# The Cosserats' Memoir of 1896 on Elasticity

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**Abstract** Nowadays the Cosserat brothers are mostly cited for their work on so-called "Cosserat continua" of 1909 that practically initiated the theory of "oriented media" as generalized continua. But in 1896 they had already published a lengthy well-structured memoir on the theory of elasticity. This memoir is often considered as a foundational work on the modern approach to elasticity as it beautifully summarizes what was achieved in the nineteenth century but with original traits that will permeate further the twentieth century developments with an emphasis on finite deformations, the interest for applying the thermodynamic laws, the allied formulation of the notion of stress (internal forces), questions of stability, and the use of curvilinear coordinates, though still without using vector and/or tensor analysis. The present contribution examines in detail the contents of this epoch-making work of 1896, its main sources (e.g. Kirchhoff, Kelvin, Saint-Venant, Boussinesq, and Poincaré) and its insertion in the then current technical literature. We try to appraise its importance and its legacy in the modern developments of continuum mechanics, especially after the revival of the field by Truesdell and others.

## 1 Introduction

At the time of writing of this contribution, the most cited work of the Cosserat brothers, Eugène and François, certainly is their book of 1909 (Cosserat and Cosserat 1909). This is due to a justified renewal of interest in continua endowed with a microstructure (in particular, so-called "micropolar continua" also rightly named "Cosserat continua"). These are not classical in the sense that such media exhibit nonsymmetric stress tensors and so-called moment (or couple) stresses. Year 2009 marked with some emphasis the hundredth anniversary of the publication of this

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famous but rarely read opus (cf. Maugin and Metrikine 2010). In the period 1896–1914, the Cosserats in fact published together no less than 21 works in the field of theoretical mechanics. Out of these, 14 were short notes—of three or four pages—to the Paris Academy of Sciences. Apart from their book of 1909, the only long original memoir they published was a long paper in a true serial scientific journal in Toulouse in 1896 (Cosserat and Cosserat 1896), while their other publications in the field are scattered in odd places, often as supplements to lecture notes or books by other authors [Koenigs, Chwolson (in French translation), Appell, Voss (also in French translation)]. This paper of 1896 is the object of the present perusal. The originality of its contents is a discussed matter, whether the paper provides a nice overview of nineteenth century continuum mechanics or it does bring a new enriching viewpoint with specific traits of the brothers' talents and rigour, a positive appraisal certainly expressed by Truesdell on different occasions (cf. Truesdell 1952a; Truesdell and Toupin 1960). Our own opinion is that the Cosserats demonstrated a deep understanding of the bases of continuum mechanics and thus clarified many points, and they exhibited a style and ideas that were to bear fruits during the following 60 years or about.

## 2 About the Cosserats and Their Scientific Environment

In order to grasp the essentials of the Cosserats' personalities and achievements, we need to comprehend their scientific formation and to appraise the scientific environment they shared at a time that may schematically be called the "Belle Epoque" (roughly, 1880–1914). In that period the two most prestigious schools in France were the *Ecole Polytechnique* and the *Ecole Normale Supérieure* (ENS), both in Paris, and accessible only after a difficult competitive entrance examination. The former was destined to form engineers essentially for the needs of the State although the programme in mathematics was the highest possible with the best available teachers. To be fully trained in more engineering matters the best alumni from *Ecole Polytechnique* had to follow an "Ecole d'application" of which the most well-known one was the *Ecole Nationale des Ponts et Chaussées* (ENPC). Students who successfully completed their study in the two schools would become members of the elitist "Corps of Engineers of *Ponts et Chaussées*", one of the most desired titles in the French Third Republic. This opened the way to both technical and managerial positions at the highest level in the State or in private companies (e.g. the newly expanding railway companies). Notice that not much was said about universities (or rather faculties) which fell under the unique directorship of the Ministry of Education. Famous French scientists, physicists and mathematicians of the nineteenth century belonged to the Corps of *Ponts et Chaussées*, among them, Cauchy, Navier, Lamé, Duhamel, Coriolis, Clapeyron, Poncelet, Liouville, Arago and Barré de Saint-Venant. "Poor" Boussisnesq who "modestly" graduated from the University of Montpellier had a much harder work to achieve to reach the same stratospheric medium. Another prestigious school of application of *Polytechnique* was the National School of Mines in Paris. Henri Poincaré thus belonged to the "Corps of Engineers of Mines"—which in time became even more prestigious than the one of *Ponts et Chaussées*—although he devoted his whole life to mathematics and mathematical physics.

The *Ecole Normale Supérieure* was initially destined to form teachers for *Lycées*, i.e. secondary high schools educating students from age 12 to 18 with a final diploma called the "Baccalauréat" with a strong emphasis on classics. Then they could attempt a university or continue to prepare for the difficult examination entrance to *Polytechnique* and ENS. Very good students were admitted to both schools and selected the one that pleased them most. Under the influence of Louis Pasteur the ENS also became a "fish tank" for creative scientists who would soon join and then surpassed the polytechnicians.

François Cosserat (1852–1914), the elder of the two brothers, graduated from the Ecole Polytechnique and became a member of the Corps of *Ponts et Chaussées*. He had a professional career in the fast growing development of railways with the North and then the East companies of Railways in France. Eugène Cosserat (1866–1931), his cadet by 14 years, was educated in mathematics at the *Ecole Normale Supérieure* in Paris and became a professional (mathematical) astronomer with a career spent almost entirely in Toulouse in the south-west of France. As such he had to teach courses in analysis, astronomy and celestial mechanics, but he also had a marked interest in differential geometry already exhibited in his doctoral thesis.

We do not know what prompted the interest of the Cosserat brothers for rational mechanics and the theory of elasticity in particular. It may be the lectures received by François at both *Polytechnique* and ENPC and then the influence of this older brother on his cadet. The cooperation of the two brothers lasted from 1896 to the death of François in 1914. Anyway, they must have been bright students to start with and endowed with some easiness to grasp fundamental concepts and a gift to expand them as neither François nor Eugène were officially professional mathematicians in the field of mechanics. But they were enlightened amateurs with all technical abilities and a background of true professionals. Both became members of the Paris Academy of Sciences (François in 1896, and Eugène in 1919). François was even elected President of the French Society of Mathematics (*Société Mathématique de France*) in 1913 one year before his death. François was certainly confronted to the works of Adhémar Barré de Saint-Venant (1797–1886) and Joseph V. Boussinesq (1842–1929) at the ENPC. In his engineering curriculum he met with the works of his great predecessors, namely, Gabriel Lamé (1795–1870) and Alfred Clebsch (1833–1872), both authors of the first comprehensive treatises on elasticity (cf. Lamé 1852; with a tremendous expansion by Barré de Saint-Venant 1883 for the latter in French translation), and also Gustav Kirchhoff (1824–1887) in Kirchhoff (1852) and James Clerk Maxwell (1831–1873) (cf. Maxwell 1853). Eugène Cosserat defended his Sorbonne thesis in mathematics before a committee formed by Gaston Darboux (1842–1917), Paul Appell (1855–1930) and Gabriel Koenigs (1858–1931)—see Lebon (1910). This thesis on geometry was published in the *Annales* of the Faculty of Sciences of Toulouse in 1885. Darboux was the author of a formidable work—in four volumes—on the theory of surfaces and an ardent propagandist of the notion of *mobile triad* that was readily adopted by the Cosserats. Paul Appell became professor of rational mechanics at the Sorbonne in 1885 and, among many creative works, published an influential encyclopaedic treatise on rational mechanics (starting in 1893 with many augmented editions) and practically became the godfather of all mechanicians in France in the period of interest. Koenigs, a student of Darboux, became professor of mechanics at the Sorbonne while publishing a successful treatise on kinematics (Lessons of 1895–1897, Koenigs 1895, see also Lovett 1900). Both Darboux and Koenigs left a strong print on the Cosserats' work of 1896 as witnessed by the large number of citations to their books. Other French contemporaries of the two brothers were Henri Poincaré (1854–1912), Pierre Duhem (1861–1916), Marcel Brillouin (1854–1948), Emile Picard (1856–1941), Emile Jouguet (1871–1943), Jacques Hadamard (1865–1963), and Paul Painlevé (1863–1933), all educated at the ENS save Poincaré. Eugène Cosserat was very close to Hadamard and Painlevé. Contemporaries outside France were Woldemar Voigt (1850–1919), August Föppl (1854–1924), Hermann von Helmholtz (1821–1894), Georg Hamel (1887–1954), and Ludwig Boltzmann (1844–1906) in Germany, Josiah Willard Gibbs (1838–1905) in the USA, and William Thomson (1824–1907; *aka* Lord Kelvin), A.E.H. Love (1863–1940), and Lord Rayleigh (1842–1919) in the UK. What are really missing in the interactions with foreign scientists are any contacts with, and citations to, Italian mechanical engineers and mathematicians. The strangest fact is the lack of connection with Gabrio Piola (1794–1850), apparently eclipsed by Kirchhoff. In all, the scientific environment of the Cosserat brothers in Paris was stupendous, and they dutifully cited all scientists—that they studied in detail—at the proper place of their works with high accuracy. Hard working in such a rich environment and equipped with knowledge of the most influential foreign languages, the Cosserats were in a most favourable frame to develop their original views although their activity in rational mechanics was only an aside to their professional occupations. The result is all the more remarkable.

## 3 The Cosserats' Paper of 1896

***Preliminary remark***: In their general kinematic description the Cosserat brothers note the direct deformation $(x, y, z) \to (x_1, y_1, z_1)$ that we note $(X^K,\ K = 1, 2, 3) \to (x^i,\ i = 1, 2, 3)$ in modern indicial notation. They note $(u, v, w)$ the components of the displacement that we would note $(u^i, i = 1, 2, 3)$. The initial density they note $\rho$ and the final one $\rho_1$ while we shall use $\rho_0$ and $\rho$ for these two, respectively. We repeatedly use the convention of summation over repeated indices. The Cosserats do not use any vector or tensor notation and thus have to give all components explicitly, but we rewrote the main cited equations in the modern outlook to help the reader. We hope that this does not create any confusion, still always referring to the original equations of the Cosserats in their text where necessary (i.e. page number and equation number). They do not treat the dynamical case.

### *3.1 Deformations*

The Cosserats define finite deformation just like Green in his celebrated memoir of 1839 (Green 1839), but they emphasize the intimate link with the use of the theory of curvilinear coordinates. That is, to be unambiguous, their formulas (3) and (4) are none other than the modern formulas for the Cauchy–Green strain of material components $E_{KL}$ and for the finite deformation gradient $\boldsymbol{F}$ of components $F^i_{.K}$ such that

$$\begin{aligned} \mathrm{d}\boldsymbol{x} &= \boldsymbol{F}\,\mathrm{d}\boldsymbol{X}, \quad F = \left\{ F^i_{.K} = \frac{\partial x^i}{\partial X^K} \right\}, \\ \boldsymbol{E} &= \tfrac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \mathbf{1}\right) = \left\{ E_{KL} = \tfrac{1}{2}\left( F^i_{.K} F^j_{.L} g_{ij} - \delta_{KL} \right) \right\}. \end{aligned} \tag{1}$$

The (relative) strain measures here introduced have also been considered by Barré de Saint-Venant, Kirchhoff, Lord Kelvin (William Thomson) and Boussinesq. The six functions given by the elements of $E_{KL}$ cannot be completely arbitrary as they must verify a system of second-order partial differential equations (known as compatibility conditions; cf. Barré de Saint-Venant 1864). Whenever all $E_{KL}$'s vanish it means that the deformed configuration is deduced form the original one by a displacement "en bloc", combined or not combined with a symmetry transformation (cf. p. I.12). This is of fundamental importance because it defines what is understood by a *rigid-body motion*. After Lord Kelvin, a homogeneous deformation is one in which the $E_{KL}$'s are all constants or they vanish identically. This allows one to introduce analytically the simple form (homographic transformation) of homogeneous deformations (Eq. (5), p. I.13), particular cases being those of linear dilatations and angular dilatations. For a sufficiently small portion of the undeformed body about a point $P$, one can substitute a homogeneous deformation to the actual deformation at $P$. In this they essentially adopt the viewpoint of W. Thomson (Lord Kelvin) with so-called linear dilatations and angular dilatations as main constructive elements. Following the original work of Cauchy (1827) they pay special attention to the notions of ellipsoids of deformation, rotation at a point, and pure deformation. The first ellipsoid $E$ of deformation clearly corresponds to a transformation of an initially spherical form into an ellipsoid. Reciprocally, the second ellipsoid $E_1$ relates to the inverse relation between a sphere in the final configuration and an ellipsoid in the initial configuration. The three axes of $E$ can be brought parallel to those of $E_1$ by an appropriate *rotation*. The vanishing of such rotation corresponds to what Thomson and Tait (1867), p. 132, call a *pure* deformation. This combination of pure deformation and rotation (pp. I.19–I.25) materializes in what is called the *polar decomposition* (attributed to, but not proved by, Cauchy) of the deformation gradient—noted $\boldsymbol{F} = \boldsymbol{RU} = \boldsymbol{VR}$ in modern treatises (e.g. Truesdell and Toupin 1960). As noted in the modern formula the rotation can be effected first and pure deformation next, or in the other order but with a different pure deformation (in fact in a different space; cf. Footnote in p. I.20). In their geometric proof the Cosserats exploit the transformation of quadratic forms and the notion of principal axes of the involved ellipsoids. They also have to introduce the cubic dilatation and the Jacobian determinant—that they note $\Delta$—of the deformation, i.e. $J = \det \boldsymbol{F}$ in modern notation. These considerations lead them directly to introduce the invariants of deformation (p. I.26) and the useful notions of simple extension (stretch) and simple shear (pp. I.25–I.28). The usefulness of the notion of *simple shear*, e.g. (with coefficient $\gamma$ characterizing the amount of shear)

$$x = X + \gamma Y, \qquad y = Y, \qquad z = Z, \tag{2}$$

had particularly been emphasized by a certain Louis Vicat (1786–1861)—cf. Vicat (1833)—and above all Barré de Saint-Venant in his lectures of 1837–1838 (Barré de Saint-Venant 1837, 1838) at the School of *Ponts et Chaussées*—see also Brillouin (1891). As we know now, the notion of simple shear is often used as a test deformation in the characterization of nonlinear elastic responses for various materials with a priori prescribed strain energy.

Infinitesimally small deformations are correctly introduced (p. I.29) by the Cosserats with the help of an ordering parameter (that they note $t$) and expansion of the displacement components in integer powers of this parameter, assuming uniform convergence of the corresponding series and of those defined by their derivatives with respect to the initial coordinates. In this context, special cases are those of linear and angular "dilatations" (following the vocabulary introduced by Cauchy). Principal dilatations (or stretches) are those expanded along the axes of the second ellipsoid of deformation. In the case where both linear dilatations and relative shears vanish, then it is shown, following a method due to Darboux, that the displacement field is one of the rigid-body types that we can write in direct notation as

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{\omega} \times \boldsymbol{X} \tag{3}$$

where both $\boldsymbol{u}_0$ and $\boldsymbol{\omega}$ are translation and rotation of constant values.

Finally, the Cosserats (pp. I.35–I.37) recall the necessary and sufficient conditions that a system of six functions of coordinates must satisfy to be that of a symmetric deformation associated with an existing displacement. These conditions form a set of six second-order partial differential equations, an auxiliary system, now called the compatibility condition of Navier and Saint-Venant, but in fact introduced by Barré de Saint-Venant (1864) in his commented edition of Navier's lectures (cf. Navier 1864). Related works by Boussinesq (1871), Beltrami (1889), Love (1892) and Cesàro (1894) are cited in this context.

Globally, in this introduction to the deformation theory of continua, the Cosserats do not innovate so much but they faithfully incorporate all progress made since Cauchy till the work of their contemporaries (Poincaré, Darboux, Koenigs, Kelvin,...). Still, they cultivate this fruitful view that general deformations must be considered first, leaving infinitesimal deformations as infinitesimally small limits in a strict mathematical vision.

Truesdell (1952a), p. 53, however, notes that the Cosserats missed the long innovative paper of Finger (1894)—obviously very recent at the time of the Cosserats' publication—where Finger introduced the spatial strain measure named after him, i.e. $(\boldsymbol{c}^{-1})^i_{.j} = (\boldsymbol{F}\boldsymbol{F}^T)^i_{.j} = F^i_{.K} F^k_{.L} \delta_{kj} \delta^{KL}$, which would have made simple the formulation of elasticity constitutive equations for finite strain in isotropic bodies.

### *3.2 Internal Forces (stresses) in a Continuum*

The Cosserats do not elaborate much about the original introduction of the notion of *stresses* (according to the coinage of Rankine), i.e. more traditionally, *internal forces* in a continuum. They skip Cauchy's classical argument to introduce (p. I.39) the stress notion at a cut at a point in a body, simply remarking in passing that the cut is tangent to an infinity of curved surfaces, so that only the *normal* to the cut at a point is involved, and stresses (as we shall call them now) are forces per unit area in contrast to body forces that are *mass* forces. The Cosserats do not refer to these internal forces as "tensors" (following Voigt 1898 or, as if they had followed Gibbs 1881–1884) and others, "linear vector functions". But here, to facilitate the reading by modern students, we denote by the Cartesian tensor components $t^{j}_{.i}$ or $t^{ij}$ the stress in the *actual* (after deformation) configuration, and will avoid any direct (no indices) notation that could create some confusion.

Cauchy's equilibrium equations are stated as (cf. Eqs. (24) and (23) in pp. I.39–I.40) in the following traditional form:

$$\frac{\partial}{\partial x_i} t^{i}_{.j} + \rho f_j = 0 \tag{4}$$

at internal points in the body and

$$t_j = n_i \, t^{i}_{.j} \tag{5}$$

at its regular boundary of unit outward normal of components $n_i$. But the Cosserats have formulated the deformation theory essentially in the undeformed reference configuration (see preceding section). They thus want to reformulate Eqs. (4) and (5) in the appropriate framework, that is, per unit undeformed volume and unit undeformed area. They rightly think that the required manoeuvre must be analogous to what is done in hydrodynamics in passing from Euler to Lagrange equations. This is called a "pull back operation" in modern treatises, and this is in fact defined by the celebrated Piola transformation (Piola 1836), but the Cosserats refer only to Kirchhoff (1852) for this operation which they achieved astutely by associating to Eqs. (4) and (5) a form of the principle of virtual work and then effecting the required transformation in this global formulation (pp. I.42–I.48). Noting $\delta u^j$ the virtual displacement, one obtains thus the global expression (Eq. I.26)

$$\int_V \rho f_j \delta u^j \, \mathrm{d}V + \int_{\partial V} t_j \delta u^j da - \int_V t^{i}_{.j} \frac{\partial}{\partial x_i} \left(\delta u^j\right) \, \mathrm{d}V = 0. \tag{6}$$

On this occasion, the Cosserats remark on the definition of a virtual "rigidifying" deformation which cancels out the last expression in the left-hand side of Eq. (6). The lengthy transformation of (6) that we do not repeat yields the following expression of the principle of virtual work:

$$\int_{V_0} \rho_0 f_j \delta u^j \,\mathrm{d}V_0 + \int_{\partial V_0} T_j \,\delta u^j da_0 - \int_{V_0} S^{KL} \delta E_{KL} \,\mathrm{d}V_0 = 0, \tag{7}$$

assuming that $J = \det \boldsymbol{F}$ is everywhere positive and the continuity equation reads $\rho_0 = \rho J$. Here $\delta E_{KL}$ is the variation of the Cauchy–Green strain measure resulting from the variation $\delta u^j$, and $S^{KL}$ is the conjugate stress (now called the second Piola–Kirchhoff stress). The Cosserats are then able to transform (7) in the form

$$\int_{\partial V_0} \left(T_j - N_K T^K_{.j}\right) \delta u^j da_0 + \int_{V_0} \left( \frac{\partial}{\partial X^K} T^K_{.j} - \rho_0 f_j \right) \delta u^j \,\mathrm{d}V_0 = 0\,, \tag{8}$$

with the definition of the object $T^K_{.j}$ (now called the first Piola–Kirchhoff stress) given by (in our notation; cf. Eq. (36) in p. I.48)

$$T^K_{.j} = \frac{\partial J}{\partial F^i_{.K}} t^i_{.j} = J X^K_{,i} t^i_{.j}\,, \tag{9}$$

and $N_K$ denotes the components of the unit outward normal to the surface body in the undeformed configuration. The localisation of (8) provides the two equations (cf. Eqs. (34) and (35) in p. I.46)

$$\frac{\partial}{\partial X^K} T^K_{.j} + \rho_0 f_j = 0 \qquad \text{in } V_0, \tag{10}$$

$$T_j = N_K T^K_{.j} \qquad \text{at } \partial V_0. \tag{11}$$

Here, as emphasized by the Cosserats (top of p. I.47), the hybrid geometrical object $T^K_{.j}$ represents a force in the direction of the actual axis noted $i$, but per unit area in the undeformed configuration. Equations (10)–(11) were given by Marcel Brillouin (1884, 1885).

On using an identity established by Carl Neumann (1860),

$$\frac{\partial}{\partial X^K} \left( \frac{\partial J}{\partial F^i_{.K}} \right) = 0\,, \tag{12}$$

one can revert to the actual (Eulerian form of the) equation of equilibrium as proved by Boussinesq (1869) since with (9) and (10) one has

$$\frac{\partial}{\partial X^K} \left( t^i_{.j} \frac{\partial J}{\partial F^i_{.K}} \right) + \rho_0 f_j = 0. \tag{13}$$

But (see p. I.49)

$$\frac{\partial}{\partial X^K}\left(t^i_{.j}\frac{\partial J}{\partial F^i_{.K}}\right) = \frac{\partial J}{\partial F^i_{.K}}\ \frac{\partial}{\partial X^K}t^i_{.j}$$
$$= J\,\frac{\partial X^K}{\partial x^i}\ \frac{\partial}{\partial X^K}t^i_{.j} = J\frac{\partial}{\partial x^i}t^i_{.j} = \frac{\rho_0}{\rho}\ \frac{\partial}{\partial x^i}t^i_{.j}, \qquad (14)$$

whence Eq. (4). Note that the virtual work of external forces can be written as (cf. Eqs. (6) and (7) above):

$$\delta T_e = \int_V t^j_{.i}\frac{\partial}{\partial x^j}\delta u^i\,\mathrm{d}V = \int_{V_0} T^K_{.i}\frac{\partial}{\partial X^K}\delta x^i\,\mathrm{d}V_0. \qquad (15)$$

The Cosserats then discuss the notion of isostatic surfaces after Lamé, Boussinesq and Weingarten (1881), a subject matter that we skip here. In concluding their chapter II the Cosserats evoke the equilibrium equations (cf. p. I.58) in a straight cylinder (before deformation), i.e. a thin rod, and mention those that would be obtained in plates of any thickness loaded on their edge. These are the equations expanded by Clebsch and Barré de Saint-Venant (1883) in the French translation of the book of Clebsch (1883).

One has to wait for the next chapter to witness an introduction of elasticity constitutive equations on thermodynamic bases.

## 3.3 *Equations of Equilibrium*

In their formulation of the equations of equilibrium for *elastic* bodies (Chapter III), the Cosserat brothers are strongly influenced by the thermodynamic works of Kelvin (Thomson 1855, 1856, 1857); also (Thomson and Tait 1867), and the recent considerations brought to the field by Pierre Duhem (1887, 1894). That means that they exploit the formulation of the first and second laws of thermodynamics, respectively, then called the *principle of equivalence of heat and work* (with the symbol $E > 0$ standing for the so-called mechanical equivalent of heat, and ignored in modern texts with appropriate physical units) and the *principle of Carnot and Clausius*. For a body in its natural state (homogeneous and without deformation), one then considers homogeneous deformations from this natural state with the same absolute temperature $T$ for all material points. The state of this body after deformation from the natural state is defined by six strains and the temperature. It is assumed (p. I.60) that these seven parameters remain within acceptable limits so that any alteration of the body may be viewed as a continuous sequence of equilibrium states and it corresponds to a reversible evolution (following Duhem). In the sequence of these states the body is maintained in such states by the application of a unique system of external forces with the external bodies kept at the same temperature as the considered body. The two principles of thermodynamics then read:

$$E\,\delta Q + \delta T_e = \delta \sum \frac{mv^2}{2} \;+ \mathrm{d}U \tag{16}$$

and

$$E\,\delta Q - \delta \sum \frac{mv^2}{2} \;- E\,T\,\mathrm{d}S \;< 0. \tag{17}$$

Here $\delta Q$ is the quantity of heat received by the system during any elementary alteration, while both the external forces have achieved a work $\delta T_e$ and $\mathrm{d}U$ is an *exact* differential of a function $U$ called the *internal energy*. In Eq. (17) $\mathrm{d}S$ denotes an exact differential of a function $S$ called the entropy. Both functions $U$ and $S$ are *functions of state* that completely define the state of the system in terms of the seven introduced parameters (six deformations and temperature).

The writing of Eqs. (16) and (17) in which there simultaneously appear variations noted "$\delta$" and exact differentials noted "d" is particularly shocking to our modern eyes and was thus forcefully criticized by supporters of rational thermodynamics in the Truesdellian School in the 1960–1970s.

If one defines Duhem's thermodynamic potential (now called free energy or Helmholtz's potential) by

$$F = \; U - E\,S\,T, \tag{18}$$

one, on account of (16), can rewrite (17) in any of the following two forms (Eq. (53), p.I.61)

$$\mathrm{d}U - E\,T\,\mathrm{d}S - \delta T_e < 0 \qquad \text{or} \qquad \mathrm{d}F + E\,S\,\mathrm{d}T - \delta_e\,T_e < 0 \tag{19}$$

In the same conditions, for a truly *reversible* evolution Eq. (17) reduces to

$$\delta Q - T\,\mathrm{d}S = 0, \tag{20}$$

and this can be rewritten as

$$\mathrm{d}U - E\,T\,\mathrm{d}S - \delta T_e = 0 \qquad \text{or} \qquad \mathrm{d}F + E\,S\,\mathrm{d}\,T - \delta_e T_e = 0. \tag{21}$$

Alterations satisfying (17) or (19) are said to be "realizable". Those satisfying (6) are said to be "reversible" in the sense of Duhem (1894). Following also this last author, the equilibrium conditions for the body under the action of a prescribed system of forces are then thus established. These conditions are to be understood as corresponding to our notions of thermodynamic equilibrium (i.e. *thermostatics*) and the absence of dissipation of mechanical origin. Indeed, the first case considered where temperature is assumed to be known ($T = T_0$) yields (Eq. (56), p.I.62)

$$\frac{\partial F}{\partial T} = -E\,S. \tag{22}$$

While the thermodynamic state being described by means of "normal variables of state" (a concept due to Duhem which isolates entropy as a specific state variable among the seven variables $e_{ij} = e_{ji}$ and $S$), for all variations of the parameters one obtains (in modern notation; cf. Eq. in p. I.62)

$$\frac{\partial F}{\partial e_{ij}}\delta e_{ij} = \delta T_e, \tag{23}$$

where the left-hand side is none other than $\delta F$ computed at $T = T_0$.

However, if it is entropy that keeps a given value ($S = S_0$), then using the first of (21), we are led to the following results (Eqs. in p. I.63):

$$\frac{\partial U}{\partial S} = E\,T, \tag{24}$$

and

$$\frac{\partial U}{\partial e_{ij}}\ \delta e_{ij} = \ \delta T_e, \tag{25}$$

where $U$ is computed at $S = S_0$. Equations (22) and (23) on the one hand and (24) and (25) on the other characterize isothermal and adiabatic elasticity evolutions, respectively. We recognize in (22) and (24), the *thermostatic* definitions of entropy and temperature. The Cosserats call "energy of deformation" $W$—per unit volume of the undeformed configuration—either $F$ or $U$, the choice being made according to circumstances. This allows the authors to deduce the general form (in the manner of George Green) for the elastic constitutive equations, i.e. (cf. Eq. (59), p. I.64) but in modern notation

$$S^{KL} = \frac{\partial W}{\partial E_{KL}} \tag{26}$$

or (cf. Eq. 60, p. I.65)

$$T^{K}_{.i} = \frac{\partial W}{\partial F^{i}_{.K}}. \tag{27}$$

Here $S^{KL}$ and $T^{K}_{.i}$ are none other than the second and first Piola–Kirchhoff stresses but the Cosserats give no name to them. Constitutive Eq. (26) is sometimes called the Kelvin–Cosserat formulation, while (27) is referred to as Kirchhoff (1852) form. Going from (26) to (27) implies the use of the Piola transformation given by the Cosserats in their component equations (31) and (33)—pp. I.44–I.45—without mention of Piola but with due citation to Kirchhoff (1852). Furthermore, the Cauchy stress in the deformed configuration is then given in a form attributed to Boussinesq (1869) that we can rewrite in condensed form as

$$t^i_{.j} = J^{-1} F^i_{.K} \, T^K_{.j} = J^{-1} x^i_{,K} S^{KL} x^p_{,L} \, g_{jp} = J^{-1} x^i_{,K} \frac{\partial W}{\partial E_{KL}} x^p_{,L} \, g_{pj}. \tag{28}$$

This, obviously, is not reported in this tensorial form, but the Cosserats give only the form taken in full by two of the components of $t^i_{.j}$ (cf. Eq. (61), or (62), p. I.65) indicating that other components are easily deduced.

In the above-specified conditions the mechanical equilibrium equations are obtained as (cf. Eq. (63)–(64), p. I.66)

$$\frac{\partial}{\partial X^K} \left( \frac{\partial W}{\partial F^i_{.K}} \right) + \rho_0 f_i = 0 \tag{29}$$

at internal points in the body and

$$T_i = N_K \frac{\partial W}{\partial F^i_{.K}} \tag{30}$$

at its regular boundary of unit outward pointing normal of components $N_K$ in the undeformed configuration.

In the rest of this chapter, the Cosserats deal with various matters that include a "paradox" previously dealt with by Poincaré, Kirchhoff and others, notions on stability, the choice of a natural state, the question of material symmetry, and the case of infinitesimal deformations. The paradox referred to by the Cosserats concerns the possible a priori existence of a function $\Phi$ of the gradient components $F^i_{.K}$ such that

$$\int_V \delta\Phi \, \mathrm{d}V - \delta T_e = 0. \tag{31}$$

This means that for an equilibrium position one must have Eqs. (27) and (30) with $W$ replaced by $\Phi$, so that, for any part of the body, Eq. (27) must be written with $W$ replaced by $\Phi$. But the quantities $F^i_{.K}$ cannot be taken arbitrarily as they must obey a set of three partial differential equations (Eq. (37) in p. I.49) of which the general integral is an arbitrary function of the six components $E_{KL}$ of the finite deformation. This is the requirement for Eq. (31) to be compatible with the existence of internal forces. This was noticed by Poincaré in his lectures on elasticity (Poincaré 1892, p. 77) but also by Kirchhoff (1852), C. Neumann (1860), and closer to the Cosserats by Cellérier (1893).

The second remark relates to the stability of equilibrium and the notions of "bifurcation" equilibrium and "limit" equilibrium of Poincaré. We must recall that the years 1890s are fruitful as regards questions of stability. This is particularly true of the works of Henri Poincaré with his marked interest in the stability of liquid masses in rotation, a subject also of interest to Paul Appell (1888) in his treatise on rational mechanics, and the original work by Aleksandr Lyapunov (1857–1918) with his Doctoral thesis (in Russian) on "The general problem of stability of motion" at Kharkov, Ukraine (Love 1892). Although the Cosserats had some knowledge of Russian, Lyapunov's work came too late to influence them, but will influence Pierre Duhem when the latter will have identified a potential akin to a Lyapunov function. Thus, the Cosserats are mostly influenced by Poincaré and his considerations on stability in his lectures on elasticity (Poincaré 1892), Chapters III and IV). Along this line, one first notes that in the absence of external forces, the function $W - T_e$ reduces to $W$. If the latter is minimum at the natural state, then one can only say that the corresponding equilibrium is stable only in so far as deformations are concerned (but it is not stable in a general way). But, now, if the externally applied forces vary in a continuous way depending on a parameter $y$, assuming that $T_e$ exits for all values of $y$, then one is led to a situation identical to that envisaged by Poincaré in his study of the equilibrium of a fluid mass in rotation (Poincaré 1885), so that one has to consider Poincaré's notions of "bifurcation" and "limit" equilibria (cf. Cosserats, p.I.69) and to imagine a linear series of equilibrium forms that correspond to a series of real values of $y$ related to the critical points of $y$ functions defined by the system of equilibrium equations. That is all for this remark.

The next remark relates to the choice of a *natural* state. In the absence of external loading, one admits the existence of a natural state that corresponds to a vanishing of the derivatives of the function $W$ with respect to the strain components. We can assume that $W$ can be expanded in the positive entire powers of the strain components, providing thus an expression of the type (Cosserats, Eq. (67), p. I.70)

$$W = W_2 + W_3 + \cdots , \tag{32}$$

where $W_k$ denotes a homogeneous polynomial of degree $k$, assuming that the constant term has been set equal to zero without loss in generality. For a natural state corresponding to a stable equilibrium from the point of view of deformations (see above), it is sufficient that $W$ be positive for all infinitesimally small components of the strain. This classically yields the definite positiveness of the quadratic form $W_2$. Following more generally Poincaré (1892), Sects. 27 through 30, one can assume that there exists a first-order contribution $W_1$ so that 27 elasticity coefficients—at most—will be defined from $W_1$ and $W_2$ in the absence of any specific symmetry. The Cosserats then turn to the special case of *isotropy* for a homogeneous body. Invoking the traditional three invariants of strains, $W_1$ contains only one coefficient $\nu$ while $W_2$ contains the famous two Lamé coefficients, $\lambda$ and $\mu$, which have to satisfy the inequality

$$3\lambda + 2\mu > 0, \quad \mu > 0, \tag{33}$$

to warrant stability about a natural state (for which $\nu = 0$).

The chapter concludes with the formulation of equations in the case of infinitesimal deformations. This brief analysis (pp. I.72–I.74) introduces an order parameter noted $t$ by the Cosserats; this leads to an expansion of the displacement field in successive positive integer powers of $t$. This is also the case of the function $W$. This follows considerations of Darboux in his theory of surfaces (Darboux 1887–1896, Vol. 4, p. 65 on) and Poincaré in his general approach to problems of mathematical physics (cf. Poincaré 1894). The study of $W_2$ in fact follows the developments offered by Poincaré (1892), pp. 46–58, that we shall not repeat. A theorem due to Kirchhoff (1852) applies when forces vanish. Finally, the standard equilibrium equations are deduced for infinitesimal strains (Cosserats, Eq. (80), p.I.77) rewritten in modern intrinsic notation for the isotropic case as

$$(\lambda + \mu)\, \nabla\theta + \mu\, \Delta \boldsymbol{u} + \rho \boldsymbol{f} = \boldsymbol{0}, \tag{34}$$

where $\theta = \nabla \cdot \boldsymbol{u}$ denotes the dilatation.

### *3.4 On Curvilinear Coordinates*

The long and final chapter IV must have been welcomed by most readers when the paper was published. It deals with the basic problem of the formulation of the equations of elasticity in curvilinear coordinates. This was approached by pioneers such as Lamé and Beltrami and other scientists before the advent of tensor calculus. But the Cosserat brothers are still living in a period where vector calculus still is in development and is rarely applied (see Crowe 1967, for a historical perspective) and tensor analysis is still in infancy with no clear-cut application but for the notion of tensor introduced by W. Voigt (1898) and that of dyadic by J.W. Gibbs (1881–1884). What the Cosserats propose is to implement the theory of the *mobile triad* introduced by G. Darboux in his general studies of surfaces (Darboux 1887–1896). This is not so surprising since this theory is in full blossom and "é la mode" in these years 1890s. Furthermore, Eugène Cosserat was a disciple of Darboux, who in fact belonged to his Doctoral thesis committee. The main point in this approach is the consideration of a displacement field that depends on three independent parameters (noted $\rho_i$ , $i = 1, 2, 3$ by the Cosserats,) and the important role played by rotations. Then one first envisages the case where the mobile system has a fixed point (it can only rotate). But the interesting case for continuum mechanics is one where the mobile triad of three rectangular axes moves in any way through space so that nine new entities (related to translation) have to be adjoined to the nine rotation parameters (director cosines). In all this is equivalent to a single motion but observed in different systems of axes. Advance in the theory (p. I.83) is made by following Gauss (1827) (in 2D) and Lamé (1859) (in 3D) in exploiting the geometric representation of a system of curvilinear coordinates by considering three families of surfaces and looking for the expression of an arc of any curve traced in space in terms of the $\rho_i$'s, with a drastic simplification if the curvilinear coordinates are rectangular. The change in surface element is evaluated in the same conditions. The consideration of a *reference* mobile triad is emphasized (cf. p. I.65). This allows one to deal with geometric questions related to surfaces and curves traced in space (problem of conjugated tangents on one of the surfaces $\rho_i = \text{const.}$ with $i$ fixed, or the problem of establishing the differential equation for curvature lines). This looks very much like exercises given in the past to students in competition for admission to *Grandes Ecoles*. But for applications to continuum mechanics in 3D, one must focus on kinematic formulas where parameters $\rho_i$ are none other than the original orthogonal coordinates (noted $x$, $y$, $z$ by the Cosserats, but simply $X^K$, $K = 1, 2, 3$ in modern indicial notation). Then translations are given by the displacement. The latter has to be projected on the mobile triad, and the strain components can be expressed in terms of this projection (cf. Eq. (98) in p. I.91). External applied forces also are reported to the mobile axes. In the end one can write down the equations of equilibrium in this framework (See Eq. (100), p. I.92). The result is a set of partial differential equations satisfied by both translations and rotations, the knowledge of which is intimately related to the triple system of surfaces in which the primitive rectangular coordinate planes have been transformed. Then the Cosserats specialize to the case of infinitesimally small deformations with corresponding expansions of various quantities in the already introduced ordering parameter $t$, resulting in fact in expressions already given by Beltrami and Barré de Saint-Venant. These are given by Eq. (I.103) for a simple natural state and an isotropic elastic body. The Cosserats mention that the case of thin straight rods and thin plates would be treated in the like manner, but the corresponding elaboration is postponed to further works. The more general case where the body before deformation is reported to an arbitrary triple system of surfaces (with parameters $\rho_i$) is then lengthily exposed in the rest of the chapter together with equilibrium equations relative to the deformed body. This is achieved with the help of the principle of virtual work (Eq. (32) in p. I.101) for both finite and infinitesimally small deformations with, we must say, rather atrocious equations in terms of the $\rho_i$ 's (for illustration, see Eqs. (116)–(117) in p. I.103 on).

In all, the contents of this chapter IV seem a bit obsolete to our modern eyes used to reasoning with tensors. But in the circumstances of the period where both vector and tensor analyses are not yet sufficiently developed and/or applied, the Cosserats' efforts are certainly justified in spite of the obvious laborious feeling that we gather from them and the somewhat old-fashioned geometric character that permeates them. These may not have been felt as such by the contemporaries of the Cosserats.

## 4 Summary and Conclusion

This brief perusal of the long paper published by the Cosserat brothers in 1896 brings us to the following general remarks and conclusion. First, the very length and detail of the paper lean towards an interpretation of this paper as an aborted series of lectures on a field of marked interest at the period. Indeed, the first chapters of the opus support this interpretation, especially in the theory of deformations. However, the detailed and accurate description with appropriate references reveals a typical trait of the brothers' style. They are clearly mathematical and, in spite of their professions, pay little attention, if any, to applications such as the strength of materials. This is borne out by the primal consideration of finite deformations, small ones being only viewed as perturbations. The original reference can only be one to the great Cauchy and his memoir of 1827 and the notion of ellipsoids of deformation. But then there are unavoidable references to more recent works, in particular by G. Green, Kirchhoff, William Thomson (Kelvin), Barré de Saint-Venant, and Boussinesq. They have clearly benefited from Poincaré's lectures (Poincaré 1892). Often enriched by astute remarks, this part is an excellent compuscus of the abstract level of description reached in the 1890s without the use of tensor analysis. With the consideration of the notion of internal forces (stresses), the Cosserats are in the main stream of the approach to continuum mechanics in the second half of the nineteenth century. Cauchy is only noted in passing while the Cosserats favour the approach advocated by Kirchhoff (1852), apparently one of their favourite sources, but also Clebsch (1883) as revised and augmented by Barré de Saint-Venant. Strangely enough, they never cite Gabrio Piola, who is now considered a precursor of Kirchhoff and a missing link between the 1820 and the 1850s. The constitutive theory for (finite strain) elasticity is fully thermodynamic with a strong influence of G. Green, Kelvin and the then new rising star in phenomenological physics, Pierre Duhem. The Cosserats kept very much aware of any recent developments in the 1880s–1890s. What is more surprising to modern readers is the frequent reference to the lectures of Poincaré on elasticity. Of course Poincaré is the acknowledged genius of the time and it seems quite natural to pay him the respectful dues he deserves. But what is less known is the nice critical view of elasticity that Poincaré offered in his lectures of 1892 (in fact redacted in a rather student style by two of the auditors; we have examined this point in Maugin 2016). As a never tired inquisitive "student" of all what was currently developed in mathematical physics, he applied in these lectures his usual dexterity and easiness in grasping the totality of a field in a short time with spot on critical comments, and this proved much useful to the brothers in their own analysis, including original considerations on stability.

In all, the Cosserat brothers seem to have been strongly influenced by their own formation, through the teaching at the School of *Ponts et Chaussées* and reference to the lectures of other great renowned past members of this Corps of engineers for the oldest brother, François, and through the works of Darboux and Koenigs for Eugène. This last influence is particularly felt in their last chapter IV on curvilinear coordinates. Their memoir is rather lengthy and one may wonder about its place of publication, in a little publicized journal, the *Annales* of the Faculty of Sciences in Toulouse—where Eugène Cosserat (1885) had already published the full text of his doctoral thesis. Although the Cosserats had a rather unpredictable policy of publication (clearly they were not preoccupied by matters of publication index and impact factor!) one explanation may be that since Eugène was teaching in Toulouse he may have felt a duty to publish something in the local *Annales* and the brothers used this opportunity to publish an unusually long memoir that could have been welcomed in a more known scientific periodical such as the *Journal de Mathématiques Pures et Appliquées* or the *Annales of the Ecole Normale Supérieure.*

Then the question remains of what was the influence of the Cosserats paper immediately and much later on. We have noticed in other studies that this memoir was dutifully cited by Pierre Duhem, Paul Appell and Ernst Hellinger who may be considered contemporaries of the brothers. It was dutifully cited by the most famous authors on finite-strain elasticity in the transitional period of the 1920–1930s, e.g. L. Brillouin, B.R. Seth, F.D. Murnaghan and A. Signorini. The most emblematic work of the period was by Murnaghan (1937). As a matter of fact, perhaps with a nasty will to belittle his work, Truesdell (1952b) claims that this work by Murnaghan was essentially a rewriting of the Cosserats' work of 1896 in the form of tensors. It was indeed Truesdell (1952a, 1984), pp.148–150, who revived this work as well as those of other scientists of the nineteenth century in his historical review. This was incorporated in the Truesdell–Toupin encyclopaedic article of the *Handbuch der Physik* (Truesdell and Toupin 1960) with now full reference to both Kirchhoff *and* Piola. From then on direct reference to the Cosserats' paper of 1896 became extremely rare, having become part of the accepted history of the field.

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