Chapter 5
CR Submanifolds in (l.c.a.) Kaehler and *S*-manifolds

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5.1 Introduction

K. Yano [21] introduced in 1963 the notion of f-structure on a $(2m+s)$-dimensional manifold as a tensor field f of type (1, 1) and rank $2m$ satisfying $f^3 + f = 0$. Almost complex ($s = 0$) and almost contact ($s = 1$) structures are well-known examples of f-structures. A Riemannian manifold endowed with an f-structure ($s \geq 2$) compatible with the Riemannian metric is called a metric f-manifold (for $s = 0$ we have almost Hermitian manifolds and for $s = 1$, metric almost contact manifolds). In this context, D.E. Blair [5] defined K-manifolds (and particular cases of S-manifolds and C-manifolds) as the analogue of Kaehlerian manifolds in the almost complex geometry and of quasi-Sasakian manifolds (and particular cases of Sasakian manifolds and cosymplectic manifolds) in the almost contact geometry.

He also showed that the curvature of S-manifolds is completely determined by their f-sectional curvatures. Later, M. Kobayashi and S. Tsuchiya [15] got expressions for the curvature tensor field of S-manifolds when their f-sectional curvature is constant depending on such a constant. Such spaces are called S-space-forms and they generalize complex and Sasakian space-forms. Nice examples of S-space-forms

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can be found in [5, 6, 8, 13]. In particular, it is proved in [5, 8] that certain principal toroidal bundles over complex-space-forms are S-space-forms and a generalization of the Hopf fibration denoted by $\mathbb{H}^{2m+s}$ is introduced as a canonical example of such manifolds playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry [5, 6].

When we want to study the submanifolds of a metric f-manifold, the natural first step is to consider such submanifolds depending on their behavior with respect to the f-structure. So, invariant and anti-invariant submanifolds (in the terminology of the complex geometry, holomorphic and totally real submanifolds) appear if all the tangent vector fields to the submanifold are transformed by f into tangent vector fields or into normal vector fields. But since an hypersurface of a metric f-manifold tangent to the structure vector fields is neither invariant nor anti-invariant, it is necessary to introduce a wider class of submanifolds: the CR-submanifolds. This work was made firstly by A. Bejancu and B.-Y. Chen [1, 10, 11] in the case $s = 0$ and by A. Bejancu and N. Papaghiuc, M. Kobayashi and K. Yano and M. Kon in the case $s = 1$ (we refer to the books [3, 22] for the background of these cases where a large list of fundamental references can be found). For $s \geq 2$, I. Mihai [16] introduced the notion of CR-submanifold in a natural way.

Many authors have studied the geometry of submanifolds of locally conformal almost Kaehler (l.c.a.K.) manifolds [10, 11, 14, 20], which are almost Hermitian manifolds $(\widetilde{M}, J, g)$ such that every $x \in \widetilde{M}$ has an open neighborhood U such that for some differentiable function $h : U \longrightarrow \mathbb{R}$, $\tilde{g}_U = e^{-h} g|_U$ is a (l.c.a.) Kaehler metric on U. If one can take $U = \widetilde{M}$, the manifold is then called globally conformal almost Kaeler (g.c.a.K) manifold. Examples of l.c.K. manifolds are provided by the Hopf manifolds. So, it seems interesting to study CR-submanifolds of l.c.a.K. manifolds.

On the other hand, M. Okumura [17, 18] studied normal real hypersurfaces of Kaehlerian manifolds and obtained nice properties. For this reason, it also seems interesting to introduce and study normal CR-submanifolds. In the cases $s = 0$ and $s = 1$, the papers [2] and [4] can be consulted.

The aim of the present work is to briefly summarize our contributions to the study of CR-submanifolds of l.c.a.K. manifolds, normal CR-submanifolds of S-manifolds. To this end, we separate them into two different sections, which can be read independently.

5.2 CR-Submanifolds of (l.c.a.) Kaehler Manifolds

Let $(\widetilde{M}, J, g)$ be an almost Hermitian manifold ($dim(\widetilde{M}) = 2m$) with almost complex structure J and Hermitian metric g and let M be a Riemannian submanifold isometrically immersed in $\widetilde{M}$.

A. Bejancu [1] introduced the notion of a CR-submanifold of $\widetilde{M}$. In fact, M is a CR-submanifold of the almost Hermitian manifold $\widetilde{M}$ if there exists on M a differentiable holomorphic distribution $\mathcal{D}$, i.e., $J(\mathcal{D}_x) \subseteq \mathcal{D}_x$ for any $x \in M$ such that its orthogonal complement $\mathcal{D}^\perp$ in M is totally real in $\widetilde{M}$, i.e., $J(\mathcal{D}_x^\perp) \subseteq T_x^\perp(M)$ for any $x \in M$, where $T_x^{\perp}(M)$ is the normal space at x. If $dim(\mathcal{D}) = 0$, M is called a *totally real* submanifold, and if $dim(\mathcal{D}^{\perp}) = 0$ M is a *holomorphic* submanifold.

We first discuss the Gauss–Weingarten equations of the submanifold with respect to the metric g and with respect to the local conformal Kaehler metrics and then we shall establish thereby the analytical conditions that characterize the important types of submanifolds.

5.2.1 Preliminaries

Let $(\widetilde{M}, J, g)$ be an almost Hermitian manifold. It is easy to see [20] that $(\widetilde{M}, J, g)$ is a l.c.(a).K. manifold if and only if there is a global closed 1-form ω on $\widetilde{M}$ (the Lee form) such that $d\Omega = \omega \wedge \Omega$ (Ω the fundamental form of the manifold) and $(\widetilde{M}, J, g)$ is a g.c.(a).K. manifold if and only if ω is also exact. In case $\omega = 0$, the manifold is an (almost) Kaehler manifold.

Let $(\widetilde{M}, J, g)$ be a l.c.(a).K. manifold and consider the Lee vector field B [20] of $(\widetilde{M}, J, g)$ defined by $g(X, B) = \omega(X)$. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of g and define

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B. \tag{5.1}$$

Then $\overline{\nabla}$ is a torsionless linear connection on $\widetilde{M}$ which is called the Weyl connection of g. It is easy to see that $\overline{\nabla}_X g = \omega(X)g$. We have

Theorem 5.1 ([20]) *The almost Hermitian manifold $(\widetilde{M}, J, g)$ is a l.cK. manifold if and only if there is a closed 1-form ω on $\widetilde{M}$ such that the Weyl connection is almost complex, That is, $\overline{\nabla} J = 0$.*

Let $(\widetilde{M}, J, g)$ be a l.c.K. manifold and M a Riemannian manifold isometrically immersed in $\widetilde{M}$. We denote by g the metric tensor of $\widetilde{M}$ as well as that induced on M, and let ∇, ∇^M be the covariant derivations on M induced by $\widetilde{\nabla}$ and $\overline{\nabla}$, respectively. Then, the Gauss–Weingarten formulas for M with respect to $\widetilde{\nabla}$ and $\overline{\nabla}$ are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \ \ \widetilde{\nabla}_X V = -A_V X + D_X V, \tag{5.2}$$

$$\overline{\nabla}_X Y = \nabla^M_X Y + \overline{\sigma}(X, Y), \ \ \overline{\nabla}_X V = -\overline{A}_V X + \overline{D}_X V, \tag{5.3}$$

for any vector fields X, Y tangent to M and V normal to M, where σ (respectively, $\overline{\sigma}$) is the second fundamental form of M with respect to $\widetilde{\nabla}(\overline{\nabla})$ and D (respectively, $\overline{D}$) is the normal connection. The formulas (5.3) are the Gauss–Weingarten equations of $M|_U$ in $(\widetilde{M}|_U, e^{-h}g|_U)$. The second fundamental tensors $A_V, \overline{A}_V$ are related to $\sigma, \overline{\sigma}$ respectively by

$$g(A_V X, Y) = g(\sigma(X, Y), V), \ \ g(\overline{A}_V X, Y) = g(\overline{\sigma}(X, Y), V). \tag{5.4}$$

For any vector X tangent to M and V normal to M write

$$JX = TX + NX, \;\; JV = tV + nV, \tag{5.5}$$

where TX and NX (respectively, tV and nV) are the tangential and normal component of $J(X)$ (respectively JV). For the Lee field B, we have

$$B_x = (B_x)_1 + (B_x)_2, \quad x \in M, \tag{5.6}$$

where $(B_x)_1$ (resp.y $(B_x)_2$) is the tangential (resp. normal) component of B_x.

If M is a CR-submanifold of an almost Hermitian manifold $(\widetilde{M}, J, g)$ let us denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^\perp$ in $T^\perp(M)$. Hence we have, $T^\perp(M) = J\mathcal{D}^\perp \oplus \nu$.

5.2.2 *Integrability Conditions of the Basic Distributions*

First we give some general identities.

Lemma 1 *Let M be a CR-submanifold of a l.c.K. manifold $(\widetilde{M}, J, g)$. Then, we have*

$$\nabla^M_X Y = \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B_1 \tag{5.7}$$

$$\overline{\sigma}(X, Y) = \sigma(X, Y) + \frac{1}{2}g(X, Y)B_2 \tag{5.8}$$

$$\overline{A}_V X = A_V X + \frac{1}{2}\omega(V)X \tag{5.9}$$

$$\overline{D}_V X = D_V X - \frac{1}{2}\omega(X)V \tag{5.10}$$

for any vector fields X, Y tangent to M and V normal to M.

Proof The assertions follow immediately from (5.1)–(5.3). ■

The following result is well known:

Theorem 5.2 ([7]) *The totally real distribution $\mathcal{D}^\perp$ of any CR-submanifold of a l.c.K. manifold is integrable.*

For the holomorphic distribution $\mathcal{D}$, we have

Theorem 5.3 *Let M be a submanifold of a l.c.K. manifold $\widetilde{M}$ and let $\mathcal{D}_x$ de maximal holomorphic subspace of $T_x(M)$ and assume dim$(\mathcal{D}_x)$ is a constant. Then, the holomorphic distribution $\mathcal{D}$ is integrable if and only if the second fundamental form $\overline{\sigma}$ satisfies $\overline{\sigma}(X, JY) = \overline{\sigma}(JX, Y)$ or, equivalently, $\sigma(X, JY) - \sigma(JX, Y) + \Omega(X, Y)B_2 = 0$, for all vector fields $X, Y \in \mathcal{D}$.*

If M is a CR-submanifold, the integrability condition on $\mathcal{D}$ in Theorem 5.3 can be replaced by a weaker condition.

Theorem 5.4 *Let M be a CR-submanifold of a l.c.K. manifold $\widetilde{M}$. The holomorphic distribution $\mathcal{D}$ is integrable if and only if*

$$g\left(\sigma(X, JY) - \sigma(JX, Y) + \Omega(X, Y)B, J\mathcal{D}^{\perp}\right) = 0,$$

for all $X, Y \in \mathcal{D}$.

Theorems 5.3 and 5.4 follow easily from similar theorems in the Kaehlerian case ([7]), from (5.8) and the fact that, locally, $\widetilde{M}$ is endowed with Kaehler metrics $\widetilde{g}_U$ whose Levi-Civita connection is $\overline{\nabla}$.

With regard to integral submanifolds of $\mathcal{D}^{\perp}$ and $\mathcal{D}$ (provided $\mathcal{D}$ is integrable), we have the following theorem.

Theorem 5.5 *For a CR-submanifold M of a l.c.K. manifold $\widetilde{M}$, the leaf $M^{\perp}$ is totally geodesic in M if and only if*

$$g\left(A_{JW}Z + \frac{1}{2}g(Z, W)JB, \mathcal{D}\right) = 0,$$

that is,

$$g(\sigma(Z, X), JW) = \frac{1}{2}g(Z, W)\omega(JW),$$

for any $X \in \mathcal{D}$, $Z, W \in \mathcal{D}^{\perp}$.

Proof From (5.1), (5.2) and $\overline{\nabla}J = 0$, for any $X \in \mathcal{D}$, $Z, W \in \mathcal{D}^{\perp}$, we obtain

$$g(J\nabla_Z W, X) + \frac{1}{2}g(Z, W)g(JB, X) = -g(A_{JW}Z, X). \tag{5.11}$$

But $M^{\perp}$ is totally geodesic in M if and only if $\nabla_Z W \in \mathcal{D}^{\perp}$ for all $Z, W \in \mathcal{D}^{\perp}$, and then (5.11) gives the theorem. ∎

Theorem 5.6 *Let M be a CR-submanifold of a l.c.K manifold $\widetilde{M}$. If the holomorphic distribution $\mathcal{D}$ is integrable and M^T is an integral submanifold of $\mathcal{D}$, then M^T is totally geodesic if and only if*

$$g\left(J\sigma(X, Y) + \frac{1}{2}g(X, Y)JB - \frac{1}{2}\Omega(X, Y)B, \mathcal{D}^{\perp}\right) = 0,$$

for any $X, Y \in \mathcal{D}$.

Proof From (5.1), (5.3), and $\overline{\nabla}J = 0$, for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$, we have

$$g(J\sigma(X,Y),Z) + \frac{1}{2}g(X,Y)g(JB,Z) = g(\nabla_X(JY),Z) + \frac{1}{2}\Omega(X,Y)g(B,Z). \tag{5.12}$$

But M^T is totally geodesic in M if and only if $\nabla_X Y \in \mathcal{D}$ for all $X, Y \in \mathcal{D}$, and hence Eq. (5.12) gives the theorem. ■

5.2.3 CR-Submanifolds of l.c.K. Manifolds

First of all, we shall give some identities for later use. Let T, N t, and n be the endomorphisms and vector-valued 1-forms defined in (5.5). The following lemma can be easily obtained from (5.3), (5.9), and $\overline{\nabla}J = 0$.

Lemma 2 *Let M be an isometrically immersed submanifold of a l.c.K. manifold $\widetilde{M}$. Then, we have*

$$\nabla^M_X(TY) - \overline{A}_{NY}X = T\nabla^M_X Y + t\overline{\sigma}(X,Y) \tag{5.13}$$

$$\overline{\sigma}(X,TY) + D_X(NY) = N\nabla^M_X Y + n\overline{\sigma}(X,Y), \tag{5.14}$$

$$\nabla^M_X(tV) - \overline{A}_{nV}X = -T\overline{A}_V X + t\overline{D}_X V, \tag{5.15}$$

$$\overline{\sigma}(X,tV) + \overline{D}_X(nV) = -N\overline{A}_V X + n\overline{D}_X V, \tag{5.16}$$

$$\left[\overline{A}_V, \overline{A}_{\overline{V}}\right] = \left[A_V, A_{\overline{V}}\right], \tag{5.17}$$

for any vector fields X, Y tangent to M and V, $\overline{V}$ normal to M.

Now, we shall study totally umbilical and totally geodesic CR-submanifolds.

Theorem 5.7 *Let M be a totally umbilical CR-submanifold of a l.c.K. manifold $\widetilde{M}$. Then, we have*

(i) Either $\dim(\mathcal{D}^{\perp}) = 1$ or the component $H_{J(TM)}$ of the mean curvature tensor H in $J(TM)$ is given by $H_{J(TM)} = -\frac{1}{2}B_2$.
(ii) If $\dim(\mathcal{D}^{\perp}) > 1$ and M is proper (neither holomorphic nor totally real) such that B is tangent to M, then M is totally geodesic.

Proof First, since M is totally umbilical, $\sigma(X,Y) = g(X,Y)H$ for any X, Y tangent to M, and hence

$$g(\sigma(X,X),JW) = g(X,X)g(H,JW). \tag{5.18}$$

From (5.3) and (5.4) it is easy to see that

$$\overline{A}_{JZ}W = \overline{A}_{JW}Z \tag{5.19}$$

and, then, if we take an unit vector field $X = Z \in \mathcal{D}^{\perp}$ orthogonal to W, (5.9), (5.18), and (5.19) give

$$\begin{aligned} g(H, JW) =& g(A_{JW}Z, Z) = g(A_{JZ}W + \frac{1}{2}\omega(JZ)W - \frac{1}{2}\omega((JW)Z, Z) \\ =& -\frac{1}{2}\omega(JW) = g(-\frac{1}{2}B_2, JW), \end{aligned}$$

so that (i) holds.

Now, since $dim(\mathcal{D}^{\perp}) > 1$, from (5.5) and assertion (i), we have $tH = 0$. Thus, (5.15) gives $t\overline{D}_Y H = \overline{A}_{nH}Y - T\overline{A}_H Y$, for any Y tangent to M. Therefore, for any Z tangent to M, from (5.8) and (5.9) we get

$$g(t\overline{D}_Y H, Z) = -g(\overline{A}_H Y, TZ) - g(\overline{\sigma}(Y, Z), nH) = -g(Y, TZ)g(H, H) \tag{5.20}$$

and, if we take $Z = TY$, we have

$$-g(Y, T^2 Y)g(H, H) = g(t\overline{D}_Y H, TY) = g(Tt\overline{D}_Y H, Y) = 0. \tag{5.21}$$

The last equation holds because $Tt = 0$ for any CR-submanifold of an almost Hermitian manifold [22]. Moreover, it is easy to see [22] that $T^2 = -I + tN$ and then (5.21) gives

$$g(Y, Y)g(H, H) - g(NY, NY)g(H, H) = 0. \tag{5.22}$$

Since M is proper, we can choose an unit vector field X in $\mathcal{D}$. Thus, $NX = 0$ and from (5.22) we have $H = 0$. ■

Theorem 5.8 *Let M be a totally geodesic CR-submanifold of a l.c.K. manifold $\widetilde{M}$. We have*

(i) If $B_x \in \mathcal{D}_x$, for all $x \in M$, then $\mathcal{D}$ is integrable and any integral submanifold M^T of $\mathcal{D}$ is totally geodesic in $\widetilde{M}$.
(ii) If B is normal to M, any integral submanifold $M^{\perp}$ of $\mathcal{D}^{\perp}$ is totally geodesic in $\widetilde{M}$. Furthermore, $\mathcal{D}$ is integrable if and only if $B_x \in \nu_x$, for any $x \in M$, and in this case any integral submanifold M^T of $\mathcal{D}$ is totally geodesic in $\widetilde{M}$.

Proof Firstly, since B is tangent to M, from Theorem 5.7 the distribution $\mathcal{D}$ is integrable. Let M^T be an integral submanifold of $\mathcal{D}$. For any vector field X tangent to M, $Y \in \mathcal{D}$, $Z \in \mathcal{D}^{\perp}$, from (5.3) and (5.4) we get $g(\nabla^M_X Z, Y) = -g(\overline{\sigma}(X, JY), JZ)$. But from (5.7) and (5.8) we find

$$g(\nabla_X Z, Y) - \frac{1}{2}\omega(Z)g(X, Y) + \frac{1}{2}g(X, Z)g(B, Y) = -g(\sigma(X, JY), JZ) = 0. \tag{5.23}$$

If $X \in \mathcal{D}$, (5.23) gives $g(\nabla_X Z, Y) = 0$, or, equivalently, $g(\nabla_X Y, Z) = 0$ and therefore, $\nabla_X Y \in \mathcal{D}$. Thus M^T is totally geodesic in M and hence in $\widetilde{M}$.

Next, if B is normal to M, from Theorem 5.5, any integral submanifold $M^\perp$ of $\mathcal{D}^\perp$ is totally geodesic in $\widetilde{M}$. The second statement follows immediately from Theorems 5.6 and 5.7. ■

Corollary 1 *Let M be a totally geodesic proper CR-submanifold of a l.c.K. manifold $\widetilde{M}$ such that $B_x \in \nu_x$, for any $x \in M$. Then, M is locally the Riemannian product of a Kaehler submanifold and a totally real submanifold of $\widetilde{M}$.*

Proof From Theorem 5.8, M is locally the product of a holomorphic submanifold M^T and a totally real submanifold $M^\perp$ of $\widetilde{M}$. But $\omega = 0$ on M, so that we have induced on M^T a Kaehlerian structure. Moreover, it can be easily seen that the projection map p (resp., q) onto $\mathcal{D}$ (resp., $\mathcal{D}^\perp$) is parallel with respect to ∇, so that this local product is actually a local Riemannian product. ■

Next, we consider the particular case when M is either holomorphic or totally real.

Lemma 3 *Let M be a holomorphic submanifold of a l.c.K. manifold $\widetilde{M}$. Then the subbundles TM and $T^\perp(M)$ are holomorphic. Moreover, we have*

$$\overline{\sigma}(JX, Y) = \overline{\sigma}(X, JY) = J\overline{\sigma}(X, Y), \tag{5.24}$$

$$\overline{A}_{JV} = J\overline{A}_V = -\overline{A}_V J, \tag{5.25}$$

$$\overline{D}_X(JV) = J\overline{D}_X V, \tag{5.26}$$

$$\nabla^M_X(JY) = J\nabla^M_X Y, \tag{5.27}$$

for any vector fields X, Y tangent to M and V normal to M.

Proof As $\widetilde{M}$ is locally endowed with Kaehler metrics $\widetilde{g}_U$ whose Levi-Civita connection is $\overline{\nabla}$, these formulas follow from similar formulas in the Kaehlerian case. ■

Theorem 5.9 *Let M be a holomorphic submanifold of a l.c.K. manifold $\widetilde{M}$. Then, we have*

(i) The mean curvature vector H of M is given by $H = -\frac{1}{2}B_2$.
(ii) M is totally umbilical if and only if the Weingarten endomorphisms are commutative.

Proof Firstly, if $dim(M) = 2k > 0$, let $\{e_1, \ldots, e_k, Je_1, \ldots, Je_k\}$ be an orthonormal basis for $T_x(M)$, $x \in M$. Then

$$2kH_x = (tr(\sigma))_x = \sum_{i=1}^{k} \sigma_x(e_i, e_i) + \sum_{i=1}^{k} \sigma_x(Je_i, Je_i). \tag{5.28}$$

But from (5.8) and (5.24), (5.28) gives $2kH_x = -k(B_2)_x$.
Next, let V be a vector field normal to M. From (5.17) and (5.25), we have

$$0 = [A_V, A_{JV}] = [\overline{A}_V, \overline{A}_{JV}] = -2J(\overline{A}_V)^2,$$

Thus $\overline{A}_V = 0$ and from (5.9), we have $A_V = -\frac{1}{2}\omega(V)I$ ■

The endomorphism n of the normal bundle $T^{\perp}M$ defined in (5.5) induces an f-structure in $T^{\perp}M$ [22]. For any vector field X tangent to M and V normal to M, we write

$$(\widetilde{\nabla}'_X n)V = D_X(nV) - nD_X V,$$

$$(\overline{\nabla}'_X n)V = \overline{D}_X(nV) - n\overline{D}_X V.$$

When $\widetilde{\nabla}'n = 0$, the f-structure n is said to be parallel [10].

Lemma 4 *Let M be an r-dimensional totally real submanifold of a $2m$-dimensional l.c.K. manifold $\widetilde{M}$. Then we have*

(i) $\overline{A}_{JX}Y = \overline{A}_{JY}X$, *for any* X, Y *tangent to* M.
(ii) *If* $r = m$, *then* $\overline{D}_X(JY) = J\nabla^M_X Y$, $\nabla^M_X(JV) = J\overline{D}_X V$, *and* $\overline{\sigma}(X, JV) = -J\overline{A}_V X$.
(iii) $\widetilde{\nabla}'n = \overline{\nabla}'n$.
(iv) *If the f-structure n is parallel, then*

$$A_V = -\frac{1}{2}\omega(V)I, \tag{5.29}$$

for any $V \in \nu$.
(v) *If the Weingarten endomorphisms are commutative, then there is an orthonormal local basis* $\{e_1, \ldots, e_r\}$ *in* M *such that with respect to this basis* $\overline{A}_{Je_i}$ *is a diagonal matrix*

$$\overline{A}_{Je_i} = (0 \ldots 0\, \lambda_i\, 0 \ldots 0)\,, \quad i = 1, \ldots, r. \tag{5.30}$$

Proof Assertions (i) and (ii) follow immediately from similar formulas in the Kaehlerian case. From Eq. (5.10), we easily obtain (iii).

In order to prove (iv), we take $V \in \nu$, and $X \in T(M)$. Then, (iii) gives $(\overline{\nabla}'_X n)V = \overline{D}_X(nV) - n\overline{D}_X V = 0$. By using (5.25) and (5.26) this yields $J\overline{A}_V X = 0$. Therefore, $\overline{A}_V = 0$ and from (5.9), we obtain (iv).

Finally, from (5.17) we have $[\overline{A}_V, \overline{A}_{\overline{V}}] = 0$, for any $V, \overline{V}$ normal to M. Then, we can find a local orthonormal basis $\{\widetilde{e}_1 \dots, \widetilde{e}_r\}$ in M (with respect to the local Kaehlerian metrics $\widetilde{g}_U = e^{-h} g|_U$) such that $\overline{A}_{Je_i} = (0 \dots \mu_i \dots 0)$, $i = 1, \dots, r$. If we start by using this basis, we can obtain an orthonormal (with respect to the metric g) local basis $\{e_1, \dots, e_n\}$ in M such that (v) holds. ∎

Theorem 5.10 *Let M be an r-dimensional totally real and minimal submanifold of a l.c.K. manifold $\widetilde{M}$ such that their Weingarten endomorphisms are commutative and the f-structure n is parallel. Then, we have*

(i) If $r \geq 2$, M is totally geodesic if and only if the Lee vector field B is tangent to M.
(ii) If $r = 1$ and B is orthogonal to ν, then M is a geodesic curve.

Proof First, since the Weingarten endomorphisms are commutative, let $\{e_1, \dots, e_r\}$ be an orthonormal local basis as in Lemma 4 (v). From Eq. (5.8), we have

$$0 = g(H, Je_i) = \frac{1}{n} \sum_{j=1}^{n} g\left(\sigma(e_j, e_j), Je_i\right) = \frac{1}{n} \sum_{j=1}^{n} g\left(\overline{A}_{Je_i} e_j, e_j\right) - \frac{1}{2}\omega(Je_i)$$
$$= \frac{1}{n}\lambda_i - \frac{1}{2}\omega(Je_i), \quad i = 1, \dots, r.$$

Therefore,

$$\overline{A}_{Je_i} e_j = \delta_{ij}\lambda_i e_j = \delta_{ij} \frac{n}{2} \omega(Je_i) e_j, \quad i = 1, \dots, r. \tag{5.31}$$

Now, from (5.9) and (5.31) we obtain,

$$A_{Je_i} e_j = \frac{1}{2}(n\delta_{ij} - 1)\omega(Je_i)e_j, \quad i = 1, \dots, r. \tag{5.32}$$

Thus, if $r \geq 2$ and B is tangent to M, Eq. (5.32) gives $A_{Je_i} = 0$, $i = 1, \dots, r$. Moreover, from (iv) in Lemma 4, $A_V = -\frac{1}{2}\omega(V)I = 0$, for any $V \in \nu$. Then, $A_{\overline{V}} = 0$, for any vector field $\overline{V}$ normal to M.

On the other hand, if there is $x \in M$ such that $(B_2)_x \neq 0$, from (5.32) and (iv) in Lemma 4, we can take a vector field V normal to M such that $A_V \neq 0$. This gives (i).

In order to prove (ii), let us take a unit vector field X tangent to M. We have $0 = g(H, JX) = g(\sigma(X, X), JX) = g(A_{JX}X, X)$, and, then, $A_{JX} = 0$. But, if B is orthogonal to ν, from (iv) in Lemma 4, $A_V = -\frac{1}{2}\omega(V) = 0$, for any $V \in \nu$. This means that $A_{\overline{V}} = 0$, for any vector field $\overline{V}$ normal to M. ∎

Theorem 5.11 *Let M be an r-dimensional $(r \geq 2)$ totally real and totally umbilical submanifold of a l.c.K. manifold $\widetilde{M}$ such that the f-structure n is parallel. Then M is totally geodesic if and only if B is tangent to M.*

Proof Let $\{u_1, \dots, u_r\}$ be an orthonormal local basis in U. Since M is totally umbilical, for any vector field X tangent to M, by using Eqs. (5.8) and (5.9), we find

$$g(\overline{A}_{JX}u_j, u_k) = \frac{1}{r}\delta_{jk}tr(\overline{A}_{JX}). \tag{5.33}$$

But from Eq. (5.9) and (iv) in Lemma 4 we also have

$$\overline{A}_V = 0, \tag{5.34}$$

for any $V \in \nu$. On the other hand $[A_{\overline{V}}, A_{\overline{\overline{V}}}] = [\overline{A}_{\overline{V}}, \overline{A}_{\overline{\overline{V}}}] = 0$, for any vector fields $\overline{V}, \overline{\overline{V}}$ normal to M. Therefore, from (v) in Lemma 4, there is an orthonormal local basis $\{e_1, \ldots, e_r\}$ in M such that, with respect to this basis, Eq. (5.30) holds. But, from Eq. (5.33), we also have

$$g(\overline{A}_{Je_i}e_j, e_j) = \frac{1}{r}\lambda_i, \quad i.j = 1, \ldots, n. \tag{5.35}$$

Since $r \geq 2$, we can take $j \neq i$ and then, Eqs. (5.30) and (5.35) give $\lambda_i = 0$, $i = 1, \ldots, r$. Thus, we get $\overline{A}_{Je_i} = 0$, $i = 1, \ldots, r$, which, together with (5.34) gives $\overline{A}_{\overline{V}} = 0$, for any vector field $\overline{V}$ normal to M. Now, if B is tangent to M, Eq. (5.9) proves that M is totally geodesic.

Conversely, if M is totally geodesic, from (5.29) we have $0 = A_V = -\frac{1}{2}\omega(V)I$, for any $V \in \nu$. This means that B is normal to ν. Furthermore, from (v) in Lemma 4, we can find an orthonormal local basis $\{e_1, \ldots, e_r\}$ in M such that $\overline{A}_{Je_i}$ has a diagonal matrix $\overline{A}_{Je_i} = (0 \ldots 0\, \lambda_i\, 0 \ldots 0) = \frac{1}{2}\omega(Je_i)I$, $i = 1, \ldots, r$. Since $r \geq 2$, this means $\omega(Je_i) = 0$, $i = 1, \ldots, r$ so that B is normal to $J(T(M))$. Thus, B is tangent to M. ∎

5.2.4 CR-products in l.c.K. Manifolds

Let T, N, t, n be the endomorphisms and vector-valued 1-forms defined by (5.5). Let us write

$$\begin{aligned}(\widetilde{\nabla}'_Z T)W &= \nabla_Z(TW) - T\nabla_Z W,\\ (\overline{\nabla}'_Z T)W &= \nabla^M_Z(TW) - T\nabla^M_Z W,\end{aligned} \tag{5.36}$$

for all Z, W tangent to M. On the other hand, T is said to be parallel if $\overline{\nabla} T = 0$. From (5.1)–(5.3) it is easy to prove that

$$\begin{aligned}(\overline{\nabla}'_Z T)W = (\widetilde{\nabla}'_Z T)W &+ \frac{1}{2}\omega(W)TZ - \frac{1}{2}\omega(TW)Z\\ &+ \frac{1}{2}g(Z, TW)B_1 - \frac{1}{2}g(Z, W)TB_1.\end{aligned} \tag{5.37}$$

But, from (5.36) we see that

$$(\overline{\nabla}'_Z T)W = t\overline{\sigma}(Z, W) + \overline{A}_{NW}Z. \tag{5.38}$$

Definition 1 A *CR*-submanifold of a l.c.K. manifold $\widetilde{M}$ is called a *CR-product* if it is locally a Riemannian product of a holomorphic submanifold M^T and a totally real submanifold $M^\perp$ of $\widetilde{M}$.

Theorem 5.12 *Let M be a CR-submanifold of a l.c.K. manifold $\widetilde{M}$ such that the Lee field B is normal to M. Then M is a CR-product if and only if T is parallel.*

Proof Since B is normal to M, from Eq. (5.37), we have $\widetilde{\nabla}'T = \overline{\nabla}'T$. If T is parallel, from (5.8), (5.9), and (5.38), we find

$$t\sigma(Z, W) + \frac{1}{2}g(Z, W)tB = -A_{NW}Z - \frac{1}{2}\omega(NW)Z. \tag{5.39}$$

But for any $X \in \mathcal{D}$, $NX = 0$, and the last equation gives

$$0 = g(A_{NW}Z, X) + \frac{1}{2}\omega(NW)g(Z, X),$$

or, equivalently, $g(\sigma(Z, X), JW) + \frac{1}{2}g(JW, B)g(Z, X) = 0$, for any W tangent to M. Therefore,

$$\sigma(Z, X) = -\frac{1}{2}g(Z, X)B. \tag{5.40}$$

If we take $Z \in \mathcal{D}$, the last equation gives $\sigma(X, JY) - \sigma(JX, Y) = -\Omega(X, Y)B$, and, from Theorem 5.3, $\mathcal{D}$ is integrable. Let M^T be an integral submanifold of $\mathcal{D}$. For any $Z \in \mathcal{D}^\perp$, Eq. (5.40) yields $g(\sigma(Z, X), JZ) = -\frac{1}{2}g(Z, Z)g(B, JZ)$ and, from Theorem 5.6, the submanifold M^T is totally geodesic in M. Now, let $M^\perp$ be an integral submanifold od $\mathcal{D}^\perp$. From (5.40), if $Z \in \mathcal{D}^\perp$, then $\sigma(Z, X) = 0$ and, from Theorem 5.5, $M^\perp$ is totally geodesic.

Conversely, assume that M is a *CR*-product. First, we prove that $\nabla^M_Z X \in \mathcal{D}$, for any $X \in \mathcal{D}$ and Z tangent to M. As M is locally a Riemannian product of M^T (holomorphic submanifold) and $M^\perp$ (totally real submanifold), it suffices to prove that $\nabla^M_Z X \in \mathcal{D}$, for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. In fact, from (5.3) we have

$$J\nabla^M_Z X = \nabla^M_Z(JX) + \overline{\sigma}(Z, JX) - J\overline{\sigma}(Z, X).$$

Thus, if $W \in \mathcal{D}^\perp$, $g(J\nabla^M_Z X, JW) = g(\overline{\sigma}(Z, JX), JW)$. Since $M^\perp$ is totally geodesic in M, from (5.8) and Theorem 5.5 we have $g(\nabla^M_Z X, W) = 0$, for any $W \in \mathcal{D}^\perp$. So, $\nabla^M_Z X \in \mathcal{D}$ and $\nabla^M_Z X \in \mathcal{D}$, for any Z tangent to M. From $\overline{\nabla}J = 0$, we find

$$J\nabla^M_Z X + J\overline{\sigma}(Z, X) = \nabla^M_Z(JX) + \overline{\sigma}(Z, JX),$$

and then, $J\nabla^M_Z X = \nabla^M_Z(JX)$, $J\overline{\sigma}(Z, X) = \overline{\sigma}(Z, JX)$. Now, from (5.36) we get

$$(\overline{\nabla}'_Z T)X = \nabla^M_Z(TX) - T\nabla^M_Z X = \nabla^M_Z(JX) - J\nabla^M_Z X = 0, \tag{5.41}$$

for any $X \in \mathcal{D}$ and Z tangent to M.

In a similar way, we prove that $\nabla^M_Z Z \in \mathcal{D}^\perp$ for any $Z \in \mathcal{D}^\perp$ and Z tangent to M. Since M is a CR-product, it suffices to show this for $Z = X \in \mathcal{D}$. In fact, from (5.3), given any $Y \in \mathcal{D}$ we find that

$$g(J\nabla^M_X Z, Y) = -g(\overline{A}_{JZ}X, Y) - g(J\overline{\sigma}(X, Z), Y) = -g(\overline{\sigma}(X, Y), JZ) = 0,$$

where the last equation holds from (5.8) and Theorem 5.6. Then, $J\nabla^M_X Z$ is orthogonal to $\mathcal{D}$. On the other hand, if $W \in \mathcal{D}^\perp$, we have

$$g\left(\nabla^M_X Z, W\right) = -g(\overline{\sigma}(X, W), JZ) + g(\overline{\sigma}(X, Z), JW).$$

But, from Theorem 5.5 we have $g(J\nabla^M_X Z, W) = 0$, That is, $J\nabla^M_X Z$ is normal to M, so that $\nabla^M_X Z \in \mathcal{D}^\perp$. Therefore, we have

$$(\overline{\nabla}'_Z T)Z = \nabla^M_Z (TZ) - T\nabla^M_Z Z = 0. \tag{5.42}$$

Now, from (5.37), (5.41), and (5.42), we have $\widetilde{\nabla}' T = 0$. ■

Theorem 5.13 *Let M be a CR-submanifold of a l.c.K. manifold $\widetilde{M}$ such that $B_x \in \mathcal{D}_x$ for each $x \in M$. If T is parallel, then M is a CR-product. The converse does not holds unless $\dim(\mathcal{D}) = 2$ or $B = 0$ on M.*

Proof Since T is parallel, Eqs. (5.37) and (5.38) give

$$\begin{aligned} t\overline{\sigma}(Z, W) + \overline{A}_{NW} Z = \frac{1}{2}\omega(W)TZ - \frac{1}{2}\omega(TW)Z \\ + \frac{1}{2}g(Z, TW)B - \frac{1}{2}g(Z, W)TB. \end{aligned} \tag{5.43}$$

If $X \in \mathcal{D}$, then $NX = 0$ and (5.43) gives

$$\begin{aligned} -g(J\overline{\sigma}(X, Z), W) = \frac{1}{2}g(B, W)g(JZ, X) + \frac{1}{2}g(W, JB)g(Z, X) \\ - \frac{1}{2}g(JZ, W)g(B, X) - \frac{1}{2}g(Z, W)g(JB, X), \end{aligned} \tag{5.44}$$

for any vector field W tangent to M. From (5.8), (5.44) yields

$$\begin{aligned} -J\sigma(X, Z) = \frac{1}{2}g(JZ, X)B + \frac{1}{2}g(Z, X)JB \\ - \frac{1}{2}g(B, X)JZ - \frac{1}{2}g(JB, X)Z. \end{aligned} \tag{5.45}$$

For any $Z \in \mathcal{D}^\perp$, Eq. (5.45) gives $g(A_{JZ}, Z) = \frac{1}{2}\omega(JX)g(Z, Z)$, for any Z tangent to M and, hence, we have

$$A_{JZ}X = \frac{1}{2}\omega(JX)Z. \tag{5.46}$$

Next, for $Y \in \mathcal{D}$, from (5.46), we have

$$g(\sigma(X,Y), JZ) = 0, \quad for\ X \in \mathcal{D},\ Z \in \mathcal{D}^\perp. \tag{5.47}$$

Therefore, $g(\sigma(X, JY) - \sigma(JX, Y, J\mathcal{D}^\perp) = 0$ and, from Theorem 5.4, the distribution $\mathcal{D}$ is integrable. Moreover, any integral submanifold $M^\perp$ od $\mathcal{D}$ is totally geodesic in M because of (5.47) and Theorem 5.6. Now, let $M^\perp$ be an integral submanifold of $\mathcal{D}^\perp$. For any $W \in \mathcal{D}^\perp$, Eq. (5.46) gives

$$g\left(A_{JZ} + \frac{1}{2}g(Z,W)JB, X\right) = 0$$

and this means that $M^\perp$ is totally geodesic in M (Theorem 5.5). Thus M is a *CR*-product. ■

In order to prove the converse, we first give the following Lemma.

Lemma 5 *If M is a CR-product in a l.c.K. manifold $\widetilde{M}$ such that $B_x \in \mathcal{D}_x$ for any $x \in M$, then*

$$\nabla_Z X \in \mathcal{D}, \tag{5.48}$$

$$\nabla_X Z \in \mathcal{D}^\perp, \tag{5.49}$$

$$J\nabla_Z X = \nabla_Z(JX), \tag{5.50}$$

for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

Proof If $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$, then from (5.7) and (5.8), we obtain

$$\begin{aligned} J\nabla_Z X = \frac{1}{2}\omega(X)JZ - J\sigma(Z,X) + \nabla_Z(JX) + \nabla_Z(JX) \\ - \frac{1}{2}\omega(JX)Z + \sigma(Z,JX). \end{aligned} \tag{5.51}$$

Now, for any $W \in \mathcal{D}^\perp$, (5.51) yields

$$g(J\nabla_Z X, JW) = g(\nabla_Z X, W) = g\left(A_{JW}Z + \frac{1}{2}g(Z,W)JB, JX\right) = 0.$$

The last equation holds because any leaf $M^\perp$ of $\mathcal{D}^\perp$ is totally geodesic in M (Theorem 5.5). Thus $\nabla_Z X \in \mathcal{D}$ and this is assertion (5.48). Now, take $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. From (5.1) and (5.2), we find that

$$g(J\nabla_X Z, Y) = -g(A_{JZ}, Y) = -g(\sigma(X,Y), JZ) = 0 \tag{5.52}$$

The last equation holds because of Theorem 5.6. If $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$, from (5.1) and (5.2) again we have

$$g(J\nabla_X Z, W) = g(A_{JW}Z, X) - g(A_{JZ}W, X).$$

But, from Theorem 5.5 we obtain

$$g(A_{JW}Z, X) - g(A_{JZ}W, X) = -\frac{1}{2}g(Z, W)g(JB, X) + \frac{1}{2}g(W, Z)g(JB, X) = 0$$

and, hence

$$g(J\nabla_X Z, W) = 0. \tag{5.53}$$

Now, (5.49) follows from (5.52) and (5.53). Finally, (5.48) and (5.51) give (5.50). ■

Now we prove the converse of Theorem 5.13. From (5.36) and (5.48), for any $X \in \mathcal{D}$ and Z tangent to M we have

$$(\widetilde{\nabla}_Z', T)X = \nabla_Z(JX) - J(\nabla_Z X).$$

On the other hand, we write $Z = Y + Z$, where $Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. Then, from (5.50) we have

$$(\widetilde{\nabla}_Z' T)X = \nabla_Y(JX) - J\nabla_Y X. \tag{5.54}$$

But (5.1)–(5.3) give

$$\nabla_Y(JX) - J\nabla_Y X = \frac{1}{2}\omega(Y)JX - \frac{1}{2}(JY)X - \frac{1}{2}g(X, Y)JB + \frac{1}{2}g(X, JY)B. \tag{5.55}$$

Now we have

(a) If $dim(\mathcal{D}) \geq 4$ and $B_x \neq 0$ for some $x \in M$, there are $X, Y \in \mathcal{D}$ such that the right-hand side of (5.55) does not vanish at x. Therefore, T is not parallel.

(b) If $dim(\mathcal{D}) = 2$, then the right-hand side of (5.55) vanishes and, hence

$$(\widetilde{\nabla}_Z' T)X = 0, \tag{5.56}$$

for any $X \in \mathcal{D}$ and Z tangent to M. But (5.49) implies $\nabla_Z Z \in \mathcal{D}^\perp$, for any $Z \in \mathcal{D}^\perp$ and Z tangent to M, so that

$$(\widetilde{\nabla}_Z' T)Z = \nabla_Z(TZ) - T\nabla_Z Z = -T\nabla_Z Z = 0. \tag{5.57}$$

Then, (5.56) and (5.57) prove that T is parallel. ■

5.3 Normal CR-Submanifolds of S-manifolds

We want to study here the normal CR-submanifolds for general S-manifolds. In fact, the normal CR-submanifolds become to be a very wide class of CR-submanifolds. Actually, either totally f-umbilical submanifolds (see [19] for more details) or CR-products (see [12]) of an S-manifold are normal CR-submanifolds. We also study normal CR-submanifolds of an S-space-form, specially in the concrete cases of $\mathbb{R}^{2m+s}$ (with constant f-sectional curvature $c = -3s$) and $\mathbb{H}^{2m+s}$ (with constant f-sectional curvature $c = 4 - 3s$).

5.3.1 Preliminaries

A $(2m + s)$-dimensional Riemannian manifold $(\wedge M, g)$ endowed with an f-structure f (that is, a tensor field of type $(1, 1)$ and rank $2m$ satisfying $f^3 + f = 0$ [21]) is said to be a *metric f-manifold* if, moreover, there exist s global vector fields $\xi_1, \dots, \xi_s$ on $\wedge M$ (called *structure vector fields*) such that, if $\eta_1, \dots, \eta_s$ are the dual 1-forms of $\xi_1, \dots, \xi_s$, then

$$f\xi_\alpha = 0;\ \eta_\alpha \circ f = 0;\ f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha;$$

$$g(X, Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{5.58}$$

for any $X, Y \in \mathcal{X}(\wedge M)$ and $\alpha = 1, \dots, s$.

Let F be the 2-form on $\wedge M$ defined by $F(X, Y) = g(X, fY)$, for any $X, Y \in \mathcal{X}(\wedge M)$. Since f is of rank $2m$, then

$$\eta_1 \wedge \cdots \wedge \eta_s \wedge F^m \neq 0$$

and, particularly, $\wedge M$ is orientable.

The f-structure f is said to be *normal* if

$$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f, f]$ is the Nijenhuis torsion of f.

A metric f-manifold is said to be a K-manifold [5] if it is normal and $\mathrm{d}F = 0$. A K-manifold is called an S-manifold if $F = \mathrm{d}\eta_\alpha$, for any α. Note that, for $s = 0$, a K-manifold is a Kaehlerian manifold and, for $s = 1$, a K-manifold is a quasi-Sasakian manifold and an S-manifold is a Sasakian manifold. When $s \geq 2$, nontrivial examples can be found in [5, 13]. Moreover, a K-manifold $\wedge M$ is an S-manifold if and only if

$$\wedge \nabla_X \xi_\alpha = -fX, \tag{5.59}$$

for any $X \in \mathcal{X}(\wedge M)$ and any $\alpha = 1, \ldots, s$, where $\wedge\nabla$ denotes the Levi-Civita connection of g. It is easy to show that in any S-manifold

$$(\wedge\nabla_X f)Y = \sum_{\alpha=1}^{s} \left\{ g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X \right\}, \tag{5.60}$$

for any $X, Y \in \mathcal{X}(\wedge M)$. A plane section π on a metric f-manifold $\wedge M$ is said to be an *f-section* if it is determined by a unit vector X, normal to the structure vector fields and fX. The sectional curvature of π is called an *f-sectional curvature*. An S-manifold is said to be an *S-space-form* if it has a constant f-sectional curvature c and then, it is denoted by $\wedge M(c)$. In such case, the curvature tensor field $\wedge R$ of $\wedge M(c)$ satisfies [15]

$$\begin{aligned} \wedge R&(X, Y, Z, W) \\ &= \sum_{\alpha,\beta} (g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) \\ &\quad + g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)) \\ &\quad + \frac{c+3s}{4}(g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)) \\ &\quad + \frac{c-s}{4}(F(X, W)F(Y, Z) - F(X, Z)F(Y, W) \\ &\qquad\qquad - 2F(X, Y)F(Z, W)), \end{aligned} \tag{5.61}$$

for any $X, Y, Z, W \in \mathcal{X}(\wedge M)$. Next, let M be a isometrically immersed submanifold of a metric f-manifold $\wedge M$ (for the general theory of submanifolds, we refer to [3, 22]). We denote by $\mathcal{X}(M)$ the Lie algebra of tangent vector fields to M and by $T(M)^\perp$ the set of tangent vector fields to $\wedge M$ which are normal to M. For any vector field $X \in \mathcal{X}(M)$, we write

$$fX = TX + NX, \tag{5.62}$$

where TX and NX are the tangential and normal components of fX, respectively. Then, T is an endomorphism of the tangent bundle of M and N is a normal bundle valued 1-form on such tangent bundle. It is easy to show that if T does not vanish, it defines an f-structure in the tangent bundle of M. The submanifold M is said to be *invariant* if N is identically zero, that is, if fX is tangent to M, for any $X \in \mathcal{X}(M)$. On the other hand, M is said to be an *anti-invariant* submanifold if T is identically zero, that is, if fX is normal to M, for any $X \in \mathcal{X}(M)$. In the same way, for any $V \in T(M)^\perp$, we write

$$fV = tV + nV, \tag{5.63}$$

where tV and nV are the tangential and normal components of fV, respectively. Then, t is a tangent bundle valued 1-form on the normal bundle of M and n is an endomorphism of the normal bundle of M. It is easy to show that if n does nor vanish, it defines an f-structure in the normal bundle of M. From now on, we suppose that all the structure vector fields are tangent to the submanifold M and so, $dim(M) \geq s$. Then, the distribution on M spanned by the structure vector fields is denoted by $\mathcal{M}$ and its complementary orthogonal distribution is denoted by $\mathcal{L}$. Consequently, if $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any $\alpha = 1, \dots, s$ and if $X \in \mathcal{M}$, then $fX = 0$. In this context, M is said to be a *CR-submanifold* of $\wedge M$ if there exist two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M satisfying

(i) $\mathcal{X}(M) = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{M}$, where $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathcal{M}$ are mutually orthogonal to each other;
(ii) The distribution $\mathcal{D}$ is invariant by f, that is, $f\mathcal{D}_x = \mathcal{D}_x$, for any $x \in M$;
(iii) The distribution $\mathcal{D}^\perp$ is anti-invariant by f, that is, $f\mathcal{D}_x^\perp \subseteq T_x(M)^\perp$, for any $x \in M$.

This definition is motivated by the following theorem.

Theorem 5.14 ([16]) *Let $\wedge M$ be an S-manifold which is the bundle space of a principal toroidal bundle over a Kaehler manifold $\wedge M'$, $\wedge\pi : \wedge M \longrightarrow \wedge M'$, M a submanifold immersed in $\wedge M$, tangent to the structure vector fields and M' a submanifold immersed in $\wedge M'$ such that there exists a fibration $\pi : M \longrightarrow M'$, the diagram*

$$\begin{array}{ccc} M & \xrightarrow{i} & \wedge M \\ \pi \downarrow & & \downarrow \wedge\pi \\ M' & \xrightarrow{i'} & \wedge M' \end{array}$$

commutes and the immersion i is a diffeomorphism on the fibers. Then, M is a CR-submanifold of $\wedge M$ if and only if M' is a CR-submanifold of $\wedge M'$.

We denote by $2p$ and q the real dimensions of $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. Then, we see that for $p = 0$ we obtain an anti-invariant submanifold tangent to the structure vector fields and for $q = 0$ an invariant submanifold. A *CR*-submanifold of an S-manifold is said to be a *generic submanifold* if given any $V \in T(M)^\perp$, there exists $Z \in \mathcal{D}^\perp$ such that $V = fZ$, a $(\mathcal{D}, \mathcal{D}^\perp)$-*geodesic submanifold* if $\sigma(X, Z) = 0$, for any $X \in \mathcal{D}$ and any $Z \in \mathcal{D}^\perp$ and a $\mathcal{D}^\perp$-*geodesic submanifold* if $\sigma(Y, Z) = 0$, for any $Y, Z \in \mathcal{D}^\perp$. As an example, it is easy to show that each hypersurface of $\wedge M$ which is tangent to the structure vector fields is a *CR*-submanifold. Now, we write by P and Q the projections morphisms of $\mathcal{X}(M)$ on $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. Thus, for any $X \in \mathcal{X}(M)$, we have that

$$X = PX + QX + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha.$$

We define the tensor field v of type (1, 1) by $vX = fPX$ and the non-null, normal bundle valued 1-form u by $uX = fQX$, for any $X \in \mathcal{X}(M)$. Then, it is easy to show that $u \circ v = 0$ and $\eta_\alpha \circ u = \eta_\alpha \circ v = 0$, for any $\alpha = 1, \ldots, s$. Moreover, a direct computation gives

$$g(X, Y) = g(uX, uY) + g(vX, vY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{5.64}$$

$$F(X, Y) = g(X, vY), \quad F(X, Y) = F(vX, vY), \tag{5.65}$$

for any $X, Y \in \mathcal{X}(M)$. From Gauss–Weingarten formulas and by using (5.59), for any $X \in \mathcal{X}(M)$, $V \in T(M)^\perp$, and $\alpha = 1, \ldots, s$, we have

$$\nabla_X \xi_\alpha = -vX, \tag{5.66}$$

$$\sigma(X, \xi_\alpha) = -uX, \tag{5.67}$$

$$A_V \xi_\alpha \in \mathcal{D}^\perp. \tag{5.68}$$

Moreover, from (5.60) and the Gauss–Weingarten formulas, if $X, Y \in \mathcal{X}(M)$, comparing the components in $\mathcal{D}$, $\mathcal{D}^\perp$ and $T(M)^\perp$ respectively, we get

$$P\nabla_X vY - PA_{uY}X = v\nabla_X Y - \sum_{\alpha=1}^{s} \eta_\alpha(Y)PX, \tag{5.69}$$

$$Q\nabla_X vY - QA_{uY}X = t\sigma(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(Y)QX, \tag{5.70}$$

$$\sigma(X, vY) + D_X uY = u\nabla_X Y + n\sigma(X, Y). \tag{5.71}$$

From the above formulas and (5.60) we obtain

$$(\nabla_X v)Y = A_{uY}X + t\sigma(X, Y) - \sum_{a=1}^{s} \{\eta_\alpha(Y)f^2X + g(fX, fY)\xi_\alpha\}, \tag{5.72}$$

$$(\nabla_X u)Y = n\sigma(X, Y) - \sigma(X, vY), \tag{5.73}$$

for any $X, Y \in \mathcal{X}(M)$. Also, from (5.60) and the Gauss–Weingarten formulas again, we have

$$\nabla_X Z = vA_{fZ}X - tD_X fZ, \tag{5.74}$$

$$tD_X fZ = -Q\nabla_X Z, \tag{5.75}$$

for any $X \in \mathcal{X}(M)$ and any $Z \in \mathcal{D}^\perp$. With regard to the integrability of the distributions involved in the definition of a CR-submanifold, I. Mihai [16] proved that the distributions $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\perp} \oplus \mathcal{M}$ are always integrable. On the other hand, if $p > 0$, the distributions $\mathcal{D}$ and $\mathcal{D} \oplus \mathcal{D}^{\perp}$ are not integrable and the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable if and only if

$$\sigma(X, fY) = \sigma(fX, Y), \tag{5.76}$$

for any $X, Y \in \mathcal{D}$. In [12], CR-products of S-manifolds are defined as CR-submanifolds such that the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable and locally they are Riemannian products $M_1 \times M_2$, where M_1 (resp., M_2) is a leaf of $\mathcal{D} \oplus \mathcal{M}$ (resp., $\mathcal{D}^{\perp}$). From Theorem 3.1 and Proposition 3.2 in [12], we know that a CR-submanifold M of an S-manifold is a CR-product if and only if one of the following assertions is satisfied:

$$A_{f\mathcal{D}^{\perp}} f\mathcal{D} = 0, \tag{5.77}$$

$$g(\sigma(X, Y), fZ) = 0, \; X \in \mathcal{D}, \; Y \in \mathcal{X}(M), \; Z \in \mathcal{D}^{\perp}, \tag{5.78}$$

$$\nabla_X Y \in \mathcal{D} \oplus \mathcal{M}, \; X \in \mathcal{D}, \; Y \in \mathcal{X}(M). \tag{5.79}$$

5.3.2 Normal CR-Submanifolds of an S-manifold

Let M be a CR-submanifold of an S-manifold $\wedge M$. We say that M is a normal CR-submanifold if

$$N_v(X, Y) = 2t\mathrm{d}u(X, Y) - 2\sum_{\alpha=1}^{s} F(X, Y)\xi_\alpha, \tag{5.80}$$

for any $X, Y \in \mathcal{X}(M)$, where N_v is denoting the Nijenhuis torsion of v, that is

$$N_v(X, Y) = (\nabla_{vX} v)Y - (\nabla_{vY} v)X + v((\nabla_Y v)X - (\nabla_X v)Y).$$

We notice that (5.80) is equivalent to

$$S^*(X, Y) = N_v(X, Y) - t((\nabla_X u)Y - (\nabla_Y u)X) + 2\sum_{\alpha=1}^{s} F(X, Y)\xi_\alpha = 0,$$

for any $X, Y \in \mathcal{X}(M)$. Now, we can prove the following characterization theorem in terms of the shape operator.

Theorem 5.15 *A CR-submanifold M of an S-manifold $\wedge M$ is normal if and only if*

$$A_{uY} vX = vA_{uY} X, \tag{5.81}$$

for any $X \in \mathcal{D}$ and any $Y \in \mathcal{D}^{\perp}$.

Proof A direct expansion by using (5.72) and (5.73) gives that

$$S^*(X, Y) = A_{uY} vX - vA_{uY} X - A_{uX} vY + vA_{uX} Y, \tag{5.82}$$

for any $X, Y \in \mathcal{X}(M)$. Now, if M is a normal CR-submanifold of $\wedge M$, (5.81) follows form (5.82) since $uX = 0$, for any $X \in \mathcal{D}$. Conversely, if (5.81) holds, we use the decomposition $\mathcal{X}(M) = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \mathcal{M}$. First, since $uX = 0$ for any $X \in \mathcal{D}$ and $v\xi_{\alpha} = 0 = u\xi_{\alpha}$, for any α, we deduce from (5.81) and (5.82) that $S^*(X, Y) = 0$, for any $X \in \mathcal{D}$ and any $Y \in \mathcal{X}(M)$. Moreover, if $Y \in \mathcal{D}^{\perp}$, from (5.68) we have $A_{uY}\xi_{\alpha} \in \mathcal{D}^{\perp}$ and so, $vA_{uY}\xi_{\alpha} = 0$ dfor any α. Consequently, $S^*(X, \xi_{\alpha}) = 0$, for any $X \in \mathcal{X}(M)$. Finally, if $X, Y \in \mathcal{D}^{\perp}$, (5.82) becomes

$$S^*(X, Y) = v(A_{fX} Y - A_{fY} X),$$

since $vX = vY = 0$ and $uX = fX$, $uY = fY$. But, from (5.60) we easily show that $A_{fX} Y = A_{fY} X$. ■

Corollary 2 *A CR-submanifold M of an S-manifold is normal if and only if*

$$g(\sigma(X, vY) + \sigma(Y, vX), fZ) = 0, \tag{5.83}$$

$$g(\sigma(X, Z) f W) = 0, \tag{5.84}$$

for any $X, Y \in \mathcal{D}$ and any $Z, W \in \mathcal{D}^{\perp}$.

Proof Since v is skew-symmetric, from (5.81) we see that M is normal if and only if

$$g(\sigma(X, vY), uZ) = -g(\sigma(Y, vX), uZ) m \tag{5.85}$$

for any $X \in \mathcal{X}(M)$, $Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$. Now, if M is normal, from (5.85) we get (5.83) taking $X \in \mathcal{D}$ and (5.84) taking $X \in \mathcal{D}^{\perp}$. Conversely, if (5.83) and (5.84) are satisfied, we observe that (5.85) is satisfied too if $X \in \mathcal{D}$ and $X \in \mathcal{D}^{\perp}$, respectively. Finally, if $X \in \mathcal{M}$, we have $vX = 0$ and, by using that $u \circ v = 0$ and (5.67), $\sigma(X, vY) = 0$, for any $Y \in \mathcal{D}$. Thus, (5.85) holds for any $X \in \mathcal{X}(M)$. ■

Corollary 3 *Any normal generic submanifold of an S-manifold is a $(\mathcal{D}, \mathcal{D}^{\perp})$-geodesic submanifold.*

From (5.60), (5.67), (5.83), and (5.84), we have

$$\sigma(fX, Z) = f\sigma(X, Z), \tag{5.86}$$

$$t\sigma(fX, fX) = t\sigma(X, X), \tag{5.87}$$

$$A_{fZ}X \in \mathcal{D}, \tag{5.88}$$

for any $X \in$ and any $Z \in \mathcal{D}^{\perp}$. On the other hand, from (5.78) and (5.83)–(5.84), we deduce

Proposition 1 *Each CR-product in an S-manifold is a normal CR-submanifold.*

For the converse we prove the following theorems.

Theorem 5.16 *Let M be a normal CR-submanifold of an S-manifold. Then, M is a CR-product if and only if the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable.*

Proof The necessary condition is obvious. Conversely, let $X \in \mathcal{D}$. If $Y \in \mathcal{D}^{\perp}$, then (5.78) is (5.84). Further, if $Y \in \mathcal{M}$, from (5.67) we get $\sigma(X, Y) = 0$. Finally, if $Y \in \mathcal{D}$, from (5.76) and (5.83) we obtain (5.78). ■

Theorem 5.17 *Let M be a normal CR-submanifold of an S-manifold such that* $\mathrm{d}u = 0$. *Then, M is a CR-product.*

Proof A straightforward computation gives, by using the hypothesis and (5.72),

$$g((\nabla_X v)Y, Z) = \sum_{\alpha=1}^{s}\{\mathrm{d}\eta_\alpha(vX, Y)\eta_\alpha(Z) - \mathrm{d}\eta_\alpha(vZ, X)\eta_\alpha(Y)\}, \tag{5.89}$$

for any $X, Y, Z \in \mathcal{X}(M)$. Now, if $Y \in \mathcal{D}$, from (5.64) and (5.65) we get $\mathrm{d}\eta_\alpha(vX, Y) = F(vX, Y) = g(vX, vY) = g(X, Y)$. So, (5.89) becomes

$$(\nabla_X v)Y = \sum_{\alpha=1}^{s} g(X, Y)\xi_\alpha$$

for any $X \in \mathcal{X}(M)$ and any $Y \in \mathcal{D}$. Comparing with (5.72) we have $\sigma(X, Y) = 0$ and so (5.78) holds. ■

We say that v is *η-parallel* if

$$(\nabla_X v)Y = \sum_{\alpha=1}^{s}\{g(PX, PY)\xi_\alpha - \eta_\alpha(Y)PX\},$$

for any $X, Y \in \mathcal{X}(M)$. Then, from (5.64), (5.65), and (5.89), we prove

Proposition 2 *Any normal CR-submanifold of an S-manifold such that* $\mathrm{d}u = 0$ *is* η*-parallel.*

Given a *CR*-submanifold M of an S-manifold, a vector field $X \in \mathcal{X}(M)$ is said to be $\mathcal{D}$*-Killing* if

$$g(P\nabla_Z X, PY) + g(P\nabla_Y X, PZ) = 0, \tag{5.90}$$

for any $Y, Z \in \mathcal{X}(M)$. We notice that it is possible to characterize normal *CR*-submanifolds in terms of $\mathcal{D}$-Killing vector fields.

Theorem 5.18 *A CR-submanifold M of an S-manifold is a normal CR-submanifold if and only if any* $Z \in \mathcal{D}^{\perp}$ *is a* $\mathcal{D}$*-Killing vector field*

Proof Given $X, Y \in \mathcal{X}(M)$ and $Z \in \mathcal{D}^{\perp}$, from (5.74) we get

$$g(\nabla_X Z, Y) + g(\nabla_Y Z, X) = g(vA_{fZ}X, Y) - g(tD_X fZ, Y) + g(vA_{fZ}Y, X) - g(tD_Y fZ, X). \tag{5.91}$$

But $g(vA_{fZ}Y, X) = -g(A_{fZ}vX, Y)$ and so, from (5.91)

$$\begin{aligned} &g(P\nabla_X Z, PY) + g(P\nabla_Y Z, PX) + g(Q\nabla_X Z, QY) + g(Q\nabla_Y Z, QX) \\ &\quad + \sum_{\alpha=1}^{s}\{\eta_\alpha(\nabla_X Z)\eta_\alpha(Y) + \eta_\alpha(\nabla_Y Z)\eta_\alpha(X)\} \\ &= g((vA_{fZ} - A_{fZ}v)X, Y) - g(tD_X fZ, Y) - g(tD_Y fZ, X). \end{aligned} \tag{5.92}$$

Now, since it is easy to show that $\eta_\alpha(\nabla_X Z) = 0$ for any $\alpha = 1, \ldots, s$, by using (5.75), we deduce that (5.92) becomes

$$g(P\nabla_X Z, PY) + g(P\nabla_Y Z, PX) = g((vA_{fZ} - A_{fZ}v)X, Y). \tag{5.93}$$

Consequently, if Z is a $\mathcal{D}$-Killing vector field, from (5.81) we obtain that M is a normal *CR*-submanifold. Conversely, if $X \in \mathcal{D}$, the right part of the equality (5.93) vanishes by using (5.81). If $X \in \mathcal{D}^{\perp}$, then $vX = 0$ and from (5.84), $A_{fZ}X \in \mathcal{D}^{\perp}$, that is, $vA_{fZ}X = 0$ and the right part of (5.93) vanishes again. Finally, if $X \in \mathcal{M}$, $vX = 0$ and from (5.68), $A_{fZ}X \in \mathcal{D}^{\perp}$. In any case, from (5.93) we have (5.90). ■

To end this section, we recall that a submanifold M of an S-manifold is said to be *totally f-umbilical* [19] if there exists a normal vector field V such that

$$\sigma(X, Y) = g(fX, fY)V + \sum_{\alpha=1}^{s}\{\eta_a(Y)\sigma(X, \xi_\alpha) + \eta_\alpha(X)\sigma(Y, \xi_\alpha)\}, \tag{5.94}$$

for any $X, Y \in \mathcal{X}(M)$. These submanifolds have been studied and classified in [9]. Since from (5.94) we easily get (5.83) and (5.84), then we have the following theorem.

Theorem 5.19 *Any totally f-umbilical CR-submanifold of an S-manifold is a normal CR-submanifold.*

5.3.3 Normal CR-Submanifolds of an S-space-form

Let $\wedge M(c)$ a $(2m+s)$-dimensional S-space-form, where c is denoting the constant f-sectional curvature and let M be a CR-submanifold. Firstly, we can prove

Proposition 3 *If M is a normal CR-submanifold, then*

$$\|A_{fZ}X\|^2 + \|\sigma(X,Z)\|^2 - g(t\sigma(Z,Z), t\sigma(X,X)) = \frac{c+3s}{4}, \tag{5.95}$$

for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

Proof From the Codazzi equation, we have

$$\begin{aligned}\wedge R(X, fX, Z, fZ) = {} & g(D_X\sigma(fX,Z) - D_{fX}\sigma(X,Z), fZ)\\ & - g(\sigma([X,fX],Z), fZ)\\ & + g(\sigma(X,\nabla_{fX}Z) - \sigma(fX,\nabla_X Z), fZ).\end{aligned} \tag{5.96}$$

Now, from (5.60), (5.84), and (5.86), a direct expansion gives

$$g(D_X\sigma(fX,Z) - D_{fX}\sigma(X,Z), fZ) = -2\|\sigma(X,Z)\|^2. \tag{5.97}$$

On the other hand, since $X \in \mathcal{D}$ is a unit vector field (and so, fX too), we see from (5.59) that $\eta_\alpha([X,fX]) = 2$ for any α and from (5.70) that $Q[X,fX] = t\sigma(X,X) + t\sigma(fX,fX)$. Thus, taking into account (5.67), (5.84), and (5.87), we get

$$g(\sigma([X,fX],Z), fZ) = 2g(\sigma(t\sigma(X,X),Z), fZ) - 2s. \tag{5.98}$$

However, since $Z \in \mathcal{D}^\perp$, by using (5.70) it is easy to show that

$$g(\sigma(t\sigma(X,X),Z), fZ) = -g(t\sigma(X,X), t\sigma(Z,Z)).$$

Therefore, from (5.98) we have

$$g(\sigma([X,fX],Z), fZ) = -2s - 2g(t\sigma(X,X), t\sigma(Z,Z)). \tag{5.99}$$

Next, since $\eta_\alpha(\nabla_{fX}Z) = \eta_\alpha(\nabla_X Z) = 0$ for any α, from (5.69), (5.83), (5.84), and (5.88), we obtain

$$g(\sigma(X,\nabla_{fX}Z) - \sigma(fX,\nabla_X Z), fZ) = -2\|A_{fZ}X\|^2. \tag{5.100}$$

Finally, from (5.61) we deduce $\wedge R(X, fX, Z, fZ) = -(c-s)/2$. Then, substituting (5.97), (5.99), and (5.100) into (5.96), we complete the proof. ■

Corollary 4 *If M is a normal $\mathcal{D}^{\perp}$-geodesic CR-submanifold of an S-space-form $\wedge M(c)$, then $c \geq -3s$.*

Proposition 4 *If M is a normal CR-submanifold of an S-space-form $\wedge M(c)$ such the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable, then $c \geq -3s$ and M is a CR-product.*

Proof It is clear that M is a CR-product due to Theorem 5.16. Moreover, from (5.78) we have $g(\sigma(X, Y), fZ) = 0$. for any $X, Y \in \mathcal{D}$. Then, if $X \in \mathcal{D}$ is a unit vector field, $t\sigma(X, X) = 0$ and, by using (5.95), $c \geq -3s$. ■

Now, we are going to study the concrete case of the $(2m+s)$-dimensional euclidean S-space-form $\mathbb{R}^{2m+s}(-3s)$ (see [13] for the details of this structure). In this context, we can prove

Theorem 5.20 *If M is a normal $(\mathcal{D}, \mathcal{D}^{\perp})$-geodesic and $\mathcal{D}^{\perp}$-geodesic CR-submanifold of $\mathbb{R}^{2m+s}(-3s)$, then it is a CR-product.*

Proof From (5.95), we have $A_{fZ}X = 0$ for any $X \in \mathcal{D}$ and any $Z \in \mathcal{D}^{\perp}$. So, from (5.77), M is a CR-product. ■

Corollary 5 *A normal $\mathcal{D}^{\perp}$-geodesic generic submanifold of $\mathbb{R}^{2m+s}(-3s)$ is a CR-product.*

Another interesting example of S-space-form is $\mathbb{H}^{2m+s}(4-3s)$, a generalization of the Hopf fibration $\pi : \S^{2m+1} \longrightarrow \mathbb{P}C^m$, introduced by Blair in [5] as a canonical example of an S-manifold playing the role of the complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. This space is given by (see [5, 6] for more details)

$$\mathbb{H}^{2m+s} = \{(x_1, \ldots, x_s) \in \S^{2m+1} \times \overset{s)}{\cdots} \times \S^{2m+1} / \pi(x_1) = \cdots = \pi(x_s)\}$$

and its f-sectional curvature is constant equal to $4-3s$. Let M be a CR-submanifold of $\mathbb{H}^{2m+s}(4-3s)$ (we always suppose $s \geq 2$). Denote by ν the orthogonal complementary distribution of $f\mathcal{D}^{\perp}$ in $T(M)^{\perp}$. Then, $f\nu \subseteq \nu$. Let

$$\{E_1, \ldots, E_{2p}\}, \quad \{F_1, \ldots, F_q\}, \quad \{V_1, \ldots, V_r, fV_1, \ldots, fV_r\},$$

be local fields of orthonormal frames on $\mathcal{D}$, $\mathcal{D}^{\perp}$ and ν, respectively, where $2r$ is the real dimension of ν. First, we prove

Lemma 6 *If M is a CR-product in $\mathbb{H}^{2m+s}(4-3s)$, then*

$$\|\sigma(X, Z)\| = 1, \tag{5.101}$$

for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

Proof We know, from Proposition 1, that M is a normal CR-submanifold. Since, $c = 4 - 3s$, from (5.77), (5.78) and (5.95) we complete the proof. ■

Lemma 7 *If M is a CR-product in $\mathbb{H}^{2m+s}(4-3s)$, the vector field $\sigma(E_i, F_a)$, $i = 1, \ldots, 2p$ and $a = 1, \ldots, q$, are $2pq$ orthonormal vector fields on ν.*

Proof From (5.101) and by the linearity, we get $g(\sigma(E_i, Z), \sigma(E_j, Z)) = 0$, for any $i, j = 1, \ldots, 2p$, $i \neq j$ and any unit vector field $Z \in \mathcal{D}^\perp$. Now, from (5.84), if $q = 1$, we complete the proof. If $q \geq 2$, by linearity again, we have $g(\sigma(E_i, F_a), \sigma(E_j, F_b)) + g(\sigma(E_i, F_b), \sigma(E_j, F_a)) = 0$, for any $i, j = 1, \ldots, 2p$, $i \neq j$, $a, b = 1, \ldots, q$, $a \neq b$. Next, by using (5.79) and the Bianchi identity, we obtain $R(X, Y, Z, W) = 0$, for any $X, Y \in \mathcal{D}$, $Z, W \in \mathcal{D}^\perp$, where R is denoting the curvature tensor field of M. But, if $i \neq j$ and $a \neq b$, (5.61) gives $\wedge R(E_i, E_j, F_a, F_b) = 0$. Then, from the Gauss equation we get

$$g(\sigma(E_i, F_a), \sigma(E_j, F_b)) - g(\sigma(E_i, F_b), \sigma(E_j, F_a)) = 0,$$

for any $i, j = 1, \ldots, 2p$, $i \neq j$, $a, b = 1, \ldots, q$, $a \neq b$ and this completes the proof. ■

Now, we study the normal CR-submanifolds of $\mathbb{H}^{2m+s}(4-3s)$.

Theorem 5.21 *Let M be a normal CR-submanifold of $\mathbb{H}^{2m+s}(4-3s)$, $s \geq 2$, such that the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable. Then*

(i) M is a CR-product $M_1 \times M_2$.
(ii) $m \geq pq + p + q$.
(iii) If $n = pq + p + q$, then M_1 is an invariant totally geodesic submanifold immersed in $\mathbb{H}^{2m+s}(4-3s)$.
(iv) $\|\sigma\|^2 \geq 2q(2p + s)$.
(v) If $\|\sigma\|^2 = 2q(2p + s)$, then M_1 is an S-space-form of constant f-sectional curvature $4 - 3s$ and M_2 has constant curvature 1.
(vi) If M is a minimal submanifold, then $\rho \leq 4p(p+1) + 2p(q+s) + q(q-1)$, where ρ denotes the scalar curvature and the equality holds if and only if $\|\sigma\|^2 = 2q(2p + s)$.

Proof (i) follows directly from Proposition 4. Now, from Lemma 7, $dim(\nu) = 2(m - p) - 2q \geq 2pq$. So (ii) holds. Next, suppose that $m = pq + p + q$. If $X, Y, Z \in \mathcal{D}$ and $W \in \mathcal{D}^\perp$, from (5.61), $\wedge R(X, Y, Z, W) = 0$ and, by using a similar proof to that one of Lemma 7, $R(X, Y, Z, W) = 0$. So, the Gauss equation gives

$$g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)) = 0. \tag{5.102}$$

Since from Proposition 3.2 of [12], $\sigma(fX, Z) = f\sigma(X, Z)$, if we put $Y = fX$, we have, by using (5.86), $g(\sigma(fX, W), (\sigma(X, Z)) = 0$. Now, if we put $Z = fY$, then $g(\sigma(X, Y), \sigma(X, W)) = 0$. Thus, by linearity, we get $g(\sigma(X, W), \sigma(Y, Z)) + g(\sigma(X, Z), \sigma(Y, W)) = 0$. Consequently, from (5.102)

$$g(\sigma(X, W), \sigma(Y, Z)) = 0, \tag{5.103}$$

for any $X, Y, Z \in \mathcal{D}$ and $W \in \mathcal{D}^{\perp}$. Since now $dim(\nu) = 2pq$, (5.103) implies that $\sigma(X, Y) = 0$, for any $X, Y \in \mathcal{D}$ and so, (iii) holds from Theorem 2.4(ii) of [12]. Assertions (iv) and (v) follow from Theorem 4.2 of [12]. Finally, if M is a minimal normal CR-submanifold of $\mathbb{H}^{2m+s}(4-3s)$, a straightforward computation gives

$$\rho = 4p(p+1) + 2s(p+q) + q(q-1) + 6pq - \|\sigma\|^2.$$

Then, by using (iv), the proof is complete. ■

Theorem 5.22 *Let M be a normal, $(\mathcal{D}, \mathcal{D}^{\perp})$-geodesic and $\mathcal{D}^{\perp}$-geodesic CR-submanifold of $\mathbb{H}^{2m+s}(4-3s)$. Then,*

(i) $\|A_{fZ}X\| = 1$, *for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.*
(ii) $\|\sigma\|^2 \geq 2q(p+s)$ *and the equality hold if and only if $\sigma(\mathcal{D}, \mathcal{D}) \in f\mathcal{D}^{\perp}$.*

Proof (i) follows immediately from (5.95). Now, considering the above-mentioned local fields of orthonormal frames for $\mathcal{D}$, $\mathcal{D}^{\perp}$, and ν, a straightforward computation using the hypothesis gives (ii). ■

Finally, from (5.84) and (5.95), we can prove

Corollary 6 *Let M be a normal $\mathcal{D}^{\perp}$-geodesic generic submanifold of $\mathbb{H}^{2m+s}(4-3s)$. Then*

(i) $\|A_{fZ}X\| = 1$, *for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.*
(ii) $\|\sigma\|^2 = 2q(p+s)$.

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