Chapter 12
CR-Submanifolds of Semi-Riemannian Kaehler Manifolds

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12.1 Introduction

An almost complex structure on a smooth manifold M is a (1, 1)-tensor field J satisfying the condition

$$J^2 = -I \tag{12.1}$$

where I is the identity operator on the tangent space at each point. M furnished with an almost complex structure is known as an almost complex manifold and is even-dimensional and orientable. An almost complex structure that comes from a complex structure is called integrable, and when one wishes to specify a complex structure as opposed to an almost complex structure, one calls it an integrable complex structure. This integrability condition is equivalent to the vanishing of the Nijenhuis' tensor $[J, J]$ defined by

$$[J, J](X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]$$

for arbitrary vector fields X and Y on M. An almost complex manifold is called an almost Hermitian manifold if there exists a Riemannian metric g such that $g(JX, JY) = g(X, Y)$. An almost Hermitian manifold is said to be Hermitian if the underlying almost complex structure is integrable.

One may note that, if g is semi-Riemannian, then its signature has even number (including 0) of positive signs and even number (including 0) of negative signs.

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A Kaehler manifold is a Hermitian manifold whose complex structure J is parallel with respect to the Levi-Civita connection ∇ of g. A holomorphic section of a Kaehler manifold is section obtained by a plane element spanned by a non-null tangent vector X (i.e. $g(X, X) \neq 0$) at a point and JX. The sectional curvature of M with respect to a holomorphic section is called the holomorphic sectional curvature. A Kaehler manifold is said to be a complex space-form if its holomorphic sectional curvature is independent of the choice of a holomorphic section at each point. A complex space-form with constant holomorphic sectional curvature c is denoted by $M(c)$ whose curvature tensor is given by

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ] \tag{12.2}$$

One can easily check that c is constant on M.

At this point, we denote an almost Hermitian manifold by $(\bar{M}, g)$ with the almost complex structure J. Generalizing the cases of invariant and anti-invariant (i.e., totally real) submanifolds, Bejancu [2] introduced the notion of a CR-submanifold as follows.

Definition 1 A CR-submanifold of a semi-Riemannian almost Hermitian manifold $(\bar{M}, g)$ is a non-degenerate submanifold (M, g) of $(\bar{M}, g)$ admitting a smooth distribution $D : p \to D_p \subset T_pM$ such that

(1) D is invariant, i.e., $JD_p = D_p$ at each point $p \in M$ and
(2) the orthogonal complementary distribution $D^\perp$ is anti-invariant, i.e. $JD_p^\perp \subset (T_pM)^\perp$ for each $p \in M$.

Remark 1 The above definition would not be feasible if D was degenerate with respect to g, because D and $D^\perp$ are not necessarily complementary in TM for degenerate D. Henceforth, we will assume D and $D^\perp$ both non-degenerate with respect to g.

Recall that a CR-structure on a smooth manifold M is a complex subbundle H of the complexified tangent bundle $C(TM)$ of M such that $(H \cap \bar{H})_p = 0$ at each $p \in M$ and H is involutive, i.e., $X, Y \in H \Rightarrow [X, Y] \in H$. It is known that, on a CR-manifold, there exist a real distribution D and a field of endomorphisms $P : D \to D$ such that $P^2 = -I$, $D = Re(H \oplus \bar{H})$ and $H_p = \{X - iPX : X \in D_p\}$. Blair and Chen [7] proved that a proper CR-submanifold M of a Hermitian manifold is a CR-manifold. This justifies the term "CR-submanifold."

If the holomorphic distribution D is equal to the tangent bundle TM, then M reduces to an invariant submanifold of $\bar{M}$, and if the totally real distribution $D^\perp$ equals TM then M reduces to a totally real submanifold of $\bar{M}$. When the dimensions of $D^\perp$ and $(TM)^\perp$ are equal, M is said to be a generic submanifold of $\bar{M}$.

For a tangential vector field X and a normal vector field V on a CR-submanifold of an almost Hermitian manifold $\bar{M}$, we have the following decomposition formulas:

$$JX = PX + FX \tag{12.3}$$

$$JV = tV + fV \tag{12.4}$$

where PX and tV are the tangential parts of JX and JV, respectively, and FX and fV are the normal parts of JX and JV, respectively. It is easy to verify from the preceding two equations that

$$g(FX, V) + g(X, tV) = 0 \tag{12.5}$$

and that $g(PX, Y)$ is skew-symmetric in X, Y, and $g(fU, V)$ is skew-symmetric in U, V. Operation of J on Eqs. (12.3) and (12.4) yields the following relations:

$$P^2 = -I - tF, \quad FP + fF = 0 \tag{12.6}$$

$$Pt + tf = 0, \quad f^2 = -I - Ft \tag{12.7}$$

Let us denote the projection operator on D by l and that on $D^{\perp}$ bt $l^{\perp}$. Then, evidently

$$l + l^{\perp} = I, \quad l^2 = l, \quad (l^{\perp})^2 = l^{\perp}, \quad ll^{\perp} = l^{\perp}l = 0$$

$$l^{\perp}Pl = 0, \quad Fl = 0, \quad Pl = P$$

Using this in the second equation of (12.6) one gets

$$FP = 0 \tag{12.8}$$

Thus, the second equation of (12.6) reduces to

$$fF = 0$$

Taking fV for V in Eq. (12.5) we find

$$tf = 0 \tag{12.9}$$

Using this in the first equation of (12.7) gives

$$Pt = 0 \tag{12.10}$$

Consequently, the first equation of (12.6) implies

$$P^3 + P = 0 \tag{12.11}$$

and the second equation of (12.7) implies

$$f^3 + f = 0 \tag{12.12}$$

Equations (12.11) and (12.12) show that P and f define f-structures (for details on an f-structure, we refer to Yano [18]) on the tangent and normal bundles of M, respectively. Setting $l = P^2$ and $l^{\perp} = I - l$, one can easily verify the following result of Yano and Kon [21].

Theorem 12.1 *A submanifold M of an almost Hermitian manifold $\bar{M}$ is a CR-submanifold if and only if $FP = 0$.*

12.2 Basic Equations and Results

According to a theorem of Flaherty [12], we know that the signature of a Hermitian metric g on an almost complex manifold has even number of positive signs and even number of negative signs. Thus, g cannot be Lorentzian which is essential for a physical space-time of relativity. For a four dimensional space-time, we can choose a coordinate system comprising two real coordinates x, y and complex null coordinates $z + it$ and $z - it$. The aforementioned facts suggest that a complex structure can be defined only on its two dimensional submanifold defined by $x =$ constant and $y =$ constant. With this motivation and the purpose of applying our results in relativity theory, we consider a class of submanifolds of a semi-Riemannian Kaehler manifold such that there may be complementary complex and real distributions. One of the settings for such a distribution can be provided by singling out holomorphic distributions of the *CR*-submanifolds (see for example, Penrose [15]).

As pointed out in the previous section, a *CR*-submanifold (M, g) has an induced f-structure defined by the (1, 1) tensor field P on M, and hence the metric g can be Lorentzian. Our study is not only applicable within the framework of general relativity, but also in the theory of semi-Riemannian manifolds whose metrics have signatures compatible with the induced f-structure. We also note in our context that indefinite Kaehler manifolds (in particular, complex space-forms) were studied by Barros and Romero in [1].

We denote the Levi-Civita connection of the induced metric g on the *CR*-submanifold (M, g) of a semi-Riemannian Kaehler manifold $(\bar{M}, g, J)$ by ∇ and that of $(\bar{M}, g)$ by $\bar{\nabla}$. The second fundamental form of M is denoted by B and the Weingarten operator by A_V for an arbitrary normal vector field V on M. They are related by $g(A_V X, Y) = g(B(X, Y), V)$. The Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)$$

$$\bar{\nabla}_X V = -A_V X + D_X V$$

The Gauss and Codazzi equations are

$$g(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) - g(B(Y,Z),B(X,W)) + g(B(X,Z),B(Y,W))$$

$$[\bar{R}(X,Y)Z]^{\perp} = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)$$

where

$$(\nabla_X B)(Y,Z) = D_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) \tag{12.13}$$

X, Y, Z, W denote arbitrary vector fields tangent to M, and D denotes the normal connection of M. Also, R and $\bar{R}$ denote the curvature tensors of ∇ and $\bar{\nabla}$, respectively.

The covariant derivatives of the operators P, F, t, f are defined along M as

$$(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y) \tag{12.14}$$

$$(\nabla_X F)Y = D_X(FY) - F(\nabla_X Y) \tag{12.15}$$

$$(\nabla_X t)V = \nabla_X(tV) - t(D_X V) \tag{12.16}$$

$$(\nabla_X f)V = D_X(fV) - f(D_X V) \tag{12.17}$$

At this point, we use the Kaehlerian condition $\bar{\nabla} J = 0$. The Gauss and Weingarten formulas provide the following equations

$$(\nabla_X P)Y = A_{FY}X + tB(X,Y) \tag{12.18}$$

$$(\nabla_X F)Y = -B(X,PY) + fB(X,Y) \tag{12.19}$$

$$(\nabla_X t)V = A_{fV}X - PA_V X \tag{12.20}$$

$$(\nabla_X f)V = -FA_V X - B(X,tV) \tag{12.21}$$

We now recall the following results and definitions from Yano and Kon [20] and Yano and Ishihara [19], that will be used later.

Lemma 1 *Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then, for any vector fields X and Y in $D^{\perp}$ we have*

$$A_{FX}Y = A_{FY}X. \tag{12.22}$$

Theorem 12.2 *Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then, the totally real distribution $D^{\perp}$ is integrable and its maximal integral submanifold $M^{\perp}$ is an anti-invariant (totally real) submanifold of M.*

Definition 2 The f-structure induced on the CR-submanifold of a Kaehler manifold is said to be partially integrable if D is integrable and the almost complex structure induced on each leaf of D is integrable.

Theorem 12.3 *Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then the f-structure induced on M is partially integrable if and only if*

$$B(PX, Y) = B(X, PY) \tag{12.23}$$

for all X and Y in D.

Definition 3 The f-structure induced on the CR-submanifold M of a Kaehler manifold is said to be normal if the (1, 2)-tensor field S defined by

$$S(X, Y) = [P, P](X, Y) - t((\nabla_X F)Y - (\nabla_Y F)X)$$

vanishes identically on M.

This normality condition is equivalent (see [20]) to $A_{FX} = PA_{FX}$ for any X tangent to M.

Definition 4 A CR-submanifold of a Kaehler manifold is said to be mixed totally geodesic if $B(X, Y) = 0$ for any vector field $X \in D$ and $Y \in D^{\perp}$.

12.3 Mixed Foliate *CR*-Submanifolds

In this section, we will deal with a subclass of mixed totally geodesic CR-submanifolds characterized by the partial integrability of f-structure induced on them.

Definition 5 A CR-submanifold of a Kaehler manifold is called mixed foliate if it is mixed totally geodesic and the f-structure induced on it is partially integrable.

Next, we recall the following lemma (see Yano and Kon [20]).

Lemma 2 *Let M be a mixed foliate CR-submanifold of a Kaehler manifold $\bar{M}$. Then, for all $V \in (TM)^{\perp}$ we have*

$$A_V P + PA_V = 0 \tag{12.24}$$

Now, we recall the following theorem of Bejancu et al. [4], which holds for a positive definite Kaehler metric.

Theorem 12.4 *If M is a mixed foliate proper CR-submanifold of a complex space-form $\bar{M}(c)$, then $c \leq 0$.*

So, the following question arises "what sort of constraint is imposed on the possible values of c when the metric of $\bar{M}(c)$ is indefinite?" Sharma and Duggal [17] provided an answer to this question in a special case in the form of the following result.

Theorem 12.5 *If the mixed foliate proper CR-submanifold of a semi-Riemannian complex space-form $\bar{M}(c)$ is such that the metric g restricted to D is definite and g restricted to $D^{\perp}$ is indefinite, then $c = 0$.*

Proof The curvature tensor of $\bar{M}(c)$ is given by Eq. (12.2). Restricting the vector fields X, Y to D and Z to $D^{\perp}$ we find that

$$[\bar{R}(X, Y)Z]^{\perp} = \frac{c}{2} g(PY, X)JZ \tag{12.25}$$

Equation (12.13) provides

$$(\nabla_X B)(Y, Z) = -B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

since M is mixed totally geodesic. Anti-symmetrizing the last equation with respect to X and Y we get

$$\begin{aligned}(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = &-B([X, Y], Z)\\ &- B(Y, \nabla_X Z) + B(X, \nabla_Y Z)\end{aligned}$$

As per our hypothesis, M is mixed foliate and hence, by the integrability of D, the above equation reduces to

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = -B(Y, \nabla_X Z) + B(X, \nabla_Y Z) \tag{12.26}$$

As Z is any vector field in $D^{\perp}$, there is a normal vector field V such that $Z = JV$. Therefore, $Z = tV$ and $fV = 0$. Consequently, we have

$$\nabla_Y Z = (\nabla_Y t)V + tD_Y V = tD_Y V - PA_V Y$$

where we used Eq. (12.20). The Eq. (12.10) shows that $tD_Y V \in D^{\perp}$. The use of Lemma 2 transforms equation (12.26) into

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(X, A_V PY) + B(PY, A_V X) \tag{12.27}$$

Taking $X = PY$, substituting the value JV for Z, and taking inner product with V provides

$$g(A_V PY, A_V PY) = -\frac{c}{4} g(PY, PY)g(V, V) \tag{12.28}$$

By hypothesis, g restricted to D is definite. By Lemma 2, $A_V PY = -PA_V Y$. Thus, the above equation implies the inequality

$$cg(V, V) = cg(Z, Z) \leq 0$$

for any $Z \in D^{\perp}$. Again, by hypothesis, g restricted to $D^{\perp}$ is indefinite, and therefore, $D^{\perp}$ does contain at least one space-like vector field Z_1 (i.e., $g(Z_1, Z_1) > 0$) and a time-like vector field Z_2 (i.e., $g(Z_2, Z_2) < 0$). Consequently, we get $c \geq 0$ and $c \leq 0$. Thus, we conclude that $c = 0$, completing the proof.

Corollary 1 *Under the hypothesis of the preceding theorem, we have*

$$A_V P = 0 \tag{12.29}$$

for every $V \in JD^{\perp}$.

Proof It follows from (12.28) and the conclusion $c = 0$ of Theorem 12.5, that $g(A_V PY, A_V PY) = 0$. Since $A_V P = -PA_V$, the vector field $A_V PY \in D$. The hypothesis that g is definite on D, implies that (12.28) holds.

Remark 2 The following result of Chen [8] "A CR-submanifold of C^n is mixed foliate if and only if it is a CR-product, i.e., the product of the leaves of D and $D^{\perp}$" can be shown to be valid for both definite and indefinite metrics.

Employing it for the mixed foliate CR-submanifold M under the hypothesis of the Theorem 12.5, it follows straightaway that M is a CR-product. This can be proved independently (without using Chen's theorem) in another way to gain more insight into the structure of M. First, let us establish the following lemma.

Lemma 3 *A necessary and sufficient condition for the integrability of the f-structure induced on a mixed foliate proper CR-submanifold M of a Kaehler manifold $\bar{M}$ is that $A_{FY} P = 0$ for any vector field Y tangent to M.*

Proof We know that a proper mixed foliate CR-submanifold M of a Kaehler manifold $\bar{M}$ has a partially integrable f-structure and integrable $D^{\perp}$. The f-structure would be completely integrable if its Nijenhuis tensor $[P, P]$ vanishes identically, i.e.,

$$[PX, PY] + P^2[X, Y] - P[PX, Y] - P[X, PY] = 0.$$

For $X, Y \in D$, the integrability of D implies $[P, P](X, Y) = 0$. For $X, Y \in D^{\perp}$, the integrability of $D^{\perp}$ implies $[P, P](X, Y) = 0$. For $X \in D$ and $Y \in D^{\perp}$ we observe that

$$\begin{aligned}[P, P](X, Y) &= (\nabla_{PX} P)Y - (\nabla_{PY} P)X - P[(\nabla_X P)Y - (\nabla_Y P)X] \\ &= A_{FY} PX - PA_{FY} X = 2A_{FY} PX\end{aligned}$$

where Eqs. (12.18) and (12.24) have been used. Hence, the f-structure induced on M is integrable if and only if $A_{FY} P = 0$ for any V in $JD^{\perp}$.

Definition 6 The f-structure induced on the CR-submanifold of a Kaehler manifold is said to be normal if the (1, 2)-tensor field S defined by

$$S(X, Y) = [P, P](X, Y) - t[(\nabla_X F)Y - (\nabla_Y F)X]$$

vanishes identically.

It has been shown in [20] that the normality of the f-structure induced on the CR-submanifold of a Kaehler manifold is equivalent to $A_{FX}P = PA_{FX}$, for any vector field X tangent to M. This holds for a definite as well as an indefinite metric. The following result characterizes the intrinsic structure of M hypothesized as in Theorem 12.5.

Theorem 12.6 *Under the hypothesis of Theorem 12.5, the f-structure induced on M is integrable and normal. Moreover, if $D^\perp$ is parallel, then M is locally a CR-product $M^T \times M^\perp$, where M^T is flat and $M^\perp$ is a totally geodesic real submanifold of M.*

Proof We conclude from Eq. (12.28) and the conclusion $c = 0$ of Theorem 12.5 that $A_V P = 0$. Hence Lemma 3 asserts that the f-structure on M is integrable. Now, the necessary and sufficient condition for the normality of the f-structure on M is $PA_V = A_V P$ for any $V \in JD^\perp$. This is automatically satisfied since we have $A_V P = 0$ (corollary to Theorem 12.5 and Lemma 2). Thus, the structure is also normal. It can be shown with the aid of Eq. (12.18) that the fundamental 2-form Ω of the f-structure is closed. It therefore follows from a result of Goldberg [13] that $(\nabla_X P)Y = 0$ for $X \in D$. The expression (12.18) for $(\nabla_X P)Y$ ensures that it lies in $D^\perp$, which is clear from the result $PA_V X = -A_V PX = 0$ so that $A_V X \in D^\perp$. Next, from Eq. (12.18) and Lemma 1 we have $(\nabla_X P)Y = (\nabla_Y P)X$ for all $X, Y \in D^\perp$. Therefore, $g((\nabla_X P)Y, Z) = g((\nabla_Y P)X, Z)$ for all $X, Y \in D^\perp$ and $Z \in T(M)$. This means, $(\nabla_X \Omega)(Y, Z) = -(\nabla_Y \Omega)(Z, X)$ whence we find $(\nabla_Z \Omega)(X, Y) = 0$. This shows that $(\nabla_Z P)X \in D$, but as shown earlier, $(\nabla_Z P)X \in D^\perp$ for any Z and X tangent to M. We had also proved that $(\nabla_Z P)X = 0$ whenever $X \in D$ and Z is tangent to M. Consequently, we obtain $(\nabla_Z P)X = 0$ for any Z and X tangent to M, i.e. $\nabla P = 0$. Applying Chen's result [8] "A CR-submanifold of a Kaehler manifold is a CR-product if and only if $\nabla P = 0$," we conclude that M is $M^T \times M^\perp$, where M^T is a leaf of D totally geodesic in M and $M^\perp$ is a leaf of $D^\perp$ totally geodesic in M. This shows that M^T is flat, and hence completes the proof.

Proposition 1 *Under the hypothesis of Theorem 12.5, if $D^\perp$ is parallelizable and the normal connection is flat, then M is locally flat.*

Proof Since $D^\perp$ is parallelizable, we can choose an orthonormal base (ξ_a) of $D^\perp$. If (η^a) denotes its dual, then one can show that $FX = \eta^a(X)J\xi_a$ and $tJ\xi_a = -\xi_a$. Hence, we have

$$S(X, Y) = N_P(X, Y) - t[(\nabla_X F)Y - (\nabla_Y F)X]$$

Therefore, as the f-structure is integrable and normal, we obtain

$$(d\eta^a)(X, Y)\xi_a - \eta^a(Y)tD_X J\xi_a + \eta^a(X)tD_Y J\xi_a = 0$$

But the normal connection is flat, and so $d\eta^a = 0$. For such a structure we know from Blair [5] that $L_{\xi_a} g = 0$. Consequently, $\nabla \xi_a = 0$ and hence $R(X, Y)\xi_a = 0$, i.e., $M^\perp$ is locally flat. Hence, M is locally flat. This completes the proof.

Remark 3 If M of the foregoing proposition was of dimension 4 and complete, and the f-structure globally framed, then M would be a quotient of the Minkowski space-time of special relativity.

12.4 Normal Mixed Totally Geodesic *CR*-Submanifolds

Let us consider a class of *CR*-submanifolds of a Kaehler manifold, which are mixed totally geodesic with distribution D not necessarily integrable (unlike that of a mixed foliate *CR*-submanifold) and the f-structure induced on M is normal.

Definition 7 A *CR*-submanifold M of a Kaehler manifold $\bar{M}$ is said to be normal mixed totally geodesic if it is mixed totally geodesic and the f-structure induced on M is normal.

We state and prove the following result.

Theorem 12.7 *Let M be a normal mixed totally geodesic CR-submanifold of a complex space-form $\bar{M}(c)$. Then,*

(1) if g and $W = A_V^2 + A_{FA_V Z}$ ($V \in JD^\perp$ and $Z = JV$) are positive definite on D, then $c \geq 0$ and
(2) if g is positive definite on D and indefinite on $D^\perp$, then W cannot be definite on D. Also, $c = 0$ if and only if $W = 0$ on D.

Proof Supposing $X, Y \in D$, using Codazzi equation and the expression (12.2) for the curvature tensor of $\bar{M}(c)$ we can show that

$$B(Y, PA_V X) - B(X, PA_V Y) - B([X, Y], Z) = \frac{c}{2} g(PY, X)JZ$$

where $Z = JV \in D^\perp$. Taking its inner product with V we get

$$g(A_V Y, PA_V X) - g(A_V X, PA_V Y) - g(A_V Z, [X, Y]) + \frac{c}{2} g(PY, X)g(V, V) = 0 \tag{12.30}$$

It can be shown by a straightforward computation that

$$B(PX, Y) - B(X, PY) = F[X, Y]$$

which, in turn, implies that

$$g(A_V PX + PA_V X, Y) = g([X, Y], Z)$$

Substituting PY for X in Eq. (12.30) gives

$$g(WY, Y) = \frac{c}{2} g(V, V) g(Y, Y) \tag{12.31}$$

where we have used the normality condition $A_V P = PA_V$. If g and W (as defined in Theorem 12.7) are positive definite on D, then (12.31) implies that $c \geq 0$, which proves part (1). Let W be definite on D. If g is positive definite on D and indefinite on $D^\perp$, then (12.31) implies $cg(V, V) = cg(Z, Z) \geq 0$. Now, Z being in $D^\perp$ could be space-like or time-like. Hence $c = 0$, and therefore the operator W vanishes on D, which contradicts our hypothesis that W is definite. The last part of (2) follows from Eq. (12.31). This completes the proof.

Remark 4 For the case when g is definite on D and indefinite on $D^\perp$, we compare Theorem 12.5 and part (2) of the Theorem 12.7. As a consequence of Theorem 12.5, M reduces to a CR-product provided the f-structure on it is globally framed. On the other hand Theorem 12.7 involves the operator W on D. The condition that c may vanish, is that W must vanish identically on D. This is quite compatible with the consequence of Theorem 12.5 in that if we assume that M of Theorem 12.7 (part (2)) is a CR-product then we must have A_V vanish on D and hence the operator W vanishes on D, thus reducing c to 0. Hence, we claim to have gotten a wider class of CR-submanifolds, as hypothesized in part (2) of Theorem 12.7, which can be embedded in C^n.

12.5 Totally Umbilical *CR*-Submanifolds

This section is devoted to totally umbilical CR-submanifolds of a Kaehler manifold. We denote the dimension of the totally real distribution $D^\perp$ by q. First, we state and prove

Proposition 2 *Let M be a CR-submanifold of a Kaehler manifold. Then both the distributions D and $D^\perp$ are non-degenerate.*

Proof Let D be degenerate. Then there exists a nonzero vector field $X \in D$ such that $g(X, Y) = 0$ for all $Y \in D$. As D and $D^\perp$ are complementary and orthogonal to each other, we conclude that $g(X, Y) = 0$ for all $Y \in TM$. Hence $X = 0$ because TM is nondegenerate. But X is nonzero. Hence, we arrive at a contradiction. This proves that D is nondegenerate. Similarly, one can show that $D^\perp$ is non-degenerate.

Next, we state and prove

Proposition 3 *The mean curvature vector μ of a totally umbilical CR-submanifold of a Kaehler manifold belongs to $JD^\perp$.*

Proof Total umbilicity of M means $B(X, Y) = g(X, Y)\mu$. Consider any $X \in D$ and V in the complementary orthogonal subbundle to $JD^\perp$ in $TM^\perp$. Then we have

$$g(J(\bar{\nabla}_X X), JV) = g(\bar{\nabla}_X JX, JV)$$
$$= g(\nabla_X JX + g(X, JX)\mu, JV) = 0$$

$$g(J(\bar{\nabla}_X X), JV) = g(\bar{\nabla}_X X, V)$$
$$= g(\nabla_X X + g(X, X)\mu, V) = g(X, X)g(\mu, V).$$

Thus we find $g(X, X)g(\mu, V) = 0$. By the preceding proposition, we conclude that $g(\mu, V) = 0$, i.e. $f\mu = 0$. Hence $\mu \in JD^\perp$, completing the proof.

Let us recall the following result of Bejancu [3] for a positive definite metric.

Theorem 12.8 *Let M be a totally umbilical proper CR-submanifold of a Kaehler manifold M. For $q > 1$, M reduces to a totally geodesic submanifold and is locally a Riemannian product of an invariant and an anti-invariant submanifold of M.*

That this theorem holds for an indefinite metric, was proved by Duggal and Sharma [11]. The proof is slightly longer than that for the positive definite case, and is given below.

Proof By virtue of Lemma 1, we have $A_{FX}Y = A_{FY}X$ for all $X, Y \in D^\perp$. As $t\mu \in D^\perp$, for any $X \in D^\perp$ we have $A_{FX}t\mu = A_{Ft\mu}X$. As M is totally umbilical, we have $B(X, Y) = g(X, Y)\mu$ and $A_V X = g(\mu, V)X$. Hence we obtain

$$g(t\mu, X)t\mu = g(t\mu, t\mu)X \tag{12.32}$$

for all $X \in D^\perp$. Since $q > 1$, it follows, upon contraction of Eq. (12.32) at X with respect to a local orthonormal basis of $D^\perp$, that $g(t\mu, t\mu) = 0$. Hence $t\mu = 0$. Now, let X be a vector field tangent to M. Then

$$(\nabla_X t)\mu = \nabla_X t\mu - tD_X\mu = -tD_X\mu$$

Using Eq. (12.20) in the above equation provides

$$-tD_X\mu = A_{f\mu}X - PA_\mu X = -g(\mu, \mu)PX$$

Using $Pt = 0$, we get $g(\mu, \mu)P^2X = 0$. As M is proper CR-submanifold, we conclude that $g(\mu, \mu) = 0$. Further, $(\nabla_X P)Y = A_{FY}X + tB(X, Y) = -g(Y, t\mu) = 0$. Thus we obtain $\nabla_X P = 0$, which implies through Chen's result [8] mentioned earlier, that M is locally a product of an invariant submanifold M^T and an anti-invariant submanifold $M^\perp$ of M. What remains to be proved is that $\mu = 0$. Suppose that $Y \in D$ so that $FY = 0$. As D is parallel, $\nabla_X Y \in D$ and hence $F(\nabla_X Y) = 0$. Consequently, $(\nabla_X F)Y = 0$ and using Eq. (12.19) we have $g(X, PY)\mu = g(X, Y)f\mu$ for every $Y \in D$. Substituting $X = PY$ and noting the skew-symmetry $g(PX, Y) = -g(PY, X)$ we find that $\mu = 0$. Hence $B = 0$, i.e., M is totally geodesic and locally a CR-product of the leaves of D and $D^\perp$. This completes the proof.

The case $q = 1$ was not covered in the preceding theorem. Chen [9] proved the following result.

Theorem 12.9 *Let M be a totally umbilical CR-submanifold of a Kaehler manifold $\bar{M}$. Then (i) M is totally geodesic, or (ii) $q = 1$, or (iii) M is totally real.*

Note that if M was a proper CR-submanifold in the above theorem, then the possibility (iii) would be ruled out. Also, note that (i) and (ii) are not mutually exclusive. The case (ii) has been investigated by Chen [9], in the context of a locally Hermitian symmetric space $\bar{M}$ with dim. $\bar{M} \geq 5$. In [11], Duggal and Sharma studied the case (ii) by relaxing these conditions and assuming M to be proper, and proved the following result.

Theorem 12.10 *Let M be a proper totally umbilical CR-submanifold of a semi-Riemannian Kaehler manifold $\bar{M}$ with $q = 1$ and g positive definite on $D^\perp$. Suppose that the mean curvature vector μ vanishes nowhere on M. Then the following statements are equivalent: (1) M has an α-Sasakian structure, (2) μ has a constant norm, (3) μ is parallel in the normal bundle, (4) second fundamental form of M is parallel.*

For α-Sasakian structures, we refer to [6, 14].

Proof As $\mu \neq 0$ and $\mu \in JD^\perp$ by Proposition 3, it follows that $t\mu \neq 0$ and lies in $D^\perp$. Now since $q = 1$, any vector field in $D^\perp$ is a scalar multiple of $t\mu$. For any X tangent to M we can show, using Eq. (12.20), that

$$g(\mu, \mu)P^2X = g(t\mu, X)t\mu - g(t\mu, t\mu)X$$

Operating P on this gives $g(\mu, \mu) = g(t\mu, t\mu)$. Hence we get

$$g(t\mu, t\mu)(P^2X + X) = g(t\mu, X)t\mu \tag{12.33}$$

In this case too, Eq. (12.32) holds, which shows ($q = 1$) that $g(t\mu, t\mu) \neq 0$ and hence $g(\mu, \mu) \neq 0$. Hence the Eq. (12.33) assumes the form

$$P^2X = -X + [g(t\mu, t\mu)]^{-1}g(t\mu, X)t\mu \tag{12.34}$$

As g is positive definite on $D^{\perp}$, and μ vanishes nowhere on M, we have $g(t\mu, t\mu) = \alpha^2$. Hence Eq. (12.34) becomes

$$P^2X = -X + \eta(X)\xi \tag{12.35}$$

where $\xi = \frac{1}{\alpha}t\mu$ is a unit vector field, and η is a 1-form on M given by $\eta(X) = g(X, \xi)$. One can easily verify that $P\xi = 0$, $\eta(PX) = 0$, rank$(P) = n - 1$, and

$$g(PX, PY) = g(X, Y) - \eta(X)\eta(Y) \tag{12.36}$$

Use of Eq. (12.18) and total umbilicity shows that

$$(\nabla_X P)Y = \alpha[g(X, Y)\xi - g(\xi, Y)X] \tag{12.37}$$

Equations (12.35)–(12.37) show that (M, g) is an α-Sasakian manifold if and only if $g(\mu, \mu)$ is constant. This proves the equivalence of (1) to (2). In virtue of the equality (whose proof is easy)

$$tD_X\mu = (Xln|g(t\mu, t\mu)|)t\mu$$

the statement (2) is equivalent to $tD_X\mu = 0$. Differentiating the result $f\mu = 0$ obtained earlier, and operating f^2 on the derived equation provides $f(D_X\mu) = 0$. Hence (2) is equivalent to $D_X\mu = 0$, i.e. the statement (3). The statement (4) means

$$D_X(B(Y, Z)) = B(\nabla_X Y, Z) + B(Y, \nabla_X Z)$$

Substituting totally umbilical condition (hypothesis) $B(X, Y) = g(X, Y)\mu$ in the preceding equation shows that (4) is equivalent to (3). This completes the proof.

Remark 5 In particular, if M was a real hypersurface of $\bar{M}$ (as hypothesized in the foregoing theorem, second case), then the statement (3) would have been automatically true, as apparent from the fact that $D_X\mu$ does not belong to $JD^{\perp}$.

12.6 Application to General Relativity

The CR-submanifolds under the hypothesis of Theorem 12.8 are locally decomposable as $M^T \times M^{\perp}$. Recall that these submanifolds carry a parallel f-structure: $P^3 + P = 0$, rank $(P) = 2p$, and $\nabla P = 0$, where $2p$ is the dimension of D. M has a pair of complementary orthogonal distributions $D^{\perp}$ (of dimension q) and D defined respectively by the projection operators $-P^2$ and $P^2 + I$ acting on the tangent space of M at every point. For simplicity, we assume that D and $D^{\perp}$ are each of dimension 2 (i.e. $2p = 2$, $q = 2$). Let $D^{\perp}$ be parallelizable so that there exist vector fields ξ_1, ξ_2 spanning $D^{\perp}$ and their duals η^1, η^2 such that

$$P^2(X) = -X + \eta^1(X)\xi_1 + \eta^2(X)\xi_2$$

$$P\xi_1 = P\xi_2 = 0, \quad P\xi_3 = \xi_4, P\xi_4 = -\xi_3$$

where ξ_3, ξ_4 is a basis of D such that $(\xi_1, \xi_2, \xi_3, \xi_4)$ is an orthonormal basis of TM. Thus, if the metric g on M is indefinite on $D^\perp$ and positive definite on D, then it can be expressed canonically as

$$g = -\eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2 + \eta^3 \otimes \eta^3 + \eta^4 \otimes \eta^4.$$

Using the condition $\nabla P = 0$, we can show that

$$\nabla_X \xi_1 = h(X)\xi_2, \nabla_X \xi_2 = h(X)\xi_1$$
$$\nabla_X \xi_3 = w(X)\xi_4, \nabla_X \xi_4 = -w(X)\xi_3$$

where h and w are smooth 1-forms on M. A straightforward computation gives the curvature tensor R, Ricci tensor Ric and scalar curvature r as follows:

$$R(X, Y)Z = 2H(\eta^2 \wedge \eta^1)(X, Y)[\eta^1(Z)\xi_2 + \eta^2(Z)\xi_1]$$
$$+ 2W(\eta^4 \wedge \eta^3)(X, Y)[\eta^3(Z)\xi_4 - \eta^4(Z)\xi_3]$$

$$Ric = H(-\eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2) + W(\eta^3 \otimes \eta^3 + \eta^4 \otimes \eta^4)$$

$$r = 2(H + W)$$

where $H = (dh)(\xi_2, \xi_1)$ and $W = (dw)(\xi_4, \xi_3)$. Let us call such a manifold (M, g) a Lorentzian Framed (LF)-manifold.

Evidently, LF-manifolds are Ricci-flat if and only if h and w are closed. Also, LF-manifolds are Einstein if and only if $4H = 4W = r$. By a straightforward calculation one can verify that LF-manifolds are conformally flat if and only if $r = 0$. Let us consider a coordinate frame $(\frac{\partial}{\partial x^i})$ (abbreviated ∂_i) compatible with the LF-structure, for a local coordinate system $(t, x, y, z) = (x^i)$ such that

$$\partial_1 = \sigma\xi_1, \partial_2 = \sigma\xi_2, \partial_3 = \tau\xi_3, \partial_4 = \tau\xi_4$$

where σ and τ are nonzero smooth functions. Under such a coordinate system, the metric g takes the form

$$ds^2 = \sigma^2(-dt^2 + dx^2) + \tau^2(dy^2 + dz^2)$$

where $\sigma = \sigma(t, x)$ and $\tau = \tau(y, z)$ and are related to H and W by partial differential equations

$$(ln\sigma)_{,tt} - (ln\sigma)_{,xx} = H\sigma^2, (ln\tau)_{,yy} + (ln\tau)_{,zz} = -W\tau^2$$

The Ricci tensor is expressed in terms of the coordinates (t, x, y, z) as

$$Ric = H(-dt \otimes dt + dx \otimes dx) + W(dy \otimes dy + dz \otimes dz).$$

Exact solutions of the Einstein's field equations

$$Ric - \frac{r}{2}g = 8\pi T$$

with a given energy-momentum tensor T, have been obtained by Duggal and Sharma in [10] under various cases such as flat (Minkowski), Einstein, Conformally flat, Scalar field and nonsingular simple electromagnetic field. For details we refer to [10] and the Ph.D. dissertation of Sharma [16].

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