Prestressed Orthotropic Material Containing an Elliptical Hole

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Abstract Based on the representation of the incremental stress fields by complex potentials and conformal mapping technique, the fundamental solutions for an unbounded, homogeneous, orthotropic elastic body containing an elliptical hole subjected to uniform remote loads are determined. The orthotropic body is under by uniform remote tensile, tangential, and antiplane shear loads—cases corresponding to Mode I, Mode II, and Mode III of fracture. The solutions are obtained in a compact and elementary form.

1 Introduction

The problem of an isotropic body with an elliptical hole was studied by many authors using Kolosov–Muskhelishvili's complex potentials Muskhelishvili (1953)–Bertoldi et al. (2007) or the integral transform method Singh et al. (2012). In what follows our results Craciun and Soós (2006), Craciun and Barbu (2015) for a prestressed elastic composite material under by uniform distributed remote loads are presented and extended. To get the complex potentials describing the incremental stress and displacement fields, $\Psi_1 = \Psi_1(z_1)$ and $\Psi_2 = \Psi_2(z_2)$ for the plane problem, and $\Psi_3 = \Psi_3(z_3)$ for the antiplane problem, a technique based on the conformal mapping of the exterior of the elliptical hole in the planes on the exterior of the unit circle is used. The unknown potentials are represented by two Laurent series in the complex planes and their coefficients are determined from the boundary conditions. The compact closed-form analytical solutions, i.e., the complex potentials, of the considered boundary value problems for an unbounded, homogeneous, prestressed orthotropic elastic composite with an elliptic hole are obtained.

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2 Representation of the Incremental Stress Fields

The representation of elastic fields by complex potentials in the classical case of anisotropic elastic bodies was given by Lekhnitski (1963). This representation was used, for instance, by Sih and Leibowitz (1968) to analyze problems concerning the existence of a crack in an anisotropic elastic solid. The results obtained by Lekhnitski (1963) were generalized for the case of a prestressed material by Guz (1983), who also has analyzed the influence of the initial applied stresses on the behavior of a solid body containing cracks. Guz's representation of the incremental stress fields by complex potentials Guz (1983), Cristescu et al. (2004) is presented. In what follows it is considered an elastic composite with an elliptical hole prestressed with the initial applied stress $\overset{\circ}{\sigma}_{11}$, in the direction of Ox_1 axis, i.e., along the major semi-axis of the ellipse. The initial deformed equilibrium configuration of the body is assumed to be homogeneous and locally stable. The paper starts with representation of the incremental stress fields corresponding to the antiplane state, by a single complex potential $\Psi_3 = \Psi_3(z_3)$ depending on the complex variable $z_3 = x_1 + \mu_3 x_2$. The complex parameter μ_3 is the root of the characteristic equation of the differential equilibrium equation and has the following form, see Guz (1983)–Cristescu et al. (2004):

$$\mu_3 = \frac{1}{\omega_{2332}}\left[-\omega_{1332} + i\sqrt{\omega_{1331}\omega_{2332} - \omega_{1332}\omega_{2331}}\right], \tag{1}$$

where ω_{klmn} $(k, l, m, n = 1, 2, 3)$ are the instantaneous elasticities of the material in its free reference configuration and can be expressed through engineering constants of the composite and initial applied stress $\overset{\circ}{\sigma}_{11}$ and i denotes the imaginary unit, see Cristescu et al. (2004).

Taking into account the antiplane state relative to the plane x_1x_2 the instantaneous elasticities of the material have to satisfy the following restrictions:

$$\sqrt{\omega_{1331}\omega_{2332} - \omega_{1332}\omega_{2331}} > 0, \omega_{2332} > 0. \tag{2}$$

The corresponding components θ_{13} and θ_{23} of the nominal stress are then given by

$$\theta_{13} = 2Re\{q\Psi_3(z_3)\}, \theta_{23} = 2Re\{\Psi_3(z_3)\}, q = \frac{\rho_1}{\rho_2},$$

$$\rho_1 = \omega_{1331} + \mu_3\omega_{1323}, \rho_2 = \omega_{2313} + \mu_3\omega_{2323}. \tag{3}$$

It is assumed that the initial deformed composite material is in *plane state* relative to the x_1x_2 plane.

The representation of the incremental stress fields by two arbitrary analytical complex potentials $\Psi_j = \Psi_j(z_j)$, $j = 1, 2$ has the following form:

$$\theta_{22} = 2Re\{\Psi_1(z_1) + \Psi_2(z_2)\}, \tag{4}$$

$$\theta_{21} = -2Re\left\{a_1\mu_1\Psi_1(z_1) + a_2\mu_2\Psi_2(z_2)\right\}, \tag{5}$$

$$a_j = \frac{\omega_{2112}\omega_{1122}\mu_j^2 - \omega_{1111}\omega_{1212}}{B_j\mu_j^2}, \tag{6}$$

$$\theta_{12} = -2Re\left\{\mu_1\Psi_1(z_1) + \mu_2\Psi_2(z_2)\right\}, \tag{7}$$

$$\theta_{11} = 2Re\left\{a_1\mu_1^2\Psi_1(z_1) + a_2\mu_2^2\Psi_2(z_2)\right\}, \tag{8}$$

$$B_j = \omega_{2222}\omega_{2112}\mu_j^2 + \omega_{1111}\omega_{2222} - \omega_{1122}(\omega_{1122} + \omega_{1212}), \tag{9}$$

where μ_1 and μ_2 are the roots of characteristic equation of equilibrium, see Cristescu et al. (2004). The instantaneous elasticities of the material ω_{klmn} $(k, l, m, n = 1, 2)$ can be expressed through engineering constants of the composite and initial applied stress $\overset{\circ}{\sigma}_{11}$, by the following relations, see Cristescu et al. (2004):

$$\omega_{1111} = \frac{1 - \nu_{23}\nu_{32}}{E_2E_3H} + \overset{\circ}{\sigma}_{11}, \quad \omega_{2222} = \frac{1 - \nu_{13}\nu_{31}}{E_1E_3H},$$

$$\omega_{1122} = \frac{\nu_{12} + \nu_{32}\nu_{13}}{E_1E_3H},$$

$$\omega_{1212} = \omega_{1221} = \omega_{2112} = G_{12},$$

with

$$H = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{21}\nu_{32}\nu_{13} - \nu_{12}\nu_{23}\nu_{31}}{E_1E_2E_3}.$$

In these relations E_1, E_2, E_3 are Young's moduli in the corresponding symmetry directions of the material, G_{12} is the shear modulus in the symmetry plane Ox_1x_2 and ν_{12}, ..., ν_{32} are the Poisson's ratios.

Also, for an orthotropic material the roots μ_1 and μ_2 usually are not equal. In what follows the case of non-equal roots is considered

$$\mu_1 \neq \mu_2.$$

3 Antiplane State

In this section the plane problem of antiplane shear loads, i.e., the case corresponding to the third mode of fracture, is studied. Let us consider an unbounded, homogeneous, prestressed orthotropic elastic composite containing an elliptical hole which is acted by an antiplane constant shear load $\tau > 0$ in the direction of the x_3 axis at large distances. The boundary of the elliptical hole is free from stresses.

Let us write the boundary conditions corresponding to the mechanical problem:

$$\lim_{|z_3|\to\infty} \theta_{13}(z_3) = 0, \quad \lim_{|z_3|\to\infty} \theta_{23}(z_3) = \tau > 0,$$

$$n_1\theta_{13}(z_3) + n_2\theta_{23}(z_3) = 0, \tag{10}$$

on the hole boundary, where n_1 and n_2 are the components of the unit exterior normal to the boundary.

In order to find the complex potential $\Psi_3 = \Psi_3(z_3)$ is considered the *conformal mapping* of the exterior of the elliptical hole onto the exterior of the unit circle, having the form

$$z = x_1 + ix_2 = \frac{a+b}{2}\zeta + \frac{a-b}{2}\frac{1}{\zeta}$$

$$z_3 = x_1 + \mu_3 x_2 = \frac{a - i\mu_3 b}{2}\zeta_3 + \frac{b + i\mu_3 b}{2}\frac{1}{\zeta_3}. \tag{11}$$

The inverse mapping is given by

$$\zeta = \frac{z + \sqrt{z^2 - a^2 + b^2}}{a+b}, \qquad \zeta_3 = \frac{z_3 + \sqrt{z_3^2 - a^2 - \mu_3 b^2}}{a - i\mu_3 b}. \tag{12}$$

Let $a, b, b \leq a$ be the two semi-axis of the elliptical hole and if $b \to 0$ the considered hole obviously becomes the mathematical model of an usual, classical Griffith-Irwin crack given by a segment of length $2a$.

Let us introduce now the complex potential $\Psi_3(\zeta_3)$ through the relation

$$\Psi_3(\zeta_3) = \Psi_3(z_3(\zeta_3)), \tag{13}$$

where for simplicity it is used the same notation Ψ_3 for the complex potential depending on z_3 or on ζ_3. The boundary conditions (10) by means of the potential $\Psi_3(\zeta_3)$ and the mapping formula (11) become

$$\lim_{|z_3|\to\infty} \theta_{13}(\zeta_3) = 0, \quad \lim_{|z_3|\to\infty} \theta_{23}(\zeta_3) = \tau \tag{14}$$

and

$$(\cos\theta)\theta_{13}(\zeta_3) + (\sin\theta)\theta_{23}(\zeta_3) = 0, \quad \zeta_3 = e^{i\theta}, 0 \leq \theta \leq 2\pi, \tag{15}$$

where $(\cos\theta, \ \sin\theta)$ is the unit exterior to the unit circle in the complex plane ζ_3, and θ is obviously the angle between this normal and the x_1 axis.

The complex potential $\Psi_3 = \Psi_3(\zeta_3)$ is an analytic function on the exterior of the unit circle, and thus we may write it as

$$\Psi_3(\zeta_3) = A_0 + \sum_{m=1}^{\infty} A_m \zeta_3^{-m}, \tag{16}$$

where $A_0,\ A_1,\ A_2, \ldots$ are unknown complex constants to be determined from the boundary conditions. The second boundary condition from (10) at large distance from the hole leads to the following restrictions on A_0:

$$qA_0 + \bar{q}\bar{A}_0 = 0,\ A_0 + \bar{A}_0 = \tau \tag{17}$$

therefore,

$$A_0 = -\tau \frac{\bar{q}}{q - \bar{q}} = \frac{\tau \bar{q} i}{2r_2},\ q = r_1 + ir_2. \tag{18}$$

The third boundary condition from (10) imposes an additional restriction on the coefficients of the potential $\Psi_3(z_3)$

$$\begin{aligned}
&(q+i)A_1 + (\bar{q}-i)\bar{A}_1 + (q+i)A_2 e^{-i\theta} + (\overline{q}-i)\ \overline{A}_2\ e^{i\theta} \\
&\quad + \sum_{m=2}^{\infty}[(q+i)A_{m+1} + (q-i)A_{m-1}]e^{-im\theta} + \sum_{m=2}^{\infty}[(\bar{q}-i)\bar{A}_{m+1} + (\bar{q}+i)\bar{A}_{m-1}]e^{im\theta} \\
&= -A_0[(q+i)e^{i\theta} + (q-i)e^{-i\theta}] - \bar{A}_0[(\bar{q}+i)e^{i\theta} + (\bar{q}-i)e^{-i\theta}],
\end{aligned} \tag{19}$$

for $0 \leq \theta \leq 2\pi$. Condition (19) is fulfilled if and only if the following relations are satisfied:

$$(q+i)A_1 + (\bar{q}-i)\bar{A}_1 = 0 \tag{20}$$

$$\begin{aligned}
(q+i)A_2 &= -[A_0(q-i) + \bar{A}_0(\bar{q}-i)], \\
(\bar{q}-i)\bar{A}_2 &= -[A_0(q+i) + \bar{A}_0(\bar{q}+i)]
\end{aligned} \tag{21}$$

$$A_{1+2m} = s^m A_1,\ A_{2+2m} = s^m A_2,\ m = 1, 2, 3... \tag{22}$$

with

$$s = -\frac{q-i}{q+i}. \tag{23}$$

Equation (21) becomes

$$A_2 = -\frac{A_0\,(q-i) + \overline{A}_0\,(\overline{q}-i)}{q+i} \tag{24}$$

and taking into account (24) the expression of the constant A_2 has the following form:

$$A_2 = i\frac{\tau}{q+i}. \tag{25}$$

Let γ and δ denote the real and the imaginary part of A_1, respectively, i.e.,

$$A_1 = \gamma + i\delta. \tag{26}$$

Now, from (20) we get the following value for A_1:

$$A_1 = \gamma\left(1 + i\frac{r_1}{r_2 + 1}\right) \tag{27}$$

with

$$q = r_1 + ir_2. \tag{28}$$

Let us remark that the real number γ remains undetermined in the expression (27) of A_1. This is not an unexpected result since we have a boundary value problem in stress where such indetermination generally occurs. After some laborious manipulations, using (22) into (14) the *final* form of the complex potential $\Psi_3(\zeta_3)$ has the following form:

$$\Psi_3(\zeta_3) = A_0 + \frac{A_1\zeta_3 + A_2}{\zeta_3^2 - s}. \tag{29}$$

The basic complex potential $\Psi_3(z_3)$ may then be obtained by introducing the expression of ζ_3 given by (12) into the right-hand side of (29) and the problem is completely solved.

4 Plane State

In this section the plane problem of a uniform distributed remote tensile load, i.e., the case corresponding to the first opening mode of fracture, is studied. Let us consider an unbounded, homogeneous, prestressed orthotropic elastic composite containing an elliptical hole which is acted by a uniform constant normal tensile load $p > 0$ in the direction of the x_2 axis at large distances. The boundary of the elliptical hole is free from stresses.

Let us write the boundary conditions corresponding to our mechanical problem:

$$\lim_{|z|\to\infty} \theta_{11}(z) = \lim_{|z|\to\infty} \theta_{12}(z) = \lim_{|z|\to\infty} \theta_{21}(z) = 0, \quad \lim_{|z|\to\infty} \theta_{22}(z) = p > 0, \tag{30}$$

$$n_1\theta_{11}(z) + n_2\theta_{21}(z) = 0, \;\; n_1\theta_{12}(z) + n_2\theta_{22}(z) = 0,$$

on the hole boundary, where n_1 and n_2 are the components of the unit exterior normal to the boundary.

The complex potentials $\Psi_j = \Psi_j\left(z_j\right)$, $j = 1, 2$, must be determined not in the region of the infinite prestressed orthotropic plate with an elliptical hole, denoted by

S, but in the regions S_j, $j = 1, 2$ obtained from S by the affine transformations:

$$x_1^j = x_1 + \alpha_j x_2,\ x_2^j = \beta_j x_2,\ j = 1, 2. \tag{31}$$

The regions S_j are also planes with elliptical holes whose contours are given by the equations

$$x_1^j = a\cos\theta + \alpha_j b\sin\theta,\ x_2^j = \beta_j \sin\theta,\ 0 \le \theta \le 2\pi,\ j = 1, 2. \tag{32}$$

The following conformal mapping of the regions S, S_1, and S_2 onto the exterior of the unit circle is used:

$$z = x_1 + ix_2 = \frac{a+b}{2}\zeta + \frac{a-b}{2}\frac{1}{\zeta},$$

$$z_j = x_1 + \mu_j x_2 = \frac{a - i\mu_j b}{2}\zeta_j + \frac{a + i\mu_j b}{2}\frac{1}{\zeta_j},\ j = 1, 2. \tag{33}$$

The inverse mapping is given by

$$\zeta = \frac{z + \sqrt{z^2 - a^2 + b^2}}{a+b}, \zeta_j = \frac{z_j + \sqrt{z_j^2 - a^2 - \mu_j b^2}}{a - i\mu_j b},\ j = 1, 2. \tag{34}$$

When the x_1 and x_2 are running along the contour of the ellipse taking the values $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$, the functions defined by the (34) take the values $\zeta = \zeta_1 = \zeta_2 = e^{i\theta}$.

Let us introduce now the complex potentials $\Psi_j\left(\zeta_j\right)$ through the relations $\Psi_j\left(\zeta_j\right) = \Psi_j\left(z_j\left(\zeta_j\right)\right),\ j = 1, 2,$ where for simplicity we use the same notation Ψ_j for the complex potentials depending on z_j or on ζ_j. The boundary conditions (30) by means of the potential $\Psi_j\left(\zeta_j\right)$ and the mapping formulae (33) become

$$\lim_{|\zeta|\to\infty} \theta_{11}(\zeta) = \lim_{|\zeta|\to\infty} \theta_{12}(\zeta) = \lim_{|\zeta|\to\infty} \theta_{21}(\zeta) = 0, \quad \lim_{|\zeta|\to\infty} \theta_{22}(\zeta) = p > 0, \tag{35}$$

$$(\cos\theta)\theta_{11}(\zeta) + (\sin\theta)\theta_{21}(\zeta) = 0,\ \ (\cos\theta)\theta_{12}(\zeta) + (\sin\theta)\theta_{22}(\zeta) = 0, \tag{36}$$

on the unit circle $\zeta = e^{i\theta}, 0 \le \theta < 2\pi$, where $(\cos\theta,\ \sin\theta)$ is the unit exterior normal to the unit circle in the complex plane ζ, and θ is obviously the angle between this normal and the x_1 axis.

The complex potentials $\Psi_j = \Psi_j\left(\zeta_j\right)$ $j = 1, 2$ are analytical functions on the exterior of the unit circle, i.e.,

$$\Psi_1(\zeta_1) = A_0 + \sum_{m=1}^{\infty} A_m \zeta_1^{-m},$$

$$\Psi_2(\zeta_2) = B_0 + \sum_{m=1}^{\infty} B_m \zeta_2^{-m}, \tag{37}$$

where A_k and $B_k,\ k = 0, 1, \ldots$ are unknown complex constants to be determined from the boundary conditions.

Using Eqs. (35)–(37) the following restrictions on $A_0, \overline{A}_0, B_0$ and $\overline{B}_0$ are imposed:

$$A_0 + \overline{A}_0 + B_0 + \overline{B}_0 = p, \tag{38}$$

$$\begin{aligned}
&\mu_1 A_0 + \overline{\mu}_1 \overline{A}_0 + \mu_2 B_0 + \overline{\mu}_2 \overline{B}_0 = 0,\\
&a_1\mu_1 A_0 + \overline{a}_1\overline{\mu}_1 \overline{A}_0 + a_2\mu_2 B_0 + \overline{a}_2\overline{\mu}_2 \overline{B}_0 = 0,\\
&a_1\mu_1^2 A_0 + \overline{a}_1\overline{\mu}_1^2 \overline{A}_0 + a_2\mu_2^2 B_0 + \overline{a}_2\overline{\mu}_2^2 \overline{B}_0 = 0.
\end{aligned} \tag{39}$$

Using the representation formulae, the expressions of the complex potentials, and boundary conditions (36), the following expressions are obtained:

$$A_{2m+1} = \xi_1^m A_1,\ \ A_{2m+2} = \xi_1^m A_2,\ B_{2m+1} = \xi_2^m B_1,\ \ B_{2m+2} = \xi_2^m B_2, \tag{40}$$

with

$$\xi_k = \frac{i - \mu_k}{\mu_k + i},\ k = 1, 2,$$

$$B_1 = 2\frac{\overline{a}_2\overline{\mu}_2 Re((\mu_1 + i)A_1) - Re(a_1\mu_1(\mu_1 + i)A_1)}{(\mu_2 + i)(a_2\mu_2 - \overline{a}_2\overline{\mu}_2)}. \tag{41}$$

Finally, from system (38), the coefficients A_2 and B_2 can be determined.

The complex coefficient A_1 remains undetermined in the expressions (37) of complex potentials of $\Psi_j = \Psi_j\left(z_j\right),\ j = 1, 2$. This is not an unexpected result since it is considered a boundary value problem in stress where such indetermination generally occurs.

The expression of the complex potentials $\Psi_j = \Psi_j\left(z_j\right),\ j = 1, 2$ may now be written using (40) into (37):

$$\begin{aligned}
\Psi_1(\zeta_1) &= A_0 + \frac{A_1\zeta_1 + A_2}{\zeta_1^2 - \xi_1},\\
\Psi_2(\zeta_2) &= B_0 + \frac{B_1\zeta_2 + B_2}{\zeta_2^2 - \xi_2}.
\end{aligned} \tag{42}$$

In the last part of the paper, the plane problem of uniform remote tangential shear loads, i.e., the case corresponding to the second mode of fracture, is studied.

Let us consider an unbounded, homogeneous, anisotropic elastic body containing an elliptical hole under by a uniform remote constant tangential shear load $h > 0$ in the direction of the x_1 axis. The boundary of the elliptical hole is free from stress. The boundary condition (36) remains unchanged and the far-field conditions (35) become

$$\lim_{|z|\to\infty} \theta_{11}(z) = \lim_{|z|\to\infty} \theta_{21}(z) = \lim_{|z|\to\infty} \theta_{22}(z) = 0, \quad \lim_{|z|\to\infty} \theta_{12}(z) = h > 0. \tag{43}$$

Using the same formalism as in the previous case, the same expressions of the complex potentials are obtained. The coefficient B_1 has the same form as before and the coefficient A_1 rests undetermined. To find the coefficients A_0, B_0, A_2, and B_2, it will use the same procedure, as for the plane problem of uniform remote tensile load. From the far-field conditions the following restrictions are obtained:

$$\begin{gathered} A_0 + \overline{A}_0 + B_0 + \overline{B}_0 = 0, \quad \mu_1 A_0 + \overline{\mu}_1 \overline{A}_0 + \mu_2 B_0 + \overline{\mu}_2 \overline{B}_0 = -h, \\ a_1 \mu_1 A_0 + \overline{a}_1 \overline{\mu}_1 \overline{A}_0 + a_2 \mu_2 B_0 + \overline{a}_2 \overline{\mu}_2 \overline{B}_0 = 0, \\ a_1 \mu_1^2 A_0 + \overline{a}_1 \overline{\mu}_1^2 \overline{A}_0 + a_2 \mu_2^2 B_0 + \overline{a}_2 \overline{\mu}_2^2 \overline{B}_0 = 0. \end{gathered} \tag{44}$$

Let us observe that the above system could be a determinate system, an indeterminate system, and in the case of resonance due to the initial applied stress $\overset{\circ}{\sigma}_{11}$ an incompatible system. Finally, the values of the complex coefficients A_2 and B_2 are obtained.

The *final* forms of the complex potentials $\Psi_j(\zeta_j)$, $j = 1, 2$ are thus determined by elementary calculus in both situations of uniform remote tensile and tangential shear loads, respectively. The basic complex potentials $\Psi_j(z_j)$ $j = 1, 2$ may be then obtained by introducing the expression of ζ_j given by (34) into the right-hand side of (42) and the problem is completely solved.

5 Final Remarks

Compact closed-form analytical solutions of the considered boundary value problems for an unbounded, homogeneous, prestressed orthotropic elastic composite containing an elliptical hole, subjected to uniform remote tensile, tangential, and antiplane shear loads (Mode I, Mode II, and Mode III of Fracture), are obtained.

The general results are practically relevant, e.g., for the study of incremental stress, strain, and displacement fields in the vicinity of the elliptical hole, can be applied to study a variety of composite mechanics problems, and can be extended for prestressed thermoelastic, ferromagnetic, or piezoelectric materials with elliptical holes.

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