Identities of Symmetry for the Generalized Degenerate Euler Polynomials

Dae San Kim and Taekyun Kim

Abstract In this paper, we give some identities of symmetry for the generalized degenerate Euler polynomials attached to χ which are derived from the symmetric properties for certain fermionic p-adic integrals on $\mathbb{Z}_p$.

Keywords Identities of symmetry · Generalized degenerate Euler polynomial · Fermionic p-adic integral

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1 Introduction and Preliminaries

Let p be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will be the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively.

The p-adic norm $|\cdot|_p$ in $\mathbb{C}_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $f(x)$ be continuous function on $\mathbb{Z}_p$. Then the fermionic p-adic integral on $\mathbb{Z}_p$ is defined as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) \tag{1.1}$$

D.S. Kim
Department of Mathematics, Sogang University, Seoul 121-742,
Republic of Korea
e-mail: dskim@sogang.ac.kr

T. Kim (✉)
Department of Mathematics, Kwangwoon University, Seoul 139-701,
Republic of Korea
e-mail: tkkim@kw.ac.kr

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$$= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad (\text{see [9]}).$$

From (1.1), we note that

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad (\text{see [7]}), \tag{1.2}$$

where $n \in \mathbb{N}$.

As is well known, the Euler polynomials are defined by the generating function

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.3}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers (see [1–19]).

For a fixed odd integer d with $(p, d) = 1$, we set

$$X = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} \left(a + dp\mathbb{Z}_p\right),$$

$$a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \pmod{dp^N}\right\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$.

It is known that

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \int_X f(x)\, d\mu_{-1}(x), \quad (\text{see [7–9]}),$$

where f is a continuous function on $\mathbb{Z}_p$.

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and let χ be a Dirichlet character with conductor d. Then the generalized Euler polynomials attached to χ are defined by the generating function

$$\left(\frac{2\sum_{a=0}^{d-1} (-1)^a \chi(a) e^{at}}{e^{dt}+1}\right) e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!}. \tag{1.4}$$

In particular, for $x = 0$, $E_{n,\chi} = E_{n,\chi}(0)$ are called the generalized Euler numbers attached to χ.

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, by (1.2), we get

$$\begin{aligned}\int_X \chi(y)\, e^{(x+y)t}\, d\mu_{-1}(y) &= \frac{2\sum_{a=0}^{d-1}(-1)^a \chi(a)\, e^{at}}{e^{dt}+1} e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\chi}(x)\frac{t^n}{n!}, \quad \text{(see [9–11])}.\end{aligned} \tag{1.5}$$

From (1.5), we have

$$\int_X \chi(y)(x+y)^n\, d\mu_{-1}(y) = E_{n,\chi}(x), \quad (n \geq 0). \tag{1.6}$$

Carlitz considered the degenerate Euler polynomials given by the generating function

$$\begin{aligned}&\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} \mathcal{E}_n(x \mid \lambda)\frac{t^n}{n!}, \quad \text{(see [3])}.\end{aligned} \tag{1.7}$$

Note that $\lim_{\lambda \to 0} \mathcal{E}_n(x \mid \lambda) = E_n(x)$, $(n \geq 0)$.
From (1.2), we note that

$$\begin{aligned}\int_X (1+\lambda t)^{\frac{x+y}{\lambda}}\, d\mu_{-1}(y) &= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} \mathcal{E}_n(x \mid \lambda)\frac{t^n}{n!}.\end{aligned} \tag{1.8}$$

Thus, by (1.8), we get

$$\int_X (y+x \mid \lambda)_n\, d\mu_{-1}(y) = \mathcal{E}_n(x \mid \lambda), \quad (n \geq 0), \tag{1.9}$$

where $(x \mid \lambda)_n = x(x-\lambda)\cdots(x-(n-1)\lambda)$, for $n \geq 1$, and $(x \mid \lambda)_0 = 1$.

From (1.2), we can derive the following equation:

$$\begin{aligned}\int_X \chi(y)(1+\lambda t)^{\frac{x+y}{\lambda}}\,d\mu_{-1}(y) \\ = \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)(1+\lambda t)^{\frac{a}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}},\end{aligned} \tag{1.10}$$

where $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$.

In view of (1.5), we define the generalized degenerate Euler polynomials attached to χ as follows:

$$\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)(1+\lambda t)^{\frac{a}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda,\chi}(x)\frac{t^n}{n!}. \tag{1.11}$$

When $x=0$, $\mathcal{E}_{n,\lambda,\chi}=\mathcal{E}_{n,\lambda,\chi}(0)$ are called the generalized degenerate Euler numbers attached to χ.

Let n be an odd natural number. Then, by (1.2), we get

$$\begin{aligned}\int_X \chi(x)(1+\lambda t)^{\frac{nd+x}{\lambda}}\,d\mu_{-1}(x)+\int_X\chi(x)(1+\lambda t)^{\frac{x}{\lambda}}\,d\mu_{-1}(x) \\ = 2\sum_{l=0}^{nd-1}(-1)^l\chi(l)(1+\lambda t)^{\frac{l}{\lambda}}.\end{aligned} \tag{1.12}$$

Now, we set

$$R_k(n,\lambda\mid x) = 2\sum_{l=0}^{n}(-1)^l\chi(l)(l\mid\lambda)_k. \tag{1.13}$$

From (1.2) and (1.12), we have

$$\begin{aligned}\int_X (1+\lambda t)^{\frac{x+dn}{\lambda}}\chi(x)\,d\mu_{-1}(x)+\int_X\chi(x)(1+\lambda t)^{\frac{x}{\lambda}}\,d\mu_{-1}(x) \\ = \frac{2\int_X(1+\lambda t)^{\frac{x}{\lambda}}\chi(x)\,d\mu_{-1}(x)}{\int_X(1+\lambda t)^{\frac{ndx}{\lambda}}\,d\mu_{-1}(x)} \\ = \sum_{k=0}^{\infty}R_k(nd-1,\lambda\mid\chi)\frac{t^k}{k!},\end{aligned} \tag{1.14}$$

where $n,d\in\mathbb{N}$ with $n\equiv 1 \pmod 2$, $d\equiv 1 \pmod 2$.

In this paper, we give some identities of symmetry for the generalized degenerate Euler polynomials attached to χ derived from the symmetric properties of certain fermionic p-adic integrals on $\mathbb{Z}_p$.

2 Identities of Symmetry for the Generalized Degenerate Euler Polynomials

Let w_1, w_2 be odd natural numbers. Then we consider the following integral equation:

$$\begin{aligned} &\frac{\int_X \int_X (1+\lambda t)^{\frac{w_1 x_1 + w_2 x_2}{\lambda}} \chi(x_1)\chi(x_2)\, d\mu_{-1}(x_1)\, d\mu_{-1}(x_2)}{\int_X (1+\lambda t)^{\frac{d w_1 w_2 x}{\lambda}}\, d\mu_{-1}(x)} \\ &= \frac{2\left((1+\lambda t)^{\frac{d w_1 w_2}{\lambda}} + 1\right)}{\left((1+\lambda t)^{\frac{w_1 d}{\lambda}} + 1\right)\left((1+\lambda t)^{\frac{w_2 d}{\lambda}} + 1\right)} \\ &\times \sum_{a=0}^{d-1} \chi(a)(1+\lambda t)^{\frac{w_1 a}{\lambda}} (-1)^a \\ &\times \sum_{b=0}^{d-1} \chi(b)(1+\lambda t)^{\frac{w_2 b}{\lambda}} (-1)^b . \end{aligned} \tag{2.1}$$

From (1.10) and (1.11), we note that

$$\int_X \chi(y)(x+y \mid \lambda)_n\, d\mu_{-1}(y) = \mathcal{E}_{n,\lambda,\chi}(x), \quad (n \geq 0). \tag{2.2}$$

By (1.14), we get

$$\int_X \chi(x)(x+dn \mid \lambda)_k\, d\mu_{-1}(x) + \int_X \chi(x)(x \mid \lambda)_k\, d\mu_{-1}(x) = R_k(nd-1, \lambda \mid x), \tag{2.3}$$

where $k \geq 0$.

Thus, by (2.2) and (2.3), we get

$$\mathcal{E}_{k,\lambda,\chi}(nd) + \mathcal{E}_{k,\lambda,\chi} = R_k(nd-1, \lambda \mid \chi), \tag{2.4}$$

where $k \geq 0$, $n, d \in \mathbb{N}$ with $n \equiv 1 \pmod 2$, $d \equiv 1 \pmod 2$.

Now, we set

$$I_{\chi}\left(w_1,w_2\mid\lambda\right)=\frac{\int_X\int_X\chi\left(x_1\right)\chi\left(x_2\right)\left(1+\lambda t\right)^{\frac{w_1x_1+w_2x_2+w_1w_2x}{\lambda}}d\mu_{-1}\left(x_1\right)d\mu_{-1}\left(x_2\right)}{\int_X\left(1+\lambda t\right)^{\frac{dw_1w_2x}{\lambda}}d\mu_{-1}\left(x\right)}. \tag{2.5}$$

From (2.5), we have

$$\begin{aligned} &I_{\chi}\left(w_1,w_2\mid\lambda\right) \\ &=\frac{2\left(\left(1+\lambda t\right)^{\frac{dw_1w_2}{\lambda}}+1\right)\left(1+\lambda t\right)^{\frac{w_1w_2x}{\lambda}}}{\left(\left(1+\lambda t\right)^{\frac{w_1d}{\lambda}}+1\right)\left(\left(1+\lambda t\right)^{\frac{w_2d}{\lambda}}+1\right)} \\ &\quad\times\sum_{a=0}^{d-1}\chi\left(a\right)\left(-1\right)^{a}\left(1+\lambda t\right)^{\frac{w_1a}{\lambda}} \\ &\quad\times\sum_{b=0}^{d-1}\chi\left(b\right)\left(-1\right)^{b}\left(1+\lambda t\right)^{\frac{w_2b}{\lambda}}. \end{aligned} \tag{2.6}$$

Thus, by (2.6), we see that $I_{\chi}\left(w_1,w_2\mid\lambda\right)$ is symmetric in w_1,w_2. By (1.12), (1.14), (2.2) and (2.5), we get

$$\begin{aligned} &2I_{\chi}\left(w_1,w_2\mid\lambda\right) \\ &=\sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\binom{l}{i}\mathcal{E}_{i,\frac{\lambda}{w_2},\chi}\left(w_1x\right)w_2^{i}w_1^{l-i}R\left(dw_2-1,\frac{\lambda}{w_1}\middle|\chi\right)\right)\frac{t^l}{l!}. \end{aligned} \tag{2.7}$$

From the symmetric property of $I_{\chi}\left(w_1,w_2\mid\lambda\right)$ in w_1 and w_2, we have

$$\begin{aligned} &2I_{\chi}\left(w_1,w_2\mid\lambda\right) \\ &=2I_{\chi}\left(w_2,w_1\mid\chi\right) \\ &=\sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\binom{l}{i}\mathcal{E}_{i,\frac{\lambda}{w_1},\chi}\left(w_2x\right)w_1^{i}w_2^{l-i}R\left(dw_1-1,\frac{\lambda}{w_2}\middle|\chi\right)\right)\frac{t^l}{l!}. \end{aligned} \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 1 *For* $w_1,w_2,d\in\mathbb{N}$ *with* $w_1\equiv w_2\equiv d\equiv 1\pmod 2$, *let* χ *be a Dirichlet character with conductor* d. *Then, we have*

$$\sum_{i=0}^{l}\binom{l}{i}\mathcal{E}_{i,\frac{\lambda}{w_1},\chi}(w_2x)\,w_1^i w_2^{l-i} R\left(dw_1-1,\frac{\lambda}{w_2}\middle|\chi\right)$$
$$=\sum_{i=0}^{l}\binom{l}{i}\mathcal{E}_{i,\frac{\lambda}{w_2},\chi}(w_1x)\,w_2^i w_1^{l-i} R\left(dw_2-1,\frac{\lambda}{w_1}\middle|\chi\right),$$

where $l \geq 0$.

When $x = 0$, by Theorem 1, we get

$$\sum_{i=0}^{l}\binom{l}{i}\mathcal{E}_{i,\frac{\lambda}{w_1},\chi}w_1^i w_2^{l-i} R\left(dw_1-1,\frac{\lambda}{w_2}\middle|\chi\right)$$
$$=\sum_{i=0}^{l}\binom{l}{i}\mathcal{E}_{i,\frac{\lambda}{w_2},\chi}w_2^i w_1^{l-i} R\left(dw_2-1,\frac{\lambda}{w_1}\middle|\chi\right),\quad (l\geq 0).$$

By (2.5), we get

$$2I_\chi(w_1,w_2\mid\lambda) \tag{2.9}$$
$$=\sum_{l=0}^{dw_2-1}(-1)^l\chi(l)\int_X(1+\lambda t)^{\frac{w_2}{\lambda}\left(w_2+w_1x+\frac{w_1}{w_2}l\right)}\chi(x_2)\,d\mu_{-1}(x)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{dw_2-1}(-1)^l\chi(l)\,\mathcal{E}_{n,\frac{\lambda}{w_2},\chi}\left(w_1x+\frac{w_1}{w_2}l\right)w_2^n\right)\frac{t^n}{n!}.$$

On the other hand,

$$2I_\chi(w_2,w_1\mid\lambda)=2I_\chi(w_1,w_2\mid\lambda) \tag{2.10}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{dw_1-1}(-1)^l\chi(l)\,\mathcal{E}_{n,\frac{\lambda}{w_1},\chi}\left(w_2x+\frac{w_2}{w_1}l\right)w_1^n\right)\frac{t^n}{n!}.$$

Therefore, by (2.9) and (2.10), we obtain the following theorem.

Theorem 2 *For* $w_1, w_2, d \in \mathbb{N}$ *with* $d \equiv 1 \pmod 2$, $w_1 \equiv 1 \pmod 2$ *and* $w_2 \equiv 1 \pmod 2$, *let* χ *be a Dirichlet character with conductor* d. *Then, we have*

$$w_2^n\sum_{l=0}^{dw_2-1}(-1)^l\chi(l)\,\mathcal{E}_{n,\frac{\lambda}{w_2},\chi}\left(w_1x+\frac{w_1}{w_2}l\right)$$
$$=w_1^n\sum_{l=0}^{dw_1-1}(-1)^l\chi(l)\,\mathcal{E}_{n,\frac{\lambda}{w_1},\chi}\left(w_2x+\frac{w_2}{w_1}l\right),\quad (n\geq 0).$$

To derive some interesting identities of symmetry for the generalized degenerate Euler polynomials attached to χ, we used the symmetric properties for certain fermionic p-adic integrals on $\mathbb{Z}_p$. When $w_2 = 1$, from Theorem 2, we have

$$\sum_{l=0}^{d-1} (-1)^l \chi(l) \mathcal{E}_{n,\lambda,\chi}(w_1 x + w_1 l) = w_1^n \sum_{l=0}^{dw_1-1} (-1)^l \chi(l) \mathcal{E}_{n,\frac{\lambda}{w_1},\chi}\left(x + \frac{1}{w_1} l\right).$$

In particular, for $x = 0$, we get

$$\sum_{l=0}^{d-1} (-1)^l \chi(l) \mathcal{E}_{n,\lambda,\chi}(w_1 l) = w_1^n \sum_{l=0}^{dw_1-1} (-1)^l \chi(l) \mathcal{E}_{n,\frac{\lambda}{w_1},\chi}\left(\frac{1}{w_1} l\right).$$

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