Generalized Absolute Convergence of Trigonometric Fourier Series

R.G. Vyas

Abstract Recently, Moricz and Veres generalized the classical results of Bernstein, Szasz, Zygmund and others related to the absolute convergence of single and multiple Fourier series. In this paper, we have extended this result for single Fourier series of functions of the classes $\Lambda BV(\overline{\mathbb{T}})$ and $\Lambda BV^{(p)}(\overline{\mathbb{T}})$.

Keywords Generalized $\beta-$absolute convergence · Fourier series · $\Lambda BV^{(p)}(\overline{\mathbb{T}})$

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1 Introduction

The classical result of Zygmund, for the absolute convergence of Fourier series if a function of bounded variation on $\overline{T}$, where $\mathbb{T} = [-\pi, \pi)$ is the torus, is generalized in many ways and many interesting results are obtained for different generalized absolute convergence of Fourier of functions of different generalized classes (see [1, 4]). In 2006, Gogoladze and Meskhia [1] obtained sufficient conditions for the generalized absolute convergence of a single Fourier series. Moricz and Veres [2] obtained sufficient conditions for the generalized absolute convergence of single and multiple Fourier series of functions of the classes $BV^{(p)}(\overline{\mathbb{T}})$ and $BV^{(p)}(\overline{\mathbb{T}}^N)$, respectively (also see [5]). In this paper, generalizing such results for single Fourier series, we have obtained sufficient conditions for the generalized absolute convergence of single Fourier series of functions of the classes $\Lambda BV(\overline{\mathbb{T}})$ and $\Lambda BV^{(p)}(\overline{\mathbb{T}})$.

In the sequel, $\mathbb{L}$ is the class of non-decreasing sequence $\Lambda = \{\lambda_i\}$ $(i = 1, 2, \ldots)$ of positive numbers such that $\sum_i \frac{1}{\lambda_i}$ diverges, a real number $p \geq 1$ and C represents a constant vary time to time.

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2 Notations and Definitions

For a complex valued, 2π-periodic, function $f \in L^1(\overline{\mathbb{T}})$, its Fourier series is defined as

$$f(x) \sim \sum_{m\in\mathbb{Z}} \hat{f}(m)e^{imx}, \quad x \in \overline{\mathbb{T}},$$

where

$$\hat{f}(m) = \left(\frac{1}{2\pi}\right)\int_{\overline{\mathbb{T}}} f(x)e^{-imx}\,dx$$

denotes the mth Fourier coefficient of f.

For $p \geq 1$, the p-integral modulus of continuity of f over $\overline{\mathbb{T}}$ is define as

$$\omega^{(p)}(f;\delta) := \underset{0<h\leq\delta}{sup} \parallel T_h f - f \parallel_p,$$

where $T_h f(x) = f(x+h)$ for all x and $\parallel (\cdot) \parallel_p$ denotes the L^p-norm over $\overline{\mathbb{T}}$. $p = \infty$ gives the modulus of continuity $\omega(f;\delta)$ of f.

Following the definition in [1], a sequence $\gamma = \{\gamma_m : m \in \mathbb{N}\}$ of nonnegative numbers is said to belongs to the class $\mathcal{A}_\alpha$ for some $\alpha \geq 1$ if

$$\left(\sum_{m\in\mathcal{D}_\mu} \gamma_m^\alpha\right)^{1/\alpha} \leq \kappa 2^{\mu(1-\alpha)/\alpha} \sum_{m\in\mathcal{D}_{\mu-1}} \gamma_m, \quad \mu \in \mathbb{N}, \tag{2.1}$$

where

$$\mathcal{D}_0 := \{1\}; \quad \mathcal{D}_\mu := \{2^{\mu-1}+1, 2^{\mu-1}+2, \ldots, 2^\mu\}, \quad \mu \in \mathbb{N}; \tag{2.2}$$

and the constant κ does not dependent on μ. Without the loss of generality, we assume that $\kappa \geq 1$.

Note that,

$$\mathcal{A}_{\alpha_2} \subset \mathcal{A}_{\alpha_1}, \quad where \quad 1 \leq \alpha_1 < \alpha_2 < \infty. \tag{2.3}$$

If a sequences γ is such that

$$\max\{\gamma_m : m \in \mathcal{D}_\mu\} \leq \kappa \,\min\{\gamma_m : m \in \mathcal{D}_{\mu-1}\}, \quad \mu \in \mathbb{N}, \tag{2.4}$$

then $\gamma \in \mathcal{A}_\alpha$ for every $\alpha \geq 1$. This inequality was introduced by Ul'yanov [3]. Moreover, Moricz and Veres [2] observed that, if a sequence $\gamma = \{\gamma_m\}$ is of the form

$$\gamma_m = m^\tau w(m), \quad m \in \mathbb{N},$$

where $\tau \in \mathbb{R}$ and $w : \mathbb{R}_+ \to \mathbb{R}_+$ is a slowly varying function, that is,

$$\lim_{x \to \infty} \frac{w(\lambda x)}{w(x)} = 1, \quad \textit{for every } 0 < \lambda < \infty, \tag{2.5}$$

then $\gamma \in \mathcal{A}_\alpha$ for every $\alpha \geq 1$.

For convenience in writing, put

$$\gamma_{-m} \ := \ \gamma_m, \quad m \in \mathbb{N}. \tag{2.6}$$

Definition 2.1 Given $\Lambda = \{\lambda_n\} \in \mathbb{L}$. A complex valued function f defined on an interval $I := [a, b]$ is said to be of $p - \Lambda$-bounded variation (that is, $f \in \Lambda BV^{(p)}(I)$) if

$$V_{\Lambda_p}(f, I) = \sup_{\{I_k\}} \left(\sum_k \frac{|f(I_k)|^p}{\lambda_k} \right)^{1/p} < \infty,$$

where $\{I_k\}$ is a finite collections of non-overlapping subintervals $I_k = [a_k, b_k] \subset [a, b]$ and $f(I_k) = f(b_k) - f(a_k)$.

Note that, for $p = 1$ and $\Lambda = \{1\}$ (that is, $\lambda_n = 1$, for all n,) the class $\Lambda BV^{(p)}(I)$ reduces to the class $BV(I)$ (the class of functions of bounded variation). For $p = 1$ the class $\Lambda BV^{(p)}(I)$ reduces to the class $\Lambda BV(I)$; and for $\Lambda = \{1\}$ the class $\Lambda BV^{(p)}(I)$ reduces to the class $BV^{(p)}(I)$ (the class of functions of p-bounded variation).

3 Results for Functions of Single Variable

Theorem 3.1 *If $f \in \Lambda BV(\overline{\mathbb{T}})$ and $\gamma = \{\gamma_m\} \in \mathcal{A}_{2/(2-\beta)}$ for some $\beta \in (0, 2)$ then*

$$\sum(\gamma; f)_\beta = \sum_{|m|\geq 1} \gamma_m |\hat{f}(m)|^\beta \leq \kappa C \sum_{\mu=0}^{\infty} 2^{-\mu\beta/2} \Gamma_{\mu-1} \left(\frac{(\omega(f; \frac{\pi}{2^\mu}))}{\sum_{i=1}^{2^\mu} \frac{1}{\lambda_i}} \right)^{\beta/2},$$

where κ is from (2.1) corresponding to $\alpha = 2/(2 - \beta)$ and C is a constant,

$$\Gamma_\mu := \sum_{m \in \mathcal{D}_\mu} \gamma_m \quad \textit{for} \quad \mu \in \mathbb{N}, \quad \textit{and} \quad \Gamma_{-1} := \Gamma_0 = \{\gamma_1\} \tag{3.1}$$

Corollary 3.2 *Under the hypothesis of Theorem 3.1, we have*

$$\sum(\gamma;f)_\beta \le \kappa C \sum_{m=1}^{\infty} m^{-\beta/2}\gamma_m \left(\frac{(\omega(f;\frac{\pi}{m}))}{\sum_{i=1}^{m}\frac{1}{\lambda_i}}\right)^{\beta/2},$$

In the case when $\gamma_m \equiv 1$, it follows from the above Corollary that $\sum(1;f)_\beta := \sum_{|m|\ge 1}|\hat{f}(m)|^\beta$

$$\le C\sum_{m=1}^{\infty} m^{-\beta/2}\left(\frac{(\omega(f;\frac{\pi}{m})}{\sum_{i=1}^{m}\frac{1}{\lambda_i}}\right)^{\beta/2}.$$

This gives the result [6, Theorem1,with$n_k = k$, forall k,] as a particular case.

Above corollary can easily follow from the Theorem 3.1.

Theorem 3.3 *If $f \in \Lambda BV^{(p)}(\overline{\mathbb{T}})$ and $\gamma = \{\gamma_m\} \in \mathcal{A}_{2/(2-\beta)}$ for some $\beta \in (0,2)$ then*

$$\sum(\gamma;f)_\beta \le \kappa C \sum_{\mu=0}^{\infty} 2^{-\mu\beta/2}\Gamma_{\mu-1}\left(\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{2^\mu}))^{2r-p}}{\sum_{i=1}^{2^\mu}\frac{1}{\lambda_i}}\right)^{1/r}\right)^{\beta/2},$$

where $\frac{1}{r}+\frac{1}{s}=1$, κ is from (2.1) corresponding to $\alpha = 2/(2-\beta)$ and C is a constant.

Corollary 3.4 *Under the hypothesis of Theorem 3.3, we have*

$$\sum(\gamma;f)_\beta \le \kappa C \sum_{m=1}^{\infty} m^{-\beta/2}\gamma_m\left(\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{m}))^{2r-p}}{\sum_{i=1}^{m}\frac{1}{\lambda_i}}\right)^{1/r}\right)^{\beta/2},$$

In the case when $\gamma_m \equiv 1$, it follows from the above Corollary that

$$\sum(1;f)_\beta := \sum_{|m|\ge 1}|\hat{f}(m)|^\beta$$

$$\le C\sum_{m=1}^{\infty} m^{-\beta/2}\left(\left(\frac{(\omega^{((2-p)s+p)}(f;\frac{\pi}{m}))^{2r-p}}{\sum_{i=1}^{m}\frac{1}{\lambda_i}}\right)^{1/r}\right)^{\beta/2}.$$

This gives the result [4, Theorem1, with $n_k = k$, forall k,] as a particular case.

Above Corollary 3.4 can be easily follows from the Theorem 3.3.

Proof of Theorem 3.1 $f \in \Lambda BV(\overline{\mathbb{T}})$ implies that f is bounded over $\overline{\mathbb{T}}$ and hence $f \in L^2(\overline{\mathbb{T}})$. For given $h > 0$, put $f_j = T_{jh}f - T_{(j-1)h}f$, then $\hat{f_j}(m) = 2i\hat{f}(m)e^{im(j-\frac{1}{2}h)}$ $\sin(\frac{mh}{2})$.

By Parseval's equality, we get

$$4\sum_{m\in\mathbb{Z}} |\hat{f}(m)|^2 \sin^2\left(\frac{mh}{2}\right) = O(||f_j||_2^2).$$

Putting $h = \frac{\pi}{2^\mu}$, $\mu \in \mathbb{N}$, and observing that

$$\frac{\pi}{4} < \frac{|m|\pi}{2^{\mu+1}} \leq \frac{\pi}{2} \quad for \quad |m| \in \mathcal{D}_\mu, \quad implies \quad \sin^2\left(\frac{mh}{2}\right) > \frac{1}{2}.$$

Thus, we have

$$\begin{aligned} B = \sum_{|m|\in\mathcal{D}_\mu} |\hat{f}(m)|^2 &= O\left(||f_j||_2^2\right) \\ &= O\left(\omega(f;h)\right)\left(\int_0^{2\pi} |f_j(x)|dx\right). \end{aligned} \tag{3.2}$$

Multiplying both the sides of the above inequality by $\frac{1}{\lambda_j}$ and then summing over $j = 1$ to $j = 2^\mu$, we have

$$B = O\left(\frac{\omega(f;h)}{\sum_{j=1}^{2^\mu}\frac{1}{\lambda_j}}\right)\left(\int_0^{2\pi}\sum_{j=1}^{2^\mu}\frac{(|f_j(x)|)}{\lambda_j}\,dx\right) = O\left(\frac{\omega(f;h)}{\sum_{j=1}^{2^\mu}\frac{1}{\lambda_j}}\right),$$

as $f \in \Lambda BV(\overline{\mathbb{T}})$ implies $\sum_{j=1}^{2^\mu}\frac{(|f_j(x)|)}{\lambda_j} = O(1)$.

Since $1 = \frac{\beta}{2} + \frac{2-\beta}{2}$, by Holder's inequality, for $\mu \geq 1$, we have

$$\begin{aligned} S_\mu := \sum_{|m|\in\mathcal{D}_\mu} \gamma_m|\hat{f}(m)|^\beta &\leq \left(\sum_{|m|\in\mathcal{D}_\mu}|\hat{f}(m)|^2\right)^{\beta/2}\left(\sum_{|m|\in\mathcal{D}_\mu}\gamma_m^{2/(2-\beta)}\right)^{(2-\beta)/2} \\ &\leq C\left(\frac{\omega(f;h)}{\sum_{j=1}^{2^\mu}\frac{1}{\lambda_j}}\right)^{\frac{\beta}{2}}\left(\sum_{|m|\in\mathcal{D}_\mu}\gamma_m^{2/(2-\beta)}\right)^{(2-\beta)/2}. \end{aligned} \tag{3.3}$$

Thus for $\mu \geq 1$,

$$S_\mu \leq C\kappa\left(2^{-\mu\beta/2}\,\Gamma_{\mu-1}\left(\frac{\omega(f;h)}{\sum_{j=1}^{2^\mu}\frac{1}{\lambda_j}}\right)^{\frac{\beta}{2}}\right).$$

If $\mu = 0$, then from (3.3) it follows that

$$S_0 := \gamma_1(|\hat{f}(-1)|^\beta + |\hat{f}(1)|^\beta) = O\left(\gamma_1\left(\frac{\omega(f;\pi)}{\frac{1}{\lambda_1}}\right)\right).$$

Hence, the result follows from

$$\sum_{|m|\geq 1} \gamma_m |\hat{f}(m)|^\beta = \sum_{\mu=0}^{\infty} S_\mu.$$

Proof of Theorem 3.3. $f \in \Lambda BV^{(p)}(\overline{\mathbb{T}})$ implies that f is bounded over $\overline{\mathbb{T}}$ [4, in view of Lemma 1, p.771] and hence $f \in L^2(\overline{\mathbb{T}})$. Proceeding as in the proof of Theorem 3.1, we get (3.2).

Since $2 = \frac{(2-p)s+p}{s} + \frac{p}{r}$, by using Holder's inequality, we have

$$||f_j||_2^2 \leq \left(||f_j||_p\right)^{p/r} \left(\int_0^{2\pi} |f_j|^{(2-p)s+p} dx\right)^{1/s} \leq \left(||f_j||_p\right)^{p/r} \Omega_h^{1/r},$$

where $\Omega_h^{1/r} = (\omega^{(2-p)s+p}(f;h))^{2r-p}$.

This together with (3.2) implies

$$B^r = \left(\sum_{|m|\in\mathcal{D}_\mu} |\hat{f}(m)|^2\right)^r = O\left(\Omega_h \int_0^{2\pi} |f_j(x)|^p dx\right).$$

Multiplying both the sides of the above inequality by $\frac{1}{\lambda_j}$ and then summing over $j = 1$ to $j = 2^\mu$, we have

$$B^r = O\left(\frac{\Omega_h}{\sum_{j=1}^{2^\mu} \frac{1}{\lambda_j}}\right)\left(\int_0^{2\pi} \sum_{j=1}^{2^\mu} \frac{(|f_j(x)|^p)}{\lambda_j}\, dx\right) = O\left(\frac{\Omega_h}{\sum_{j=1}^{2^\mu} \frac{1}{\lambda_j}}\right).$$

Thus

$$B = O\left(\frac{\Omega_h}{\sum_{j=1}^{2^\mu} \frac{1}{\lambda_j}}\right)^{1/r}.$$

Now, proceeding as in the proof of the Theorem 3.1 the result follows.

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