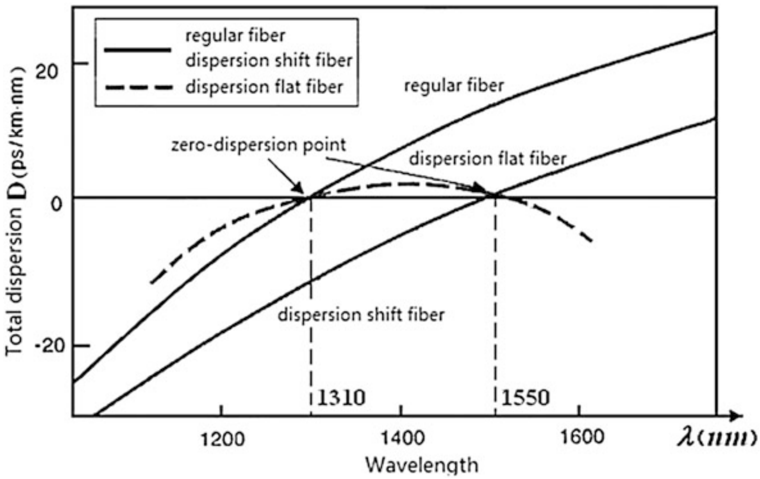

Fig. 9.2 Dispersion–wavelength curves for three kinds of single-mode fibers: the regular fiber, the dispersion shift fiber and the dispersion flat fiber

There are common three kinds of single-mode fibers: the regular fiber (λ_D = 1.31 μm),the dispersion shift fiber (λ_D = 1.55 μm) and the dispersion flat fiber (λ_D = 1.31 μm and 1.55 μm), their dispersion-wavelength characteristic curves are shown in Fig. 9.2.

9.2 Group Velocity Dispersion and Self-phase Modulation [1]

In two sides of Eq. (9.1.40) multiply by i, then we obtain

$$i\frac{\partial A}{\partial z}+i\beta_1\frac{\partial A}{\partial t}=-\frac{i\alpha}{2}A+\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2}-\gamma|A|^2A. \tag{9.2.1}$$

If we use the time coordinate of motion reference system with group velocity v_g (the space coordinate is no change),

$$T=t-z/v_g=t-\beta_1 z, \tag{9.2.2}$$

and using the derivation formula for the function of functions, we obtain

$$\begin{aligned}\frac{\partial A(z,T)}{\partial z}&=\frac{\partial}{\partial z}A(z,t=T+\frac{z}{v_g})=\frac{\partial A(z,t)}{\partial z}+\frac{\partial A(z,t)}{\partial t}\cdot\frac{\partial t}{\partial z}\\&=\frac{\partial A(z,t)}{\partial z}+\frac{\partial A(z,t)}{\partial t}\cdot\frac{1}{v_g}=\frac{\partial A(z,t)}{\partial z}+\beta_1\frac{\partial A(z,t)}{\partial t}.\end{aligned}$$

Equation (9.2.1) becomes

$$i\frac{\partial A}{\partial z} = -\frac{i\alpha}{2}A + \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \gamma|A|^2 A. \tag{9.2.3}$$

This is NLS equation for describing the propagation of light pulse in the fiber. $A(z, T)$ is the amplitude of pulse envelop. The first item of right side describes the absorption of light pulse in the fiber, the second item describes the group velocity dispersion (GVD) of light pulse, the third item describes self-phase modulation (SPM) of light pulse (the nonlinear optics effect). Actually, this equation is neglected the high-order nonlinear effects. This equation is suitable to use for describing the propagation of light pulse with initial pulsewidth $T_0 > 5ps$ in the fiber.

When the light pulse propagates in the fiber for a certain length L, in order to estimate which effect: the dispersion effect or the self-phase modulation, plays main role, we respectively introduce two physical quantities: the dispersion length L_D and the nonlinear length L_{NL}, according to the comparison of the length of L_D or L_{NL} relative to fiber length L, to determine which effect has more important contribution.

For realizing the normalization of the equation, we introduce a normalized time τ, which is relative to the initial pulsewidth T_0:

$$\tau = \frac{T}{T_0} = \frac{t - z/v_g}{T_0}. \tag{9.2.4}$$

In the same time, we introduce a normalized light filed amplitude $U(z, \tau)$, which is proportional to the amplitude $A(z, \tau)$, the proportionality coefficient includes the input light power and the absorption loss related with propagation distance:

$$A(z, \tau) = \sqrt{P_0}\exp(-\alpha z/2)U(z, \tau), \tag{9.2.5}$$

where P_0 is the peak power of input light pulse; the exponent factor is for measuring the fiber loss. So the Eq. (9.2.3) is rewritten to

$$i\frac{\partial U}{\partial z} = \frac{\mathrm{sgn}(\beta_2)}{2L_D}\frac{\partial^2 U}{\partial \tau^2} - \frac{\exp(-\alpha z)}{L_{NL}}|U|^2 U, \tag{9.2.6}$$

where $\mathrm{sgn}(\beta_2) = \pm 1$ is determined by the symbol of GVD parameter $\beta_2; L_D$ is dispersion length, which is defined as

$$L_D = \frac{T_0^2}{|\beta_2|}; \tag{9.2.7}$$

L_{NL} is nonlinear length, which is defined as

$$L_{NL} = \frac{1}{\gamma P_0}. \tag{9.2.8}$$

We can see that L_D is related with the fiber group velocity dispersion parameter β_2 and the initial width of input light pulse T_0; and L_{NL} is related with the nonlinear parameter of fiber γ and the peak power of input light pulse P_0. Namely β_2 and T_0 determine the group velocity dispersion GVD. And γ and P_0 determine the self-phase modulation SPM. According to comparison of L_D and L_{NL} with the actual length of fiber L in numerical relative size, we can put the light pulse propagation in optical fiber into four cases. Below we start from Eq. (9.2.6) to study these four cases respectively.

9.2.1 Pulse Propagation Excluding Dispersion and Nonlinearity

When the length of fiber is very short, i.e., $L << L_D$ and $L << L_{NL}$, whether dispersion or nonlinearity both not play important role. In Eq. (9.2.6), two items of right side are all equal to zero, i.e., $\frac{\partial U(z,\tau)}{\partial z} = 0$, then $U(z,\tau) = U(0,\tau)$, namely in propagation process the shape of light pulse keeps no change (except the absorption induces slightly decrease of power). This case is favorable to the optical communication.

For example, for the standard fiber at wavelength $\lambda = 1.55\,\mu\text{m}$, $|\beta_2| \approx 20\,\text{ps}^2/\text{km}$, and $\gamma \approx 2\,\text{W}^{-1}\text{km}^{-1}$, if the light pulse with $T_0 \geq 100\,\text{ps}$ and $P_0 \leq 1\,\text{mW}$, in this case L_D and $L_{NL} \geq 500\,\text{km}$, for the fiber with length of $L \leq 50\,\text{km}$, the dispersion and the nonlinearity all can be neglected. But when the pulsewidth of incident pulse is narrow down, and the power of incident pulse increase, L_D and L_{NL} become smaller, such as $T_0 \approx 1\,\text{ps}$ and $P_0 \approx 1\,\text{W}$, then L_D and $L_{NL} \approx 100\,\text{m}$, for this fiber, when its length exceeds 10 m, we should consider the influences of the dispersion effect and the nonlinearity effect in the same time.

9.2.2 Influence of Dispersion to Pulse Propagation

If $L << L_{NL}$, but $L \approx L_D$, the second item of right side of Eq. (9.2.6) can be neglected, the pulse variation mainly depends on the group velocity dispersion (GVD). In this case L_D is much smaller than L_{NL}:

$$\frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|} << 1. \tag{9.2.9}$$

Roughly estimating, for the standard fiber at $\lambda = 1.55\,\mu\text{m}$, taking the typical values of γ and $|\beta_2|$, this case is suitable to the light pulse with power of $P_0 << 1\,\text{W}$ and pulsewidth of 1 ps.

Below we will discuss the influence of group velocity dispersion to the light pulse. Assuming in Eq. (9.2.3), $\gamma = 0$, using the normalized amplitude defined by Eq. (9.2.5) $U(z, T)$, then $U(z, T)$ is satisfied the following linear partial differential equation:

$$i\frac{\partial U}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 U}{\partial T^2}. \tag{9.2.10}$$

This equation is easy to solve by using Fourier transformation method. Suppose $\tilde{U}(z, \omega)$ is the Fourier transformation of $U(z, T)$, namely

$$U(z, T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(z, \omega)\exp(-i\omega T)d\omega, \tag{9.2.11}$$

Then it satisfies the ordinary differential equation

$$i\frac{\partial \tilde{U}}{\partial z} = -\frac{1}{2}\beta_2\omega^2\tilde{U}. \tag{9.2.12}$$

The solution is

$$\tilde{U}(z, \omega) = \tilde{U}(0, \omega)\exp(\frac{i}{2}\beta_2\omega^2 z). \tag{9.2.13}$$

Equation (9.2.13) shows that GVD changes the phase of each frequency spectrum component in the pulse, the size of change is different according to the different frequency ω and the transmission distance z. Substituting Eq. (9.2.13) into Eq. (9.2.11), we can obtain the general solution of Eq. (9.2.10):

$$U(z, T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(0, \omega)\exp(\frac{i}{2}\beta_2\omega^2 z - i\omega T)dT, \tag{9.2.14}$$

where $\tilde{U}(0, \omega)$ is the Fourier transformation of incident light filed at $z = 0$, namely

$$\tilde{U}(0, \omega) = \int_{-\infty}^{\infty} U(0, T)\exp(i\omega T)dT. \tag{9.2.15}$$

Equations (9.2.14) and (9.2.15) are suitable to the arbitrary shape incident pulse. If the incident pulse is a Gaussian pulse (as shown in Fig. 9.3):

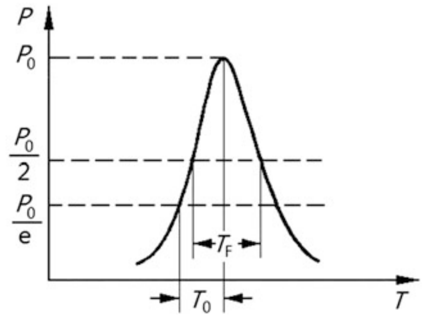

Fig. 9.3 Gaussian pulse waveform and the relation of T_0 and T_{FWHM}

$$U(0,T) = \exp\left(-\frac{T^2}{2T_0^2}\right), \tag{9.2.16}$$

where T_0 is the same as the pulsewidth of light pulse introduced by Eq. (9.2.4), its strict definition is the half width of the light pulse at $1/e$ of peek value. Actually in common use, T_0 is replaced by full width at half maximum T_{FWHM}. For Gaussian light pulse, the relationship between T_{FWHM} and T_0 is

$$T_{\mathrm{FWHM}} = 2(\ln 2)^{1/2} T_0 \approx 1.665 T_0. \tag{9.2.17}$$

Using Eqs. (9.2.14)–(9.2.16), and to integral, we obtain the amplitude at any point z along the fiber:

$$U(z,T) = \frac{T_0}{\left(T_0^2 - i\beta_2 z\right)^{1/2}} \exp\left(-\frac{T^2}{2\left(T_0^2 - i\beta_2 z\right)}\right). \tag{9.2.18}$$

So the light pulse in the propagation keeps its Gaussian shape, but its pulsewidth T_1 is increased with increasing of z as follows:

$$T_1(z) = T_0\left[1 + (z/L_D)^2\right]^{1/2}. \tag{9.2.19}$$

Equation (9.2.19) shows that except the broaden factor T_1/T_0 is related with z, it also depends on the dispersion length $L_D = T_0^2/|\beta_2|$. For a certain fiber length, if the pulsewidth T_0 is shorter and dispersion $|\beta_2|$ is lager, the dispersion length L_D is shorter, than the pulse broaden is larger at $z = L_D$, the broaden of Gaussian pulse is $\sqrt{2}$ times of that of the incident pulse. Figure 9.4 shows the broaden situations of Gaussian pulse $|U(z,T)|^2 – T/T_0$ curves at $z/L_D = 0, 2$, and 4 induced by dispersion. It is clear that propagation distance is longer, the pulse broaden is larger.

To comparison Eqs. (9.2.16) and (9.2.18) we can see, although incident pulse is not chirped (the frequency is not modulated), the transmitted pulse becomes chirped (the frequency is modulated). For clear, we rewrite the Eq. (9.2.18) to be following form:

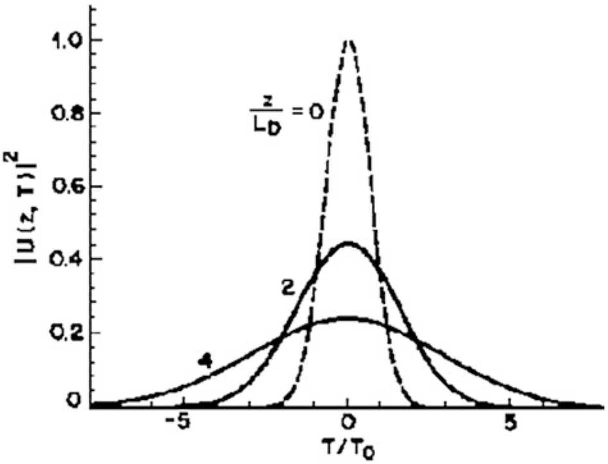

Fig. 9.4 Broadening of Gaussian pulse due to dispersion effect at $z = 2L_D, 4L_D$. The vertical coordinate is the normalized light intensity, the transverse coordinate is the normalized time, and the imaginary line denotes the incident pulse waveform at $z = 0$

$$U(z,T) = |U(z,T)|\exp[i\phi(z,T)], \tag{9.2.20}$$

where

$$\phi(z,T) = -\frac{\mathrm{sgn}(\beta_2)(z/L_D)}{1+(z/L_D)^2}\frac{T^2}{T_0^2} + \frac{1}{2}\tan^{-1}\left(\frac{z}{L_D}\right). \tag{9.2.21}$$

It is dispersion induced phase variation (phase shift) of light pulse with time. It means that in the two sides of center frequency ω_0, there are different frequency difference between the frequency at each moment ω and the center frequency ω_0, namely $\delta\omega = \omega - \omega_0$, this frequency difference is equal to a negative number of the time derivative of phase shift, i.e., $\delta\omega(T) = -\partial\phi/\partial T$ [the negative sign is due to selection of factor $\exp(-i\omega_0 t)$ in the light-filed expression (9.1.4)]:

$$\delta\omega(T) = -\frac{\partial\phi}{\partial T} = \frac{\mathrm{sgn}(\beta_2)(2z/L_D)}{1+(z/L_D)^2}\frac{T}{T_0^2}. \tag{9.2.22}$$

It means that the fiber dispersion applies a time-dependent frequency to the light pulse, that frequency variation with time is frequency chirp. Because the relation between the chirp and the time is a linear relation, it is called linear frequency chirp. The plus or minus of chirp $\delta\omega$ depends on the sign of β_2, in the normal dispersion region ($\beta_2 > 0$), in the pulse leading edge ($T<0$), $\delta\omega$ is minus (red shift), but in the pulse tailing edge ($T > 0$), $\delta\omega$ is plus (blue shift), i.e., red head and violet tail; in the anomalous dispersion region ($\beta_2<0$), the contrary is the case, $\delta\omega$ of the pulse leading edge is plus (blue shift), and $\delta\omega$ of the pulse tailing edge is

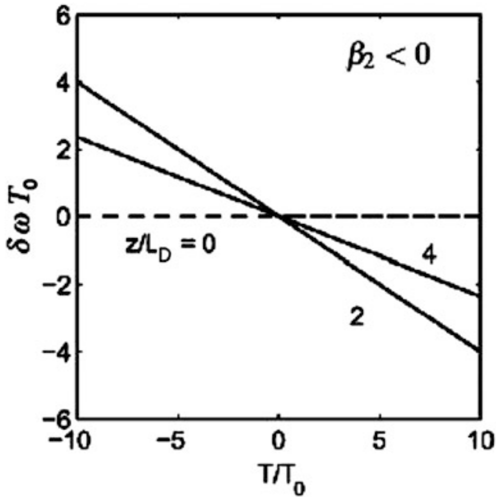

Fig. 9.5 Normalized frequency chirp of Gaussian light pulse $\delta\omega T_0$ as a function of T/T_0 (*straight line* relation) at $z = 2L_D, 4L_D$, the imaginary line denotes that the incident pulse at $z = 0$ is no chirp

minus (red shift), i.e., violet head and red tail. The curves of $\delta\omega T_0 - (T/T_0)$ is shown in Fig. 9.5. Form the figure we can see, when $z = 0$, the Gaussian light pulse has no chirp $\delta\omega$; the chirp becomes larger with increase of propagation distance of light pulse; when $z = L_{\mathrm{D}}$, the chirp is maximum, after that the chirp gradually becomes smaller, until disappears.

We can understand the dispersion induced light-pulse broadening in this way: because of GVD effect, the different frequency component has different propagation velocity in the fiber. In the normal dispersion region $(\beta_2 > 0)$, the red light component is going faster than the blue light component, but in the anomalous dispersion region $(\beta_2 < 0)$, the blue light component is going faster than red light component. Any relative decay of different frequency component all leads the pulse broadening. Only when all frequency components arrive at the same time, the pulsewidth is possible to keep no change.

9.2.3 Influence of Self-phase Modulation to Pulse Propagation

If $L << L_D$, but $L \approx L_{NL}$, The first item of left of Eq. (9.2.6) can be neglected, the pulse change mainly depends on the self-phase modulation (SPM). In this case L_{NL} is much smaller than L_D:

$$\frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|} \gg 1. \tag{9.2.23}$$

This condition is suitable the light pulse with more wide pulsewidth $T_0 > 100\,\text{ps}$ and power $P_0 \approx 1\,\text{W}$ propagates in the regular fiber at $\lambda = 1.55\,\mu\text{m}$. Omitting the dispersion item, Eq. (9.2.6) becomes

$$\frac{\partial U}{\partial z} = \frac{ie^{-\alpha z}}{L_{NL}} |U|^2 U, \tag{9.2.24}$$

where α is the loss of fiber; $L_{NL} = (\gamma P_0)^{-1}$; $\gamma = n_2\omega_0/cS_{eff}$.

Setting the trying solution of Eq. (9.2.24) is

$$U = V\,\exp(i\phi_{NL}), \tag{9.2.25}$$

Substituting Eq. (9.2.25) into Eq. (9.2.24), than divided it into the real part (amplitude) and imaginary part (nonlinear phase shift) two equations:

$$\frac{\partial V}{\partial z} = 0, \tag{9.2.26}$$

$$\frac{\partial \phi_{NL}}{\partial z} = \frac{e^{-\alpha z}}{L_{NL}} V^2. \tag{9.2.27}$$

Because from Eq. (9.2.26) we can know that the amplitude V is without change with z, we can directly integral to Eq. (9.2.27), than obtain the general solution:

$$U(L,T) = U(0,T)\exp[i\phi_{NL}(L,T)], \tag{9.2.28}$$

where $U(0,T)$ is the light filed amplitude at $z = 0$, the nonlinear phase shift $\phi_{NL}(L,T)$ is:

$$\phi_{NL}(L,T) = |U(0,T)|^2 (L_{eff}/L_{NL}), \tag{9.2.29}$$

where L_{eff} is the effective length related with the absorption loss:

$$L_{eff} = [1 - \exp(-\alpha L)]/\alpha. \tag{9.2.30}$$

From Eq. (9.2.29) one can see, SPM induced nonlinear phase shift is proportional to the intensity of incident light in every moment, so the law of the phase-shift variation with time is the same as the law of the incident-light-pulse intensity variation with time, from Eq. (9.2.30) one can see, the increase of phase shift with increasing of fiber length. Due to the absorption loss of fiber, the efficient length L_{eff} is smaller than the length of fiber L. But when without loss, i.e., $\alpha = 0$, than $L_{eff} = L$. Because U is normalized, $U(0,0) = 1$ is maximum value, therefore the maximum phase shift $\phi_{\max}$ appears at $T = 0$ of the center of pulse. From Eq. (9.2.29), we obtain

$$\phi_{\max} = L_{eff}/L_{NL} = \gamma P_0 L_{eff}. \tag{9.2.31}$$

Equation (9.2.31) shows the physical meaning of nonlinear length is the efficient propagation length under $\phi_{\max} = 1$. If taking the typical nonlinear parameter $\gamma = 2\,\mathrm{W}^{-1}\mathrm{km}^{-1}$ for fiber at $\lambda = 1.55\,\mu\mathrm{m}$, when $P_0 = 10\,\mathrm{mW}$, we have $L_{NL} = 50\,\mathrm{km}$. L_{NL} will shorten with increase of P_0.

SPM induced frequency-spectrum broadening is coming from the variation of $\phi_{NL}(L,T)$ with time, it means that there are different frequency difference between the instantaneous frequency at two sides of center frequency in every moment and the center frequency. Using Eq. (9.2.29), we can calculate to get the frequency difference:

$$\delta\omega(T) = -\frac{\partial\phi_{NL}}{\partial T} = -\left(\frac{L_{eff}}{L_{NL}}\right)\frac{\partial}{\partial T}|U(0,T)|^2. \tag{9.2.32}$$

The variation of $\delta\omega$ with time is called the frequency chirp. This SPM induced chirp is increased with increasing of the propagation distance. In other word, as the light pulse propagation in optical fiber, the new frequency components produced continuously, cause frequency spectrum broadening continuously.

From Eq. (9.2.32) one can see, the frequency chirp is related with the pulse waveform. If the light pulse is Gaussian type, than SPM induced chirp is

$$\delta\omega(T) = 2\frac{L_{eff}}{L_{NL}}\frac{T}{T_0^2}\exp\left[-\frac{T^2}{T_0}\right]. \tag{9.2.33}$$

Figure 9.6 shows when $L_{eff} = L_{NL}$, the action of self-phase modulation to the Gaussian-pulse propagation produced characteristic curves: the nonlinear phase shift ϕ_{NL} change with time (above) and the frequency chirp $\delta\omega$ change with time (below). According to Eq. (9.2.29), the curve of ϕ_{NL} change with time in the figure above is the same as the curve of pulse-intensity change with time. From the figure below we can see, the curve of frequency chirp $\delta\omega$ change with time has following characteristic: at the pulse leading edge, $\delta\omega$ is negative (red shift); to reach the pulse tailing edge, $\delta\omega$ becomes positive (blue shift), i.e., it appears red head and violet tail phenomenon; in the wide range of the pulse center region the chirp is linear and upward (up chirp); at the steepest inflection point of pulse leading edge and pulse tailing edge, there is maximum chirp value.

9.2.4 Combined Action of Dispersion and Self-phase Modulation

When $L \geq L_D$ and $L \geq L_{NL}$, GVD and SPM combined action to the light pulse. The interaction between both generates entirely different influence to the behavior of light pulse. In the fiber anomalous dispersion region ($\beta_2 < 0$), the actions of GVD

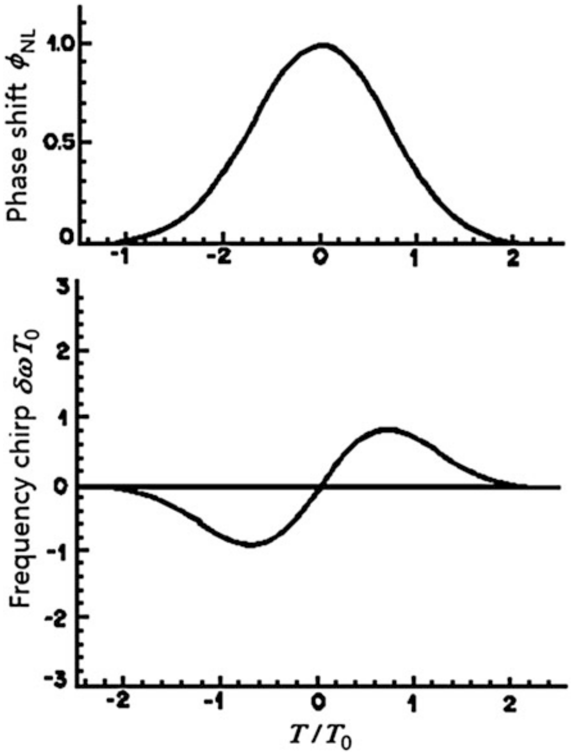

Fig. 9.6 Curves of phase shift ϕ_{NL} (*power*) and frequency chirp $\delta\omega$ as a function of time for Gaussian pulse propagation characteristic produced by self-phase modulation

and SPM are opposite: GVD generates violet head and red trail chirp; but SPM generates red head and violet trail chirp. When both reach the balance, it can eliminate the chirp in the optical fiber, the pulse shape will remain unchanged, thus the optical soliton will be produced.

Now we rewrite the nonlinear Schrodinger Eq. (9.2.6) into following normalized form:

$$i\frac{\partial U}{\partial \xi} = \mathrm{sgn}(\beta_2)\frac{1}{2}\frac{\partial^2 U}{\partial \tau^2} - N^2 e^{-\alpha z}|U|^2 U, \tag{9.2.34}$$

where ξ and τ denote the normalized variables distance and time respectively, which are defined as

$$\xi = z/L_D, \quad \tau = T/T_0. \tag{9.2.35}$$

The definition of parameter N is

$$N^2 = \frac{L_D}{L_{NL}} \equiv \frac{\gamma P_0 T_0^2}{|\beta_2|}. \tag{9.2.36}$$

In Eq. (9.2.34), $\mathrm{sgn}(\beta_2) = \pm$ depend on that the GVD is in normal dispersion region ($\beta_2 > 0$) or in anomalous dispersion region ($\beta_2 > 0$). The integer value N depends on the relative strength of SPM and GVD in the evolution process of light pulse in the fiber. When $N \ll 1$, the action of GVD is dominating; and when $N \gg 1$, the action of SPM is dominating, but when $N \approx 1$, the two actions of SPM and GVD are equal.

For a specific N, there are many practical groups of pulsewidth and power suitable Eq. (9.2.36), for example, if $N = 1$, one can select: $T_0 = 1\,\mathrm{ps}$ and $P_0 = 1\,\mathrm{W}$; $T_0 = 10\,\mathrm{ps}$ and $P_0 = 10\,\mathrm{mW}$; $T_0 = 0.1\,\mathrm{ps}$ and $P_0 = 100\,\mathrm{W}$, etc.

NLS Eq. (9.2.34) is a nonlinear partial differential equation. In general it cannot obtain the analytical solution. In order to obtain the numerical solution of the NLS equation, one can employ the split-step Fourier method, namely use of different differential operator to denote the linear and nonlinear effects of dispersion and absorption respectively, applied to the different segment of fiber, replacing the differential operator by the Fourier frequency, using finite Fourier transform (FFT) algorithm to numerical calculation.

Figure 9.7 draws the evolution process of pulsewidth and frequency spectrum for an non-initial chirp Gaussian pulse propagating in the fiber with length of $z = 5L_D$ in fiber normal dispersion region ($\beta_2 > 0$), and in the condition of $N = 1$ and $\alpha = 0$. Because in the normal dispersion region SPM makes the pulse leading

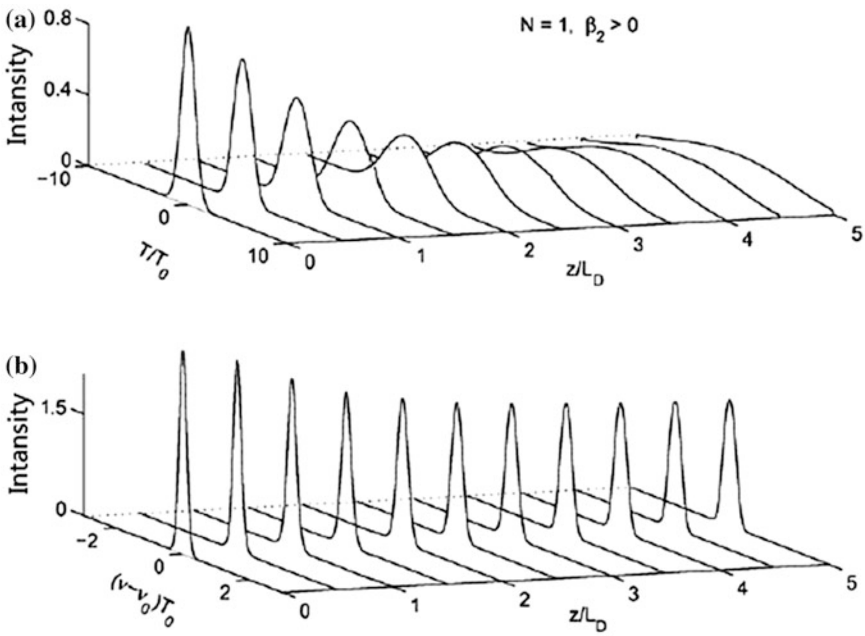

Fig. 9.7 When a Gaussian pulse without initial chirp propagates in a fiber with length of $z = 5L_D$, in the fiber normal dispersion region ($\beta_2 > 0$) and condition of $N = 1$ and $\alpha = 0$, the curves of **a** pulse shape versus distance; **b** frequency spectrum versus distance

edge to red shift and pulse trailing edge to blue shift, its action is the same as the GVD, therefore SPM induced pulse broadening speed is faster than the case only has the action of GVD.

In the anomalous dispersion region ($\beta_2 > 0$), the evolution process of pulse-width and frequency spectrum of Gaussian pulse under other same condition ($N = 1$ and $\alpha = 0$) is shown in Fig. 9.8. In the beginning, the speed of pulse broadening is slower than the speed in the case without SPM (it only has GVD), until when $z > 4L_D$, than basically achieve the unchanged state. But the frequency spectrum width is much narrower than the width in the case without SPM (it only has GVD). It is not hard to understand, according to Eq. (9.2.32), SPM generated chirp is possible; but according to Eq. (9.2.22), GVD generated chirp in $\beta_2 < 0$ region is negative. When $N = 1$, the actions of two chirps in the nearby the center of Gaussian pulse cancel each other out. In the pulse propagation process, though the combined action of GVD and SPM, self-regulation of the pulse shape, as far as possible entirely offset these two inverse chirps, to maintain non-chirped pulse propagation. The generation process of optical soliton is similar to this situation: in the beginning the Gaussian pulse inputs, it is not fundamental-state soliton, so it has a certain broadening. However the combined action of GVD and SPM leads the pulse shaping, finally the pulse evolves into a hyperbolic secant type fundamental-state optical soliton, as shown in Fig. 9.8.

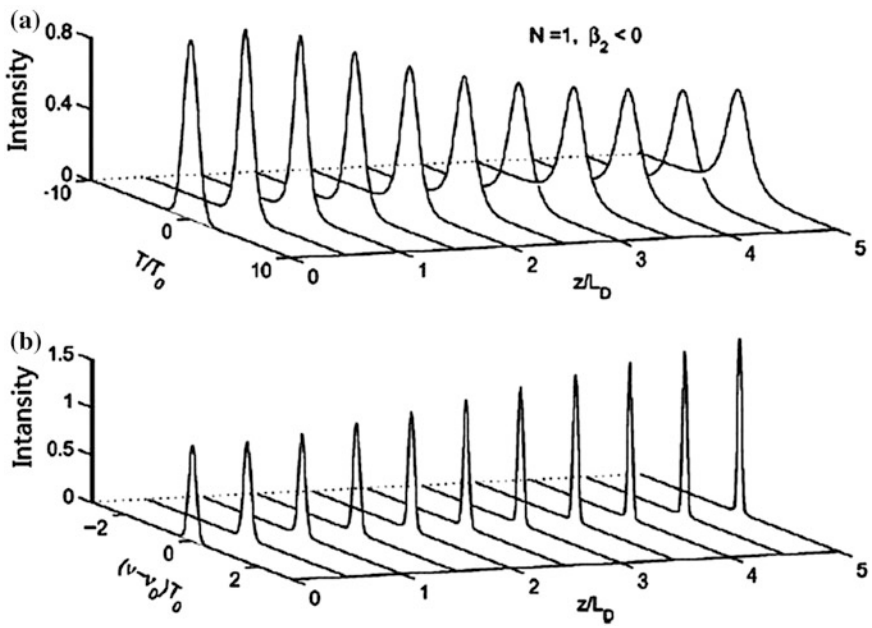

Fig. 9.8 Non-chirp Gaussian pulse propagation curves: **a** pulse shape and **b** frequency spectrum in anomalous dispersion region ($\beta_2 > 0$), and in condition of $N = 1$ and $\alpha = 0$

Actually, although many lasers launched laser pulses are all Gaussian type, only some specially-made mode-locked laser generates the hyperbolic secant type optical soliton, namely

$$U(0,T) = \operatorname{sec h}\left(\frac{T}{T_0}\right)\exp\left(-\frac{iCT^2}{2T_0^2}\right), \tag{9.2.37}$$

where C is the chirp parameter, it depends on the initial chirp of pulse. T_0 is the half width at 1/e of peak intensity, for hyperbolic secant type optical pulse, the relation between T_0 and the full width at half maximum intensity T_{FWHM} is

$$T_{FWHM} = 2\ln(1+\sqrt{2})T_0 \approx 1.763T_0. \tag{9.2.38}$$

In comparison with Eq. (9.2.17), we can see that the difference of T_{FWHM} between the hyperbolic secant type light pulse and the Gaussian type light pulse is not too big, but the hyperbolic secant type light pulsewidth is narrower than the Gaussian type light pulsewidth.

If light pulse is non-chirp pulse, $C = 0$, i.e., it is the fundamental wave optical soliton. In the propagation process, its waveform and optical spectrum all keep no change. If the incident light pulse deviate the hyperbolic secant pulse waveform, the combined action of GVD and SPM can make the light pulse to evolve to be hyperbolic secant pulse.

9.3 Time Soliton and Space Soliton

Optical soliton can be interpreted as a matter state when both of linear effect and nonlinear effect achieve a balance in the propagation of light wave. Optical soliton in general is classified into the time optical soliton and the space optical soliton. The time soliton is a balance state of light pulse when two opposite chirp effects induced by the dispersion and the nonlinear self-phase modulation achieve a balance. The space soliton is a balance state of light pulse when the diffraction and the nonlinear self-focusing achieve a balance.

9.3.1 Time Soliton

In 1834 S. Russell, a shipbuilder of the United Kingdom observed a circular smooth wave peak of water wave in a narrow river channel, this phenomenon is called solitary wave or soliton by later generations. In 1895, Korteweg and De Vries proposed KDV equation to explain it. Until the 70s of 20th century the development of fiber communication, the optical soliton study brought to attention.

In 1971 Zeldovich and Sobelman at the first time proposed the theory of compressing light pulse with SPM by using GVD effect [2]; in 1972 Zakharov and Shabat based on analysis of nonlinear wave equation, obtained the conclusion of existence of time soliton wave solution with hyperbolic secant form [3]. In 1973, Bell Labs A. Hasegawa and F. Tappert firstly propose the idea for application of optical soliton in the optical fiber communication [4]. Because the formation of stable optical soliton requires very high technology, until to1980, Mollenauer et al. firstly observed optical soliton in Bell laboratory [5]. In 1984 Mollenauer et al. successfully developed the color center soliton laser [6], in 1991, Smith et al. successfully developed all-fiber integrated erbium-doped fiber soliton laser [7], in 1999, P. Andrekson et al. completed several field investigations of soliton propagation in fiber [8].

Time soliton can be described by nonlinear Schrodinger Eq. (9.2.3). If neglecting the absorption loss, the nonlinear Schrodinger Equation can be simplified as

$$i\frac{\partial A}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \gamma|A|^2 A, \tag{9.3.1}$$

where $A = A(z, T)$ is amplitude of pulse envelop (wave packet); β_2 is GVD parameter; γ is SPM nonlinear parameter.

In order to normalize Eq. (9.3.1), we introduce three dimensionless variables:

$$U = \frac{A}{\sqrt{P_0}},\ \xi = \frac{z}{L_D},\ \tau = \frac{T}{T_0}. \tag{9.3.2}$$

Then the equation becomes

$$i\frac{\partial U}{\partial \xi} = \mathrm{sgn}(\beta_2)\frac{1}{2}\frac{\partial^2 U}{\partial \tau^2} - N^2|U|^2 U, \tag{9.3.3}$$

where P_0 is the pulse peak power; T_0 is the incident pulsewidth; Parameter N is defined as

$$N^2 = \frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|}. \tag{9.3.4}$$

Through the definition of

$$u = NU = \sqrt{\gamma L_D} A, \tag{9.3.5}$$

The parameter N can be eliminated from Eq. (9.3.3). Considering in anomalous dispersion GVD case, we take $\mathrm{sgn}(\beta_2) = -1$, the equation becomes following standard form of nonlinear Schrodinger (NLS) Equation:

$$i\frac{\partial u}{\partial \xi} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = 0. \tag{9.3.6}$$

Equation (9.3.6) can be solved by using inverse scattering method. Actually, it is a method similar to Fourier transform. This method is using the incident filed at $z = 0$ to obtain the initial scattering data, than through solving lineal scattering problem to obtain the change of propagation field along z, than from the changed scattering data to rebuild the new propagation field. The process of this method is too complicate; here we just introduce a simpler method to solve the fundamental state soliton.

Suppose NLS Equation has a solution maintaining a no-changed shape, namely

$$u(\xi, \tau) = V(\tau)\exp[i\phi(\xi, \tau)], \tag{9.3.7}$$

where V is independent with ξ, Eq. (9.3.7) denotes the fundamental state soliton with no-changed shape in propagation process. The phase ϕ is a function of ξ and τ.

Substituting Eq. (9.3.7) into Eq. (9.3.6), and then separating into real part and imaginary part, one can obtained the two equations related with amplitude V and phase ϕ, respectively. The phase equation shows that ϕ should adopt the form of $\phi(\xi, \tau) = K\xi - \delta\tau$, in which K and δ are constant. If taking $\delta = 0$ (without frequency shift), then V should satisfy

$$\frac{d^2 V}{d\tau^2} = 2V(K - V^2). \tag{9.3.8}$$

In the two sides of Eq. (9.3.8) multiply by $2(dV/d\tau)$, and integral to τ, we can obtain

$$(dV/d\tau)^2 = 2KV^2 - V^4 + C, \tag{9.3.9}$$

where C is integration constant. Using boundary condition, namely when $|\tau| \to \infty$, V and $dV/d\tau$ are equal to zero, so $C = 0$. Constant K is depended on the condition that at soliton peak value, $V = 1$ and $dV/d\tau = 0$. And assuming the peak value appears at $\tau = 0$, thus we can get $K = 1/2$, then $\phi = \xi/2$. Simple integral to the equation, to obtain $V(\tau) = \mathrm{sec\,h}\,\tau$, from this we can obtain the following soliton solution:

$$u(\xi, \tau) = \mathrm{sec\,h}(\tau)\exp(i\xi/2). \tag{9.3.10}$$

This is a standard form of fundamental state soliton. Equation (9.3.10) indicates that if light pulse is a hyperbolic secant pulse with pulsewidth T_0, peak power P_0 satisfied Eq. (9.3.4) with $N = 1$. If this pulse inputs into a lossless idea fiber, the pulse will be non-distortion propagation, never change its shape in any distance. Setting $N = 1$ in Eq. (9.3.4), we can obtain the peak power required for propagating fundamental state soliton, i.e.,

$$P_0 = \frac{|\beta_2|}{\gamma T_0^2} \approx \frac{3.11|\beta_2|}{\gamma T_{FWHM}^2}, \tag{9.3.11}$$

where the relation of $T_{FWHM} \approx 1.76T_0$ is used.

For the dispersion shift fiber at 1.55 μm, the typical values are $\beta_2 = -1\,\text{ps}^2/\text{km}$ and $\gamma = 3\,\text{W}^{-1}/\text{km}$. When $T_0 = 1\,\text{ps}$, P_0 is about 1 W; when $T_0 = 10\,\text{ps}$, P_0 is reduced to 10 mW. Therefore even for 20 Gb/s high bit rate transmission system, it can be reached by semiconductor laser at this power level, also can form the fundamental state soliton in the fiber.

The solution satisfied NLS Eq. (9.3.10) is not only above described so called "bright soliton" solution, but also having many other solutions. For example the "dark soliton" is also a solution. Its intensity profile is a caving on the uniform background. We only change the symbol of time-differential item in Eq. (9.3.6), the NLS equation for describing the dark soliton can be obtained:

$$i\frac{\partial u}{\partial \xi} - \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = 0. \tag{9.3.12}$$

We can assume the form of solution is

$$u(\xi, \tau) = V(\tau)\exp[i\phi(\xi, \tau)]. \tag{9.3.13}$$

Then substrate it into V and ϕ satisfied differential equations, thus we can obtain the solution of dark soliton. The difference compared with the bright soliton is: when $|\tau| \to \infty$, $V(\tau)$ becomes a non-zero constant, its general solution can be written as

$$|V(\xi, \tau)| = V(\tau) = \eta\{1 - B^2 \sec\text{h}^2[\eta B(\tau - \tau_s)]\}^{1/2}. \tag{9.3.14}$$

The phase is

$$\phi(\xi, \tau) = \frac{1}{2}\eta^2(3 - B^2)\xi + \eta\sqrt{1 - B^2}\tau + \arctan\left[\frac{B\tanh(\eta B\tau)}{\sqrt{1 - B^2}}\right], \tag{9.3.15}$$

where parameters η and τ_s are denoted the amplitude of soliton and the location of caving, respectively. B is denoted the deep of caving ($|B| \leq 1$). For $|B| = 1$, the intensity of caving center drops to zero; for other value of B, caving not tends to zero. So the dark soliton for $|B| < 1$ is called gray soliton. The parameter B determines the dark degree of gray soliton. The gray soliton for $|B| = 1$ is called dark soliton. In Eq. (9.3.14), to set $\eta = 1$ and $B = 1$, we can obtain the standard form of dark soliton:

$$u(\xi, \tau) = \tanh(\tau)\exp(i\xi). \tag{9.3.16}$$

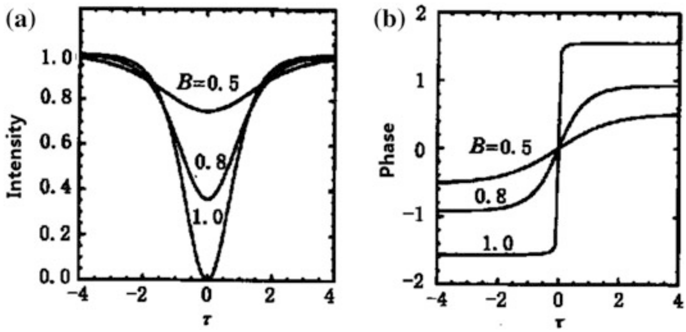

Fig. 9.9 Characteristic curves of dark soliton for different parameter B: **a** the intensity versus the time; **b** the phase versus the time

Therefore the dark soliton has the hyperbolic tangent type amplitude. The dark soliton is a hyperbolic tangent pulse with a sunk at the center, it keeps no change when propagation in normal dispersion region.

Figure 9.9 gives the intensity-time curve and the phase-time curve for dark soliton when Btakes different values. At $\tau = 0$, the dark soliton is a hyperbolic tangent pulse with a center sink, the sink of black soliton is maximum, $|B|$ is smaller, the sink is smaller; at $\tau = 0$, also has a phase break, the phase break of black soliton is $\pi.|B|$ is smaller, the phase break is smaller, and change slower.

9.3.2 Space Soliton

In comparison with time soliton, the conception of space soliton is proposed later. So called space soliton is that the light pulse propagates in nonlinear medium, when the linear diffraction and the nonlinear self-focusing effect both reach a balance, the pulse propagation forwards ahead with a stable space form. The study related to space soliton should date back to 1964, Chiao et al. [9] started to study the self-focusing, they proposed the description of light beam "self-trapping" effect. But until 1972 Zakharov and Shabat [10] given the soliton theory by using classical nonlinear Schrodinger equation, after that people gradually aware the self-focusing filament is just a kind of space bright optical soliton.

In the situation of neglecting the medium loss, the propagation of space soliton can be described by using following NLS equation:

$$i\frac{\partial q}{\partial \xi} + \frac{\partial^2 q}{\partial \zeta^2} - 2|q|^2 q = 0. \qquad (9.3.17)$$

where $q = E/E_0$ is normalized filed amplitude; $\xi = z/z_0$ is the normalized vertical coordinate; $\zeta = x/x_0$ is the normalized transverse coordinate. Here E_0 is maximum amplitude of filed, $z_0 = 2n_0/(\beta n_2|E_0|^2)$ is the characteristic scale of space soliton propagation, $x_0 = \sqrt{n_0}/(\beta\sqrt{n_2}|E_0|^2)$ is the characteristic scale of space soliton width. Here n_0 is linear refractive index, β is light propagation constant in the medium.

The space optical soliton also can be divided into two kinds of the bright soliton and the dark soliton. In 1985 A. Barthelemy et al. in the first experimentally proved that in the Kerr medium, the diffraction effect and the nonlinear self-focusing effect reach a balance to form a space bright soliton. Actually, any medium with self-focusing effect all can observe the space bright soliton, the materials including various three-order nonlinear optical medium, two-order nonlinear crystal, liquid crystal composed with anisotropic molecules, and photorefractive materials.

In 1987, P.A. Belanger and P. Mathieu, at the first time, started from NLS equation to prove that the existence of the space dark soliton in self-focusing Kerr medium is possible. In 1988, Maneuf et al. [11] observed the change from fundamental soliton to three–order soliton in KDP crystal. After that people observed space dark optical soliton in different mediums, and tried to apply the space dark soliton in design of controllable flexibility optical waveguide, X and Y-type directional couplers, and all-optical switches, etc. in 1996, Li's [12] research group experimentally proved the existence of space dark soliton in C_{60} solution and demonstrated the space dark soliton induced flexibility optical waveguide. In 1997 they also experimentally demonstrated the thermal induced space dark soliton in chlorophyll-acetone solution [13].

In general, the experimental studies of space dark soliton are used the material with minus nonlinear refractive index $(\Delta n < 0)$, including various isotopic three-order nonlinear medium, and the wavelength of incident laser are selected nearby the single photon and two-photon resonant frequency; in addition the anisotropic two-order nonlinear crystal and photorefractive materials, etc. can also produce the refractive index change $\Delta n < 0$, to demonstrate the space dark soliton.

Review Questions of Chapter 9

1. There are which steps to deduce the Nonlinear Schrodinger Eq. (9.1.38) for describing the propagation of picosecond light pulse in the single mode fiber; what physical meaning in that equation?
2. Illustrate the source, meaning, significance and application of following forms of NLS equations: (1) Eq. (9.2.3) using group-velocity motion time coordinate; (2) Eq. (9.2.6) containing normalized time, dispersion length and nonlinear length; (3) Eq. (9.2.34) containing normalized time, normalized distance and parameter *N*; (4) Standard Eq. (9.3.6) using normalized time, distance, and amplitude.
3. Please deduce the material dispersion formula (9.1.43). What are regular fiber, dispersion shift fiber and dispersion flat fiber? Draw the dispersion-wavelength

characteristic curves. What is normal dispersion and anomalous dispersion? In which kind fiber for generation of optical soliton?

4. Discuss the propagation of light pulse in following four cases: (1) excluding dispersion and nonlinearity; (2) consider the influence of dispersion; (3) consider the influence of self-phase modulation; (4) consider the combined action of dispersion and self-phase modulation. Explain the formation of time optical soliton.
5. From NLS Eq. (9.3.6) to find the solution of fundamental wave time optical soliton. Write down dark soliton equation and its fundamental state solution. What is the characteristic of dark soliton, what is difference between bright soliton and dark soliton?
6. What is space optical soliton? Illustrate its generation mechanism. What is nonlinear Schrodinger equation for describing space optical soliton?

References

1. G.P. Agrawal, *Nonlinear Fiber Optics*, 4th edn. (Academic Press, New York, 2006)
2. B. Zeldovich, I.I. Sobelman, Possibility of shortening light pulses in alkali-metal vapor. JETP **13**(3), 129–131 (1971)
3. V.E. Zekharov, A.B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of wave in nonlinear media. JETP **34**(1), 62–65 (1972)
4. A. Hasegawa, F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion. Appl. Phys. Lett. **23**(3), 142–144 (1973)
5. L.F. Mollenauer, R.H. Stolen, J.P. Dorden, Experimental observation of picosecond pulse narrowing and solitons in optical fibers. Phys. Rev. Lett. **45**(13), 1095–1098 (1980)
6. L.F. Mollenauer, R.H. Stolen, The soliton laser. Opt. Lett. **9**(1), 13–15 (1984)
7. K. Smith. E.J. Greer, R. Wyatt, et. al., Totally integrated erbium fibre soliton laser pumped by laser diode. Electron Lett. **27**(3), 244–246 (1991)
8. P.A. Andrekson, Laser focus world. Feature Article **35**(5), 145–153 (1999)
9. R.Y. Chiao, E. Garmire, C.H. Townes, Self-trapping of optical beams. Phys. Rev. Lett. **13**(15), 479–482 (1964)
10. V.E. Zakharov, A.B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of wave in nonlinear media. Sovi. Phys. JETP **34**(1), 62–69 (1972)
11. S. Maneuf, R. Desailly, C. Froehly, Stable self-trapping of laser beams: observation in a nonlinear planar waveguide. Opt. Commun. **65**(3), 193–198 (1988)
12. C. Li, T. Hong, Space dark soliton and its induced feasible optical waveguide in C_{60} solution. Acta Phys. Sini. **45**(10), 1671–1975 (1996)
13. L. Hong, X. Jiandong, S. Liu, C. Li, J. Hong, Experiments of thermal induced space dark soliton in chlorophyll-acetone solution. Acta Optica Sinica **18**(1), 37–40 (1998)