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Preface

The present volume gathers contributions by several experts in the theory of isometric immersions between Riemannian manifolds, and focuses on the geometry of CR structures on submanifolds in Hermitian manifolds. CR structures are a bundle theoretic recast of the tangential Cauchy–Riemann equations in complex analysis in several complex variables. Let $\Omega \subset \mathbb{C}^n$ $(n \geq 2)$ be an open set and let

$$\overline{\partial} f \equiv \frac{\partial f}{\partial \overline{z}^j} d\overline{z}^j = 0 \tag{1}$$

be the ordinary Cauchy–Riemann equations in $\mathbb{C}^n$. A function $f \in C^1(\Omega, \mathbb{C})$ is *holomorphic* in Ω if f satisfies (1) everywhere in Ω. Let M be an embedded real hypersurface in $\mathbb{C}^n$ such that $U = M \cap \Omega \neq \emptyset$ and let us set

$$T_{1,0}(M)_x = [T_x(M) \otimes_{\mathbb{R}} \mathbb{C}] \cap T^{1,0}(\mathbb{C}^n)_x, \quad x \in M, \tag{2}$$

where $T^{1,0}(\mathbb{C}^n)$ is the holomorphic tangent bundle over $\mathbb{C}^n$ (the span of $\{\partial/\partial z^j : 1 \leq j \leq n\}$). Then $T_{1,0}(M)$ is a rank $n-1$ complex vector bundle over M, referred to as the *CR structure* of M (induced on M by the complex structure of the ambient space $\mathbb{C}^n$) and one may consider the first order differential operator

$$\overline{\partial}_b : C^1(M, \mathbb{C}) \to C\big(T_{0,1}(M)^*\big), \tag{3}$$

$$(\overline{\partial}_b u)\overline{Z} = \overline{Z}(u), \quad u \in C^1(M, \mathbb{C}), \quad Z \in T_{1,0}(M),$$

(the *tangential Cauchy–Riemann operator*) where $T_{0,1}(M) = \overline{T_{1,0}(M)}$ (overbars denote complex conjugates). A function $u \in C^1(M, \mathbb{C})$ is a *CR function* on M if u satisfies

$$\overline{\partial}_b u = 0 \tag{4}$$

(the *tangential Cauchy–Riemann equations*) everywhere on M. Let $\mathrm{CR}^1(M)$ be the space of all CR functions on M. The trace on U of any holomorphic function $f \in \mathcal{O}(\Omega)$ is a CR function $u \in \mathrm{CR}^1(U)$. In other words, the Cauchy–Riemann equations (1) induce on M the first order partial differential system (4). A sufficiently small open piece U of M may be described by a smooth defining function $\rho : \Omega \to \mathbb{R}$ i.e.,

$$U = \{x \in \Omega : \rho(x) = 0\}$$

such that $D\rho(x) \neq 0$ for any $x \in U$. By eventually restricting the open set U we may assume that $\rho_{z^n}(x) \neq 0$ for any $x \in U$. Here $\rho_{z^j} \equiv \partial\rho/\partial z^j$ for $1 \le j \le n$. The portion of $T_{1,0}(M)$ over U is then the span of

$$Z_\alpha \equiv \rho_{z^n}\frac{\partial}{\partial z^\alpha} - \rho_{z^\alpha}\frac{\partial}{\partial z^n}, \quad 1 \le \alpha \le n-1,$$

and the tangential Cauchy–Riemann equations (4) on U may be written as

$$Z_{\overline{\alpha}}(u) = 0, \quad 1 \le \alpha \le n-1, \tag{5}$$

where $Z_{\overline{\alpha}} \equiv \overline{Z_\alpha}$. As such the tangential Cauchy–Riemann equations may be seen to be a first order overdetermined PDEs system with smooth complex valued coefficients. While constant coefficient equations are nowadays fairly well understood, there is still much work to do on variable coefficient PDEs such as (5).

The geometric approach to the study of (local and global properties of) solutions to (4) or (5) is to study the complex vector bundle $T_{1,0}(M)$. This is commonly accomplished by introducing additional geometric objects, familiar within differential geometry. For instance, should one need to compute the Chern classes of $T_{1,0}(M)$, one would need a connection in $T_{1,0}(M)$. Indeed it is rather well known (cf. e.g., [10]) that Tanaka and Webster built (cf. [14] and [15]) a linear connection ∇ on any nondegenerate real hypersurface $M \subset \mathbb{C}^n$, uniquely determined by a fixed contact from θ on M [the *Tanaka–Webster connection* of (M, θ)] and such that ∇ descends to a connection in $T_{1,0}(M)$ as a vector bundle. Chern classes of $T_{1,0}(M)$ may then be computed in terms of the curvature of the Tanaka–Webster connection, in the presence of a fixed contact form on M.

CR structures induced on real hypersurfaces of $\mathbb{C}^n$ are but a particular instance of a more general notion, that of an abstract CR structure on a $(2n+k)$-dimensional manifold M. A complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$, of complex rank n, of the complexified tangent bundle, is said to be an (*abstract*) *CR structure* on M if

$$T_{1,0}(M)_x \cap T_{0,1}(M)_x = (0), \quad x \in M, \tag{6}$$

$$Z, W \in C^\infty(U, T_{1,0}(M)) \Longrightarrow [Z, W] \in C^\infty(U, T_{1,0}(M)), \tag{7}$$

for any open subset $U \subset M$. The integers n and k are respectively the *CR dimension* and *CR codimension* of $T_{1,0}(M)$ while the pair (n,k) is its type. A complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ satisfying only axiom (6) is an *almost* CR structure on M. Axiom (7) is often referred to as the *formal* (or *Frobenius*) *integrability* property. An almost CR structure satisfying the formal integrability property (7) is a CR structure. CR structures on real hypersurfaces $M \subset \mathbb{C}^n$ have type $(n-1, 1)$.

A large portion of the mathematical literature devoted to the study of CR structures is confined to the case of CR codimension 1 in the presence of additional nondegeneracy assumptions (cf. [16]). Let $(M, T_{1,0}(M))$ be a CR manifold of type (n, k). The *Levi distribution* is the real rank $2n$ distribution

$$H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}.$$

It carries the complex structure

$$J: H(M) \to H(M), \quad J(Z+\bar{Z}) = i(Z-\bar{Z}),\ Z \in T_{1,0}(M),$$

$(i = \sqrt{-1})$. The *Levi form* of $(M, T_{1,0}(M))$ is

$$L_x : T_{1,0}(M)_x \times T_{0,1}(M)_x \to \frac{T_x(M) \otimes_{\mathbb{R}} \mathbb{C}}{H(M)_x \otimes_{\mathbb{R}} \mathbb{C}}, \quad x \in M,$$

$$L_x(z,w) = i\pi_x\left[Z, \overline{W}\right]_x, \quad z, w \in T_{1,0}(M)_x,$$

where $Z, W \in C^\infty(T_{1,0}(M))$ are arbitrary globally defined smooth sections such that $Z_x = z$ and $W_x = w$. Also $\pi : T(M) \otimes \mathbb{C} \to [T(M) \otimes \mathbb{C}]/[H(M) \otimes \mathbb{C}]$ is the natural projection. The CR structure $T_{1,0}(M)$ [or the CR manifold $(M, T_{1,0}(M))$] is *nondegenerate* if L_x is nondegenerate for any $x \in M$. Assuming that $k = 1$ there is yet another customary description of the Levi form and of nondegeneracy, as understood in complex analysis. Let

$$H(M)_x^\perp = \{\omega \in T_x^*(M) : \mathrm{Ker}(\omega) \supset H(M)_x\}, \quad x \in M,$$

be the conormal bundle associated to $H(M)$. Assume that M is oriented, so that $T(M)$ is oriented as a vector bundle. The Levi distribution $H(M)$ is oriented by its complex structure J. Hence the quotient $T(M)/H(M)$ is oriented. There are (noncanonical) bundle isomorphisms $H(M)^\perp \approx T(M)/H(M)$, hence $H(M)^\perp$ is oriented, as well. Any oriented real line bundle over a connected manifold is trivial, hence $H(M)^\perp \approx M \times \mathbb{R}$ (a vector bundle isomorphism). Hence globally defined nowhere zero smooth sections $\theta \in C^\infty(H(M)^\perp)$ [referred to as *pseudohermitian structures* on M] do exist. Let $\mathcal{P}$ be the set of all pseudohermitian structures on M. Given $\theta \in \mathcal{P}$ one may set

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M),$$

and one may easily check that L_θ and L agree. As it turns out, if $T_{1,0}(M)$ is nondegenerate then each $\theta \in \mathcal{P}$ is a contact form, i.e., $\theta \wedge (d\theta)^n$ is a volume form on M.

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold, of type (n, 1), and let $\theta \in \mathcal{P}$. The *Reeb vector* of (M, θ) is the unique globally defined nowhere zero tangent vector field $\xi \in \mathfrak{X}(M)$ determined by

$$\theta(\xi) = 1, \quad \xi \rfloor d\theta = 0.$$

The *Webster metric* is the semi-Riemannian metric g_θ on M given by

$$g_\theta(X, Y) = (d\theta)(X, JY), \quad g_\theta(X, \xi) = 0, \ g_\theta(\xi, \xi) = 1, \tag{8}$$

for any $X, Y \in H(M)$. Axioms (8) uniquely determine g_θ because of $T(M) = H(M) \oplus \mathbb{R}\xi$. For any nondegenerate CR manifold $(M, T_{1,0}(M))$, on which a contact form $\theta \in \mathcal{P}$ has been specified, there is a unique linear connection ∇ on M [the *Tanaka–Webster connection* of (M, θ)] obeying to the following axioms (i) $H(M)$ is parallel with respect to ∇, (ii) $\nabla J = 0$ and $\nabla g_\theta = 0$, and (iii) the torsion tensor field T_∇ is pure, i.e.,

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2iL_\theta(Z, \overline{W})\xi,$$

$$\tau \circ J + J \circ \tau = 0,$$

for any $Z, W \in T_{1,0}(M)$, where $\tau(X) = T_\nabla(\xi, X)$ for any $X \in \mathfrak{X}(M)$. One should notice that the existence of ∇ is tied to that of ξ, which in turn is a direct consequence of nondegeneracy and orientability.

We say $(M, T_{1,0}(M))$ is *strictly pseudoconvex* if L_θ is positive definite for some $\theta \in \mathcal{P}$. To emphasize on the role play by Chern classes $c_j\big(T_{1,0}(M)\big)$, let us recall (cf. [13, 14]) the *Lee conjecture* according to which any abstract strictly pseudoconvex CR manifold M with $c_1\big(T_{1,0}(M)\big) = 0$ should admit a contact form θ such that (M, θ) is pseudo-Einstein [i.e., the pseudohermitian Ricci tensor (of the Tanaka–Webster connection of (M, θ)) is proportional to the Levi form]. The sphere $S^{2n-1} \subset \mathbb{C}^n$ is pseudo-Einstein with the canonical contact form $\theta = \frac{i}{2}\big(\overline{\partial} - \partial\big)|z|^2$ [and of course $c_1\big(T_{1,0}(S^{2n-1})\big) = 0$].

From the definition of the notion of an (abstract) CR structure, it is manifest that the prospective study of the CR structure $T_{1,0}(M)$ of a real hypersurface $M \subset \mathbb{C}^n$ ignores the metric structure (the canonical Euclidean structure of $\mathbb{C}^n \approx \mathbb{R}^{2n}$) and only takes into consideration the complex structure on $\mathbb{C}^n$. Whatever metric structure M is seen to possess *a posteriori*, such as the Levi form L_θ, springs from the CR structure (from the complex structure J along $H(M)$) and is determined by it only up to a "conformal factor", very much as in the theory of Riemann surfaces.

Indeed if $\theta,\hat{\theta} \in \mathcal{P}$ then $\hat{\theta} = \lambda\theta$ for some C^∞ function $\lambda : M \to \mathbb{R}\backslash\{0\}$, implying that $L_{\hat{\theta}} = \lambda L_\theta$. However, the fact that the Webster metric g_θ is semi-Riemannian is tied to nondegeneracy (g_θ is actually Riemannian when $(M, T_{1,0}(M))$ is strictly pseudoconvex) and none of these objects, including of course the Tanaka–Webster connection, is available on a CR manifold whose Levi form has a degeneracy locus (for instance in the extreme case where $(M, T_{1,0}(M))$ is *Levi flat*, i.e., $L_\theta = 0$ for some $\theta \in \mathcal{P}$, and thus for all). We also underline that everything said and done in pseudohermitian geometry is confined to the starting assumption that the given CR manifold has CR codimension $k = 1$.

Examples of real hypersurfaces $M \subset \mathbb{C}^n$ which are not nondegenerate, abound (for instance, boundaries of worm domains, cf. [12]). Also, CR manifolds of higher CR codimension ($k \geq 2$) are frequently met, as submanifolds of certain Hermitian manifolds. The absence of an analog to pseudohermitian geometry in these cases may be compensated by making full use of the additional metric structure on M, as the first fundamental form of a given immersion $M \hookrightarrow \tilde{M}$, where $\tilde{M}$ is a Hermitian manifold. This became apparent, and came as a surprise to the CR community, with the work of A. Bejancu at the end of the 1970s (cf. [2–3]).

Let $\tilde{M}$ be a Hermitian manifold, of complex dimension N, with the complex structure J and the Hermitian metric $\tilde{g}$. Let M be a real m-dimensional submanifold, i.e., the inclusion $\iota : \mathrm{M} \hookrightarrow \tilde{M}$ is an embedding. Let us assume that $N = m + p$ and $m = 2n + k$ with $p \geq 1$ $n \geq 1$ and $k \geq 1$. Let $\mathcal{D}$ be a smooth real rank $2n$ distribution on M such that (i) $J_x(\mathcal{D}_x) = \mathcal{D}_x$ and (ii) $J_x(\mathcal{D}_x^\perp) \subset T(M)_x^\perp$, for any $x \in M$. Here $\mathcal{D}_x^\perp$ is the g_x-orthogonal complement of $\mathcal{D}_x$ in $T_x(M)$ and $T(M)^\perp \to M$ is the normal bundle of the given immersion ι [so that $T(M)_x^\perp$ is the $\tilde{g}_x$-orthogonal complement of $T_x(M)$ in $T_x(\tilde{M})$]. Also $g = \iota^*\tilde{g}$ is the induced metric (the first fundamental form of ι). A pair $(M, \mathcal{D})$, consisting of a $(2n + k)$-dimensional submanifold of $\tilde{M}$ and of a distribution $\mathcal{D}$ as above, is called a *CR submanifold* of the Hermitian manifold $(\tilde{M}, J, \tilde{g})$. This is the notion of a CR submanifold as introduced by Bejancu (cf. [2]) except for Bejancu's original request that the ambient space be a Kählerian manifold, i.e., that $\tilde{g}$ be a Kähler metric. Orientable real hypersurfaces $M \subset \mathbb{C}^n$ fit into this category, for one may choose a unit normal vector field N on M, take a rotation of N of angle $\pi/2$ so that to get $\xi = J_0 N \in \mathfrak{X}(M)$, and set $\eta(X) = g_0(X, \xi)$ for any $X \in \mathfrak{X}(M)$, where J_0 and g_0 are respectively the canonical complex structure and (flat) Kählerian metric on $\mathbb{C}^n$. Then $\mathcal{D} = \mathrm{Ker}(\eta)$ organizes M as a CR submanifold of $(\mathbb{C}^n, J_0, g_0)$.

A CR submanifold $(M, \mathcal{D})$ with $\mathcal{D} = T(M)$ is a complex submanifold of $\tilde{M}$, while one with $\mathcal{D} = (0)$ is totally real (such submanidolds are also referred to as anti-invariant). It has been argued by a number of authors that A. Bejancu has introduced the notion of a CR submanifold in an attempt to unify the notions of complex (or invariant) and totally real submanifolds, and of course that of a generic submanifold (where $J(\mathcal{D}^\perp) = T(M)^\perp$). Whether or not A. Bejancu had the insight that his notion was tied to the theory of tangential Cauchy–Riemann equations (as understood in complex analysis) became soon irrelevant, with the nice and elementary discovery by Blair and Chen (cf. [5]) that each CR submanifold $(M, \mathcal{D})$ of a Hermitian manifold may be organized as a CR manifold with the CR structure

$$T_{1,0}(M) = \{X - iJX : X \in \mathcal{D}\}.$$

With respect to this CR structure the inclusion is a CR immersion [i.e., $(d_x \iota) T_{1,0}(M)_x \subset T^{1,0}(\tilde{M})_{\iota(x)}$ for any $x \in M$, where $T^{1,0}(\tilde{M})$ is the holomorphic tangent bundle over the complex manifold $(\tilde{M}, J)$] so that Bejancu's CR submanifolds appear as *embedded* CR manifolds.

A CR manifold $(M, T_{1,0}(M))$ is *locally embeddable* if there is $N > \dim(M)$ such that for any point $x_0 \in M$ there is an open neighborhood $U \subset M$ and a C^∞ immersion $\Psi : U \to \mathbb{C}^N$ such that

$$(d_x \Psi) T_{1,0}(M)_x = \left[T_{\Psi(x)}(\Psi(U)) \otimes_{\mathbb{R}} \mathbb{C}\right] \cap T^{1,0}(\mathbb{C}^N)_{\Psi(x)}, \quad x \in U.$$

At the time A. Bejancu introduced the notion of a CR submanifold L. Nirenberg's problem (i.e., whether a given abstract CR manifold may embed, even if just locally, cf. e.g., [6]) was far less popular than nowadays, and perhaps unknown to Riemannian geometers, who embraced early Bejancu's notion and produced a significant amount of work (cf. e.g., [16]). In the meanwhile it became classical mathematics that real analytic CR manifolds are always locally embeddable (cf. [1]) while in the C^∞ category all strictly pseudoconvex CR manifolds of dimension $\dim(M) \geq 7$ embed locally (and there are known counterexamples in dimension 3, while the case $\dim(M) = 5$ is open). The positive embeddability results close a circle of ideas and incorporating the Hermitian manifold $\tilde{M}$ in the definition of a CR (sub) manifold may not any longer be seen as a limitation of sorts. It should also be noticed that $\tilde{M}$ in Bejancu's definition is an arbitrary Hermitian manifold, and not necessarily $\tilde{M} = \mathbb{C}^N$ for some N (e.g., $\tilde{M}$ may be the complex projective space $\mathbb{C}P^N$, or the complex hyperbolic space $\mathbb{C}H^N$).

A comment is due on Bejancu's motivation[1] for fixing a complement to the holomorphic, or invariant, distribution $\mathcal{D}$ (its orthogonal complement $\mathcal{D}^\perp$, with respect to the induced metric $g = \iota^* \tilde{g}$). As it appears, inspiration was drawn from the work by S. Greenfield (cf. [11]) where a complement to the Levi distribution $H(M)$ is fixed to start with. S. Greenfield's choice may be criticized as non-canonical. Indeed, to make such a choice within pseudohermitian geometry ($k = 1$) one first requires nondegeneracy of the given CR structure and then chooses $\mathbb{R}\xi$ as a complement to $H(M)$, where ξ is the Reeb vector associated to a fixed contact form θ. This choice is perhaps natural enough, yet certainly has one leave the realm of CR geometry (and confines oneself to pseudohermitian geometry). In turn

[1]Of course one may interview Professor Bejancu on the argument. Here we adopt the historian perspective giving preference to historical reconstructions based on documents rather than testimonies.

Bejancu's choice of $\mathcal{D}^\perp$ as a complement to $\mathcal{D}$ uses the metric structure [say $g = \iota^* g_0$ for a given real hypersurface $\iota : M \hookrightarrow \mathbb{C}^n$] and, be it canonical or not, it fits nicely into the adopted philosophy [which is to exploit the additional metric structure to compensate for the eventual lack of nondegeneracy]. We end this remark by recalling that $\mathbb{R}\xi$ is certainly orthogonal to $H(M)$ with respect to the Webster metric g_θ, but in general it cannot be expected to be orthogonal to $H(M)$ with respect to the induced metric. For only in rare occasions [e.g., for the sphere $S^{2n-1} \hookrightarrow \mathbb{C}^n$] does the Webster metric (associated to some contact form) coincide with the induced metric [in fact, all the (infinitely many) Webster metrics of the boundary of the Siegel domain $\Omega = \{(z,w) \in \mathbb{C}^2 : \mathrm{Im}(w) > |z|^2\}$ are distinct from the first fundamental form of the immersion $\partial\Omega \hookrightarrow \mathbb{C}^2$]. This is a point of divergence between Bejancu's theory and pseudohermitian geometry (Webster's theory) yet hasn't proved to be counterproductive so far. On the contrary, one was led to study the geometry of the foliation tangent to $\mathcal{D}^\perp$ [$\mathcal{D}^\perp$ is always Frobenius integrable, provided that the ambient space is Kählerian or locally conformal Kähler] resulting into a deeper understanding of the geometry of the CR submanifold $(M, \mathcal{D})$ itself.

Let $(M, \mathcal{D})$ be a CR submanifold of the Hermitian manifold $(\tilde{M}, J, \tilde{g})$. When $\tilde{g}$ is a Kählerian, or a locally conformal Kähler, metric, the Levi form L of M as a CR manifold, and the second fundamental form h of the given immersion $M \hookrightarrow \tilde{M}$, are related in a nice computable manner. Studying the geometry of h is then closely related to the study of CR geometry (and pseudohermitian geometry, in the CR codimension case) on M. Only now no nondegeneracy assumptions are needed to start with, and the classical machinery in the theory of isometric immersions of Riemannian manifolds (e.g., the Gauss–Ricci–Codazzi equations) becomes available.

Besides from Nirenberg's CR embedding problem mentioned above, one should recall the equally classical CR extension problem. As already seen at the beginning of this preface, traces of holomorphic functions on real hypersurfaces of $\mathbb{C}^n$ are solutions to the tangential Cauchy–Riemann equations. Conversely, the CR extension problem is whether CR functions (on embedded CR manifolds) extend, at least locally, to holomorphic functions on (some open subset of) $\mathbb{C}^n$. An interesting feature of the generic CR submanifolds of higher CR codimension is that positive CR extension results depend on the presence of particular cones in the normal space at each point of the submanifold, and the metric structure of the ambient space may not be ignored any longer (cf. [6]).

The past 30 years have seen a great increase in the volume of research devoted to the geometry of CR submanifolds, from the point of view of Riemannian geometry. The results at the level of K. Yano and M. Kon's monograph mentioned above were integrated by the publication of a book by Bejancu himself (cf. [4]) and by the monograph [9] reporting on the case where the ambient metric $\tilde{g}$ is locally conformal Kähler (and the list of monographs devoted to CR submanifolds is by far not complete).

Given the huge amount of work on CR submanifolds produced since the appearance of the last monograph (i.e., [7]) on the argument, the editors thought it appropriate to invite a number of specialists to contribute one or more papers, perhaps of partially expositive nature, illustrating the state of the art in the theory. The following colleagues answered our call (and are listed here in alphabetical order).

Bang-Yen Chen contributes two papers (cf. Chaps. 1 and 2 in this book), one about his theory of δ-invariants as it applies to CR geometry, and another on the geometry of CR-warped products in Kählerian manifolds.

Miroslava Antić and **Luc Vrancken** contribute a study (cf. Chap. 3 in this book) of CR submanifolds of the nearly Kähler 6-sphere.

Elisabetta Barletta and **Sorin Dragomir** contribute new results (cf. Chap. 4 in this book) on the interplay between the geometry of the second fundamental form of a CR submanifold and the tangential Cauchy–Riemann equations.

J.L. Cabrerizo, **A. Carriazo** and **L.M. Fernándes** report on their work (cf. Chap. 5 in this book) on CR submanifolds of (locally conformal) Kähler manifolds and normal CR submanifolds of S-manifolds.

Krishan Lal Duggal surveys some of his work (cf. Chap. 6 in this book) relating Lorentzian and Cauchy–Riemann geometry.

Hitoshi Furuhata and **Izumi Hasegawa** contribute their work (cf. Chap. 7 in this book) on CR submanifolds of holomorphic statistical manifolds.

Ion Mihai and **Adela Mihai** contribute their work (cf. Chap. 8 in this book) concerning minimality of warped product CR submanifolds in complex space forms, and estimates on the scalar curvature of such submanifolds.

Andrea Olteanu presents results (cf. Chap. 9 in this book) on geometric inequalities occurring on CR-doubly warped product submanifolds.

Toru Sasahara surveys the known results (cf. Chap. 10 in this book) on δ-ideal submanifolds in complex space forms, the nearly Kähler 6-sphere, and odd dimensional spheres.

Mohammad Hasan Shahid, **Falleh R. Al-Solamy** and **Mohammed Jamali** contribute a survey (cf. Chap. 11 in this book) on the geometry of submersions from a CR submanifold.

Ramesh Sharma's contribution (cf. Chap. 12 in this book) regards the geometry of CR submanifolds in semi-Kählerian manifolds, hinting to possible applications to space–time physics.

Gabriel-Eduard Vîlcu presents a generalization (cf. Chap. 13 in this book) of CR submanifold theory to paraquaternionic geometry.

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About the Editors

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Chapter 1
CR-Warped Submanifolds in Kaehler Manifolds

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1.1 Introduction

Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F, respectively, and let f be a positive function on B. Consider the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The *warped product* $M = B \times_f F$ is the manifold $B \times F$ equipped with the warped product Riemannian metric given by

$$g = g_B + f^2 g_F \tag{1.1}$$

We call the function f the *warping function* of the warped product [2]. The notion of warped products plays important roles in differential geometry as well as in physics, especially in the theory of general relativity (cf. [26, 40]).

A submanifold M of a Kaehler manifold $(\tilde{M}, \tilde{g}, J)$ is called a *CR-submanifold* if there exist a holomorphic distribution $\mathcal{D}$ and a totally real distribution $\mathcal{D}^\perp$ on M such that $TM = \mathcal{D} \oplus \mathcal{D}^\perp$, where TM denotes the tangent bundle of M. The notion of *CR*-submanifolds was introduced by A. Bejancu (cf. [1]).

On the other hand, the author proved in [14] that there do not exist warped product submanifolds of the form $M_\perp \times_f N_T$ in any Kaehler manifold $\tilde{M}$ such that N_T is a holomorphic submanifold and $N_\perp$ is a totally real submanifold of $\tilde{M}$. Moreover, the author introduced the notion of *CR-warped products* in [14] as follows. A submanifold M of a Kaehler manifold $\tilde{M}$ is called a *CR*-warped product if it is a warped

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product $M_T \times_f N_\perp$ of a complex submanifold M_T and a totally real submanifold $M_\perp$ of $\tilde{M}$.

A famous embedding theorem of J.F. Nash [39] states that every Riemannian manifold can be isometrically imbedded in a Euclidean space with sufficiently high codimension. In particular, the Nash theorem implies that *every warped product manifold* $N_1 \times_f N_2$ *can be isometrically embedded as a Riemannian submanifold in a Euclidean space.*

In view of Nash's theorem, the author asked at the beginning of this century the following two fundamental questions (see [17, 18, 26]).

Fundamental Question A. *What can we conclude from an arbitrary isometric immersion of a warped product manifold into a Euclidean space or more generally, into an arbitrary Riemannian manifold?*

Fundamental Question B. *What can we conclude from an arbitrary CR-warped product manifold into an arbitrary complex-space-form or more generally, into an arbitrary Kaehler manifold?*

The study of these two questions was initiated by the author in a series of his articles [11, 13–15, 17–25, 29, 34]. Since then the study of warped product submanifolds has become an active research subject in differential geometry of submanifolds.

The purpose of this article is to survey recent results on warped product and *CR*-warped product submanifolds in Kaehler manifolds.

1.2 Preliminaries

In this section, we provide some basic notations, formulas, definitions, and results.

For the submanifold M we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are given respectively by (see, for instance, [3, 26, 27])

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{1.2}$$

$$\tilde{\nabla}_X \xi = A_\xi X + D_X \xi \tag{1.3}$$

for any vector fields X, Y tangent to M and vector field ξ normal to M, where σ denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold.

Let M be an n-dimensional submanifold of a Riemannian m-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ in $\tilde{M}^m$ such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M and hence $e_{n+1}, \ldots, e_m$ are normal to M. Let $\{\sigma_{ij}^r\}$, $i, j = 1, \ldots, n;\ r = n+1, \ldots, m$, denote the coefficients of the second fundamental form h with respect to $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$. Then, we have

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle = \langle A_{e_r} e_i, e_j \rangle,$$

where $\langle\ ,\ \rangle$ denotes the inner product.

The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n}\,\mathrm{trace}\,\sigma = \frac{1}{n}\sum_{i=1}^{n} \sigma(e_i, e_i), \tag{1.4}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle TM of M. The squared mean curvature is then given by

$$H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle.$$

A submanifold M is called *minimal* in $\tilde{M}^m$ if its mean curvature vector vanishes identically.

Let R and $\tilde{R}$ denote the Riemann curvature tensors of M and $\tilde{M}^m$, respectively. The *equation of Gauss* is given by

$$\begin{aligned} R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) &+ \langle \sigma(X, W), \sigma(Y, Z) \rangle \\ &- \langle \sigma(X, Z), \sigma(Y, W) \rangle \end{aligned} \tag{1.5}$$

for vectors X, Y, Z, W tangent to M. For a submanifold of a Riemannian manifold of constant curvature c, we have

$$\begin{aligned} R(X, Y; Z, W) = c\{&\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle\} \\ &+ \langle \sigma(X, W), \sigma(Y, Z)\rangle - \langle \sigma(X, Z), \sigma(Y, W)\rangle. \end{aligned} \tag{1.6}$$

Let M be a Riemannian p-manifold and $e_1, \ldots, e_p$ be an orthonormal frame fields on M. For differentiable function φ on M, the Laplacian $\Delta\varphi$ of φ is defined by

$$\Delta\varphi = \sum_{j=1}^{p} \{(\nabla_{e_j} e_j)\varphi - e_j e_j \varphi\}. \tag{1.7}$$

For any orthonormal basis $e_1, \ldots, e_n$ of the tangent space T_pM at a point $p \in M$, the scalar curvature τ of M at p is defined to be (cf. [9, 10, 26])

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j), \tag{1.8}$$

where $K(e_i \wedge e_j)$ denotes the sectional curvature of the plane section spanned by e_i and e_j.

1.3 Warped Products in Space Forms

Let $M_1, \ldots, M_k$ be k Riemannian manifolds and let

$$f : M_1 \times \cdots \times M_k \to \mathbb{E}^N$$

be an isometric immersion of the Riemannian product $M_1 \times \cdots \times M_k$ into the Euclidean N-space $\mathbb{E}^N$. J.D. Moore [38] proved that if the second fundamental form σ of f has the property that $\sigma(X, Y) = 0$ for X tangent to M_i and Y tangent to M_j, $i \neq j$, then f is a product immersion, that is, there exist isometric immersions $f_i : M_i \to E^{m_i}$, $1 \leq i \leq k$, such that

$$f(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_k)) \tag{1.9}$$

when $x_i \in M_i$ for $1 \leq i \leq k$.

Let $\phi : N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian manifold with constant sectional curvature c. Denote by σ the second fundamental form of ϕ. The immersion $\phi : N_1 \times_f N_2 \to R^m(c)$ is called *mixed totally geodesic* if $\sigma(X, Z) = 0$ for any X in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$.

The next theorem provides a solution to Fundamental Question A.

Theorem 1.1 ([17]) *For any isometric immersion $\phi : N_1 \times_f N_2 \to R^m(c)$ of a warped product $N_1 \times_f N_2$ into a Riemannian manifold of constant curvature c, we have*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1 c, \tag{1.10}$$

where $n_i = \dim N_i$, $n = n_1 + n_2$, H^2 is the squared mean curvature of ϕ, and Δf is the Laplacian of f on N_1.

The equality sign of (1.10) holds identically if and only if $\iota : N_1 \times_f N_2 \to R^m(c)$ is a mixed totally geodesic immersion satisfying trace h_1 = trace h_2, *where* trace h_1 *and* trace h_2 *denote the trace of σ restricted to N_1 and N_2, respectively.*

By making a minor modification of the proof of Theorem 1.1 in [17], using the method of [25], we also have the following general solution from [35] to the Fundamental Question A.

Theorem 1.2 *If $\tilde{M}^m_c$ is a Riemannian manifold with sectional curvatures bounded from above by a constant c, then for any isometric immersion $\phi : N_1 \times_f N_2 \to \tilde{M}^m_c$ from a warped product $N_1 \times_f N_2$ into $\tilde{M}^m_c$ the warping function f satisfies*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} H^2 + n_1 c, \tag{1.11}$$

where $n_1 = \dim N_1$ and $n_2 = \dim N_2$.

An immediate consequence of Theorem 1.2 is the following.

Corollary 1 *There do not exist minimal immersions of a Riemannian product* $N_1 \times N_2$ *of two Riemannian manifolds into a negatively curved Riemannian manifold* $\tilde{M}$.

For arbitrary warped products submanifolds in complex hyperbolic spaces, we have the following general results from [20].

Theorem 1.3 *Let* $\phi : N_1 \times_f N_2 \to CH^m(4c)$ *be an arbitrary isometric immersion of a warped product* $N_1 \times_f N_2$ *into the complex hyperbolic m-space* $CH^m(4c)$ *of constant holomorphic sectional curvature* $4c$*. Then, we have*

$$\frac{\Delta f}{f} \leq \frac{(n_1+n_2)^2}{4n_2} H^2 + n_1 c. \tag{1.12}$$

The equality sign of (1.12) holds if and only if the following three conditions hold.

(1) ϕ *is mixed totally geodesic,*
(2) $\operatorname{trace} h_1 = \operatorname{trace} h_2$*, and*
(3) $J\mathcal{D}_1 \perp \mathcal{D}_2$*, where* J *is the almost complex structure of* CH^m.

Some interesting immediate consequences of Theorem 1.3 are the following nonexistence results [20].

Corollary 2 *Let* $N_1 \times_f N_2$ *be a warped product whose warping function* f *is harmonic. Then* $N_1 \times_f N_2$ *does not admit an isometric minimal immersion into any complex hyperbolic space.*

Corollary 3 *If* f *is an eigenfunction of Laplacian on* N_1 *with eigenvalue* $\lambda > 0$, *then* $N_1 \times_f N_2$ *does not admit an isometric minimal immersion into any complex hyperbolic space.*

Corollary 4 *If* N_1 *is compact, then every warped product* $N_1 \times_f N_2$ *does not admit an isometric minimal immersion into any complex hyperbolic space.*

For arbitrary warped products submanifolds in the complex projective m-space $CP^m(4c)$ with constant holomorphic sectional curvature $4c$, we have the following results from [22].

Theorem 1.4 *Let* $\phi : N_1 \times_f N_2 \to CP^m(4c)$ *be an arbitrary isometric immersion of a warped product into the complex projective m-space* $CP^m(4c)$ *of constant holomorphic sectional curvature* $4c$*. Then, we have*

$$\frac{\Delta f}{f} \leq \frac{(n_1+n_2)^2}{4n_2} H^2 + (3+n_1)c. \tag{1.13}$$

The equality sign of (1.13) holds identically if and only if we have

(1) $n_1 = n_2 = 1$,
(2) f *is an eigenfunction of the Laplacian of* N_1 *with eigenvalue* $4c$, *and*
(3) ϕ *is totally geodesic and holomorphic.*

An immediate application of Theorem 1.4 is the following non-immersion result.

Corollary 5 *If* f *is a positive function on a Riemannian* n_1*-manifold* N_1 *such that* $(\Delta f)/f > 3 + n_1$ *at some point* $p \in N_1$, *then, for any Riemannian manifold* N_2, *the warped product* $N_1 \times_f N_2$ *does not admit any isometric minimal immersion into* $CP^m(4)$ *for any m.*

Theorem 1.4 can be sharpened as the following theorem for totally real minimal immersions.

Theorem 1.5 *If* f *is a positive function on a Riemannian* n_1*-manifold* N_1 *such that* $(\Delta f)/f > n_1$ *at some point* $p \in N_1$, *then, for any Riemannian manifold* N_2, *the warped product* $N_1 \times_f N_2$ *does not admit any isometric totally real minimal immersion into* $CP^m(4)$ *for any m.*

The following examples illustrate that Theorems 1.3–1.5 are sharp.

Example 1 Let $I = (-\frac{\pi}{4}, \frac{\pi}{4})$, $N_2 = S^1(1)$ and $f = \frac{1}{2}\cos 2s$. Then the warped product

$$N_1 \times_f N_2 =: I \times_{(\cos 2s)/2} S^1(1)$$

has constant sectional curvature 4. Clearly, we have $(\Delta f)/f = 4$. If we define the complex structure J on the warped product by $J\left(\frac{\partial}{\partial s}\right) = 2(\sec 2s)\frac{\partial}{\partial t}$, then $(I \times_{(\cos 2s)/2} S^1(1), g, J)$ is holomorphically isometric to a dense open subset of $CP^1(4)$.

Let $\phi : CP^1(4) \to CP^m(4)$ be a standard totally geodesic embedding of $CP^1(4)$ into $CP^m(4)$. Then the restriction of ϕ to $I \times_{(\cos 2s)/2} S^1(1)$ gives rise to a minimal isometric immersion of $I \times_{(\cos 2s)/2} S^1$ into $CP^m(4)$ which satisfies the equality case of inequality (1.13) on $I \times_{(\cos 2s)/2} S^1(1)$ identically.

Example 2 Consider $N_1 \times_f N_2 = I \times_{(\cos 2s)/2} S^1(1)$ and let $\phi : CP^1(4) \to CP^m(4)$ be the totally geodesic holomorphic embedding of $CP^1(4)$ into $CP^m(4)$. Then the restriction of ϕ to $N_1 \times_f N_2$ is an isometric minimal immersion of $N_1 \times_f N_2$ into $CP^m(4)$ which satisfies $(\Delta f)/f = 3 + n_1$ identically. This example shows that the assumption "$(\Delta f)/f > 3 + n_1$ at some point in N_1" given in Theorem 1.4 is best possible.

Example 3 Let g_1 be the standard metric on $S^{n-1}(1)$. Denote by $N_1 \times_f N_2$ the warped product given by $N_1 = (-\pi/2, \pi/2)$, $N_2 = S^{n-1}(1)$ and $f = \cos s$. Then the warping function of this warped product satisfies $\Delta f/f = n_1$ identically. Moreover, it is easy to verify that this warped product is isometric to a dense open subset of S^n. Let

$$\phi\ : S^n(1) \xrightarrow[2:1]{\text{projection}} RP^n(4) \xrightarrow[\text{totally real}]{\text{totally geodesic}} CP^n(4)$$

be a standard totally geodesic Lagrangian immersion of $S^n(1)$ into $CP^n(4)$. Then the restriction of ϕ to $N_1 \times_f N_2$ is a totally real minimal immersion. This example illustrates that the assumption "$(\Delta f)/f > n_1$ at some point in N_1" given in Theorem 1.5 is also sharp.

1.4 Segre Imbedding and Its Converse

For simplicity, we denote $S^n(1)$, $RP^n(1)$, $CP^n(4)$ and $CH^n(-4)$ by S^n, RP^n, CP^n and CH^n, respectively.

Let $(z_0^i, \dots, z_{\alpha_i}^i)$, $1 \le i \le s$, be homogeneous coordinates of CP^{α_i}. Define a map:

$$S_{\alpha_1 \dots \alpha_s} : CP^{\alpha_1} \times \cdots \times CP^{\alpha_s} \to CP^N, \quad N = \prod_{i=1}^{s} (\alpha_i + 1) - 1, \tag{1.14}$$

which maps a point $((z_0^1, \dots, z_{\alpha_1}^1), \dots, (z_0^s, \dots, z_{\alpha_s}^s))$ in $CP^{\alpha_1} \times \cdots \times CP^{\alpha_s}$ to the point $(z_{i_1}^1 \dots z_{i_j}^s)_{1 \le i_1 \le \alpha_1, \dots, 1 \le i_s \le \alpha_s}$ in CP^N. The map $S_{\alpha_1 \dots \alpha_s}$ is a Kaehler embedding which is known as the *Segre embedding*. The Segre embedding was constructed by C. Segre in 1891.

The following results from [4, 31] established in 1981 can be regarded as the "converse" to Segre embedding constructed in 1891.

Theorem 1.6 *Let $M_1^{\alpha_1}, \dots, M_s^{\alpha_s}$ be Kaehler manifolds of dimensions $\alpha_1, \dots, \alpha_s$, respectively. Then every holomorphically isometric immersion*

$$f : M_1^{\alpha_1} \times \cdots \times M_s^{\alpha_s} \to CP^N, \quad N = \prod_{i=1}^{s} (\alpha_i + 1) - 1,$$

of $M_1^{\alpha_1} \times \cdots \times M_s^{\alpha_s}$ into CP^N is locally the Segre embedding, i.e., $M_1^{\alpha_1}, \dots, M_s^{\alpha_s}$ are open portions of $CP^{\alpha_1}, \dots, CP^{\alpha_s}$, respectively. Moreover, f is congruent to the Segre embedding.

Let $\bar\nabla^k \sigma$, $k = 0, 1, 2, \dots$, denote the kth covariant derivative of the second fundamental form. Denoted by $||\bar\nabla^k \sigma||^2$ the squared norm of $\bar\nabla^k \sigma$.

The following result was proved in [31].

Theorem 1.7 *Let $M_1^{\alpha_1} \times \cdots \times M_s^{\alpha_s}$ be a product Kaehler submanifold of CP^N. Then*

$$||\bar{\nabla}^{k-2}\sigma||^2 \geq k!\, 2^k \sum_{i_1 < \cdots < i_k} \alpha_1 \ldots \alpha_k, \tag{1.15}$$

for $k = 2, 3, \ldots$.

The equality sign of (1.15) holds for some k if and only if $M_1^{\alpha_1}, \ldots, M_s^{\alpha_s}$ are open parts of $CP^{\alpha_1}, \ldots, CP^{\alpha_s}$, respectively, and the immersion is congruent to the Segre embedding.

If $k = 2$, Theorem 1.7 reduces to the following result of [4].

Theorem 1.8 *Let $M_1^h \times M_2^p$ be a product Kaehler submanifold of CP^N. Then we have*

$$||\sigma||^2 \geq 8hp. \tag{1.16}$$

The equality sign of inequality (1.16) holds if and only if M_1^h and M_2^p are open portions of CP^h and CP^p, respectively, and moreover the immersion is congruent to the Segre embedding $S_{h,p}$.

We may extend Theorem 1.8 to the following for warped products.

Theorem 1.9 *Let (M_1^h, g_1) and (M_2^p, g_2) be two Kaehler manifolds of complex dimension h and p respectively and let f be a positive function on M_1^h. If $\phi : M_1^h \times_f M_2^p \to CP^N$ is a holomorphically isometric immersion of the warped product manifold $M_1^h \times_f M_2^p$ into CP^N. Then f is a constant, say c. Moreover, we have*

$$||\sigma||^2 \geq 8hp. \tag{1.17}$$

The equality sign of (1.17) holds if and only if (M_1^h, g_1) and (M_2^p, cg_2) are open portions of CP^h and CP^p, respectively, and moreover the immersion ϕ is congruent to the Segre embedding.

Proof Under the hypothesis, the warped product manifold $M_1^h \times_f M_2^p$ must be a Kaehler manifold. Therefore, the warping function f must be a positive constant. Consequently, the theorem follows from Theorem 1.8.

1.5 CR-Products in Kaehler Manifolds

A submanifold N in a Kaehler manifold $\tilde{M}$ is called a *totally real submanifold* if the almost complex structure J of $\tilde{M}$ carries each tangent space T_xN of N into its corresponding normal space $T_x^{\perp}N$ [12, 16, 33]. The submanifold N is called a holomorphic submanifold (or Kaehler submanifold) if J carries each T_xN into itself. The submanifold N is called *slant* [8] if for any nonzero vector X tangent to N the angle $\theta(X)$ between JX and T_pN does not depend on the choice of the point $p \in N$ and of the choice of the vector $X \in T_pN$. On the other hand, it depends only on the point p, then the submanifold N is called *pointwise slant* [30].

Let M be a submanifold of a Kaehler manifold $\tilde{M}$. For each point $p \in M$, put

$$\mathcal{H}_p = T_pM \cap J(T_pM),$$

i.e., $\mathcal{H}_p$ is the maximal holomorphic subspace of the tangent space T_pM. If the dimension of $\mathcal{H}_p$ remains the same for each $p \in M$, then M is called a *generic submanifold* [7].

A CR-submanifold of a Kaehler manifold $\tilde{M}$ is called a *CR-product* [4, 5] if it is a Riemannian product $N_T \times N_\perp$ of a Kaehler submanifold N_T and a totally real submanifold $N_\perp$. A CR-submanifold is called *mixed totally geodesic* if the second fundamental form of the CR-submanifold satisfying

$$\sigma(X, Z) = 0$$

for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

For CR-products in complex space forms, the following result are known.

Theorem 1.10 ([4]) *We have*

(i) A CR-submanifold in the complex Euclidean m-space $\mathbb{C}^m$ is a CR-product if and only if it is a direct sum of a Kaehler submanifold and a totally real submanifold of linear complex subspaces.
(ii) There do not exist CR-products in complex hyperbolic spaces other than Kaehler submanifolds and totally real submanifolds.

CR-products $N_T \times N_\perp$ in CP^{h+p+hp} are obtained from the Segre embedding $S_{h,p}$; namely, we have the following results.

Theorem 1.11 ([4]) *Let $N_T^h \times N_\perp^p$ be the CR-product in CP^m with constant holomorphic sectional curvature* 4*. Then*

$$m \geq h + p + hp. \tag{1.18}$$

The equality sign of (1.18) holds if and only if

(a) N_T^h is a totally geodesic Kaehler submanifold,
(b) $N_\perp^p$ is a totally real submanifold, and
(c) the immersion is given by

$$N_T^h \times N_\perp^p \to CP^h \times CP^p \xrightarrow[\text{Segre imbedding}]{S_{hp}} CP^{h+p+hp}.$$

Theorem 1.12 ([4]) *Let $N_T^h \times N_\perp^p$ be the CR-product in CP^m. Then the squared norm of the second fundamental form satisfies*

$$||\sigma||^2 \geq 4hp. \tag{1.19}$$

The equality sign of (1.19) holds if and only if

(a) N_T^h is a totally geodesic Kaehler submanifold,
(b) $N_\perp^p$ is a totally geodesic totally real submanifold, and
(c) the immersion is given by

$$N_T^h \times N_\perp^p \xrightarrow{\text{totally geodesic}} CP^h \times CP^p \xrightarrow[\text{Segre imbedding}]{S_{hp}} CP^{h+p+hp} \subset CP^m.$$

1.6 Warped Product CR-Submanifolds

In this section we present known results on warped product *CR*-submanifold in Kaehler manifolds. First, we mention the following result.

Theorem 1.13 ([14]) *If $N_\perp \times_f N_T$ is a warped product CR-submanifold of a Kaehler manifold $\tilde{M}$ such that $N_\perp$ is a totally real and N_T a Kaehler submanifold of $\tilde{M}$, then it is a CR-product.*

Theorem 1.13 shows that there does not exist warped product *CR*-submanifolds of the form $N_\perp \times_f N_T$ other than *CR*-products. So, we only need to consider warped product *CR*-submanifolds of the form: $N_T \times_f N_\perp$, *by reversing the two factors N_T and $N_\perp$* of the warped product. The author simply calls such *CR*-submanifolds *CR-warped products* in [14].

CR-warped products are simply characterized as follows.

Proposition 1.1 ([14]) *A proper CR-submanifold M of a Kaehler manifold $\tilde{M}$ is locally a CR-warped product if and only if the shape operator A satisfies*

$$A_{JZ}X = ((JX)\mu)Z, \quad X \in \mathcal{D}, \quad Z \in \mathcal{D}^\perp, \tag{1.20}$$

for some function μ on M satisfying $W\mu = 0$, $\forall W \in \mathcal{D}^\perp$.

A fundamental result on *CR*-warped products in arbitrary Kaehler manifolds is the following theorem.

Theorem 1.14 ([14]) *Let $N_T \times_f N_\perp$ be a CR-warped product submanifold in an arbitrary Kaehler manifold $\tilde{M}$. Then the second fundamental form σ satisfies*

$$||\sigma||^2 \geq 2p\,||\nabla(\ln f)||^2, \tag{1.21}$$

where $\nabla(\ln f)$ *is the gradient of* $\ln f$ *on* N_T *and* $p = \dim N_\perp$.

If the equality sign of (1.21) holds identically, then N_T *is a totally geodesic Kaehler submanifold and* $N_\perp$ *is a totally umbilical totally real submanifold of* $\tilde{M}$. *Moreover,* $N_T \times_f N_\perp$ *is minimal in* $\tilde{M}$.

When M *is anti-holomorphic, i.e., when* $J\mathcal{D}_x^\perp = T_x^\perp N$, *and* $p > 1$. *The equality sign of (1.21) holds identically if and only if* $N_\perp$ *is a totally umbilical submanifold of* $\tilde{M}$.

If M *is anti-holomorphic and* $p = 1$, *then the equality sign of (1.21) holds identically if and only if the characteristic vector field* $J\xi$ *of* M *is a principal vector field with zero as its principal curvature. (Notice that in this case,* M *is a real hypersurface in* $\tilde{M}$.*) Also, in this case, the equality sign of (1.21) holds identically if and only if* M *is a minimal hypersurface in* $\tilde{M}$.

CR-warped products in complex space forms satisfying the equality case of (1.21) have been completely classified in [14, 15].

Theorem 1.15 *A CR-warped product* $N_T \times_f N_\perp$ *in* $\mathbb{C}^m$ *satisfies*

$$||\sigma||^2 = 2p||\nabla(\ln f)||^2 \tag{1.22}$$

identically if and only if the following four statements hold:

(i) N_T *is an open portion of a complex Euclidean h-space* $\mathbb{C}^h$,
(ii) $N_\perp$ *is an open portion of the unit p-sphere* S^p,
(iii) there exists $a = (a_1, \ldots, a_h) \in S^{h-1} \subset \mathbb{E}^h$ *such that* $f = \sqrt{\langle a, z\rangle^2 + \langle ia, z\rangle^2}$ *for* $z = (z_1, \ldots, z_h) \in \mathbb{C}^h$, $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1}$, *and*
(iv) up to rigid motions, the immersion is given by

$$\mathbf{x}(z, w) = \left(z_1 + (w_0 - 1)a_1 \sum_{j=1}^h a_j z_j, \ldots, z_h + a(w_0 - 1)a_h \sum_{j=1}^h a_j z_j, \right.$$
$$\left. w_1 \sum_{j=1}^h a_j z_j, \ldots, w_p \sum_{j=1}^h a_j z_j, 0, \ldots, 0 \right).$$

A *CR*-warped product $N_T \times_f N_\perp$ is said to be *trivial* if its warping function f is constant. A trivial *CR*-warped product $N_T \times_f N_\perp$ is nothing but a *CR*-product $N_T \times N_\perp^f$, where $N_\perp^f$ is the manifold with metric $f^2 g_{N_\perp}$ which is homothetic to the original metric $g_{N_\perp}$ on $N_\perp$.

The following result completely classifies *CR*-warped products in complex projective spaces satisfying the equality case of (1.21) identically.

Theorem 1.16 ([15]) *A non-trivial CR-warped product* $N_T \times_f N_\perp$ *in the complex projective m-space* $CP^m(4)$ *satisfies the basic equality* $||\sigma||^2 = 2p||\nabla(\ln f)||^2$ *if and only if we have*

(1) N_T *is an open portion of complex Euclidean h-space* $\mathbb{C}^h$,
(2) $N_\perp$ *is an open portion of a unit p-sphere* S^p, *and*
(3) *up to rigid motions, the immersion* $\mathbf{x}$ *of* $N_T \times_f N_\perp$ *into* CP^m *is the composition* $\pi \circ \breve{\mathbf{x}}$, *where*

$$\breve{\mathbf{x}}(z,w) = \Bigg(z_0 + (w_0 - 1)a_0 \sum_{j=0}^{h} a_j z_j, \ldots, z_h + (w_0 - 1)a_h \sum_{j=0}^{h} a_j z_j, \\ w_1 \sum_{j=0}^{h} a_j z_j, \ldots, \; w_p \sum_{j=0}^{h} a_j z_j, 0, \ldots, 0 \Bigg),$$

π *is the projection* $\pi : \mathbb{C}_*^{m+1} \to CP^m$, $a_0, \ldots, a_h$ *are real numbers satisfying* $a_0^2 + a_1^2 + \cdots + a_h^2 = 1$, $z = (z_0, z_1, \ldots, z_h) \in \mathbb{C}^{h+1}$ *and* $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1}$.

The following result completely classifies *CR*-warped products in complex hyperbolic spaces satisfying the equality case of (1.21) identically.

Theorem 1.17 ([15]) *A CR-warped product* $N_T \times_f N_\perp$ *in the complex hyperbolic m-space* $CH^m(-4)$ *satisfies the basic equality*

$$||\sigma||^2 = 2p||\nabla(\ln f)||^2$$

if and only if one of the following two cases occurs:

(1) N_T *is an open portion of complex Euclidean h-space* $\mathbb{C}^h$, $N_\perp$ *is an open portion of a unit p-sphere* S^p *and, up to rigid motions, the immersion is the composition* $\pi \circ \breve{\mathbf{x}}$, *where* π *is the projection* $\pi : \mathbb{C}_{*1}^{m+1} \to CH^m$ *and*

$$\breve{\mathbf{x}}(z,w) = \Bigg(z_0 + a_0(1 - w_0) \sum_{j=0}^{h} a_j z_j, z_1 + a_1(w_0 - 1) \sum_{j=0}^{h} a_j z_j, \ldots, z_h \\ + a_h(w_0 - 1) \sum_{j=0}^{h} a_j z_j, \; w_1 \sum_{j=0}^{h} a_j z_j, \ldots, \; w_p \sum_{j=0}^{h} a_j z_j, 0, \ldots, 0 \Bigg)$$

for some real numbers $a_0, \ldots, a_h$ *satisfying* $a_0^2 - a_1^2 - \cdots - a_h^2 = -1$, *where* $z = (z_0, \ldots, z_h) \in \mathbb{C}_1^{h+1}$ *and* $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1}$.
(2) $p = 1$, N_T *is an open portion of* $\mathbb{C}^h$ *and, up to rigid motions, the immersion is the composition* $\pi \circ \breve{\mathbf{x}}$, *where*

$$\breve{\mathbf{x}}(z,t) = \Bigg(z_0 + a_0(\cosh t - 1)\sum_{j=0}^{h} a_j z_j,\, z_1 + a_1(1-\cosh t)\sum_{j=0}^{h} a_j z_j, \ldots, z_h + a_h(1-\cosh t)\sum_{j=0}^{h} a_j z_j,\, \sinh t \sum_{j=0}^{h} a_j z_j, 0, \ldots, 0 \Bigg)$$

for some real numbers $a_0, a_1, \ldots, a_{h+1}$ satisfying $a_0^2 - a_1^2 - \cdots - a_h^2 = 1$.

A multiply warped product $N_T \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ in a Kaehler manifold $\tilde{M}$ is called a *multiply CR-warped product* if N_T is a holomorphic submanifold and $N_\perp = {}_{f_2}N_2 \times \cdots \times_{f_k} N_k$ is a totally real submanifold of $\tilde{M}$.

The next theorem extends (1.14) for multiply *CR*-warped products.

Theorem 1.18 ([28]) *Let $N = N_T \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ be a multiply CR-warped product in an arbitrary Kaehler manifold $\tilde{M}$. Then the second fundamental form σ and the warping functions $f_2, \ldots, f_k$ satisfy*

$$||\sigma||^2 \geq 2\sum_{i=2}^{k} n_i ||\nabla(\ln f_i)||^2. \tag{1.23}$$

The equality sign of inequality (1.23) holds identically if and only if the following four statements hold:

(a) N_T is a totally geodesic holomorphic submanifold of $\tilde{M}$;
(b) For each $i \in \{2, \ldots, k\}$, N_i is a totally umbilical submanifold of $\tilde{M}$ with $-\nabla(\ln f_i)$ as its mean curvature vector;
(c) ${}_{f_2}N_2 \times \cdots \times_{f_k} N_k$ is immersed as mixed totally geodesic submanifold in $\tilde{M}$; and
(d) For each point $p \in N$, the first normal space $\operatorname{Im} h_p$ *is a subspace of $J(T_pN_\perp)$.*

Remark 1 B. Sahin [41] extends Theorem 1.13 to the following.

Theorem 1.19 *There exist no warped product submanifolds of the type $M_\theta \times_f M_T$ and $M_T \times_f M_\theta$ in a Kaehler manifold, where M_θ is a proper slant submanifold and M_T is a holomorphic submanifold of $\tilde{M}$.*

Remark 2 As an extension of Theorem 1.19 the following nonexistence result was proved by K.A. Khan, S. Ali and N. Jamal.

Theorem 1.20 ([37]) *There do not exist proper warped product submanifolds of the type $N \times_f N_T$ and $N_T \times_f N$ in a Kaehler manifold, where N_T is a complex submanifold and N is any non-totally real generic submanifold of a Kaehler manifold $\tilde{M}$.*

Remark 3 B. Sahin proved the following.

Theorem 1.21 ([42]) *There do not exist doubly warped product CR-submanifolds which are not (singly) warped product CR-submanifolds in the form ${}_{f_1}M_T \times_{f_2} M_\perp$, where M_T is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of a Kaehler manifold $\tilde{M}$.*

1.7 CR-Warped Products with Compact Holomorphic Factor

When the holomorphic factor N_T of a CR-warped product $N_T \times_f N_\perp$ is compact, we have the following sharp results.

Theorem 1.22 ([24]) *Let $N_T \times_f N_\perp$ be a CR-warped product in the complex projective m-space $CP^m(4)$ of constant holomorphic sectional curvature 4. If N_T is compact, then we have*

$$m \geq h + p + hp.$$

Remark 4 The example mentioned in Statement (c) of Theorem 1.11 shows that Theorem 1.22 is sharp.

Theorem 1.23 ([24]) *If $N_T \times_f N_\perp$ is a CR-warped product in $CP^{h+p+hp}(4)$ with compact N_T, then N_T is holomorphically isometric to CP^h.*

Theorem 1.24 ([24]) *For any CR-warped product $N_T \times_f N_\perp$ in $CP^m(4)$ with compact N_T and any $q \in N_\perp$, we have*

$$\int_{N_T \times \{q\}} ||\sigma||^2 dV_T \geq 4hp \operatorname{vol}(N_T), \tag{1.24}$$

where $||\sigma||$ is the norm of the second fundamental form, dV_T is the volume element of N_T, and $\operatorname{vol}(N_T)$ is the volume of N_T.

The equality sign of (1.24) holds identically if and only if we have:

(1) The warping function f is constant.
(2) (N_T, g_{N_T}) is holomorphically isometric to $CP^h(4)$ and it is isometrically immersed in CP^m as a totally geodesic complex submanifold.
(3) $(N_\perp, f^2 g_{N_\perp})$ is isometric to an open portion of the real projective p-space $RP^p(1)$ of constant sectional curvature one and it is isometrically immersed in CP^m as a totally geodesic totally real submanifold.
(4) $N_T \times_f N_\perp$ is immersed linearly fully in a complex subspace $CP^{h+p+hp}(4)$ of $CP^m(4)$; and moreover, the immersion is rigid.

Theorem 1.25 ([24]) *Let $N_T \times_f N_\perp$ be a CR-warped product with compact N_T in $CP^m(4)$. If the warping function f is a non-constant function, then for each $q \in N_\perp$ we have*

$$\int_{N_T \times \{q\}} ||\sigma||^2 dV_T \geq 2p\lambda_1 \int_{N_T} (\ln f)^2 dV_T + 4hp \operatorname{vol}(N_T), \tag{1.25}$$

where λ_1 is the first positive eigenvalue of the Laplacian Δ of N_T.

Moreover, the equality sign of (1.25) holds identically if and only if we have

(1) $\Delta \ln f = \lambda_1 \ln f$.
(2) The CR-warped product is both N_T*-totally geodesic and* $N_\perp$*-totally geodesic.*

The following example shows that Theorems 1.24 and 1.25 are sharp.

Example 4 Let ι_1 be the identity map of $CP^h(4)$ and let

$$\iota_2 : RP^p(1) \to CP^p(4)$$

be a totally geodesic Lagrangian embedding of $RP^p(1)$ into $CP^p(4)$. Denote by

$$\iota = (\iota_1, \iota_2) : CP^h(4) \times RP^p(1) \to CP^h(4) \times CP^p(4)$$

the product embedding of ι_1 and ι_2. Moreover, let $S_{h,p}$ be the Segre embedding of $CP^h(4) \times CP^p(4)$ into $CP^{hp+h+p}(4)$. Then the composition

$$\phi = S_{h,p} \circ \iota : CP^h(4) \times RP^p(1) \xrightarrow[\text{totally geodesic}]{(\iota_1,\iota_2)} CP^h(4) \times CP^p(4)$$
$$\xrightarrow[\text{Segre embedding}]{S_{h,p}} CP^{hp+h+p}(4)$$

is a CR-warped product in $CP^{h+p+hp}(4)$ whose holomorphic factor $N_T = CP^h(4)$ is a compact manifold. Since the second fundamental form of ϕ satisfies the equation: $||\sigma||^2 = 4hp$, we have the equality case of inequality (1.24) identically.

The next example shows that the assumption of compactness in Theorems 1.24 and 1.25 cannot be removed.

Example 5 Let $\mathbb{C}^* = \mathbb{C} - \{0\}$ and $\mathbb{C}^{m+1}_* = \mathbb{C}^{m+1} - \{0\}$. Denote by $\{z_0, \ldots, z_h\}$ a natural complex coordinate system on $\mathbb{C}^{m+1}_*$.

Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{m+1}_*$ given by

$$\lambda \cdot (z_0, \ldots, z_m) = (\lambda z_0, \ldots, \lambda z_m)$$

for $\lambda \in \mathbb{C}_*$. Let $\pi(z)$ denote the equivalent class containing z under this action. Then the set of equivalent classes is the complex projective m-space $CP^m(4)$ with the complex structure induced from the complex structure on $\mathbb{C}^{m+1}_*$.

For any two natural numbers h and p, we define a map:

$$\breve{\phi} : \mathbb{C}^{h+1}_* \times S^p(1) \to \mathbb{C}^{h+p+1}_*$$

by

$$\breve{\phi}(z_0, \ldots, z_h; w_0, \ldots, w_p) = \big(w_0 z_0, w_1 z_0, \ldots, w_p z_0, z_1, \ldots, z_h\big)$$

for $(z_0, \ldots, z_h)$ in $\mathbb{C}^{h+1}_*$ and $(w_0, \ldots, w_p)$ in S^p with $\sum_{j=0}^p w_j^2 = 1$.

Since the image of $\breve{\phi}$ is invariant under the action of $\mathbb{C}_*$, the composition:

$$\pi \circ \breve{\phi} : \mathbb{C}_*^{h+1} \times S^p \xrightarrow{\breve{\phi}} \mathbb{C}_*^{h+p+1} \xrightarrow{\pi} CP^{h+p}(4)$$

induces a *CR*-immersion of the product manifold $N_T \times S^p$ into $CP^{h+p}(4)$, where

$$N_T = \big\{(z_0, \ldots, z_h) \in CP^h : z_0 \neq 0\big\}$$

is a proper open subset of $CP^h(4)$. Clearly, the induced metric on $N_T \times S^p$ is a warped product metric and the holomorphic factor N_T is non-compact.

Notice that the complex dimension of the ambient space is $h+p$; far less than $h+p+hp$.

1.8 Another Optimal Inequality for CR-Warped Products

All *CR*-warped products in complex space forms also satisfy another general optimal inequality obtained in [21].

Theorem 1.26 *Let $N = N_T^h \times_f N_\perp^p$ be a CR-warped product in a complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature c. Then we have*

$$||\sigma||^2 \geq 2p\big\{||\nabla(\ln f)||^2 + \Delta(\ln f) + 2hc\big\}. \tag{1.26}$$

If the equality sign of (1.26) holds identically, then N_T is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold. Moreover, N is a minimal submanifold in $\tilde{M}(4c)$.

The following three theorems completely classify all *CR*-warped products which satisfy the equality case of (1.26) identically.

Theorem 1.27 ([21]) *Let $\phi : N_T^h \times_f N_\perp^p \to \mathbb{C}^m$ be a CR-warped product in $\mathbb{C}^m$. Then we have*

$$||\sigma||^2 \geq 2p\big\{||\nabla(\ln f)||^2 + \Delta(\ln f)\big\}. \tag{1.27}$$

The equality case of inequality (1.27) holds identically if and only if the following four statements hold.

(1) N_T is an open portion of $\mathbb{C}_*^h := \mathbb{C}^h - \{0\}$;
(2) $N_\perp$ is an open portion of S^p;
(3) There is α, $1 \leq \alpha \leq h$, and complex Euclidean coordinates $\{z_1, \ldots, z_h\}$ on $\mathbb{C}^h$ such that $f = \sqrt{\sum_{j=1}^{\alpha} z_j \bar{z}_j}$;
(4) Up to rigid motions, the immersion ϕ is given by

$$\phi = \big(w_0 z_1, \ldots, w_p z_1, \ldots, w_0 z_\alpha, \ldots, w_p z_\alpha, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0\big)$$

for $z = (z_1, \ldots, z_h) \in \mathbb{C}^h_*$ *and* $w = (w_0, \ldots, w_p) \in S^p(1) \subset \mathbb{E}^{p+1}$.

Theorem 1.28 ([21]) *Let* $\phi : N_T \times_f N_\perp \to CP^m(4)$ *be a CR-warped product with* $\dim_{\mathbb{C}} N_T = h$ *and* $\dim_{\mathbb{R}} N_\perp = p$. *Then we have*

$$||\sigma||^2 \geq 2p\big\{||\nabla(\ln f)||^2 + \Delta(\ln f) + 2h\big\}. \tag{1.28}$$

The CR-warped product satisfies the equality case of inequality (1.28) identically if and only if the following three statements hold.

(a) N_T *is an open portion of complex projective h-space* $CP^h(4)$*;*
(b) $N_\perp$ *is an open portion of unit p-sphere* $S^p(1)$*; and*
(c) There exists a natural number $\alpha \leq h$ *such that, up to rigid motions,* ϕ *is the composition* $\pi \circ \breve{\phi}$, *where*

$$\breve{\phi}(z, w) = \big(w_0 z_0, \ldots, w_p z_0, \ldots, w_0 z_\alpha, \ldots, w_p z_\alpha, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0\big)$$

for $z = (z_0, \ldots, z_h) \in \mathbb{C}^{h+1}_*$ *and* $w = (w_0, \ldots, w_p) \in S^p(1) \subset \mathbb{E}^{p+1}$, *where* π *is the projection* $\pi : \mathbb{C}^{m+1}_* \to CP^m$.

Theorem 1.29 ([21]) *Let* $\phi : N_T \times_f N_\perp \to CH^m(-4)$ *be a CR-warped product with* $\dim_{\mathbb{C}} N_T = h$ *and* $\dim_{\mathbb{R}} N_\perp = p$. *Then we have*

$$||\sigma||^2 \geq 2p\big\{||\nabla(\ln f)||^2 + \Delta(\ln f) - 2h\big\}. \tag{1.29}$$

The CR-warped product satisfies the equality case of (1.29) identically if and only if the following three statements hold.

(a) N_T *is an open portion of complex hyperbolic h-space* $CH^h(-4)$*;*
(b) $N_\perp$ *is an open portion of unit p-sphere* $S^p(1)$ *(or* $\mathbb{R}$, *when* $p = 1$*); and*
(c) up to rigid motions, ϕ *is the composition* $\pi \circ \breve{\phi}$, *where either* $\breve{\phi}$ *is given by*

$$\breve{\phi}(z, w) = \big(z_0, \ldots, z_\beta, w_0 z_{\beta+1}, \ldots, w_p z_{\beta+1}, \ldots, w_0 z_h, \ldots, w_p z_h, 0, \ldots, 0\big)$$

for $0 < \beta \leq h$, $z = (z_0, \ldots, z_h) \in \mathbb{C}^{h+1}_{*1}$ *and* $w = (w_0, \ldots, w_p) \in S^p(1)$, *or* $\breve{\phi}$ *is given by*

$$\breve{\phi}(z, u) = \big(z_0 \cosh u, z_0 \sinh u, z_1 \cos u, z_1 \sin u, \ldots,$$
$$z_\alpha \cos u, z_\alpha \sin u, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0\big)$$

for $z = (z_0, \ldots, z_h) \in \mathbb{C}^{h+1}_{*1}$, *where* π *is the projection* $\pi : \mathbb{C}^{m+1}_{*1} \to CH^m(-4)$.

1.9 Warped Product Real Hypersurfaces

For real hypersurfaces, we have the following nonexistence theorem.

Theorem 1.30 ([32]) *There do not exist real hypersurfaces in complex projective and complex hyperbolic spaces which are Riemannian products of two or more Riemannian manifolds of positive dimension.*

In other words, every real hypersurface in a non-flat complex space form is irreducible.

A contact manifold is an odd-dimensional manifold M^{2n+1} with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. A curve $\gamma = \gamma(t)$ in a contact manifold is called a *Legendre curve* if $\eta(\beta'(t)) = 0$ along β. Let $S^{2n+1}(c)$ denote the hypersphere in $\mathbb{C}^{n+1}$ with curvature c centered at the origin. Then $S^{2n+1}(c)$ is a contact manifold endowed with a canonical contact structure which is the dual 1-form of the characteristic vector field $J\xi$, where J is the complex structure and ξ the unit normal vector on $S^{2n+1}(c)$.

Legendre curves are known to play an important role in the study of contact manifolds, e.g. a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves.

Contrast to Theorem 1.30, there exist many warped product real hypersurfaces in complex space forms as given in the following three theorems from [19].

Theorem 1.31 *Let a be a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ be a unit speed Legendre curve $\gamma : I \to S^3(a^2) \subset \mathbb{C}^2$ defined on an open interval I. Then*

$$\mathbf{x}(z_1, \ldots, z_n, t) = \big(a\Gamma_1(t)z_1, a\Gamma_2(t)z_1, z_2, \ldots, z_n\big), \quad z_1 \neq 0 \tag{1.30}$$

defines a real hypersurface which is the warped product $\mathbb{C}^n_{**} \times_{a|z_1|} I$ of a complex n-plane and I, where $\mathbb{C}^n_{**} = \{(z_1, \ldots, z_n) : z_1 \neq 0\}$.

Conversely, up to rigid motions of $\mathbb{C}^{n+1}$, every real hypersurface in $\mathbb{C}^{n+1}$ which is the warped product $N \times_f I$ of a complex hypersurface N and an open interval I is either obtained in the way described above or given by the product submanifold $\mathbb{C}^n \times C \subset \mathbb{C}^n \times \mathbb{C}^1$ of $\mathbb{C}^n$ and a real curve C in $\mathbb{C}^1$.

Let $S^{2n+3}(1)$ denote the unit hypersphere in $\mathbb{C}^{n+2}$ centered at the origin and put

$$U(1) = \{\lambda \in \mathbb{C} : \lambda\bar{\lambda} = 1\}.$$

Then there is a $U(1)$-action on $S^{2n+3}(1)$ defined by $z \mapsto \lambda z$. At $z \in S^{2n+3}(1)$ the vector $V = iz$ is tangent to the flow of the action. The quotient space $S^{2n+3}(1)/\sim$, under the identification induced from the action, is a complex projective space CP^{n+1} which endows with the canonical Fubini-Study metric of constant holomorphic sectional curvature 4.

The almost complex structure J on $CP^{n+1}(4)$ is induced from the complex structure J on $\mathbb{C}^{n+2}$ via the Hopf fibration: $\pi : S^{2n+3}(1) \to CP^{n+1}(4)$. It is well-known that the Hopf fibration π is a Riemannian submersion such that $V = iz$ spans the vertical subspaces. Let $\phi : M \to CP^{n+1}(4)$ be an isometric immersion. Then $\hat{M} = \pi^{-1}(M)$ is a principal circle bundle over M with totally geodesic fibers. The lift $\hat{\phi} : \hat{M} \to S^{2n+3}(1)$ of ϕ is an isometric immersion so that the diagram:

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\phi}} & S^{2n+3}(1) \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\phi} & CP^{n+1}(4) \end{array}$$

commutes.

Conversely, if $\psi : \hat{M} \to S^{2n+3}(1)$ is an isometric immersion which is invariant under the $U(1)$-action, then there is a unique isometric immersion $\psi_\pi : \pi(\hat{M}) \to CP^{n+1}(4)$ such that the associated diagram commutes. We simply call the immersion $\psi_\pi : \pi(\hat{M}) \to CP^{n+1}(4)$ *the projection* of $\psi : \hat{M} \to S^{2n+3}(1)$.

For a given vector $X \in T_z(CP^{n+1})(4)$ and a point $u \in S^{2n+2}(1)$ with $\pi(u) = z$, we denote by X^*_u the horizontal lift of X at u via π. There exists a canonical orthogonal decomposition:

$$T_u S^{2n+3}(1) = (T_{\pi(u)} CP^{n+1}(4))^*_u \oplus \text{Span}\,\{V_u\}. \tag{1.31}$$

Since π is a Riemannian submersion, X and X^*_u have the same length.

We put

$$\begin{aligned} S^{2n+1}_*(1) &= \left\{ (z_0, \ldots, z_n) : \sum_{k=0}^{n} z_k \bar{z}_k = 1,\ z_0 \neq 0 \right\}, \\ CP^n_0 &= \pi(S^{2n+1}_*(1)). \end{aligned} \tag{1.32}$$

The next theorem classifies all warped products hypersurfaces of the form $N \times_f I$ in complex projective spaces.

Theorem 1.32 ([19]) *Suppose that a is a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ is a unit speed Legendre curve $\gamma : I \to S^3(a^2) \subset \mathbb{C}^2$ defined on an open interval I. Let $\mathbf{x} : S^{2n+1}_* \times I \to \mathbb{C}^{n+2}$ be the map defined by*

$$\mathbf{x}(z_0, \ldots, z_n, t) = \big(a\Gamma_1(t)z_0, a\Gamma_2(t)z_0, z_1, \ldots, z_n\big), \quad \sum_{k=0}^{n} z_k \bar{z}_k = 1. \tag{1.33}$$

Then

(1) $\mathbf{x}$ induces an isometric immersion $\psi : S^{2n+1}_*(1) \times_{a|z_0|} I \to S^{2n+3}(1)$.
(2) The image $\psi(S^{2n+1}_*(1) \times_{a|z_0|} I)$ in $S^{2n+3}(1)$ is invariant under the action of $U(1)$.
(3) the projection

$$\psi_\pi : \pi(S^{2n+1}_*(1) \times_{a|z_0|} I) \to CP^{n+1}(4)$$

of ψ via π is a warped product hypersurface $CP^n_0 \times_{a|z_0|} I$ in $CP^{n+1}(4)$.

Conversely, if a real hypersurface in $CP^{n+1}(4)$ *is a warped product* $N \times_f I$ *of a complex hypersurface N of* $CP^{n+1}(4)$ *and an open interval I, then, up to rigid motions, it is locally obtained in the way described above.*

In the complex pseudo-Euclidean space $\mathbb{C}_1^{n+2}$ endowed with pseudo-Euclidean metric

$$g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^{n+1} dz_j d\bar{z}_j, \tag{1.34}$$

we define the anti-de Sitter space-time by

$$H_1^{2n+3}(-1) = \big\{(z_0, z_1, \ldots, z_{n+1}) : \langle z, z\rangle = -1\big\}. \tag{1.35}$$

It is known that $H_1^{2n+3}(-1)$ has constant sectional curvature -1. There is a $U(1)$-action on $H_1^{2n+3}(-1)$ defined by $z \mapsto \lambda z$. At a point $z \in H_1^{2n+3}(-1)$, iz is tangent to the flow of the action. The orbit is given by $z_t = e^{it} z$ with $\frac{dz_t}{dt} = iz_t$ which lies in the negative-definite plane spanned by z and iz.

The quotient space $H_1^{2n+3}(-1)/\sim$ is the complex hyperbolic space $CH^{n+1}(-4)$ which endows a canonical Kaehler metric of constant holomorphic sectional curvature -4. The complex structure J on $CH^{n+1}(-4)$ is induced from the canonical complex structure J on $\mathbb{C}_1^{n+2}$ via the totally geodesic fibration: $\pi : H_1^{2n+3} \to CH^{n+1}(-4)$.

Let $\phi : M \to CH^{n+1}(-4)$ be an isometric immersion. Then $\hat{M} = \pi^{-1}(M)$ is a principal circle bundle over M with totally geodesic fibers. The lift $\hat{\phi} : \hat{M} \to H_1^{2n+3}(-1)$ of ϕ is an isometric immersion such that the diagram:

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\phi}} & H_1^{2n+3}(-1) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\phi} & CH^{n+1}(-4) \end{array}$$

commutes.

Conversely, if $\psi : \hat{M} \to H_1^{2n+3}(-1)$ is an isometric immersion which is invariant under the $U(1)$-action, there is a unique isometric immersion $\psi_\pi : \pi(\hat{M}) \to CH^{n+1}(-4)$, called the *projection of* ψ so that the associated diagram commutes.

We put

$$H_{1*}^{2n+1}(-1) = \big\{(z_0, \ldots, z_n) \in H_1^{2n+1}(-1) : z_n \neq 0\big\}, \tag{1.36}$$

$$CH_*^n(-4) = \pi(H_{1*}^{2n+1}(-1)). \tag{1.37}$$

The following theorem classifies all warped products hypersurfaces of the form $N \times_f I$ in complex hyperbolic spaces.

Theorem 1.33 ([19]) *Suppose that a is a positive number and* $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ *is a unit speed Legendre curve* $\gamma : I \to S^3(a^2) \subset \mathbb{C}^2$*. Let*

$$\mathbf{y} : H^{2n+1}_{1*}(-1) \times I \to \mathbb{C}^{n+2}_1$$

be the map defined by

$$\mathbf{y}(z_0, \ldots, z_n, t) = (z_0, \ldots, z_{n-1}, a\Gamma_1(t)z_n, a\Gamma_2(t)z_n), \tag{1.38}$$

$$z_0\bar{z}_0 - \sum_{k=1}^{n} z_k\bar{z}_k = 1. \tag{1.39}$$

Then we have

(1) $\mathbf{y}$ *induces an isometric immersion* $\psi : H^{2n+1}_{1*}(-1) \times_{a|z_n|} I \to H^{2n+3}_1(-1)$.
(2) The image $\psi(H^{2n+1}_{1*}(-1) \times_{a|z_n|} I)$ *in* $H^{2n+3}_1(-1)$ *is invariant under the* $U(1)$*-action.*
(3) the projection

$$\psi_\pi : \pi(H^{2n+1}_{1*}(-1) \times_{a|z_n|} I) \to CH^{n+1}(-4)$$

of ψ *via* π *is a warped product hypersurface* $CH^n_*(-1) \times_{a|z_n|} I$ *in* $CH^{n+1}(-4)$.

Conversely, if a real hypersurface in $CH^{n+1}(-4)$ *is a warped product* $N \times_f I$ *of a complex hypersurface N and an open interval I, then, up to rigid motions, it is locally obtained in the way described above.*

1.10 Twisted Product CR-Submanifolds

Twisted products $B \times_\lambda F$ are natural extensions of warped products, namely the function may depend on both factors (cf. [6, p. 66]). When λ depends only on B, the twisted product becomes a warped product. If B is a point, the twisted product is nothing but a conformal change of metric on F.

The study of twisted product *CR*-submanifolds was initiated by the author in 2000 (see [11]). In particular, the following results are obtained.

Theorem 1.34 ([11]) *If* $M = N_\perp \times_\lambda N_T$ *is a twisted product CR-submanifold of a Kaehler manifold* $\tilde{M}$ *such that* $N_\perp$ *is a totally real submanifold and* N_T *is a holomorphic submanifold of* $\tilde{M}$*, then M is a CR-product.*

Theorem 1.35 ([11]) *Let* $M = N_T \times_\lambda N_\perp$ *be a twisted product CR-submanifold of a Kaehler manifold* $\tilde{M}$ *such that* $N_\perp$ *is a totally real submanifold and* N_T *is a holomorphic submanifold of* $\tilde{M}$*. Then we have*

(1) *The squared norm of the second fundamental form of* M *in* $\tilde{M}$ *satisfies*

$$||\sigma||^2 \geq 2p\,||\nabla^T(\ln\lambda)||^2,$$

where $\nabla^T(\ln\lambda)$ *is the* N^T*-component of the gradient* $\nabla(\ln\lambda)$ *of* $\ln\lambda$ *and* p *is the dimension of* $N_\perp$*.*

(2) *If* $||\sigma||^2 = 2p\,||\nabla^T \ln\lambda||^2$ *holds identically, then* N_T *is a totally geodesic submanifold and* $N_\perp$ *is a totally umbilical submanifold of* $\tilde{M}$*.*

(3) *If* M *is anti-holomorphic in* $\tilde{M}$ *and* $\dim N_\perp > 1$*, then* $||\sigma||^2 = 2p\,||\nabla^T \ln\lambda||^2$ *holds identically if and only if* N_T *is a totally geodesic submanifold and* $N_\perp$ *is a totally umbilical submanifold of* $\tilde{M}$*.*

For mixed foliate twisted product *CR*-submanifolds of Kaehler manifolds, we have the following result.

Theorem 1.36 ([11]) *Let* $M = N_T \times_\lambda N_\perp$ *be a twisted product CR-submanifold of a Kaehler manifold* $\tilde{M}$ *such that* $N_\perp$ *is a totally real submanifold and* N_T *is a holomorphic submanifold of* $\tilde{M}$*. If* M *is mixed totally geodesic, then we have*

(1) *The twisted function* λ *is a function depending only on* $N_\perp$*.*

(2) $N_T \times N_\perp^\lambda$ *is a CR-product, where* $N_\perp^\lambda$ *denotes the manifold* $N_\perp$ *equipped with the metric* $g^\lambda_{N_\perp} = \lambda^2 g_{N_\perp}$*.*

Next, we provide ample examples of twisted product *CR*-submanifolds in complex Euclidean spaces which are not *CR*-warped product submanifolds.

Let $z : N_T \to \mathbb{C}^m$ be a holomorphic submanifold of a complex Euclidean m-space $\mathbb{C}^m$ and $w : N_\perp^1 \to \mathbb{C}^\ell$ be a totally real submanifold such that the image of $N_T \times N_\perp^1$ under the product immersion $\psi = (z, w)$ does not contain the origin $(0, 0)$ of $\mathbb{C}^m \oplus \mathbb{C}^\ell$.

Let $j : N_\perp^2 \to S^{q-1} \subset \mathbb{E}^q$ be an isometric immersion of a Riemannian manifold $N_\perp^2$ into the unit hypersphere S^{q-1} of $\mathbb{E}^q$ centered at the origin.

Consider the map

$$\phi = (z, w) \otimes j : N_T \times N_\perp^1 \times N_\perp^2 \to (\mathbb{C}^m \oplus \mathbb{C}^\ell) \otimes \mathbb{E}^q$$

defined by

$$\phi(p_1, p_2, p_3) = (z(p_1), z(p_2)) \otimes j(p_3), \tag{1.40}$$

for $p_1 \in N_T,\ p_2 \in N_\perp^1,\ p_3 \in N_\perp^2$.

On $(\mathbb{C}^m \oplus \mathbb{C}^\ell) \otimes \mathbb{E}^q$ we define a complex structure J by

$$J((B, E) \otimes F) = (\mathrm{i}B, \mathrm{i}E) \otimes F, \quad \mathrm{i} = \sqrt{-1},$$

for any $B \in \mathbb{C}^m$, $E \in \mathbb{C}^\ell$ and $F \in \mathbb{E}^q$. Then $(\mathbb{C}^m \oplus \mathbb{C}^\ell) \otimes \mathbb{E}^q$ becomes a complex Euclidean $(m + \ell)q$-space $\mathbb{C}^{(m+\ell)q}$.

Let us put $N_\perp = N_\perp^1 \times N_\perp^2$. We denote by $|z|$ the distance function from the origin of $\mathbb{C}^m$ to the position of N_T in $\mathbb{C}^m$ via z; and denote by $|w|$ the distance function from the origin of $\mathbb{C}^\ell$ to the position of $N_\perp^1$ in $\mathbb{C}^\ell$ via w. We define a function λ by $\lambda = \sqrt{|z|^2 + |w|^2}$. Then $\lambda > 0$ is a differentiable function on $N_T \times N_\perp$, which depends on both N_T and $N_\perp = N_\perp^1 \times N_\perp^2$.

Let M denote the twisted product $N_T \times_\lambda N_\perp$ with twisted function λ. Clearly, M is not a warped product.

For such a twisted product $N_T \times_\lambda N_\perp$ in $\mathbb{C}^{(m+\ell)q}$ defined above we have the following.

Proposition 1.2 ([11]) *The map $\phi = (z, w) \otimes j : N_T \times_\lambda N_\perp \to \mathbb{C}^{(m+\ell)q}$ defined by (1.40) satisfies the following properties:*

(1) $\phi = (z, w) \otimes j : N_T \times_\lambda N_\perp \to \mathbb{C}^{(m+\ell)q}$ is an isometric immersion.
(2) $\phi = (z, w) \otimes j : N_T \times_\lambda N_\perp \to \mathbb{C}^{(m+\ell)q}$ is a twisted product CR-submanifold such that N_T is a holomorphic submanifold and $N_\perp$ is a totally real submanifold of $\mathbb{C}^{(m+\ell)q}$.

Proposition 1.2 shows that there are many twisted product *CR*-submanifolds $N_T \times_\lambda N_\perp$ such that N_T are holomorphic submanifolds and $N_\perp$ are totally real submanifolds. Moreover, such twisted product *CR*-submanifolds are not warped product *CR*-submanifolds.

Let (B, g_B) and (F, g_F) be Riemannian manifolds and let $\pi_B : B \times F \to B$ and $\pi_F : B \times F \to F$ be the canonical projections. Also let b, f be smooth real-valued functions on $B \times F$. Then the *doubly twisted product* of (B, g_B) and (F, g_F) with twisting functions b and f is defined to be the product manifold $M = B \times F$ with metric tensor

$$g = f^2 g_B + b^2 g_F.$$

We denote this kind manifolds by ${}_fB \times_b F$.

Remark 5 B. Sahin proved the following.

Theorem 1.37 ([42]) *There does not exist doubly twisted product CR-submanifolds in a Kaehler manifold which are not (singly) twisted product CR-submanifolds in the form ${}_{f_1}M_T \times_{f_2} M_\perp$, where M_T is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of the Kaehler manifold $\tilde{M}$.*

An almost Hermitian manifold (M, g, J) with almost complex structure J is called a *nearly Kähler* manifold provided that [36]

$$(\nabla_X J)X = 0, \quad \forall X \in TM. \tag{1.41}$$

Remark 6 Theorem 1.37 was extended by S. Uddin in [43] to doubly twisted product *CR*-submanifolds in a nearly Kaehler manifold.

Remark 7 For further results on *CR*-submanifolds in nearly Kaehler manifolds, see Luc Vrancken's article "Nearly Kaehler 6-sphere and its CR-submanifold" to appear in this volume.

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Chapter 2
CR-Submanifolds and δ-Invariants

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2.1 Introduction

In 1956, John F. Nash proved in [34] the following famous embedding theorem.

Theorem 2.1 *Every Riemannian n-manifold can be isometrically embedded in a Euclidean m-space* $\mathbb{E}^m$ *with* $m = \frac{n}{2}(n+1)(3n+11)$.

For example, Nash's theorem implies that every Riemannian 3-manifold can be isometrically embedded in $\mathbb{E}^{120}$ with codimension 117.

The Nash embedding theorem was aimed for in the hope that if Riemannian manifolds could always be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help. Till when observed in [32] as such by M.L. Gromov, this hope had not been materialized however.

There were several reasons why it is so difficult to apply Nash's theorem. One reason is that it requires very large codimension for a Riemannian manifold to admit an isometric embedding in Euclidean spaces in general. On the other hand, submanifolds of higher codimension are very difficult to understand, e.g., there are no general results for arbitrary Riemannian submanifolds, except the three fundamental equations of Gauss, Codazzi, and Ricci. Another reason for this is lack of controls of the extrinsic invariants of the submanifold by the known classical intrinsic invariants.

In order to overcome such difficulties as well as to provide answers to an open question on minimal immersions proposed by S.S. Chern in the 1960s, the author

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introduced in the early 1990s a new type of Riemannian invariants; namely, the δ-invariants. At the same time, the author was able to establish universal optimal inequalities for Riemannian submanifolds involving the intrinsic δ-invariants and the most important extrinsic invariant; namely, the squared mean curvature.

The δ-invariants are very different in nature from the "classical" Ricci and scalar curvatures; simply due to the fact that both scalar and Ricci curvatures are "total sum" of sectional curvatures on a Riemannian manifold. In contrast, the author's δ-invariants are obtained from the scalar curvature by throwing away a certain amount of sectional curvatures. After δ-invariants were introduced and the corresponding inequalities were established, δ-invariants were investigated by many mathematicians in the past two decades (see [21, 22] for recent surveys on δ-invariants and their applications).

Let N be a Riemannian manifold isometrically immersed in a Kähler manifold $\tilde{M}$ with complex structure J. For each point $x \in N$, let $\mathcal{D}_x$ denote the maximal complex subspace $T_xN \cap J(T_xN)$ of the tangent space T_xN, $x \in N$. If the dimension of $\mathcal{D}_x$ is the same for all $x \in N$, then $\{\mathcal{D}_x, x \in N\}$ defines a complex distribution $\mathcal{D}$ on N. A submanifold N in a Kähler manifold $\tilde{M}$ is called a *CR-submanifold* if there exists on N a totally real distribution $\mathcal{D}^\perp$ whose orthogonal complement is $\mathcal{D}$, i.e., $TN = \mathcal{D} \oplus \mathcal{D}^\perp$ and $J\mathcal{D}_x^\perp \subset T_x^\perp N$, $x \in N$ (cf. [3]).

A Riemannian submersion $\pi : M \to B$ is an everywhere surjective map from a Riemannian manifold M onto another Riemannian manifold B such that the differential π_* preserves lengths of horizontal vectors.

The main purpose of this article is to present recent results on *CR*-submanifolds in complex space forms which are closely related to δ-invariants and Riemannian submersions.

2.2 Preliminaries

Let M be an n-dimensional submanifold of a Riemannian m-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ in $\tilde{M}^m$ such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M and hence $e_{n+1}, \ldots, e_m$ are normal to M.

For the submanifold M in $\tilde{M}^m$, we denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are given, respectively, by (see, for instance, [22])

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for any vector fields X, Y tangent to M and vector field ξ normal to M, where σ denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold.

Let $\{\sigma_{ij}^r\}$, $i, j = 1, \ldots, n;\ r = n+1, \ldots, m$, denote the coefficients of the second fundamental form h with respect to $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$. Then, we have

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle = \langle A_{e_r} e_i, e_j \rangle,$$

where $\langle\ ,\ \rangle$ denotes the inner product.

The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n} \operatorname{trace} \sigma = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i). \tag{2.3}$$

The squared mean curvature is then given by

$$H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle.$$

The submanifold M is called minimal in $\tilde{M}^m$ if its mean curvature vector vanishes identically. It is called totally geodesic if its second fundamental form σ vanishes identically.

Denote by R and $\tilde{R}$ the Riemann curvature tensors of M and $\tilde{M}^m$, respectively. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) &+ \langle \sigma(X, W), \sigma(Y, Z) \rangle \\ &- \langle \sigma(X, Z), \sigma(Y, W) \rangle \end{aligned} \tag{2.4}$$

for vectors X, Y, Z, W tangent to M. In particular, for a submanifold of a Riemannian manifold of constant sectional curvature c, we have

$$\begin{aligned} R(X, Y; Z, W) = c\{&\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle\} \\ &+ \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(X, Z), \sigma(Y, W) \rangle . \end{aligned} \tag{2.5}$$

Let M be a Riemannian p-manifold and $e_1, \ldots, e_p$ be an orthonormal frame fields on M. For differentiable function φ on M, the Laplacian $\Delta\varphi$ of φ is defined by

$$\Delta\varphi = \sum_{j=1}^{p} \{(\nabla_{e_j} e_j)\varphi - e_j e_j \varphi\}. \tag{2.6}$$

We simply call a Kähler manifold of constant holomorphic sectional curvature a *complex space form*. In this article, we denote a complete simply connected complex m-dimensional complex space form of constant holomorphic sectional curvature $4c$ by $\tilde{M}^m(4c)$.

The curvature tensor $\tilde{R}$ of a complex space form $\tilde{M}^m(4c)$ satisfies

$$\tilde{R}(U,V,W) = c\{\langle V,W\rangle U - \langle X,W\rangle V + \langle JV,W\rangle JU - \langle JU,W\rangle JV + 2\langle U,JV\rangle JW\}. \tag{2.7}$$

It is well-known that $\tilde{M}^m(4c)$ is holomorphically isometric to the complex projective m-space $CP^m(4c)$, the complex Euclidean m-space $\mathbf{C}^m$, or the complex hyperbolic m-space $CH^m(4c)$ according to $c > 0$, $c = 0$, or $c < 0$, respectively.

2.3 CR-Submanifolds and CR-Warped Products

A submanifold N of a Kähler manifold $\tilde{M}$ is called *totally real* if the complex structure J of $\tilde{M}$ carries each tangent space T_xN of N into the normal space $T_x^{\perp}N$ for each $x \in N$ (cf. [25]). A totally real submanifold N is called *Lagrangian* if $\dim_{\mathbf{C}} \tilde{M} = \dim N$. For a point $x \in N$, we denote by $\mathcal{D}_x$ the maximal complex subspace $T_xN \cap J(T_xN)$ of the tangent space T_xN. If the dimension of $\mathcal{D}_x$ is the same for all $x \in N$, then $\{\mathcal{D}_x, x \in N\}$ defines a complex distribution $\mathcal{D}$ on N.

A submanifold N in a Kähler manifold $\tilde{M}$ is called a *CR*-submanifold if there exists on N a totally real distribution $\mathcal{D}^{\perp}$ whose orthogonal complement is $\mathcal{D}$, i.e., $TN = \mathcal{D} \oplus \mathcal{D}^{\perp}$ and $J\mathcal{D}_x^{\perp} \subset T_x^{\perp}N, x \in N$ (cf. [3]).

For a *CR*-submanifold N with the complex distribution $\mathcal{D}$ and the totally real distribution $\mathcal{D}^{\perp}$, we denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^{\perp}$ in the normal bundle $T^{\perp}N$ of N. Then, we have the following orthogonal direct sum decomposition:

$$T^{\perp}N = J\mathcal{D}^{\perp} \oplus \nu, \quad J\mathcal{D}^{\perp} \perp \nu. \tag{2.8}$$

A *CR*-submanifold is called *proper* if it is neither holomorphic nor totally real. Throughout this article, let h denote the complex rank of the complex distribution $\mathcal{D}$ and p the real rank of the totally real distribution $\mathcal{D}^{\perp}$.

A *CR*-submanifold N is called a *CR-product* if it is a Riemannian product of a holomorphic submanifold N^T and a totally real submanifold $N^{\perp}$ of $\tilde{M}$. It was proved in [8, Theorem 4.6] that every *CR*-product in a complex Euclidean space is a direct product of a holomorphic submanifold of a linear complex subspace and a totally real submanifold of another linear complex subspace. Also, it was proved in [8, Theorem 4.4] that there do not exist proper *CR*-products in complex hyperbolic spaces. Furthermore, it was known in [8, Theorem 5.3] that *CR*-products in complex projective space CP^{h+p+hp} are obtained from the Segre imbedding in a natural way.

Let B and F be two Riemannian manifolds endowed with Riemannian metrics g_B and g_F, respectively, and let f be a positive differentiable function on B. The warped product $B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian metric

$$g = g_B + f^2 g_F.$$

The function f is called the *warping function* of the warped product (cf. [36]).

It was shown in [16, Theorem 3.1] that there do not exist warped products of the form: $N^{\perp} \times_f N^T$ in any Kähler manifold besides CR-products, where $N^{\perp}$ is a totally real submanifold and N^T is a holomorphic submanifold.

By contrast, it was also shown in [16] that there exist many CR-submanifolds which are warped products of the form $N^T \times_f N^{\perp}$. Such a warped product CR-submanifold is simply called a *CR-warped product* (see [16]). Furthermore, it was proved in [16, Theorem 5.1] that every CR-warped product in an arbitrary Kähler manifold satisfies the following optimal universal inequality:

$$||\sigma||^2 \geq 2p||\nabla(\ln f)||^2, \tag{2.9}$$

where $\nabla(\ln f)$ denotes the gradient of $\ln f$ and σ is the second fundamental form of the CR-warped product.

CR-warped products in complex space forms satisfying the equality case of (2.9) have been completely classified in [16, 17]. Further results on CR-warped products in complex space forms were obtained in [19, 20].

Lemma 1 ([8]) *Let N be a CR-submanifold in a* $CP^m(4)$. *Then, we have*
(a) $\langle \nabla_U Z, X\rangle = \langle JA_{JZ}U, X\rangle$,
(b) $A_{JZ}W = A_{JW}Z$,
(c) $A_{J\eta}X = -A_{\eta}JX$, *and*
(d) $\langle D_U JZ, JW\rangle = \langle \nabla_U Z, W\rangle$, $\langle D_U JZ, J\eta\rangle = \langle h(U, Z), \eta\rangle$,
for any vector fields $U \in TN$; $X, Y \in \mathcal{D}$; $Z, W \in \mathcal{D}^{\perp}$ *and* $\eta \in \nu$, *where* ν *is defined by (2.8).*

Lemma 2 *The totally real distribution* $\mathcal{D}^{\perp}$ *of a CR-submanifold of a Kähler manifold is an integrable distribution.*

Lemma 3 *Let N be a CR-submanifold of a Kähler manifold* $\tilde{M}$. *Then, we have:*

(1) the complex distribution $\mathcal{D}$ *is integrable if and only if*

$$\langle \sigma(X, JY), JZ\rangle = \langle \sigma(JX, Y), JZ\rangle \tag{2.10}$$

holds for any $X, Y \in \mathcal{D}$ *and* $Z \in \mathcal{D}^{\perp}$,

(2) the leaves of the totally real distribution $\mathcal{D}^{\perp}$ *are totally geodesic in N if and only if* $\langle \sigma(X, Z), JW\rangle = 0$ *holds for any* $X \in \mathcal{D}$ *and* $Z, W \in \mathcal{D}^{\perp}$.

A CR-submanifold N of a Kähler manifold $\tilde{M}$ is called *anti-holomorphic* if we have $J\mathcal{D}^{\perp}_x = T^{\perp}_x N,\ x \in N$. And it is called *mixed totally geodesic* if its second fundamental form σ satisfies $\sigma(X, Z) = 0$ for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$. A mixed totally geodesic CR-submanifold is called *mixed foliate* if its complex distribution $\mathcal{D}$ is also integrable. Obviously, real hypersurfaces of a Kähler manifold are anti-holomorphic submanifolds with $p = \operatorname{rank} \mathcal{D}^{\perp} = 1$.

Lemma 4 *A complex space form* $\tilde{M}^m(4c)$ *with* $c \neq 0$ *admits no mixed foliate proper CR-submanifolds.*

Lemma 4 is due to [4] for $c > 0$ and due to [29] for $c < 0$.

For mixed foliate CR-submanifolds in a complex Euclidean space, we have the following results from [8].

Lemma 5 *Let N be a CR-submanifold of $\mathbf{C}^m$. Then N is mixed foliate if and only if N is a CR-product.*

Lemma 6 *Every CR-product in a complex Euclidean m-space $\mathbf{C}^m$ is a direct product of a holomorphic submanifold of a linear complex subspace and a totally real submanifold of another linear complex subspace.*

Lemma 7 *Let N be a mixed foliate CR-submanifold of a Kähler manifold $\tilde{M}^m(4c)$ of constant holomorphic sectional curvature $4c$. Then, for any unit vectors $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$, we have $||A_{JZ}X||^2 = -c$.*

Lemma 8 *Let N be a CR-submanifold of a Kähler manifold $\tilde{M}$ with totally real distribution $\mathcal{D}^\perp$. Then at each point $x \in N$ there exists an orthonormal basis $\{e_1, \ldots, e_p\}$ of $\mathcal{D}^\perp_x$ such that the second fundamental form h of N in $\tilde{M}$ satisfies*

$$\langle A_{Je_1} e_1, e_i \rangle = 0, \quad i = 2, \ldots, p. \tag{2.11}$$

Lemma 8 extends a result of [31].

For CR-warped products, we have [16]

Lemma 9 *For a CR-warped product $N^T \times_f N^\perp$ in a Kähler manifold $\tilde{M}$, we have*

(1) $\left\langle \sigma(\mathcal{D}, \mathcal{D}), J\mathcal{D}^\perp \right\rangle = 0$*;*
(2) $\nabla_X Z = \nabla_Z X = (X \ln f) Z$*;*
(3) $\langle \sigma(JX, Z), JW \rangle = (X \ln f) \langle Z, W \rangle$*;*
(4) $D_X(JZ) = J\nabla_X Z$*, whenever* $\sigma(\mathcal{D}, \mathcal{D}^\perp) \subset J\mathcal{D}^\perp$*;*
(5) $\left\langle \sigma(\mathcal{D}, \mathcal{D}^\perp), J\mathcal{D}^\perp \right\rangle = 0$ *if and only if* $N^T \times_f N^\perp$ *is a trivial CR-warped product in* $\tilde{M}$*,*

where $X, Y \in \mathcal{D}$ *and* $Z, W \in \mathcal{D}^\perp$*.*

2.4 Riemannian Submersions

A *Riemannian submersion* $\pi : M \to B$ is a map from a Riemannian manifold M onto another Riemannian manifold B such that π has maximal rank and the differential π_* preserves lengths of horizontal vectors. Throughout this article, we only consider Riemannian submersions $\pi : M \to B$ with $m > b > 0$, where $m = \dim M$ and $b = \dim B$.

For each $x \in B$, $\pi^{-1}(x)$ is an $(m - b)$-dimensional submanifold of M. The submanifolds $\pi^{-1}(x)$, $x \in B$, are called *fibers*. A vector field on M is called *vertical* if it is always tangent to fibers; and *horizontal* if it is always orthogonal to fibers. We use corresponding terminology for individual tangent vectors as well. A vector field on M is called *basic* if X is horizontal and π-related to a vector field X_* on B, i.e., $\pi_* X_u = X_{*\pi(u)}$, for all $u \in M$.

Let $\mathcal{H}$ and $\mathcal{V}$ denote the projections of tangent spaces of M onto the subspaces of horizontal and vertical vectors, respectively. We use the same letters to denote the horizontal and vertical distributions.

Let g and g_B be the metric tensors of M and B, respectively, and g_F the induced metric on fibers. Denote by R_M, R_B and R_F the Riemann curvature tensors of the metrics g, g_B and g_F, respectively.

Associated with a Riemannian submersion $\pi : M \to B$, there exists a natural $(1, 2)$-tensor $\mathcal{A}$ on M, known as the O'Neill's integrability tensor, defined by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F) \tag{2.12}$$

for vector fields E, F tangent to M. In particular, for a horizontal vector field X and a vertical vector field V, we have

$$\mathcal{A}_X V = \mathcal{H}\nabla_X V. \tag{2.13}$$

For horizontal vector fields X, Y, the tensor $\mathcal{A}$ has the alternation property

$$\mathcal{A}_X Y = -\mathcal{A}_Y X. \tag{2.14}$$

Associated with $\pi : M \to B$, the invariants $\breve{A}_\pi$ and $\mathring{A}_\pi$ on M are defined by (cf. [22])

$$\breve{A}_\pi = \sum_{i=1}^{b} \sum_{s=b+1}^{m} ||\mathcal{A}_{X_i} V_s||^2, \quad \mathring{A}_\pi = \sum_{1 \le i < j \le b} ||\mathcal{A}_{X_i} X_j||^2, \tag{2.15}$$

where $X_1, \ldots, X_b$ are orthonormal basic horizontal vector fields and $V_{b+1}, \ldots, V_m$ are orthonormal vertical vector fields.

The following lemma can be found in [22, 36].

Lemma 10 *For vector fields X, Y tangent to B, we have*
(1) $\left\langle \bar{X}, \bar{Y} \right\rangle = \langle X, Y \rangle \circ \pi$,
(2) $\mathcal{H}[\bar{X}, \bar{Y}] = [X, Y]^-$,
(3) $\mathcal{H}\nabla_{\bar{X}} \bar{Y} = (\nabla'_X Y)^-$, *where ∇' is the Levi-Civita connection of B,*
where $\bar{X}, \bar{Y}$ and $[X, Y]^-$ are the horizontal lifts of X, Y and $[X, Y]$, respectively.

Lemma 11 *Let X, Y be horizontal vector fields and E, F be vector fields on M. Then, each of the following holds:*
(a) $\mathcal{A}_X Y = -\mathcal{A}_Y X$, *or equivalently,* $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$,
(b) $\mathcal{A}_{\mathcal{H}E} F = \mathcal{A}_E F$,
(c) $\mathcal{A}_E$ *maps the horizontal subspace into the vertical one and the vertical subspace into the horizontal one.*

Lemma 12 *Let $\pi : M \to B$ be a pseudo-Riemannian submersion. Then*

$$R_B(\pi_*X, \pi_*Y; \pi_*Y, \pi_*X) = R_M(X, Y; Y, X) + 3||\mathcal{A}_X Y||^2. \tag{2.16}$$

Moreover, if π has totally geodesic fibers, then we also have
(1) $R_M(U, V; V, U) = R_F(U, V; V, U)$,
(2) $R_M(X, U; U, X) = ||\mathcal{A}_X U||^2$,
for horizontal vector filed X, Y and vertical vector field U.

Lemma 13 *For a Riemannian submersion $\pi : M \to B$, we have*

$$K_M(\bar{X}, \bar{Y}) = K_B(X, Y) - 3\,||\mathcal{A}_{\bar{X}}\bar{Y}||^2$$

for orthonormal vector fields X, Y on B.

Let $\mathbb{C}^{m+1}$ denote the complex Euclidean $(m + 1)$-space and let

$$S^{2m+1} = \{z = (z_1, \ldots, z_{m+1}) \in \mathbb{C}^{m+1} : \ \langle z, z\rangle = 1\}$$

be the unit hypersphere of $\mathbb{C}^{m+1}$. Consider the Hopf fibration

$$\pi\colon\ S^{2m+1}(c) \to CP^m(4c). \tag{2.17}$$

Then, π is a Riemannian submersion with totally geodesic fibers.

Given $z \in S^{2m+1}$, the vector $\xi = \mathrm{i}z$ is tangent to the fibers and the horizontal space at z is the orthogonal complement of $\mathrm{i}z$ with respect to the induced metric on S^{2m+1} from the standard metric on $\mathbb{C}^{m+1}$. Moreover, given a horizontal vector X, then $\mathrm{i}X$ is again horizontal and $\pi_*(\mathrm{i}X) = J(\pi_*(X))$, where J is the complex structure on $CP^m(4)$. It is well-known that S^{2m+1} is a Sasakian manifold with characteristic vector field ξ and with the contact structure obtained from the projection of the complex structure J of $\mathbb{C}^{m+1}$.

Let $\phi : N \to CP^m(4)$ be an isometric immersion of a Riemannian n-manifold N into $CP^m(4)$. Then, $\tilde{N} = \pi^{-1}(N)$ is a principal circle bundle over N with totally geodesic fibers and the lift $\tilde{\phi} : \tilde{N} \to S^{2m+1}$ of ϕ is an isometric immersion such that

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\ \tilde{\phi}\ } & S^{2m+1} \\ {\scriptstyle\tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ N & \xrightarrow{\ \phi\ } & CP^m(4) \end{array}$$

commutes. Since ξ generate the vertical subspaces of $\pi\colon\ S^{2m+1}(c) \to CP^m(4c)$, we have the orthogonal decomposition

$$T_z\tilde{N} = \overline{T_{\pi(z)}N} \oplus Span\{\xi\}.$$

2.5 Warped Product CR Submanifolds and δ-Invariants

Let M be a Riemannian n-manifold. Let $K(\pi)$ be the sectional curvature associated with a plane section $\pi \subset T_pM$, $p \in M$. For an orthonormal basis $e_1, \ldots, e_n$ of T_pM, the scalar curvature τ at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of L by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

Let N be a CR-submanifold of a Kähler manifold. Denote by $\mathcal{D}$ and $\mathcal{D}^\perp$ the complex and the totally real distributions of N as before. The *CR δ-invariant* $\delta(\mathcal{D})$ is then defined by

$$\delta(\mathcal{D})(x) = \tau(x) - \tau(\mathcal{D}_x), \tag{2.18}$$

where $\tau(x)$ and $\tau(\mathcal{D}_x)$ denote the scalar curvature of N at $x \in N$ and the scalar curvature of $\mathcal{D}_x \subset T_xN$, respectively (see [23] for details).

For a CR-warped product $N^T \times_f N^\perp$ in the complex space form $\tilde{M}^{h+p}(4c)$ with $h = \dim_C N^T$ and $p = \dim N^\perp$, let us choose a local orthonormal frame $\{e_1, \ldots, e_{2h+p}\}$ on N such that $e_1, \ldots, e_h, e_{h+1} = Je_1, \ldots, e_{2h} = Je_h$ are in $\mathcal{D}$ and $e_{2h+1}, \ldots, e_{2h+p}$ are in $\mathcal{D}^\perp$.

In the following, we shall use the following convention on the range of indices *unless mentioned otherwise*:

$$i, j, k = 1, \ldots, 2h; \quad \alpha, \beta, \gamma = 1, \ldots, h,$$
$$r, s, t = 2h+1, \ldots, 2h+p; \quad A, B, C = 1, \ldots, 2h+p.$$

Let us put $\sigma^r_{AB} = \langle \sigma(e_A, e_B), Je_r \rangle$.

It follows from Lemma 9(2) that we have

$$\frac{\Delta f}{f} = \sum_{j=1}^{2h} K(e_j \wedge e_r) \tag{2.19}$$

for each $r \in \{2h+1, \ldots, 2h+p\}$.

The next theorem provides an optimal inequality for CR-warped submanifolds in complex space forms involving the CR δ-invariant.

Theorem 2.2 ([23]) *Let $N = N^T \times_f N^\perp$ be a CR-warped product in a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \dim_C N^T \geq 1$ and $p = \dim N^\perp \geq 2$. Then, we have*

$$H^2 \geq \frac{2(p+2)}{(2h+p)^2(p-1)}\left\{\delta(\mathcal{D}) - \frac{p\Delta f}{f} - \frac{p(p-1)c}{2}\right\}, \tag{2.20}$$

where Δf is the Laplacian of the warping function f and H^2 is the squared mean curvature.

The equality sign of (2.20) holds at a point $x \in N$ if and only if there exists an orthonormal basis $\{e_{2h+1}, \dots, e_n\}$ of $\mathcal{D}_x^\perp$ such that the coefficients of the second fundamental σ with respect to $\{e_{2h+1}, \dots, e_n\}$ satisfy

$$\begin{aligned} &\sigma_{rr}^r = 3\sigma_{ss}^r, && \text{for } 2h+1 \leq r \neq s \leq 2h+p, \\ &\sigma_{st}^r = 0, && \text{for distinct } r, s, t \in \{2h+1, \dots, 2h+p\}. \end{aligned} \tag{2.21}$$

Proof Let $N = N^T \times_f N^\perp$ be a *CR*-warped product in a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \dim_C N^T \geq 1$ and $p = \dim N^\perp \geq 2$. Let us choose an orthonormal frame $\{e_1, \dots, e_{2h+p}\}$ on N as above.

It follows from Gauss' equation, (2.18), and (2.19) that $\delta(\mathcal{D})$ satisfies

$$\begin{aligned} \delta(\mathcal{D}) &= \sum_{i,r} K(e_i, e_r) + \sum_{r<s} K(e_r, e_s) \\ &= \frac{p\Delta f}{f} + \sum_{r<s} \langle\sigma(e_r, e_r), \sigma(e_s, e_s)\rangle - \sum_{r<s} ||\sigma(e_r, e_s)||^2 + \frac{p(p-1)}{2}c. \end{aligned} \tag{2.22}$$

On the other hand, it follows from Lemma 9(1) and $J\mathcal{D}^\perp = T^\perp N$ that

$$\sum_{r<s} \langle\sigma(e_r, e_r), \sigma(e_s, e_s)\rangle - \sum_{r<s} ||\sigma(e_r, e_s)||^2 = \frac{n^2}{2}||H||^2 - \frac{1}{2}||\sigma_\perp||^2, \tag{2.23}$$

where $n = 2h + p$ and $||\sigma_\perp||^2$ is the squared norm of σ restricted to $\mathcal{D}^\perp$, i.e.

$$||\sigma_\perp||^2 = \sum_{r,s} ||\sigma(e_r, e_s)||^2. \tag{2.24}$$

By combining (2.22) and (2.23) we find

$$\delta(\mathcal{D}) = \frac{p\Delta f}{f} + \frac{p(p-1)}{2}c + \frac{n^2}{2}||H||^2 - \frac{1}{2}||\sigma_\perp||^2. \tag{2.25}$$

Thus we obtain

$$n^2||H||^2 + \frac{2(p+2)}{p-1}\left(\frac{p\Delta f}{f} - \delta(\mathcal{D})\right) + p(p+2)c$$
$$= \frac{3n^2}{1-p}||H||^2 + \frac{p+2}{p-1}||\sigma_\perp||^2. \tag{2.26}$$

From Lemma 1(a) we find $\sigma^r_{st} = \sigma^s_{rt} = \sigma^t_{rs}$. Now, we derive from (2.26) and Lemma 9(1) that

$$\begin{aligned}
&n^2||H||^2 + \frac{2(p+2)}{p-1}\left(\frac{p\Delta f}{f} - \delta(\mathcal{D})\right) + p(p+2)c \\
&= \sum_r \Big(\sum_s \sigma^r_{ss}\Big)^2 + \frac{3(p+1)}{p-1}\sum_{r\neq s}(\sigma^r_{ss})^2 + \frac{6(p+2)}{p-1}\sum_{r<s<t}(\sigma^r_{st})^2 \\
&\quad + \frac{2(p+2)}{p-1}\sum_r\sum_{s<t}\sigma^r_{ss}\sigma^r_{tt} \\
&= \sum_r (\sigma^r_{rr})^2 + \frac{3(p+1)}{p-1}\sum_{r\neq s}(\sigma^r_{ss})^2 + \frac{6(p+2)}{p-1}\sum_{r<s<t}(\sigma^r_{st})^2 \\
&\quad - \frac{6}{p-1}\sum_r\sum_{s<t}\sigma^r_{ss}\sigma^r_{tt} \\
&= \frac{6(p+2)}{p-1}\sum_{r<s<t}(\sigma^r_{st})^2 + \frac{3}{p-1}\sum_{r\neq s,t}\sum_{s<t}(\sigma^r_{ss} - \sigma^r_{tt})^2 \\
&\quad + \frac{1}{p-1}\sum_{s\neq r}(\sigma^r_{rr} - 3\sigma^r_{ss})^2 \\
&\geq 0.
\end{aligned} \tag{2.27}$$

Consequently, inequality (2.20) follows from (2.27). Moreover, it is easy to verify that the equality sign of (2.20) holds if and only if (2.21) holds.

All *CR*-warped products in the complex Euclidean $(h+p)$-space $\mathbb{C}^{h+p}$ satisfying the equality case of inequality (2.20) identically have been completely classified in [23] as follows.

Theorem 2.3 *Let* $\psi : N^T \times_f N^\perp \to \mathbb{C}^{h+p}$ *be a CR-warped product in* $\mathbb{C}^{h+p}$ *with* $h = \dim_C N^T \geq 1$ *and* $p = \dim N^\perp \geq 2$. *Then*

$$H^2 \geq \frac{2(p+2)}{(2h+p)^2(p-1)}\left\{\delta(\mathcal{D}) - \frac{p\Delta f}{f}\right\}. \tag{2.28}$$

The equality sign of (2.28) holds identically if and only if, up to dilations and rigid motions of $\mathbb{C}^{h+p}$, *one of the following three cases occurs:*

(a) *The CR-warped product is an open part of the CR-product* $\mathbb{C}^h \times W^p \subset \mathbb{C}^h \times \mathbb{C}^p$, *where* W^p *is the Whitney p-sphere in* $\mathbb{C}^p$;

(b) N^T *is an open part of* $\mathbb{C}^h$, $N^\perp$ *is an open part of the unit p-sphere* S^p, $f = |z_1|$ *and* ψ *is the minimal immersion defined by*

$$\big(z_1 w_0, \cdots, z_1 w_p, z_2, \ldots, z_h\big),$$

where $z = (z_1, \ldots, z_h) \in \mathbb{C}^h$ *and* $w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1}$;

(c) N^T *is an open part of* $\mathbb{C}^h$, $N^\perp$ *is the warped product of a curve and an open part of* S^{p-1} *with warping function* $\varphi = (\sqrt{c^2-1}/\sqrt{2})\mathrm{cn}\big(c\,t, \sqrt{c^2-1}/\sqrt{2}c\big)$, $c > 1$, $f = |z_1|$, *and* ψ *is the non-minimal immersion defined by*

$$\left(z_1 e^{\int \frac{\varphi(\varphi' + \mathrm{i}k\varphi^2)}{\varphi^2 - 1}dt}, z_1 \varphi e^{\mathrm{i}k\int \varphi dt} w_1, \cdots z_1 \varphi e^{\mathrm{i}k\int \varphi dt} w_p, z_2, \ldots, z_h\right),$$

with $z = (z_1, \ldots, z_h) \in \mathbb{C}^h$, $(w_1, \ldots, w_p) \in S^{p-1}(1) \subset \mathbb{E}^p$, *and* $k = \sqrt{c^4 - 1}/2$, *where* cn *is a Jacobi's elliptic function.*

For the proof of Theorem 2.3, see [23].

2.6 Anti-holomorphic Submanifolds with $p \geq 2$

For a *CR*-submanifold N of a Kähler manifold, the two partial mean curvature vectors $\overrightarrow{H}_{\mathcal{D}}$ and $\overrightarrow{H}_{\mathcal{D}^\perp}$ of N are defined by

$$\overrightarrow{H}_{\mathcal{D}} = \frac{1}{2h}\sum_{i=1}^{2h} \sigma(e_i, e_i), \qquad \overrightarrow{H}_{\mathcal{D}^\perp} = \frac{1}{p}\sum_{r=2h+1}^{2h+p} \sigma(e_r, e_r). \tag{2.29}$$

An anti-holomorphic submanifold N of a Kähler manifold $\tilde{M}$ is called *minimal* (resp., $\mathcal{D}$*-minimal* or $\mathcal{D}^\perp$*-minimal*) if $H = 0$ holds identical (resp., $\overrightarrow{H}_{\mathcal{D}} = 0$ or $\overrightarrow{H}_{\mathcal{D}^\perp} = 0$ hold identically).

For anti-holomorphic submanifolds with $p = \mathrm{rank}\,\mathcal{D}^\perp \geq 2$, we have the following optimal inequality.

Theorem 2.4 ([2]) *Let N be an anti-holomorphic submanifold of a complex space form* $\tilde{M}^{h+p}(4c)$ *with* $h = \mathrm{rank}_{\mathbf{C}}\,\mathcal{D} \geq 1$ *and* $p = \mathrm{rank}\,\mathcal{D}^\perp \geq 2$. *Then we have*

$$\delta(\mathcal{D}) \leq \frac{(p-1)(2h+p)^2}{2(p+2)} H^2 + \frac{p}{2}(4h + p - 1)c. \tag{2.30}$$

The equality sign of inequality (2.30) holds identically if and only if the following three conditions are satisfied:

(a) *N is $\mathcal{D}$-minimal, i.e.,* $\overrightarrow{H}_{\mathcal{D}} = 0$,
(b) *N is mixed totally geodesic, and*
(c) *there exist an orthonormal frame* $\{e_{2h+1}, \ldots, e_n\}$ *of* $\mathcal{D}^{\perp}$ *such that the second fundamental* σ *of N satisfies*

$$\begin{cases} \sigma^r_{rr} = 3\sigma^r_{ss}, & \text{for } 2h+1 \leq r \neq s \leq 2h+p, \\ \sigma^r_{st} = 0, & \text{for distinct } r, s, t \in \{2h+1, \ldots, 2h+p\}. \end{cases} \tag{2.31}$$

Proof Let N be an anti-holomorphic submanifold in a complex space form $\tilde{M}^{h+p}(4c)$. Let us choose an orthonormal frame $\{e_1, \ldots, e_{2h+p}\}$ on N as above.

It follows from the equation of Gauss and the definition of *CR* δ-invariant that $\delta(\mathcal{D})$ satisfies

$$\begin{aligned} \delta(\mathcal{D}) &= \sum_{i=1}^{2h} \sum_{r=2h+1}^{2h+p} K(e_i, e_r) + \sum_{2h+1 \leq r \neq s \leq 2h+p} \frac{1}{2} K(e_r, e_s) \\ &= \sum_{i=1}^{2h} \sum_{r=2h+1}^{2h+p} \langle \sigma(e_i, e_i), \sigma(e_r, e_r) \rangle + \sum_{r,s=2h+1}^{2h+p} \frac{1}{2} \langle \sigma(e_r, e_r), \sigma(e_s, e_s) \rangle \\ &\quad - \sum_{i=1}^{2h} \sum_{r=2h+1}^{2h+p} ||\sigma(e_i, e_r)||^2 - \sum_{r,s=2h+1}^{2h+p} \frac{1}{2} ||\sigma(e_r, e_s)||^2 + \frac{p}{2}(4h+p-1)c. \end{aligned} \tag{2.32}$$

On the other hand, we have

$$\begin{aligned} &\sum_{i=1}^{2h} \sum_{r=2h+1}^{2h+p} \langle \sigma(e_i, e_i), \sigma(e_r, e_r) \rangle + \sum_{r,s=2h+1}^{2h+p} \frac{1}{2} \langle \sigma(e_r, e_r), \sigma(e_s, e_s) \rangle \\ &\quad - \sum_{r,s=2h+1}^{2h+p} \frac{1}{2} ||\sigma(e_r, e_s)||^2 \\ &\quad = \frac{(2h+p)^2}{2} H^2 - 2h^2 |\overrightarrow{H}_{\mathcal{D}}|^2 - \frac{1}{2} ||\sigma_{\mathcal{D}^{\perp}}||^2, \end{aligned} \tag{2.33}$$

where $||\sigma_{\mathcal{D}^{\perp}}||^2$ is defined by

$$||\sigma_{\perp}||^2 = \sum_{r,s=2h+1}^{2h+p} ||\sigma(e_r, e_s)||^2. \tag{2.34}$$

By combining (2.32) and (2.33) we find

$$\delta(\mathcal{D}) = \frac{(2h+p)^2}{2}H^2 + \frac{p}{2}(4h+p-1)c - 2h^2|\overrightarrow{H}_{\mathcal{D}}|^2 - \sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2 - \frac{1}{2}||\sigma_{\mathcal{D}^\perp}||^2. \tag{2.35}$$

It follows from statement (2) of Lemma 1 the coefficients of the second fundamental form satisfy

$$\sigma^r_{st} = \sigma^s_{rt} = \sigma^t_{rs}. \tag{2.36}$$

We find from (2.29), (2.34), and (2.36) that

$$\begin{aligned}
&(p+2)||\sigma_{\mathcal{D}^\perp}||^2 - 3p^2|H_{\mathcal{D}^\perp}|^2 \\
&= (p-1)\sum_{r=2h+1}^{2h+p}\Big(\sum_{s=2h+1}^{2h+p}\sigma^r_{ss}\Big)^2 \\
&\quad + \sum_{2h+1\le r\ne s\le 2h+p} 3(p+1)(\sigma^r_{ss})^2 + \sum_{2h+1\le r<s<t\le 2h+p} 6(p+2)(\sigma^r_{st})^2 \\
&\quad + \sum_{r=2h+1}^{2h+p}\sum_{2h+1\le s<t\le 2h+p} 2(p+2)\sigma^r_{ss}\sigma^r_{tt} \\
&= \sum_{r=2h+1}^{2h+p}(p-1)(\sigma^r_{rr})^2 + \sum_{2h+1\le r\ne s\le 2h+p} 3(p+1)(\sigma^r_{ss})^2 \\
&\quad + \sum_{2h+1\le r<s<t\le 2h+p} 6(p+2)(\sigma^r_{st})^2 - \sum_{r=2h+1}^{2h+p}\sum_{2h+1\le s<t\le 2h+p} 6\sigma^r_{ss}\sigma^r_{tt} \\
&= \sum_{2h+1\le r<s<t\le 2h+p} 6(p+2)(\sigma^r_{st})^2 + \sum_{2h+1\le s\ne r\le 2h+p}(\sigma^r_{rr} - 3\sigma^r_{ss})^2 \\
&\quad + \sum_{r\ne s,t}\sum_{2h+1\le s<t\le 2h+p} 3(\sigma^r_{ss} - \sigma^r_{tt})^2 \\
&\ge 0.
\end{aligned} \tag{2.37}$$

Thus, we get

$$||\sigma_{\mathcal{D}^\perp}||^2 \ge \frac{3p^2}{p+2}|H_{\mathcal{D}^\perp}|^2, \tag{2.38}$$

with equality holding if and only if

$$\begin{aligned}
&\sigma^r_{rr} = 3\sigma^r_{ss}, && \text{for } 2h+1 \le r \ne s \le 2h+p, \\
&\sigma^r_{st} = 0, && \text{for distinct } r, s, t \in \{2h+1, \ldots, 2h+p\}.
\end{aligned} \tag{2.39}$$

Now, by combining (2.35) and (2.38), we obtain

$$
\begin{aligned}
&\frac{(2h+p)^2}{2}H^2 + \frac{p}{2}(4h+p-1)c - \delta(\mathcal{D}) \\
&\geq 2h^2|\overrightarrow{H}_{\mathcal{D}}|^2 + \sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2 + \frac{3p^2}{2(p+2)}|H_{\mathcal{D}^\perp}|^2 \\
&= \frac{3}{2(p+2)}\left\{(2h+p)^2H^2 - 4h^2|\overrightarrow{H}_{\mathcal{D}}|^2 - 2\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2\right\} \\
&\quad + 2h^2|\overrightarrow{H}_{\mathcal{D}}|^2 + \sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2. \\
&= \frac{3(2h+p)^2}{2(p+2)}H^2 + \frac{2h^2(p-1)}{p+2}|\overrightarrow{H}_{\mathcal{D}}|^2 + \frac{p-1}{p+2}\sum_{i=1}^{2h}\sum_{r=2h+1}^{2h+p}||\sigma(e_i,e_r)||^2 \\
&\geq \frac{3(2h+p)^2}{2(p+2)}H^2. \qquad (2.40)
\end{aligned}
$$

It is obvious that the equality of the last inequality in (2.40) holds if and only if N is $\mathcal{D}$-minimal and mixed totally geodesic. Consequently, we may obtain inequality (2.30) from (2.40).

It is straightforward to verify that the equality sign of (2.30) holds identically if and only if conditions (a), (b) and (c) of Theorem 2.4 are satisfied.

2.7 Anti-holomorphic Submanifolds with Equality in (2.30)

The notion of H-umbilical Lagrangian submanifolds was introduced in [12, 13].

Definition 1 A Lagrangian submanifold is said to be *H-umbilical* if its second fundamental form satisfies the following simple form:

$$
\begin{aligned}
&\sigma(e_1,e_1) = \lambda Je_1, \quad \sigma(e_1,e_j) = \mu Je_j, \\
&\sigma(e_2,e_2) = \cdots = \sigma(e_n,e_n) = \mu Je_1, \qquad (2.41) \\
&\sigma(e_j,e_k) = 0, \quad j \neq k, \quad j,k = 2,\ldots,n,
\end{aligned}
$$

for some suitable functions φ and ψ with respect to some suitable orthonormal local frame field $\{e_1,\ldots,e_n\}$.

Since there do not exist umbilical Lagrangian submanifold in Kähler manifolds other than totally geodesic ones (cf. [26]), H-umbilical Lagrangian submanifolds are the simplest Lagrangian submanifolds next to totally geodesic one (cf. [12, 13]).

Let $G : N^{p-1} \to \mathbb{E}^p$ be an isometric immersion of a Riemannian $(p-1)$-manifold into the Euclidean p-space $\mathbb{E}^p$ and let $F : I \to \mathbb{C}^*$ be a unit speed curve in $\mathbb{C}^* = \mathbb{C} - \{0\}$. We extend $G : N^{p-1} \to \mathbb{E}^p$ to an immersion of $I \times N^{p-1}$ into $\mathbb{C}^p$ as

$$F \otimes G : I \times N^{p-1} \to \mathbb{C} \otimes \mathbb{E}^p = \mathbb{C}^p, \tag{2.42}$$

where $(F \otimes G)(s, q) = F(s) \otimes G(q)$ for $s \in I,\ q \in N^{p-1}$. This extension $F \otimes G$ of G via tensor product is called the *complex extensor* of G via F.

Example 1 (*Whitney sphere*) Let $w : S^p(1) \to \mathbb{C}^p$ be the map of the unit p-sphere into $\mathbb{C}^p$ defined by

$$w(y_0, y_1, \ldots, y_p) = \frac{1 + \mathrm{i}y_0}{1 + y_0^2}(y_1, \ldots, y_p), \quad y_0^2 + y_1^2 + \cdots + y_p^2 = 1.$$

The map w is a (non-isometric) Lagrangian immersion with one self-intersection point which is called the *Whitney p-sphere.* The Whitney p-sphere is a complex extensor $\phi = F \otimes \iota$ of $\iota : S^{p-1}(1) \subset \mathbb{E}^p$ via F, where $F = F(s)$ is an arclength reparametrization of the curve $f : I \to \mathbb{C}$ defined by

$$f(t) = \frac{\sin t + \mathrm{i} \sin t \cos t}{1 + \cos^2 t}.$$

Up to rigid motions and dilations, the Whitney sphere is the only Lagrangian H-umbilical submanifold in $\mathbb{C}^p$ which satisfies (2.41) with $\lambda = 3\mu$ (see [12]).

Consider the product immersion

$$\phi : \mathbf{C}^h \times S^p(1) \to \mathbf{C}^h \oplus \mathbf{C}^p = \mathbf{C}^{h+p}$$

defined by

$$\phi(z, x) = (z, w(x)), \quad \forall z \in \mathbf{C}^h, \quad \forall x \in S^p(1). \tag{2.43}$$

It is straightforward to verify that ϕ is an anti-holomorphic isometric immersion which satisfies the equality sign of (2.30) identically.

Anti-holomorphic submanifolds satisfying the equality case of inequality (2.30) were classified by the following two theorems.

Theorem 2.5 ([2]) *Let N be an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \mathrm{rank}_{\mathbf{C}}\, \mathcal{D} \geq 1$ and $p = \mathrm{rank}\, \mathcal{D}^{\perp} \geq 2$. If N satisfies the equality case of (2.30) identically and if the complex distribution $\mathcal{D}$ is integrable, then $c = 0$ so that $\tilde{M}^{h+p}(4c) = \mathbf{C}^{h+p}$. Moreover, we have either*

(i) *N is a totally geodesic anti-holomorphic submanifold of $\mathbf{C}^{h+p}$ or,*
(ii) *up to dilations and rigid motions of $\mathbf{C}^{h+p}$, N is given by an open portion of the following product immersion:*

$$\phi : \mathbf{C}^h \times S^p(1) \to \mathbf{C}^{h+p}; \quad (z, x) \mapsto (z, w(x)), \quad z \in \mathbf{C}^h,\ x \in S^p(1),$$

where $w : S^p(1) \to \mathbf{C}^p$ is the Whitney p-sphere.

Proof Assume that N is an anti-holomorphic submanifold of a complex space form $\tilde{M}^{h+p}(4c)$ with $h = \text{rank}_{\mathbf{C}}\,\mathcal{D} \geq 1$ and $p = \text{rank}\,\mathcal{D}^\perp \geq 2$. If N satisfies the equality case of (2.30) and if the complex distribution $\mathcal{D}$ is integrable, then it follows from Theorem 2.4 that N is mixed foliate. Hence Lemma 4 implies that $c = 0$. Thus, according to Lemma 5, N is a CR-product. Therefore, N is locally a CR-product given by

$$\mathbf{C}^h \times N^\perp \subset \mathbf{C}^h \times \mathbf{C}^p,$$

where $\mathbf{C}^h$ is a complex Euclidean h-subspace and $N^\perp$ is a Lagrangian submanifold of $\mathbf{C}^p$. Consequently, condition (c) of Theorem 2.4 implies that $N^\perp$ is a Lagrangian H-umbilical submanifold in $\mathbf{C}^p$ whose second fundamental form satisfying

$$\begin{aligned}
&\sigma(e_{2h+1}, e_{2h+1}) = 3\lambda J e_{2h+1},\\
&\sigma(e_{2h+1}, e_s) = \lambda J e_s,\\
&\sigma(e_{2h+2}, e_{2h+2}) = \cdots = \sigma(e_{2h+p}, e_{2h+p}) = \lambda J e_{2h+1},\\
&\sigma(e_r, e_s) = 0, \quad 2h+2 \leq r \neq s \leq 2h+p,
\end{aligned} \tag{2.44}$$

for some suitable function λ with respect to some suitable orthonormal local frame field $\{e_{2h+1}, \ldots, e_{2h+p}\}$ of $TN^\perp$.

If $\lambda = 0$, then $N^\perp$ is an open portion of a totally geodesic totally real p-plane in $\mathbf{C}^p$. Hence, in this case, N is a totally geodesic anti-holomorphic submanifold.

If $\lambda \neq 0$, it follows from (2.44) that, up to dilations and rigid motions, $N^\perp$ is an open part of the Whitney p-sphere in $\mathbf{C}^p$ (cf. [7, 22]). Consequently, up to dilations and rigid motions of $\mathbf{C}^{h+p}$, the anti-holomorphic submanifold is locally given by the product immersion

$$\phi : \mathbf{C}^h \times S^p(1) \to \mathbf{C}^{h+p}; \qquad (z, x) \mapsto (z, w(x)), \tag{2.45}$$

for $z \in \mathbf{C}^h$ and $x \in S^p(1)$, where $w : S^p(1) \to \mathbf{C}^p$ is the Whitney p-sphere.

The converse is easy to verify.

Theorem 2.6 ([2]) *Let N be an anti-holomorphic submanifold in a complex space form $\tilde{M}^{1+p}(4c)$ with $h = \text{rank}_{\mathbf{C}}\,\mathcal{D} = 1$ and $p = \text{rank}\,\mathcal{D}^\perp \geq 2$. Then, we have*

$$\delta(\mathcal{D}) \leq \frac{(p-1)(p+2)^2}{2(p+2)} H^2 + \frac{p}{2}(p+3)c. \tag{2.46}$$

The equality case of (2.46) holds identically if and only if $c = 0$ *and either*

(i) *N is a totally geodesic anti-holomorphic submanifold of* $\mathbf{C}^{h+p}$ *or,*
(ii) *up to dilations and rigid motions, N is given by an open portion of the following product immersion:*

$$\phi : \mathbf{C} \times S^p(1) \to \mathbf{C}^{1+p}; \qquad (z, x) \mapsto (z, w(x)), \quad z \in \mathbf{C}, \; x \in S^p(1),$$

where $w : S^p(1) \to \mathbf{C}^p$ *is the Whitney p-sphere.*

Proof Let N be an anti-holomorphic submanifold in a complex space form $\tilde{M}^{1+p}(4c)$. Then we have inequality (2.44) from inequality (2.30).

Assume that N satisfies the equality case of (2.46) identically. Then, Theorem 2.4 implies that N satisfies conditions (a), (b), and (c) of Theorem 2.4.

By condition (a), N is $\mathcal{D}$-minimal. Thus, we find

$$\sigma(Je_1, Je_1) = -\sigma(e_1, e_1) \tag{2.47}$$

for any unit vector $e_1 \in \mathcal{D}$. It is direct to verify from (2.47) and polarization that the second fundamental form satisfies the following condition:

$$\sigma(X, JY) = \sigma(JX, Y), \quad \forall X, Y \in \mathcal{D}.$$

Therefore, according to Lemma 2(1), we may conclude that $\mathcal{D}$ is integrable. Consequently, we obtain Theorem 2.6 from Theorem 2.5.

2.8 An Optimal Inequality for Real Hypersurfaces

Obviously, anti-holomorphic submanifolds with rank $\mathcal{D}^\perp = 1$ are nothing but real hypersurfaces. A real hypersurface N of a Kähler manifold $\tilde{M}$ is called a *Hopf hypersurface* if $J\xi$ is a principal curvature vector, i.e., an eigenvector of the shape operator A_ξ, where ξ is a unit normal vector of N. In the following, we call a Hopf hypersurface N *special* if the vector field $J\xi$ is an eigenvector field of A_ξ with eigenvalue 0, i.e., $A_\xi(J\xi) = 0$.

For real hypersurfaces, we have the following:

Theorem 2.7 ([2]) *If N is a real hypersurface of a complex space form* $\tilde{M}^{h+1}(4c)$*, then the Ricci tensor Ric of N satisfies*

$$Ric(J\xi, J\xi) \leq \frac{(2h+1)^2}{2} H^2 + 2hc. \tag{2.48}$$

where ξ *is a unit normal vector field of N in* $\tilde{M}^{h+1}(4c)$.

The equality sign of inequality (2.48) holds identically if and only if N is a minimal special Hopf hypersurface.

Proof Let N be a real hypersurface of a complex space form $\tilde{M}^{h+1}(4c)$. Then it follows from the definition of $\delta(\mathcal{D})$ that

$$\delta(\mathcal{D}) = Ric(J\xi, J\xi). \tag{2.49}$$

Let us choose an orthonormal frame

$$\{e_1, \ldots, e_h, e_{h+1} = Je_1, \ldots, e_{2h} = Je_h\}$$

for the complex distribution $\mathcal{D}$ and let e_{2h+1} be a unit vector field in $\mathcal{D}^\perp$.

We put

$$\sigma_{a,b} = \langle \sigma(e_a, e_b), Je_{2h+1} \rangle, \quad a, b = 1, \ldots, 2h+1. \tag{2.50}$$

Let us define the connection forms by

$$\begin{aligned} \nabla_X e_i &= \sum_{j=1}^{2h} \omega_i^j(X) e_j + \omega_i^{2h+1}(X) e_{2h+1}, \\ \nabla_X e_{2h+1} &= \sum_{j=1}^{2h} \omega_{2h+1}^j(X) e_j, \end{aligned} \tag{2.51}$$

for $i = 1, \ldots, 2h$. It follows from (2.18) and the equation of Gauss that

$$\delta(\mathcal{D}) = \sum_{i=1}^{2h} \sigma_{i,i}\sigma_{2h+1,2h+1} - \sum_{i=1}^{2h} (\sigma_{i,2h+1})^2 + 2hc. \tag{2.52}$$

On the other hand, we have

$$\sum_{i=1}^{2h} \sigma_{i,i}\sigma_{2h+1,2h+1} = \frac{(2h+1)^2}{2} H^2 - \frac{1}{2}(\sigma_{2h+1,2h+1})^2 - 2h^2 |\overrightarrow{H}_{\mathcal{D}}|^2. \tag{2.53}$$

By combining (2.52) and (2.53) we obtain

$$\begin{aligned} \delta(\mathcal{D}) &= \frac{(2h+1)^2}{2} H^2 + 2hc - 2h^2 |\overrightarrow{H}_{\mathcal{D}}|^2 - \frac{1}{2}(\sigma_{2h+1,2h+1})^2 \\ &\quad - \sum_{i=1}^{2h} (\sigma_{i,2h+1})^2 \\ &\leq \frac{(2h+1)^2}{2} H^2 + 2hc. \end{aligned} \tag{2.54}$$

It follows from (2.54) and Lemma 3(2) that the equality sign of inequality (2.48) holds identically if and only if the following two statements hold:

(i) N is a special Hopf hypersurface and
(ii) N is $\mathcal{D}$-minimal in $\tilde{M}^{h+1}(4c)$.

Obviously, conditions (i) and (ii) imply that N is a minimal real hypersurface of $\tilde{M}^{h+1}(4c)$.

The converse is easy to verify.

The following corollary follows from Theorem 2.7.

Corollary 6 ([2]) *Let N be a real hypersurface of a complex space form $\tilde{M}^{h+1}(4c)$. If N satisfies the equality case of (2.48) identically, then the complex distribution of N is non-integrable, unless $c = 0$ and N is totally geodesic.*

Proof Under the hypothesis, if N satisfies the equality case of (2.48) identically and if the complex distribution $\mathcal{D}$ is integrable, then Theorem 2.7 implies that N is mixed foliate. So, it follows from Lemmas 4 and 5 that $c = 0$ and N is a CR-product of a complex h-subspace in $\mathbf{C}^h$ and an open portion of line in $\mathbf{C}$. Consequently, N must be totally geodesic.

The following results are some further applications of Theorem 2.7

Theorem 2.8 ([2]) *If N is a real hypersurface of $\tilde{M}^2(4c)$, then we have*

$$Ric(J\xi, J\xi) \leq \frac{9}{2}H^2 + 2c. \tag{2.55}$$

The equality sign of inequality (2.55) holds identically if and only if $c = 0$ and N is totally geodesic.

Theorem 2.9 ([2]) *Let N be a real hypersurface of $\mathbf{C}^3$. We have*

$$Ric(J\xi, J\xi) \leq \frac{25}{2}H^2. \tag{2.56}$$

If the equality case of inequality (2.56) holds identically, then N is a totally real 3-ruled minimal submanifold of $\mathbf{C}^3$.

Theorem 2.10 ([2]) *If N is a real hypersurface of $CP^3(4)$, then we have*

$$Ric(J\xi, J\xi) \leq \frac{25}{2}H^2 + 4. \tag{2.57}$$

The equality sign of inequality (2.57) holds identically if and only if locally there exists an orthonormal frame $\{e_1, e_2, e_3 = Je_1, e_4 = Je_2, e_5\}$ such that

$$\sigma(e_1, e_1) = \lambda\xi,\ \sigma(e_2, e_2) = -\lambda\xi,$$
$$\sigma(e_3, e_3) = \frac{1}{\lambda}\xi,\ \sigma(e_4, e_4) = -\frac{1}{\lambda}\xi,$$
$$\sigma(e_a, e_b) = 0\ \ otherwise,$$

where λ is a nowhere zero function.

Theorem 2.11 ([2]) *If N is a real hypersurface of $CH^3(-4)$, then we have*

$$Ric(J\xi, J\xi) \leq \frac{25}{2}H^2 - 4. \tag{2.58}$$

The equality sign of inequality (2.58) holds identically if and only if locally there exists an orthonormal frame $\{e_1, e_2, e_3 = Je_1, e_4 = Je_2, e_5\}$ on N such that

$$\sigma(e_1, e_1) = \lambda\xi,\ \sigma(e_2, e_2) = -\lambda\xi,$$
$$\sigma(e_3, e_3) = -\frac{1}{\lambda}\xi,\ \sigma(e_4, e_4) = \frac{1}{\lambda}\xi,$$
$$\sigma(e_a, e_b) = 0\ \ otherwise,$$

where λ is a nowhere zero function.

Corollary 7 ([2]) *Every real hypersurface of $CP^3(4)$ (resp., of $CH^3(-4)$) satisfying the equality case of (2.57) (resp., the equality case of (2.58)) is $\delta(2, 2)$-ideal in the sense of [15, 22].*

For the proofs of the above, see [2].

2.9 An Inequality Involving a Submersion δ-Invariant

Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers and let N be a Riemannian n-manifold isometrically immersed in B. Denote the pre-image $\pi^{-1}(N)$ of N in M by $\tilde{N}$. Then $\tilde{\pi} : \tilde{N} \to N$ is also a Riemannian submersion with totally geodesic fibers, where $\tilde{\pi}$ is the restriction $\pi|_{\tilde{N}}$.

For a horizontal 2-plane $P_x \subset T_x\tilde{N}$ we denote the $(m-b+2)$–subspace spanned by P_x and the vertical $\mathcal{V}_x$ by $\bar{P}_x$. The *submersion δ-invariant* δ^H on $\widetilde{N}$ is defined by (cf. [1]):

$$\delta^H(x) = \tau_{\tilde{N}}(x) - \inf_{\bar{P}_x} \tau_{\tilde{N}}(\bar{P}_x), \tag{2.59}$$

where $\bar{P}_x$ runs over $(m-b+2)$–subspaces associated with all horizontal 2-planes P_x at $x \in \tilde{N}$. Obviously, we have $\delta_{\tilde{N}}(r) \geq \delta^H$ with $r = 2 + m - b$.

Lemma 14 ([1]) *Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers. Then the scalar curvature τ_M of M and the scalar curvature τ_B of B satisfy*

$$\tau_M = \tau_B + \breve{A}_\pi - 3\mathring{A}_\pi + \tau_F, \tag{2.60}$$

where τ_F is the scalar curvature of fibers.

Proof Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers. For orthonormal basic horizontal vector fields $X_1, \ldots, X_b$ and orthonormal vertical vector fields $V_{b+1}, \ldots, V_m$ on M, it follows from Lemma 12(2) that

$$K_M(X_i \wedge V_\alpha) = ||\mathcal{A}_{X_i} V_\alpha||^2. \tag{2.61}$$

Also, it follows from Lemma 4 and (2.15) that the scalar curvature $\tau(\mathcal{H})$ of the horizontal space satisfies

$$\tau(\mathcal{H}) = \tau_B - 3\mathring{A}_\pi. \tag{2.62}$$

Moreover, since π has totally geodesic fibers, the scalar curvature τ_F equals the scalar curvature $\tau(\mathcal{V})$ of the vertical distribution. Consequently, we obtain (2.60) from (2.7), (2.61) and (2.62).

Lemma 15 *Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers and N be a submanifold of B. Then, for orthonormal vectors e_1, e_2 at $\pi(x) \in N$, $x \in \tilde{N}$, we have*

$$\begin{aligned}\tau_{\tilde{N}}(x) - \tau_{\tilde{N}}(\bar{P}_x) = \tau_N - K_N(e_1, e_2) - 3(\mathring{A}_{\tilde{\pi}} - ||\mathcal{A}_{\bar{e}_1}\bar{e}_2||^2) \\ + \breve{A}_{\tilde{\pi}} - \sum_{i=1}^{2}\sum_{\alpha=1}^{m-b} K_{\tilde{N}}(\bar{e}_i, v_\alpha),\end{aligned} \tag{2.63}$$

where $\bar{e}_1, \bar{e}_2$ are horizontal vectors at x, $\{v_1, \ldots, v_{m-b}\}$ is an orthonormal basis of the vertical space $\mathcal{V}_x$, and $\bar{P}_x$ is the subspace spanned by $\bar{e}_1, \bar{e}_2$ and $\mathcal{V}_x$.

Proof Under the hypothesis, $\tilde{\pi} : \tilde{N} = \pi^{-1}(N) \to N$ is a Riemannian submersion with totally geodesic fibers. Thus it follows from Lemma 14 that

$$\tau_{\tilde{N}} = \tau_N + \breve{A}_{\tilde{\pi}} - 3\mathring{A}_{\tilde{\pi}} + \tau_F. \tag{2.64}$$

Let $x \in \tilde{N}$ and e_1, e_2 orthonormal vectors at $\pi(x) \in N$. Denote by $\bar{e}_1, \bar{e}_2$ the horizontal lifts of e_1, e_2 at $x \in \tilde{N}$. As before let $\bar{P}_x$ denote the subspace of $T_x\tilde{N}$ spanned by $\bar{e}_1, \bar{e}_2$ and $\mathcal{V}_x$. Then we have

$$\tau_{\tilde{N}}(\bar{P}_x) = \tau_F + K_{\tilde{N}}(\bar{e}_1, \bar{e}_2) + \sum_{i=1}^{2}\sum_{\alpha=1}^{m-b} K_{\tilde{N}}(\bar{e}_i, v_\alpha). \tag{2.65}$$

From Lemma 4 we find

$$K_{\tilde{N}}(\bar{e}_1, \bar{e}_2) = K_N(e_1, e_2) - 3\,||\mathcal{A}_{\bar{e}_1}\bar{e}_2||^2. \tag{2.66}$$

By combining (2.64)–(2.66) and Lemma 10(2), we obtain (2.63).

Let N be a Riemannian submanifold of a Kähler manifold. For $X \in TN$ we put

$$JX = PX + FX, \tag{2.67}$$

where PX and FX are the tangential and normal components of JX, respectively. It follows from $J^2 = -I$ and (2.67) that

$$\langle PX, Y \rangle = -\langle X, PY \rangle \tag{2.68}$$

for X, Y tangent to N.

Let ψ be a 2-plane section of $T_{\bar{x}}N$, $\bar{x} \in N$, spanned by two orthonormal vectors $e_1, e_2 \in T_{\bar{x}}N$. We put

$$\Theta(\psi) = \langle Pe_1, e_2 \rangle^2. \tag{2.69}$$

If $\{e_1, \ldots, e_n\}$ is an orthonormal frame of N, then the squared norms $||P||^2$ and $||F||^2$ of P and F are defined, respectively, by

$$||P||^2 = \sum_{i=1}^{n} ||Pe_i||^2, \quad ||F||^2 = \sum_{i=1}^{n} ||Fe_i||^2. \tag{2.70}$$

Lemma 16 ([11]) *Let $\phi : N \to CP^m(4)$ be an isometric immersion from a Riemannian n-manifold N into the complex projective m-space $CP^m(4)$. Then, for any 2-plane section $\psi \subset T_yN$, $y \in N$, we have*

$$\tau_N - K_N(\psi) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{(n+1)(n-2)}{2} + \frac{3}{2}||P||^2 - 3\Theta(\psi). \tag{2.71}$$

The equality of inequality (2.71) holds at a point $y \in N$ if and only if there is an orthonormal basis $e_1, \ldots, e_m$ at y such that
(i) $\psi = \mathrm{Span}\{e_1, e_2\}$ *and*
(ii) *the shape operator A at y satisfies*

$$A_{e_s} = \begin{pmatrix} B_s & 0 \\ 0 & \mu_s I \end{pmatrix}, \quad s = n+1, \ldots, 2m, \tag{2.72}$$

where I is an identity $(n-2) \times (n-2)$-submatrix and B_s are symmetric 2×2 submatrices with $\mu_s = \mathrm{trace}\, B_s$, $s = n+1, \ldots, 2m$.

An important application of Lemma 15 is the following.

Theorem 2.12 ([1]) *Let $\pi : S^{2m+1} \to CP^m(4)$ be the Hopf fibration and let N be an n-dimensional submanifold of $CP^m(4)$. Then we have*

$$\delta^H \leq \frac{n^2(n-2)}{2(n-1)}||H||^2 + ||P||^2 + \frac{1}{2}(n^2 - n - 2), \tag{2.73}$$

where $||H||^2$ is the squared mean curvature of N in $CP^m(4)$.

The equality sign of (2.73) holds identically if and only if there is an orthonormal frame $e_1, \ldots, e_m$ such that
(a) *the shape operator A of N in $CP^m(4)$ satisfies*

$$A_{e_s} = \begin{pmatrix} B_s & 0 \\ 0 & \mu_s I \end{pmatrix}, \quad s = n+1, \ldots, 2m, \tag{2.74}$$

where I is an identity $(n-2) \times (n-2)$ matrix and B_s are symmetric 2×2 submatrices satisfying $\mu_s = trace\, B_s$, $s = n+1, \ldots, 2m$, and
(b) $Pe_1 = Pe_2 = 0$.

Proof Let $\pi : S^{2m+1} \to CP^m(4)$ be the Hopf fibration and put $\xi = \mathrm{i}z$ as before. Denote by $\hat{\nabla}$ and $\check{\nabla}$ the Levi-Civita connections of S^{2m+1} and $CP^m(4)$, respectively. For vector fields X, Y tangent to $CP^m(4)$, we have

$$\hat{\nabla}_{\bar{X}}\bar{Y} = \overline{\check{\nabla}_X Y} - \langle JX, Y\rangle\, \xi, \tag{2.75}$$

$$\hat{\nabla}_{\bar{X}}\xi = \hat{\nabla}_{\xi}\bar{X} = \overline{JX}. \tag{2.76}$$

Let N be an n-dimensional submanifold of $CP^m(4)$. Denote by $\tilde{N}$ the pre-image of N via the Hopf fibration $\pi : S^{2m+1} \to CP^m(4)$. Let P_y be a 2-plane section of a tangent space T_yN of N spanned by two orthonormal vectors e_1, e_2. As before, we denote by $\bar{P}_x$ the 3-plane spanned by ξ_x and the horizontal lifts $\bar{e}_1, \bar{e}_2$ of e_1, e_2 at a point x with $\pi(x) = y$. Then it follows from Lemma 15 that

$$\begin{aligned}\tau_{\tilde{N}}(x) - \tau_{\tilde{N}}(\bar{P}_x) = \tau_N - K_N(e_1, e_2) - 3\mathring{A}_{\tilde{\pi}} + 3||\mathcal{A}_{\bar{e}_1}\bar{e}_2||^2 \\ + \breve{A}_{\tilde{\pi}} - \sum_{i=1}^{2} K_{\tilde{N}}(\bar{e}_i, \xi),\end{aligned} \tag{2.77}$$

where $\xi = \mathrm{i}z$ is the characteristic vector field of the Sasakian space form S^{2m+1}.

If η is a normal vector field of N in $CP^m(4)$, then by using $\xi = \mathrm{i}z$ we find

$$\hat{\nabla}_{\bar{X}}\bar{\eta} = \overline{\check{\nabla}_X \eta} - \langle FX, \eta\rangle\, \xi. \tag{2.78}$$

Hence, Weingarten's formula yields

$$\hat{A}_{\bar{\xi}}\bar{X} = \overline{A_{\xi}X} + \langle FX, \eta\rangle\,\xi, \quad \hat{D}_{\bar{X}}\bar{\eta} = \overline{D_X\eta}, \tag{2.79}$$

where $A, \hat{A}$ are the shape operators of N in $CP^m(4)$ and $\tilde{N}$ in S^{2m+1}, respectively, and D and $\hat{D}$ are the corresponding normal connections.

From (2.75) we get

$$\hat{h}(\bar{X}, \bar{Y}) = \overline{h(X, Y)}, \tag{2.80}$$

where $\hat{h}$ is the second fundamental form of $\tilde{N}$ in S^{2m+1}.

By using (2.75) we find

$$\tilde{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} - \langle JX, Y\rangle\,\xi, \tag{2.81}$$

where $\tilde{\nabla}$ and ∇ are the Levi-Civita connections of $\tilde{N}$ and N, respectively. Also, it follows from (2.76) that

$$\hat{h}(\bar{X}, \xi) = \overline{FX}, \quad \tilde{\nabla}_{\bar{X}}\xi = \tilde{\nabla}_{\xi}\bar{X} = \overline{FX} \tag{2.82}$$

for $X \in TN$. Moreover, since Hopf's fibration has totally geodesic fibers, we get

$$\hat{h}(\xi, \xi) = 0. \tag{2.83}$$

Now, it follows from (2.82), (2.83) and Gauss' equation that

$$K_{\tilde{N}}(\bar{X}, \xi) = 1 - ||FX||^2 \tag{2.84}$$

for each unit tangent vector X of N.

By applying (2.12), (2.15) and (2.84) we find

$$\breve{A}_{\tilde{\pi}} = n - ||F||^2. \tag{2.85}$$

For an orthonormal frame $\{e_1, \ldots, e_n\}$, we find from $\xi = \mathrm{i}z$ and Lemma 11(a) that

$$\begin{aligned} 2\mathcal{A}_{\bar{e}_i}\bar{e}_j = \mathcal{V}[\bar{e}_i, \bar{e}_j] &= \left\langle \breve{\nabla}_{\bar{e}_i}\bar{e}_j - \breve{\nabla}_{\bar{e}_j}\bar{e}_i, \xi \right\rangle \xi \\ &= \left\langle \bar{e}_i, \mathrm{i}\breve{\nabla}_{\bar{e}_j}z \right\rangle \xi - \left\langle \bar{e}_j, \mathrm{i}\breve{\nabla}_{\bar{e}_i}z \right\rangle \xi \\ &= 2\left\langle \bar{e}_i, \mathrm{i}\bar{e}_j \right\rangle \xi = 2\left\langle e_i, Pe_j \right\rangle \xi, \end{aligned} \tag{2.86}$$

where $\breve{\nabla}$ is the Levi-Civita connection of $\mathbf{C}^{m+1}$. Combining (2.15) and (2.86) gives

$$\mathring{A}_{\tilde{\pi}} = \frac{1}{2}||P||^2. \tag{2.87}$$

By applying (2.77), (2.84), (2.86) and (2.87), we obtain

$$\begin{aligned} \tau_{\tilde{N}}(x) - \tau_{\tilde{N}}(\bar{P}_x) = \tau_N - K_N(e_1, e_2) - \frac{3}{2}||P||^2 + 3\langle Pe_1, e_2\rangle^2 \\ + n - 2 - ||F||^2 + \sum_{i=1}^{2} ||Fe_i||^2. \end{aligned} \tag{2.88}$$

Since $||PX||^2 + ||FX||^2 = ||X||^2$, we derive from Lemma 16 and (2.88) that

$$\begin{aligned} \tau_{\tilde{N}}(x) - \tau_{\tilde{N}}(\bar{P}_x) &\le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{n^2+n-6}{2} - ||F||^2 + \sum_{i=1}^{2}||Fe_i||^2 \\ &= \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{n^2-n-6}{2} + ||P||^2 + \sum_{i=1}^{2}||Fe_i||^2 \\ &= \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{n^2-n-2}{2} + ||P||^2 - \sum_{i=1}^{2}||Pe_i||^2 \\ &\le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{n^2-n-2}{2} + ||P||^2, \end{aligned} \tag{2.89}$$

which gives inequality (2.73).

From Lemma 16 and (2.79) we conclude that the equality sign of (2.73) holds identically if and only if there exists an orthonormal frame $e_1, \ldots, e_m$ such that statements (a) and (b) of the theorem hold.

Let N be a CR-submanifold of $CP^m(4)$. The next result from [1] provides the necessary and sufficient condition for the pre-image $\pi^{-1}(N)$ to satisfy the equality case of the inequality (2.73).

Theorem 2.13 ([1]) *Let N be a CR-submanifold of the complex projective m-space $CP^m(4)$. Then, N satisfies the equality case of (2.73) identically if and only if N is a totally real submanifold satisfying*

$$\delta(2) = \frac{n^2(n-2)}{2(n-1)}||H||^2 + \frac{1}{2}(n^2 - n - 2) \tag{2.90}$$

identically.

For the proof of this theorem, see [1].

Finally, we provide many examples of totally real submanifolds of $CP^m(4)$ which satisfy the equality case of inequality (2.73).

Theorem 2.14 *If N is a totally real totally geodesic submanifold of* $CP^m(4)$*, then the equality sign of (2.73) holds identically.*

Proof Let N be an n-dimensional totally real totally geodesic submanifold of $CP^m(4)$ with $n \geq 3$. In view of (2.84), we have

$$\tau_{\tilde{N}} = \frac{n(n-1)}{2}, \quad \tau_{\tilde{N}}(\bar{P}_x) = 1.$$

Thus, $\delta^H = \frac{1}{2}(n^2 - n - 2)$. Hence, we obtain the equality sign of (2.73) identically due to $||H|| = P = 0$.

Theorem 2.15 *There exist many non-totally geodesic totally real submanifolds of* $CP^m(4)$ *which satisfy the equality case of inequality (2.73) identically.*

Proof Let N be an n-dimensional submanifold in the unit m-sphere S^m satisfying

$$\delta(2) = \frac{n^2(n-2)}{2(n-1)}||H||^2 + \frac{1}{2}(n^2 - n - 2). \tag{2.91}$$

Then, N can be isometrically immersed as a totally real submanifold of $CP^m(4)$ satisfying the equality case of (2.73) via the following standard isometric immersion:

$$S^m \xrightarrow[\text{covering}]{\text{2 to 1}} RP^m(1) \xrightarrow[\text{totally real}]{\text{totally geodesic}} CP^m(4). \tag{2.92}$$

Since N in S^m satisfies equality (2.91), the shape operator of N in S^m satisfies statement (a) of Theorem 2.12, the shape operator of N in $CP^m(4)$ satisfies (a) as well. Because N is totally real in $CP^m(4)$, it also satisfies statement (b) of Theorem 2.12. It is known that there exist ample submanifolds in spheres which satisfy equality (2.91) identically. Consequently, there exist many non-totally geodesic, totally real submanifolds of $CP^m(4)$ which satisfy the equality case of inequality (2.73) identically according to Theorem 2.12.

Remark 8 For further results on the *CR*-submanifolds in Kaehler manifolds related to δ-invariants, in particular for *CR*-submanifolds in complex hyperbolic spaces, see [14, 27, 28, 37, 38].

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Chapter 3
CR-Submanifolds of the Nearly Kähler 6-Sphere

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3.1 Introduction

Considering $\mathbb{R}^7$ as the imaginary Cayley numbers, it is possible to introduce a vector cross product $\times$ on $\mathbb{R}^7$, which in its turn induces an almost complex structure J on the standard unit sphere $S^6(1)$ in $\mathbb{R}^7$ which is compatible with the standard metric. It was shown by Calabi and Gluck, see [9], that this structure, from a geometric viewpoint, is the best possible almost complex structure on $S^6(1)$. Details about this construction are recalled in the next section.

With respect to the almost complex structure J, it is natural to study submanifolds for which J maps the tangent space into the tangent space (and hence also the normal space into the normal space) and those for which J maps the tangent into the normal space. The first class are called almost complex submanifolds and the second class of submanifolds mentioned are called totally real submanifolds.

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One of the natural generalization of almost complex and totally real submanifolds are CR-submanifolds and there are two different notions of this therm. By the first one, if the dimension of the holomorphic tangent space, the maximal J-invariant subspace $H_xM = JT_xM \cap T_xM$, $x \in M$ is independent on the choice of $x \in M$ then the submanifold M is called the Cauchy-Riemann submanifold, or briefly CR-submanifold with the CR dimension being the constant complex dimension of H_xM. By the definition of Bejancu, see [4], a submanifold M is called a CR submanifold if there exists on M a differentiable holomorphic distribution $\mathcal{H}$ such that its orthogonal complement $\mathcal{H}^\perp \subset TM$ is a totally real distribution. It is clear that the CR-submanifold by Bejancu's definition it is also CR by the other definition. The converse is true for submanifolds of the maximal CR dimension $\dfrac{m-1}{2}$, where m is the dimension of the submanifold. Note that in this survey, we will focus on CR-submanifolds in the sense Bejancu. A CR submanifold is called proper if it is neither totally real (i.e., $\mathcal{H}^\perp = TM$) nor almost complex (i.e., $\mathcal{H} = TM$).

From the definition, we see that the dimension of a proper CR-submanifold can be either three, four, or five. Note, however, that from the definition it immediately follows that any five-dimensional submanifold of the six-sphere is automatically a CR-submanifold. In view of this, we will restrict ourselves in this survey to the three dimensional and four dimensional case, which are, respectively, treated in Sects. 3.3 and 3.4.

3.2 Preliminaries

We give a brief exposition of how the standard nearly Kähler structure on $S^6(1)$ arises in a natural manner from the Cayley multiplication. For further details about the Cayley numbers and their automorphism group G_2, we refer the reader to [28] and [21].

The multiplication on the Cayley numbers $\mathcal{O}$ may be used to define a vector cross product $\times$ on the purely imaginary Cayley numbers $\mathbb{R}^7$ using the formula

$$u \times v = \frac{1}{2}(uv - vu), \tag{3.1}$$

while the standard inner product on $\mathbb{R}^7$ is given by

$$\langle u, v\rangle = -\frac{1}{2}(uv + vu). \tag{3.2}$$

It is now elementary [21] to show that

$$u \times (v \times w) + (u \times v) \times w = 2\langle u, w\rangle v - \langle u, v\rangle w - \langle w, v\rangle u, \tag{3.3}$$

and that the triple scalar product $\langle u \times v, w\rangle$ is skew symmetric in u, v, w. From this it also follows that

$$\langle u \times v, u \times w\rangle = \langle u, u\rangle\langle v, w\rangle - \langle u, v\rangle\langle u, w\rangle. \tag{3.4}$$

The Cayley multiplication on $\mathcal{O}$ is given in terms of the vector cross product and the inner product by

$$(r + u)(s + v) = rs - \langle u, v\rangle + rv + su + u \times v, \quad r, s \in Re(\mathcal{O}), u, v \in Im(\mathcal{O}). \tag{3.5}$$

In view of (3.1), (3.2) and (3.5), it is clear that the group G_2 of automorphisms of $\mathcal{O}$ is precisely the group of isometries of $\mathbb{R}^7$ preserving the vector cross product.

An ordered basis e_0, ..., e_6 is said to be a G_2-frame if

$$e_2 = e_0 \times e_1, \quad e_4 = e_0 \times e_3, \quad e_5 = e_1 \times e_3, \quad e_6 = e_2 \times e_3. \tag{3.6}$$

For example, the standard basis $e_0, \ldots, e_6$ of $\mathbb{R}^7$ is a G_2-frame. Two G_2-frames are related by a unique element of G_2. Moreover, if e_0, e_1, e_3 are mutually orthogonal unit vectors with e_3 orthogonal to $e_0 \times e_1$, then e_0, e_1, e_3 determine a unique G_2-frame $e_0, \ldots, e_6$ and $(\mathbb{R}^7, \times)$ is generated by e_0, e_1, e_3 subject to the relations:

$$e_i \times (e_j \times e_k) + (e_i \times e_j) \times e_k = 2\delta_{ik}e_j - \delta_{ij}e_k - \delta_{jk}e_i. \tag{3.7}$$

Therefore, for any G_2-frame, we have the following very useful multiplication table [28]:

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6
e_0	0	e_2	$-e_1$	e_4	$-e_3$	$-e_6$	e_5
e_1	$-e_2$	0	e_0	e_5	e_6	$-e_3$	$-e_4$
e_2	e_1	$-e_0$	0	e_6	$-e_5$	e_4	$-e_3$
e_3	$-e_4$	$-e_5$	$-e_6$	0	e_0	e_1	e_2
e_4	e_3	$-e_6$	e_5	$-e_0$	0	$-e_2$	e_1
e_5	e_6	e_3	$-e_4$	$-e_1$	e_2	0	$-e_0$
e_6	$-e_5$	e_4	e_3	$-e_2$	$-e_1$	e_0	0

The standard nearly Kähler structure on $S^6(1)$ is then obtained as follows:

$$Ju = x \times u, \quad u \in T_xS^6(1), \quad x \in S^6(1).$$

It is clear that J is an orthogonal almost complex structure on $S^6(1)$. In fact, J is a nearly Kähler structure in the sense that the $(2, 1)$-tensor field G on $S^6(1)$ defined by

$$G(X, Y) = (\tilde{\nabla}_X J)Y,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $S^6(1)$ is skew symmetric. A straightforward computation also shows that

$$G(X, Y) = X \times Y - \langle x \times X, Y \rangle x, \quad X, Y \in T_x S^6(1).$$

Let M be a Riemannian submanifold of $\widetilde{M}$. If we denote by $\langle i, i \rangle$, $\overline{D}$ and $\widetilde{D}$ metric and Levi-Civita connections on M and $\widetilde{M}$, respectively, and by $D^\perp$ the corresponding normal connection of the immersion $M \to \widetilde{M}$ then the formulas of Gauss and Weingarten are given by

$$\widetilde{D}_X Y = \overline{D}_X Y + h(X, Y), \tag{3.8}$$

$$\widetilde{D}_X \xi = -A_\xi X + D_X^\perp \xi, \tag{3.9}$$

where X and Y are vector fields on M and ξ is a normal vector field on M, and h and A are the second fundamental form and the shape operator, respectively. The second fundamental form and the shape operator are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle. \tag{3.10}$$

Let us denote by ∇, $\widetilde{\nabla}$ and D the Levi-Civita connections on M, $S^6(1)$ and $\mathbb{R}^7$, respectively. Let h and $\widetilde{h}$ be the second fundamental forms corresponding to the immersions $M \to S^6(1)$ and $S^6(1) \to \mathbb{R}^7$, respectively. Let p be the position vector field of the immersion of M into $\mathbb{R}^7$. Then, the following equations hold

$$\widetilde{h}(X, Y) = -\langle X, Y \rangle p, \tag{3.11}$$

$$D_X p = X, \tag{3.12}$$

where $X, Y \in TM$. Considering (3.8), (3.9) and (3.11) we get for $X, Y \in TM$ and $\xi \in T^\perp M, \xi \in TS^6(1)$

$$D_X Y = \widetilde{\nabla}_X Y + \widetilde{h}(X, Y) = \nabla_X Y + h(X, Y) - \langle X, Y \rangle p, \tag{3.13}$$

$$D_X \xi = \widetilde{\nabla}_X \xi + \widetilde{h}(X, \xi) = \widetilde{\nabla}_X \xi - \langle X, \xi \rangle p = -A_\xi X + \nabla_X^\perp \xi, \tag{3.14}$$

where $\nabla^\perp$ denotes the normal connection corresponding to the immersion of M into $S^6(1)$. Also, we can denote

$$(\nabla h)(X, Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{3.15}$$

for $X, Y, Z \in T(M)$.

Then Gauss, Codazzi, and Ricci equations state that

$$R(X, Y, Z, W) = \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle + \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \tag{3.16}$$

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z), \tag{3.17}$$

$$\langle R^{\perp}(X, Y)\xi, \mu\rangle = \langle [A_\xi, A_\mu]X, Y\rangle. \tag{3.18}$$

Also, the following lemma holds

Lemma 1 $D_X(Y \times Z) = D_X Y \times Z + Y \times D_X Z.$

3.3 Three Dimensional CR-Submanifolds of $S^6(1)$

3.3.1 An Existence and Uniqueness Theorem

In this section, we consider M to be a three-dimensional orientable CR submanifold of the sphere $S^6(1)$. Then, there exist the following local orthonormal vector fields: the position vector field p, E_1 and $E_2 = JE_1$ which span the almost complex distribution $\mathcal{H}$, E_3 which spans the totally real distribution $\mathcal{H}^{\perp}$, and the normal vector fields are obtained by $E_4 = JE_3$, $E_5 = E_1 \times E_3$ and $E_6 = E_2 \times E_3$.

Note that by assuming that E_1, E_2 and E_3 are positively oriented, we have that the choice of E_3 is unique. Nevertheless, we still have the following freedom of rotation of E_1 in the holomorphic distribution, with corresponding rotation in the normal bundle:

$$\begin{aligned} \tilde{E}_1 &= \cos\theta E_1 + \sin\theta E_2, & \tilde{E}_2 &= J\tilde{E}_1 = -\sin\theta E_1 + \cos\theta E_2,\\ \tilde{E}_3 &= E_3, & \tilde{E}_4 &= E_4,\\ \tilde{E}_5 &= \cos\theta E_5 + \sin\theta E_6, & \tilde{E}_6 &= -\sin\theta E_5 + \cos\theta E_6. \end{aligned}$$

As M is a CR-submanifold we already have that $TM = \mathcal{H} \oplus \mathcal{H}^{\perp}$, where $\mathcal{H}$ and $\mathcal{H}^{\perp}$ are respectively the almost complex and the totally real distribution. It is immediately clear that the restriction of the nearly Kähler structure to the tangent space automatically induces an almost contact structure (φ, ξ, η) on the three dimensional CR-submanifold where φ is a $(1, 1)$ tensor field, ξ is a vector field and η a 1-form in the following way:

$$\xi = E_3, \quad \eta(X) = \langle X, \xi\rangle, \quad \varphi(xE_1 + yE_2 + zE_3) = -yE_1 + xE_2.$$

Note that the above structure is independent of the choice of E_1 and E_2 and satisfies

$$\langle \varphi(X), \varphi(Y) \rangle = \langle X, Y \rangle - \eta(X)\eta(Y),$$

showing that the induced metric is compatible with the almost contact structure. This makes M an almost contact metric manifold. For more details about almost contact structures we refer to [6].

Moreover, from the above formulas it also follows that the map $\Lambda : TM \to NM$ given by

$$\Lambda(xE_1 + yE_2 + zE_3) = xE_5 + yE_6 + zE_4,$$

is well defined and describes a natural identification between the normal and the tangent bundle. This identification can be expressed without the use of a frame by the condition that

$$\Lambda(X) = G(X, E_3) + \eta(X)JE_3 = G(X, \xi) + \eta(X)J\xi,$$

for any tangent vector X. It this identification maps an orthonormal frame to an orthonormal frame it is an isometry. Note that also the normal bundle can be splitted in a two-dimensional part which is J invariant and a one-dimensional part which by J is mapped to the tangent space.

We will now first show an existence and uniqueness theorem for CR-submanifolds, using the notion of the almost contact metric manifold. We already have seen that a CR-submanifold M admits a natural almost contact metric structure (φ, ξ, η) described earlier.

Denote by

$$\omega_{ij}^k = \langle D_{E_i}E_j, E_k \rangle, \quad h_{ij}^k = \langle D_{E_i}E_j, E_{k+3} \rangle, \quad \beta_{ij}^k = \langle D_{E_i}E_{j+3}, E_{k+3} \rangle,$$

for $1 \leq i, j, k \leq 3$. Using the standard symmetries for a connection and for the second fundamental form, we find that

$$\omega_{ij}^k = -\omega_{ik}^j, \quad h_{ij}^k = h_{ik}^j, \quad \beta_{ij}^k = -\beta_{ik}^j.$$

Taking $X \in \{E_1, E_2, E_3\}$ and $Y, Z \in \{p, E_1, \ldots, E_6\}$ in Lemma 1 we get

Lemma 2 *For the previously defined coefficient the following equations hold*

$$\begin{aligned}
&\beta_{11}^3 = -h_{13}^2, \quad \beta_{11}^2 = 1 + h_{13}^3, \quad h_{11}^1 = -\omega_{12}^3, \quad h_{12}^1 = \omega_{11}^3, \quad \beta_{21}^3 = 1 - h_{23}^2, \\
&\beta_{21}^2 = h_{23}^3, \quad h_{22}^1 = \omega_{21}^3, \quad \omega_{22}^3 = -\omega_{11}^3, \quad \beta_{31}^2 = h_{33}^3, \quad \beta_{31}^3 = -h_{33}^2, \\
&h_{23}^1 = \omega_{31}^3, \quad h_{13}^1 = -\omega_{32}^3, \quad h_{11}^3 = h_{12}^2, \quad h_{11}^2 = -h_{12}^3, \quad h_{22}^2 = h_{12}^3, \\
&h_{22}^3 = -h_{12}^2, \quad h_{23}^2 = h_{13}^3 - 1, \quad h_{23}^3 = -h_{13}^2, \quad \beta_{12}^3 = \omega_{11}^2 - \omega_{32}^3, \\
&\beta_{22}^3 = \omega_{21}^2 + \omega_{31}^3, \quad \beta_{32}^3 = \omega_{31}^2 + h_{33}^1.
\end{aligned}$$

We now define $\sigma(X, Y)$ such that for any $X, Y \in \mathcal{H}$ we have that

$$\Lambda(\sigma(X, Y)) = \langle h(X, Y), E_5\rangle E_5 + \langle h(X, Y), E_6\rangle E_6,$$
$$\Lambda(\sigma(X, \xi)) = \Lambda(\sigma(\xi, \xi)) = 0.$$

In components, writing $\sigma_{ij}^k = \langle \sigma(E_i, E_j), E_k\rangle$, this means that the only possible non vanishing components are

$$\sigma_{ij}^1 = h_{ij}^2, \quad \sigma_{ij}^2 = h_{ij}^3.$$

where $1 \leq i, j \leq 2$.

It now follows that the symmetric bilinear form σ, satisfies:

$$\sigma(\varphi(X), Y) = -\varphi\sigma(X, Y), \qquad X, Y \in \varphi(TM), \tag{3.19}$$
$$\sigma(X, \xi) = 0, \qquad X \in \varphi(TM) \tag{3.20}$$

and by denoting $\sigma_{11} = \sigma_{11}^1, \sigma_{12} = \sigma_{12}^1$ we can write

$$\sigma(E_1, E_1) = \sigma_{11}E_1 + \sigma_{12}E_2,$$
$$\sigma(E_1, E_2) = \sigma_{12}E_1 - \sigma_{11}E_2,$$
$$\sigma(E_2, E_2) = -\sigma_{11}E_1 - \sigma_{12}E_2.$$

In the same way, we define a 1-1 tensor field S, by

$$\Lambda(SX) = \langle h(X, \xi), E_5\rangle E_5 + \langle h(X, \xi), E_6\rangle E_6,$$
$$\Lambda(S\xi) = 0.$$

From the previous lemma, we get that

$$S\varphi(X) + \varphi(SX) + X = 0, \qquad X \in \mathcal{H}, \tag{3.21}$$
$$S\xi = 0, \tag{3.22}$$

so we can write

$$SE_1 = s_1E_5 + s_2E_6, \quad SE_2 = (s_2 - 1)E_1 - s_1E_6,$$

for some functions s_1 and s_2. We notice also that the previous lemma implies that

$$\langle h(X, Y), E_4\rangle = -\langle \nabla_X(\varphi(Y)), \xi\rangle,$$

for $X, Y \in \mathcal{H}$. Therefore, for $X, Y \in \mathcal{H}$ we have that

$$h(X, Y) = \Lambda(\sigma(X, Y) - \langle \nabla_X(\varphi(Y)), \xi\rangle\xi). \tag{3.23}$$

Similarly, for $X \in \mathcal{H}$, we get that

$$h(X,\xi) = \Lambda(SX + \langle \nabla_\xi \xi, \varphi(X)\rangle \xi). \tag{3.24}$$

So, if we define a vector field W such that

$$\Lambda(W) = h(\xi,\xi),$$

we see that we can write the second fundamental form as

$$\begin{aligned} h(X,Y) = &\Lambda(\sigma(X,Y) + \eta(Y)SX + \eta(X)SY) \\ &+ \Lambda(\eta(Y)\langle \nabla_\xi \xi, \varphi(X)\rangle \xi - \langle \nabla_X(\varphi(Y)), \xi\rangle \xi) + \eta(X)\eta(Y)W). \end{aligned}$$

Denote $w_i = \langle W, E_i\rangle$, $i = 1, 2$.
We put:

$$\begin{aligned} \alpha(X,Y) = &(\sigma(X,Y) + \eta(Y)SX + \eta(X)SY + \eta(Y)) \\ &+ \langle \nabla_\xi \xi, \varphi(X)\rangle \xi - \langle \nabla_X(\varphi(Y)), \xi\rangle \xi) + \eta(X)\eta(Y)W. \end{aligned}$$

Similarly, we can again use the correspondence between the tangent bundle and the normal bundle to define a metric connection $\tilde{\nabla}^\perp$ with torsion on M by the relation

$$\Lambda(\tilde{\nabla}^\perp_X(Y)) = \nabla^\perp_X \Lambda(Y). \tag{3.25}$$

This gives for vector fields X, Y, and Z orthogonal to ξ that

$$\tilde{\nabla}^\perp_Y \xi = -\varphi(SY) + Y, \tag{3.26}$$

$$\tilde{\nabla}^\perp_\xi \xi = -\varphi(W), \tag{3.27}$$

$$\langle \tilde{\nabla}^\perp_X Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle \nabla_\xi \xi, \varphi(X)\rangle \langle \varphi(Y), Z\rangle, \tag{3.28}$$

$$\langle \tilde{\nabla}^\perp_\xi Y, Z\rangle = \langle \nabla_\xi Y, Z\rangle + \langle W, \xi\rangle \langle \varphi(Y), Z\rangle. \tag{3.29}$$

Conversely, we can now formulate an existence and uniqueness theorem for CR submanifolds of the nearly Kähler 6-sphere.

Theorem 3.1 *Let M be a simply connected three dimensional, oriented Riemannian manifold with almost contact metric structure* (φ, ξ, η)*. Let $\sigma : TM \times TM \to TM$, $S : TM \to TM$ and W be respectively a symmetric bilinear form, a 1-1 tensor field and a vector field which satisfy* (3.19) *upto* (3.22)*. Define*

$$\begin{aligned} \alpha(X,Y) = &(\sigma(X,Y) + \eta(Y)SX + \eta(X)SY + \eta(Y)) \\ &+ \langle \nabla_\xi \xi, \varphi(X)\rangle \xi - \langle \nabla_X(\varphi(Y)), \xi\rangle \xi) + \eta(X)\eta(Y)W \end{aligned}$$

and a metric connection with torsion on M by (3.26) upto (3.29). Suppose that

$$(\tilde{\nabla}\alpha)(Z, X, Y) = \tilde{\nabla}^{\perp}_{Z}\alpha(X, Y) - \alpha(\nabla_Z X, Y) - \alpha(X, \nabla_Z Y),$$

is symmetric in X, Y and Z and that

$$\begin{aligned}\langle R(X, Y)Z, W\rangle = \langle Y, Z\rangle\langle X, W\rangle - \langle Y, Z\rangle\langle X, W\rangle \\ + \langle\alpha(Y, Z), \alpha(X, W)\rangle - \langle\alpha(X, Z), \alpha(Y, W)\rangle.\end{aligned}$$

Then, there exist a CR immersion of M into $S^6(1)$ with (φ, ξ, η) as induced metric contact structure and $\Lambda(\alpha(X, Y))$ as second fundamental form.

Proof We take $NM = TM$. We define a bundle E over M such that the fibre over the point p satisfies $E_p = \mathbb{R} \oplus TM_p \oplus NM_p$. We take a local unit length tangent vector field V, orthogonal to ξ and define a local frame along M by

$$\begin{aligned}&E_0 = (1, 0, 0), \qquad E_1 = (0, V, 0), \quad E_2 = (0, \varphi(V), 0),\\ &E_3 = (0, \xi, 0), \qquad E_4 = (0, 0, \xi), \quad E_5 = (0, 0, V),\\ &E_6 = (0, 0, \varphi(V)).\end{aligned}$$

Let Λ be an isomorphism of bundles TM and NM defined by $\Lambda(E_3) = E_4$, $\Lambda(E_1) = E_5$, $\Lambda(E_2) = E_6$. If we denote by ∇ the Levi-Civita connection of the manifold M, we define the second fundamental form on NM by

$$h(X, Y) = \Lambda(\alpha(X, Y)),$$

and the normal connection on NM by

$$\nabla^{\perp}_{X}\Lambda(Y) = \Lambda(\tilde{\nabla}^{\perp}_{X}(Y)), \tag{3.30}$$

where $\tilde{\nabla}^{\perp}$ is defined by (3.26) upto (3.29).

Note that, by definition, the second fundamental form and the normal connection satisfy the equations of Gauss and Codazzi. Whereas, a straightforward calculation shows that the Ricci equations are a consequence of the Gauss and Codazzi equations. Hence, by the standard existence and uniqueness theorem for submanifolds in space forms we know that there exist an immersion $F : M \to S^6(1) \subset \mathbb{R}^7$. We now define a vector cross product along M by using the previously constructed multiplication table. It is then straightforward to check that for $X \in \{E_1, E_2, E_3\}$ and $Y, Z \in \{E_0, E_1, E_2, E_3, E_4, E_5, E_6\}$ we have that

$$D_X(Y \times Z) = (D_X Y) \times Z + Y \times (D_X Z).$$

Hence, the product $\times$ is parallel along M, implying that we get a CR immersion. By construction, we have (φ, ξ, η) as the induced metric contact structure and $\Lambda(\alpha(X, Y))$ as the second fundamental form.

Theorem 3.2 *Let M be a metric almost contact manifold and let $f_1 : M \to S^6(1)$ and $f_2 : M \to S^6(1)$ be two isometric CR immersions of M into $S^6(1)$ which induce the given metric almost contact structure. Suppose moreover, that for both immersions the previously introduced invariants σ, S and W coincide. Then both immersions are G_2 congruent.*

Proof It follows immediately from Lemma 2 that there exist a normal bundle isomorphism such that for both immersions the second fundamental form and the normal connections coincide. Hence, see Spivak [27], both immersions are congruent by an element $A \in SO(7)$. Moreover, as A maps the G_2-frame of the first immersion to the G_2-frame of the second immersion, it follows that A actually preserves the vector product. Hence, $A \in G_2$.

Note that if we assume that the immersion is minimal, it follows from Lemma 2 that the vector field W can be determined by the induced connection. Indeed we have that

Lemma 3 *Let M be a proper minimal orientable three dimensional CR-submanifold of $S^6(1)$. Then,*

$$W = (\omega_{12}^3 - \omega_{21}^3)E_3.$$

Proof As M is minimal we have that

$$h(E_1, E_1) + h(E_2, E_2) + h(E_3, E_3) = 0.$$

Using (3.23) this implies that

$$\begin{aligned}
&\langle \sigma(E_1, E_1) + \sigma(E_2, E_2), E_1\rangle + \langle W, E_1\rangle = 0,\\
&\langle \sigma(E_1, E_1) + \sigma(E_2, E_2), E_2\rangle + \langle W, E_2\rangle = 0,\\
&-\langle \nabla_{E_1}\varphi(E_1), \xi\rangle - \langle \nabla_{E_2}\varphi(E_2), \xi\rangle + \langle W, \xi\rangle = 0.
\end{aligned}$$

Note that the first two equations imply that $W \in \mathcal{H}^\perp$, whereas the last one implies that

$$w_3 = \langle W, \xi\rangle = (\omega_{12}^3 - \omega_{21}^3).$$

3.3.2 Properties of Distributions and Elementary Examples

Before introducing some more properties of CR-submanifolds, we recall the following definitions.

Definition 1 Let M be a proper orientable CR-submanifold. Then,

(1) M is called $\mathcal{H}$ geodesic if and only if $h(X, Y) = 0$ for all $X, Y \in \mathcal{H}$,
(2) M is called $\mathcal{H}^\perp$ geodesic if and only if $h(\xi, \xi) = 0$,
(3) M is called mixed totally geodesic if and only if $h(X, \xi) = 0$ for all $X \in \mathcal{H}$.

Note that a totally geodesic submanifold satisfies all 3 conditions in the previous definition. Whereas, a totally umbilical submanifold, defined by

$$\langle h(X, Y), N\rangle = \lambda_N \langle X, Y\rangle,$$

for an arbitrary normal vector field N and tangent vector fields X and Y is automatically mixed geodesic.

Notice also that from the way the second fundamental form is decomposed, we have the following lemmas.

Lemma 4 *Let M be a proper three dimensional orientable CR-submanifold of $S^6(1)$. Then, M is $\mathcal{H}$ totally geodesic if and only if*

(1) the symmetric bilinear form σ vanished identically,
(2) the distribution $\mathcal{H}$ is totally geodesic.

Lemma 5 *Let M be a proper three dimensional orientable CR-submanifold of $S^6(1)$. Then M is $\mathcal{H}^\perp$ totally geodesic if and only if the vector field W vanished.*

Lemma 6 *There does not exist a proper three dimensional orientable CR-submanifold of $S^6(1)$ which is mixed totally geodesic.*

Proof From (3.24) we see that M is mixed totally geodesic if and only if S vanished identically and the integral curves of ξ are geodesics.

However from (3.21) we deduce that the vanishing of S leads to a contradiction.

As corollaries we have

Theorem 3.3 *There does not exist a proper three dimensional orientable CR-submanifold of $S^6(1)$ which is totally geodesic.*

and

Theorem 3.4 *There does not exist a proper three dimensional orientable CR-submanifold of $S^6(1)$ which is totally umbilical.*

As there are no totally geodesic or totally umbilical examples of three dimensional proper CR-submanifolds it was a nontrivial question to obtain examples of CR-submanifolds. The first such example was discovered in [26]. The example constructed was both $\mathcal{H}$ totally geodesic and $\mathcal{H}^\perp$ totally geodesic. It was later generalised in [22] where they constructed a whole family of such examples. They are immersions of $S^2 \times \mathbb{R}$ and are given by

$$F_\lambda((y_1, y_2, y_3), s) = y_1(\cos se_0 + \sin se_4) + y_2(\cos(\lambda s)e_1 + \sin(\lambda s)e_5) + y_3(\cos((1+\lambda)s)e_2 - \sin((1+\lambda)s)e_6),$$

where $y_1^2 + y_2^2 + y_3^2 = 1$ and $\{e_0, \ldots, e_6\}$ is a G_2-frame. Note that in [22] these examples were only defined for $\lambda \neq 0$ and $\lambda \neq -1$. However, it is easy to check that also for $\lambda \in \{0, -1\}$, the resulting immersion is a CR immersion with the same properties.

Note that the parameter λ can also be represented projectively by the relation

$$(1 : \lambda : -1 - \lambda) = (\mu_1 : \mu_2 : \mu_3),$$

with $\mu_1 + \mu_2 + \mu_3 = 0$. Rescaling the constants μ_i simply corresponds to a rescaling of the parameter s.

Note, moreover, that all of these examples satisfy

(1) the immersion is minimal
(2) the immersion is contained in a totally geodesic hypersphere
(3) the immersion is $\mathcal{H}$ totally geodesic
(4) the immersion is $\mathcal{H}^\perp$ totally geodesic.

Conversely from [3], we recall that

Theorem 3.5 *Let M be a minimal proper three dimensional CR-submanifold of $S^6(1)$ which is not linearly full in $S^6(1)$. Then, M is locally congruent to the immersion*

$$\begin{aligned} F(s, x_1, x_2) = &\cos x_1 \ \cos x_2 \ [\cos(\mu_1 s)e_0 + \sin(\mu_1 s)e_4] \\ &+ \sin x_1 \ \cos x_2 \ [\cos(\mu_2 s)e_1 + \sin(\mu_2 s)e_5] \\ &+ \sin x_2 \ [\cos(\mu_3 s)e_2 + \sin(\mu_3 s)e_6], \\ &\mu_1 + \mu_2 + \mu_3 = 0, \quad \mu_1^2 + \mu_2^2 + \mu_3^2 \neq 0, \end{aligned}$$

where $e_0, \ldots, e_6$ is a standard G_2-basis of the space $\mathbb{R}^7$.

and

Theorem 3.6 *Let M be a proper three dimensional CR-submanifold of $S^6(1)$. Assume that it is both $\mathcal{H}$ and $\mathcal{H}^\perp$ totally geodesic. Then, M is locally congruent to the immersion*

$$\begin{aligned} F(s, x_1, x_2) = &\cos x_1 \ \cos x_2 \ [\cos(\mu_1 s)e_0 + \sin(\mu_1 s)e_4] \\ &+ \sin x_1 \ \cos x_2 \ [\cos(\mu_2 s)e_1 + \sin(\mu_2 s)e_5] \\ &+ \sin x_2 \ [\cos(\mu_3 s)e_2 + \sin(\mu_3 s)e_6], \\ &\mu_1 + \mu_2 + \mu_3 = 0, \quad \mu_1^2 + \mu_2^2 + \mu_3^2 \neq 0, \end{aligned}$$

where $e_0, \ldots, e_6$ is a standard G_2-basis of the space $\mathbb{R}^7$.

Note that for a special value of $\lambda = -1$, or equivalently $\mu_3 = -1$, this example already appeared in [14] were it was characterised using the δ-invariants of B.Y. Chen. Chen's inequality for submanifolds of real space forms relates the main intrinsic invariants of the submanifold M^n of a real space form $\bar{M}^m(c)$, being its *sectional curvature* function K, its *scalar curvature* function τ, and its main extrinsic invariant: the *mean curvature* function $\|H\|$ (H being the mean curvature vector field of M in $\widetilde{M}$). For doing so, it is convenient to define a Riemannian invariant δ_M of M^n by

$$\delta_M(2) = \tau_p - \inf K_p,$$

where $\inf K$ is the function assigning to each $p \in M^n$ the infimum of $K(\pi)$, where π runs over all planes in T_pM and τ is defined by $\tau = \sum_{i<j} K(e_i \wedge e_j)$. The inequality can be written as follows.

$$\delta_M(2) \leq \tfrac{n^2(n-2)}{2(n-1)}\|H\|^2 + \tfrac{1}{2}(n+1)(n-2)c.$$

Note that later, see [12], B.Y. Chen introduced a whole series of such invariants. Note, however, that for a three dimensional manifold $\delta(2)$ is the only nontrivial invariant. Then, we have

Theorem 3.7 ([14]) *Let M be a three dimensional minimal CR-submanifold in $S^6(1)$ realising the equality in Chen's equality. Then, M is a totally real submanifold or locally M is congruent with the immersion*

$$\begin{aligned} f(t,u,v) = (&\cos t\ \cos u\ \cos v,\quad \sin t,\quad \cos t\ \sin u\ \cos v,\\ &\cos t\ \cos u\ \sin v,\quad 0,\quad -\cos t\ \sin u\ \sin v,\quad 0). \end{aligned} \tag{3.31}$$

Later in [16] it was shown that

Theorem 3.8 ([16]) *Let M be a three dimensional CR- submanifold in $S^6(1)$ realising the equality in Chen's equality. Then M is minimal and therefore congruent to the example obtained in the previous theorem.*

Note that as mentioned in a previous theorem, there does not exist a proper three dimensional totally geodesic CR-submanifold. In the previous examples and theorems, we have introduced some notions of $\mathcal{H}$ totally geodesic and $\mathcal{H}^\perp$ geodesic CR-submanifolds. These conditions characterised the class of examples introduced by Hashimoto and Mashimo.

In order to obtain yet another class of submanifolds, which can be considered to be close to totally geodesic submanifolds, we can use the nullity distribution. This distribution is defined by

$$\mathcal{D} = \{X | \forall Y \in TM : h(X,Y) = 0\}.$$

Note that for a totally geodesic submanifold the nullity distribution coincides with TM.

Proper CR-submanifolds with a nullity distribution of dimension at least one were investigated in [15]. In that paper such submanifolds are separated into two types, depending whether they are $\mathcal{H}$ geodesic (Type 1) or not (Type 2).

As far as nullity type distributions of Type 1 are concerned, the following results were obtained:

Theorem 3.9 *Let γ be a spherical curve parameterized by arc length in $S^6(1)$ such that at every point $\gamma''(v)$, $\gamma(v)$, $\gamma'(v)$ and $\gamma(v) \times \gamma'(v)$ are linearly independent. Then the map*

$$\mathcal{F}(t, u, v) = -\sin t\ \ \gamma(v) - \cos t \sin u\ \ \gamma(v) \times \gamma'(v) - \cos t \cos u\ \ \gamma'(v)$$

defines a proper three-dimensional CR submanifold M which admits a one-dimensional nullity distribution of Type 1.

and

Theorem 3.10 *Let α be a spherical curve parameterized by arc length which lies in a totally geodesic $S^5(1)$ in $S^6(1)$. Denote by γ a unit normal of $S^5(1)$ in $S^6(1)$. Then the map*

$$\mathcal{F}(t, u, v) = -\sin t\ \ \gamma + \cos t \sin u\ \ \alpha(v) - \cos t \cos u\ \ \gamma \times \alpha(v)$$

defines a proper three dimensional CR-submanifold which admits an one-dimensional nullity distribution of Type 1.

As the examples of Hashimoto and Mashimo, the example

$$\mathcal{F}(t, u, v) = -\sin t\ \ \gamma(v) - \cos t \sin u\ \ \gamma(v) \times \gamma'(v) - \cos t \cos u\ \ \gamma'(v)$$

can also be seen as an immersion of $S^2 \times \mathbb{R}$ into $S^6(1)$ by putting

$$(y_1, y_2, y_3) = (-\sin t, -\cos t \sin u, -\cos t \cos u).$$

By a straightforward computation, we get that

$$\begin{aligned}
\mathcal{F}_t &= -\cos t\ \ \gamma + \sin t \sin u\ \ \gamma \times \gamma' + \sin t \cos u\ \ \gamma' \\
\mathcal{F}_u &= -\cos t \cos u\ \ \gamma \times \gamma' + \cos t \sin u\ \ \gamma' \\
\mathcal{F}_v &= -\sin t\ \ \gamma' - \cos t \sin u\ \ \gamma \times \gamma'' - \cos t \cos u\ \ \gamma''.
\end{aligned}$$

As γ'' has a component which is orthogonal to γ, γ' and $\gamma \times \gamma'$, it is clear that $\mathcal{F}$ defines an immersion for almost every value of u and t. As

$$\mathcal{F} \times \mathcal{F}_t = -\sin u\ \ \gamma' + \cos u\ \ \gamma \times \gamma' = -\tfrac{1}{\cos t}\mathcal{F}_u,$$

it follows that M is a proper CR submanifold. Moreover, we get that

$$\mathcal{F}_{tt} = -\mathcal{F}$$
$$\mathcal{F}_{tu} = -\tan t\mathcal{F}_u.$$

As far as the invariants are concerned these last two equations imply that $\mathcal{H}$ is totally geodesic. This means that the symmetric bilinear form σ vanishes identically and the almost contact structure satisfies $\nabla_X \xi = 0$, for $X \in \mathcal{H}$. We also have that

$$\mathcal{F}_{tv} = -\cos t\ \gamma' + \sin t \sin u\ \gamma \times \gamma'' + \sin t \cos u\ \gamma'',$$
$$\mathcal{F}_{uv} = -\cos t \cos u\ \gamma \times \gamma'' + \cos t \sin u\ \gamma''.$$

Therefore, we get that

$$\mathcal{F}_{tv} = -\tan t\mathcal{F}_v - \tfrac{1}{\cos t}\gamma'.$$

However, from the equations for $\mathcal{F}$, $\mathcal{F}_u$ and $\mathcal{F}_t$, we see that we can write γ' as a linear combination of $\mathcal{F}$, $\mathcal{F}_u$ and $\mathcal{F}_t$. Hence, $\mathcal{F}_{tv}$ does not have a component which is normal to M but tangent to the sphere.

The geometric characteristics of the second class of examples of Type 1 can be deduced in the following way. We write

$$\mathcal{F}(t, u, v) = -\sin t\ \gamma + \cos t \sin u\ \alpha(v) - \cos t \cos u\ \gamma \times \alpha(v).$$

By a straightforward computation, we get that

$$\mathcal{F}_t = -\cos t\ \gamma - \sin t \sin u\ \alpha + \sin t \cos u\ \gamma \times \alpha$$
$$\mathcal{F}_u = \cos t \cos u\ \alpha + \cos t \sin u\ \gamma \times \alpha$$
$$\mathcal{F}_v = \cos t \sin u\ \alpha' - \cos t \cos u\ \gamma \times \alpha'.$$

As α' has a component which is orthogonal to $\gamma \times \alpha$, it is clear that $\mathcal{F}$ defines an immersion for almost every value of u and t. As

$$\mathcal{F} \times \mathcal{F}_t = \cos u\ \alpha + \sin u\ \gamma \times \alpha = \tfrac{1}{\cos t}\mathcal{F}_u,$$

it follows that M is a proper CR submanifold. Moreover, we get that

$$\mathcal{F}_{tt} = -\mathcal{F}$$
$$\mathcal{F}_{tu} = -\tan t\mathcal{F}_u.$$

This implies again that the symmetric bilinear form σ vanishes identically and the almost contact structure satisfies $\nabla_X \xi = 0$, for $X \in \mathcal{H}$. Finally, we have that

$$\mathcal{F}_{tv} = -\sin t \sin u \ \alpha' + \sin t \cos u \ \gamma \times \alpha' = -\tan t \mathcal{F}_v.$$

As far as the Type 2 ones are concerned, they are described by

Theorem 3.11 *Let M be a proper CR-submanifold with an one dimensional nullity distribution of Type 2. Then, there exist spherical curves γ and α and a constant H such that M can be written as*

$$\begin{aligned} \mathcal{F} = &-\sin t \ \left[\gamma(v) \times \gamma'(v)\right] \\ &- \cos t \ \Big[\alpha(v) \sin u - (\tfrac{2}{\sqrt{4+H^2}} \alpha(v) \times \gamma'(v) \\ &+ \tfrac{H}{\sqrt{4+H^2}} \alpha(v) \times (\gamma(v) \times \gamma'(v))) \cos u\Big] \end{aligned} \tag{3.32}$$

so that the frame γ, γ', $\gamma \times \gamma'$, α, $\gamma \times \alpha$, $\gamma' \times \alpha$ and $-\gamma \times (\gamma' \times \alpha) = -\alpha \times (\gamma \times \gamma')$ describes a G_2-frame and such that the curves α and γ satisfy

$$\begin{aligned} \gamma'' &= -\gamma + k_1 \gamma \times \alpha + k_2 \left(\tfrac{H}{\sqrt{4+H^2}} \gamma' \times \alpha + \tfrac{2}{\sqrt{4+H^2}} \alpha \times (\gamma \times \gamma')\right) \\ \alpha' &= k_1 \gamma \times \gamma' + \tfrac{1}{\sqrt{4+H^2}} \left(\tfrac{H}{\sqrt{4+H^2}} \gamma' \times \alpha + \tfrac{2}{\sqrt{4+H^2}} \alpha \times (\gamma \times \gamma')\right). \end{aligned}$$

Conversely, a surface as described in (3.32), with α and γ satisfying the previously mentioned conditions is a proper CR-submanifold of Type 2.

3.3.3 Frame Equations

If we now use the previously constructed G_2-frame, which is determined upto a rotation, and write the Gauss and Codazzi equations in terms of the previously defined components, we obtain the following Gauss and Codazzi equations:

$$\begin{aligned} -E_1(\omega_{21}^2) + E_2(\omega_{11}^2) - {\omega_{11}^2}^2 + 2{\omega_{11}^3}^2 + 2\omega_{12}^3 \omega_{21}^3 + \omega_{12}^3 \omega_{31}^2 - {\omega_{21}^2}^2 \\ - \omega_{21}^3 \omega_{31}^2 + 2{\sigma_{11}}^2 + 2{\sigma_{12}}^2 - 1 = 0 \end{aligned} \tag{3.33}$$

$$\begin{aligned} -E_1(\omega_{21}^3) + E_2(\omega_{11}^3) + \omega_{11}^3(\omega_{33}^2 - 2\omega_{11}^2) - \omega_{21}^2(\omega_{12}^3 + \omega_{21}^3) \\ + \omega_{33}^1(\omega_{21}^3 - 2\omega_{12}^3) + 2s_1\sigma_{12} - 2s_2\sigma_{11} + \sigma_{11} = 0 \end{aligned} \tag{3.34}$$

$$\begin{aligned} -E_1(\omega_{11}^3) - E_2(\omega_{12}^3) + \omega_{11}^2(\omega_{12}^3 + \omega_{21}^3) - \omega_{11}^3(2\omega_{21}^2 + \omega_{33}^1) \\ + \omega_{33}^2(\omega_{12}^3 - 2\omega_{21}^3) + 2s_1\sigma_{11} + 2s_2\sigma_{12} - \sigma_{12} = 0 \end{aligned} \tag{3.35}$$

$$\begin{aligned} -E_1(\sigma_{12}) + E_2(\sigma_{11}) - \sigma_{11}(3\omega_{11}^2 + \omega_{33}^2) - 2\omega_{11}^3 + 2s_1(\omega_{12}^3 - \omega_{21}^3) \\ + \sigma_{12}(\omega_{33}^1 - 3\omega_{21}^2) = 0 \end{aligned} \tag{3.36}$$

$$E_1(\sigma_{11}) + E_2(\sigma_{12}) - \sigma_{12}(3\omega_{11}^2 + \omega_{33}^2) + 2\omega_{12}^3(s_2 - 1) + 3\omega_{21}^2\sigma_{11} - 2\omega_{21}^3 s_2 - \omega_{33}^1\sigma_{11} = 0 \tag{3.37}$$

$$E_1(\omega_{33}^1) + E_2(\omega_{33}^2) - \omega_{11}^2\omega_{33}^2 - 2{\omega_{11}^3}^2 - 2\omega_{12}^3\omega_{21}^3 + \omega_{12}^3 w_3 + \omega_{21}^2\omega_{33}^1 - \omega_{21}^3 w_3 + 2{s_1}^2 + 2(s_2 - 1)s_2 - 1 = 0 \tag{3.38}$$

$$-E_1(s_2) + E_2(s_1) - 2s_1(\omega_{11}^2 + \omega_{33}^2) - 2\omega_{11}^3\sigma_{12} + \sigma_{11}(\omega_{12}^3 + \omega_{21}^3) + w_1(\omega_{12}^3 - \omega_{21}^3) - 2\omega_{21}^2 s_2 + \omega_{21}^2 + 2\omega_{33}^1 s_2 + \omega_{33}^1 = 0 \tag{3.39}$$

$$E_1(s_1) + E_2(s_2) - 2s_2(\omega_{11}^2 + \omega_{33}^2) + \omega_{11}^2 + 2\omega_{11}^3\sigma_{11} + \sigma_{12}(\omega_{12}^3 + \omega_{21}^3) + w_2(\omega_{12}^3 - \omega_{21}^3) + 2s_1(\omega_{21}^2 - \omega_{33}^1) + 3\omega_{33}^2 = 0 \tag{3.40}$$

$$-E_1(\omega_{31}^2) + E_3(\omega_{11}^2) - \omega_{11}^2\omega_{11}^3 + 2\omega_{11}^3\omega_{33}^2 - \omega_{21}^2(\omega_{12}^3 + \omega_{31}^2) + \omega_{33}^1(\omega_{31}^2 - 2\omega_{12}^3) + 2s_1\sigma_{12} - 2s_2\sigma_{11} + \sigma_{11} = 0 \tag{3.41}$$

$$-E_2(\omega_{33}^2) + E_3(\omega_{11}^3) + {\omega_{11}^3}^2 + \omega_{12}^3\omega_{21}^3 - \omega_{12}^3\omega_{31}^2 - \omega_{33}^1(\omega_{21}^2 + \omega_{33}^1) - \omega_{21}^3\omega_{31}^2 + \omega_{21}^3 w_3 + {\omega_{33}^2}^2 - {s_1}^2 - (s_2 - 2)s_2 - \sigma_{11}w_1 - \sigma_{12}w_2 = 0 \tag{3.42}$$

$$-E_1(\omega_{33}^2) - E_3(\omega_{12}^3) - \omega_{11}^2\omega_{33}^1 - 2\omega_{11}^3\omega_{31}^2 + \omega_{11}^3 w_3 + 2\omega_{33}^1\omega_{33}^2 + s_1 - \sigma_{11}w_2 + \sigma_{12}w_1 = 0 \tag{3.43}$$

$$E_2(s_2) + E_3(\sigma_{11}) + \omega_{11}^3(\sigma_{11} + w_1) + 2\omega_{21}^2 s_1 + \omega_{21}^3(\sigma_{12} - w_2) - \sigma_{12}(3\omega_{31}^2 + w_3) - 2\omega_{33}^2(s_2 - 1) = 0 \tag{3.44}$$

$$-E_2(s_1) + E_3(\sigma_{12}) + \omega_{11}^3(\sigma_{12} + w_2) + \omega_{21}^2(2s_2 - 1) - \omega_{21}^3\sigma_{11} + \omega_{21}^3 w_1 + 3\omega_{31}^2\sigma_{11} - \omega_{33}^1 + 2\omega_{33}^2 s_1 + \sigma_{11}w_3 = 0 \tag{3.45}$$

$$-E_1(w_3) + E_3(\omega_{33}^2) - 3\omega_{11}^3\omega_{33}^2 + \omega_{33}^1(3\omega_{12}^3 + \omega_{31}^2 + w_3) - 2s_1 w_2 + 2s_2 w_1 + w_1 = 0 \tag{3.46}$$

$$-E_1(w_1) + E_3(s_1) + \omega_{11}^2 w_2 - 2\omega_{11}^3 s_1 - 2(\omega_{12}^3(s_2 - 1) + s_2(\omega_{31}^2 + w_3)) + \omega_{31}^2 - \omega_{33}^1\sigma_{11} + \omega_{33}^1 w_1 - \omega_{33}^2\sigma_{12} + 2\omega_{33}^2 w_2 - w_3 = 0 \tag{3.47}$$

$$-E_1(w_2) + E_3(s_2) - \omega_{11}^2 w_1 - 2\omega_{11}^3 s_2 + 2s_1(\omega_{12}^3 + \omega_{31}^2 + w_3) - \omega_{33}^1\sigma_{12} + \omega_{33}^1 w_2 + \omega_{33}^2\sigma_{11} - 2\omega_{33}^2 w_1 = 0 \tag{3.48}$$

$$-E_2(\omega_{31}^2) + E_3(\omega_{21}^2) - \omega_{11}^2\omega_{21}^3 + \omega_{11}^2\omega_{31}^2 + \omega_{11}^3\omega_{21}^2 + 2\omega_{11}^3\omega_{33}^1 + 2\omega_{21}^3\omega_{33}^2 + \omega_{31}^2\omega_{33}^2 - 2s_1\sigma_{11} - 2s_2\sigma_{12} + \sigma_{12} = 0 \tag{3.49}$$

$$E_2(\omega_{33}^1) + E_3(\omega_{21}^3) + 2\omega_{11}^3\omega_{31}^2 - \omega_{11}^3 w_3 - \omega_{21}^2\omega_{33}^2 - 2\omega_{33}^1\omega_{33}^2 - s_1 + \sigma_{11}w_2 - \sigma_{12}w_1 = 0 \tag{3.50}$$

$$-E_2(w_3) - E_3(\omega_{33}^1) - 3\omega_{11}^3\omega_{33}^1 + \omega_{33}^2(-3\omega_{21}^3 + \omega_{31}^2 + w_3) - 2s_1 w_1 - 2s_2 w_2 + 3w_2 = 0 \tag{3.51}$$

$$
\begin{aligned}
E_1(w_2) - E_2(w_1) + \omega_{11}^2 w_1 + \omega_{11}^3(4s_2 - 2) - 2s_1(\omega_{12}^3 + \omega_{21}^3)& \\
+ \omega_{21}^2 w_2 - 3\omega_{33}^1 w_2 + 3\omega_{33}^2 w_1 = 0& \qquad (3.52)\\
-E_1(w_1) - E_2(w_2) + \omega_{11}^2 w_2 - 4\omega_{11}^3 s_1 - 2s_2(\omega_{12}^3 + \omega_{21}^3)& \\
+ 2\omega_{12}^3 - \omega_{21}^2 w_1 + 3\omega_{33}^1 w_1 + 3\omega_{33}^2 w_2 - 4w_3 = 0.& \qquad (3.53)
\end{aligned}
$$

Note that on p. 191 of [16], the last two equations, which can be obtained, respectively, from the Codazzi equations of

$$
\begin{aligned}
(\nabla h)(E_3, E_2, E_3) &= (\nabla h)(E_2, E_3, E_3),\\
(\nabla h)(E_3, E_1, E_3) &= (\nabla h)(E_1, E_3, E_3)
\end{aligned}
$$

are missing.

3.3.4 Exampleslong Based on Special Almost Contact Manifolds

In the previous examples, we gave several examples and their characterisations of proper three dimensional CR-submanifolds for which the second fundamental form had special properties. Another question one can ask is whether a given three dimensional manifold can be isometrically immersed as a proper CR-submanifold. As a proper CR-submanifold admits a canonical almost contact structure it is natural to ask whether some special almost contact manifolds can be realised as CR-submanifolds.

An almost contact structure is called *contact metric* if the additional property

$$d\eta(X, Y) = \tfrac{1}{2}(X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])) = g(X, \varphi(Y)) \qquad (3.54)$$

holds for all vector fields X and Y on M. Then the integral curves of the characteristic vector field ξ are geodesics.

Further, if ξ is a Killing vector field with respect to g, then the manifold is *K-contact*. Equivalently, it holds

$$\nabla_X \xi = -\varphi(X) \qquad (3.55)$$

where ∇ is the Levi-Civita connection associated to g. Finally, if the Riemann curvature tensor R of g satisfies

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi = \eta(Y)X - \eta(X)Y \qquad (3.56)$$

for all vector fields X and Y, then the contact metric manifold is *Sasakian*. In this case, ξ is necessarily a Killing vector field, hence any Sasakian manifold is K-contact. In the three dimensional case, the converse also holds.

As far as a Sasakian manifold is concerned, we recall from [16] the following:

Theorem 3.12 *Let M be a three dimensional Sasakian manifold. Then M can not be immersed as a proper CR-submanifolds such that the induced contact structure corresponds with the Sasakian structure.*

Note, however, that there exist three dimensional CR-submanifolds of $S^6(1)$ which at the same time are Sasakian manifolds. More precisely, whereas the contact structure φ is the naturally induced contact structure by the almost complex structure on $S^6(1)$, the corresponding metric is not the induced metric, but is a constant multiple of the induced metric. The examples are constructed as follows.

We follow the approach by [17] and [23] and we start with an almost complex curve in $S^6(1)$ which is superminimal in $S^6(1)$. Following the notation of [7] such a complex curve is called of Type I. We also recall that Bryant [8] obtained a Weierstrass representation for such surfaces and showed that for every genus examples do exist.

We then look at a tube around such a surface in the direction of the second normal bundle, i.e., we define:

$$F : UN^2 \to S^6(1) : v \mapsto \cos\gamma\ f + \sin\gamma\ v \times \frac{\alpha(v, v)}{\|\alpha(v, v)\|},$$

where γ is a constant, $f : N^2 \to S^6(1)$ is a complex curve of Type I, α denotes the second fundamental form of the immersion f and UN^2 denotes the unit tangent bundle.

In order to determine for which value of γ the tube is a CR-submanifold, we introduce the notation given in the Sect. 5 of [13]. Let V be an arbitrary unit tangent vector field defined on a neighborhood of the point p. Denote by $U = JV$. Using the properties of the vector cross product, we see that $F_1 = f$, $F_2 = V$, $F_3 = JV$, $F_4 = \alpha(V, V)/\|\alpha(V, V)\|$, $F_5 = \alpha(V, JV)/\|\alpha(V, V)\| = J\alpha(V, V)/\|\alpha(V, V)\| = F_1 \times F_4$, $F_6 = F_2 \times \alpha(V, V)/\|\alpha(V, V)\|$ and $F_7 = F_3 \times \alpha(V, V)/\|\alpha(V, V)\|$ form a G_2-frame and hence satisfy the corresponding multiplication table relations. Denote also $a_i = \langle(\nabla\alpha)(V, V, V), F_{i+3}\rangle/\|\alpha(V, V)\|$, $i \in \{1, 2, 3, 4\}$ and $\mu_1 = \langle\nabla_V V, U\rangle$ and $\mu_2 = \langle\nabla_U U, V\rangle$.

Assuming that the surface is superminimal, we have $a_3 = 0$ and $a_4 = -\frac{1}{2}$. Note that the immersion F can be parameterized by

$$F(t, q) = \cos\gamma f(q) + \sin\gamma(\cos t F_6(q) + \sin t F_7(q)),$$

where q denotes a point of the surface. Using Lemma 5.1 of [13], we then get

$$\begin{aligned}
D_V F &= \cos\gamma F_2 + \tfrac{1}{2}\sin t\sin\gamma F_4 - \tfrac{1}{2}\cos t\sin\gamma F_5 - (a_2 + 3\mu_1)\sin t\sin\gamma F_6\\
&\quad + (a_2 + 3\mu_1)\cos t\sin\gamma F_7,\\
D_U F &= \cos\gamma F_3 - \tfrac{1}{2}\cos t\sin\gamma F_4 - \tfrac{1}{2}\sin t\sin\gamma F_5 - (a_1 - 3\mu_2)\sin t\sin\gamma F_6\\
&\quad + (a_1 - 3\mu_2)\cos t\sin\gamma F_7,\\
D_{\frac{\partial}{\partial t}} F &= \sin\gamma(-\sin t F_6 + \cos t F_7),
\end{aligned}$$

and a straightforward computation shows that

$$F \times D_{\frac{\partial}{\partial t}} F = -\sin\gamma^2 F + \cos t \cos\gamma \sin\gamma F_6 + \cos\gamma \sin t \sin\gamma F_7$$

is orthogonal to $D_V F$, $D_U F$ and $D_{\frac{\partial}{\partial t}} F$. Denoting by V_1 and V_2 the component of $D_U F$ and $D_V F$, respectively, which is orthogonal to $D_{\frac{\partial}{\partial t}} F$, we compute

$$\begin{aligned} F \times V_1 &= \tfrac{1}{4}(1 + 3\cos 2\gamma)F_3 + \tfrac{3}{2}\cos t \cos\gamma \sin\gamma F_4 + \tfrac{3}{2}\cos\gamma \sin t \sin\gamma F_5, \\ V_2 &= \cos\gamma F_3 - \tfrac{1}{2}\cos t \sin\gamma F_4 - \tfrac{1}{2}\sin t \sin\gamma F_5. \end{aligned}$$

Therefore, we conclude that for the tube with radius $\gamma = \arccos\frac{1}{3}$ it follows

$$F \times V_1 = -V_2$$

and consequently M is a CR submanifold.

By choosing $V_3 = D_{\frac{\partial}{\partial t}} F$, we compute

$$\begin{aligned} \langle V_1, V_1\rangle &= \langle V_2, V_2\rangle = \tfrac{1}{3}, \\ \langle V_1, V_2\rangle &= 0 = \langle V_1, V_3\rangle = \langle V_2, V_3\rangle, \\ \langle V_3, V_3\rangle &= \tfrac{8}{9}, \end{aligned}$$

and

$$\nabla_{V_3} V_3 = 0, \qquad \nabla_{V_1} V_3 = -\tfrac{2}{3} V_2.$$

In particular, taking E_1, E_2 and E_3 to be the normalized vector fields corresponding to V_1, V_2 and V_3 respectively, we conclude

$$\nabla_{E_3} E_3 = 0, \qquad \nabla_{E_1} E_3 = -\tfrac{1}{\sqrt{2}} E_2,$$

which yields that the manifold M is Sasakian with respect to metric $\frac{1}{2}\langle ., .\rangle$.

The above manifolds can be seen as belonging to a larger class of manifolds, the so-called trans-Sasakian spaces, see [6], p. 99. In the classification of Gray and Hervella (1980) of almost Hermitian manifolds there appears a class, $\mathcal{W}_4$, of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. Start with an almost contact metric space M and consider on the product manifold $M \times \mathbb{R}$ the almost complex structure $J(X, f\frac{d}{dt})) = (\varphi(X) - f\xi, \eta(X)\frac{d}{dt})$ and product metric. The notion of a trans-Sasakian structure is then introduced as an almost contact metric structure for which associated almost Hermitian manifold belongs to the class $\mathcal{W}_4$. This may be expressed by the condition that

$$(\nabla\varphi)(X,Y) = \alpha(\langle X,Y\rangle\xi - \eta(Y)X) + \beta(\langle\varphi(X),Y\rangle\xi - \eta(Y)\varphi(X)),$$

and such manifolds are also called the (α, β)-trans-Sasakian manifolds. From this it follows in an elementary way that

$$\nabla_X\xi = -\alpha\varphi(X) + \beta(X - \eta(X))\xi.$$

Note that in the three-dimensional case, the latter equation is sufficient in order to determine that the almost contact manifold is a trans-Sasakian manifold.

Note that if a proper CR-submanifold is a trans-Sasakian space, the previous formula together with the fact that

$$\omega_{11}^3 + \omega_{22}^3 = 0,$$

implies that the function β has to vanish.

Theorem 3.13 *Let M be a (α, β)-trans-Sasakian manifold immersed as a proper CR-submanifold in $S^6(1)$. If the induced contact structure coincides with the trans-Sasakian structure then $\beta = 0$.*

As in the case of the previous tube example, we have that

$$\nabla_{E_3}E_3 = 0, \qquad \nabla_{E_1}E_3 = -\tfrac{1}{\sqrt{2}}E_2,$$

we deduce that this tube is a $(\frac{1}{\sqrt{2}}, 0)$-trans-Sasakian manifold.

Theorem 3.14 *Let M be a (α, β)-trans-Sasakian manifold immersed as a proper CR-submanifold in $S^6(1)$. Assume that the induced contact structure coincides with the trans-Sasakian structure and that the mean curvature vector is a multiple of $J\xi$. Then, M is congruent to a tube with radius $\gamma = \arccos\frac{1}{3}$ in the direction of the second normal bundle on a superminimal almost complex surface in $S^6(1)$.*

Proof As the mean curvature vector is in the direction of $JE_3 = J\xi$, we have that $w_1 = w_2 = 0$. Note also that if necessary by replacing ξ by $-\xi$ we may assume that $\alpha \geq 0$. As M is trans-Sasakian, we know that $\beta = 0$. So, we have that $\omega_{12}^3 = \alpha = -\omega_{21}^3$, $\omega_{33}^1 = \omega_{33}^2 = 0$ and $\omega_{11}^3 = \omega_{22}^3 = 0$. By applying a rotation of E_1 and E_2, we may also assume that $\sigma_{12} = 0$. As $\sigma_{12} = 0$, it follows from (3.34) and (3.35) that

$$\begin{aligned} E_1(\alpha) &= \sigma_{11}(2s_2 - 1), \\ E_2(\alpha) &= 2s_1\sigma_{11}. \end{aligned}$$

Similarly, it follows from (3.36) and (3.37) that

$$\begin{aligned} E_1(\sigma_{11}) &= -3\omega_{21}^2\sigma_{11} + 2\alpha(1 - 2s_2), \\ E_2(\sigma_{11}) &= 3\omega_{11}^2\sigma_{11} - 4s_1\alpha, \end{aligned}$$

and from (3.43) and (3.50) we obtain that $E_3(\alpha) = s_1 = 0$.

Note also that (3.38), (3.42) and (3.53) reduce to an algebraic equations, not containing any derivatives, namely they yield

$$\begin{aligned}
&2(s_2-1)s_2 - 1 + 2\alpha(w_3+\alpha) = 0,\\
&-\alpha^2 - \alpha w_3 - s_2^2 + 2s_2 = 0,\\
&2\alpha - 4w_3 = 0.
\end{aligned}$$

We have that $s_1 = 0$, $w_3 = \frac{1}{2}\alpha$, $s_2 = \frac{1}{2}$ and $\alpha^2 = \frac{1}{2}$. As we assume α to be positive, we deduce that $\alpha = \frac{1}{\sqrt{2}}$. We have

$$\begin{aligned}
&\nabla_{E_1}E_1 = \omega_{11}^2 E_2, && \nabla_{E_1}E_2 = -\omega_{11}^2 E_1 + \frac{1}{\sqrt{2}}E_3, && \nabla_{E_1}E_3 = -\frac{1}{\sqrt{2}}E_2,\\
&\nabla_{E_2}E_1 = \omega_{21}^2 E_2 - \frac{1}{\sqrt{2}}E_3, && \nabla_{E_2}E_2 = -\omega_{21}^2 E_1, && \nabla_{E_2}E_3 = \frac{1}{\sqrt{2}}E_1,\\
&\nabla_{E_3}E_1 = \omega_{31}^2 E_2, && \nabla_{E_3}E_2 = -\omega_{31}^2 E_1, && \nabla_{E_3}E_3 = 0,
\end{aligned}$$

and also

$$\begin{aligned}
&h(E_1,E_1) = \sigma_{11}E_5 - \frac{1}{\sqrt{2}}E_4, && h(E_2,E_2) = -\sigma_{11}E_5 - \frac{1}{\sqrt{2}}E_4,\\
&h(E_3,E_3) = \frac{1}{2\sqrt{2}}E_4, && h(E_1,E_2) = -\sigma_{11}E_6,\\
&h(E_1,E_3) = \frac{1}{2}E_6, && h(E_2,E_3) = -\frac{1}{2}E_5.
\end{aligned}$$

From (3.45), we then obtain that either $\sigma_{11} = 0$ or $\omega_{31}^2 = -\frac{1}{2}\alpha$. Now, we deal with two subcases. First, we deal with the case that $\sigma_{11} \neq 0$ and hence $\omega_{31}^2 = -\frac{1}{2\sqrt{2}}$. In that case the remaining equations are

$$\begin{aligned}
&E_3(\sigma_{11}) = 0,\\
&E_3(\omega_{21}^2) = -\frac{\omega_{11}^2\alpha}{2},\\
&E_3(\omega_{11}^2) = \frac{\omega_{21}^2\alpha}{2},\\
&E_2(\omega_{11}^2) - E_1(\omega_{21}^2) - (\omega_{11}^2)^2 - (\omega_{21}^2)^2 - \frac{5}{2} = 0.
\end{aligned}$$

Let us denote the immersion by f and by $G : U \to S^6(1)$ a mapping of neighborhood U of a point p, given by

$$G = \frac{1}{3}f + \frac{2\sqrt{2}}{3}f \times E_3.$$

By a straightforward computation we obtain that

$$D_{E_1}G = \frac{1}{3}E_1 + \frac{2\sqrt{2}}{3}(E_1 \times E_3 + f \times D_{E_1}E_3) = E_1 + \sqrt{2}E_5,$$
$$D_{E_2}G = \frac{1}{3}E_2 + \frac{2\sqrt{2}}{3}(E_2 \times E_3 + f \times D_{E_2}E_3) = E_2 + \sqrt{2}E_6,$$
$$D_{E_3}G = 0.$$

We can take such a neighborhood U, that we can identify it with a neighborhood $W_1 \times I$ of the origin in R^3, with coordinates (u, v, t), taking $p = (0, 0, 0)$ and $E_3 = \frac{\partial}{\partial t}$. Then, there are local functions α_1 and α_2 such that the vectors fields $X = E_1 + \alpha_1 E_3$ and $Y = E_2 + \alpha_2 E_3$ define a basis of the space tangent to W_1 at every point $(u, v, 0)$. Since we have $G_*X = D_X G = E_1 + \sqrt{2}E_5$ and $G_*Y = D_Y G = E_2 + \sqrt{2}E_6$ it follows that the restriction of G to W_1 is an immersion. Moreover, we have that $G \times G_*X = G_*Y$ and therefore the immersion is an almost complex one. The vector fields G_*X and G_*Y are orthogonal and of the same length $\sqrt{3}$. Further we get,

$$D_X(E_1 + \sqrt{2}E_5) = \left(\omega_{11}^2 + \frac{\alpha_1}{2\sqrt{2}}\right) G_*Y + \sigma_{11}(E_5 - \sqrt{2}E_1) - 3G,$$

and since $E_5 - \sqrt{2}E_1$ is orthogonal to G, G_*X and G_*Y, and therefore a section of the normal bundle of the immersion, we have $h(X, X) = \sigma_{11}(E_5 - \sqrt{2}E_1)$, and the minimality of the almost complex immersion into $S^6(1)$ implies $h(Y, Y) = -\sigma_{11}(E_5 - \sqrt{2}E_1)$. Similarly,

$$D_X(E_2 + \sqrt{2}E_6) = -\left(\omega_{11}^2 + \frac{\alpha_2}{2\sqrt{2}}\right) G_*X - \sigma_{11}(E_6 - \sqrt{2}E_2),$$

and we obtain that $h(X, Y) = -\sigma_{11}(E_6 - \sqrt{2}E_2)$. Now, the fact that $h(X, X)$ and $h(X, Y)$ are orthogonal and of the same length $\sqrt{3}|\sigma_{11}|$ yields that the immersion is superminimal.

Now we can write the immersion f as the union of the integral curves of E_3 through the points $(u, v, 0)$, i.e.,

$$f(u, v, t) = \frac{1}{3}f(u, v, 0) + \frac{3}{2\sqrt{2}}E_4(u, v, 0)$$
$$- \frac{2}{3}\cos\left(\sqrt{\frac{3}{2}}t\right)\left(\frac{1}{\sqrt{2}}E_4((u, v, 0) - f(u, v, 0)\right)$$

$$+\sqrt{\frac{2}{3}}\sin\left(\sqrt{\frac{3}{2}}t\right)E_1(u,v,0)$$

and represents a tube on the almost complex immersion.

Next we deal with the case that σ_{11} vanishes which then implies that we still have the freedom of rotation in the moving frame. In that case, the remaining equations are

$$\begin{aligned}
&E_3(\omega_{21}^2) - E_2(\omega_{31}^2) = -\omega_{11}^2(\omega_{31}^2 + \alpha),\\
&E_3(\omega_{11}^2) - E_1(\omega_{31}^2) = \omega_{21}^2(\omega_{31}^2 + \alpha),\\
&E_2(\omega_{11}^2) - E_1(\omega_{21}^2) - (\omega_{11}^2)^2 - (\omega_{21}^2)^2 + 2\alpha\omega_{31}^2 - \frac{5}{2} = 0.
\end{aligned}$$

A straightforward computation shows that the one form μ given on some neighborhood of a point by $\mu(E_1) = -\omega_{11}^2, \mu(E_2) = -\omega_{21}^2, \mu(E_3) = -\omega_{31}^2 + \frac{5}{2\sqrt{2}}$ is closed and therefore exact. Let $\mu = d\theta$ and denote by

$$\begin{aligned}
\widetilde{E}_1 &= \cos\theta E_1 + \sin\theta E_2,\\
\widetilde{E}_2 &= -\sin\theta E_1 + \cos\theta E_2.\\
\widetilde{E}_5 &= \cos\theta E_5 + \sin\theta E_6,\\
\widetilde{E}_6 &= -\sin\theta E_5 + \cos\theta E_6.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\nabla_{\widetilde{E}_1}\widetilde{E}_1 = 0, && \nabla_{\widetilde{E}_1}\widetilde{E}_2 = \frac{1}{\sqrt{2}}E_3, && \nabla_{\widetilde{E}_1}E_3 = -\frac{1}{\sqrt{2}}\widetilde{E}_2,\\
&\nabla_{\widetilde{E}_2}\widetilde{E}_1 = -\frac{1}{\sqrt{2}}E_3, && \nabla_{\widetilde{E}_2}\widetilde{E}_2 = 0, && \nabla_{\widetilde{E}_2}E_3 = \frac{1}{\sqrt{2}}\widetilde{E}_1,\\
&\nabla_{E_3}\widetilde{E}_1 = \frac{5}{2\sqrt{2}}\widetilde{E}_2, && \nabla_{E_3}\widetilde{E}_2 = -\frac{5}{2\sqrt{2}}\widetilde{E}_1, && \nabla_{E_3}E_3 = 0,
\end{aligned}$$

and

$$\begin{aligned}
&h(\widetilde{E}_1,\widetilde{E}_1) = -\frac{1}{\sqrt{2}}E_4, && h(\widetilde{E}_2,\widetilde{E}_2) = -\frac{1}{\sqrt{2}}E_4, && h(E_3,E_3) = \frac{1}{2\sqrt{2}}E_4,\\
&h(\widetilde{E}_1,\widetilde{E}_2) = 0, && h(\widetilde{E}_1,E_3) = \frac{1}{2}\widetilde{E}_6, && h(\widetilde{E}_2,E_3) = -\frac{1}{2}\widetilde{E}_5.
\end{aligned}$$

Theorem 3.15 *Let M be a $(\frac{1}{\sqrt{2}},0)$-trans-Sasakian manifold immersed as a proper CR-submanifold in $S^6(1)$. Assume that the induced contact structure coincides with the trans-Sasakian structure. Then M is congruent to a tube with radius $\gamma = \arccos\frac{1}{3}$*

in the direction of the second normal bundle on a superminimal almost complex surface in $S^6(1)$.

Proof It follows from (3.34) and (3.35) that $s_1\sigma_{11} = 0$ and $(2s_2 - 1)\sigma_{11} = 0$. So let us assume first that $\sigma_{11} \neq 0$ and therefore $s_1 = 0$ and $s_2 = \frac{1}{2}$. It then follows from (3.43) that $w_2 = 0$. From (3.39) we then deduce that $w_1 = 0$. Therefore, we can apply the result of the previous theorem which completes the proof. If $\sigma_{11} = 0$, we recuperate once more the rotation freedom in E_1 and E_2. We may therefore assume that $w_2 = 0$. If w_1 vanishes as well, we are in the case of the previous theorem and therefore the proof is completed. Therefore, we may assume that $w_1 \neq 0$. From (3.36) and (3.37), we get now that $s_1 = 0$ and $s_2 = \frac{1}{2}$.

3.4 Four Dimensional CR-Submanifolds of $S^6(1)$

3.4.1 Construction of the Moving Frame

Now, let M be a four dimensional CR-submanifold of $S^6(1)$. Although we have already decided to deal with proper CR-submanifolds, let us mention that in [20] A. Gray showed that there are no four dimensional almost complex submanifolds of the sphere $S^6(1)$. Since the dimension of the almost complex distribution is even it follows that the dimensions of the almost complex and totally real distributions, $\mathcal{H}$ and $\mathcal{H}^\perp$, both have to be two.

Let us show the construction of the local orthonormal moving frame of the submanifold M and its normal bundle that is particulary convenient to work with. Denote by ξ and η the local orthonormal vector fields spanning the $T^\perp M$ and by p the position vector field of the submanifold. Then, the vector fields $J\xi = p \times \xi$ and $J\eta = p \times \eta$ belong to the totally real distribution by its definition, and since the almost complex structure J is Hermitian it follows $\langle J\xi, J\eta\rangle = 0$ so $J\xi, J\eta$ span the totally real distribution $\mathcal{H}^\perp$.

Further, we have

$$\langle p, \xi \times \eta\rangle = -\langle \xi, p \times \eta\rangle = -\langle \xi, J\eta\rangle = 0,$$

implying that $\xi \times \eta$ is a vector field tangent to the sphere $S^6(1)$. Moreover, we have

$$\langle \xi \times \eta, \xi\rangle = 0 = \langle \xi \times \eta, \eta\rangle,$$

so $\xi \times \eta$ belongs to the tangent bundle TM, while $\langle \xi \times \eta, p \times \eta\rangle = \langle \xi, p\rangle = 0$ and $\langle \xi \times \eta, p \times \xi\rangle = -\langle \xi \times \eta, \xi \times p\rangle = -\langle \eta, p\rangle = 0$ imply that $\xi \times \eta \in \mathcal{H}$.

Similarly, from (3.3) we have,

$$\langle p \times (\xi \times \eta), \xi\rangle = \langle (p \times \eta) \times \xi + 2\langle p, \xi\rangle\eta - \langle p, \eta\rangle\xi - \langle \xi, \eta\rangle p, \xi\rangle$$

$$= \langle (p \times \eta) \times \xi, \xi \rangle = 0,$$

so $J(\xi \times \eta)$ is orthogonal to ξ and in the same manner we find that $J(\xi \times \eta)$ is also orthogonal to η and obviously to p, which implies $J(\xi \times \eta) \in TM$. Then, from

$$\langle J(\xi \times \eta), J\xi \rangle = \langle \xi \times \eta, \xi \rangle = 0, \quad \langle J(\xi \times \eta), J\eta \rangle = 0$$

we conclude that $J(\xi \times \eta) \in \mathcal{H}$. Also, it holds $\langle \xi \times \eta, J(\xi \times \eta) \rangle = 0$. Therefore, the almost complex distribution $\mathcal{H}$ is spanned by $\xi \times \eta$ and $J(\xi \times \eta)$.

Also, it is clear that the frame $\{p, \xi, J\xi, \eta, J\eta, \xi \times \eta, -J(\xi \times \eta)\}$ is a G_2-frame.

However, it is obvious that we have a certain freedom in choice of the initial orthonormal frame $\{\xi, \eta\}$ of the normal bundle. If $\{\widetilde{\xi}, \widetilde{\eta}\}$ is another such frame then there exists a locally defined differential function α such that it holds

$$\begin{aligned} \widetilde{\xi} &= \cos \alpha \xi + \sin \alpha \eta, \\ \widetilde{\eta} &= \pm(-\sin \alpha \xi + \cos \alpha \eta), \end{aligned}$$

and further

$$\begin{aligned} &\widetilde{\xi} \times \widetilde{\eta} = \pm(\cos \alpha \xi + \sin \alpha \eta) \times (-\sin \alpha \xi + \cos \alpha \eta) = \pm \xi \times \eta, \\ &J(\widetilde{\xi} \times \widetilde{\eta}) = \pm J(\xi \times \eta), \\ &J\widetilde{\xi} = \cos \alpha J\xi + \sin \alpha J\eta, \\ &J\widetilde{\eta} = \pm(-\sin \alpha J\xi + \cos \alpha J\eta), \end{aligned}$$

so the vector field $\xi \times \eta$ remains invariant under rotations in the normal bundle. Let us denote

$$F_1 = \xi \times \eta, \quad F_2 = J(\xi \times \eta), \quad F_3 = J\xi, \quad F_4 = J\eta.$$

Now we shall give the conditions that each such local frame

$$\{F_1, \ldots, F_4, \quad \xi, \quad \eta\}$$

has to satisfy. We denote by

$$\omega_{ij}^k = \langle D_{F_i} F_j, F_k \rangle, \quad h_{ij}^n = \langle D_{F_i} F_j, n \rangle, \quad \alpha_i = \langle D_{F_i} \xi, \eta \rangle,$$

where $1 \leq i, j, k \leq 4$ and $n \in \{\xi, \eta\}$. The scalar products of the frame vector fields are constant, so by taking their covariant derivatives in the directions of F_i, $i = 1, \ldots, 4$ we get that $\omega_{ij}^k = -\omega_{ik}^j$ and also, that the following lemma holds.

Lemma 7 *If α_i, $i = 1, \ldots, 4$ are previously defined functions, we have*

$$\nabla^\perp_{F_1} \xi = \alpha_1 \eta, \quad \nabla^\perp_{F_1} \eta = -\alpha_1 \xi, \quad \nabla^\perp_{F_2} \xi = \alpha_2 \eta, \quad \nabla^\perp_{F_2} \eta = -\alpha_2 \xi,$$

$$\nabla^{\perp}_{F_3}\xi = \alpha_3\eta, \quad \nabla^{\perp}_{F_3}\eta = -\alpha_3\xi, \quad \nabla^{\perp}_{F_4}\xi = \alpha_4\eta, \quad \nabla^{\perp}_{F_4}\eta = -\alpha_4\xi.$$

Also, the symmetry of the second fundamental form implies that $h^{\xi}_{ij} = h^{\xi}_{ji}$, $h^{\eta}_{ij} = h^{\eta}_{ji}$ and we also have

$$h(F_i, F_j) = h^{\xi}_{ij}\xi + h^{\eta}_{ij}\eta. \tag{3.57}$$

Lemma 8 *Let ∇ be the Levi-Civita connection of M. Then we have*

$$\begin{aligned}
\nabla_{F_1}F_1 &= (h^{\xi}_{13} + h^{\eta}_{14})F_2 - h^{\xi}_{12}F_3 - h^{\eta}_{12}F_4,\\
\nabla_{F_1}F_2 &= -(h^{\xi}_{13} + h^{\eta}_{14})F_1 + h^{\xi}_{11}F_3 + h^{\eta}_{11}F_4,\\
\nabla_{F_1}F_3 &= h^{\xi}_{12}F_1 - h^{\xi}_{11}F_2 + \alpha_1F_4,\\
\nabla_{F_1}F_4 &= h^{\eta}_{12}F_1 - h^{\eta}_{11}F_2 - \alpha_1F_3,\\
\nabla_{F_2}F_1 &= (h^{\xi}_{23} + h^{\eta}_{24})F_2 - h^{\xi}_{22}F_3 - h^{\eta}_{22}F_4,\\
\nabla_{F_2}F_2 &= -(h^{\xi}_{23} + h^{\eta}_{24})F_1 + h^{\xi}_{12}F_3 + h^{\eta}_{12}F_4,\\
\nabla_{F_2}F_3 &= h^{\xi}_{22}F_1 - h^{\xi}_{12}F_2 + (-1 + \alpha_2)F_4,\\
\nabla_{F_2}F_4 &= h^{\xi}_{22}F_1 - h^{\eta}_{12}F_2 - (-1 + \alpha_2)F_3,\\
\nabla_{F_3}F_1 &= (h^{\xi}_{33} + h^{\eta}_{34})F_2 - h^{\xi}_{23}F_3 - h^{\eta}_{23}F_4,\\
\nabla_{F_3}F_2 &= -(h^{\xi}_{33} + h^{\eta}_{34})F_1 + h^{\xi}_{13}F_3 + (1 + h^{\eta}_{13})F_4,\\
\nabla_{F_3}F_3 &= h^{\xi}_{23}F_1 - h^{\xi}_{13}F_2 + \alpha_3F_4,\\
\nabla_{F_3}F_4 &= h^{\eta}_{23}F_1 - (1 + h^{\eta}_{13})F_2 - \alpha_3F_3,\\
\nabla_{F_4}F_1 &= (h^{\xi}_{34} + h^{\eta}_{44})F_2 - h^{\xi}_{24}F_3 - h^{\eta}_{24}F_4,\\
\nabla_{F_4}F_2 &= -(h^{\xi}_{34} + h^{\eta}_{44})F_1 + (-1 + h^{\xi}_{14})F_3 + h^{\eta}_{14}F_4,\\
\nabla_{F_4}F_3 &= h^{\xi}_{24}F_1 - (-1 + h^{\xi}_{14})F_2 + \alpha_4F_4,\\
\nabla_{F_4}F_4 &= h^{\eta}_{24}F_1 - h^{\eta}_{14}F_2 - \alpha_4F_3,
\end{aligned}$$

and
$h^{\xi}_{14} + 1 = h^{\eta}_{13}, \quad = h^{\xi}_{24} = h^{\eta}_{23}, \quad = h^{\xi}_{34} = h^{\eta}_{33}, \quad = h^{\xi}_{44} = h^{\eta}_{34}.$

Proof Using (3.13) and (3.57) we get

$$D_{F_1}F_2 = \nabla_{F_1}F_2 + h^{\xi}_{12}\xi + h^{\eta}_{12}\eta$$

and using $F_2 = JF_1 = p \times F_1$, Lemma 1 and the multiplication table, we get

$$\begin{aligned}
D_{F_1}F_2 &= D_{F_1}p \times F_1 + p \times D_{F_1}F_1\\
&= F_1 \times F_1 + p \times [\nabla_{F_1}F_1 + h(F_1, F_1) - \langle F_1, F_1\rangle p]\\
&= -\omega^2_{11}F_1 - \omega^3_{11}\xi - \omega^4_{11}\eta + h^{\xi}_{11}F_3 + \omega^{\eta}_{11}F_4
\end{aligned}$$

and we get

$$h^{\xi}_{12} = -\omega^{3}_{11}, \quad h^{\eta}_{12} = -\omega^{4}_{11}, \quad h^{\xi}_{11} = \omega^{3}_{12}, \quad h^{\eta}_{11} = \omega^{4}_{12}.$$

Analogously, the other equations also hold.

Lemma 9 *If A_ξ, A_η are the shape operators with respect to the vector fields ξ and η, it holds $A_\eta(J\xi) - A_\xi(J\eta) = \xi \times \eta$.*

Proof Since we have $\langle \xi, J\eta \rangle = 0$, we straightforwardly obtain that

$$\begin{aligned} &\langle D_X\xi\,,\, J\eta\rangle + \langle \xi\,,\, D_X(p \times \eta)\rangle \\ &\quad = \langle D_X\xi\,,\, J\eta\rangle + \langle \xi\,,\, X \times \eta\rangle + \langle \xi\,,\, p \times D_X\eta\rangle = 0 \end{aligned}$$

implying

$$-\langle A_\xi X, J\eta\rangle + \langle A_\eta X, J\xi\rangle - \langle X, \xi \times \eta\rangle = 0$$

which concludes the proof.

3.4.2 Some Examples

The investigation of four dimensional CR-submanifolds has not been as wide as it was for three dimensional ones. In [26] Sekigawa showed that there are no CR product submanifolds of the sphere $S^6(1)$. In [24] were given some topological restrictions on such submanifolds. Namely, it was shown that the first Pontrjagin class of the oriented CR-submanifold M vanishes, and moreover, that if it is in addition compact its Euler number $\chi(M)$ is zero. Therefore, S^4, $S^2 \times S^2$ and CP^2 cannot be immersed into $S^6(1)$ as a CR-submanifold. It is also not hard to show that there are no totally geodesic submanifolds of $S^6(1)$, regardless of the compactness condition. Also, regarding the properties of the almost complex and totally real distribution, in the same paper, the following theorem was proven.

Theorem 3.16 *The totally real distribution $\mathcal{H}^\perp$ of M is not involutive. If the almost complex distribution $\mathcal{H}$ is involutive then each compact leaf of $\mathcal{H}$ is homeomorphic to a two dimensional torus.*

The first examples of four dimensional CR-submanifolds of the sphere $S^6(1)$, which we here present, were given in [24], see also [25].

Example 1 Denote by $\mathbb{H}$ the space of quaternions. Then, we can put $\mathcal{O} = \mathbb{H} \oplus \mathbb{H}$ and write the an octonion in the following way

$$a \cdot 1 + \sum_{i=0}^{6} a_i e_i = a \cdot 1 + \sum_{i=0}^{2} a_i e_i + \left(a_3 \cdot 1 + \sum_{i=4}^{6} a_i e_{i-3} \right) e_3 = q_1 + q_2 e_3,$$

where $q_1, q_2 \in \mathbb{H}$.

Let $\gamma : I \to S^2 \subset \Im\mathbb{H}$ be a regular curve parametrized by its arclength, and $S^3 \subset \mathbb{H}$ the space of unit length quaternion. Then the immersion $\psi : I \times S^3 \to S^6(1)$ given by

$$\psi(t, q) = a\gamma(t) - b\overline{q}e_3, \quad a, b > 0, \quad a^2 + b^2 = 1$$

gives rise to a four dimensional CR-submanifold of $S^6(1)$. In particular, one orthonormal frame of the normal bundle is given by

$$\xi = \gamma' \times \gamma, \quad \eta = b\gamma - a\overline{q}e_3.$$

and further, the local frame of the tangent bundle by

$$\begin{aligned} J\xi &= -a\gamma' + b(\gamma' \times \gamma)\overline{q}e_3, \\ J\eta &= \gamma \times \overline{q}e_3, \\ \xi \times \eta &= -(b\gamma' + a(\gamma' \times \gamma)\overline{q}e_3), \\ J(\xi \times \eta) &= -\gamma'\overline{q}e_3. \end{aligned}$$

Example 2 The immersion $\phi : S^1 \times S^3 \to S^6(1)$ given by

$$\phi(\theta, q) = a(qe_0\overline{q}) + b[(t(-\sin\theta + \cos\theta e_0) + s(\cos\theta e_1 + \sin\theta e_2))\overline{q}]e_3,$$

where $a, b, t, s > 0, \quad a^2 + b^2 = 1, \quad t^2 + s^2 = 1$ gives rise to a CR-submanifold of $S^6(1)$. Note that ϕ is not an injection since $\phi(\theta + \pi, -q) = \phi(\theta, q)$. We can denote by

$$\tau(\theta) = t(-\sin\theta + \cos\theta e_0) + s(\cos\theta e_1 + \sin\theta e_2),$$

and write

$$\phi(\theta, q) = a(qe_0\overline{q}) + b(\tau(\theta)\overline{q})e_3.$$

Here, one frame of the normal bundle is then given by

$$\begin{aligned} \xi &= b(qe_0\overline{q}) - a(\tau(\theta)\overline{q})e_3, \\ \eta &= \frac{1}{\sqrt{1 + 3a^2}} \left\{ b(qe_1\overline{q}) + 2a(\tau(\theta)e_2\overline{q})e_3 \right\}, \end{aligned}$$

and, further we have

$$J\xi = (\tau(\theta)e_0\overline{q})e_3,$$
$$J\eta = \frac{1}{1+3a^2}(-3abqe_2\overline{q} + (1-3a^2)\tau(\theta)e_1\overline{q}e_3),$$
$$\xi \times \eta = -\frac{1}{\sqrt{1+3a^2}}((3a^2-1)qe_2\overline{q} - 3ab\tau(\theta)e_2\overline{q}e_3),$$
$$J(\xi \times \eta) = -\frac{1}{\sqrt{1+3a^2}}(2aqe_1\overline{q} - b\tau(\theta)e_2\overline{q}e_3).$$

It is also interesting to note some of the properties of these immersions

(1) immersion ϕ is minimal if and only if $a = \sqrt{(3+\sqrt{57})/24}$, $t = 1/\sqrt{2}$;
(2) if $a = 1/\sqrt{3}$, $t = 1/\sqrt{2}$ then the almost complex distribution $\mathcal{H}$ is integrable;
(3) for arbitrary a, b, t, s the immersion ϕ is full in $S^6(1)$.

In [2], the following classification result was given.

Theorem 3.17 *Let M be a four dimensional minimal CR-submanifold in $S^6(1)$ which satisfies Chen's equality. Then, M is locally congruent with the immersion*

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = \ & (\cos x_4 \cos x_1 \cos x_2, \sin x_4 \sin x_1 \cos x_2, \\ & \sin 2x_4 \sin x_3 \cos x_2 + \cos 2x_4 \sin x_2, 0, \sin x_4 \cos x_1 \cos x_2, \\ & \cos x_4 \sin x_1 \cos x_2, \cos 2x_4 \sin x_3 \cos x_2 - \sin 2x_4). \end{aligned} \tag{3.58}$$

Namely, let us denote by

$$\mathcal{D}(p) = \{X \in T_pM \mid (n-1)h(X,Y) = ng(X,Y)H, \ \text{ for all } \ Y \in T_pM\}, \tag{3.59}$$

the distribution of a n-dimensional submanifold of a real space form.

Then, it is well known that if a submanifold M satisfies the equality sign in the Chen's inequality then one of the following holds:

(1) $\mathcal{D}$ is a $(n-2)$-dimensional distribution;
(2) $\mathcal{D} = TM$ and M is a totally geodesic submanifold;
(3) the bundle Imh is one dimensional.

Conversely, we also have the following lemma.

Lemma 10 *If M is a n-dimensional ($n > 2$) submanifold of a real space form such that the dimension of $\mathcal{D}$ is greater or equal ($n-2$) then M satisfies Chen's equality.*

Here, for a four-dimensional minimal CR submanifold of the $S^6(1)$ this distribution is given by

$$\mathcal{D} = \{Z \in TM | h(X,Z) = 0, \forall X \in TM\},$$

and since submanifold satisfies Chen's equality, $\mathcal{D}$ has to be at least two-dimensional. Also, the minimality implies that

$$h^{\xi}_{11} + h^{\xi}_{22} + h^{\xi}_{33} + h^{\xi}_{44} = h^{\eta}_{11} + h^{\eta}_{22} + h^{\eta}_{33} + h^{\eta}_{44} = 0. \tag{3.60}$$

It is interesting to note that, by using Lemma 9, straightforwardly we obtain that for all $V \in \mathcal{D}$ it holds

$$\begin{aligned}\langle \xi \times \eta, V\rangle &= \langle A_\eta(J\xi), V\rangle - \langle A_\xi(J\eta), V\rangle \\ &= \langle \eta, h(J\xi, V)\rangle - \langle \xi, h(V, J\eta)\rangle = 0\end{aligned}$$

which means that such vector field V is in the space spanned by F_2, F_3 and F_4.

Let us find an orthonormal local frame for distribution $\mathcal{D}$. Since $\mathcal{D}$ is two-dimensional, there exists a $V_1 \in \mathcal{D}$ which is orthogonal to F_2 (i.e., $V_1 \in \mathcal{L}(F_3, F_4)$). Recall that we can make a suitable choice of the basis of $T^\perp(M)$ so we may assume that $V_1 = F_3$. Next, we take $V_2 \in \mathcal{D}$ which is orthogonal to V_1. Then, we can write

$$V_2 = \cos\phi F_2 + \sin\phi F_4.$$

Further, since V_1, V_2 belong to distribution $\mathcal{D}$ we obtain the following proposition.

Proposition 1 *If M satisfies the Chen's equality then there exists locally defined differentiable function t such that following holds.*

$$\begin{aligned}&h^{\xi}_{13} = h^{\eta}_{13} = h^{\xi}_{23} = h^{\eta}_{23} = h^{\xi}_{24} = h^{\xi}_{33} = h^{\eta}_{33} = h^{\xi}_{34} = h^{\eta}_{34} = h^{\xi}_{44} = 0, \quad h^{\xi}_{14} = -1, \\ &h^{\xi}_{12} = t, \quad h^{\xi}_{11} = h^{\xi}_{22} = 0, \quad h^{\eta}_{12} = -th^{\eta}_{14} \quad h^{\eta}_{24} = -th^{\eta}_{44}, \quad h^{\eta}_{22} = t^2 h^{\eta}_{44}, \\ &h^{\eta}_{11} = -h^{\eta}_{22} - h^{\eta}_{44} = -(1+t^2)h^{\eta}_{44}.\end{aligned} \tag{3.61}$$

Also, we have

Proposition 2

$$\begin{aligned}&\alpha_3 = 0, \quad \alpha_1 = 0, \quad F_3(t) + 1 + t^2 = 0, \quad F_3(h^{\eta}_{14}) = 0, \quad \alpha_2 = 1 - 2t^2, \quad \alpha_4 = 3t, \\ &h^{\eta}_{44} = 0, \quad F_1(h^{\eta}_{14}) = 0, \quad F_2(t) = 2t\gamma_4(1+t^2), \quad F_2(th^{\eta}_{14}) = 3t{h^{\eta}_{14}}^2 + t(1-2t^2), \\ &F_1(t) = 0, \quad F_4(v) = 3t{h^{\eta}_{14}}^2 + 3t, \quad F_4(t) = -2h^{\eta}_{14}(t^2+1), \\ &F_2(h^{\eta}_{14}) = (1-2t^2)({h^{\eta}_{14}}^2 + 1).\end{aligned}$$

These relations follow directly from Gauss, Codazzi, and Ricci equations and moreover, they satisfy the integrability conditions. By integrating, the immersion (3.58) is obtained. Note that this immersion is also not linearly full in $S^6(1)$, meaning, it is an immersion into a totally geodesic sphere $S^5 \subset S^6(1)$. In [1] minimal, non linearly full CR-submanifolds were investigated.

When dealing with a nonlinearly full submanifold of the sphere $S^6(1)$, one can observe the hyperplane through the origin of $\mathbb{R}^7$ that also contains the image of the immersion and its unit normal vector. Then the unit normal vector field induced by it in the points of the submanifold M is orthogonal to M, and to the position vector field p, so it is tangent to the sphere $S^6(1)$. Therefore, it belongs to $T^\perp M = \mathcal{L}(\xi, \eta)$, for arbitrary local frame $\{\xi, \eta\}$.

Therefore, keeping in mind the freedom of rotation in the normal bundle of the submanifold, one can assume that the unit normal vector field orthogonal to that hyperplane is η.

Moreover, the covariant derivatives of η in the directions of F_i, $i = 1, \ldots, 4$ are then identically equal to zero, so, along with the minimality condition (3.60), we have the following lemma.

Lemma 11

$$h^\eta_{11} = h^\eta_{12} = h^\eta_{14} = h^\eta_{22} = h^\eta_{24} = h^\eta_{33} = h^\eta_{44} = 0, \quad h^\xi_{11} + h^\xi_{22} + h^\xi_{33} = 0,$$
$$h^\xi_{24} = h^\xi_{34} = h^\xi_{44} = 0, \quad h^\xi_{14} = -1, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.$$

The Gauss, Codazzi and Ricci equations here yield the following relations.

Lemma 12

$$F_4(h^\xi_{13}) = 3h^\xi_{12}, \quad F_4(h^\xi_{23}) = 3(h^\xi_{11} + 2h^\xi_{22}), \quad F_4(h^\xi_{12}) = -3h^\xi_{13},$$
$$F_4(h^\xi_{22}) = -6h^\xi_{23}, \quad F_4(h^\xi_{11}) = 0, \quad F_1(h^\xi_{22}) = -3h^\xi_{12}h^\xi_{13} + 3h^\xi_{11}h^\xi_{23} + F_2(h^\xi_{12}),$$
$$F_2(h^\xi_{11}) = -3h^\xi_{13}h^\xi_{22} + 3h^\xi_{12}h^\xi_{23} + F_1(h^\xi_{12}),$$
$$F_1(h^\xi_{11}) = -3h^\xi_{12}h^\xi_{13} + 3h^\xi_{11}h^\xi_{23} - F_3(h^\xi_{13}) - F_1(h^\xi_{22}),$$
$$F_2(h^\xi_{22}) = 3(h^\xi_{12}h^\xi_{23} - h^\xi_{13}h^\xi_{22}) - F_2(h^\xi_{11}) - F_3(h^\xi_{23}),$$
$$F_3(h^\xi_{11}) = -3(h^\xi_{11}h^\xi_{12} + h^\xi_{12}h^\xi_{22} + h^\xi_{13}h^\xi_{23}) + F_1(h^\xi_{13}),$$
$$F_2(h^\xi_{23}) = -3(h^\xi_{11}h^\xi_{12} + h^\xi_{12}h^\xi_{22} + h^\xi_{13}h^\xi_{23}) + F_3(h^\xi_{22}),$$
$$F_2(h^\xi_{13}) = F_1(h^\xi_{23}) + 1 + (h^\xi_{11})^2 - 2(h^\xi_{12})^2 + (h^\xi_{13})^2 + 4h^\xi_{11}h^\xi_{22} + (h^\xi_{22})^2 + (h^\xi_{23})^2,$$
$$F_1(h^\xi_{23}) = F_3(h^\xi_{12}) + 1 - 2(h^\xi_{11})^2 + (h^\xi_{12})^2 - 2(h^\xi_{13})^2 - 2h^\xi_{11}h^\xi_{22} + (h^\xi_{22})^2 + (h^\xi_{23})^2.$$

These equations, along with the integrability conditions which are equivalent to a very large system of equations, give the following result, see [1].

Theorem 3.18 *Let M be a four dimensional, minimal CR-submanifold of the sphere $S^6(1)$ contained in a totally geodesic sphere $S^5(1)$. Then, M is locally congruent*
(1) to the immersion (3.58) and it satisfies Chen's equality.
(2) to the immersion

$$f_2(x_1, x_2, x_3, x_4) = \frac{\sqrt{2}}{8}(3(\cos x_1 \cos x_2 + \cos x_3 \sin x_1) + \sqrt{2}\cos(\sqrt{3}x_4))e_0$$

$$
\begin{aligned}
&+ \frac{\sqrt{6}}{8}(\sqrt{2}\cos(\sqrt{3}x_4) - \cos x_1 \cos x_2 - \cos x_3 \sin x_1)e_1 \\
&+ \frac{\sqrt{6}}{4}(\sin x_1 \sin x_3 - \cos x_1 \sin x_2)e_2 \\
&+ \frac{\sqrt{2}}{8}(-3(\cos x_1 \sin x_2 + \sin x_1 \sin x_3) + \sqrt{2}\sin(\sqrt{3}x_4))e_4 \\
&+ \frac{\sqrt{6}}{8}(\cos x_1 \sin x_2 + \sin x_1 \sin x_3 + \sqrt{2}\sin(\sqrt{3}x_4))e_5 \\
&+ \frac{\sqrt{6}}{4}(\cos x_3 \sin x_1 - \cos x_1 \cos x_2)e_6.
\end{aligned}
$$

(3) to the immersion $f_3(y_1, y_2, y_3, y_4) = \sum_{i=0}^{6} a_i e_i$ *where*

$$
\begin{aligned}
a_0 &= \frac{1}{4}(((1+\sqrt{2})\cos y_2 + \cos y_3)\cos(y_1 - y_4) + (\cos y_2 + \cos y_3 - \sqrt{2}\cos y_3) \\
&\quad \cdot \cos(y_1 + y_4) + \sin y_2 \sin y_3((\sqrt{2}-1)\sin(y_1 + y_4) - \sin(y_1 - y_4))), \\
a_1 &= \frac{1}{4}(((1+\sqrt{2})\cos y_3 - \cos y_2)\cos(y_1 - y_4) - ((\sqrt{2}-1)\cos y_2 + \cos y_3) \\
&\quad \cdot \cos(y_1 + y_4) + \sin y_2 \sin y_3(\sin(y_1 + y_4) - (1+\sqrt{2})\sin(y_1 - y_4))), \\
a_2 &= \sqrt{8\sqrt{2} - 8}((1+\sqrt{2})(\cos y_1 \cos y_3 \sin y_2 - \sin y_1 \sin y_3)\sin y_4 \\
&\quad - \cos y_4(\cos y_3 \sin y_1 \sin y_2 + \cos y_1 \sin y_3)), \\
a_3 &= 0, \\
a_4 &= (\cos y_2 + (1-\sqrt{2})\cos y_3)\sin(y_1 + y_4) - ((1+\sqrt{2})\cos y_2 + \cos y_3) \\
&\quad \cdot \sin(y_1 - y_4) - (\cos(y_1 - y_4) + (\sqrt{2}-1)\cos(y_1 + y_4))\sin y_2 \sin y_3, \\
a_5 &= -((1+\sqrt{2})\cos(y_1 - y_4) + \cos(y_1 + y_4))\sin y_2 \sin y_3 \\
&\quad + (\cos y_2 - (1+\sqrt{2})\cos y_3)\sin(y_1 - y_4) \\
&\quad - ((\sqrt{2}-1)\cos y_2 + \cos y_3)\sin(y_1 + y_4), \\
a_6 &= -4\sqrt{\sqrt{2}+2}(\cos y_4(\cos y_1 \cos y_3 \sin y_2 - \sin y_1 \sin y_3) \\
&\quad + (\sqrt{2}-1)(\cos y_3 \sin y_1 \sin y_2 + \cos y_1 \sin y_3)\sin y_4)).
\end{aligned}
$$

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Chapter 4
CR Submanifolds of Hermitian Manifolds and the Tangential CR Equations

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4.1 Bejancu's CR Submanifolds and CR Analysis

The purpose of this (partially expository) paper is to set the basis for the mathematical analysis of solutions to the tangential Cauchy–Riemann equations $\overline{\partial}_M u = 0$ on CR submanifolds M of Hermitian (e.g., Kählerian, locally conformal Kähler, etc.) manifolds M^{2N}, as introduced by A. Bejancu in his celebrated work [5], in an attempt to fill in a gap between the differential geometric side of the subject (e.g., devoted to the geometry of the second fundamental form of M in M^{2N}) and the analysis problems related to the (local) properties of (weak) solutions to $\overline{\partial}_M u = 0$ or the (local or global) holomorphic extension of CR functions on M.

Dedicated to Aurel Bejancu—A. Bejancu (**B**.): Romanian mathematician (b. 1946). The scientific creation of **B**. is mainly devoted to the theory of isometric immersions in (semi) Riemannian and Finslerian geometry.

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4.1.1 CR Submanifolds

Let M^{2N} be a Hermitian manifold, of complex dimension N, with the complex structure J and the Hermitian metric G. Let M be a real m-dimensional manifold, $m = 2n + k < 2N$, $n \geq 0$, $k \geq 0$, and $\Psi : M \to M^{2N}$ a C^∞ immersion. For every $x \in M$, the differential $d_x\Psi : T_x(M) \to T_{\Psi(x)}(M^{2N})$ has rank m hence (by the classical rank theorem) each point $x_0 \in M$ admits an open neighborhood $x_0 \in U \subset M$ such that $\Psi : U \to M^{2n}$ is an injective map. Therefore, through these notes, we shall assume that $\Psi : M \to M^{2N}$ is an *injective* immersion, rather than restrict the discussion to certain open sets (which would complicate the notation). Let $\Psi^{-1}T(M^{2N}) \to M$ be the pullback bundle [the pullback of the tangent bundle $T(M^{2n}) \to M^{2n}$ by Ψ], i.e., $\left(\Psi^{-1}T(M^{2n})\right)_x = T_{\Psi(x)}(M^{2n})$ for any $x \in M$. For each tangent vector field $X \in \mathfrak{X}(M)$, we denote by $\Psi_* X$ the section in the pullback bundle is given by

$$(\Psi_* X)(x) = (d_x\Psi)X_x \in T_{\Psi(x)}(M^{2n}), \quad x \in M.$$

Let $E(\Psi) \to M$ be the normal bundle of the given immersion, so that

$$T_{\Psi(x)}(M^{2n}) = [(d_x\Psi)T_x(M)] \oplus E(\Psi)_x, \quad x \in M.$$

Let d be the codimension of M in M^{2N}, so that $\dim_{\mathbb{R}} E(\Psi)_x = d$ for any $x \in M$, and assume that $d \geq k$. Let $g = \Psi^* G$ be the *first fundamental form* of Ψ.

Definition 1 A pair $(M, \mathcal{D})$ consisting of a manifold M and a C^∞ distribution $\mathcal{D}$ of real rank $2n$ on M is a *CR submanifold* of type (n, k) of M^{2n} if (i) $\mathcal{D}$ is J-invariant, i.e., $J_{\Psi(x)}(d_x\Psi)\,\mathcal{D}_x = (d_x\Psi)\mathcal{D}_x$ for any $x \in M$, and (ii) the orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$ in $(T(M),\ g)$ is J-anti-invariant, i.e., $J_{\Psi(x)}(d_x\Psi)\,\mathcal{D}_x^\perp \subset E(\Psi)_x$ for any $x \in M$. The integers n and k are the *CR dimension* and *CR codimension* of $(M, \mathcal{D})$ respectively. □

The notion of CR submanifold was introduced by A. Bejancu, [5], in an attempt to unify the notions of complex, totally real, and generic submanifolds of a Hermitian manifold. Let ∇^g and ∇^G be the Levi-Civita connections of the Riemannian manifolds (M, g) and (M^{2N}, G) and let $\Psi^{-1}\nabla^G \equiv \left(\nabla^G\right)^\Psi$ be the pullback connection [the pullback of ∇^G by Ψ, a connection in the pullback bundle $\Psi^{-1}T(M^{2N}) \to M$]. The *second fundamental form* of Ψ is

$$h(X, Y) = \left(\nabla^G\right)^\Psi_X \Psi_* Y - \Psi_* \nabla^g_X Y, \quad X, Y \in \mathfrak{X}(M).$$

The main purpose of A.Bejancu's work (cf. *op. cit.*) is essentially confined to the theory of isometric immersions among Riemannian manifolds, i.e., the study of the geometry of the second fundamental form h of Ψ. The theory of CR manifolds is in turn older (cf. e.g., S. Greenfield, [17]) and the interconnection between the two theories was observed somewhat later by D.E. Blair and B-Y. Chen (cf. [11]).

Definition 2 A complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ of the complexified tangent bundle, of complex rank n, is a *CR structure* on M if (i) $T_{1,0}(M)_x \cap T_{0,1}(M)_x = (0)$ for any $x \in M$, and (ii) if $Z, W \in C^\infty(U, T_{1,0}(M))$ then $[Z, W] \in C^\infty(U, T_{1,0}(M))$ for any open set $U \subset M$. Here $T_{0,1}(M) = \overline{T_{1,0}(M)}$ and overbars denote complex conjugation. A pair $(M, T_{0,1}(M))$ consisting of a real $(2n+k)$-dimensional manifold M and a CR structure $T_{1,0}(M)$ on M is a *CR manifold*. The integers n and k are the *CR dimension* and *CR codimension*, while (n, k) is the *type* of the CR manifold. □

Given two CR manifolds $(M, T_{1,0}(M))$ and $(N, T_{1,0}(N))$, a C^∞ map $\phi : M \to N$ is a *CR map* if

$$(d_x\phi)T_{1,0}(M)_x \subset T_{1,0}(N)_{\phi(x)}, \quad x \in M.$$

A *CR isomorphism* is a C^∞ diffeomorphism and a CR map. CR manifolds and CR maps form a small category (the *CR category*).

Let $(M, \mathcal{D})$ be a CR submanifold of the Hermitian manifold M^{2N} endowed with the Hermitian structure (J, G). Let $J_M : \mathcal{D} \to \mathcal{D}$ be induced by J, i.e.,

$$(d_x\Psi)J_{M,x}v = J_{\Psi(x)}(d_x\Psi)v, \quad v \in \mathcal{D}_x, \quad x \in M.$$

Let $J_M^{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of J_M to $\mathcal{D} \otimes \mathbb{C}$. As $J_M^2 = -I$ one has $\left(J_M^{\mathbb{C}}\right)^2 = -I$ hence $\text{Spec}\left(J_M^{\mathbb{C}}\right) = \{\pm i\}$ (with $i = \sqrt{-1}$). Next let us set $T_{1,0}(M)_x = \text{Eigen}\left(J_x^{\mathbb{C}};\ i\right)$ for any $x \in M$. By a result of D.E. Blair and B-Y. Chen (cf. [11]) $T_{1,0}(M)$ is a CR structure on M, of CR dimension n, so that $(M, T_{1,0}(M))$ is a CR manifold, of type (n, k). CR structures such as in Definition 2 above are referred to as *abstract*, while CR structures occurring (via D.E. Blair and B-Y. Chen's result) on a CR submanifold of a Hermitian manifold are referred to as *embedded*.[1]

4.1.2 CR Functions

CR structures, abstract or embedded, are a bundle theoretic recast of tangential Cauchy–Riemann equations.

Definition 3 The first-order differential operator

$$\overline{\partial}_M : C^1(M, \mathbb{C}) \to C(T_{0,1}(M)^*),$$

$$(\overline{\partial}_M u)\overline{Z} = \overline{Z}(u), \quad u \in C^1(M, \mathbb{C}), \quad Z \in T_{1,0}(M),$$

is the *tangential Cauchy–Riemann* operator. Also

$$\overline{\partial}_M u = 0 \tag{4.1}$$

[1]The term is often reserved for CR submanifolds of $\mathbb{C}^N$ for some $N \geq 2$.

are the *tangential Cauchy–Riemann equations* and a C^1 solution to (4.1) is a *CR function* on M. □

Weakly CR functions may be introduced as follows. The *divergence* of a C^1 vector field X on M is defined by $\mathcal{L}_X \, d\, v_g = \mathrm{div}(X)\, d\, v_g$ where $\mathcal{L}_X$ is the Lie derivative at the direction X and $d\, v_g$ is the volume form associated to the induced metric $g = \Psi^* G$. The *formal adjoint* of X is given by $X^*(v) = -X(v) - v\,\mathrm{div}(X)$ for any $v \in C^1_0(M)$. A function $u \in L^1_{\mathrm{loc}}(M)$ is said to be a *weakly CR function* if

$$\int_M u\, \overline{Z}^*(\varphi)\, d\, v_g = 0 \tag{4.2}$$

for any $\varphi \in C^\infty_0(M)$ and any $Z \in C^\infty(T_{1,0}(M))$. Integration by parts shows that any weakly CR function $u \in C^1(M, \mathbb{C})$ is also a CR function in the sense of Definition 3 (a *strongly CR* function).

Let $T^{1,0}(M^{2N}) \subset T(M^{2N}) \otimes \mathbb{C}$ be the holomorphic tangent bundle over M^{2N}. If $(V,\, Z^1, \ldots, Z^N)$ is a local system of complex coordinates on M^{2N} then $T^{1,0}(M^{2N})_x$ is the span of $\left\{\left(\partial/\partial Z^j\right)_x : 1 \le j \le N\right\}$ over $\mathbb{C}$, for any $x \in V$. Let

$$\overline{\partial} : C^1(M^{2N}, \mathbb{C}) \to C(T^{1,0}(M^{2N})^*),$$

$$(\overline{\partial} f)\overline{Z} = \overline{Z}(f), \quad f \in C^1(M^{2N}, \mathbb{C}), \quad Z \in T^{1,0}(M^{2N}),$$

be the Cauchy–Riemann operator on M^{2N}. A C^1 function $f : M^{2N} \to \mathbb{C}$ such that $\overline{\partial} f = 0$ is *holomorphic*. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, defined on the open subset $\Omega \subset M^{2N}$ such that $\Omega \cap \Psi(M) \neq \emptyset$. Let us assume that $\Psi : M \to M^{2N}$ is an embedding, i.e., for any open set $\Omega \subset M^{2N}$ there is an open set $U \subset M$ such that $\Omega \cap \Psi(M) = \Psi(U)$. Then $u : U \to \mathbb{C}$, $u(x) = f(\Psi(x))$, $x \in U$, is a CR function on U. Indeed $(d_x\Psi)T_{1,0}(M)_x \subset T^{1,0}(M^{2N})_{\Psi(x)}$, $x \in M$, hence for any $Z \in C^\infty(T_{1,0}(M))$

$$\overline{Z}(u)_x = \overline{Z}_x(f \circ \Psi) = \left[(d_x\Psi)\overline{Z}_x\right](f) = 0.$$

A fundamental problem in complex analysis is whether CR functions extend to holomorphic functions, at least locally. Let $\mathrm{CR}^1(M)$ denote the space of all CR functions $u : M \to \mathbb{C}$ of class C^1. Let $u \in \mathrm{CR}^1(M)$. The problem is then, given $x_0 \in M$, whether an open set $\Omega \subset M^{2N}$ exists such that $\Psi(x_0) \in \Omega \cap \Psi(M)$ and there is a holomorphic function $f \in \mathcal{O}(\Omega)$ such that $f \circ \Psi = u$ on U [where $U \subset M$ is an open set such that $\Psi(U) = \Omega \cap \Psi(M)$]. Open subsets $U \subset M$ of a CR manifold are CR manifolds, so the extension problem makes sense for each $u \in \mathrm{CR}^1(U)$. The *CR extension problem* is however to *simultaneously* extend all CR functions $u \in \mathrm{CR}^1(M)$, at least locally. Let $r : \mathcal{O}(\Omega) \to \mathrm{CR}^1(U)$ be the restriction map, i.e., $r(f) = f \circ \Psi$ for any $f \in \mathcal{O}(\Omega)$. A formal recast of the CR extension problem is, given $x_0 \in M$, whether an open set $\Omega \subset M^{2N}$ as above exists such that $r : \mathcal{O}(\Omega) \to \mathrm{CR}^1(U)$ is surjective.

The two decades after the publishing of [5] registered an impressive amount of work devoted to the differential geometric properties (as understood within Riemannian geometry) of CR submanifolds, mainly in Kählerian manifolds yet including contact analogs (cf. [29]) such as CR submanifolds of Sasakian manifolds. The progress in this direction up to 1982 is reported on in [30]. Findings up to 1986 are described in [7]. With the advance of locally conformal Kähler geometry (cf. L. Ornea et al., [13]), the study of CR submanifolds in a l.c.K. manifold (e.g., a complex Hopf manifold with the Boothby metric) has known a similar impetus (cf. E. Barletta, [3], S. Dragomir, [12, 13], pp. 147–275). The CR extension problem is at least 20 years older[2] and the first relevant results go back to the work by H. Lewy, [19]. The main findings are described by A. Boggess (cf. [10]) and are confined to CR submanifolds of $\mathbb{C}^N$. No tentative to study the interrelation among the two arguments was performed, except for the mild attempt in [14] to exhibit the relationship between pseudohermitian geometry (as built by S.M. Webster, [26]) and contact geometry (as a segment of Riemannian geometry, cf. e.g., D.E. Blair, [8]).

4.2 Real Hypersurfaces

4.2.1 Oriented Real Hypersurfaces

Let $M \subset M^{2(n+1)}$ be a real hypersurface, i.e., the inclusion $j : M \to M^{2N}$ is an immersion of codimension $d = 1$. Then

$$T_{1,0}(M)_x = [T_x(M) \otimes_{\mathbb{R}} \mathbb{C}] \cap T^{1,0}(M^{2(n+1)})_x, \quad x \in M, \tag{4.3}$$

is a CR structure on M, of CR dimension n and CR codimension $k = 1$. Hence $(M, T_{1,0}(M))$ is a CR manifold of type $(n, 1)$. Let us assume that M is orientable and choose a unit normal field $N \in C^\infty(E(j))$, i.e.,

$$G(X, N) = 0, \quad G(N, N) = 1, \quad X \in \mathfrak{X}(M).$$

Let $\xi = -JN$. Then $\xi \in \mathfrak{X}(M)$. Indeed

$$G(\xi, N) = -G(JN, N) = 0$$

as $\Omega(V, W) = G(V, JW)$ is a differential 2-form on $M^{2(n+1)}$. Let $\eta \in \Omega^1(M)$ be the real differential 1-form on M given by

$$\eta(X) = g(X, \xi), \quad X \in \mathfrak{X}(M).$$

[2]The first appearance of tangential Cauchy–Riemann equations goes back to the 1907 paper by H. Poincaré, [20]. Cf. also [27], p. 189.

If $\mathcal{D} = \text{Ker}(\eta)$ then $(M, \mathcal{D})$ is a CR submanifold of the Hermitian manifold $M^{2(n+1)}$. Indeed if $X \in \mathcal{D}$ then

$$G(JX, N) = -G(X, JN) = g(X, \xi) = \eta(X) = 0$$

implies that JX is tangent to M. On the other hand

$$\eta(JX) = g(JX, \xi) = -G(JX, JN) = -G(X, N) = 0$$

so that $JX \in \mathcal{D}$ as well. Therefore $\mathcal{D}$ is J-invariant. Next if $\mathcal{D}^\perp$ is the orthogonal complement of $\mathcal{D}$ in $(T(M), g)$ then $\mathcal{D}^\perp$ is the span of ξ hence for any $Y \in \mathcal{D}^\perp$ there is $f \in C^\infty(M)$ such that $Y = f\xi$. Finally for any $V \in \mathfrak{X}(M)$

$$G(JY, V) = f\, G(J\xi, V) = f\, G(N, V) = 0$$

so that $J(\mathcal{D}^\perp) \subset E(j)$. In particular, if $J_M : \mathcal{D} \to \mathcal{D}$ is the restriction of J to $\mathcal{D} = \text{Ker}(\eta)$, then (by D.E. Blair and B-Y. Chen's theorem) the eigenbundle $\text{Eigen}(J_M^{\mathbb{C}}; i)$ is a CR structure on M coinciding with (4.3).

4.2.2 Boundary of Siegel Domain

For instance, let

$$\Omega_{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \text{Im}(w) > |z|^2\}$$

be the *Siegel domain* in $\mathbb{C}^{n+1}$. Its boundary $\partial\Omega_{n+1}$ is an orientable real hypersurface in $\mathbb{C}^{n+1}$ hence may be organized as a CR submanifold of $(\mathbb{C}^{n+1}, J_0, G_0)$ and as a CR manifold of type $(n, 1)$. Here J_0 and G_0 are the canonical complex and (flat) Riemannian structures of $\mathbb{C}^{n+1}$. Let us consider the Dirichlet problem for the Cauchy–Riemann equations on the Siegel domain

$$\overline{\partial} f = 0 \quad \text{in} \quad \Omega_{n+1}, \tag{4.4}$$

$$f = u \quad \text{on} \quad \partial\Omega_{n+1}, \tag{4.5}$$

for a given function $u \in C^\infty(\partial\Omega_{n+1}, \mathbb{C})$. We shall prove the following

Theorem 4.1 *Assume that the Dirichlet problem* (4.4) and (4.5) *admits a solution f smooth up to the boundary, i.e., $f \in C^\infty(\overline{\Omega}_{n+1}, \mathbb{C})$. Then u is a CR function on $M = \partial\Omega_{n+1}$, i.e., $\overline{\partial}_M u = 0$.*

Proof A complex tangent vector field of type $(1, 0)$

$$Z = \lambda^\alpha \frac{\partial}{\partial z^\alpha} + \mu \frac{\partial}{\partial w} \in C^\infty(T^{1,0}(\mathbb{C}^{n+1}))$$

is tangent to $M = \partial\Omega_{n+1}$ if $Z(\rho) = 0$ where $\rho(z, w) = |z|^2 - \text{Im}(w)$ [so that $\Omega_{n+1} = \{(z, w) \in \mathbb{C}^{n+1} : \rho(z, w) < 0\}$]. Hence $Z \in C^\infty(T_{1,0}(M))$ if $\mu = 2i\,\overline{z}^\alpha \lambda_\alpha$ where $\lambda_\alpha = \lambda^\alpha$. Consequently, the CR structure $T_{1,0}(M)$ is (globally) the span of $\{Z_\alpha : 1 \leq \alpha \leq n\}$ where

$$(d_x j) Z_{\alpha,x} = \left(\frac{\partial}{\partial z^\alpha} + 2i\,\overline{z}^\alpha \frac{\partial}{\partial w} \right)_x, \quad 1 \leq \alpha \leq n,$$

for any $x = (z, w) \in M$. Here $j : M \to \mathbb{C}^{n+1}$ is the inclusion. As $f \in \mathcal{O}(\Omega_{n+1})$ one has

$$\frac{\partial f}{\partial \overline{z}^\alpha}(z, w) = 0, \quad \frac{\partial f}{\partial \overline{w}}(z, w) = 0, \quad (z, w) \in \Omega_{n+1},$$

hence

$$\frac{\partial f}{\partial \overline{z}^\alpha} - 2i\, z^\alpha \frac{\partial f}{\partial \overline{w}} = 0 \quad \text{in} \quad \Omega_{n+1}.$$

Yet f is C^∞ up to the boundary which means that f and its derivatives of any order stay bounded at the boundary. As known from calculus in several real variables, for domains with smooth boundary such as Ω_{n+1}, smoothness up to the boundary is equivalent to the existence of an open set $U \subset \mathbb{C}^{n+1}$ and of a function $F \in C^\infty(U)$ such that $\overline{\Omega}_{n+1} \subset U$ and $F|_{\overline{\Omega}_{n+1}} = f$. In particular $F \circ j = u$. So

$$\frac{\partial F}{\partial \overline{z}^\alpha}(z, w) - 2i\, z^\alpha \frac{\partial F}{\partial \overline{w}}(z, w) = 0$$

for any $(z, w) \in \Omega_{n+1}$. Let $(z_0, w_0) \in \partial\Omega_{n+1}$ and let $\{(z_\nu, w_\nu)\}_{\nu \geq 1}$ be a sequence of points in Ω_{n+1} such that $(z_\nu, w_\nu) \to (z_0, w_0)$ as $\nu \to \infty$. Then, as $\partial F/\partial \overline{z}^j$ and $\partial F/\partial \overline{w}$ are continuous functions

$$\begin{aligned} 0 &= \lim_{\nu \to \infty} \left[\frac{\partial F}{\partial \overline{z}^\alpha}(z_\nu, w_\nu) - 2i\, z_\nu^\alpha \frac{\partial F}{\partial \overline{w}}(z_\nu, w_\nu) \right] \\ &= \frac{\partial F}{\partial \overline{z}^\alpha}(z_0, w_0) - 2i\, z_0^\alpha \frac{\partial F}{\partial \overline{w}}(z_0, w_0) \\ &= \left[(d_{(z_0,w_0)} j) \overline{Z}_{\alpha,(z_0,w_0)} \right](F) = \overline{Z}_{\alpha,(z_0,w_0)}(F \circ j) \overline{Z}_{\alpha,(z_0,w_0)}(u). \end{aligned}$$

Q.e.d.

Theorem 4.2 *Let $M = \partial\Omega_{n+1}$ be the boundary of the Siegel domain. The tangent bundle $T(M)$ is the span of $\{X_i, Y_i, \partial/\partial u : 1 \leq i \leq n\}$ where*

$$X_i = \frac{\partial}{\partial x^i} + 2x_i \frac{\partial}{\partial v}, \quad Y_i = \frac{\partial}{\partial y^i} + 2y_i \frac{\partial}{\partial v}, \quad 1 \leq i \leq n,$$

and $z^i = x^i + \sqrt{-1}\, y^i$, $w = u + \sqrt{-1}\, v$. *Consequently,*

$$N = \frac{1}{\sqrt{1+4|z|^2}} \left(\frac{\partial}{\partial v} - 2x^i \frac{\partial}{\partial x^i} - 2y^i \frac{\partial}{\partial y^i} \right)$$

is a unit normal vector field on M. *The first fundamental form* $g = j^* G_0$, *the induced connection* ∇, *and the second fundamental form* h *of the given immersion* $j : M \subset \mathbb{C}^{n+1}$ *are*

$$g : \begin{pmatrix} \delta_{ik} + 4x_i x_k & 4x_i y_k & 0 \\ 4y_i x_k & \delta_{ik} + 4y_i y_k & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\nabla_{X_i} X_k = \nabla_{Y_i} Y_k = \frac{4\delta_{ik}}{1+4|z|^2} \left(x^\ell X_\ell + y^\ell Y_\ell \right), \tag{4.6}$$

$$\begin{aligned} \nabla_{X_i} Y_k &= \nabla_{Y_i} X_k = \nabla_{X_i} \frac{\partial}{\partial u} = \nabla_{Y_i} \frac{\partial}{\partial u} \\ &= \nabla_{\partial/\partial u} X_k = \nabla_{\partial/\partial u} Y_k = \nabla_{\partial/\partial u} \frac{\partial}{\partial u} = 0, \qquad (4.7) \\ h(X_i,\, X_k) &= h(Y_i,\, Y_k) = \frac{2\,\delta_{ik}}{\sqrt{1+4|z|^2}} N, \\ h(X_i,\, \partial/\partial u) &= h(Y_i, \partial/\partial u) = h(\partial/\partial u,\, \partial/\partial u) = 0, \end{aligned}$$

hence j *is not totally geodesic. In particular, the mean curvature vector* $H = \mathrm{Trace}(h)$ *is*

$$H = \frac{2}{\sqrt{1+4|z|^2}} \left[\frac{1}{1+4|z|^2} + 2n - 1 \right] N$$

hence j *is not minimal. The Weingarten operator* A_N *is*

$$\begin{pmatrix} \frac{2}{\sqrt{1+4|z|^2}} \left(\delta_i^k - \frac{4x_i x^k}{1+4|z|^2} \right) & -\frac{8x_i y^k}{\left(1+4|z|^2\right)^{3/2}} & 0 \\ -\frac{8y_i x^k}{\left(1+4|z|^2\right)^{3/2}} & \frac{2}{\sqrt{1+4|z|^2}} \left(\delta_i^k - \frac{4y_i y^k}{1+4|z|^2} \right) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hence j *is not totally umbilical.*

Proof Taking into account the direct sum decomposition

$$j^{-1} T(\mathbb{C}^{n+1}) = T(M) \oplus E(j)$$

the tangent vector fields $\{\partial/\partial x^i,\ \partial/\partial y^i,\ \partial/\partial v\}$ decompose (into tangential and normal components) as

$$\frac{\partial}{\partial x^i} = \left(\delta_i^k - \frac{4x_i x^k}{1+4|z|^2}\right) X_k - \frac{4x_i y^k}{1+4|z|^2} Y_k - \frac{2x_i}{\sqrt{1+4|z|^2}} N,$$
$$\frac{\partial}{\partial y^j} = -\frac{4y_i x^k}{1+4|z|^2} x_k + \left(\delta_i^k - \frac{4y_i y^k}{1+4|z|^2}\right) Y_k - \frac{2y_i}{\sqrt{1+4|z|^2}} N,$$
$$\frac{\partial}{\partial v} = \frac{2x^k}{1+4|z|^2} X_k + \frac{2y^k}{1+4|z|^2} Y_k + \frac{1}{\sqrt{1+4|z|^2}} N,$$

for any $1 \le i \le n$. Let G_0 be the Euclidean metric on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ and let ∇^0 be its Levi-Civita connection. The Gauss–Weingarten formulae

$$\nabla^0_X Y = \nabla_X Y + h(X,Y), \quad \nabla_X N = -A_N X \quad X, Y \in \mathfrak{X}(M),$$

yield Theorem 4.2. Q.e.d.

Theorem 4.1 exhibits CR functions (on the quadric $\partial\Omega_{n+1}$ of equation $v = |z|^2$ in $\mathbb{C}^{n+1}$) as boundary values of holomorphic functions on the Siegel domain. Theorem 4.2 shows that an exhaustive study of the geometry of the second fundamental form (of the isometric immersion $\partial\Omega_{n+1} \hookrightarrow \mathbb{C}^{n+1}$) may be performed by means of the embedding equations (here Gauss and Weingarten formulae). Can the two be made to merge into a unifying theory?

4.3 Levi Form

4.3.1 Vector-Valued Levi Form

Let $(M, T_{1,0}(M))$ be an orientable (abstract) CR manifold, of type (n,k). The *Levi distribution* is

$$\mathcal{D} = \mathrm{Re}\left\{T_{1,0}(M) \oplus T_{0,1}(M)\right\}.$$

It is a real rank $2n$ subbundle of $T(M)$. The integrability property of $T_{1,0}(M)$ [cf. Definition 2] is equivalent to the requirement

$$X, Y \in \mathcal{D} \Longrightarrow [J_M X,\ Y] + [X,\ J_M Y] \in \mathcal{D}, \tag{4.8}$$

$$[J_M X,\ J_M Y] - [X,\ Y] = J_M \{[J_M X,\ Y] + [X,\ J_M Y]\}. \tag{4.9}$$

For each $x \in M$, let

$$\pi_x : T_x(M) \otimes_{\mathbb{R}} \mathbb{C} \to \frac{T_x(M) \otimes_{\mathbb{R}} \mathbb{C}}{\mathcal{D}_x \otimes_{\mathbb{R}} \mathbb{C}}$$

be the canonical projection. The *Levi form* is

$$L_x : T_{1,0}(M)_x \times T_{0,1}(M)_x \to \frac{T_x(M) \otimes_{\mathbb{R}} \mathbb{C}}{\mathcal{D}_x \otimes_{\mathbb{R}} \mathbb{C}},$$

$$L_x(v, \overline{w}) = \frac{i}{2} \pi_x \left[V, \overline{W}\right]_x, \quad v, w \in T_{1,0}(M)_x, \quad x \in M. \tag{4.10}$$

Here V and W are C^∞ sections in $T_{1,0}(M)$ on M extending v and w, i.e., $V_x = v$ and $W_x = w$. The definition of $L_x(v, \overline{w})$ does not depend upon the choice of extensions of v and w. The CR structure $T_{1,0}(M)$ is *nondegenerate* if L is nondegenerate, i.e., for any $x \in M$ the knowledge that $L_x(v, \overline{w}) = 0$ for any $w \in T_{1,0}(M)_x$ implies $v = 0$. As opposed to the nondegenerate case, the CR structure $T_{1,0}(M)$ is *Levi flat* [and $(M, T_{1,0}(M))$ is a *Levi-flat* CR manifold] if $L = 0$. For any Levi-flat CR manifold, the distribution $\mathcal{D}$ is involutive, by the very definition of L together with

$$\mathcal{D} \otimes \mathbb{C} = T_{1,0}(M) \oplus T_{0,1}(M).$$

Then (by the classical Frobenius theorem) $\mathcal{D}$ is completely integrable, giving rise to a foliation $\mathcal{F}$ of M whose leaves are the maximal integral manifolds of $\mathcal{D}$. For each $x_0 \in M$, let $S \in M/\mathcal{F}$ be the unique leaf of $\mathcal{F}$ (the maximal integral manifold of $\mathcal{D}$) passing through x_0. Then $\mathcal{D}_x = T_x(S)$ for any $x \in S$ so that J_M descends to an almost complex structure J_S on S

$$J_{S,x} v = J_{M,x} v, \quad v \in T_x(S), \quad x \in S.$$

The integrability property of $T_{1,0}(M)$ [in its form (4.8) and (4.9)] implies that $N_{J_S} = 0$, i.e., J_S is integrable. By Newlander–Nirenberg's theorem, S admits a complex manifold structure inducing the almost complex structure J_S. Thus $\mathcal{F}$ is a foliation of M by complex n-dimensional manifolds, the *Levi foliation* of M. Levi foliations are discussed at some extent in [4].

4.3.2 Scalar Levi Form

Let us assume from now on that $k = 1$, i.e., $T_{1,0}(M)$ has CR codimension 1. Let $E \subset T^*(M)$ be the conormal bundle associated to $\mathcal{D}$, i.e.,

$$E_x = \left\{\omega \in T_x^*(M) : \mathrm{Ker}(\omega) \supseteq \mathcal{D}_x\right\}, \quad x \in M.$$

Then $E \to M$ is a real line bundle, isomorphic to the quotient $T(M)/\mathcal{D}$. As M is orientable, one may fix an orientation of M, so that its tangent bundle is an oriented vector bundle. Also $\mathcal{D}$ is oriented by its complex structure hence the quotient $T(M)/\mathcal{D}$, and then the conormal bundle E, is an oriented bundle. Any oriented line bundle over a connected manifold is trivial, so that $E \approx M \times \mathbb{R}$ (a vector bundle isomorphism). Hence, globally defined nowhere zero sections $\theta \in C^\infty(E)$ do exist and are referred to as *pseudohermitian structures* on M (by adopting a terminology due to S.M. Webster, [26]). Cf. also [16] for an extensive treatment of pseudohermitian geometry. Let $\mathcal{P}$ be the linear space of all pseudohermitian structures on M. Note that $\eta \in \mathcal{P}$. Given a pseudohermitian structure $\theta \in \mathcal{P}$, the *Levi form* G_θ is

$$G_\theta(X, Y) = (d\theta)(X, J_M Y), \quad X, Y \in \mathcal{D}.$$

If $\hat{\theta} \in \mathcal{P}$ is another pseudohermitian structure, then $\hat{\theta} = \lambda\,\theta$ for some C^∞ function $\lambda : M \to \mathbb{R}\backslash\{0\}$. In particular $d\hat{\theta} = d\lambda \wedge \theta + \lambda\, d\theta$ hence $G_{\hat{\theta}} = \lambda\, G_\theta$. In particular, the CR structure $T_{1,0}(M)$ is nondegenerate if and only if G_θ is nondegenrate for some $\theta \in \mathcal{P}$ (and thus for all). If $T_{1,0}(M)$ is nondegenerate, then each $\theta \in \mathcal{P}$ is a *contact form*, i.e., $\theta \wedge (d\theta)^n$ is a *volume form* (a nowhere zero top degree form) on M. An equivalent approach, frequently adopted in the mathematical literature devoted to CR geometry, is to define the Levi form as

$$L_\theta(V, \overline{W}) = -i\,(d\theta)(V, \overline{W}), \quad V, W \in T_{1,0}(M). \tag{4.11}$$

Indeed L_θ coincides with the $\mathbb{C}$-linear extension of G_θ to $T_{1,0}(M) \otimes T_{0,1}(M)$. For any fixed $\theta \in \mathcal{P}$, the bundle map

$$\frac{T_x(M) \otimes_{\mathbb{R}} \mathbb{C}}{\mathcal{D}_x \otimes_{\mathbb{R}} \mathbb{C}} \xrightarrow{\Phi_\theta} E_x \otimes_{\mathbb{R}} \mathbb{C}, \quad \Phi_{\theta,x} : v + \mathcal{D}_x \otimes_{\mathbb{R}} \mathbb{C} \longmapsto \theta_x(v)\, v, \quad x \in M,$$

is an isomorphism of complex vector bundles. Also

$$\begin{aligned}\Phi_{\theta,x}\left(L_x(v, \overline{w})\right) &= \frac{i}{2}\,\Phi_{\theta,x}\left(\pi_x\left[V, \overline{W}\right]_x\right) = \frac{i}{2}\,\theta_x\left(\left[V, \overline{W}\right]_x\right)\,\theta_x \\ &= -i\,(d\theta)_x(v, \overline{w}) = L_{\theta,x}(v, \overline{w})\,\theta_x\end{aligned}$$

and the two approaches to the Levi form [as a vector-valued form (4.10) or as a scalar form (4.11)] are seen to coincide (for CR codimension 1).

A CR structure $T_{1,0}(M)$ of CR codimension $k = 1$ is said to be *strictly pseudoconvex* if G_θ is positive definite [i.e., $G_{\theta,x}(v, v) > 0$ for any $v \in \mathcal{D}_x\backslash\{0\}$ and any $x \in M$] for some $\theta \in \mathcal{P}$. Clearly strict pseudoconvexity implies nondegeneracy. If $T_{1,0}(M)$ is strictly pseudoconvex, then $\mathcal{P}$ admits a natural orientation $\mathcal{P}_+$ consisting of all $\theta \in \mathcal{P}$ such that G_θ is positive definite. A contact form $\theta \in \mathcal{P}_+$ is said to be *positively oriented*.

Real Hypersurfaces in Kählerian Manifolds

The Levi form is related to the second fundamental form of the given immersion, as follows. Let $M \subset M^{2(n+1)}$ be an orientable real hypersurface in the Kählerian manifold $M^{2(n+1)}$, endowed with the complex structure J and the Kählerian metric G. We need the Gauss and Weingarten formulae

$$\nabla^G_X Y = \nabla^g_X Y + h(X, Y), \tag{4.12}$$

$$\nabla^G_X V = -A_V X + \nabla^\perp_X V, \tag{4.13}$$

for any $X, Y \in \mathfrak{X}(M)$ and any $V \in C^\infty(E(j))$. Here h and A are, respectively, the second fundamental form and shape (or Weingarten) operator of $j : M \subset M^{2(n+1)}$, while $\nabla^\perp \in \mathcal{C}(E(j))$ is the normal connection. For any $X, Y \in \mathcal{D}$

$$\begin{aligned} 2\, G_\eta(X, Y) &= 2\,(d\eta)(X, JY) = -\eta([X, JY]) \\ &= -g(\xi,\ [X, JY]) = G(JN,\ [X, JY]) = -G(N,\ J[X, JY]) \end{aligned}$$

(as ∇^G is torsion-free)

$$= -G(N,\ J\left\{\nabla^G_X JY - \nabla^G_{JY} X\right\})$$

(by $\nabla^G J = 0$ and $J^2 = -I$)

$$= G(N,\ \nabla^G_X Y + \nabla^G_{JY} JX)$$

(by Gauss formula)

$$= G(N,\ h(X, Y)) + G(N, h(JY, JX))$$

yielding (as h is symmetric)

$$G_\eta(X, Y)N = \frac{1}{2}\,\{h(X, Y) + h(JX, JY)\} \tag{4.14}$$

for any $X, Y \in C^\infty(\mathcal{D})$. A few properties of the second fundamental form, useful in the sequel, themselves consequences of the Kähler condition $\nabla^G J = 0$, may be derived as follows. Let $(M, \mathcal{D})$ be a CR submanifold of the Kählerian manifold (M^{2N}, G, J). Assume for simplicity that the given immersion is the inclusion $j : M \subset M^{2N}$. We set

$$PX = \tan(JX), \quad F = \operatorname{nor}(JX), \quad tV = \tan(JV), \quad fV = \operatorname{nor}(JV),$$

for any $X \in \mathfrak{X}(M)$ and $V \in C^\infty(E(j))$, where $\tan_x : T_x(M^{2(n+1)}) \to T_x(M)$ and $\operatorname{nor}_x : T_x(M^{2(n+1)}) \to E(j)_x$ are the projections associated to the direct sum decomposition

$$T_x(M^{2(n+1)}) = T_x(M) \oplus E(j)_x, \quad x \in M.$$

Then for any $X, Y \in C^\infty(\mathcal{D})$ (by Gauss formula)

$$\nabla^G_X JY = \nabla^g_X JY + h(X, JY),$$

$$\nabla^G_X JY = J\nabla^G_X Y = J\,\nabla^g_X Y + J\,h(X, Y),$$

hence

$$\nabla^g_X JY = P\,\nabla^g_X Y + t\,h(X, Y), \tag{4.15}$$

$$h(X, JY) = F\,\nabla^g_X Y + f\,h(X, Y). \tag{4.16}$$

As a corollary of (4.16)

$$h(X, JY) - h(JX, Y) = F[X,\ Y] \tag{4.17}$$

for any $X, Y \in C^\infty(\mathcal{D})$. For the boundary $M = \partial\Omega_{n+1}$ of the Siegel domain, the tangent vector field $\xi = -J_0 N$ is given by

$$\begin{aligned}\xi &= \frac{1}{\sqrt{1+4|z|^2}}\left(\frac{\partial}{\partial u} + 2x^k\,\frac{\partial}{\partial y^k} - 2y^k\,\frac{\partial}{\partial x^k}\right)\\ &= \frac{1}{\sqrt{1+4|z|^2}}\left(\frac{\partial}{\partial u} + 2x^k\,Y_k - 2y^k\,X_k\right)\end{aligned}$$

where $\{X_k,\ Y_k,\ \partial/\partial u\}$ is the frame in Theorem 4.2. When looking at the Levi distribution it is convenient to use the adapted frame $\{E_a,\ \xi : 1 \le a \le 2n\}$ where

$$\overline{Z}_\alpha = \frac{1}{2}\,(E_\alpha + i\,E_{\alpha+n})\,, \quad 1 \le \alpha \le n,$$

are the real and imaginary parts of the Lewy operators, i.e.,

$$E_\alpha = \frac{\partial}{\partial x^\alpha} + 2y^\alpha\,\frac{\partial}{\partial u} + 2x^\alpha\,\frac{\partial}{\partial v}, \quad E_{\alpha+n} = \frac{\partial}{\partial y^\alpha} - 2x^\alpha\,\frac{\partial}{\partial u} + 2y^\alpha\,\frac{\partial}{\partial v},$$

or

$$\begin{aligned}&(E_1, \dots, E_n,\ E_{n+1}, \dots, E_{2n},\ \xi)\\ &\quad = a\ (X_1, \dots, X_n,\ Y_1, \dots, Y_n,\ \partial/\partial u)\end{aligned}$$

with $a : M \to \mathrm{GL}(2n+1, \mathbb{R})$ given by

$$a = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \\ -\frac{2y}{\sqrt{1+4|z|^2}} & \frac{2x}{\sqrt{1+4|z|^2}} & \frac{1}{\sqrt{1+4|z|^2}} \end{pmatrix}.$$

Then (by Theorem 4.2)

$$h(E_\alpha,\, E_\beta) = h(E_{\alpha+n},\, E_{\beta+n}) = \frac{2\delta_{\alpha\beta}}{\sqrt{1+4|z|^2}}\, N,$$
$$h(E_\alpha,\, \xi) = -\frac{4y_\alpha}{1+4|z|^2}\, N, \quad h(E_{\alpha+n},\, \xi) = \frac{4x_\alpha}{1+4|z|^2}\, N,$$
$$h(\xi, \xi) = \frac{8|z|^2}{\left(1+4|z|^2\right)^{3/2}}\, N,$$

so that [by (4.14) and $J_M E_\alpha = E_{\alpha+n}$]

$$G_\eta(E_\alpha,\, E_\beta) = G_\eta(E_{\alpha+n},\, E_{\beta+n}) = \frac{2\delta_{\alpha\beta}}{\sqrt{1+4|z|^2}} \tag{4.18}$$

and [by (4.17) and (4.6)–(4.7) in the proof of Theorem 4.2]

$$G_\eta(E_\alpha,\, E_{\beta+n})\, N = \frac{1}{2}\left\{h(E_\alpha,\, J_M E_\beta) - h(J_M E_\alpha,\, E_\beta)\right\}$$
$$= F\left(\nabla^g_{E_\alpha} E_\beta - \nabla^g_{E_\beta} E_\alpha\right),$$
$$\nabla_{E_\alpha} E_\beta = \frac{4\delta_{\alpha\beta}}{1+4|z|^2}\left(x^\mu X_\mu + y^\mu Y_\mu\right),$$

so that

$$G_\eta(E_\alpha,\, E_{\beta+n}) = 0. \tag{4.19}$$

In particular, $T_{1,0}(\partial\Omega_{n+1})$ is strictly pseudoconvex. For the choice of contact form $\theta = \sqrt{1+4|z|^2}\,\eta$ one then has

$$G_\theta(E_\alpha,\, E_\beta) = G_\theta(E_{\alpha+n},\, E_{\beta+n}) = 2\,\delta_{\alpha\beta}, \quad G_\theta(E_\alpha,\, E_{\beta+n}) = 0.$$

Real Hypersurfaces in l.c.K. Manifolds

Let M be an oriented real hypersurface of the l.c.K. manifold $M^{2(n+1)}$, carrying the complex structure J and the l.c.K. metric G. Let $X, Y \in \mathcal{D}$. Then [cf. (1.8) in [13], p. 4]

$$\nabla^G_X JY = J\nabla^G_X Y + \frac{1}{2}\{\theta(Y)X - \omega(Y)JX - g(X,Y)A - \Omega(X,Y)B\} \tag{4.20}$$

where ω and θ (respectively B and A) are the *Lee* and *anti-Lee forms* (the *Lee* and *anti-Lee vector fields*) cf., e.g., [13], p. 1–4 for basic definitions in l.c.K. geometry. We recall that $\theta = \omega \circ J$ and $A = -JB$. Let us apply J to (4.20) and take the inner product with the unit normal vector N in the resulting equation. We obtain (by Gauss formula)

$$G(N,\, J\nabla^G_X JY) = -G(N, h(X,Y)) - \frac{1}{2}\{g(X,Y)\,\omega(N) + \Omega(X,Y)\,\omega(\xi)\}. \tag{4.21}$$

Here one made use of

$$G(N, A) = -G(N, JB) = G(JN, B) = -G(\xi, B) = -\omega(\xi).$$

Similarly,

$$\nabla^G_{JY} JX = J\nabla^G_{JY} X - $$

$$-\frac{1}{2}\{\omega(Y)X + \theta(Y)JX + \Omega(X,Y)A - g(X,Y)B\}$$

yields

$$G(N,\, J\nabla^G_{JY} X) = G(N,\, h(JX, JY)) - \frac{1}{2}\{\Omega(X,Y)\omega(\xi) + g(X,Y)\omega(N)\}. \tag{4.22}$$

Finally, for any $X, Y \in \mathcal{D}$

$$2\, G_\eta(X,Y) = -G\left(N,\, J\left\{\nabla^G_X JY - \nabla^G_{JY} X\right\}\right) \quad \text{[by (4.21) and (4.22)]}$$
$$= G(N, h(X,Y)) + G(N, h(JY, JX))$$

leading to (4.14) [as well as for a real hypersurface in a Kählerian manifold].

4.3.3 Extrinsic Levi Form

Let $(M, \mathcal{D})$ be a CR submanifold of the Hermitian manifold M^{2N} endowed with the Hermitian structure (J, G) and let $\Psi : M \to M^{2N}$ be the given immersion. Let $\pi_{\mathcal{D}^\perp} : T(M) \to \mathcal{D}^\perp$ be the projection associated to the decomposition $T(M) = \mathcal{D} \oplus \mathcal{D}^\perp$. It extends (by $\mathbb{C}$-linearity) to a map $T(M) \otimes \mathbb{C} \to \mathcal{D}^\perp \otimes \mathbb{C}$ denoted by the same symbol. Following the conventions in [10], p. 159, let us consider the map

$$\mathcal{L}_x : T_{1,0}(M)_x \to \mathcal{D}_x^{\perp} \otimes_{\mathbb{R}} \mathbb{C}, \quad x \in M,$$

$$\mathcal{L}_x(v) = \frac{1}{2i}\, \pi_{\mathcal{D}^{\perp},\, x} \left[\overline{V},\ V\right]_x, \quad v \in T_{1,0}(M)_x,$$

where $V \in C^{\infty}(T_{1,0}(M))$ is an arbitrary smooth extension of v, i.e., $V_x = v$. Then $\mathcal{L}$ is the *Levi form* of the CR submanifold $(M, \mathcal{D})$ and it is related to the Levi form L of M as a CR manifold, as follows. Let $\Phi : [T(M) \otimes \mathbb{C}]/[\mathcal{D} \otimes \mathbb{C}] \to \mathcal{D}^{\perp} \otimes \mathbb{C}$ be the complex vector bundle isomorphism given by

$$\Phi_x : \frac{T_x(M) \otimes_{\mathbb{R}} \mathbb{C}}{\mathcal{D}_x \otimes_{\mathbb{R}} \mathbb{C}} \to \mathcal{D}_x^{\perp} \otimes_{\mathbb{R}} \mathbb{C}, \quad x \in M,$$

$$\Phi_x : v + \mathcal{D}_x \otimes_{\mathbb{R}} \mathbb{C} \longmapsto \pi_{\mathcal{D}^{\perp},\, x}(v), \quad v \in T_x(M) \otimes_{\mathbb{R}} \mathbb{C}.$$

Then $\Phi \circ \pi = \pi_{\mathcal{D}^{\perp}}$ by the very definitions. Finally,

$$\Phi_x\left(L_x(v, \overline{v})\right) = \frac{1}{2i}\, \Phi_x\, \pi_x \left[\overline{V},\ V\right]_x = \frac{1}{2i} \pi_{\mathcal{D}^{\perp},\, x} \left[\overline{V},\ V\right]_x = \mathcal{L}_x(v)$$

for any $v \in T_{1,0}(M)_x$. The *extrinsic Levi form* of $(M, \mathcal{D})$ is (cf. [10], p. 160)

$$\tilde{\mathcal{L}}_x : T_{1,0}(M)_x \to E(\Psi)_x \otimes_{\mathbb{R}} \mathbb{C}, \quad \tilde{\mathcal{L}}_x = J_{\Psi(x)} \circ (d_x \Psi) \circ \mathcal{L}_x, \quad x \in M.$$

Let $\{\xi_1, \ldots, \xi_k\}$ be a local g-orthonormal frame of $\mathcal{D}^{\perp}$, defined on the open set $U \subset M$, and let us set

$$N_j(x) = J_{\Psi(x)}(d_x \Psi)\xi_{j,\, x}, \quad x \in U, \quad 1 \le j \le k,$$

so that $N_j \in C^{\infty}(E(\Psi))$. Let us assume that the ambient metric G is Kählerian. Let $v \in T_{1,0}(M)_p$ with $x \in U$ and let $V \in C^{\infty}(U, T_{1,0}(M))$ such that $V_x = v$. Then

$$\begin{aligned} \tilde{\mathcal{L}}_x(v) &= J_{\Psi(x)}(d_x \Psi)\mathcal{L}_x(V_x) = \frac{1}{2i}\, J_{\Psi(x)}(d_x \Psi)\pi_{\mathcal{D}^{\perp},\, x} \left[\overline{V},\ V\right]_x \\ &= \frac{1}{2i} \sum_{j=1}^{k} g\left(\left[\overline{V}, V\right],\ \xi_j\right)_x N_j(x) \end{aligned}$$

and (as $g = \Psi^* G$ and ∇^G is torsion-free)

$$\begin{aligned} g\left(\left[\overline{V}, V\right],\ \xi_j\right) &= G\left(\nabla^G_{\overline{V}} V - \nabla^G_V \overline{V},\ \Psi_* \xi_j\right) \\ &= -G\left(\nabla^G_{\overline{V}} V - \nabla^G_V \overline{V},\ J N_j\right) \end{aligned}$$

(as $G(JX, JY) = G(X, Y)$ and $\nabla^G J = 0$)

$$= G\left(\nabla^G_{\overline{V}} J_M V - \nabla^G_V J_M \overline{V},\ N_j\right)$$

(as $V \in T_{1,0}(M)$ and $T_{1,0}(M)$ is the eigenbundle of $J_M^{\mathbb{C}}$ corresponding to the eigenvalue i)

$$= i\, G\left(\nabla^G_{\overline{V}} V + \nabla^G_V \overline{V},\ N_j\right)$$

(by Gauss formula and the symmetry of h)

$$= 2i\, G\left(h(V,\ \overline{V}),\ N_j\right).$$

It should be observed that, in the preceding calculation, we used $\mathbb{C}$-linear extensions of various objects, such as J_M, ∇^G, and h (and denoted the extensions by the same symbols). Also use was made of the $\mathbb{C}$-linear extension of the Gauss formula. Precisely, one may extend (by $\mathbb{C}$-linearity) both sides of (4.12). Since (4.12) holds for any real vector fields X and Y it will continue to hold for any complex vector fields X and Y. As $J_{\Psi(x)}(d_x\Psi)\mathcal{D}_x^{\perp} \subset E(\Psi)_x$ one may consider its $G_{\Psi(x)}$-orthogonal complement $\left(J\Psi_*\mathcal{D}^{\perp}\right)_x^{\perp}$ in $E(\Psi)_x$ so that

$$E(\Psi)_x = \left[J_{\Psi(x)}(d_x\Psi)\mathcal{D}_x^{\perp}\right] \oplus \left(J\Psi_*\mathcal{D}^{\perp}\right)_x^{\perp}, \quad x \in M. \tag{4.23}$$

If $Z \in E(\Psi)_x$ let $Z_{J\Psi_*\mathcal{D}^{\perp}}$ be the $J_{\Psi(x)}(d_x\Psi)\mathcal{D}_x^{\perp}$-component of Z (with respect to the decomposition (4.23)). We may conclude that

$$\tilde{\mathcal{L}}_x(v) = h_x(v,\ \overline{v})_{J\Psi_*\mathcal{D}^{\perp}}, \quad v \in T_{1,0}(M)_x,$$

which is the relationship between the extrinsic Levi form and the second fundamental form of $\Psi : M \to M^{2N}$ (provided that M^{2N} is Kählerian).

4.3.4 CR Extension from a Hyperplane

By Theorem 4.1 the boundary values of a solution $f \in C^{\infty}(\overline{\Omega}_{n+1},\ \mathbb{C})$ to the Dirichlet problem (4.4) and (4.5) must be a CR function, so that the tangential Cauchy–Riemann equations may be looked at as compatibility equations along $\partial\Omega_{n+1}$, that the boundary data must satisfy (for a solution smooth up to the boundary to exist). Viceversa, if $u \in \mathrm{CR}^{\omega}(\partial\Omega_{n+1})$ is a real analytic CR function then, by a result of G. Tomassini, [23], there is an open neighborhood $U \subset \mathbb{C}^{n+1}$ of $\partial\Omega_{n+1}$ and a holomorphic function $f \in \mathcal{O}(U)$ such that $f|_{\partial\Omega_{n+1}} = u$. In general, however, real analyticity (of the given CR manifold M and of the CR functions on M) does not suffice for simultaneously extending *all* CR functions from a neighborhood of a point in M.

The geometric properties of M, as tied to the Levi form (and hence to the second fundamental form of the given immersion) are expected to play a role, as anticipated by the following example.

Theorem 4.3 *Let* $M = \{(z, w) \in \mathbb{C}^2 : \text{Im}(w) = 0\}$. *For any open set* $\Omega \subset \mathbb{C}^2$ *such that* $\Omega \cap M \neq \emptyset$ *there is a CR function* $u : U \to \mathbb{C}$ *of class* C^ω *on* $U \subset \Omega \cap M$ *admitting no holomorphic extension to* Ω.

On the other hand, M is a hyperplane in $\mathbb{C}^2$ hence it is totally geodesic in $(\mathbb{C}^2, G_0)$ and in particular Levi flat. The given CR structure should then possess some nondegenericity property, for $r : \mathcal{O}(\Omega) \to \text{CR}^\omega(U)$ to be a surjective morphism.

To prove Theorem 4.3, we need to recall a few notions of complex analysis in several complex variables. An open set $\Omega \subset \mathbb{C}^n$ is a *domain of holomorphy* if there are no open sets Ω_1 and Ω_2 in $\mathbb{C}^n$ such that (a) $\Omega_2 \cap \Omega \supset \Omega_1 \neq \emptyset$, (b) Ω_2 is connected and not contained in Ω, (c) for every $f \in \mathcal{O}(\Omega)$ there is a function $f_2 \in \mathcal{O}(\Omega_2)$ such that $f = f_2$ in Ω_1. By the principle of analytic continuation such f_2 is necessarily uniquely determined. Any convex open set in $\mathbb{C}^n$ is a domain of holomorphy (cf., e.g., Corollary 2.5.6 in [18], p. 39).

At this point, we may attack the proof of Theorem 4.3. The restriction to $U \subset \Omega \cap M$ of any holomorphic function $f \in \mathcal{O}(\Omega)$ is a real analytic CR function. We ought to show that the restriction morphism $\mathcal{O}(\Omega) \to \text{CR}^\omega(U)$ is not surjective, for any open set $\Omega \subset \mathbb{C}^n$. Let $p \in U$ and let $r > 0$ such that $B(p, r) \subset \Omega$. Since M is an embedded submanifold of $\mathbb{C}^n$, the set $U_r = B(p, r) \cap M$ is open in M. For every $\epsilon > 0$, we set

$$\Omega_\epsilon = \left\{(z, w) \in \mathbb{C}^2 : (z, \text{ Re}(w)) \in U_r, \quad |\text{Im}(w)| < \epsilon\right\}$$

so that $U_r = \bigcap_{\epsilon > 0} \Omega_\epsilon$. Actually, $\overline{\Omega}_\epsilon \subset U_r$ for any $\epsilon > 0$ [since for any $\epsilon > 0$ one may consider $\delta > \epsilon$ so that $\overline{\Omega}_\epsilon \subset \Omega_\delta \subset U_r$]. Both M and the ball $B(p, r)$ are convex sets, so U_r is a convex set, as well. Consequently Ω_ϵ is convex, for any $\epsilon > 0$. Then Ω_ϵ is a domain of holomorphy so that there is a holomorphic function $f_\epsilon \in \mathcal{O}(\Omega_\epsilon)$ which cannot be continued holomorphically beyond Ω_ϵ. Let u_ϵ be the restriction of f_ϵ to U_r. Then $u_\epsilon \in \text{CR}^\omega(U_r)$ and u_ϵ admits no holomorphic extension to Ω. For given $g \in \mathcal{O}(\Omega)$ such that $g|_{U_r} = u_\epsilon$ it follows that

$$F_\epsilon = f_\epsilon - \hat{f}_\epsilon \in \mathcal{O}(\Omega_\epsilon), \quad \hat{f}_\epsilon \equiv g|_{\Omega_\epsilon}.$$

Hence $F_\epsilon(z, w) = 0$ for any $(z, w) \in U_r$. Yet Ω_ϵ is a connected neighborhood in $\mathbb{C}^n$ of the real hypersurface U_r hence (by Lemma 2 in [10], p. 142) $F_\epsilon = 0$ identically on Ω_ϵ hence $g|_{\Omega_\epsilon} = f_\epsilon$, i.e., g continues f_ϵ beyond Ω_ϵ, a contradiction. Q.e.d.

Of course the title of the current section should better be "failure of CR extension from a hyperplane". A very general nonextendibility result is due to M.S. Baouendi and L.P. Rothschild (cf. Theorem 2 in [2], p. 46), i.e., for any submanifold $M \subset \mathbb{C}^N$, for which the ambient complex structure induces a *generic* CR structure, and for any non *minimal* point $x_0 \in M$, there is $u \in \text{CR}^\infty(U)$ defined on a neighborhood $U \subset M$

of x_0 such that u does not extend holomorphically to any *wedge* with edge U. No generalizations of the quoted result, to CR submanifolds (in the sense of Definition 1) of a complex space form $M^{2N}(c)$, $c \neq 0$, are known in the present day mathematical literature.

4.4 CR Functions of Teodorescu Class B^1

Let M be a CR submanifold in the Hermitian manifold M^{2N} and let $\Omega^{0,1}(M) = C(M, T_{0,1}(M)^*)$. Throughout $C(E)$ denotes the space of all continuous globally defined sections in the vector bundle $E \to M$. Let us set

$$\|\omega_x\| = \sup\left\{\left|\omega_x\left(\overline{Z}\right)\right| : Z \in T_{1,0}(M)_x, \quad \|Z\| \leq 1\right\},$$

$$\|\omega\|_K = \sup_{x \in K} \|\omega_x\|,$$

for any $\omega \in \Omega^{0,1}(M)$, any point $x \in M$, and any compact subset $K \subset M$. Here $\|Z\| = g_x(Z, \overline{Z})^{1/2}$ and $g = \Psi^* G$. Then $\{\| \cdot \|_K : K \subset\subset M\}$ is a family of semi-norms on $\Omega^{1,0}(M)$ organizing it as a complex Fréchet space. By slightly adapting the proof of Lemma 1 in [15], p. 64, the tangential Cauchy–Riemann operator

$$\overline{\partial}_M : C^\infty(M, \mathbb{C}) \subset C(M, \mathbb{C}) \to \Omega^{0,1}(M) \tag{4.24}$$

is a preclosed operator of complex Fréchet spaces, i.e., for any sequence $\{u_\nu\}_{\nu \geq 1} \subset \mathcal{D}(\overline{\partial}_M) = C^\infty(M, \mathbb{C})$ such that $u_\nu \to 0$ and $\overline{\partial}_M u_\nu \to \omega$ for some $\omega \in \Omega^{0,1}(M)$, respectively in $C(M, \mathbb{C})$ and $\Omega^{0,1}(M)$ as $\nu \to \infty$, it follows that $\omega = 0$ as well. Let then D_M be the minimal closed extension of (4.24) and let us denote its domain by $B^1(M) = \mathcal{D}(D_M)$. Here D_M extends $\overline{\partial}_M$ (and one writes as customary $\overline{\partial}_M \subset D_M$), i.e., $\mathcal{D}(\overline{\partial}_M) \subset \mathcal{D}(D_M)$ and $D_M u = \overline{\partial}_M u$ for any $u \in C^\infty(M, \mathbb{C})$. Also D_M is a closed operator, i.e., its graph is a closed subset of $C(M, \mathbb{C}) \times \Omega^{0,1}(M)$ and if $T : \mathcal{D}(T) \subset C(M, \mathbb{C}) \to \Omega^{0,1}(M)$ is another closed extension of (4.24) then $D_M \subset T$. We recall that the domain $B^1(M)$ of D_M consists of all $u \in C(M, \mathbb{C})$ such that $\lim_{\nu\to\infty} u_\nu = u$ in $C(M, \mathbb{C})$ for some $\{u_\nu\}_{\nu \geq 1} \subset \mathcal{D}(\overline{\partial}_M)$ such that $\lim_{\nu\to\infty} \overline{\partial}_M u_\nu = \omega$ in $\Omega^{0,1}(M)$, for some $\omega \in \Omega^{0,1}(M)$. Moreover, to define $D_M u$ for $u \in \mathcal{D}(D_M)$ one picks up a sequence $\{u_\nu\}_{\nu \geq 1} \subset C^\infty(M, \mathbb{C})$ as above and sets $D_M u = \lim_{\nu\to\infty} \overline{\partial}_M u_\nu$. A function $u \in B^1(M)$ is said to be of *Teodorescu class* B^1 (cf. [22, 25]). Also an element $u \in \mathrm{Ker}(D_M)$ is a CR function of (*Teodorescu*) *class* B^1 (cf. [15]). It should be mentioned that in [15] one works with an *abstract* CR manifold M and then strict pseudoconvexity of $T_{1,0}(M)$ is needed (e.g., to make sense of the semi-norms $\| \cdot \|_K$). Of course this also prompts CR codimension $k = 1$. Our point in this section is that the considerations above still hold true with only minor modifications (of the arguments in [15]) when the Webster metric g_θ (the Levi form G_θ) is replaced with the first fundamental form g (with the restriction of g to the invariant distribution $\mathcal{D}$), a situation which also allows for arbitrary CR codimension.

Proposition 1 *Let M be a CR submanifold of a Hermitian manifold. Then any CR function of Teodorescu class B^1 on M is a weakly CR function.*

Proof Let $u \in \text{Ker}(D_M) \subset B^1(M)$ and let us consider a sequence $\{u_\nu\}_{\nu \geq 1} \subset C^\infty(M, \mathbb{C})$ such that $\lim_{\nu \to \infty} u_\nu = u$ in $C(M, \mathbb{C})$ and the sequence of (0, 1)-forms $\{\overline{\partial}_M u_\nu\}_{\nu \geq 1}$ is convergent in $\Omega^{0,1}(M)$. As u is continuous on M one has $u \in L^1_{\text{loc}}(M)$. For any $\varphi \in C_0^\infty(M)$ [as Supp $\left[\overline{Z}^*(\varphi)\right] \subset \text{Supp}(\varphi)$]

$$\left| \int_M u_\nu \overline{Z}^*(\varphi)\, d\,v_g - \int_M u \overline{Z}^*(\varphi)\, d\,v_g \right|$$
$$\leq \text{Vol}(K) \sup_K \left| \overline{Z}^*(\varphi) \right| \, \|u_\nu - u\|_K \to 0, \quad \nu \to \infty,$$

where $K = \text{Supp}(\varphi)$ and $\text{Vol}(K) = \int_K d\,v_g$. Also $\|v\| = \sup_{x \in K} |v(x)|$ for any $v \in C(M, \mathbb{C})$. Hence,

$$\int_M u \overline{Z}^*(\varphi)\, d\,v_g = \int_M u_\nu \overline{Z}^*(\varphi)\, d\,v_g.$$

On the other hand,

$$\left| \int_M u_\nu \overline{Z}^*(\varphi)\, d\,v_g \right| = \left| \int_M \overline{Z}\,(u_\nu)\ \varphi\, d\,v_g \right|$$
$$\leq \text{Vol}(K) \sup_K |\varphi| \, \left\| \left(\overline{\partial}_M u_\nu\right) \overline{Z} \right\|_K \to 0, \quad \nu \to \infty.$$

We may conclude that (4.2) holds, i.e., u is weakly CR. Q.e.d.

A study of (weakly) CR functions on CR submanifolds of complex space forms $M^{2N}(c)$ (similar to that in [21], which is confined to real hypersurfaces in $\mathbb{C}^{n+1}$) with $c \neq 0$ is missing from the present day mathematics literature.

4.5 The Hans Lewy Extension Phenomenon

4.5.1 CR Submanifolds in $\mathbb{C}^N$

Let $M = \{z \in \mathbb{C}^{n+1} : \rho(z) = 0\}$ be a real hypersurface and $x_0 \in M$ a point. Let us set $\Omega^+ = \{z \in \mathbb{C}^{n+1} : \rho(z) > 0\}$ and $\Omega^- = \{z \in \mathbb{C}^{n+1} : \rho(z) < 0\}$. The following result by H. Lewy (cf. [19]) is by now classical.

Theorem 4.4 (i) *If $\tilde{\mathcal{L}}_{x_0}$ has at least one positive eigenvalue then for each open set $x_0 \in U \subset M$ there is an open set $x_0 \in \Omega \subset \mathbb{C}^{n+1}$ such that for any $u \in \text{CR}^1(U)$ there is a unique $f \in \mathcal{O}(\Omega \cap \Omega_+) \cap C(\Omega \cap \overline{\Omega}_+)$ such that $f = u$ on $\Omega \cap M$.*

(ii) *If* $\tilde{\mathcal{L}}_{x_0}$ *has at least one negative eigenvalue then the conclusion in part* (i) *holds with* Ω_+ *replaced by* Ω_-.

(iii) *If* $\tilde{\mathcal{L}}_{x_0}$ *has eigenvalues of opposite sign then for each open set* $x \in U \subset M$ *there is an open set* $x_0 \in \Omega \subset \mathbb{C}^{n+1}$ *such that for each* $u \in \mathrm{CR}^1(U)$ *there is a unique* $f \in \mathcal{O}(\Omega)$ *such that* $f = u$ *on* $\Omega \cap U$.

An *eigenvalue* of $\tilde{\mathcal{L}}_{x_0}$ is an eigenvalue of the matrix

$$\left[\frac{\partial^2 \rho}{\partial z^j \, \partial \overline{z}^k}(x_0)\, w^j_\alpha \, \overline{w^k_\beta}\right]_{1 \le \alpha, \beta \le n}$$

for some local frame $\{T_\alpha : 1 \le \alpha \le n\}$ in $T_{1,0}(M)$ defined on an open neighborhood of x_0 such that

$$T_{\alpha, x_0} = w^j_\alpha \left(\frac{\partial}{\partial z^j}\right)_{x_0}, \quad T_\alpha(\rho) = 0, \quad w^j_\alpha \in \mathbb{C}, \quad 1 \le \alpha \le n, \quad 1 \le j \le n+1.$$

As to the CR codimension $k \ge 2$ case, the image of the extrinsic Levi form $\tilde{\mathcal{L}}_x$ provides information about the second-order concavity of M near x. Indeed, for each $x \in M$ let

$$\Gamma_x = \mathrm{co}\left[\tilde{\mathcal{L}}_x \, T_{1,0}(M)_x\right]$$

be the convex hull of

$$\tilde{\mathcal{L}}_x \, T_{1,0}(M)_x = \left\{h_x(v, \overline{v})_{J\Psi_* \mathcal{D}^\perp} : v \in T_{1,0}(M)_x\right\}$$

[a convex subset of $E(\Psi)_x$]. Then, Γ_x is a cone, i.e., $v \in \Gamma_x$ and $\lambda \ge 0$ imply $\lambda v \in \Gamma_x$. When $M^{2N} = \mathbb{C}^N$, the cone Γ_x is known to determine the shape and size of the open set to which CR functions extend (holomorphically). Let $S_x = \{Z \in E(\Psi)_x : G_{\Psi(x)}(Z, Z) = 1\}$ be the unit sphere in $E(\Psi)_x$. Let Γ_a be two ($a \in \{1, 2\}$) cones in $E(\Psi)_x$. We say Γ_1 is *smaller* than Γ_2, and write $\Gamma_1 < \Gamma_2$, if $\Gamma_1 \cap S_x$ is a compact subset of the interior of $\Gamma_2 \cap S_x$. Let $B_\epsilon = \{Z \in E(\Psi)_x : G_{\Psi(x)}(Z, Z) < \epsilon^2\}$ be the ball of radius $\epsilon > 0$ and origin 0_x in $E(\Psi)_x$. By a result of A. Boggess and J.C. Polking (cf. [9])

Theorem 4.5 *Let* $M \subset \mathbb{C}^N$ *be a real* $(2N - d)$*-dimensional generic CR submanifold with* $1 \le d \le N - 1$. *Let* $x_0 \in M$ *such that* Γ_{x_0} *has nonempty interior, with respect to* $E(j)_{x_0}$. *For any neighborhood* $U \subset M$ *of* x_0, *there exist open sets* $U' \subset M$ *and* $\Omega \subset \mathbb{C}^N$ *such that*

(i) $x_0 \in U' \subset \overline{\Omega} \cap M \subset U$,

(ii) *for each open cone* $\Gamma < \Gamma_{x_0}$ *there is a connected neighborhood* $U_\Gamma \subset M$ *of* x_0 *and a number* $\epsilon > 0$ *such that* $U_\Gamma + (\Gamma \cap B_\epsilon) \subset \Omega$,

(iii) *for each CR function* $u \in \mathrm{CR}^1(U)$ *there is a unique holomorphic function* $f \in \mathcal{O}(\Omega) \cap C(\Omega \cup U')$ *such that* $f = u$ *on* U'.

Here *generic* means that $\dim_{\mathbb{R}} \mathcal{D}_x = 2(N-d)$ for any $x \in M$ (equivalently $J(\mathcal{D}^\perp) = E(j)$ as in [5]). Following the philosophy outlined in § 1, that is aiming at the unification of the geometry of the second fundamental form of $j : M \to M^{2N}$ and the mathematical analysis of the solutions to the tangential Cauchy–Riemann equations on M, one should seek for generalizations of A. Boggess and J.C. Polking's result (Lewy's result for $k = 1$) to CR submanifolds $(M, \mathcal{D})$ of a complex space form $M^{2N}(c)$ of (constant) holomorphic sectional curvature $c \neq 0$. On the differential geometric side of the subject, CR submanifolds of a complex space form were studied by A. Bejancu et al. (cf. [6]). A major ingredient in the proof of Theorem 4.5 is the following uniform approximation result by M.S. Baouendi and F. Tréves (cf. [1])

Theorem 4.6 *Let $M \subset \mathbb{C}^N$ be a real $(2N-d)$-dimensional generic CR submanifold and $x_0 \in M$ a point. Given an open neighborhood $x_0 \in U \subset M$ there is an open neighborhood $x_0 \in V \subset U$ such that each CR function $u \in \mathrm{CR}^1(U)$ can be uniformly approximated on V by a sequence of entire functions on $\mathbb{C}^N$.*

Other ingredients required by CR extension theory are the Fourier transform technique (cf. [10], p. 229–250) and analytic discs (cf. [10], p. 207–228). A generalization of Theorem 4.6 to the case of vector-valued CR functions was given in [15]. Going back to M.S. Baouendi and L.P. Rothschild's result (cf. [2]) quoted in Sect. 3.4, we recall that a point $x_0 \in M$ (on a CR manifold M) is *minimal* (cf. A.E. Tumanov, [24]) if there is no CR immersion[3] $N \hookrightarrow M$, from some CR manifold N, such that $x_0 \in N$ and $\dim_{\mathbb{R}} N < \dim_{\mathbb{R}} M$. Is the notion of minimality in complex analysis related[4] to minimality as understood in Riemannian geometry? Let $M \subset \mathbb{C}^{n+1}$ be a *generic* submanifold, i.e., M is locally defined, near a point $x_0 \in M$, by the equations $\rho_j = 0$, $1 \leq j \leq k$, where ρ_j are smooth, real-valued functions such that their complex differentials $\partial \rho_j$ are linearly independent [such M is, together with the CR structure induced by the complex structure of $\mathbb{C}^{n+1}$, a CR manifold of type (n, k)]. A *wedge of edge* M is an open subset of $\mathbb{C}^{n+1}$ of the form $\mathcal{W}(\Omega, \Gamma) = \{\zeta \in \Omega : \rho(\zeta) \in \Gamma\}$ where $\Omega \subset \mathbb{C}^{n+1}$ is a neighborhood of $x_0 \in M$, Γ is an open cone in $\mathbb{R}^k$, and $\rho = (\rho_1, \dots, \rho_k)$ with ρ_j defining functions of M near x_0. By a result in [24] if $M \subset \mathbb{C}^{n+1}$ is a generic submanifold and $x_0 \in M$ is minimal then every CR function defined on a neighborhood of x_0 is the boundary values of a holomorphic function defined in an open wedge of $\mathbb{C}^{n+1}$ of edge M. Notions such as minimality, wedges of a given edge, finite type, etc. should be absorbed into Kählerian, or locally conformal Kähler, geometry as a step towards extending results as Tumanov's (cf. *op. cit.*) to CR functions on Bejancu's CR sumanifolds in Hermitian manifolds.

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[3] A *CR immersion* is a C^∞ immersion and a CR map.

[4] CR immersions are CR analogs to holomorphic immersions and any isometric holomorphic immersion into a Kählerian manifold is known to be minimal (has vanishing mean curvature).

4.5.2 CR Submanifolds in $\mathbb{C}P^N$

Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit sphere, thought of as a Sasakian manifold with the standard Sasakian structure (ϕ, ξ, η, G), cf., e.g., [8]. A submanifold $\iota : M \subset S^{2n+1}$ is a *contact CR submanifold* (cf. [30], p. 48) if M is tangent to the Reeb vector field $\xi = -J_0\mathcal{N}$ of S^{2n+1}, i.e., $\xi \in \mathfrak{X}(M)$ and M is endowed with a distribution $\mathcal{D} \subset T(M)$ such that i) $\mathcal{D}$ is ϕ-invariant, i.e., $\phi_x(\mathcal{D})_x \subset \mathcal{D}_x$ for any $x \in M$ and ii) the g-orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$ in $T(M)$ is ϕ-anti-invariant, i.e., $\phi_x(\mathcal{D}^\perp_x) \subset E(\iota)_x$ for any $x \in M$. Here $\mathcal{N}$ is a unit normal field on S^{2n+1} and $g = \iota^* G$. Also $E(\iota) \to M$ is the normal bundle of the immersion $\iota : M \to S^{2n+1}$. As $\xi \in T(M) = \mathcal{D} \oplus \mathcal{D}^\perp$ one has an orthogonal decomposition

$$\xi = \xi_{\mathcal{D}} + \xi_{\mathcal{D}^\perp}, \quad \xi_{\mathcal{D}} \in \mathcal{D}, \quad \xi_{\mathcal{D}^\perp} \in \mathcal{D}^\perp.$$

We shall only examine the case[5] where $\xi_{\mathcal{D}}(x) \neq 0$ for any $x \in M$. Let $H(M)$ be the g-orthogonal complement of $\mathbb{R}\,\xi_{\mathcal{D}}$ (the distribution spanned by $\xi_{\mathcal{D}}$) in $\mathcal{D}$ so that

$$\mathcal{D} = H(M) \oplus \mathbb{R}\,\xi_{\mathcal{D}}.$$

Lemma 1 *Let $(M, \mathcal{D})$ be a contact CR submanifold of S^{2n+1} such that $\xi_{\mathcal{D}}(x) \neq 0$ for any $x \in M$. The pair $(M, H(M))$ is a CR submanifold $j : M \subset \mathbb{C}^{n+1}$ of type $(p-1, q)$ and codimension $d = 2(n+1) - p - q$, where $p = \dim_{\mathbb{R}} \mathcal{D}_x$ and $q = \dim_{\mathbb{R}} \mathcal{D}^\perp_x$ for any $x \in M$.*

Proof We recall (cf., e.g., [8]) that $\phi X = \tan\,(J_0 X)$ for any $X \in \mathfrak{X}(S^{2n+1})$ where $\tan_x : T_x(\mathbb{C}^{n+1}) \to T_x(S^{2n+1})$ is the projection associated to the decomposition

$$T_x(\mathbb{C}^{n+1}) = T_x(S^{2n+1}) \oplus T_x(S^{2n+1})^\perp, \quad x \in S^{2n+1}.$$

Consequently, $J_0 X = \phi X + \eta(X)\,\mathcal{N}$ for each $X \in \mathfrak{X}(S^{2n+1})$. Let $X \in H(M)$. Then

$$\begin{aligned} \eta(X) &= G(X, \xi) \qquad [\text{as} \quad X \perp \mathbb{R}\,\xi_{\mathcal{D}}] \\ &= G(X,\ \xi_{\mathcal{D}^\perp}) = 0 \qquad [\text{as}\ \ X \in H(M) \subset \mathcal{D} \perp \mathcal{D}^\perp] \end{aligned}$$

yielding $J_0 X = \phi X \in \mathcal{D}$. Moreover,

$$G_0(J_0 X,\ \xi_{\mathcal{D}}) = -G_0\,(X,\ J_0\,\xi_{\mathcal{D}})$$

[as $X \in H(M) \subset T(M)$ and $J_0\xi_{\mathcal{D}^\perp} \in E(\iota) \Longrightarrow G(X, J_0\xi_{\mathcal{D}^\perp}) = 0$]

$$= -G_0\,(X,\ J_0\,\xi_{\mathcal{D}}) = -G_0\,(X,\ J_0\xi) = -G_0(X, \mathcal{N}) = 0$$

[5] The particular case $\xi_{\mathcal{D}} = 0$ (equivalently $\xi \in \mathcal{D}^\perp$) may be treated in a similar manner.

implying that $J_0 X \in (\mathbb{R}\,\xi_{\mathcal{D}})^{\perp} = H(M)$, i.e., $H(M)$ is J_0-invariant. Let $H(M)^{\perp}$ be the g-orthogonal complement of $H(M)$ in $T(M)$ so that $T(M) = H(M) \oplus H(M)^{\perp}$. We wish to check that $H(M)^{\perp}$ is J_0-anti-invariant, i.e., $J_0 H(M)^{\perp} \subset E(j)$, where $E(j) \to M$ is the normal bundle of $j : M \subset \mathbb{C}^{n+1}$. Let $Y \in H(M)^{\perp}$ and $X \in T(M)$. As

$$T(M) = H(M) \oplus \mathbb{R}\,\xi_{\mathcal{D}} \oplus \mathcal{D}^{\perp}$$

one has a decomposition

$$X = X_{H(M)} + f\,\xi_{\mathcal{D}} + X_{\mathcal{D}^{\perp}},$$

$$X_{H(M)} \in H(M), \quad f \in C^{\infty}(M), \quad X_{\mathcal{D}^{\perp}} \in \mathcal{D}^{\perp}.$$

Therefore,

$$\begin{aligned} G_0\,(J_0 Y,\, X) &= G_0\left(J_0 Y,\, X_{H(M)} + f\,\xi_{\mathcal{D}} + X_{\mathcal{D}^{\perp}}\right) \\ &= -G_0\left(Y,\, J_0 X_{H(M)}\right) - f\,G_0\,(Y,\, J_0\,\xi_{\mathcal{D}}) - G_0\,(Y,\, J_0 X_{\mathcal{D}^{\perp}}) \end{aligned}$$

and

$$\begin{aligned} &Y \in H(M)^{\perp}, \;\; J_0 X_{H(M)} \in H(M) \Longrightarrow g(Y,\, J_0 X_{H(M)}) = 0, \\ &Y \in T(M), \;\; J_0\,X_{\mathcal{D}^{\perp}} = \phi\,X_{\mathcal{D}^{\perp}} + \eta(X_{\mathcal{D}^{\perp}})\,\mathcal{N} \in E(\iota) \oplus T(S^{2n+1})^{\perp} \\ &\quad \Longrightarrow G_0\,(Y,\, J_0 X_{\mathcal{D}^{\perp}}) = 0. \end{aligned}$$

It follows that

$$G_0\,(J_0 Y,\, X) = -f\,G_0\,(Y,\, J_0\,\xi_{\mathcal{D}})$$

[as $Y \in H(M) \subset T(M)$ and $J_0\,\xi_{\mathcal{D}^{\perp}} = \phi(\xi_{\mathcal{D}^{\perp}}) + \eta(\xi_{\mathcal{D}^{\perp}})\,\mathcal{N} \in E(\iota) \oplus T(S^{2n+1})^{\perp} \Longrightarrow G_0(Y,\, J_0\,\xi_{\mathcal{D}^{\perp}}) = 0$]

$$= -f\,G_0\,(Y,\, J_0\,\xi_{\mathcal{D}} + J_0\,\xi_{\mathcal{D}^{\perp}}) = -f\,G_0\,(Y,\, J_0\,\xi) = -sf\,G_0(Y, \mathcal{N}) = 0$$

i.e., $J_0 Y \perp T(M)$. Q.e.d.

Let $\mathbb{C}P^n$ be the complex projective space, endowed with the standard complex structure $\mathcal{J}$ and the Fubini–Study metric $\mathcal{G}$. Let $S^1 \to M \to N$ be a principal subbundle of the Hopf bundle $S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$ such that N is a CR submanifold of $\mathbb{C}P^n$ endowed with the ($\mathcal{J}$-invariant) distribution $\mathcal{D}_N$. Then (cf., e.g., Proposition 7.1 in [30], p. 102) M is a contact CR submanifold of S^{2n+1}. However, the construction of the ϕ-invariant distribution $\mathcal{D}$ (organizing M as a contact CR submanifold) isn't given in [30]. In the sequel, we provide the definition of $\mathcal{D}$ and relate $\mathrm{CR}^1(M)$ (the space of all CR functions of class C^1 on M, as a CR submanifold $j : M \subset \mathbb{C}^{n+1}$) to $\mathrm{CR}^1(N)$.

The differential 1-form η is a connection 1-form on the Hopf bundle (as a principal S^1-bundle). If $X \in \mathfrak{X}(\mathbb{C}P^n)$ then $X^\uparrow \in \mathfrak{X}(S^{2n+1})$ denotes the horizontal lift of X with respect to η, i.e.,

$$X_x^\uparrow \in \text{Ker}\,(\eta_x)\,, \quad (d_x\pi)X_x^\uparrow = X_{\pi(x)}, \quad x \in S^{2n+1}.$$

Then [by (7.1) in [30], p. 100]

$$(\mathcal{J}X)^\uparrow = \phi X^\uparrow, \quad \mathcal{G}(X, Y) \circ \pi = G\left(X^\uparrow,\ Y^\uparrow\right), \tag{4.25}$$

for any $X, Y \in \mathfrak{X}(S^{2n+1})$.

Theorem 4.7 *Let $S^1 \to M \to N$ be a principal subbundle of the Hopf bundle $S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$, over a CR submanifold $N \subset \mathbb{C}^n$.* (i) *The Reeb vector $\xi \in \mathfrak{X}(S^{2n+1})$ is tangent to M, i.e., $\xi \in \mathfrak{X}(M)$. Let $\mathcal{D}_N$ be the $\mathcal{J}$-invariant distribution of N.* (ii) *If we set*

$$\mathcal{D} = \mathcal{D}_N^\uparrow \oplus \mathbb{R}\,\xi \tag{4.26}$$

then $(M, \mathcal{D})$ is a contact CR submanifold of S^{2n+1}. In particular (iii) *$(M, \mathcal{D}_N^\uparrow)$ is a CR submanifold of $\mathbb{C}^{n+1}$ and then a CR manifold whose CR structure $T_{1,0}(M)$ is given by*

$$T_{1,0}(M) = T_{1,0}(N)^\uparrow. \tag{4.27}$$

Consequently, (iv) *vertical lifting of functions from N induces an isomorphism $\text{CR}^1(N) \approx \text{CR}^1(M)^{S^1}$.* (v) *If $N \subset \mathbb{C}P^n$ is generic then $M \subset \mathbb{C}^{n+1}$ is generic.* (vi) *The extrinsic Levi forms of the CR immersions $N \hookrightarrow \mathbb{C}P^n$ and $M \hookrightarrow \mathbb{C}^{n+1}$ are related by*

$$\tilde{\mathcal{L}}_{N,p}(w)^\uparrow = \tilde{\mathcal{L}}_{M,x}\left(w^\uparrow\right) + \mathcal{G}_p\left(w, \overline{w}\right)\mathcal{N}_x \tag{4.28}$$

for any $w \in T_{1,0}(N)_p$ and $p \in N$, where $\uparrow$ is the complexification of the horizontal lift, i.e., $\beta_x : T_p(\mathbb{C}P^n) \otimes_{\mathbb{R}} \mathbb{C} \to \text{Ker}(\eta_x) \otimes_{\mathbb{R}} \mathbb{C}$ with $x \in M$ such that $\pi(x) = p$. Also $\mathcal{N} = \sum_{j=1}^{2(n+1)} x_j\, \partial/\partial x_j$ (the radial vector field) and $\left(x_1, \ldots, x_{2(n+1)}\right)$ are the Cartesian coordinates on $\mathbb{C}^{n+1} \approx \mathbb{R}^{2(n+1)}$.

Here $\text{CR}^1(M)^{S^1}$ is the space of all S^1-invariant CR functions $u : M \to \mathbb{C}$ of class C^1. To prove Theorem 4.7 note that

(i) $M = \pi^{-1}(N)$ (a saturated subset of S^{2n+1}, with respect to the foliation of S^{2n+1} be maximal integral curves of ξ). In particular ξ is tangent to M.

(ii) Let $X \in T_p(N)$ be a tangent vector. There is a smooth curve $\gamma : (-\epsilon, \epsilon) \to N$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X$. Let $\gamma^\uparrow : (-\epsilon, \epsilon) \to S^{2n+1}$ be the unique horizontal lift (with respect to η) such that $\gamma^\uparrow(0) = x$ for some fixed $x \in M$ such that $\pi(x) = p$. Let $\beta_x : T_{\pi(x)}(N) \to \text{Ker}(\eta_x)$ be the horizontal lift isomorphism [the inverse of $d_x\pi : \text{Ker}(\eta_x) \to T_p(N)$]. Then

$$\frac{d\gamma^\uparrow}{dt}(0) = \beta_x\,(X)$$

and $\gamma^\uparrow(t) \in M$ for any $|t| < \epsilon$ because $\pi \circ \gamma^\uparrow = \gamma$ and again $M = \pi^{-1}(N)$. Consequently the horizontal lift $X^\uparrow = \beta_x(X)$ of X is tangent to M so that

$$T(N)^\uparrow \subset T(M). \tag{4.29}$$

Precisely if $\eta_M = \iota^*\eta$ then $T(N)^\uparrow = \text{Ker}(\eta_M)$. One also has

$$T(M) = T(N)^\uparrow \oplus \mathbb{R}\,\xi. \tag{4.30}$$

Indeed if $V \in T(N)^\uparrow \cap \mathbb{R}\,\xi$, then $V = X^\uparrow = f\,\xi$ for some $X \in T(N)$ and $f \in C^\infty(M)$ hence

$$0 = \eta\left(X^\uparrow\right) = f\,\eta(\xi) = f$$

hence $V = 0$. Consequently, the sum $T(N)^\uparrow + \mathbb{R}\,\xi$ is direct and [by (4.29) and $\xi \in T(M)$] contained in $T(M)$. Finally, $\dim_{\mathbb{R}}(M) = \dim_{\mathbb{R}}(N) + 1$ yields (4.30). Let $T(N)^\perp \to N$ be the normal bundle of the immersion $N \hookrightarrow \mathbb{C}P^n$. Then

$$E(\iota) = \left[T(N)^\perp\right]^\uparrow \tag{4.31}$$

where $E(\iota) \to M$ is the normal bundle of $\iota : M \hookrightarrow S^{2n+1}$. Indeed, let $V \in T(N)^\perp$ and $W \in T(M)$ so that [by (4.30)] $W = X^\uparrow + f\,\xi$ for some $X \in T(N)$. Then [by (4.25)]

$$G\left(V^\uparrow,\ W\right) = \mathcal{G}(V, X) \circ \pi + f\,\eta\left(V^\uparrow\right) = 0$$

so that $\left[T(N)^\perp\right]^\uparrow \subset E(\iota)$ and (4.31) follows from

$$\text{codim}\left(M \hookrightarrow S^{2n+1}\right) = 2n + 1 - \dim(M)$$

$$= 2n - \dim(N) = \text{codim}\left(N \hookrightarrow \mathbb{C}P^n\right).$$

As $\mathcal{D}_N^\uparrow \subset \text{Ker}(\eta)$ it follows that $\mathcal{D}_N^\uparrow$ and ξ are orthogonal, hence $\mathcal{D}$ is well defined [by (4.26)]. Let us show that $\mathcal{D}$ is ϕ-invariant. To this end, let $V = X^\uparrow + f\,\xi \in \mathcal{D}$ with $X \in \mathcal{D}_N$. Then [by (4.25)]

$$\phi(V) = (\mathcal{J}X)^\uparrow + f\,\phi(\xi) = (\mathcal{J}X)^\uparrow \in \mathcal{D}_N^\uparrow \subset \mathcal{D}$$

as $\mathcal{D}_N$ is $\mathcal{J}$-invariant and $\phi(\xi) = 0$. Let $\mathcal{D}^\perp$ be the g-orthogonal complement of $\mathcal{D}$ in $T(M)$ [where $g = \iota^*G$]. Let us show that $\mathcal{D}^\perp$ is ϕ-anti-invariant. One ought to check first that

$$\mathcal{D}^\perp = \left(\mathcal{D}_N^\perp\right)^\uparrow. \tag{4.32}$$

Indeed if $Y \in \mathcal{D}_N^\perp$ and $V = X^\uparrow + f\,\xi \in \mathcal{D}$ (with $X \in \mathcal{D}_N$) then

$$G\left(Y^\uparrow,\ V\right) = \mathcal{G}(Y, X) \circ \pi + f\,\eta\left(Y^\uparrow\right) = 0$$

hence $\left(\mathcal{D}_N^\perp\right)^\uparrow \subset \mathcal{D}^\perp$ so that (4.32) follows by comparing ranks. Moreover, given $W \in \mathcal{D}^\perp$ and $V \in T(M)$ one has [by (4.32) and (4.30)]

$$W = Y^\uparrow, \quad V = X^\uparrow + f\,\xi,$$

$$Y \in \mathcal{D}_N^\perp, \quad X \in T(N), \quad f \in C^\infty(M),$$

hence

$$G(\phi W, V) = \mathcal{G}(\mathcal{J}Y, X) \circ \pi + f\,\eta\left(\phi Y^\uparrow\right) = 0$$

as $\mathcal{J}\mathcal{D}_N^\perp \subset T(N)^\perp$ and $\eta \circ \phi = 0$. We may conclude that $\phi(\mathcal{D}^\perp)$ is orthogonal to $T(M)$, i.e., $\phi(\mathcal{D}^\perp) \subset E(\iota)$. We may conclude that $(M, \mathcal{D})$ is a contact CR submanifold of the sphere S^{2n+1} as a Sasakian manifold.

(iii) With the notations in Lemma 1 one has

$$H(M) = \mathcal{D}_N^\uparrow. \tag{4.33}$$

By Lemma 1 (with $\xi_{\mathcal{D}} = \xi$) the pair $(M, H(M))$ is a CR submanifold of $\mathbb{C}^{n+1}$ (not passing through the origin, i.e., $M \subset \mathbb{C}^{n+1}\setminus\{0\}$). Let $J_M : H(M) \to H(M)$ be the restriction of J_0 to $H(M)$. Then (by the Blair–Chen theorem, [11]) $T_{1,0}(M) = \text{Eigen}(J_M^{\mathbb{C}}; i)$ is a CR structure on M. Similarly, let $J_N : \mathcal{D}_N \to \mathcal{D}_N$ be the restriction of $\mathcal{J}$ to $\mathcal{D}_N$, so that $T_{1,0}(N) = \text{Eigen}(J_N^{\mathbb{C}}; i)$ is a CR structure on N. Then [by (4.25)]

$$J_M^{\mathbb{C}}\left(W^\uparrow\right) = \left(J_N^{\mathbb{C}}\, W\right)^\uparrow, \quad W \in \mathcal{D} \otimes \mathbb{C},$$

hence $T_{1,0}(N)^\uparrow \subset T_{1,0}(M)$ and (4.27) follows by comparing dimensions.

(iv) Let $u \in C^1(N, \mathbb{C})$ and $W \in T_{1,0}(M)$ so that [by (4.27)] $W = Z^\uparrow$ for some $Z \in T_{1,0}(N)$. Then

$$\overline{\partial}_M\,(u \circ \pi)\;\overline{W} = \overline{Z}^\uparrow(u \circ \pi) = \overline{Z}(u) \circ \pi = \left[(\overline{\partial}_N u)\overline{Z}\right] \circ \pi.$$

Here horizontal lifting $\beta : \pi^{-1}T(\mathbb{C}P^n) \to \text{Ker}(\eta)$ was tacitly extended by $\mathbb{C}$-linearity. As β is a real vector bundle isomorphism, its $\mathbb{C}$-linear extension commutes with complex conjugation. In particular, the vertical lift of any CR function on N is a CR function on M and conversely, i.e., for any S^1-invariant CR function on M the corresponding base function is CR.

(v) If the base manifold N is a codimension d generic CR submanifold in $\mathbb{C}P^n$, i.e., $\dim_{\mathbb{R}} \mathcal{D}_{N,p} = d$ for any $p \in N$, then

$$\dim_{\mathbb{R}} H(M)_x^\perp = 2n - d + 1 - \dim_{\mathbb{R}} H(M)_x$$

$$= 2n - d + 1 - \dim_{\mathbb{R}} \mathcal{D}_{N,\pi(x)} = d + 1,$$

$$\text{codim}\left(M \hookrightarrow \mathbb{C}^{n+1}\right) = d + 1,$$

hence $j : M \subset \mathbb{C}^{n+1}$ is generic, as well.

(vi) Let B, h, and h_S be, respectively, the second fundamental forms of the immersions $N \hookrightarrow \mathbb{C}P^n$, $M \hookrightarrow \mathbb{C}^{n+1}$ and $M \hookrightarrow S^{2n+1}$. Then [by (7.5) in [30], p. 101]

$$B(V, W)^{\uparrow} = h_S\left(V^{\uparrow},\ W^{\uparrow}\right), \quad V, W \in \mathfrak{X}(N). \tag{4.34}$$

We need the Gauss formulae

$$\nabla^0_X Y = \nabla_X Y + h(X, Y), \quad \nabla^S_X Y = \nabla_X Y + h_S(X, Y), \tag{4.35}$$

$$\nabla^0_{\mathcal{X}} \mathcal{Y} = \nabla^S_{\mathcal{X}} \mathcal{Y} - G(\mathcal{X}, \mathcal{Y})\, \mathcal{N}, \tag{4.36}$$

with $X, Y \in \mathfrak{X}(M)$ and $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}(S^{2n+1})$. Here ∇^0, ∇^S and ∇ are the Levi-Civita connections of $(\mathbb{C}^{n+1}, G_0)$, (S^{2n+1}, G) and (M, g). Formulae (4.35)–(4.36) yield

$$h(X, Y) = h_S(X, Y) - G(X, Y)\mathcal{N}, \quad X, Y \in \mathfrak{X}(M). \tag{4.37}$$

Therefore, [by (4.34) and (4.37)]

$$B(V, W)^{\uparrow} = h\left(V^{\uparrow},\ W^{\uparrow}\right) + [\mathcal{G}(V, W) \circ \pi]\ \mathcal{N} \tag{4.38}$$

for any $V, W \in \mathfrak{X}(N)$. We claim that (4.38) implies (4.28) in Theorem 4.7. To prove the claim, we need the decompositions

Lemma 2 *Let $H(M)^{\perp}$ be the orthogonal complement of $H(M)$ in $(T(M), g)$. Then*

$$JH(M)^{\perp} = \left(\mathcal{J}\mathcal{D}_N^{\perp}\right)^{\uparrow} \oplus \mathbb{R}\mathcal{N}. \tag{4.39}$$

Let $\left(JH(M)^{\perp}\right)^{\perp}$ [respectively $\left(\mathcal{J}\mathcal{D}_N^{\perp}\right)^{\perp}$] be the G_0-orthogonal complement of $JH(M)^{\perp}$ [respectively the $\mathcal{G}$-orthogonal complement of $\mathcal{J}\mathcal{D}_N^{\perp}$] in $E(M \hookrightarrow \mathbb{C}^{n+1})$ [respectively in $E(N \hookrightarrow \mathbb{C}P^n) = T(N)^{\perp}$]. Then

$$\left(JH(M)^{\perp}\right)^{\perp} = \left[\left(\mathcal{J}\mathcal{D}_N^{\perp}\right)^{\perp}\right]^{\uparrow}. \tag{4.40}$$

Proof We start by establishing

$$H(M)^{\perp} = \left(\mathcal{D}_N^{\perp}\right)^{\uparrow} \oplus \mathbb{R}\xi. \tag{4.41}$$

First let $V \in \left(\mathcal{D}_N^{\perp}\right)^{\uparrow} \cap \mathbb{R}\xi$, i.e., $V = Y^{\uparrow} = f\,\xi$ for some $Y \in \mathcal{D}_N^{\perp}$ and $f \in C^{\infty}(M)$. Then $V \in \mathrm{Ker}(\eta) + \mathrm{Ker}(d\pi) = (0)$ hence the sum $\left(\mathcal{D}_N^{\perp}\right)^{\uparrow} + \mathbb{R}\xi$ is direct. Next let $Y \in \mathcal{D}_N^{\perp}$ and $V \in H(M)$, i.e., $V = X^{\uparrow}$ for some $X \in \mathcal{D}_N$ [as a consequence of (4.33)]. Then

$$G\left(Y^{\uparrow},\ V\right) = \mathcal{G}(Y, X) \circ \pi = 0$$

so that $Y^\uparrow$ is orthogonal to $H(M)$, i.e., $\left(\mathcal{D}_N^\perp\right)^\uparrow \subset H(M)^\perp$. The same manner

$$G(\xi, V) = G(\xi, X^\uparrow) = \eta(X^\uparrow) = 0$$

so that $\xi \in H(M)^\perp$. We proved $\left(\mathcal{D}_N^\perp\right)^\uparrow \oplus \mathbb{R}\xi \subset H(M)^\perp$ and equality (4.41) follows by comparing ranks. Decomposition (4.39) then follows by applying J to (4.41) and using $\xi = -J\mathcal{N}$. Finally, we ought to check (4.40). To this end let $V \in JH(M)^\perp$ and $W \in \left(\mathcal{J}\mathcal{D}_N^\perp\right)^\perp$. Then [by (4.39)]

$$V = (\mathcal{J}Y)^\uparrow + f\mathcal{N}, \quad Y \in \mathcal{D}_N^\perp, \quad f \in C^\infty(M),$$

so that

$$G_0\left(V,\ W^\uparrow\right) = \mathcal{G}(\mathcal{J}Y,\ W)\circ\pi + f\,G_0\left(\mathcal{N},\ W^\uparrow\right) = 0.$$

Q.e.d. Let $w \in T_{1,0}(N)_p$. Then

$$B_p(w,\ \overline{w}) = \tilde{\mathcal{L}}_{N,p}(w) + W$$

for some $W \in \left(\mathcal{J}\mathcal{D}_N^\perp\right)_p^\perp \otimes_{\mathbb{R}} \mathbb{C}$, so that [by (4.38)]

$$\begin{aligned}\tilde{\mathcal{L}}_{N,p}(w)^\uparrow + W^\uparrow &= h_x\left(w^\uparrow,\ \overline{w}^\uparrow\right) + \mathcal{G}_p\left(w,\ \overline{w}\right)\mathcal{N}_x\\ &= \tilde{\mathcal{L}}_{M,x}\left(w^\uparrow\right) + \mathcal{V} + \mathcal{G}_p\left(w,\ \overline{w}\right)\mathcal{N}_x\end{aligned}$$

for some $\mathcal{V} \in \left(JH(M)^\perp\right)_x^\perp$. Let us use (4.39) and (4.40) and identify the $J_xH(M)_x^\perp$-components. We obtain (4.28). Q.e.d.

Proposition 2 *Let $S^1 \to M \xrightarrow{\pi} N$ be a principal subbundle of the Hopf bundle $S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$ such that N is a generic CR submanifold in $\mathbb{C}P^n$. Let $p_0 \in N$ be a point and let $0 \le j \le n$ such that $p_0 \in U_j$. There is an open neighborhood $p_0 \in \mathcal{U} \subset N \cap U_j$ such that for every $u \in \mathrm{CR}^1(N \cap U_j)$ there is a sequence of holomorphic functions $\{f_\nu\}_{\nu\ge1} \subset \mathcal{O}(U_j)$ such that $f_\nu \to u$ as $\nu \to \infty$ uniformly on $\mathcal{U}$.*

Here $U_j = \{\pi_0(\zeta) : \zeta_j \neq 0\}$ for any $0 \le j \le n$, where $\pi_0 : \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{C}P^n$ is the projection. To prove Corollary 2 let $u \in \mathrm{CR}^1(N \cap U_0)$ be a CR function of class C^1 on $N \cap U_0$. For simplicity, let us assume that $j = 0$. Next let us set $v = u \circ \pi$ so that $v \in \mathrm{CR}^1(U)$ where $U = \pi^{-1}(N \cap U_0) \subset M$. Let us choose a point $x_0 \in U$ such that $\pi(x_0) = p_0$. For generic CR submanifolds such as $U \subset \mathbb{C}^{n+1}$, one may apply the Baouendi–Tréves approximation theorem (cf. Theorem 4.6 in Sect. 4.1) hence there is an open neighborhood $x_0 \in V \subset U$ and a sequence of entire functions $\{g_\nu\}_{\nu\ge1} \subset \mathcal{O}(\mathbb{C}^{n+1})$ such that $\lim_{\nu\to\infty} g_\nu(\zeta) = v(\zeta)$ uniformly in $\zeta \in V$. Let us consider the holomorphic map

$$s_0 : U_0 \to \mathbb{C}^{n+1}, \quad s_0(\pi_0(\zeta)) = \left(1,\ \frac{\zeta_1}{\zeta_0}, \ldots, \frac{\zeta_n}{\zeta_0}\right),$$

for any $\zeta = (\zeta_0, \dots \zeta_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. The map s_0 is also a section in $\mathbb{C}^* \to \mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{\pi_0} \mathbb{C}P^n$ (with $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$) and in particular $s_0(N \cap U_0) \subset U$. Finally, let us set $f_\nu = g_\nu \circ s_0$ so that $f_\nu \in \mathcal{O}(U_0)$. Every submersion is an open map, so that $\pi(V)$ is an open neighborhood of p_0 in M and $\lim_{\nu\to\infty} f_\nu(p) = u(p)$ uniformly in $p \in \pi(V)$. Q.e.d.

Corollary 1 *Let $m_1, \dots, m_k \in \mathbb{Z}$ be odd positive integers and let $n + 1 = \sum_{i=1}^k m_i$. Let $r_1, \dots, r_k \in \mathbb{R}$ be positive numbers such that $\sum_{i=1}^k r_i^2 = 1$. Let us consider the CR submanifold*

$$N_{m_1 \dots m_k} = \pi\left(S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k)\right) \subset \mathbb{C}P^m, \quad 2m + 1 > n + k.$$

Then each CR function on $N_{m_1 \dots m_k}$ may be uniformly approximated, in some neighborhood of a point $p_0 \in N_{m_1 \dots m_k}$, by a sequence of holomorphic functions defined on some neighborhood of p_0 in $\mathbb{C}P^m$.

To prove Corollary 1, we need to consider the immersion

$$M_{m_1 \dots m_k} = S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k) \to S^{n+k}, \quad n + 1 = \sum_{i=1}^k m_i,$$

where $m_1, \dots, m_k$ are odd and $\sum_{i=1}^k r_i^2 = 1$. Then $M_{m_1 \dots m_k}$ has parallel second fundamental form and flat normal connection (cf., e.g., [30], p. 60). Also if $2m + 1 > n + k$ then $M_{m_1 \dots m_k}$ is a contact CR submanifold in S^{2m+1}. In particular $N_{m_1 \dots m_k} = \pi\left(M_{m_1 \dots m_k}\right)$ is a generic CR submanifold of $\mathbb{C}P^m$ [and $S^1 \to M_{m_1 \dots m_k} \xrightarrow{\pi} N_{m_1 \dots m_k}$ is a principal subbundle of the Hopf bundle $S^1 \to S^{2m+1} \xrightarrow{\pi} \mathbb{C}P^m$] hence Proposition 2 applies.

Positive results on the CR extension problem, from a CR submanifold M in a complex space form $M^{2N}(c)$, should depend (in view of our experience regarding the case $c = 0$, cf. Theorems 4.4 and 4.5 in Sect. 5.1) on the definiteness properties of the second fundamental form. A condition similar to strict pseudoconvexity (yet weaker) was considered by K. Yano and M. Kon (cf. [28]). Let M be a submanifold of $\mathbb{C}P^N$. The second fundamental form of $\Psi : M \to \mathbb{C}P^N$ is said to be *semidefinite* if for any $V \in C^\infty(E(\Psi))$ either $g(A_V X, X) \geq 0$ everywhere on M for any $X \in \mathfrak{X}(M)$, or $g(A_V X, X) \leq 0$ everywhere on M for any $X \in \mathfrak{X}(M)$. As a consequence of (4.14) for any real hypersurface $M \subset \mathbb{C}P^N$ with semidefinite second fundamental form the Levi form G_η is semidefinite (yet may degenerate in certain directions). The second fundamental form of a strictly pseudoconvex hypersurface is not semidefinite in general. The main finding in [28] is

Theorem 4.8 *Let M^m be a compact orientable m-dimensional generic CR submanifold of $\mathbb{C}P^{(m+p)/2}$ with semidefinite second fundamental form and flat normal connection. If $\sum_a \operatorname{Tr} A_a^2 \leq (m-1)p$ then $p = 1$ and M^m is isometric to the geodesic hypersphere $\pi\left(S^m(r) \times S^1(r)\right)$ in $\mathbb{C}P^{(m+1)/2}$ with $r = 1/\sqrt{2}$.*

Here $\{V_a\}$ is a parallel frame of the normal bundle and $A_a = A_{V_a}$. It is unknown whether the isometry prompted by Theorem 4.8 is also a CR isomorphism. If that were the case, then Proposition 2 would apply to any compact generic CR submanifold $M^m \hookrightarrow \mathbb{C}P^{(m+p)/2}$ obeying to the geometric requirements in Theorem 4.8.

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Chapter 5
CR Submanifolds in (l.c.a.) Kaehler and *S*-manifolds

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5.1 Introduction

K. Yano [21] introduced in 1963 the notion of f-structure on a $(2m + s)$-dimensional manifold as a tensor field f of type (1, 1) and rank $2m$ satisfying $f^3 + f = 0$. Almost complex ($s = 0$) and almost contact ($s = 1$) structures are well-known examples of f-structures. A Riemannian manifold endowed with an f-structure ($s \geq 2$) compatible with the Riemannian metric is called a metric f-manifold (for $s = 0$ we have almost Hermitian manifolds and for $s = 1$, metric almost contact manifolds). In this context, D.E. Blair [5] defined K-manifolds (and particular cases of S-manifolds and C-manifolds) as the analogue of Kaehlerian manifolds in the almost complex geometry and of quasi-Sasakian manifolds (and particular cases of Sasakian manifolds and cosymplectic manifolds) in the almost contact geometry.

He also showed that the curvature of S-manifolds is completely determined by their f-sectional curvatures. Later, M. Kobayashi and S. Tsuchiya [15] got expressions for the curvature tensor field of S-manifolds when their f-sectional curvature is constant depending on such a constant. Such spaces are called S-space-forms and they generalize complex and Sasakian space-forms. Nice examples of S-space-forms

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can be found in [5, 6, 8, 13]. In particular, it is proved in [5, 8] that certain principal toroidal bundles over complex-space-forms are S-space-forms and a generalization of the Hopf fibration denoted by $\mathbb{H}^{2m+s}$ is introduced as a canonical example of such manifolds playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry [5, 6].

When we want to study the submanifolds of a metric f-manifold, the natural first step is to consider such submanifolds depending on their behavior with respect to the f-structure. So, invariant and anti-invariant submanifolds (in the terminology of the complex geometry, holomorphic and totally real submanifolds) appear if all the tangent vector fields to the submanifold are transformed by f into tangent vector fields or into normal vector fields. But since an hypersurface of a metric f-manifold tangent to the structure vector fields is neither invariant nor anti-invariant, it is necessary to introduce a wider class of submanifolds: the *CR*-submanifolds. This work was made firstly by A. Bejancu and B.-Y. Chen [1, 10, 11] in the case $s = 0$ and by A. Bejancu and N. Papaghiuc, M. Kobayashi and K. Yano and M. Kon in the case $s = 1$ (we refer to the books [3, 22] for the background of these cases where a large list of fundamental references can be found). For $s \geq 2$, I. Mihai [16] introduced the notion of *CR*-submanifold in a natural way.

Many authors have studied the geometry of submanifolds of locally conformal almost Kaehler (l.c.a.K.) manifolds [10, 11, 14, 20], which are almost Hermitian manifolds $(\widetilde{M}, J, g)$ such that every $x \in \widetilde{M}$ has an open neighborhood U such that for some differentiable function $h : U \longrightarrow \mathbb{R}$, $\tilde{g}_U = e^{-h} g|_U$ is a (l.c.a.) Kaehler metric on U. If one can take $U = \widetilde{M}$, the manifold is then called globally conformal almost Kaeler (g.c.a.K) manifold. Examples of l.c.K. manifolds are provided by the Hopf manifolds. So, it seems interesting to study *CR*-submanifolds of l.c.a.K. manifolds.

On the other hand, M. Okumura [17, 18] studied normal real hypersurfaces of Kaehlerian manifolds and obtained nice properties. For this reason, it also seems interesting to introduce and study normal *CR*-submanifolds. In the cases $s = 0$ and $s = 1$, the papers [2] and [4] can be consulted.

The aim of the present work is to briefly summarize our contributions to the study of *CR*-submanifolds of l.c.a.K. manifolds, normal *CR*-submanifolds of S-manifolds. To this end, we separate them into two different sections, which can be read independently.

5.2 CR-Submanifolds of (l.c.a.) Kaehler Manifolds

Let $(\widetilde{M}, J, g)$ be an almost Hermitian manifold $(dim(\widetilde{M}) = 2m)$ with almost complex structure J and Hermitian metric g and let M be a Riemannian submanifold isometrically immersed in $\widetilde{M}$.

A. Bejancu [1] introduced the notion of a *CR*-submanifold of $\widetilde{M}$. In fact, M is a *CR*-submanifold of the almost Hermitian manifold $\widetilde{M}$ if there exists on M a differentiable holomorphic distribution $\mathcal{D}$, i.e., $J(\mathcal{D}_x) \subseteq \mathcal{D}_x$ for any $x \in M$ such that its orthogonal complement $\mathcal{D}^{\perp}$ in M is totally real in $\widetilde{M}$, i.e., $J(\mathcal{D}^{\perp}_x) \subseteq T^{\perp}_x(M)$ for any $x \in M$, where $T_x^{\perp}(M)$ is the normal space at x. If $dim(\mathcal{D}) = 0$, M is called a *totally real* submanifold, and if $dim(\mathcal{D}^{\perp}) = 0$ M is a *holomorphic* submanifold.

We first discuss the Gauss–Weingarten equations of the submanifold with respect to the metric g and with respect to the local conformal Kaehler metrics and then we shall establish thereby the analytical conditions that characterize the important types of submanifolds.

5.2.1 Preliminaries

Let $(\widetilde{M}, J, g)$ be an almost Hermitian manifold. It is easy to see [20] that $(\widetilde{M}, J, g)$ is a l.c.(a).K. manifold if and only if there is a global closed 1-form ω on $\widetilde{M}$ (the Lee form) such that $d\Omega = \omega \wedge \Omega$ (Ω the fundamental form of the manifold) and $(\widetilde{M}, J, g)$ is a g.c.(a).K. manifold if and only if ω is also exact. In case $\omega = 0$, the manifold is an (almost) Kaehler manifold.

Let $(\widetilde{M}, J, g)$ be a l.c.(a).K. manifold and consider the Lee vector field B [20] of $(\widetilde{M}, J, g)$ defined by $g(X, B) = \omega(X)$. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of g and define

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B. \tag{5.1}$$

Then $\overline{\nabla}$ is a torsionless linear connection on $\widetilde{M}$ which is called the Weyl connection of g. It is easy to see that $\overline{\nabla}_X g = \omega(X)g$. We have

Theorem 5.1 ([20]) *The almost Hermitian manifold $(\widetilde{M}, J, g)$ is a l.cK. manifold if and only if there is a closed 1-form ω on $\widetilde{M}$ such that the Weyl connection is almost complex, That is, $\overline{\nabla} J = 0$.*

Let $(\widetilde{M}, J, g)$ be a l.c.K. manifold and M a Riemannian manifold isometrically immersed in $\widetilde{M}$. We denote by g the metric tensor of $\widetilde{M}$ as well as that induced on M, and let ∇, ∇^M be the covariant derivations on M induced by $\widetilde{\nabla}$ and $\overline{\nabla}$, respectively. Then, the Gauss–Weingarten formulas for M with respect to $\widetilde{\nabla}$ and $\overline{\nabla}$ are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \ \ \widetilde{\nabla}_X V = -A_V X + D_X V, \tag{5.2}$$

$$\overline{\nabla}_X Y = \nabla^M_X Y + \overline{\sigma}(X, Y), \ \ \overline{\nabla}_X V = -\overline{A}_V X + \overline{D}_X V, \tag{5.3}$$

for any vector fields X, Y tangent to M and V normal to M, where σ (respectively, $\overline{\sigma}$) is the second fundamental form of M with respect to $\widetilde{\nabla}(\overline{\nabla})$ and D (respectively, $\overline{D}$) is the normal connection. The formulas (5.3) are the Gauss–Weingarten equations of $M|_U$ in $(\widetilde{M}|_U, e^{-h}g|_U)$. The second fundamental tensors $A_V, \overline{A}_V$ are related to $\sigma, \overline{\sigma}$ respectively by

$$g(A_V X, Y) = g(\sigma(X, Y), V), \ \ g(\overline{A}_V X, Y) = g(\overline{\sigma}(X, Y), V). \tag{5.4}$$

For any vector X tangent to M and V normal to M write

$$JX = TX + NX, \ JV = tV + nV, \tag{5.5}$$

where TX and NX (respectively, tV and nV) are the tangential and normal component of $J(X)$ (respectively JV). For the Lee field B, we have

$$B_x = (B_x)_1 + (B_x)_2, \quad x \in M, \tag{5.6}$$

where $(B_x)_1$ (resp.y $(B_x)_2$) is the tangential (resp. normal) component of B_x.

If M is a CR-submanifold of an almost Hermitian manifold $(\widetilde{M}, J, g)$ let us denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^\perp$ in $T^\perp(M)$. Hence we have, $T^\perp(M) = J\mathcal{D}^\perp \oplus \nu$.

5.2.2 Integrability Conditions of the Basic Distributions

First we give some general identities.

Lemma 1 *Let M be a CR-submanifold of a l.c.K. manifold $(\widetilde{M}, J, g)$. Then, we have*

$$\nabla^M_X Y = \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B_1 \tag{5.7}$$

$$\overline{\sigma}(X, Y) = \sigma(X, Y) + \frac{1}{2}g(X, Y)B_2 \tag{5.8}$$

$$\overline{A}_V X = A_V X + \frac{1}{2}\omega(V)X \tag{5.9}$$

$$\overline{D}_V X = D_V X - \frac{1}{2}\omega(X)V \tag{5.10}$$

for any vector fields X, Y tangent to M and V normal to M.

Proof The assertions follow immediately from (5.1)–(5.3). ∎

The following result is well known:

Theorem 5.2 ([7]) *The totally real distribution $\mathcal{D}^\perp$ of any CR-submanifold of a l.c.K. manifold is integrable.*

For the holomorphic distribution $\mathcal{D}$, we have

Theorem 5.3 *Let M be a submanifold of a l.c.K. manifold $\widetilde{M}$ and let $\mathcal{D}_x$ de maximal holomorphic subspace of $T_x(M)$ and assume dim$(\mathcal{D}_x)$ is a constant. Then, the holomorphic distribution $\mathcal{D}$ is integrable if and only if the second fundamental form $\overline{\sigma}$ satisfies $\overline{\sigma}(X, JY) = \overline{\sigma}(JX, Y)$ or, equivalently, $\sigma(X, JY) - \sigma(JX, Y) + \Omega(X, Y)B_2 = 0$, for all vector fields $X, Y \in \mathcal{D}$.*

If M is a CR-submanifold, the integrability condition on $\mathcal{D}$ in Theorem 5.3 can be replaced by a weaker condition.

Theorem 5.4 *Let M be a CR-submanifold of a l.c.K. manifold $\widetilde{M}$. The holomorphic distribution $\mathcal{D}$ is integrable if and only if*

$$g\left(\sigma(X, JY) - \sigma(JX, Y) + \Omega(X, Y)B, J\mathcal{D}^{\perp}\right) = 0,$$

for all $X, Y \in \mathcal{D}$.

Theorems 5.3 and 5.4 follow easily from similar theorems in the Kaehlerian case ([7]), from (5.8) and the fact that, locally, $\widetilde{M}$ is endowed with Kaehler metrics $\widetilde{g}_U$ whose Levi-Civita connection is $\overline{\nabla}$.

With regard to integral submanifolds of $\mathcal{D}^{\perp}$ and $\mathcal{D}$ (provided $\mathcal{D}$ is integrable), we have the following theorem.

Theorem 5.5 *For a CR-submanifold M of a l.c.K. manifold $\widetilde{M}$, the leaf $M^{\perp}$ is totally geodesic in M if and only if*

$$g\left(A_{JW}Z + \frac{1}{2}g(Z, W)JB, \mathcal{D}\right) = 0,$$

that is,

$$g(\sigma(Z, X), JW) = \frac{1}{2}g(Z, W)\omega(JW),$$

for any $X \in \mathcal{D}$, $Z, W \in \mathcal{D}^{\perp}$.

Proof From (5.1), (5.2) and $\overline{\nabla}J = 0$, for any $X \in \mathcal{D}$, $Z, W \in \mathcal{D}^{\perp}$, we obtain

$$g(J\nabla_Z W, X) + \frac{1}{2}g(Z, W)g(JB, X) = -g(A_{JW}Z, X). \tag{5.11}$$

But $M^{\perp}$ is totally geodesic in M if and only if $\nabla_Z W \in \mathcal{D}^{\perp}$ for all $Z, W \in \mathcal{D}^{\perp}$, and then (5.11) gives the theorem. ■

Theorem 5.6 *Let M be a CR-submanifold of a l.c.K manifold $\widetilde{M}$. If the holomorphic distribution $\mathcal{D}$ is integrable and M^T is an integral submanifold of $\mathcal{D}$, then M^T is totally geodesic if and only if*

$$g\left(J\sigma(X, Y) + \frac{1}{2}g(X, Y)JB - \frac{1}{2}\Omega(X, Y)B, \mathcal{D}^{\perp}\right) = 0,$$

for any $X, Y \in \mathcal{D}$.

Proof From (5.1), (5.3), and $\overline{\nabla} J = 0$, for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$, we have

$$g(J\sigma(X,Y),Z) + \frac{1}{2}g(X,Y)g(JB,Z) = g(\nabla_X(JY),Z) + \frac{1}{2}\Omega(X,Y)g(B,Z). \tag{5.12}$$

But M^T is totally geodesic in M if and only if $\nabla_X Y \in \mathcal{D}$ for all $X, Y \in \mathcal{D}$, and hence Eq. (5.12) gives the theorem. ■

5.2.3 CR-Submanifolds of l.c.K. Manifolds

First of all, we shall give some identities for later use. Let T, N t, and n be the endomorphisms and vector-valued 1-forms defined in (5.5). The following lemma can be easily obtained from (5.3), (5.9), and $\overline{\nabla} J = 0$.

Lemma 2 *Let M be an isometrically immersed submanifold of a l.c.K. manifold $\widetilde{M}$. Then, we have*

$$\nabla^M_X(TY) - \overline{A}_{NY}X = T\nabla^M_X Y + t\overline{\sigma}(X,Y) \tag{5.13}$$

$$\overline{\sigma}(X,TY) + D_X(NY) = N\nabla^M_X Y + n\overline{\sigma}(X,Y), \tag{5.14}$$

$$\nabla^M_X(tV) - \overline{A}_{nV}X = -T\overline{A}_V X + t\overline{D}_X V, \tag{5.15}$$

$$\overline{\sigma}(X,tV) + \overline{D}_X(nV) = -N\overline{A}_V X + n\overline{D}_X V, \tag{5.16}$$

$$\left[\overline{A}_V, \overline{A}_{\overline{V}}\right] = \left[A_V, A_{\overline{V}}\right], \tag{5.17}$$

for any vector fields X, Y tangent to M and V, $\overline{V}$ normal to M.

Now, we shall study totally umbilical and totally geodesic CR-submanifolds.

Theorem 5.7 *Let M be a totally umbilical CR-submanifold of a l.c.K. manifold $\widetilde{M}$. Then, we have*

(i) Either $\dim(\mathcal{D}^{\perp}) = 1$ or the component $H_{J(TM)}$ of the mean curvature tensor H in $J(TM)$ is given by $H_{J(TM)} = -\frac{1}{2}B_2$.
(ii) If $\dim(\mathcal{D}^{\perp}) > 1$ and M is proper (neither holomorphic nor totally real) such that B is tangent to M, then M is totally geodesic.

Proof First, since M is totally umbilical, $\sigma(X,Y) = g(X,Y)H$ for any X, Y tangent to M, and hence

$$g(\sigma(X,X),JW) = g(X,X)g(H,JW). \tag{5.18}$$

From (5.3) and (5.4) it is easy to see that

$$\overline{A}_{JZ}W = \overline{A}_{JW}Z \tag{5.19}$$

and, then, if we take an unit vector field $X = Z \in \mathcal{D}^{\perp}$ orthogonal to W, (5.9), (5.18), and (5.19) give

$$\begin{aligned} g(H, JW) =& g(A_{JW}Z, Z) = g(A_{JZ}W + \frac{1}{2}\omega(JZ)W - \frac{1}{2}\omega((JW)Z, Z) \\ =& -\frac{1}{2}\omega(JW) = g(-\frac{1}{2}B_2, JW), \end{aligned}$$

so that (i) holds.

Now, since $dim(\mathcal{D}^{\perp}) > 1$, from (5.5) and assertion (i), we have $tH = 0$. Thus, (5.15) gives $t\overline{D}_Y H = \overline{A}_{nH}Y - T\overline{A}_H Y$, for any Y tangent to M. Therefore, for any Z tangent to M, from (5.8) and (5.9) we get

$$g(t\overline{D}_Y H, Z) = -g(\overline{A}_H Y, TZ) - g(\overline{\sigma}(Y, Z), nH) = -g(Y, TZ)g(H, H) \tag{5.20}$$

and, if we take $Z = TY$, we have

$$-g(Y, T^2Y)g(H, H) = g(t\overline{D}_Y H, TY) = g(Tt\overline{D}_Y H, Y) = 0. \tag{5.21}$$

The last equation holds because $Tt = 0$ for any CR-submanifold of an almost Hermitian manifold [22]. Moreover, it is easy to see [22] that $T^2 = -I + tN$ and then (5.21) gives

$$g(Y, Y)g(H, H) - g(NY, NY)g(H, H) = 0. \tag{5.22}$$

Since M is proper, we can choose an unit vector field X in $\mathcal{D}$. Thus, $NX = 0$ and from (5.22) we have $H = 0$. ■

Theorem 5.8 *Let M be a totally geodesic CR-submanifold of a l.c.K. manifold $\widetilde{M}$. We have*

(i) If $B_x \in \mathcal{D}_x$, for all $x \in M$, then $\mathcal{D}$ is integrable and any integral submanifold M^T of $\mathcal{D}$ is totally geodesic in $\widetilde{M}$.
(ii) If B is normal to M, any integral submanifold $M^{\perp}$ of $\mathcal{D}^{\perp}$ is totally geodesic in $\widetilde{M}$. Furthermore, $\mathcal{D}$ is integrable if and only if $B_x \in \nu_x$, for any $x \in M$, and in this case any integral submanifold M^T of $\mathcal{D}$ is totally geodesic in $\widetilde{M}$.

Proof Firstly, since B is tangent to M, from Theorem 5.7 the distribution $\mathcal{D}$ is integrable. Let M^T be an integral submanifold of $\mathcal{D}$. For any vector field X tangent to M, $Y \in \mathcal{D}$, $Z \in \mathcal{D}^{\perp}$, from (5.3) and (5.4) we get $g(\nabla^M_X Z, Y) = -g(\overline{\sigma}(X, JY), JZ)$. But from (5.7) and (5.8) we find

$$g(\nabla_X Z, Y) - \frac{1}{2}\omega(Z)g(X, Y) + \frac{1}{2}g(X, Z)g(B, Y) = -g(\sigma(X, JY), JZ) = 0. \tag{5.23}$$

If $X \in \mathcal{D}$, (5.23) gives $g(\nabla_X Z, Y) = 0$, or, equivalently, $g(\nabla_X Y, Z) = 0$ and therefore, $\nabla_X Y \in \mathcal{D}$. Thus M^T is totally geodesic in M and hence in $\widetilde{M}$.

Next, if B is normal to M, from Theorem 5.5, any integral submanifold $M^\perp$ of $\mathcal{D}^\perp$ is totally geodesic in $\widetilde{M}$. The second statement follows immediately from Theorems 5.6 and 5.7. ∎

Corollary 1 *Let M be a totally geodesic proper CR-submanifold of a l.c.K. manifold $\widetilde{M}$ such that $B_x \in \nu_x$, for any $x \in M$. Then, M is locally the Riemannian product of a Kaehler submanifold and a totally real submanifold of $\widetilde{M}$.*

Proof From Theorem 5.8, M is locally the product of a holomorphic submanifold M^T and a totally real submanifold $M^\perp$ of $\widetilde{M}$. But $\omega = 0$ on M, so that we have induced on M^T a Kaehlerian structure. Moreover, it can be easily seen that the projection map p (resp., q) onto $\mathcal{D}$ (resp., $\mathcal{D}^\perp$) is parallel with respect to ∇, so that this local product is actually a local Riemannian product. ∎

Next, we consider the particular case when M is either holomorphic or totally real.

Lemma 3 *Let M be a holomorphic submanifold of a l.c.K. manifold $\widetilde{M}$. Then the subbundles TM and $T^\perp(M)$ are holomorphic. Moreover, we have*

$$\overline{\sigma}(JX, Y) = \overline{\sigma}(X, JY) = J\overline{\sigma}(X, Y), \tag{5.24}$$

$$\overline{A}_{JV} = J\overline{A}_V = -\overline{A}_V J, \tag{5.25}$$

$$\overline{D}_X(JV) = J\overline{D}_X V, \tag{5.26}$$

$$\nabla^M_X(JY) = J\nabla^M_X Y, \tag{5.27}$$

for any vector fields X, Y tangent to M and V normal to M.

Proof As $\widetilde{M}$ is locally endowed with Kaehler metrics $\widetilde{g}_U$ whose Levi-Civita connection is $\overline{\nabla}$, these formulas follow from similar formulas in the Kaehlerian case. ∎

Theorem 5.9 *Let M be a holomorphic submanifold of a l.c.K. manifold $\widetilde{M}$. Then, we have*

(i) The mean curvature vector H of M is given by $H = -\frac{1}{2}B_2$.
(ii) M is totally umbilical if and only if the Weingarten endomorphisms are commutative.

Proof Firstly, if $dim(M) = 2k > 0$, let $\{e_1, \ldots, e_k, Je_1, \ldots, Je_k\}$ be an orthonormal basis for $T_x(M)$, $x \in M$. Then

$$2kH_x = (tr(\sigma))_x = \sum_{i=1}^{k} \sigma_x(e_i, e_i) + \sum_{i=1}^{k} \sigma_x(Je_i, Je_i). \tag{5.28}$$

But from (5.8) and (5.24), (5.28) gives $2kH_x = -k(B_2)_x$.
Next, let V be a vector field normal to M. From (5.17) and (5.25), we have

$$0 = [A_V, A_{JV}] = [\overline{A}_V, \overline{A}_{JV}] = -2J(\overline{A}_V)^2,$$

Thus $\overline{A}_V = 0$ and from (5.9), we have $A_V = -\frac{1}{2}\omega(V)I$ ■

The endomorphism n of the normal bundle $T^{\perp}M$ defined in (5.5) induces an f-structure in $T^{\perp}M$ [22]. For any vector field X tangent to M and V normal to M, we write

$$(\widetilde{\nabla}'_X n)V = D_X(nV) - nD_X V,$$

$$(\overline{\nabla}'_X n)V = \overline{D}_X(nV) - n\overline{D}_X V.$$

When $\widetilde{\nabla}'n = 0$, the f-structure n is said to be parallel [10].

Lemma 4 *Let M be an r-dimensional totally real submanifold of a $2m$-dimensional l.c.K. manifold $\widetilde{M}$. Then we have*

(i) $\overline{A}_{JX}Y = \overline{A}_{JY}X$, *for any X, Y tangent to M.*
(ii) *If $r = m$, then* $\overline{D}_X(JY) = J\nabla^M_X Y$, $\nabla^M_X(JV) = J\overline{D}_X V$, *and* $\overline{\sigma}(X, JV) = -J\overline{A}_V X$.
(iii) $\widetilde{\nabla}'n = \overline{\nabla}'n$.
(iv) *If the f-structure n is parallel, then*

$$A_V = -\frac{1}{2}\omega(V)I, \tag{5.29}$$

for any $V \in \nu$.
(v) *If the Weingarten endomorphisms are commutative, then there is an orthonormal local basis $\{e_1, \ldots, e_r\}$ in M such that with respect to this basis $\overline{A}_{Je_i}$ is a diagonal matrix*

$$\overline{A}_{Je_i} = (0 \ldots 0\, \lambda_i\, 0 \ldots 0)\,, \quad i = 1, \ldots, r. \tag{5.30}$$

Proof Assertions (i) and (ii) follow immediately from similar formulas in the Kaehlerian case. From Eq. (5.10), we easily obtain (iii).

In order to prove (iv), we take $V \in \nu$, and $X \in T(M)$. Then, (iii) gives $(\overline{\nabla}'_X n)V = \overline{D}_X(nV) - n\overline{D}_X V = 0$. By using (5.25) and (5.26) this yields $J\overline{A}_V X = 0$. Therefore, $\overline{A}_V = 0$ and from (5.9), we obtain (iv).

Finally, from (5.17) we have $[\overline{A}_V, \overline{A}_{\overline{V}}] = 0$, for any $V, \overline{V}$ normal to M. Then, we can find a local orthonormal basis $\{\widetilde{e}_1 \dots, \widetilde{e}_r\}$ in M (with respect to the local Kaehlerian metrics $\widetilde{g}_U = e^{-h} g|_U$) such that $\overline{A}_{Je_i} = (0 \dots \mu_i \dots 0)$, $i = 1, \dots, r$. If we start by using this basis, we can obtain an orthonormal (with respect to the metric g) local basis $\{e_1, \dots, e_n\}$ in M such that (v) holds. ■

Theorem 5.10 *Let M be an r-dimensional totally real and minimal submanifold of a l.c.K. manifold $\widetilde{M}$ such that their Weingarten endomorphisms are commutative and the f-structure n is parallel. Then, we have*

(i) If $r \geq 2$, M is totally geodesic if and only if the Lee vector field B is tangent to M.
(ii) If $r = 1$ and B is orthogonal to ν, then M is a geodesic curve.

Proof First, since the Weingarten endomorphisms are commutative, let $\{e_1, \dots, e_r\}$ be an orthonormal local basis as in Lemma 4 (v). From Eq. (5.8), we have

$$0 = g(H, Je_i) = \frac{1}{n}\sum_{j=1}^{n} g\left(\sigma(e_j, e_j), Je_i\right) = \frac{1}{n}\sum_{j=1}^{n} g\left(\overline{A}_{Je_i} e_j, e_j\right) - \frac{1}{2}\omega(Je_i)$$
$$= \frac{1}{n}\lambda_i - \frac{1}{2}\omega(Je_i), \quad i = 1, \dots, r.$$

Therefore,

$$\overline{A}_{Je_i} e_j = \delta_{ij}\lambda_i e_j = \delta_{ij}\frac{n}{2}\omega(Je_i)e_j, \quad i = 1, \dots, r. \tag{5.31}$$

Now, from (5.9) and (5.31) we obtain,

$$A_{Je_i} e_j = \frac{1}{2}(n\delta_{ij} - 1)\omega(Je_i)e_j, \quad i = 1, \dots, r. \tag{5.32}$$

Thus, if $r \geq 2$ and B is tangent to M, Eq. (5.32) gives $A_{Je_i} = 0$, $i = 1, \dots, r$. Moreover, from (iv) in Lemma 4, $A_V = -\frac{1}{2}\omega(V)I = 0$, for any $V \in \nu$. Then, $A_{\overline{V}} = 0$, for any vector field $\overline{V}$ normal to M.

On the other hand, if there is $x \in M$ such that $(B_2)_x \neq 0$, from (5.32) and (iv) in Lemma 4, we can take a vector field V normal to M such that $A_V \neq 0$. This gives (i).

In order to prove (ii), let us take a unit vector field X tangent to M. We have $0 = g(H, JX) = g(\sigma(X, X), JX) = g(A_{JX}X, X)$, and, then, $A_{JX} = 0$. But, if B is orthogonal to ν, from (iv) in Lemma 4, $A_V = -\frac{1}{2}\omega(V) = 0$, for any $V \in \nu$. This means that $A_{\overline{V}} = 0$, for any vector field $\overline{V}$ normal to M. ■

Theorem 5.11 *Let M be an r-dimensional $(r \geq 2)$ totally real and totally umbilical submanifold of a l.c.K. manifold $\widetilde{M}$ such that the f-structure n is parallel. Then M is totally geodesic if and only if B is tangent to M.*

Proof Let $\{u_1, \dots, u_r\}$ be an orthonormal local basis in U. Since M is totally umbilical, for any vector field X tangent to M, by using Eqs. (5.8) and (5.9), we find

$$g(\overline{A}_{JX}u_j, u_k) = \frac{1}{r}\delta_{jk}tr(\overline{A}_{JX}). \tag{5.33}$$

But from Eq. (5.9) and (iv) in Lemma 4 we also have

$$\overline{A}_V = 0, \tag{5.34}$$

for any $V \in \nu$. On the other hand $[A_{\overline{V}}, A_{\overline{\overline{V}}}] = [\overline{A}_{\overline{V}}, \overline{A}_{\overline{\overline{V}}}] = 0$, for any vector fields $\overline{V}, \overline{\overline{V}}$ normal to M. Therefore, from (v) in Lemma 4, there is an orthonormal local basis $\{e_1, \ldots, e_r\}$ in M such that, with respect to this basis, Eq. (5.30) holds. But, from Eq. (5.33), we also have

$$g(\overline{A}_{Je_i}e_j, e_j) = \frac{1}{r}\lambda_i, \quad i.j = 1, \ldots, n. \tag{5.35}$$

Since $r \geq 2$, we can take $j \neq i$ and then, Eqs. (5.30) and (5.35) give $\lambda_i = 0$, $i = 1, \ldots, r$. Thus, we get $\overline{A}_{Je_i} = 0$, $i = 1, \ldots, r$, which, together with (5.34) gives $\overline{A}_{\overline{V}} = 0$, for any vector field $\overline{V}$ normal to M. Now, if B is tangent to M, Eq. (5.9) proves that M is totally geodesic.

Conversely, if M is totally geodesic, from (5.29) we have $0 = A_V = -\frac{1}{2}\omega(V)I$, for any $V \in \nu$. This means that B is normal to ν. Furthermore, from (v) in Lemma 4, we can find an orthonormal local basis $\{e_1, \ldots, e_r\}$ in M such that $\overline{A}_{Je_i}$ has a diagonal matrix $\overline{A}_{Je_i} = (0 \ldots 0\, \lambda_i\, 0 \ldots 0) = \frac{1}{2}\omega(Je_i)I$, $i = 1, \ldots, r$. Since $r \geq 2$, this means $\omega(Je_i) = 0$, $i = 1, \ldots, r$ so that B is normal to $J(T(M))$. Thus, B is tangent to M. ■

5.2.4 CR-products in l.c.K. Manifolds

Let T, N, t, n be the endomorphisms and vector-valued 1-forms defined by (5.5). Let us write

$$\begin{aligned}(\widetilde{\nabla}'_Z T)W &= \nabla_Z(TW) - T\nabla_Z W,\\ (\overline{\nabla}'_Z T)W &= \nabla^M_Z(TW) - T\nabla^M_Z W,\end{aligned} \tag{5.36}$$

for all Z, W tangent to M. On the other hand, T is said to be parallel if $\overline{\nabla} T = 0$. From (5.1)–(5.3) it is easy to prove that

$$\begin{aligned}(\overline{\nabla}'_Z T)W = (\widetilde{\nabla}'_Z T)W &+ \frac{1}{2}\omega(W)TZ - \frac{1}{2}\omega(TW)Z\\ &+ \frac{1}{2}g(Z, TW)B_1 - \frac{1}{2}g(Z, W)TB_1.\end{aligned} \tag{5.37}$$

But, from (5.36) we see that

$$(\overline{\nabla}'_Z T)W = t\overline{\sigma}(Z, W) + \overline{A}_{NW}Z. \tag{5.38}$$

Definition 1 A *CR*-submanifold of a l.c.K. manifold $\widetilde{M}$ is called a *CR-product* if it is locally a Riemannian product of a holomorphic submanifold M^T and a totally real submanifold $M^\perp$ of $\widetilde{M}$.

Theorem 5.12 *Let M be a CR-submanifold of a l.c.K. manifold $\widetilde{M}$ such that the Lee field B is normal to M. Then M is a CR-product if and only if T is parallel.*

Proof Since B is normal to M, from Eq. (5.37), we have $\widetilde{\overline{\nabla}}'T = \overline{\nabla}'T$. If T is parallel, from (5.8), (5.9), and (5.38), we find

$$t\sigma(Z, W) + \frac{1}{2}g(Z, W)tB = -A_{NW}Z - \frac{1}{2}\omega(NW)Z. \tag{5.39}$$

But for any $X \in \mathcal{D}$, $NX = 0$, and the last equation gives

$$0 = g(A_{NW}Z, X) + \frac{1}{2}\omega(NW)g(Z, X),$$

or, equivalently, $g(\sigma(Z, X), JW) + \frac{1}{2}g(JW, B)g(Z, X) = 0$, for any W tangent to M. Therefore,

$$\sigma(Z, X) = -\frac{1}{2}g(Z, X)B. \tag{5.40}$$

If we take $Z \in \mathcal{D}$, the last equation gives $\sigma(X, JY) - \sigma(JX, Y) = -\Omega(X, Y)B$, and, from Theorem 5.3, $\mathcal{D}$ is integrable. Let M^T be an integral submanifold of $\mathcal{D}$. For any $Z \in \mathcal{D}^\perp$, Eq. (5.40) yields $g(\sigma(Z, X), JZ) = -\frac{1}{2}g(Z, Z)g(B, JZ)$ and, from Theorem 5.6, the submanifold M^T is totally geodesic in M. Now, let $M^\perp$ be an integral submanifold od $\mathcal{D}^\perp$. From (5.40), if $Z \in \mathcal{D}^\perp$, then $\sigma(Z, X) = 0$ and, from Theorem 5.5, $M^\perp$ is totally geodesic.

Conversely, assume that M is a *CR*-product. First, we prove that $\nabla^M_Z X \in \mathcal{D}$, for any $X \in \mathcal{D}$ and Z tangent to M. As M is locally a Riemannian product of M^T (holomorphic submanifold) and $M^\perp$ (totally real submanifold), it suffices to prove that $\nabla^M_Z X \in \mathcal{D}$, for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. In fact, from (5.3) we have

$$J\nabla^M_Z X = \nabla^M_Z (JX) + \overline{\sigma}(Z, JX) - J\overline{\sigma}(Z, X).$$

Thus, if $W \in \mathcal{D}^\perp$, $g(J\nabla^M_Z X, JW) = g(\overline{\sigma}(Z, JX), JW)$. Since $M^\perp$ is totally geodesic in M, from (5.8) and Theorem 5.5 we have $g(\nabla^M_Z X, W) = 0$, for any $W \in \mathcal{D}^\perp$. So, $\nabla^M_Z X \in \mathcal{D}$ and $\nabla^M_Z X \in \mathcal{D}$, for any Z tangent to M. From $\overline{\nabla}J = 0$, we find

$$J\nabla^M_Z X + J\overline{\sigma}(Z, X) = \nabla^M_Z (JX) + \overline{\sigma}(Z, JX),$$

and then, $J\nabla^M_Z X = \nabla^M_Z (JX)$, $J\overline{\sigma}(Z, X) = \overline{\sigma}(Z, JX)$. Now, from (5.36) we get

$$(\overline{\nabla}'_Z T)X = \nabla^M_Z (TX) - T\nabla^M_Z X = \nabla^M_Z (JX) - J\nabla^M_Z X = 0, \tag{5.41}$$

for any $X \in \mathcal{D}$ and Z tangent to M.

In a similar way, we prove that $\nabla^M_Z Z \in \mathcal{D}^\perp$ for any $Z \in \mathcal{D}^\perp$ and Z tangent to M. Since M is a CR-product, it suffices to show this for $Z = X \in \mathcal{D}$. In fact, from (5.3), given any $Y \in \mathcal{D}$ we find that

$$g(J\nabla^M_X Z, Y) = -g(\overline{A}_{JZ}X, Y) - g(J\overline{\sigma}(X, Z), Y) = -g(\overline{\sigma}(X, Y), JZ) = 0,$$

where the last equation holds from (5.8) and Theorem 5.6. Then, $J\nabla^M_X Z$ is orthogonal to $\mathcal{D}$. On the other hand, if $W \in \mathcal{D}^\perp$, we have

$$g\left(\nabla^M_X Z, W\right) = -g(\overline{\sigma}(X, W), JZ) + g(\overline{\sigma}(X, Z), JW).$$

But, from Theorem 5.5 we have $g(J\nabla^M_X Z, W) = 0$, That is, $J\nabla^M_X Z$ is normal to M, so that $\nabla^M_X Z \in \mathcal{D}^\perp$. Therefore, we have

$$(\overline{\nabla}'_Z T)Z = \nabla^M_Z(TZ) - T\nabla^M_Z Z = 0. \tag{5.42}$$

Now, from (5.37), (5.41), and (5.42), we have $\widetilde{\nabla}' T = 0$. ■

Theorem 5.13 *Let M be a CR-submanifold of a l.c.K. manifold $\widetilde{M}$ such that $B_x \in \mathcal{D}_x$ for each $x \in M$. If T is parallel, then M is a CR-product. The converse does not holds unless $\dim(\mathcal{D}) = 2$ or $B = 0$ on M.*

Proof Since T is parallel, Eqs. (5.37) and (5.38) give

$$\begin{aligned} t\overline{\sigma}(Z, W) + \overline{A}_{NW}Z = \frac{1}{2}\omega(W)TZ - \frac{1}{2}\omega(TW)Z \\ + \frac{1}{2}g(Z, TW)B - \frac{1}{2}g(Z, W)TB. \end{aligned} \tag{5.43}$$

If $X \in \mathcal{D}$, then $NX = 0$ and (5.43) gives

$$\begin{aligned} -g(J\overline{\sigma}(X, Z), W) = \frac{1}{2}g(B, W)g(JZ, X) + \frac{1}{2}g(W, JB)g(Z, X) \\ - \frac{1}{2}g(JZ, W)g(B, X) - \frac{1}{2}g(Z, W)g(JB, X), \end{aligned} \tag{5.44}$$

for any vector field W tangent to M. From (5.8), (5.44) yields

$$\begin{aligned} -J\sigma(X, Z) = \frac{1}{2}g(JZ, X)B + \frac{1}{2}g(Z, X)JB \\ - \frac{1}{2}g(B, X)JZ - \frac{1}{2}g(JB, X)Z. \end{aligned} \tag{5.45}$$

For any $Z \in \mathcal{D}^\perp$, Eq. (5.45) gives $g(A_{JZ}, Z) = \frac{1}{2}\omega(JX)g(Z, Z)$, for any Z tangent to M and, hence, we have

$$A_{JZ}X = \frac{1}{2}\omega(JX)Z. \tag{5.46}$$

Next, for $Y \in \mathcal{D}$, from (5.46), we have

$$g(\sigma(X, Y), JZ) = 0, \quad \textit{for } X \in \mathcal{D},\ Z \in \mathcal{D}^{\perp}. \tag{5.47}$$

Therefore, $g(\sigma(X, JY) - \sigma(JX, Y, J\mathcal{D}^{\perp}) = 0$ and, from Theorem 5.4, the distribution $\mathcal{D}$ is integrable. Moreover, any integral submanifold $M^{\perp}$ od $\mathcal{D}$ is totally geodesic in M because of (5.47) and Theorem 5.6. Now, let $M^{\perp}$ be an integral submanifold of $\mathcal{D}^{\perp}$. For any $W \in \mathcal{D}^{\perp}$, Eq. (5.46) gives

$$g\left(A_{JZ} + \frac{1}{2}g(Z, W)JB, X\right) = 0$$

and this means that $M^{\perp}$ is totally geodesic in M (Theorem 5.5). Thus M is a *CR*-product. ■

In order to prove the converse, we first give the following Lemma.

Lemma 5 *If M is a CR-product in a l.c.K. manifold $\widetilde{M}$ such that $B_x \in \mathcal{D}_x$ for any $x \in M$, then*

$$\nabla_Z X \in \mathcal{D}, \tag{5.48}$$

$$\nabla_X Z \in \mathcal{D}^{\perp}, \tag{5.49}$$

$$J\nabla_Z X = \nabla_Z(JX), \tag{5.50}$$

for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

Proof If $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$, then from (5.7) and (5.8), we obtain

$$\begin{aligned} J\nabla_Z X = &\frac{1}{2}\omega(X)JZ - J\sigma(Z, X) + \nabla_Z(JX) + \nabla_Z(JX) \\ &- \frac{1}{2}\omega(JX)Z + \sigma(Z, JX). \end{aligned} \tag{5.51}$$

Now, for any $W \in \mathcal{D}^{\perp}$, (5.51) yields

$$g(J\nabla_Z X, JW) = g(\nabla_Z X, W) = g\left(A_{JW}Z + \frac{1}{2}g(Z, W)JB, JX\right) = 0.$$

The last equation holds because any leaf $M^{\perp}$ of $\mathcal{D}^{\perp}$ is totally geodesic in M (Theorem 5.5). Thus $\nabla_Z X \in \mathcal{D}$ and this is assertion (5.48). Now, take $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$. From (5.1) and (5.2), we find that

$$g(J\nabla_X Z, Y) = -g(A_{JZ}, Y) = -g(\sigma(X, Y), JZ) = 0 \tag{5.52}$$

The last equation holds because of Theorem 5.6. If $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$, from (5.1) and (5.2) again we have

$$g(J\nabla_X Z, W) = g(A_{JW}Z, X) - g(A_{JZ}W, X).$$

But, from Theorem 5.5 we obtain

$$g(A_{JW}Z, X) - g(A_{JZ}W, X) = -\frac{1}{2}g(Z, W)g(JB, X) + \frac{1}{2}g(W, Z)g(JB, X) = 0$$

and, hence

$$g(J\nabla_X Z, W) = 0. \tag{5.53}$$

Now, (5.49) follows from (5.52) and (5.53). Finally, (5.48) and (5.51) give (5.50). ■

Now we prove the converse of Theorem 5.13. From (5.36) and (5.48), for any $X \in \mathcal{D}$ and Z tangent to M we have

$$(\widetilde{\nabla}_Z{}', T)X = \nabla_Z(JX) - J(\nabla_Z X).$$

On the other hand, we write $Z = Y + Z$, where $Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. Then, from (5.50) we have

$$(\widetilde{\nabla}_Z{}'T)X = \nabla_Y(JX) - J\nabla_Y X. \tag{5.54}$$

But (5.1)–(5.3) give

$$\nabla_Y(JX) - J\nabla_Y X = \frac{1}{2}\omega(Y)JX - \frac{1}{2}(JY)X - \frac{1}{2}g(X, Y)JB + \frac{1}{2}g(X, JY)B. \tag{5.55}$$

Now we have

(a) If $dim(\mathcal{D}) \geq 4$ and $B_x \neq 0$ for some $x \in M$, there are $X, Y \in \mathcal{D}$ such that the right-hand side of (5.55) does not vanish at x. Therefore, T is not parallel.

(b) If $dim(\mathcal{D}) = 2$, then the right-hand side of (5.55) vanishes and, hence

$$(\widetilde{\nabla}_Z{}'T)X = 0, \tag{5.56}$$

for any $X \in \mathcal{D}$ and Z tangent to M. But (5.49) implies $\nabla_Z Z \in \mathcal{D}^\perp$, for any $Z \in \mathcal{D}^\perp$ and Z tangent to M, so that

$$(\widetilde{\nabla}_Z{}'T)Z = \nabla_Z(TZ) - T\nabla_Z Z = -T\nabla_Z Z = 0. \tag{5.57}$$

Then, (5.56) and (5.57) prove that T is parallel. ■

5.3 Normal CR-Submanifolds of S-manifolds

We want to study here the normal CR-submanifolds for general S-manifolds. In fact, the normal CR-submanifolds become to be a very wide class of CR-submanifolds. Actually, either totally f-umbilical submanifolds (see [19] for more details) or CR-products (see [12]) of an S-manifold are normal CR-submanifolds. We also study normal CR-submanifolds of an S-space-form, specially in the concrete cases of $\mathbb{R}^{2m+s}$ (with constant f-sectional curvature $c = -3s$) and $\mathbb{H}^{2m+s}$ (with constant f-sectional curvature $c = 4 - 3s$).

5.3.1 Preliminaries

A $(2m + s)$-dimensional Riemannian manifold $(\wedge M, g)$ endowed with an f-structure f (that is, a tensor field of type $(1, 1)$ and rank $2m$ satisfying $f^3 + f = 0$ [21]) is said to be a *metric f-manifold* if, moreover, there exist s global vector fields $\xi_1, \ldots, \xi_s$ on $\wedge M$ (called *structure vector fields*) such that, if $\eta_1, \ldots, \eta_s$ are the dual 1-forms of $\xi_1, \ldots, \xi_s$, then

$$f\xi_\alpha = 0;\ \eta_\alpha \circ f = 0;\ f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha;$$

$$g(X, Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{5.58}$$

for any $X, Y \in \mathcal{X}(\wedge M)$ and $\alpha = 1, \ldots, s$.

Let F be the 2-form on $\wedge M$ defined by $F(X, Y) = g(X, fY)$, for any $X, Y \in \mathcal{X}(\wedge M)$. Since f is of rank $2m$, then

$$\eta_1 \wedge \cdots \wedge \eta_s \wedge F^m \neq 0$$

and, particularly, $\wedge M$ is orientable.

The f-structure f is said to be *normal* if

$$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f, f]$ is the Nijenhuis torsion of f.

A metric f-manifold is said to be a K-manifold [5] if it is normal and $\mathrm{d}F = 0$. A K-manifold is called an S-manifold if $F = \mathrm{d}\eta_\alpha$, for any α. Note that, for $s = 0$, a K-manifold is a Kaehlerian manifold and, for $s = 1$, a K-manifold is a quasi-Sasakian manifold and an S-manifold is a Sasakian manifold. When $s \geq 2$, nontrivial examples can be found in [5, 13]. Moreover, a K-manifold $\wedge M$ is an S-manifold if and only if

$$\wedge \nabla_X \xi_\alpha = -fX, \tag{5.59}$$

for any $X \in \mathcal{X}(\wedge M)$ and any $\alpha = 1, \ldots, s$, where $\wedge\nabla$ denotes the Levi-Civita connection of g. It is easy to show that in any S-manifold

$$(\wedge\nabla_X f)Y = \sum_{\alpha=1}^{s} \left\{ g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X \right\}, \tag{5.60}$$

for any $X, Y \in \mathcal{X}(\wedge M)$. A plane section π on a metric f-manifold $\wedge M$ is said to be an *f-section* if it is determined by a unit vector X, normal to the structure vector fields and fX. The sectional curvature of π is called an *f-sectional curvature*. An S-manifold is said to be an *S-space-form* if it has a constant f-sectional curvature c and then, it is denoted by $\wedge M(c)$. In such case, the curvature tensor field $\wedge R$ of $\wedge M(c)$ satisfies [15]

$$\begin{aligned} \wedge\, & R(X, Y, Z, W) \\ &= \sum_{\alpha,\beta} (g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) \\ &\quad + g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)) \\ &\quad + \frac{c+3s}{4}(g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)) \\ &\quad + \frac{c-s}{4}(F(X, W)F(Y, Z) - F(X, Z)F(Y, W) \\ &\qquad\qquad - 2F(X, Y)F(Z, W)), \end{aligned} \tag{5.61}$$

for any $X, Y, Z, W \in \mathcal{X}(\wedge M)$. Next, let M be a isometrically immersed submanifold of a metric f-manifold $\wedge M$ (for the general theory of submanifolds, we refer to [3, 22]). We denote by $\mathcal{X}(M)$ the Lie algebra of tangent vector fields to M and by $T(M)^\perp$ the set of tangent vector fields to $\wedge M$ which are normal to M. For any vector field $X \in \mathcal{X}(M)$, we write

$$fX = TX + NX, \tag{5.62}$$

where TX and NX are the tangential and normal components of fX, respectively. Then, T is an endomorphism of the tangent bundle of M and N is a normal bundle valued 1-form on such tangent bundle. It is easy to show that if T does not vanish, it defines an f-structure in the tangent bundle of M. The submanifold M is said to be *invariant* if N is identically zero, that is, if fX is tangent to M, for any $X \in \mathcal{X}(M)$. On the other hand, M is said to be an *anti-invariant* submanifold if T is identically zero, that is, if fX is normal to M, for any $X \in \mathcal{X}(M)$. In the same way, for any $V \in T(M)^\perp$, we write

$$fV = tV + nV, \tag{5.63}$$

where tV and nV are the tangential and normal components of fV, respectively. Then, t is a tangent bundle valued 1-form on the normal bundle of M and n is an endomorphism of the normal bundle of M. It is easy to show that if n does nor vanish, it defines an f-structure in the normal bundle of M. From now on, we suppose that all the structure vector fields are tangent to the submanifold M and so, $dim(M) \geq s$. Then, the distribution on M spanned by the structure vector fields is denoted by $\mathcal{M}$ and its complementary orthogonal distribution is denoted by $\mathcal{L}$. Consequently, if $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any $\alpha = 1, \dots, s$ and if $X \in \mathcal{M}$, then $fX = 0$. In this context, M is said to be a *CR-submanifold* of $\wedge M$ if there exist two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M satisfying

(i) $\mathcal{X}(M) = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{M}$, where $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathcal{M}$ are mutually orthogonal to each other;
(ii) The distribution $\mathcal{D}$ is invariant by f, that is, $f\mathcal{D}_x = \mathcal{D}_x$, for any $x \in M$;
(iii) The distribution $\mathcal{D}^\perp$ is anti-invariant by f, that is, $f\mathcal{D}_x^\perp \subseteq T_x(M)^\perp$, for any $x \in M$.

This definition is motivated by the following theorem.

Theorem 5.14 ([16]) *Let $\wedge M$ be an S-manifold which is the bundle space of a principal toroidal bundle over a Kaehler manifold $\wedge M'$, $\wedge\pi : \wedge M \longrightarrow \wedge M'$, M a submanifold immersed in $\wedge M$, tangent to the structure vector fields and M' a submanifold immersed in $\wedge M'$ such that there exists a fibration $\pi : M \longrightarrow M'$, the diagram*

$$\begin{array}{ccc} M & \stackrel{i}{\longrightarrow} & \wedge M \\ \pi \downarrow & & \downarrow \wedge\pi \\ M' & \stackrel{i'}{\longrightarrow} & \wedge M' \end{array}$$

commutes and the immersion i is a diffeomorphism on the fibers. Then, M is a CR-submanifold of $\wedge M$ if and only if M' is a CR-submanifold of $\wedge M'$.

We denote by $2p$ and q the real dimensions of $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. Then, we see that for $p = 0$ we obtain an anti-invariant submanifold tangent to the structure vector fields and for $q = 0$ an invariant submanifold. A CR-submanifold of an S-manifold is said to be a *generic submanifold* if given any $V \in T(M)^\perp$, there exists $Z \in \mathcal{D}^\perp$ such that $V = fZ$, a $(\mathcal{D}, \mathcal{D}^\perp)$-*geodesic submanifold* if $\sigma(X, Z) = 0$, for any $X \in \mathcal{D}$ and any $Z \in \mathcal{D}^\perp$ and a $\mathcal{D}^\perp$-*geodesic submanifold* if $\sigma(Y, Z) = 0$, for any $Y, Z \in \mathcal{D}^\perp$. As an example, it is easy to show that each hypersurface of $\wedge M$ which is tangent to the structure vector fields is a CR-submanifold. Now, we write by P and Q the projections morphisms of $\mathcal{X}(M)$ on $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. Thus, for any $X \in \mathcal{X}(M)$, we have that

$$X = PX + QX + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha.$$

We define the tensor field v of type (1, 1) by $vX = fPX$ and the non-null, normal bundle valued 1-form u by $uX = fQX$, for any $X \in \mathcal{X}(M)$. Then, it is easy to show that $u \circ v = 0$ and $\eta_\alpha \circ u = \eta_\alpha \circ v = 0$, for any $\alpha = 1, \ldots, s$. Moreover, a direct computation gives

$$g(X, Y) = g(uX, uY) + g(vX, vY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{5.64}$$

$$F(X, Y) = g(X, vY), \quad F(X, Y) = F(vX, vY), \tag{5.65}$$

for any $X, Y \in \mathcal{X}(M)$. From Gauss–Weingarten formulas and by using (5.59), for any $X \in \mathcal{X}(M)$, $V \in T(M)^\perp$, and $\alpha = 1, \ldots, s$, we have

$$\nabla_X \xi_\alpha = -vX, \tag{5.66}$$

$$\sigma(X, \xi_\alpha) = -uX, \tag{5.67}$$

$$A_V \xi_\alpha \in \mathcal{D}^\perp. \tag{5.68}$$

Moreover, from (5.60) and the Gauss–Weingarten formulas, if $X, Y \in \mathcal{X}(M)$, comparing the components in $\mathcal{D}$, $\mathcal{D}^\perp$ and $T(M)^\perp$ respectively, we get

$$P\nabla_X vY - PA_{uY}X = v\nabla_X Y - \sum_{\alpha=1}^{s} \eta_\alpha(Y)PX, \tag{5.69}$$

$$Q\nabla_X vY - QA_{uY}X = t\sigma(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(Y)QX, \tag{5.70}$$

$$\sigma(X, vY) + D_X uY = u\nabla_X Y + n\sigma(X, Y). \tag{5.71}$$

From the above formulas and (5.60) we obtain

$$(\nabla_X v)Y = A_{uY}X + t\sigma(X, Y) - \sum_{a=1}^{s} \{\eta_\alpha(Y)f^2X + g(fX, fY)\xi_\alpha\}, \tag{5.72}$$

$$(\nabla_X u)Y = n\sigma(X, Y) - \sigma(X, vY), \tag{5.73}$$

for any $X, Y \in \mathcal{X}(M)$. Also, from (5.60) and the Gauss–Weingarten formulas again, we have

$$\nabla_X Z = vA_{fZ}X - tD_X fZ, \tag{5.74}$$

$$tD_X fZ = -Q\nabla_X Z, \tag{5.75}$$

for any $X \in \mathcal{X}(M)$ and any $Z \in \mathcal{D}^\perp$. With regard to the integrability of the distributions involved in the definition of a *CR*-submanifold, I. Mihai [16] proved that the distributions $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\perp} \oplus \mathcal{M}$ are always integrable. On the other hand, if $p > 0$, the distributions $\mathcal{D}$ and $\mathcal{D} \oplus \mathcal{D}^{\perp}$ are not integrable and the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable if and only if

$$\sigma(X, fY) = \sigma(fX, Y), \tag{5.76}$$

for any $X, Y \in \mathcal{D}$. In [12], CR-products of S-manifolds are defined as CR-submanifolds such that the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable and locally they are Riemannian products $M_1 \times M_2$, where M_1 (resp., M_2) is a leaf of $\mathcal{D} \oplus \mathcal{M}$ (resp., $\mathcal{D}^{\perp}$). From Theorem 3.1 and Proposition 3.2 in [12], we know that a CR-submanifold M of an S-manifold is a CR-product if and only if one of the following assertions is satisfied:

$$A_{f\mathcal{D}^{\perp}} f\mathcal{D} = 0, \tag{5.77}$$

$$g(\sigma(X, Y), fZ) = 0, \ X \in \mathcal{D}, \ Y \in \mathcal{X}(M), \ Z \in \mathcal{D}^{\perp}, \tag{5.78}$$

$$\nabla_X Y \in \mathcal{D} \oplus \mathcal{M}, \ X \in \mathcal{D}, \ Y \in \mathcal{X}(M). \tag{5.79}$$

5.3.2 Normal CR-Submanifolds of an S-manifold

Let M be a CR-submanifold of an S-manifold $\wedge M$. We say that M is a normal CR-submanifold if

$$N_v(X, Y) = 2t\mathrm{d}u(X, Y) - 2\sum_{\alpha=1}^{s} F(X, Y)\xi_\alpha, \tag{5.80}$$

for any $X, Y \in \mathcal{X}(M)$, where N_v is denoting the Nijenhuis torsion of v, that is

$$N_v(X, Y) = (\nabla_{vX} v)Y - (\nabla_{vY} v)X + v((\nabla_Y v)X - (\nabla_X v)Y).$$

We notice that (5.80) is equivalent to

$$S^*(X, Y) = N_v(X, Y) - t((\nabla_X u)Y - (\nabla_Y u)X) + 2\sum_{\alpha=1}^{s} F(X, Y)\xi_\alpha = 0,$$

for any $X, Y \in \mathcal{X}(M)$. Now, we can prove the following characterization theorem in terms of the shape operator.

Theorem 5.15 *A CR-submanifold M of an S-manifold $\wedge M$ is normal if and only if*

$$A_{uY}vX = vA_{uY}X, \tag{5.81}$$

for any $X \in \mathcal{D}$ and any $Y \in \mathcal{D}^{\perp}$.

Proof A direct expansion by using (5.72) and (5.73) gives that

$$S^*(X, Y) = A_{uY}vX - vA_{uY}X - A_{uX}vY + vA_{uX}Y, \tag{5.82}$$

for any $X, Y \in \mathcal{X}(M)$. Now, if M is a normal CR-submanifold of $\wedge M$, (5.81) follows form (5.82) since $uX = 0$, for any $X \in \mathcal{D}$. Conversely, if (5.81) holds, we use the decomposition $\mathcal{X}(M) = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \mathcal{M}$. First, since $uX = 0$ for any $X \in \mathcal{D}$ and $v\xi_{\alpha} = 0 = u\xi_{\alpha}$, for any α, we deduce from (5.81) and (5.82) that $S^*(X, Y) = 0$, for any $X \in \mathcal{D}$ and any $Y \in \mathcal{X}(M)$. Moreover, if $Y \in \mathcal{D}^{\perp}$, from (5.68) we have $A_{uY}\xi_{\alpha} \in \mathcal{D}^{\perp}$ and so, $vA_{uY}\xi_{\alpha} = 0$ dfor any α. Consequently, $S^*(X, \xi_{\alpha}) = 0$, for any $X \in \mathcal{X}(M)$. Finally, if $X, Y \in \mathcal{D}^{\perp}$, (5.82) becomes

$$S^*(X, Y) = v(A_{fX}Y - A_{fY}X),$$

since $vX = vY = 0$ and $uX = fX$, $uY = fY$. But, from (5.60) we easily show that $A_{fX}Y = A_{fY}X$. ■

Corollary 2 *A CR-submanifold M of an S-manifold is normal if and only if*

$$g(\sigma(X, vY) + \sigma(Y, vX), fZ) = 0, \tag{5.83}$$

$$g(\sigma(X, Z)fW) = 0, \tag{5.84}$$

for any $X, Y \in \mathcal{D}$ and any $Z, W \in \mathcal{D}^{\perp}$.

Proof Since v is skew-symmetric, from (5.81) we see that M is normal if and only if

$$g(\sigma(X, vY), uZ) = -g(\sigma(Y, vX), uZ)m \tag{5.85}$$

for any $X \in \mathcal{X}(M)$, $Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$. Now, if M is normal, from (5.85) we get (5.83) taking $X \in \mathcal{D}$ and (5.84) taking $X \in \mathcal{D}^{\perp}$. Conversely, if (5.83) and (5.84) are satisfied, we observe that (5.85) is satisfied too if $X \in \mathcal{D}$ and $X \in \mathcal{D}^{\perp}$, respectively. Finally, if $X \in \mathcal{M}$, we have $vX = 0$ and, by using that $u \circ v = 0$ and (5.67), $\sigma(X, vY) = 0$, for any $Y \in \mathcal{D}$. Thus, (5.85) holds for any $X \in \mathcal{X}(M)$. ■

Corollary 3 *Any normal generic submanifold of an S-manifold is a $(\mathcal{D}, \mathcal{D}^{\perp})$-geodesic submanifold.*

From (5.60), (5.67), (5.83), and (5.84), we have

$$\sigma(fX, Z) = f\sigma(X, Z), \tag{5.86}$$

$$t\sigma(fX, fX) = t\sigma(X, X), \tag{5.87}$$

$$A_{fZ}X \in \mathcal{D}, \tag{5.88}$$

for any $X \in$ and any $Z \in \mathcal{D}^{\perp}$. On the other hand, from (5.78) and (5.83)–(5.84), we deduce

Proposition 1 *Each CR-product in an S-manifold is a normal CR-submanifold.*

For the converse we prove the following theorems.

Theorem 5.16 *Let M be a normal CR-submanifold of an S-manifold. Then, M is a CR-product if and only if the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable.*

Proof The necessary condition is obvious. Conversely, let $X \in \mathcal{D}$. If $Y \in \mathcal{D}^{\perp}$, then (5.78) is (5.84). Further, if $Y \in \mathcal{M}$, from (5.67) we get $\sigma(X, Y) = 0$. Finally, if $Y \in \mathcal{D}$, from (5.76) and (5.83) we obtain (5.78). ■

Theorem 5.17 *Let M be a normal CR-submanifold of an S-manifold such that* $\mathrm{d}u = 0$. *Then, M is a CR-product.*

Proof A straightforward computation gives, by using the hypothesis and (5.72),

$$g((\nabla_X v)Y, Z) = \sum_{\alpha=1}^{s}\{\mathrm{d}\eta_\alpha(vX, Y)\eta_\alpha(Z) - \mathrm{d}\eta_\alpha(vZ, X)\eta_\alpha(Y)\}, \tag{5.89}$$

for any $X, Y, Z \in \mathcal{X}(M)$. Now, if $Y \in \mathcal{D}$, from (5.64) and (5.65) we get $\mathrm{d}\eta_\alpha(vX, Y) = F(vX, Y) = g(vX, vY) = g(X, Y)$. So, (5.89) becomes

$$(\nabla_X v)Y = \sum_{\alpha=1}^{s} g(X, Y)\xi_\alpha$$

for any $X \in \mathcal{X}(M)$ and any $Y \in \mathcal{D}$. Comparing with (5.72) we have $\sigma(X, Y) = 0$ and so (5.78) holds. ■

We say that v is *η-parallel* if

$$(\nabla_X v)Y = \sum_{\alpha=1}^{s}\{g(PX, PY)\xi_\alpha - \eta_\alpha(Y)PX\},$$

for any $X, Y \in \mathcal{X}(M)$. Then, from (5.64), (5.65), and (5.89), we prove

Proposition 2 *Any normal CR-submanifold of an S-manifold such that* $\mathrm{d}u = 0$ *is η-parallel.*

Given a CR-submanifold M of an S-manifold, a vector field $X \in \mathcal{X}(M)$ is said to be $\mathcal{D}$-*Killing* if

$$g(P\nabla_Z X, PY) + g(P\nabla_Y X, PZ) = 0, \tag{5.90}$$

for any $Y, Z \in \mathcal{X}(M)$. We notice that it is possible to characterize normal CR-submanifolds in terms of $\mathcal{D}$-Killing vector fields.

Theorem 5.18 *A CR-submanifold M of an S-manifold is a normal CR-submanifold if and only if any $Z \in \mathcal{D}^{\perp}$ is a $\mathcal{D}$-Killing vector field*

Proof Given $X, Y \in \mathcal{X}(M)$ and $Z \in \mathcal{D}^{\perp}$, from (5.74) we get

$$\begin{aligned} g(\nabla_X Z, Y) + g(\nabla_Y Z, X) = g(vA_{fZ}X, Y) - g(tD_X fZ, Y) \\ + g(vA_{fZ}Y, X) - g(tD_Y fZ, X). \end{aligned} \tag{5.91}$$

But $g(vA_{fZ}Y, X) = -g(A_{fZ}vX, Y)$ and so, from (5.91)

$$\begin{aligned} &g(P\nabla_X Z, PY) + g(P\nabla_Y Z, PX) + g(Q\nabla_X Z, QY) + g(Q\nabla_Y Z, QX) \\ &\quad + \sum_{\alpha=1}^{s}\{\eta_\alpha(\nabla_X Z)\eta_\alpha(Y) + \eta_\alpha(\nabla_Y Z)\eta_\alpha(X)\} \\ &= g((vA_{fZ} - A_{fZ}v)X, Y) - g(tD_X fZ, Y) - g(tD_Y fZ, X). \end{aligned} \tag{5.92}$$

Now, since it is easy to show that $\eta_\alpha(\nabla_X Z) = 0$ for any $\alpha = 1, \ldots, s$, by using (5.75), we deduce that (5.92) becomes

$$g(P\nabla_X Z, PY) + g(P\nabla_Y Z, PX) = g((vA_{fZ} - A_{fZ}v)X, Y). \tag{5.93}$$

Consequently, if Z is a $\mathcal{D}$-Killing vector field, from (5.81) we obtain that M is a normal CR-submanifold. Conversely, if $X \in \mathcal{D}$, the right part of the equality (5.93) vanishes by using (5.81). If $X \in \mathcal{D}^{\perp}$, then $vX = 0$ and from (5.84), $A_{fZ}X \in \mathcal{D}^{\perp}$, that is, $vA_{fZ}X = 0$ and the right part of (5.93) vanishes again. Finally, if $X \in \mathcal{M}$, $vX = 0$ and from (5.68), $A_{fZ}X \in \mathcal{D}^{\perp}$. In any case, from (5.93) we have (5.90). ∎

To end this section, we recall that a submanifold M of an S-manifold is said to be *totally f-umbilical* [19] if there exists a normal vector field V such that

$$\sigma(X, Y) = g(fX, fY)V + \sum_{\alpha=1}^{s}\{\eta_a(Y)\sigma(X, \xi_\alpha) + \eta_\alpha(X)\sigma(Y, \xi_\alpha)\}, \tag{5.94}$$

for any $X, Y \in \mathcal{X}(M)$. These submanifolds have been studied and classified in [9]. Since from (5.94) we easily get (5.83) and (5.84), then we have the following theorem.

Theorem 5.19 *Any totally f-umbilical CR-submanifold of an S-manifold is a normal CR-submanifold.*

5.3.3 Normal CR-Submanifolds of an S-space-form

Let $\wedge M(c)$ a $(2m+s)$-dimensional S-space-form, where c is denoting the constant f-sectional curvature and let M be a CR-submanifold. Firstly, we can prove

Proposition 3 *If M is a normal CR-submanifold, then*

$$\|A_{fZ}X\|^2 + \|\sigma(X,Z)\|^2 - g(t\sigma(Z,Z), t\sigma(X,X)) = \frac{c+3s}{4}, \tag{5.95}$$

for any unit vector fields $X \in \mathcal{D}$ *and* $Z \in \mathcal{D}^\perp$.

Proof From the Codazzi equation, we have

$$\begin{aligned}\wedge R(X,fX,Z,fZ) = g(D_X\sigma(fX,Z) - D_{fX}\sigma(X,Z), fZ)\\ - g(\sigma([X,fX],Z),fZ)\\ + g(\sigma(X,\nabla_{fX}Z) - \sigma(fX,\nabla_X Z), fZ).\end{aligned} \tag{5.96}$$

Now, from (5.60), (5.84), and (5.86), a direct expansion gives

$$g(D_X\sigma(fX,Z) - D_{fX}\sigma(X,Z),fZ) = -2\|\sigma(X,Z)\|^2. \tag{5.97}$$

On the other hand, since $X \in \mathcal{D}$ is a unit vector field (and so, fX too), we see from (5.59) that $\eta_\alpha([X,fX]) = 2$ for any α and from (5.70) that $Q[X,fX] = t\sigma(X,X) + t\sigma(fX,fX)$. Thus, taking into account (5.67), (5.84), and (5.87), we get

$$g(\sigma([X,fX],Z),fZ) = 2g(\sigma(t\sigma(X,X),Z),fZ) - 2s. \tag{5.98}$$

However, since $Z \in \mathcal{D}^\perp$, by using (5.70) it is easy to show that

$$g(\sigma(t\sigma(X,X),Z),fZ) = -g(t\sigma(X,X), t\sigma(Z,Z)).$$

Therefore, from (5.98) we have

$$g(\sigma([X,fX],Z),fZ) = -2s - 2g(t\sigma(X,X), t\sigma(Z,Z)). \tag{5.99}$$

Next, since $\eta_\alpha(\nabla_{fX}Z) = \eta_\alpha(\nabla_X Z) = 0$ for any α, from (5.69), (5.83), (5.84), and (5.88), we obtain

$$g(\sigma(X,\nabla_{fX}Z) - \sigma(fX,\nabla_X Z),fZ) = -2\|A_{fZ}X\|^2. \tag{5.100}$$

Finally, from (5.61) we deduce $\wedge R(X, fX, Z, fZ) = -(c - s)/2$. Then, substituting (5.97), (5.99), and (5.100) into (5.96), we complete the proof. ■

Corollary 4 *If M is a normal $\mathcal{D}^{\perp}$-geodesic CR-submanifold of an S-space-form $\wedge M(c)$, then $c \geq -3s$.*

Proposition 4 *If M is a normal CR-submanifold of an S-space-form $\wedge M(c)$ such the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable, then $c \geq -3s$ and M is a CR-product.*

Proof It is clear that M is a CR-product due to Theorem 5.16. Moreover, from (5.78) we have $g(\sigma(X, Y), fZ) = 0$. for any $X, Y \in \mathcal{D}$. Then, if $X \in \mathcal{D}$ is a unit vector field, $t\sigma(X, X) = 0$ and, by using (5.95), $c \geq -3s$. ■

Now, we are going to study the concrete case of the $(2m + s)$-dimensional euclidean S-space-form $\mathbb{R}^{2m+s}(-3s)$ (see [13] for the details of this structure). In this context, we can prove

Theorem 5.20 *If M is a normal $(\mathcal{D}, \mathcal{D}^{\perp})$-geodesic and $\mathcal{D}^{\perp}$-geodesic CR-submanifold of $\mathbb{R}^{2m+s}(-3s)$, then it is a CR-product.*

Proof From (5.95), we have $A_{fZ}X = 0$ for any $X \in \mathcal{D}$ and any $Z \in \mathcal{D}^{\perp}$. So, from (5.77), M is a CR-product. ■

Corollary 5 *A normal $\mathcal{D}^{\perp}$-geodesic generic submanifold of $\mathbb{R}^{2m+s}(-3s)$ is a CR-product.*

Another interesting example of S-space-form is $\mathbb{H}^{2m+s}(4 - 3s)$, a generalization of the Hopf fibration $\pi : \S^{2m+1} \longrightarrow \mathbb{P}C^m$, introduced by Blair in [5] as a canonical example of an S-manifold playing the role of the complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. This space is given by (see [5, 6] for more details)

$$\mathbb{H}^{2m+s} = \{(x_1, \ldots, x_s) \in \S^{2m+1} \times \overset{s)}{\cdots} \times \S^{2m+1} / \pi(x_1) = \cdots = \pi(x_s)\}$$

and its f-sectional curvature is constant equal to $4 - 3s$. Let M be a CR-submanifold of $\mathbb{H}^{2m+s}(4 - 3s)$ (we always suppose $s \geq 2$). Denote by ν the orthogonal complementary distribution of $f\mathcal{D}^{\perp}$ in $T(M)^{\perp}$. Then, $f\nu \subseteq \nu$. Let

$$\{E_1, \ldots, E_{2p}\}, \quad \{F_1, \ldots, F_q\}, \quad \{V_1, \ldots, V_r, fV_1, \ldots, fV_r\},$$

be local fields of orthonormal frames on $\mathcal{D}$, $\mathcal{D}^{\perp}$ and ν, respectively, where $2r$ is the real dimension of ν. First, we prove

Lemma 6 *If M is a CR-product in $\mathbb{H}^{2m+s}(4 - 3s)$, then*

$$\|\sigma(X, Z)\| = 1, \tag{5.101}$$

for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.

Proof We know, from Proposition 1, that M is a normal CR-submanifold. Since, $c = 4 - 3s$, from (5.77), (5.78) and (5.95) we complete the proof. ∎

Lemma 7 *If M is a CR-product in $\mathbb{H}^{2m+s}(4-3s)$, the vector field $\sigma(E_i, F_a)$, $i = 1, \ldots, 2p$ and $a = 1, \ldots, q$, are $2pq$ orthonormal vector fields on ν.*

Proof From (5.101) and by the linearity, we get $g(\sigma(E_i, Z), \sigma(E_j, Z)) = 0$, for any $i, j = 1, \ldots, 2p$, $i \neq j$ and any unit vector field $Z \in \mathcal{D}^\perp$. Now, from (5.84), if $q = 1$, we complete the proof. If $q \geq 2$, by linearity again, we have $g(\sigma(E_i, F_a), \sigma(E_j, F_b)) + g(\sigma(E_i, F_b), \sigma(E_j, F_a)) = 0$, for any $i, j = 1, \ldots, 2p$, $i \neq j$, $a, b = 1, \ldots, q$, $a \neq b$. Next, by using (5.79) and the Bianchi identity, we obtain $R(X, Y, Z, W) = 0$, for any $X, Y \in \mathcal{D}$, $Z, W \in \mathcal{D}^\perp$, where R is denoting the curvature tensor field of M. But, if $i \neq j$ and $a \neq b$, (5.61) gives $\wedge R(E_i, E_j, F_a, F_b) = 0$. Then, from the Gauss equation we get

$$g(\sigma(E_i, F_a), \sigma(E_j, F_b)) - g(\sigma(E_i, F_b), \sigma(E_j, F_a)) = 0,$$

for any $i, j = 1, \ldots, 2p$, $i \neq j$, $a, b = 1, \ldots, q$, $a \neq b$ and this completes the proof. ∎

Now, we study the normal CR-submanifolds of $\mathbb{H}^{2m+s}(4-3s)$.

Theorem 5.21 *Let M be a normal CR-submanifold of $\mathbb{H}^{2m+s}(4-3s)$, $s \geq 2$, such that the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable. Then*

(i) M is a CR-product $M_1 \times M_2$.
(ii) $m \geq pq + p + q$.
(iii) If $n = pq + p + q$, then M_1 is an invariant totally geodesic submanifold immersed in $\mathbb{H}^{2m+s}(4-3s)$.
(iv) $\|\sigma\|^2 \geq 2q(2p+s)$.
(v) If $\|\sigma\|^2 = 2q(2p+s)$, then M_1 is an S-space-form of constant f-sectional curvature $4-3s$ and M_2 has constant curvature 1.
(vi) If M is a minimal submanifold, then $\rho \leq 4p(p+1) + 2p(q+s) + q(q-1)$, where ρ denotes the scalar curvature and the equality holds if and only if $\|\sigma\|^2 = 2q(2p+s)$.

Proof (i) follows directly from Proposition 4. Now, from Lemma 7, $dim(\nu) = 2(m - p) - 2q \geq 2pq$. So (ii) holds. Next, suppose that $m = pq + p + q$. If $X, Y, Z \in \mathcal{D}$ and $W \in \mathcal{D}^\perp$, from (5.61), $\wedge R(X, Y, Z, W) = 0$ and, by using a similar proof to that one of Lemma 7, $R(X, Y, Z, W) = 0$. So, the Gauss equation gives

$$g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)) = 0. \tag{5.102}$$

Since from Proposition 3.2 of [12], $\sigma(fX, Z) = f\sigma(X, Z)$, if we put $Y = fX$, we have, by using (5.86), $g(\sigma(fX, W), (\sigma(X, Z)) = 0$. Now, if we put $Z = fY$, then $g(\sigma(X, Y), \sigma(X, W)) = 0$. Thus, by linearity, we get $g(\sigma(X, W), \sigma(Y, Z)) + g(\sigma(X, Z), \sigma(Y, W)) = 0$. Consequently, from (5.102)

$$g(\sigma(X, W), \sigma(Y, Z)) = 0, \tag{5.103}$$

for any $X, Y, Z \in \mathcal{D}$ and $W \in \mathcal{D}^{\perp}$. Since now $dim(\nu) = 2pq$, (5.103) implies that $\sigma(X, Y) = 0$, for any $X, Y \in \mathcal{D}$ and so, (iii) holds from Theorem 2.4(ii) of [12]. Assertions (iv) and (v) follow from Theorem 4.2 of [12]. Finally, if M is a minimal normal CR-submanifold of $\mathbb{H}^{2m+s}(4-3s)$, a straightforward computation gives

$$\rho = 4p(p+1) + 2s(p+q) + q(q-1) + 6pq - \|\sigma\|^2.$$

Then, by using (iv), the proof is complete. ∎

Theorem 5.22 *Let M be a normal, $(\mathcal{D}, \mathcal{D}^{\perp})$-geodesic and $\mathcal{D}^{\perp}$-geodesic CR-submanifold of $\mathbb{H}^{2m+s}(4-3s)$. Then,*

(i) $\|A_{fZ}X\| = 1$, *for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.*
(ii) $\|\sigma\|^2 \geq 2q(p+s)$ *and the equality hold if and only if $\sigma(\mathcal{D}, \mathcal{D}) \in f\mathcal{D}^{\perp}$.*

Proof (i) follows immediately from (5.95). Now, considering the above-mentioned local fields of orthonormal frames for $\mathcal{D}$, $\mathcal{D}^{\perp}$, and ν, a straightforward computation using the hypothesis gives (ii). ∎

Finally, from (5.84) and (5.95), we can prove

Corollary 6 *Let M be a normal $\mathcal{D}^{\perp}$-geodesic generic submanifold of $\mathbb{H}^{2m+s}(4-3s)$. Then*

(i) $\|A_{fZ}X\| = 1$, *for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$.*
(ii) $\|\sigma\|^2 = 2q(p+s)$.

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Chapter 6
Lorentzian Geometry and CR-Submanifolds

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6.1 Introduction

Since the second half of the twentieth century, the Riemannian and semi-Riemannian geometries have been active areas of research in differential geometry and its applications to a variety of subjects in mathematics and physics. A survey in Marcel Berger's book [5] includes the major developments of Riemannian geometry, citing the works of differential geometers of that time. Along with that, the interest also shifted towards Lorentzian geometry, the mathematical theory used in general relativity. Since then there has been an amazing leap in the depth of the connection between modern differential geometry and mathematical relativity, both from the local and the global point of view.

Motivation of my this paper comes from the historical development of the general theory of Cauchy–Riemann (CR) manifolds and their use in mathematical physics, as follows: In the early 1930, the Riemannian geometry and the theory of complex variables were synthesized by Kähler which developed (during 1950) into the complex manifold theory. A Riemann surface, C^n and its projective space CP^{n-1} are simple examples of the complex manifolds. This interrelation between the above two main branches of mathematics developed into what is now known as Kählerian and Sasakian geometry. Almost complex [35], almost contact [8] and their complex, totally real, CR and slant submanifolds [13] are some of the most

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interesting topics of Riemannian geometry. By a CR-submanifold we mean a real submanifold M of an almost Hermitian manifold $(\bar{M}, \bar{g}, J)$, carrying a J-invariant distribution D (i.e., $JD = D$) and whose $\bar{g}$-orthogonal complement is J-anti-invariant (i.e., $JD^{\perp} \subseteq T(M)^{\perp}$), where $T(M)^{\perp} \to M$ is the normal bundle of M in $\bar{M}$. The CR-submanifolds were introduced as an umbrella of a variety (such as invariant, anti-invariant, semi-invariant and generic) of submanifolds. Details on these may be seen in [4, 35]. On the other hand, a CR-manifold (independent of its landing space) is a C^{∞} differentiable manifold M with a holomorphic subbundle H of its complexified tangent bundle $CT(M)$, such that $H \cap \bar{H} = \{0\}$ and H is involutive (i.e., $[X, Y] \in H$ for every $X, Y \in H$). For an update on the analysis of CR-manifolds (which is out of the scope of this paper), we refer a recent book [6] by Barletta, Dragomir and Duggal. Here we highlight that Blair and Chen [9] were the first to interrelate these two concepts by proving that *proper CR-submanifolds, of a Hermitian manifold, are CR-manifolds*. The study on above-mentioned variety of geometric structures was primarily confined to Riemannian manifolds and their submanifolds, which carry a positive-definite metric tensor, until in early 1980, when Beem–Ehrlich [3] published a book on *Global Lorentzian Geometry* and a book by O'Neill [30] on *Semi-Riemannian Geometry with Applications to Relativity*. Since then considerable amount of work has been done on the study of semi-Riemannian geometry and its CR-submanifolds (see Sharma [33] and Duggal [15–18]). As a result, we know that there are similarities and differences between the Riemannian and the semi-Riemannian geometries, in particular, reference to Lorentzian case used in relativity.

The objective of this paper is to provide up-to-date information on the Lorentzian geometry of CR-submanifolds, contact CR-submanifolds and globally framed CR-submanifolds (M, g) of an indefinite semi-Riemannian manifold $(\bar{M}, \bar{g})$. We focus on those key results whose Lorentzian geometry is different than their corresponding Riemannian geometry. Observe that, contrary to the case of Riemannian CR-submanifolds and above-stated two other classes, the induced metric $g_{|D}$, where D is a distribution of M, has three subcases, namely, (a) $g_{|D}$ is spacelike or (b) $g_{|D}$ is Lorentzian or (c) $g_{|D}$ is lightlike. For the first two subcases, D is an invariant submanifold of M, but, the third subcase need not be invariant. We notice that the geometry of the subcase (a) is mostly similar with the Riemannian case, but, the subcase (b) still remains an open problem since $g_{|D}$ Lorentzian is not compatible with the required Hermitian structure of D (see explanation given in Open problem 1). One needs to modify the Riemannian definition of CR-submanifolds in order to deal with the subcase (b). Moreover, the geometry of subcase (c) is quite different. As this last subcase may be quite new to the readers, we reproduce some of its physical examples taken from [16, 20]. Also, we refer two papers of Penrose [31, 32] on physical applications of CR-structures in relativity.

6.2 Cauchy Riemann(CR) Structures

In general, any n-dimensional complex manifold can be considered as a $2n$-dimensional real manifold with complex coordinates $z_i = x_i + y_i$ for x_i, y_i as real coordinates. For any complex manifold there is associated a complex (also called holomorphic) tangent bundle of the form $V = v^i \partial_{z_i}$, as opposed to the general form of a complex valued vector, $V = v^i \partial_{z_i'} + v^{\bar{i}} \partial_{\bar{z_i}}$. Let M be a real $2n$-dimensional manifold, then M is said to have a CR-structure if in the tangent space T_x, at each point $x \in M$, a $2r$-real-dimensional subspace H_x of T_x is singled out, called the holomorphic tangent space. H_x regarded as r-dimensional complex space and spanned by the vectors $Z_a = X_a + iY_a$, for every $(1 \leq a \leq n)$ provides a linear operator J satisfying $J^2 = -1$. Explicitly, $JZ_a = iZ_a$. Such a CR-structure can be realized only if one can complete a basis for the entire T_x with a complementary set of $2n - 2r$ vectors. In 1957, Newlander and Nirenberg [27] proved that a real-analytic CR-structure can be realized in above way provided following integrability relations hold:

$$[Z_a, Z_{a'}] = \quad \text{complex linear combinations of} \quad Z's, \qquad (1 \leq a, a' \leq r)$$

However, in 1973, Nirenberg [29] proved (by citing some counter examples) that for a C^∞ CR-structure above relations are not sufficient and in this case a non-realizable CR-structure may arise (for details see Penrose [31]). Here we will only present results on realizable CR-structures for which we say that (M, J, H) is called a CR-manifold if it admits a real distribution D of the subspaces $D_x = Re(H \oplus \bar{H})_x$ such that D is invariant $(JD = D)$ with respect to the structure tensor J where $H \cap \bar{H} = \{0\}$ and H is involutive, that is, $([X, Y] \in H$ for every $X, Y \in H)$. The theory of CR-manifolds (independent of its landing space) has two main branches, namely, analysis and geometry of CR-manifolds for which we refer two books [14] by Dragomir and Tomassini and [6] by Barletta et al. in which they also have discussed the interrelation of these two branches (also see Boggess [10]. In this section, we focus on the geometry branch of a CR-structure.

6.2.1 CR–Lorentzian Structures

Let (M, g, J) be a real $2n$-dimensional almost complex manifold with J, its almost complex structure tensor $(J^2 = -I)$, and g a metric tensor on M. The metric g is either positive-definite or indefinite (in particular Lorentzian). In order not to overlap with the contributions of other authors in this monograph, here we assume that the metric g of M is Lorentzian. With this assumption, we say that the pair (g, J) is an almost Hermitian structure and M is an almost Hermitian manifold if

$$g(JX, JY) = g(X, Y), \quad \forall X, Y \in T(M).$$

Moreover, if J defines a complex structure on M, then (g, J) and M are called Hermitian structure and Hermitian manifold, respectively. Hermitian condition is needed to assure the existence of a holomorphic subbundle H of the complexified tangent bundle $CT(M)$ of M which is required to construct a CR-structure. Unfortunately, in a 1976 book, Flaherty [23] was the first to show that for a real J its almost complex structure is not Hermitian. Indeed, for a real J, satisfying $J^2 = -I$, we observe that the eigenvalues of J are $i = \sqrt{-1}$ and $-i$ each one of multiplicity n. What we have then is 2n linearly independent null vectors in complex conjugate pair. None of these null vectors can be real because the eigenvalues are not real. As J is real, the only possible signatures of g are either $(0, 2n)$ for positive-definite g or of type $(2p, 2q)$ with $p + q = n$ for an indefinite g. Thus, although the use of real J has been effective in the study of Riemannian (with above-stated restrictions for semi-Riemannian) CR-manifolds and submanifolds, it is not suitable for the class of Lorentzian manifolds. To explain these two negative results, we take some material from Flaherty's book [23] using a local complex coordinates system which has been used in some problems of general relativity. We know that there do exist complex Lorentz transformations whose square is minus one. Using this idea, Flaherty [23] modified the Hermitian structure by replacing real J with a complex valued operator (we denote it $\mathcal{J}$) compatible with the Lorentzian metric which also assures the existence of a holomorphic subbundle H for M. Following is his approach:

For simplicity, take $\dim(M) = 4$ with the Lorentz metric g of signature $(- + + +)$ (although the mathematical results do hold for higher dimensional Lorentzian manifolds) and expressed in terms of a general coordinate system (x^a), where $(0 \leq a \leq 3)$. Suppose $(e_a) = \{e_o, e_1, e_2, e_3\}$ is a local orthonormal real frames field on M. Recall the Newman–Penrose (NP) [28] null tetrad $T = \{\ell, m, \bar{m}, k\}$ at each point of M such that ℓ, k are real null vectors, m is a complex null vector and $\bar{m}$ is its conjugate complex null vector with $g(\ell, k) = -1$ and $g(m, \bar{m}) = 1$ and all other products are zero. Let $\{\omega^a\} = \{\omega^0, \omega^1, \omega^2, \omega^3\}$ be its dual basis. It is always possible to introduce a NP tetrad locally. Globally, the existence of an NP tetrad is equivalent to the existence of a global orthonormal basis. The null tetrad T is associated with the orthonormal basis (e_a) as follows:

$$\ell = \frac{1}{\sqrt{2}}(e_o + e_1), \quad k = \frac{1}{\sqrt{2}}(e_o - e_1),$$
$$m = \frac{1}{\sqrt{2}}(e_2 + i\, e_3), \quad \bar{m} = \frac{1}{\sqrt{2}}(e_2 - i\, e_3).$$

Therefore, the canonical form of the matrix of g is expressed by

$$[g] = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We know from above discussion that a Lorentzian metric cannot admit a Hermitian structure defined by a real endomorphism J such that $J^2 = -I$. For this reason, Flaherty [23] modified the Hermitian structure using a complex valued endomorphism $\mathcal{J}$ defined by

$$\mathcal{J} = i(\omega^0 \otimes E_0 - \omega^1 \otimes E_1 + \omega^2 \otimes E_2 - \omega^3 \otimes E_3),$$

where $\{E_a\} = \{\ell, k, m, \bar{m}\}$. It is easy to see that $\mathcal{J}^2 = -I$ and $(g, \mathcal{J})$ satisfies the condition of an almost Hermitian structure of M. Flaherty also derived the modified integrability conditions ($\mathbf{N} = 0$), where $\mathbf{N}$ is the Nijenhuis tensor field with respect to the complex valued $\mathcal{J}$. If the modified integrability conditions are satisfied then the Lorentz metric g can be locally expressed as

$$g = A\, d\, z^0\, d\, \bar{z}^0 + B\, d\, z^1\, d\, \bar{z}^1 + C\, d\, z^0\, d\, \bar{z}^1 + D\, d\, z^1\, d\, \bar{z}^0,$$

for a complex coordinates system (z^0, z^1, $\bar{z}^0$, $\bar{z}^1$) and for some functions A, B, C and D. Examples are vacuum spacetimes, which include Schwarzschild and Kerr solutions (see Hawking-Ellis [24, pp. 149, 161]).

To recover CR–Lorentzian structure, we let $(M, g, \mathcal{J})$ be endowed with a modified Hermitian structure defined by $(g, \mathcal{J})$ as explained above. Suppose M has a q-dimensional real distribution D, following are three mutual exclusive cases of the causal character (see O'Neill [30]) of D.

(a) $g_{|D}$ is positive $\Rightarrow$ D is spacelike.
(b) $g_{|D}$ is Lorentz $\Rightarrow$ D is timelike.
(c) $g_{|D}$ is degenerate $\Rightarrow$ D is lightlike.

Assume that M has a conformal structure which is needed to preserve the causal character of D. If (a) or (b) holds then a conformal structure defines another distribution $D^\perp$ of $(2n - q)$-dimensional subspaces and

$$TM = D \oplus D^\perp \quad \text{and} \quad D \cap D^\perp = \{0\}.$$

The modified Hermitian structure by $(g, \mathcal{J})$ assures the existence of a holomorphic subbundle H of $CT(M)$ where we set $D = Re(H + \bar{H})$. Then, $(M, g, \mathcal{J}, H, D)$ has a Lorentzian CR–Lorentzian structure if $H \cap \bar{H} = \{0\}$ and H is involutive, that is, ($[X, Y] \in H$ for every $X, Y \in H$). Observe that since any almost complex manifold is even dimensional for these two cases, the CR–Lorentzian manifold M and its distribution D are even dimensional. In next section, we give some examples of CR–Lorentzian structures.

If (c) holds, then we first need the following:

Proposition 1 ([30]) *For a q-dimensional lightlike subspace D_x of a Lorentzian space T_xM, the following are equivalent:*

(1) D_x is lightlike.
(2) D_x contains a null vector but not any timelike vector.

(3) $D_x \cap \wedge_x = L_x - 0_x$, *where* $L_x = D_x \cap D_x^{\perp}$, *the* 1 *-dimensional null space and* $\wedge_x$ *is the null cone of* $T_x\bar{M}$.

Thus, D lightlike implies that we cannot call $D^{\perp}$ the orthogonal complement D since $D = D^{\perp}$ so $D + D^{\perp} \neq TM$. In view of this, for lightlike D we consider $\tilde{D}$, the complementary distribution of D in TM so that

$$T(M) = D \oplus \tilde{D}, \quad D \cap \tilde{D} = \{0\}.$$

Also D is not invariant by J which further means that we fail to get a holomorphic distribution. Thus, using above procedure one cannot realize a CR-structure for M. For this case we proceed as follows:

Since the lightlike D fails to recover a Hermitian structure, its dimension need not be even. With this set up we form the following exact sequence:

$$0 \to D \to D^{\perp} \to D^{\perp}/L \to 0, \quad L = D \cap D^{\perp},$$

where the fibres of quotient bundle $D^{\perp}/L$ are $(2n - q - 1)$-dimensional, well known as spacelike screen spaces which we denote by S_x. For physical reason, here we construct a CR-lightlike structure for an oriented 4-dimensional Lorentzian (called space-time) manifold M by setting $(n = 2, q = 1)$. Therefore, $D = L$ and the 2-dimensional screen S_x is an oriented plane. As S_x is Riemannian, we can work with real operator J on S_x. Let J act as a rotation in a chosen plane through 90°. Then, $J^2 = -1$, and J defines a complex structure on S_x. The complexified space $C(S)_x$ can be represented as a direct sum $S_x^+ \oplus S_x^-$, where

$$S_x^{\pm} = \{u \in C(S)_x : Ju = \pm iu\}.$$

Let $H_x^{\pm}$ be the subspaces of $C(D)_x$ projecting onto $S_x^{\pm}$ by a canonical map $C(D)_x \to C(S)_x$. It is easy to see that

$$H_x^+ \cap H_x^- = C(L)_x, \quad H_x^+ + H_x^- = C(D)_x.$$

Each $H_x^{\pm}$ is (maximal) 2-dimensional lightlike subspace of $CT_x(M)$. The fact that a CR-structure, with a lightlike distribution D and a maximal holomorphic space, say H_x^+, can be locally realized on M, comes from the following famous Riemann mapping theorem:

Any smooth bounded simply connected region in the Argand plane C_x^1 *is holomorphically identical with a unit disc.*

Note that Poincare, back in 1907, pointed out that the Riemann mapping theorem for C^1 has no analogue in higher complex dimensions. This is why, in general, non-realizable CR-structures may exist [31] for the case of higher dimensional complex manifolds. See next section for a mathematical model of $2n$-dimensional CR-manifolds with a lightlike real distribution.

6.2.2 Contact CR–Lorentzian Structures

A $(2n+1)$-dimensional differentiable manifold M is called a contact manifold if it has a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. For a given contact form η, there exists a unique global vector field ξ, called the characteristic vector field, satisfying

$$\eta(\xi) = 1, \qquad (d\,\eta)(\xi,\, X) = 0, \qquad \forall X \in T(M).$$

On a contact manifold M there exists a distribution D (called contact distribution), given by $\eta = 0$, which is far from being integrable. It is known (see Blair [7]) that the maximum dimension of an integral submanifold of D is n. A Riemannian metric g is said to be an associated metric of M if there exists a tensor field ϕ, of type $(1,\,1)$ such that

$$d\,\eta(X,\,Y) = g(X,\,\phi Y), \qquad g(X,\,\xi) = \eta(X),$$
$$\phi^2\,(X) = -X + \eta\,(X)\,\xi, \qquad \forall X,\,Y \in T(M).$$

The structure $(\phi,\,\eta,\,\xi,\,g)$ on M is called a contact metric structure and its associated manifold is called a contact metric manifold which is orientable and odd dimensional with $n \geq 3$. Standard examples of contact manifolds are (i) the odd dimensional spheres, (ii) tangent or cotangent sphere bundles, (iii) the 3-dimensional Lie-groups $SU(2)$ and $SL(2,\,R)$.

In Thermodynamics, we have an example due to Gibbs which is given by the contact form $d\,u - T\,d\,s + p\,d\,v$ (u is the energy, T is the temperature, s is the entropy, p is the pressure and v is the volume) whose zeros define the laws of thermodynamics. Details may be seen in Arnold [1].

Related to the focus of this subsection, we now recall that, in 1990, Duggal [17] introduced a larger class of contact manifolds as follows. Using the same notations of geometric objects as above, we say that a (2n + 1)-dimensional smooth manifold $(M,\,g)$ is called an (ϵ)-almost contact metric manifold if

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta\,(\xi) = 1,$$
$$g(\xi,\,\xi) = \epsilon, \quad g(\phi X,\,\phi Y) = g(X,\,Y) - \epsilon\,\eta(X)\,\eta(Y),$$

where $\epsilon = 1$ or -1 according as ξ is spacelike or timelike and $rank(\phi) = 2\,n$. It is important to mention that in above definition ξ is never a lightlike (null) vector field. If $(d\,\eta)\,(X,\,Y) = g(X,\,Y)$, for every $X, Y \in T(M)$, then we say that M is an (ϵ)-contact metric manifold. If $\epsilon = -1$ and the contact distribution $D(\eta = 0)$ is positive-definite, then g is Lorentzian and since M is oriantable we say that the underlying almost contact manifold M is an almost contact spacetime.

Example 1 Let $(R_1^{2n+1},\,g)$ be a$(2n+1)$-dimensional Minkowski spacetime with local coordinates $(x^i,\,y^i,\,t)$ and $i = 1, \ldots, n$. M being time oriented admits a global timelike vector field, say ξ. Define a 1-form $\eta = \frac{1}{2}\,(dt - \sum_1^n y^i\, dx^i)$ so that $\xi = 2\,\partial_t$ is the characteristic vector field. With respect to the natural field of frames $\{\partial_{x^i};\ \partial_{y^i},\ \partial_t\}$, define a tensor field ϕ of type $(1,\ 1)$ by its matrix

$$(\phi) = \begin{pmatrix} 0_{n,\,n} & I_n & 0_{n,\,1} \\ -I_n & 0_{n,\,n} & 0_{n,\,1} \\ 0_{1,\,n} & -y^i & 0 \end{pmatrix}$$

Define a Lorentzian metric g with line element given by

$$ds^2 = \frac{1}{4}\left\{\sum_1^n (\,(dx^i)^2 + (dy^i)^2) - \eta\otimes\eta\right\}.$$

Then, with respect to an orthonormal basis $\{E_i,\ E_{n+i},\ \xi\}$ such that

$$\begin{aligned} E_i &= 2\,\partial_i, \quad E_{n+i} = 2\,\partial_{n+i}, \\ \phi\, E_i &= 2\,(\partial_i - y^i\,\partial_t), \\ \phi\, E_{n+i} &= 2\,(\partial_i + y^i\,\partial_t), \end{aligned}$$

it is easy to verify that $(M,\ g)$ is an almost contact spacetime.

Physical Model. Let $\bar{M} = (M^{2n}, \bar{g}, J^2 = -I)$ be an almost Hermitian manifold with Hermitian structure defined by $\bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y})$, for every vector field $\bar{X}$, $\bar{Y}$ of $\bar{M}$. Construct product manifold defined by $M = (R\times\bar{M}, g = -dt^2 + \bar{g})$. Let $X = (\eta(X)\frac{d}{dt}, \bar{X})$ be a vector field on M where $\bar{X}$ is tangent to M, t is a tiemlike coordinate of R and $\eta(X)$ is a smooth function on M. Set $eta = dt$ so that $xi = (\frac{d}{dt}, 0)$ is a timelike global vector field tangent to M. Let ϕ be a $(1, 1)$ tensor field on M defined by

$$\phi(X) = \phi\left(\eta(X)\frac{d}{dt}, \bar{X}\right) = (0, JX).$$

Then, we obtain

$$\begin{aligned} &\phi^2(X) = -X + \eta(X)\xi, \quad \phi(\xi) = 0, \quad \eta\phi = 0, \quad \eta(\xi) = 1, \\ &g(\xi,\xi) = -1, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \end{aligned}$$

which shows that (M, g, ϕ) is an almost contact spacetime manifold. Following is a physical example extensively used in general relativity: Let S_1^n be a pseudo-Euclidean sphere of Lorentzian signature $(-1, +\cdots+)$ defined by

$$S_1^n = \{x\in R_1^{n+1} : -x_1^2 + x_1^2 + \cdots + x_{n+1}^2 = r^2 : r > 0\}.$$

Topologically S_1^n is $R^1 \times S^{n-1}$ and is a Lorentzian analogue of the sphere whose curvature is $-(1/r^2)$ and it is physically known as de-Sitter spacetime. Thus, as explained above we say that

Odd dimensional de-Sitter spacetimes can carry a contact structure.

Now we show how to recover a CR-structure for an almost contact spacetime (M, g). For this we need the following four tensors $N^{(1)}, N^{(2)}, N^{(3)}, N^{(4)}$ defined by Blair [8, Chap. 4] which will also hold for our class of almost contact spacetime (M, g):

$$N^{(1)} = [\phi, \phi](X, Y) + 2d\eta \otimes \xi,$$
$$N^{(2)} = (\pounds_{\phi X}\eta)(Y) - (\pounds_{\phi Y}\eta)(X),$$
$$N^{(3)} = (\pounds_{\xi}\phi)X, \quad N^{(4)} = (\pounds_{\xi}\eta)X.$$

Moreover, the vanishing of $N^{(1)}$ implies the vanishing of other three tensors. Based on this we say that an almost contact manifold is said to be normal if the tensor $N^{(1)}$ vanishes, that is, if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

which also holds for an almost contact spacetime. Since $\phi\xi = 0$ and $\phi^2 = -I + \eta \otimes \xi$, the eigenvalues of ϕ are 0 and $\pm i$, $\pm i$ each having multiplicity n. As D is Riemannian, $(\phi, g/D)$ defines an almost complex structure on the distribution D. Thus the complexification of D in $CT(M)$ can be decomposed at a point $x \in M$ into say $H'_x \oplus H''_x$ where $H'_x = \{X - i\phi X : X \in D_x\}$ and $H''_x = \{X + i\phi X : X \in D_x\}$. Then the following result (proved by Ianus [25] for Riemannian contact manifolds) will also hold for the contact spacetime since the distribution D for both cases is Riemannian. We reproduce the proof as given in [7].

Theorem 6.1 *Let (M, g) be a $(2n+1)$-dimensional normal almost contact spacetime whose characteristic vector field ξ is timelike. Then, (M, g, H') is a CR-manifold, where H' is a holomorphic subbundle of $CT(M)$*

Proof Since $\bar{H}'_x = H'_x$ and $H'_x \cap H''_x = \{0\}$, it is sufficient to show that for any $X, Y \in D$, $[X - i\phi X, Y - i\phi Y] \in H'$. Now for any $X, Y \in D$ the Normality condition $[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$ becomes

$$-[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi y] = 0.$$

Also from the second normality condition $N^2 = 0$, it is easy to see that

$$\eta([\phi X, Y] - [\phi Y, X]) = 0.$$

Using all this and denoting $L(X) = X - \phi X, \forall X \in D$ we obtain

$$
\begin{aligned}
[L(X), L(Y)] &= [X, Y] - [\phi X, \phi Y] - i[\phi X, Y] - i[\phi Y, X] \\
&= -\phi[\phi X, Y] - \phi[X, \phi Y] + i\phi^2[\phi X, Y] - i\eta([\phi X, Y])\xi \\
&\quad + i\phi^2[X, \phi Y] - i\eta([X, \phi Y]\xi \\
&= -\phi([\phi X, Y] - i\phi[\phi X, Y]) - \phi([X, \phi Y] - i\phi[X, \phi Y]) \in H',
\end{aligned}
$$

which completes the proof.

Example 2 Recall from previous example the construction of an almost contact metric structure (ϕ, ξ, η, g) on a Minkowski spacetime (R_1^{2n+1}, g) with local coordinates (x^i, y^i, t) and $i = 1, \dots, n$, where M admits a global timelike vector field ξ and the 1-form $\eta = \frac{1}{2}(dt - \sum_1^n y^i \, dx^i)$. From the matrix expression of ϕ, one can easily verify that this almost contact spacetime satisfies the normality condition. Following the proof of above theorem, it is straightforward to show that (R_1^{2n+1}, g) is a CR-manifold.

Open problem 1. If ξ is spacelike, then, the contact distribution D must be timelike for the almost contact spacetime (M, g). This case was not discussed by Duggal in [17] and, to the best of our knowledge, this is still an open problem. Observe that due to timelike contact distribution D, as explained in the case of CR–Lorentzian structures, the real ϕ cannot act as an almost complex structure operator on D unless $\dim(D) = 2$. Also, for this reason Ianus's theorem will not hold for a real ϕ. Thus, to define a complex structure (needed to have a holomorphic subspace) and then to recover a CR-structure for this class, one must either use Flaherty's method of replacing real ϕ with a complex valued operator and follow as explained in previous subsection or try some other way to show the existence of CR-structure.

6.3 CR–Lorentzian Submanifolds

In differential geometry, CR-structures have a key role in submanifolds theory primarily developed by Chen [11] (1973), Bejancu [4] (1986), Yano [35] (1983) and for physical applications, Duggal [15] introduced, in 1986, the concept of Lorentzian CR-submanifolds and established its fruitful interplay with general relativity (see [16, 17, 19]). Also see two books [20, 21] on the geometry of lightlike (also called null) submanifolds, including CR-submanifolds. In this section, we use some results of the works of these researchers as applicable to the Lorentzian CR-submanifolds of indefinite Hermitian and Kähler manifolds by focusing on those results which are different than the large number of known results on CR–Riemannian submanifolds, update on the main results of [16, 19] and propose some open problems.

Indefinite Hermitian and Kähler manifolds. Let $(\bar{M}, \bar{g}, \bar{J})$ be a $2m$-dimensional semi-Riemannian manifold $(m > 1)$ with an almost Hermitian structure. It is known that every almost complex manifold with a Riemannian metric admits a Hermitian metric, but as we explained in previous section, for real $\bar{J}$ the signature of $\bar{g}$ must be of the type $(2p, 2q)$ with $p + q = m$ for an indefinite metric $\bar{g}$, which we now assume. We highlight that this restriction on the metric of a semi-Riemannian almost Hermitian manifold (which, unfortunately, is either ignored or not explicitly stated in some research papers) is an essential requirement. Also note that, in particular, $\bar{g}$ Lorentzian is ruled out for real $\bar{J}$. An almost Hermitian $\bar{M}$ is called Hermitian if the Nijenhuis tensor N of $\bar{J}$ vanishes i.e.,

$$N(X, Y) = [\bar{J}X, \bar{J}Y] - [X, Y] - \bar{J}[X, \bar{J}Y] - \bar{J}[\bar{J}X, Y] = 0,$$

for all $X, Y \in T(\bar{M})$. The 2-form Ω of the Hermitian $\bar{M}$ is defined by $\Omega(X, Y) = \bar{g}(X, \bar{J}Y)$. We say that $\bar{M}$ is a Kählar manifold if Ω is closed, i.e., $d(\Omega) = 0$. It is easy to show that a Hermitian manifold is a Kählar manifold if $\bar{\nabla}\bar{J} = 0$, where we denote by $\bar{\nabla}$ the Levi-Civita connection on $\bar{M}$. Simple example is complex manifold C^n. Soon we will explain the important use of Kähler structure. Finally, an indefinite complex space form is a connected indefinite Kähler manifold of constant holomorphic sectional curvature c denoted by $\bar{M}(c)$ whose curvature tensor field is given by

$$\begin{aligned}\bar{R}(X, Y) = \frac{c}{4}\{&\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(\bar{J}Y, Z)\bar{J}X \\ &- \bar{g}(\bar{J}X, Z)\bar{J}Y + 2\bar{g}(X, \bar{J}Y)\bar{J}Z\},\end{aligned}$$

for all $X, Y, Z \in T\bar{M})$. Barros and Romero [2] constructed $\mathbf{C}^m_q$, $\mathbf{CP^m_q(c)}$ and $\mathbf{CH}^m_q(c)$, as representative examples of simply connected indefinite complex space forms according as $c = 0$, $c > 0$ and $c < 0$, respectively.

Let (M, g) be an n-dimensional submanifold of a $2m = (n + r)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$, where g is the non-degenerate metric of M induced from the metric $\bar{g}$ of $\bar{M}$. Then

$$T\bar{M} = TM \perp TM^{\perp}, \quad TM \cap TM^{\perp} = \{0\}.$$

Let ∇ be the induced Levi-Civita connection on M. Then there exists a uniquely defined unit normal vector field, say $\mathbf{n} \in \Gamma(T\bar{M})$ and the Gauss–Weingarten formulas are

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + B(X, Y)\mathbf{n}, \\ \bar{\nabla}_X \mathbf{n} &= -\epsilon A_{\mathbf{n}} X,\end{aligned}$$

and $g(\mathbf{n}, \mathbf{n}) = \epsilon \in \{\pm 1\}$. Assume that (M, g) is Lorentzian and $(\bar{M}, \bar{g}, \bar{J})$ is an indefinite almost Hermitian manifold. Suppose M has a q-dimensional real distribution D. As explained in previous section, D is either spacelike or timelike or lightlike. We discuss these three subcases. If D is spacelike, then following definition of Bejancu [4] for Riemannian case will also hold for our Lorentzian submanifold (M, g):

Definition 1 Let (M, g, J) be a Lorentzian submanifold of an indefinite almost Hermitian manifold $(\bar{M}, \bar{g}, \bar{J})$ such that M admits a spacelike distribution D. Then M is called a CR–Lorentzian submanifold of $\bar{M}$ if

(1) $D : x \to D_x \subset T_x(M)$ is invariant (holomorphic), i.e., $J(D) = D$.
(2) The complementary orthogonal distribution
$D^{\perp} : x \to D^{\perp} \subset T_x(M)$ is anti-invariant (real), i.e., $JD^{\perp} \subset T(M)^{\perp}$.

It is important to mention that in previous two papers [16, 19], above definition was used without stating that for a real J it is valid only if the distribution D is spacelike. Indeed, if D is timelike, then, as explained in previous section, it cannot be endowed with an induced Hermitian structure with respect to real J so the condition (1) of above definition will not hold. We highlight this correction, for this reason we need an amended definition of CR–Lorentzian submanifolds with a timelike D. Later on we discuss this case as an open problem.

Example 3 ([16]) Let (M, g) be a 4-dimensional spacetime embedded in a 6-dimensional Hermitian manifold $(\bar{M}, \bar{g}, J)$ with the metric $\bar{g}$ of signature $(3, 3)$ and J its almost complex structure tensor. Then for a local coordinate system $(x^r; y^r)$ of $\bar{M}$, $(r = 1, 2, 3)$, there exists an orthonormal basis $(\partial_{x^r}; \partial_{y^r})$ such that $J\partial_{x^r} = \partial_{y^r}$ and $J\partial_y^r = -\partial_{x^r}$, $J^2 = -I$. Let $\{e_a\} = (m, \bar{m}, \ell, k)$ a null tetrad at each point of M such that $m, \bar{m}$ are conjugate complex null vectors and (ℓ, k) are real null vectors with $\ell \cdot k = -1$ and $\{w^a\}$ their dual basis. Construct the embedding of M in such a way that $\{\xi_a\} = (\partial_{x^1}, \partial_{x^2}, \partial_{x^3}, \partial_{y^1})$ is an orthonormal basis of $T_x(M)$ with its dual $\{\eta^a\}$. Then,

$$\sqrt{2}\xi_1 = m + \bar{M}, \quad \sqrt{2}\xi_2 = i(\bar{m} - m), \quad \sqrt{2}\xi_3 = k - \ell, \quad \sqrt{2}\xi_4 = k + \ell,$$
$$\eta^1 = w^1 + w^2, \quad \eta^2 = i(w^2 - w^1), \quad \eta^3 = w^4 - w^3, \quad \eta^4 = w^3 + w^4,$$

where $g(\xi_a, X) = \eta^i(X)$ for $i = 1, 2, 3$ and $g(\xi_4, X) = -\eta^4(X)$ and

$$g(X, Y) = \eta^i(X)\eta^i(Y) - \eta^4(X)\eta^4(Y).$$

Then, the complexified space $CT_x(M)$ is generated by $(\partial_{z^1}, \partial_{\bar{z}^1}, \partial_{z^2}, \partial_{\bar{z}^2})$ where $\partial_{z^1} = m$ and $\partial_{z^2} = k(1 - i) - \ell(1 + i)$. Therefore, $CT_x(M)$ has a holomorphic space $H = \{m, \bar{m}\}$. This, M is a CR–Lorentzian submanifold of $\bar{M}$ with real (invariant) spacelike distribution $D = \{\xi_1, \xi_2\}$ and $D^{\perp} = \{\xi_3, \xi_4\}$ is timelike.

There is an interesting relation of above example with an important class of the Einstein–Maxwell theory. For details we refer [16].

It is well known that for a Riemannian $\bar{M}$ there exist two subcases, namely invariant (totally holomorphic) and non-invariant (totally real) CR-submanifolds according as $\dim(D^{\perp}) = 0$ and $\dim(D) = 0$, respectively. Contrary to this, it was proved in [19] that "*There exists no totally holomorphic CR–Lorentzian submanifold of an indefinite almost Hermitian manifold*". We also make a correction for the same reason and say that above result holds only if the distribution D is spacelike because $D = TM$

is not possible for a Lorentzian M. We also know that for a Riemennian case, a CR-submanifold is called proper(nontrivial) if $D \neq 0$ and $D \neq T(M)$. Recalling the notion of a CR-manifold as explained in previous section, Blair–Chen [9] has proved the following result.

Theorem 6.2 *Let M be a CR-submanifold of a Hermitian manifold $\bar{M}$. If M is nontrivial ($D \neq 0$ and $D \neq T(M)$), then M is a CR-manifold.*

For a proper CR–Lorentzian submanifold with a spacelike invariant distribution D, the condition $D \neq 0$ is sufficient as $D = T(M)$ is ruled out. Therefore, for this class of CR–Lorentzian submanifolds, it is sufficient to say that Blair–Chen's above result will hold if the spacelike $D \neq 0$. Also, just like a Riemannian case, each real Lorentzian hypersurface M of $\bar{M}$ is a proper CR-submanifold.

Now we assume that the Hermitian structure of $(\bar{M}, \bar{g})$ is also *Kähler*ian. Before proceeding further, we first explain why one needs to impose a Kähler structure. Flaherty [23] has shown that a Hermitian $\bar{M}$ is Kählerian if and only if, with respect to a complex coordinate system the Hermitian metric $\bar{g}$ is locally derivable from a real scalar potential K. This means that

$$\bar{g}_{a\bar{b}} = \partial^2 K / \partial z^a \partial \bar{z}^b,$$

where $z^a = x^a + iy^a$ are complex coordinates $(\geq 1a, b \leq m)$. The same is true for the Ricci tensor, i.e.,

$$R_{ab} = \frac{\partial^2}{\partial z^a \partial \bar{z}^b}[\ln \det(\bar{g}_{c\bar{d}})].$$

Thus, the computation of the curvature quantities is much simplified using Kähler structure on $\bar{M}$.

Definition 2 A CR–submanifold of an almost Hermitian manifold $\bar{M}$ is called a CR-product if its both distributions D and $D^\perp$ are integrable and their respective leaves S_1 and S_2 are totally geodesic in M.

This definition will also hold for any CR–Lorentzian submanifold, with non-null distributions D and $D^\perp$, of an indefinite almost Hermitian manifold $\bar{M}$. We call its product a CR–Lorentzian product. For the Riemannian–CR-product case, in 1981, Chen proved following characterization theorem:

Chen [12]: *Let $\bar{M}$ be a Kähler manifold with negative holomorphic bisectional curvature. Then every CR-product in $\bar{M}$ is either holomorphic submanifold or totally real submanifold.*

It follows from above discussion that Chen's above theorem will not hold for a spacelike invariant distribution D as $D \neq T(M)$. Thus, for the spacelike case there does not exist any totally holomorphic CR–Lorentzian product submanifold. We leave it as an exercise to verify that several other results of CR–Lorentzian submanifolds, with spacelike invariant distributions will be different than their corresponding Riemannian CR-submanifolds.

Two Physical Models of CR–Lorentzian Products

Model 1. A product manifold ($M = M_1 \times M_2$, $g = g_1 \oplus g_2$) is called locally decomposable manifold if there exists a local coordinate $x^i = (x^a, x^A)$ in terms of which the line element of the metric g has the form:

$$ds^2 = g_{ab}(x^c)dx^a dx^b + g_(AB)(x^C)dx^A dx^B,$$

where $(1 \leq i, j, k \leq n)$; $(1 \leq a, b, c \leq p)$ and $(p + 1 \leq A, B, C \leq n)$. A 4 -dimensional decomposable spacetime is either the product of a (1, 3)-dimensional spaces or a product of (2, 2)-dimensional spaces. If (M, g) is a CR–Lorentzian product spacetime, then, since its non-null invariant distribution D must be even dimensional, (M, g) must be a product of (2, 2)-spaces. Such spacetimes are candidates of a physical model of CR–Lorentzian products. Following are two specific examples of (2, 2) spacetimes taken from Kramer et al. [26]

The only Einstein–Maxwell field which is homogeneous and has a homogeneous non-singular Maxwell field is the Bertotti–Robinson solution

$$ds^2 = A^2(d\,\theta^2 + \sin^2\,\theta\,d\,\phi^2 + d\,x^2 + \sinh^2\,x\,dt^2)$$

where (t, x, θ, ϕ) and A are local coordinates and an arbitrary constant, respectively. This solution has two families of orthogonal 2-surfaces having equal and opposite curvatures. Following is second example:

The only conformally flat spacetimes, with non-null electromagnetic field, are embedding class two (i.e., co-dimension 2) decomposable spacetimes of the product of two-dimensional spaces of constant curvature. Both the curvature tensor and electromagnetic tensor field are constant.

Model 2. A spacetime $(M,\ g)$ is said to be globally hyperbolic if there exists a spacelike hypersurface Σ such that every endless causal curve intersects Σ once and only once. Such a hypersurface (if it exists) is called a Cauchy surface. If M is globally hyperbolic, then (a) M is homeomorphic to a product manifold $\mathbf{R} \times \Sigma$, where Σ is a hypersurface of M, and for each t, $\{t\} \times \Sigma$ is a Cauchy surface, (b) if Σ' is any compact hypersurface without boundary, of M, then Σ' must be a Cauchy surface. A simple example is Minkowski spacetime. It has been shown in the works of Beem–Ehrlich [3] that the globally hyperbolic spacetimes are physically important spacetimes. They have constructed a product space of a globally hyperbolic spacetime and a Riemannin manifolds as follows:

Let $(M_1,\ g_1)$ and $(M_2,\ g_2)$ be Lorentz and Riemannian manifolds, respectively. Let $h : M_1 \to (0, \infty)$ be a C^∞ function and $\pi : M_1 \times M_2 \to M_1$, $\sigma : M_1 \times M_2 \to M_2$ the projection maps given by $\pi(x,\ y) = x$ and $\sigma(x,\ y) = y$ for every $(x,\ y) \in M_1 \times M_2$. Then, define the metric g given by

$$g(X,\ Y) = g_1(\pi_\star X,\ \pi_\star Y) + h(\pi(x,\ y))\, g_2(\sigma_\star X,\ \sigma_\star Y), \quad \forall X, Y \in \Gamma(TM)$$

where $\pi_\star$ and $\sigma_\star$ are, respectively, tangent maps. Now we quote:

Beem–Ehrlich [3] *Let $(M_1,\ g_1)$ and $(M_2,\ g_2)$ be Lorentzian and Riemannian manifolds, respectively. Then, the Lorentzian warped product manifold* ($M = M_1 \times_h$

$M_2,\ g = g_1 \oplus_h g_2$) is globally hyperbolic if and only if both the following conditions hold:

(1) $(M_1,\ g_1)$*is globally hyperbolic.*
(2) $(M_2,\ g_2)$ *is a complete Riemannian manifold.*

Physical examples of warped product spacetimes are: Robertson–Walker, Schwarzschild and Reissner–Nordström spacetimes. For details on these spacetimes and their significant physical use in general relativity we refer [3]. Subject to the conditions for a CR-product space, we leave it as an exercise to show that a Lorentzian warped product manifold is a physical model of CR–Lorentzian submanifolds.

Open problem 2. If the invariant distribution D of CR–Lorentzian submanifold is timelike, then, using the information of Sect. 2, there is a need to modify the Hermitian structure as Bejancu's definition of CR–submanifolds will not hold for real J since it is not compatible with the Lorentzian metric $g_{|D}$ if D is timelike. This case was not discussed in [16, 19] and, to the best of our knowledge, this is still an unsolved problem. To define a complex structure (needed to have a holomorphic subspace) and then to recover a CR–Lorentzian structure for the timelike class one may either use Flaherty's method of replacing real J with a complex valued operator and follow as explained in Sect. 6.2.1 or try some other suitable way to modify Bejancu's definition of CR-submanifolds.

Now we construct a mathematical model of a CR–Lorentzian submanifold having a q-dimensional lightlike distribution.

Theorem 6.3 *Let (M, g) be an oriented n-dimensional Lorentzian submanifold of a 2m-dimensional almost Hermitian manifold $(\bar{M}, \bar{g}, J)$. Suppose there exists a q-dimensional lightlike distribution D on TM. Then, (M, g) is a CR–Lorentzian submanifold of $(\bar{M}, \bar{g}, J)$ with an invariant lightlike distribution $\mathcal{D}$ such that*

$$(A) \quad \dim(M) \geq 4, \quad \dim(\mathcal{D}) = \dim(M) - \dim(D) - 1 \geq 2, \quad \dim(\tilde{D}) \geq 2,$$

where $\mathcal{D} = Re(H + \bar{H})$ and $\tilde{\mathcal{D}}$ is a complementary distribution to $\mathcal{D}$ of TM and H is the associated holomorphic subbundle of $CT(M)$. Moreover, if the dim(H) is constant on M, then, M is a CR-manifold.

Proof Following as explained in the 4-dimensional case (see Sect. 2), for a q-dimensional lightlike distribution D there exist $D^{\perp}$ and $L = D \cap D^{\perp}$ which are $n - q$ and one-dimensional lightlike distributions on TM. The fibres of the quotient bundle $D^{\perp}/L$ are $(n - q - 1)$-dimensional Riemannian screen subspaces denoted by S_x at each $x \in M$. Then,

$$(B): \qquad TM = D \oplus_{orth} S \oplus_{orth} \tilde{L}, \quad D \cap \tilde{D} = \{0\},$$

where $\oplus_{orth}$ stands for orthogonal direct sum, $\tilde{L}$ is complementary to L in D and S denote the quotient subbundle of M.Since S is Riemannian, we endow it with a natural canonical automorphism J, satisfying $J^2 = -I_S$. This provides a complex structure J on S such that

$$C(S) = S^+ \oplus S^-, \quad S^{\pm} = \{z \in C(S) : Jz = \pm iz\},$$

where $C(S)$ denotes the complexified space of S. Let $H^{\pm}$ be the subbundles of $C(D^{\perp})$ projecting onto $C(S^{\pm})$ by a canonical map $C(D)^{\perp} \to C(S)$. It is easy to see that

$$H^+ \cap H^- = C(D), \quad H_x^+ + H_x^- = C(D^{\perp}).$$

Using the projection $C(H^{\pm}) \to C(S^{\pm})$, one can show that each $H^{\pm}$ is a degenerate holomorphic subbundle of $C(TM)$ with the induced complex structure J. Indeed, since J is an isometry, the square of z is equal to that of $Jz = \pm iz$ for any $z \in H^{\pm}$. The latter square is opposite to that of z; thus the square of z is zero for any $z \in H^{\pm}$. Single out a holomorphic subbundle (say $H^+ = H$) and a real J-invariant lightlike distribution

$$\mathcal{D} = Re(H + \bar{H}), \quad H = \{X - iJX : X \in \mathcal{D}\}.$$

Then, using (B) and the projection $C(H^{\pm}) \to C(S^{\pm})$, one can show that each $H^{\pm}$ is a degenerate holomorphic subbundle of $C(TM)$ with the induced complex structure J. Indeed, since J is an isometry, the square of z is equal to that of $Jz = \pm iz$ for any $z \in H^{\pm}$. The latter square is opposite to that of z; thus the square of z is zero for any $z \in H^{\pm}$. Now using hypothesis, we choose the dimension of $\dim(\mathcal{D})$ such that $\dim(H)$ is constant on M. Based on this and a paper by Wells [34], we say that M is a CR–Lorentzian submanifold of $\bar{M}$, with a real lightlike distribution $\mathcal{D}$. Also $\dim(\mathcal{D}) \geq 2$ It follows from the relation (B) we say that (A) holds which completes the proof.

Also recall from [34] that for a CR-submanifold M of $\bar{M}$ it is well known that $\max(n-m, 0) \leq \dim(H) \leq \frac{n}{2}$. If $\dim(H) = \max(n-m, 0)$ at each $x \in M$, then M is called a generic submanifold. Furthermore, note that above proof is much simple and easy to read that the one given in an earlier paper [19] of Sect. 3 (Singular CR-structures).

Now we show that some of the key results of the Riemannian CR-submanifolds will not hold for CR–Lorentzian submanifolds having a lightlike distribution. First we recall that a submanifold (M, g) is said to be totally umbilical if $B(X, Y) = \mu g(X, Y)$ for any vector fields X, Y of M. Moreover, M is totally geodesic if and only if B vanishes, where B denotes the second fundamental form of M. Examine the following two well known Theorems

Bejancu ([35, p. 96]). *Any totally umbilical proper CR-submanifold of a Kähler manifold $\bar{M}$, with $\dim(D^{\perp}) > 1$, is totally geodesic in $\bar{M}$.*

Blair–Chen ([35, p. 98]). *Any totally umbilical proper CR-submanifolds of a Kähler manifold $\bar{M}$, with the $\dim(D^{\perp}) > 1$, is locally a direct product of totally geodesic invariant and anti-invariant submanifolds of $\bar{M}$.*

Above results will also hold for a totally umbilical CR–Lorentzian submanifold with non-null distribution D for which we quote the following result

Duggal-Sharma [22]. *Let (M, g) be a totally umbilical CR–Lorentzian submanifold of a Kähler manifold $\bar{M}$. If $\dim(D^{\perp}) > 1$, then, M is totally geodesic in $\bar{M}$ if both* *the distributions D and $D^{\perp}$ are non-null. M is then locally a product of the leaves of D and $D^{\perp}$, i.e., M is a CR–Lorentzian product manifold.*

The case of lightlike distribution D is different for which we quote:

Duggal [15]. *Let M be a totally umbilical proper CR–Lorentzian submanifold of a Kähler manifold $\bar{M}$, with a lightlike distribution D. Then, M is not necessarily totally geodesic in $\bar{M}$.*

Finally, we refer [16, 19] for some physical examples of CR–Lorentzian submanifolds with spacelike and lightlike distributions.

Open problems 3. (a) As Bejancu's above result is the root of several other results on the totally umbilical CR-submanifolds, as a chain reaction there may be more results which will not hold for the CR–Lorentzian submanifolds with a lightlike distribution. We leave this as an open problem.

(b) We propose the following fundamental problem:

Characterize all totally umbilical (respectively, geodesic) Lorentzian submanifolds of an indefinite Kähler manifold.

For information on a similar problem proposed and worked by Chen we cite [12] and some other references therein.

(c) The integrability conditions for the existence of non-degenerate distributions D and $D^{\perp}$ of a CR–Lorentzian submanifolds follow from the results of Bejancu [4] on the CR–Riemannian submanifolds. However, the case of lightlike distributions still remains as an open problem.

6.4 Contact CR–Lorentzian Submanifolds

As explained in previous section, let $(\bar{M}, \bar{g}, \phi, \xi, \eta)$ be a real $(2n+1)$-dimensional (ϵ)-almost contact metric manifold whose metric $\bar{g}$ is semi-Riemannian. Denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ which is either timelike or spacelike according as $\epsilon = -1$ or $+1$, respectively, and ξ is never null. The $rank(\phi) = 2\,n$. The index q of g is odd or even number according as ξ is timelike or spacelike. This holds since on $\bar{M}$ one can consider an orthonormal field of frame $\{E_1, \dots, E_n, \phi E_1, \dots, \phi E_n, \xi\}$ with $E_i \in \Gamma(\bar{D})$ and $\bar{g}(E_i, E_i) = \bar{g}(\phi E_i, \phi E_i)$, where $\bar{D}(\eta = 0)$ denotes the contact distribution.

Unfortunately, contrary to the Riemannian case $(\epsilon = 1, q = 0)$ for which there always exists a Riemannian metric satisfying the structure equations of $\bar{M}$, there is no such guarantee for the existence of a non-degenerate metric for a proper semi-Riemannian manifold $\bar{M}$. At best the following is known.

Duggal-Bejancu [20]. *Let (ϕ, ξ, η) be an almost contact structure and h_0 a metric on a semi-Riemannian manifold $\bar{M}$ such that ξ is non-null. Then there exists on $\bar{M}$ a $(1, 2)$ type symmetric tensor field $\bar{g}$ satisfying its almost contact structure equations.*

However, on the brighter side, as we have discussed in previous section on *Contact CR–Lorentzian structures*, there always exists a Lorentzian metric $\bar{g}$ on a $\bar{M}$ satisfying its almost contact structure equations if ξ is timelike.

Moreover, there are restrictions on the signature of $\bar{g}$ as follows. For an almost contact metric manifold $\bar{M}$, its $2n$-dimensional contact distribution $\bar{D}$ has an indefinite *Hermitian structure*, defined by $J = \phi/\bar{D}$, i.e., we have

$$\bar{g}(JX, JY) = \bar{g}(X, Y), \forall X, Y \in T(\bar{D}).$$

For the *endomorphism* J, satisfying $J^2 = -I$ on $\bar{D}$, as explained in previous section, the only possible signatures of $\bar{g}/\bar{D}$ are $(2p, 2s)$ with $p + s = n$. In particular, $(\bar{D}, \bar{g}/\bar{D}, J)$ satisfying above condition, cannot carry a Lorentz metric, if J is real. Subject to these restrictions on $\bar{g}$, we have the following two classes of (ϵ)-almost contact metric manifolds.

(1) $\epsilon = 1,\ q = 2r$. $\bar{M}$ is spacelike almost contact metric manifold.
(2) $\epsilon = -1,\ q = 2r + 1$. $\bar{M}$ is timelike almost contact metric manifold.

The fundamental 2-form Ω is defined by

$$\Omega(X,\ Y) = \bar{g}(X,\ \phi Y), \quad \forall X,\ Y \in T\bar{M}).$$

Since Ω also satisfies $\eta \wedge \Omega^n \neq 0$, this (ϵ)-almost contact metric manifold $\bar{M}$ is orientable. Then, $\bar{M}$ is called a contact manifold if there exists a $1-$form η on $\bar{M}$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. The (ϵ)-almost contact structure (ϕ, ξ, η) on $\bar{M}$ is said to be normal if

$$N_\phi + 2\, d\, \eta \otimes \xi = 0,$$

where $N_\phi = [\phi, \phi]$ is the Nijenhuis tensor field of ϕ. A normal $\bar{M}$ is called an indefinite Sasakian manifold. Moreover, we say that an (ϵ)-almost contact metric manifold $\bar{M}$ is an indefinite Sasakian manifold if only if

$$(\bar{\nabla}_X\, \phi)\, Y = g\,(X,\ Y)\, \xi - \epsilon\, \eta(Y)X, \quad \forall X,\ Y \in T\bar{M}.$$

By replacing Y by ξ in above we get

$$\bar{\nabla}_X \xi = -\epsilon\phi X, \quad \forall X \in T\bar{M},$$

which implies that $\nabla_\xi \xi = 0$, i.e., the characteristic vector field ξ on an indefinite Sasakian manifold is a Killing vector field.

Finally, we present the following examples of (ϵ)-Sasakian structures on a semi-Euclidean manifold $\mathbf{R}_q^{2n+1}$ of index q, taken from [21, p. 309]:

We need the following notations: $0_{p,k} = p \times k$ null matrix; $I_k = k \times k$ unit matrix. For any nonnegative integer $s \leq n$ we put

$$\epsilon^a = \begin{cases} -1 & \text{for} \quad a \in \{1, \ldots, s\} \\ 1 & \text{for} \quad a \in \{s+1, \ldots, n\}, \end{cases}$$

in case $s \neq 0$ and $\epsilon^a = 1$ in case $s = 0$. Consider (x^i, y^i, z) as Cartesian coordinates on $\mathbf{R}_q^{2n+1}$ and define with respect to the natural field of frames $\{\partial_{x^i}, \partial_{y^i}, \partial_z\}$ a (1, 1) tensor field ϕ by its matrix

$$(\phi) = \begin{pmatrix} 0_{n,\,n} & I_n & 0_{n,\,1} \\ -I_n & 0_{n,\,n} & 0_{n,\,1} \\ 0_{1,\,n} & \epsilon^a y^a & 0 \end{pmatrix}.$$

The 1-form η is defined by

$$\eta = \frac{\epsilon}{2}\left(dz + \sum_{i=1}^{s} y^i dx^i - \sum_{i'=r+1}^{n} y^{i'}\, dx^{i'}\right), \quad \text{if} \quad s \neq 0$$

and $\eta = \frac{\epsilon}{2}\left(dz - \sum_1^n y^i\, dx^i\right)$, if $s = o$. The characteristic vector field is $\xi = 2\epsilon\partial_z$. It is easy to check that (ϕ, ξ, η) is an (ϵ)-almost contact structure on $\mathbf{R}_q^{2n+1}$ for each s. Define a semi-Riemannian metric $\bar{g}$ by the matrix

$$(\bar{g}) = \frac{\epsilon}{4}\begin{pmatrix} -\delta_{ij} + y^i y^j & -y^i y^{j'} & 0_{s,s} & O_{s,n-s} & y^i \\ -y^i y^{j'} & \delta_{i'j'} + y^{i'} y^{j'} & 0_{n-s,s} & O_{n-s,n-s} & -y^{i'} \\ O_{s,s} & O_{s,n-s} & -I_s & O_{s,n-s} & O_{s,1} \\ 0_{n-s,s} & O_{n-s,n-s} & 0_{n-s,s} & I_{n-s} & O_{n-s,1} \\ y^i & -y^{i'} & O_{1,s} & O_{1,n-s} & I \end{pmatrix}$$

for $s \neq 0$, and for $s = 0$ we get

$$(\bar{g}) = \frac{\epsilon}{4}\begin{pmatrix} -\delta_{ij} + y^i y^j & 0_{n,n} & y^i \\ O_{n,n} & I_n & O_{n,1} \\ y^i & O_{1,n} & I \end{pmatrix}.$$

An orthonormal field of frames with respect to above metric is

$$\left\{\begin{array}{l} E_i \ = 2\,\partial_{y^i}, \quad E_{i'} = 2\,\partial_{y^{i'}}, \\ \phi E_i = \ \ 2\,(\partial_{x^i} - y^i\,\partial_z), \\ \phi E_{i'} = \ 2\,(\partial_{x^i} + y^i\,\partial_z), \xi \end{array}\right\}.$$

It is easy to check that above data provides an (ϵ)-Sasakian structure on $\mathbf{R}_q^{2n+1}$ for any $s \in \{0, \ldots, n\}$. In case $s = 0$ and $\epsilon = 1$ we get the classical Sasakian structure on $\mathbf{R}^{2n+1}$. If $s \neq 0$, then, we either get a spacelike Sasakian structure on $\mathbf{R}_{2s}^{2n+1}$ for $\epsilon = 1$ or a timelike Sasakian structure on $\mathbf{R}_{2(n-s)+1}^{2n+1}$ for $\epsilon = -1$. In particular, for $s = n$ and $\epsilon = -1$ we get Lorentzian Sasakian structure.

Definition 3 Let (M, g) be a real $(n+1)$-dimensional Lorentzian submanifold of $\bar{M}$ such that M is tangent to the structure tensor ξ. We say that M is a contact CR–Lorentzian submanifold of $\bar{M}$ if there exist two differentiable distributions D and $\tilde{D}$ on M satisfying

(1) $TM = D \oplus (\tilde{D} \oplus \{\xi\})$ where $(\tilde{D} \oplus \{\xi\})$ is complement to D in TM.
(2) D is invariant with respect to ϕ, i.e., $\phi(D) = D$.
(3) $\tilde{D}$ is anti-invariant with respect to ϕ, i.e., $\phi(\tilde{D}) \subset T(M)^{\perp}$.

Following are three mutual exclusive cases of the causal character of D.

(a) $g_{|D}$ is positive $\Rightarrow D$ is spacelike and ξ is timelike.
(b) $g_{|D}$ is Lorentz $\Rightarrow D$ is timelike and ξ is spacelike.
(c) $g_{|D}$ is degenerate $\Rightarrow D$ is lightlike and ξ is spacelike.

Assume that M has a conformal structure which is needed to preserve the causal character of D. If (a) or (b) holds then D, $\tilde{D} = D^{\perp}$ and $\{\xi\}$ are mutually orthogonal to each other and above definition agrees with Bejancu's [4, p. 100] concept of "semi-invariant submanifolds" and Yano-Kon's [35] definition of "contact CR-submanifolds, both of a Riemannian $\bar{M}$. Here we are considering a larger class to accommodate the indefinite metric. We discuss these three cases separately.

Class (a) Denote by $2p$ and h the real dimensions of D_x and $\tilde{D}_x = D^{\perp}$, respectively, $x \in M$ and let codim $M = 2m - n = q$. Then, just like the Riemannian case, we say that M is invariant or anti-invariant (totally real) submanifold of $\bar{M}$ according as $h = 0$ or $p = 0$, respectively. If $q = h$, then M is called a generic submanifold of $\bar{M}$. If p and h are both nonzero, then M is said to be nontrivial (proper).

Example 4 [16] Let $\pi : (M, g) \to (\bar{M}, \bar{g})$ be a Lorentzian hypersurface of a $(2m + 2)$-dimensional almost Hermitian manifold $(\bar{M}, \bar{g}, J)$. Choose a unit timelike vector field $\mathbf{t}$ normal to M so that $J\mathbf{t}$ is tangent to M. Then, there exists a vector field ξ on M such that $J\mathbf{t} = \pi_\star \xi$. Define a tensor field ϕ of type (1, 1) and a 1-form η on M such that

$$J\pi_\star X = \pi_\star \phi X + \eta(X)\mathbf{t}.$$

Then operating above relation with J we get

$$-\pi_\star X = \pi_\star \phi^2 X + \eta(\phi X)\mathbf{t} - \eta(X)\pi_\star \xi$$

and, therefore, the following will hold

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta\phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1.$$

Thus, (M, ϕ, η, ξ) has an almost contact structure such that its induced metric $g(X, Y) = \bar{g}(\pi_\star X, \pi_\star Y)$ is compatible with Lorentzian contact structure, with ξ its timelike characteristic vector field. Indeed, we have

$$g(X, Y) = \bar{g}(\pi_\star X, \pi_\star Y) = g(\phi X, \phi Y) - \eta(X)\eta(Y)$$

where we have used $g(\xi, X) = -\eta(X)$. Moreover, ϕ acts as an almost complex structure on the Riemannian distribution C defined by $(\phi, g/C, \eta = 0)$. Thus the complexification of C in $CT(M)$ can be decomposed at a point $x \in M$ into say $H'_x \oplus H''_x$

where $H'_x = \{X - i\phi X : X \in C_p\}$ and $H''_x = \{X + i\phi X : X \in C_x\}$. Assume M is nornal. Then, following a result of Ianus [25] (see proof of Theorem 3 in Sect. 2.2) we say that M a contact spacetime manifold.

One can verify that the geometric results of class (a) contact CR–Lorentzian submanifolds will be similar with the Riemannian case since for this class both the distributions D and $D^{\perp}$ are spacelike. For example, the integrability results of the triplet $(D, D^{\perp}, \xi)$ (as discussd in [4] for the Riemannian case) will be same. Moreover, following results (taken from Yano-Kon's [35] book) on the totally umbilical or totally geodesic submanifolds will also hold for the contact CR–Lorentzian submanifolds.

(1) *Let M be a submanifold, tangent to the structure tensor field ξ, of a Sasakian manifold $\bar{M}$. If M is totally umbilical, then M is totally geodesic and invariant submanifold of $\bar{M}$.*

(2) *If M is anti-invariant submanifold, tangent to the structure tensor field ξ, of a Sasakian manifold $\bar{M}$, then, M is not totally umbilical in $\bar{M}$*

Along with the geometry of totally umbilical or totally geodesic submanifolds, for contact CR-submanifolds there is another related concept defined as follows: A contact CR-submanifold M is said to be totally contact umbilical if there exists a normal vector field H such that

$$B(X, Y) - g(\phi X, \phi Y)H + \eta X B(Y, \xi) + \eta(Y) B(X, \xi)$$

for any vector fields X, Y tangent to M, where we denote B the second fundamental form. In particular, M is totally contact geodesic if $H = 0$ in above relation. Following main result on totally contact umbilical submanifolds will also hold for this class of contact Lorentzian submanifolds:

Bejancu [4]. *Any proper totally contact umbilical contact CR-submanifold M of a Sasakian manifold $\bar{M}$ is a totally contact geodesic submanifold of $\bar{M}$ if $dim(D^{\perp}) > 1$. M is then, locally, a product $M_1 \times M_2$, where M_1 (resp. M_2) is totally geodesic invariant (resp. anti-invariant) submanifold of M and ξ is normal to M_2.*

The new information is use of some results on contact CR–Lorentzian submanifolds in other branches of mathematical physics where the metric is indefinite, in particular, general relativity. For example, above result of Bejancu serves as a link between a class of decomposable Lorentzian manifolds with the contact CR–Lorentzian submanifolds of dimension ≥ 5.

Open problem 4. If the invariant distribution D of contact CR–Lorentzian submanifold is timelike, then, using the information of Sect. 2, there is a need to modify the induced Hermitian structure of $g_{|D}$ as the Riemannian definition of contact CR-submanifolds will not hold for real ϕ since it is not compatible with the induced Lorentzian metric on timelike D. This case was not discussed in [16] and, to the best of our knowledge, is still an unsolved problem. To define a complex structure (needed to have a holomorphic subspace) for recovering a contact CR–Lorentzian structure for this class (b) one may either use Flaherty's method of replacing real ϕ with a complex valued operator and follow as explained in Sect. 2 or try some other suitable way to modify the definition of contact CR-submanifolds.

Now we deal with class (c) of a contact CR–Lorentzian submanifold for which $g_{|D}$ is lightlike. The following theorem will hold (proof is common with the proof of Theorem 3).

Theorem 6.4 *Let (M, g) be an orientable Lorentzian submanifold of an almost contact manifold $(\bar{M}, \bar{g}, \phi)$, with its structure vector field ξ tangent to M. Suppose there exists a lightlike distribution D on TM. Then, (M, g) is a a contact CR–Lorentzian submanifold of $(\bar{M}, \bar{g}, J)$ with an invariant lightlike distribution $\mathcal{D}$ such that*

$$(\mathbf{A}): \quad \dim(M) \geq 4, \quad \dim(\mathcal{D}) = \dim(M) - \dim(D) - 1 \geq 2, \quad \dim(\tilde{D}) \geq 1,$$

where $\mathcal{D} = Re(H + \bar{H})$, $(\tilde{\mathcal{D}} + \{\xi\})$ is complementary distribution to $\mathcal{D}$ of TM, H is the associated holomorphic subbundle of $CT(M)$ and ξ is spacelike.

Example 5 Let (M, g) be a 4-dimensional Lorentzian manifold embedded in a 5-dimensional Minkowski space R^5_1. With respect to a local coordinate system (x, s, y, z, t) on R^5_1, we construct an almost contact structure (ϕ, η, ξ) with $\eta = dz - ydx - t$ and $\xi = \partial_z$ and ∂_t timelike. Thus, there exists an orthonormal ϕ-basis $(X_1, X_2, Y_1, Y_2, \xi)$ for R^5_1 such that

$$X_1 = \partial_y, \quad X_2 = \partial_t, \quad Y_1 = \partial_x + y\partial_z, \quad Y_2 = \partial_s + t\partial_z.$$

Then, the restriction J of ϕ to the contact distribution D (defined by $\eta = 0$) generates a Riemannian almost complex structure on D such that $JX_i = Y_i$ and $JY_i = -X_i$, $(i = 1, 2)$. Now we construct the embedding so that, with respect to a quasi-orthonormal basis (k, X, ξ, ℓ) of $T_x(M)$, the following holds

$$\sqrt{2}X = X_1 + Y_1, \quad \sqrt{2}k = \partial_t - Y_2, \quad \sqrt{2}\ell = \partial_t + Y_2, \quad \xi = \partial_z,$$

where k and ℓ are real null vectors of M. Then M can be realized as a contact CR–Lorentzian submanifold of R^5_1 with a real lightlike distribution $\mathcal{D} = \{k, X\}$. Indeed, $\mathcal{D}$ is lightlike as it has a null vector k and no timelike vector (see Proposition 1) with spacelike ξ.

As we noticed in previous subsection, the local geometry of the lightlike case is quite different from the Riemannian case. Indeed, observe that it is obvious from above theorem that there exists no invariant (resp. anti-invariant) contact Lorentzian CR-submanifolds of an indefinite almost contact manifold, with a lightlike distribution. Therefore, these submanifolds are a subclass of CR-manifolds. Moreover, we refer [16] for two results on the non-existence of totally umbilical or totally geodesic submanifolds with lightlike ditribution.

6.4.1 Lorentz Framed CR-manifolds

A real $(2n+q)$-dimensional smooth manifold M admits an f-structure [35] if there exists a non-null smooth (1,1) tensor field ϕ, of the tangent bundle TM, satisfying

$$\phi^3 + \phi = 0, \qquad rank\,(\phi) = 2n.$$

An f-structure is a generalization of almost complex $(q=0)$ and an orientable almost contact $(q=1)$ structure. Corresponding to two projection operators P and Q applied to TM, defined by

$$\text{(i)} \quad P = -\phi^2, \qquad \text{(ii)} \quad Q = \phi^2 + I,$$

where I is the identity operator, there exist two complementary distributions D and $\tilde{D}$ such that $\dim(D) = 2n$ and $\dim(\tilde{D}) = q$. The following relations hold

$$\phi P = P\phi = \phi, \quad \phi Q = Q\phi = 0, \quad \phi^2 P = -P, \quad \phi^2 Q = 0.$$

Thus, we have an almost complex distribution $(D,\ J = \phi/D,\ J^2 = -I)$ and ϕ acts on $\tilde{D}$ as a null operator. Assume that $\tilde{D}_x$ is spanned by q globally defined orthonormal vectors $\{U_a\}$ at each point $x \in M$, $(1 \leq a,\ b, \ldots \leq q)$, with its dual set $\{u^a\}$. Then

$$\phi^2 = -I + u^a \otimes U_a.$$

In the above case, M is called a globally framed (or simply a framed) manifold and we denote its framed structure by (M, ϕ, U_a). The following holds

$$\phi U_a = 0, \qquad u^a \circ \phi = 0, \qquad u^a(U_b) = \delta^a_b.$$

A framed structure (M, ϕ, U_a) is said to be normal if the torsion tensor S_ϕ of ϕ is zero, i.e., if

$$S_\phi \equiv N_\phi + d\,u^a \otimes U_a = 0,$$

where N_ϕ is the Nijenhuis tensor field of ϕ. Now consider a semi-Riemannian metric g of index $0 < v < 2m + n$, on M with an f-structure. We say that the pair (ϕ, g) has an indefinite metric structure if

$$g(\phi X, \phi Y) = \bar{g}(X, Y) - \epsilon_a\, u^a(X_)u^a(Y),$$
$$g(X, U_a) = \epsilon_a\, u^a(X),$$

where $\epsilon_a = +1$ or -1 or 0 according as the corresponding U_a is spacelike or timelike or null, respectively. In the above case, we say that M is a metric framed manifold and its associated structure will be denoted by (M, ϕ, g, U_a). Now, the question is whether there exists an arbitrary semi-Riemannian metric g for a framed M with above metric condition. Unfortunately, contrary to the case of Riemannian framed manifolds (cf.

Blair [8]), the answer to the above question is negative. For details on this see [20, pp. 213–215]. On the brighter side, in the same reference it has been proved that a metric framed manifold (M, ϕ, g, U_a) can carry a Lorentzian metric, which we assume. Following the terminology introduced by Duggal [18], we say that (M, ϕ, g, U_a) is a Lorentz framed structure and M a Lorentz framed spacetime if g is Lorentzian. Also, on a metric framed manifold M, its $2n$-dimensional distribution $(D, g/D, J_{|\phi})$ has indefinite Hermitian structure. Since J is real, as discussed before, the only possible signatures of g/D are are either $(0, 2m)$ or $(2p, 2q)$ with $p + q = m$. In particular, $(D, g/D, J)$ satisfying Hermitian condition cannot carry a Lorentz metric unless $\dim(D) = 2$.

We list the following results taken from [18].

(A) For a metric framed manifold (M, ϕ, g, U_a) with exactly one U_a tiemlike, the following are equivalent:

(1) M is Lorentz framed manifold.

(2) D and $\tilde{D} = D^{\perp}$ are spacelike and timelike, respectively, except when $\dim(D) = 2$, then D timelike and $D^{\perp}$ is possible.

(B) For a metric framed manifold (M, ϕ, g, U_a) with exactly one null U_a, the following are equivalent:

(1) M is Lorentz framed manifold with $\dim(\tilde{D}) \geq 2$.

(2) D and $\tilde{D} \neq D^{\perp}$ are both lightlike, i.e., they each contain exactly one null vector and no timelike vector.

Finally, we quote the following result as a characterization of CR-submanifolds in terms of framed structures:

Theorem 6.5 [35, p. 87] *In order for a submanifold M of a Kählerian manifold $\bar{M}$ to be a CR-submanifold, it is necessary and sufficient that M and the normal bundle of M, both, have a framed structure*

This result will also hold for a CR–Lorentzian or contact CR–Lorentzian M of an indefinite Kählerian or Sasakian manifold, respectively.

Following the terminology introduced by Blair [8], we say that a normal Lorentz framed manifold is a K-manifold if its 2-form $\bar{\Omega}$ is closed (i.e., $d\bar{\Omega} = 0$). Since $u^1 \wedge \cdots \wedge u^n \wedge \bar{\Omega}^m \neq 0$, a K-manifold is orientable. Furthermore, we say that a K-manifold is a C-manifold if each $du^a = 0$.

Physical applications: For this purpose- we need the following results on Riemannian framed manifolds which will also hold for a lorentz framed manifold(details on their proofs may be seen in [8]).

• On a K-manifold, the vector fields $U_1, \ldots, U_n$ are Killing.

• A $(2m + n)$-dimensional C-manifold M is locally decomposable manifold of the product $M = N^{2m} \times L^n$, where N^{2m} is a Kaehler manifold and L^n is an Abelian group manifold (i.e., n one-dimensional manifolds).

Observe that above two results also hold for a Lorentz framed manifold (proofs are common). Also note that for a Lorentz framed M, the Abelian group manifold L^n will have a Minkowski metric with $n \geq 2$. It follows from above two results that a C-manifold admits an n-parameter Abelian isometry group structure, generated by its

n Killing vector fields $U_1, \ldots, U_n$. Physically, such a symmetry has been extremely useful in several aspects of general relativity theory. For example, since the Einstein's field equations are a complicated set of nonlinear partial differential equations, most explicit solutions have been found by assuming one or more Killing vector fields. In particular, the existence of Killing symmetry has been in the study of null(lightlike) hypersurface which may be models of a Killing horizon.

Now we let (M, g) belong to a class of Einstein–Maxwell spacetimes for which the electromagnetic field F_{ab} is non-singular, which is a skew-symmetric (2-form) tensor of type (0, 2). Consider the following three complex functions, called Maxwell scalar fields (see details in [18]):

$$\phi_0 = 2F_{ab}\ell^a m^b, \quad \phi_1 = F_{ab}(\ell^a k^b + \bar{m}^a m^b), \quad \phi_2 = 2F_{ab}\bar{m}^a m^b,$$

Then, the general form of F_{ab} is given by

$$\begin{aligned} F_{ab} = &-2Re\phi_1 \ell_{[a}k_{b]} + 2iIm\phi_1 m_{[a}\bar{m}_{b]} + \phi_2 \ell_{[a}m_{b]} \\ &+ \bar{\phi}_2 \ell_{[a}\bar{m}_{b]} - \phi_0 k_{[a}\bar{m}_{b]} - \bar{\phi}_0 k_{[a}m_{b]}. \end{aligned}$$

For this class we know from [18] that ϕ_1 is the only surviving Maxwell scalar. We are interested in a simple F_{ab} and, therefore, ϕ_1 is either real or pure imaginary. For this subcase, it follows from the general expresion of F that its canonical form is given by

$$F_{ab} = -2\,Re(\phi_1)\,\ell_{[a}n_{b]} \quad \text{or} \quad 2\,i\,Im(\phi_1)\,m_{[a}\bar{m}_{b]}, \quad det(F_{ab}) = 0.$$

Consider a homogeneous spacetimes for which ϕ_1 is constant. Set $|{\phi_1}^2| = 1$ for both the real or pure imaginary cases. Now define a (1, 1) tensor field $f \equiv (f^a_b)$, on the tangent space $T_p(M)$, at each point $p \in M$, such that $f^a_b = g^{ac}\,F_{cb}$, *i.e.*, $F(X, Y) = g(X, fY)$, for any vector fields X, Y of M. It follows from the well known Cayley–Hamilton theorem, that f satisfies its own minimum characteristic polynomial equation: $f^3 \pm f = 0$, where the sign $\pm$ depends on the choice of $Im(\phi)$ or $Re(\phi)$. We choose

$$f^3 + f = 0, \qquad rank(f) = 2.$$

Thus M admits a Lorentz framed structure (g, f). It is important to mention that a homogeneous spacetime (M, g), with simple F, inherits a metric f-structure without imposing any geometric condition. Finally we quote the following result

Theorem 6.6 (Duggal [18]) *Let (M, g) be a homogeneous spacetime with a simple electromagnetic field F and a normal f-structure (g, f, N_f). Then, M admits a 2-parameter group of Killing vector fields*

As a final remark we say that, based on Theorem 6.5 and above Theorem 6.6, there is link between a class of homogeneous spacetimes with a non-singular electromagnetic field F and CR–Lorentzian submanifolds and the existence of a 2-parameter group of Killing vector fields.

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Chapter 7
Submanifold Theory in Holomorphic Statistical Manifolds

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7.1 Introduction

In 1980s, the history of statistical manifolds has started from investigations of geometric structures on sets of certain probability distributions to be applied to the theory of statistical inference as information geometry. Let us consider the set of all normal distributions, for example. This set can be identified as the upper half plane, because it is parametrized by the mean and variance of each element. What geometric structure is useful for statistics on this set? An answer is given as a pair of a Riemannian metric called the Fisher information metric and an affine connection called the Chentsov–Amari connection. Roughly speaking, we may consider that such a pair in this case gives rise to the Poincaré metric and the standard flat connection of this upper half plane. By generalizing important relations, we have reached the notion of a statistical manifold, that is, a manifold with a statistical structure (Definition 1). The point is that we equip a Riemannian manifold with an affine connection besides the Levi–Civita connection. In this case, we naturally get another affine connection called the *dual* connection, from which various dualistic geometric objects arise. Hence this establishes a new aspect of geometry. We remark that geometry of statistical manifolds should be reduced to one of the Riemannian manifolds if the attached connection coincides with the Levi–Civita connection.

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Beyond expectations, statistical manifolds are familiar to geometers because they have appeared with alternate names in various research fields in differential geometry, for example, in the submanifold theory and in Hessian geometry (see [13, 18] and references therein). For geometers, it is natural to try to build the submanifold theory and the complex manifold theory of statistical manifolds. In fact, we have several researches concerning on statistical submanifolds, for example, [2, 8, 9, 21], and on "complex" statistical manifolds, for example, [16, 19].

Let us treat here a holomorphic statistical manifold (Definition 5). By definition, it can be considered as a generalization of a special Kähler manifold (see [1]). We will define the holomorphic sectional curvature for holomorphic statistical structures (Definition 6). It is an interesting and important problem to classify the holomorphic statistical manifolds of constant holomorphic sectional curvature. Although it has not been sufficiently studied, we will proceed to their submanifold theory here. The goal of this article is to give the basics for statistical submanifolds in holomorphic statistical manifolds apart from information geometry.

The study on CR-submanifolds in Kähler manifolds has long history (see Bejancu [3]). We can find many theorems in the textbook [22], for example. In this article, we generalize such classical theorems to our setting. In this direction, Milijević obtained several results [14, 15]. Let us now glance one of our theorems (see Theorem 7.6 for a precise statement). We consider a Lagrangian submanifold in a holomorphic statistical manifold of constant holomorphic sectional curvature. If the shape operator and the *dual* shape operator commute, then the submanifold is of constant sectional curvature. If the attached affine connection of the ambient space is the Levi–Civita connection, then the dual shape operator coincides with the original one, and Theorem 7.6 is reduced to a well-known property for a Lagrangian submanifold in a complex space form. The readers will find that it can be proved just like the classical case, and that many theorems can be modified to our setting. This article is written for beginners at the CR-submanifold theory. If the readers are familiar with it, they can reconsider their favorite theorems in our statistical submanifold setting. By contrast we would like to remark here that it is important to proceed to the study on properties of statistical submanifolds which cannot be obtained by modifications of the classical setting.

7.2 Statistical Manifolds

Throughout this paper, M denotes a smooth manifold of dimension $n \geq 2$, and all the objects are assumed to be smooth. $\Gamma(E)$ denotes the set of sections of a vector bundle $E \to M$. For example, $\Gamma(TM^{(p,q)})$ means the set of all the tensor fields on M of type (p, q).

In this section, we review the intrinsic theory of statistical manifolds. Especially, we give a definition of the sectional curvature for a statistical manifold.

Definition 1 Let ∇ be an affine connection of M, and g a Riemannian metric on M.

(1) The affine connection ∇^* of M defined by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z), \quad X, Y, Z \in \Gamma(TM), \tag{7.1}$$

is called the *dual connection* of ∇ with respect to g.

(2) The triplet (M, ∇, g) is called a *statistical manifold* if the torsion tensor field of ∇ vanishes and $\nabla g \in \Gamma(TM^{(0,3)})$ is symmetric.

If (M, ∇, g) is a statistical manifold, so is (M, ∇^*, g).

Example 1 (1) Let (M, g) be a Riemannian manifold, and denote the Levi–Civita connection of g by ∇^g or $\widehat{\nabla}$. Then $(M, \widehat{\nabla}, g)$ is a statistical manifold. We have $(\widehat{\nabla})^* = \widehat{\nabla}$.

(2) Let $M \hookrightarrow \mathbb{R}^{n+1}$ be a locally convex hypersurface in the $(n+1)$-dimensional Euclidean space, and g, h the first and second fundamental forms, respectively. Then (M, ∇^g, h) is a statistical manifold. In fact, h is a Riemannian metric on M, and the Codazzi equation $(\nabla^g_X h)(Y, Z) = (\nabla^g_Y h)(X, Z)$ holds.

Example 2 The triple $((\mathbb{R}^+)^n, D, g_0)$ defined below is a statistical manifold.

$$\begin{aligned}
&(\mathbb{R}^+)^n := \{y = {}^t(y^1, \ldots, y^n) \in \mathbb{R}^n \mid y^1 > 0, \cdots, y^n > 0\},\\
&g_0 := \sum_{j=1}^{n} (dy^j)^2 = (\text{the Euclidean metric})|_{(\mathbb{R}^+)^n},\\
&D: \qquad D_{\partial_i} \partial_j = -\delta_{ij} (y^j)^{-1} \partial_j,
\end{aligned}$$

where $\partial_i := \partial / \partial y^i$.

Example 3 For $x \in \Omega := \{1, \ldots, n+1\} \subset \mathbb{Z}$ and

$$\eta \in \Delta^n := \left\{ \eta = {}^t(\eta^1, \ldots, \eta^n) \in \mathbb{R}^n \;\middle|\; \eta^i > 0, \; \sum_{l=1}^{n} \eta^l < 1 \right\},$$

we set

$$p(x, \eta) := \begin{cases} \eta^i, & x = i \in \{1, \ldots, n\} \subset \mathbb{Z}, \\ 1 - \displaystyle\sum_{l=1}^{n} \eta^l, & x = n+1, \end{cases}$$

which is a positive probability density function on Ω parametrized by η. Thus Δ^n can be considered as the parameter space of the family of all the positive probability densities on the finite set Ω.

The Fisher information metric g^F for $\Delta^n \ni \eta \mapsto p(\cdot, \eta)$ is defined as

$$g^F{}_\eta(\partial_i, \partial_j) = \sum_{x\in\Omega}\{\partial_i \log p(x,\eta)\}\{\partial_j \log p(x,\eta)\}p(x,\eta)$$
$$= (\eta^i)^{-1}\delta_{ij} + \left(1 - \sum_{l=1}^{n}\eta^l\right)^{-1},$$

where $\partial_i := \partial/\partial\eta^i$. The exponential connection $\nabla^{(e)}$ (the Chentsov–Amari connection) for $\Delta^n \ni \eta \mapsto p(\cdot,\eta)$ is defined by

$$g^F_\eta(\nabla^{(e)}_{\partial_i}\partial_j, \partial_k) = \sum_{x\in\Omega}\{\partial_i\partial_j \log p(x,\eta)\}\{\partial_k \log p(x,\eta)\}p(x,\eta)$$
$$= -(\eta^i)^{-2}\delta_{ij}\delta_{jk} - \left(1 - \sum_{l=1}^{n}\eta^l\right)^{-2}.$$

Then $(\Delta^n, \nabla^{(e)}, g^F)$ is a statistical manifold.

For an affine connection ∇, we set

$$R^\nabla(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for $X, Y, Z \in \Gamma(TM)$, and denote R^∇ by R, R^{∇^*} by R^* for short. We denote g by $\langle\ ,\ \rangle$ if there is no danger of confusion.

A statistical manifold is called a *Hessian* manifold if ∇ is flat, that is, the curvature tensor field R identically vanishes. We remark that the statistical manifolds given in Examples 2 and 3 are Hessian manifolds.

Lemma 1 *For a statistical manifold* (M, ∇, g)*, the following hold for* $X, Y, Z, W \in \Gamma(TM)$*:*

$$\langle R(W,Z)Y, X\rangle = -\langle R(Z,W)Y, X\rangle, \tag{7.2}$$
$$\langle R^*(W,Z)Y, X\rangle = -\langle R^*(Z,W)Y, X\rangle, \tag{7.3}$$
$$\langle R(Z,W)X, Y\rangle = -\langle R^*(Z,W)Y, X\rangle, \tag{7.4}$$
$$R(Z,W)Y + R(W,Y)Z + R(Y,Z)W = 0, \tag{7.5}$$
$$R^*(Z,W)Y + R^*(W,Y)Z + R^*(Y,Z)W = 0. \tag{7.6}$$

Proof The formulas (7.2) and (7.3) follow directly from the definition of the curvature tensor field, and (7.4) from (7.1). The first Bianchi identity implies the formulas (7.5) and (7.6) because ∇ and ∇^* are of torsion free. ■

Definition 2 For a statistical manifold (M, ∇, g), we define

$$S^{(\nabla,g)}(X,Y)Z = \frac{1}{2}\{R^\nabla(X,Y)Z + R^{\nabla^*}(X,Y)Z\}$$

for $X, Y, Z \in \Gamma(TM)$, and denote $S^{(\nabla,g)}$ by S for short. We temporarily call $S \in \Gamma(TM^{(1.3)})$ the *statistical curvature tensor field* of (M, ∇, g).

Then the following formulas hold:

$$\langle S(W, Z)Y, X\rangle = -\langle S(Z, W)Y, X\rangle, \tag{7.7}$$

$$\langle S(Z, W)X, Y\rangle = -\langle S(Z, W)Y, X\rangle, \tag{7.8}$$

$$S(Z, W)Y + S(W, Y)Z + S(Y, Z)W = 0, \tag{7.9}$$

$$\langle S(X, Y)W, Z\rangle = \langle S(Z, W)Y, X\rangle. \tag{7.10}$$

Proof Lemma 1 implies the formulas (7.7)–(7.9), from which (7.10) follows because the same argument for Riemannian connections works (see [11, Vol.1]). ■

Definition 3 Let (M, ∇, g) be a statistical manifold. For $x \in M$ and a two-dimensional subspace $\Pi = \mathrm{span}_{\mathbb{R}}\{v, w\}$ of T_xM,

$$\frac{\langle S_x(v, w)w, v\rangle}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2}$$

is called the *sectional curvature* of (M, ∇, g) for Π. A statistical manifold (M, ∇, g) is said to be *of constant sectional curvature* $k(\in \mathbb{R})$ if it is constant k for x and Π.

The proof of the well-definedness is similar to the Riemannian geometric case, because it is given in linear algebra and the conditions (7.7)–(7.10) are the same as the ones for Riemannian case.

Remark 1 (1) The sectional curvature of a statistical manifold (M, ∇, g) is constant k if and only if

$$S(X, Y)Z = k\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}$$

for $X, Y, Z \in \Gamma(TM)$.

(2) If (M, ∇, g) is of constant curvature k in Kurose's sense (see [12]), that is,

$$R(X, Y)Z = k\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}, \quad X, Y, Z \in \Gamma(TM),$$

then the sectional curvature is constant k. In fact, by (7.4) if it is of constant curvature k, so is the dual statistical manifold (M, ∇^*, g).

(3) If ∇ is the Levi–Civita connection of g, the definition of sectional curvature coincides with the standard one.

Definition 4 Let (M, ∇, g) be an n-dimensional statistical manifold and $S \in \Gamma(TM^{(1,3)})$ the statistical curvature tensor field. We define $L \in \Gamma(TM^{(0,2)})$ and $\rho \in C^\infty(M)$ by

$$L(Y, Z) = \operatorname{tr}\{X \mapsto S(X, Y)Z\} = \sum_{i=1}^{n} \langle S(e_i, Y)Z, e_i \rangle,$$

$$\rho = \operatorname{tr}_g L = \sum_{i=1}^{n} L(e_i, e_i)$$

for any $Y, Z \in \Gamma(TM)$, where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. Furthermore, we write

$$\|S\|^2 = \sum_{k,j,i,h=1}^{n} \langle S(e_k, e_j)e_i, e_h \rangle^2, \quad \|L\|^2 = \sum_{j,i=1}^{n} (L(e_j, e_i))^2.$$

Remark 2 For an n-dimensional statistical manifold (M, ∇, g) the following inequalities hold:

$$\|L\|^2 \geq \frac{1}{n}\rho^2, \quad \|S\|^2 \geq \frac{2}{n(n-1)}\rho^2.$$

In the first inequality, the equality holds if and only if $L(X, Y) = \frac{\rho}{n}\langle X, Y \rangle$ for any $X, Y \in \Gamma(TM)$. In the second inequality, the equality holds if and only if $S(X, Y)Z = \frac{\rho}{n(n-1)}\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\}$ for any $X, Y, Z \in \Gamma(TM)$.

In fact, the vector space $M_n(\mathbb{R})$ of all $n \times n$-matrices over $\mathbb{R}$ can be identified n^2-dimensional Euclidean space $(\mathbb{R}^{n^2}, \langle \cdot, \cdot \rangle)$. For $E = \left(\delta_{ji}\right),\ L = \left(L(e_j, e_i)\right) \in M_n(\mathbb{R})$ we have

$$\rho^2 = \langle E, L \rangle^2 \leq \|E\|^2 \|L\|^2 = n\|L\|^2.$$

Similarly, for $F = \left(\delta_{ji}\delta_{kh} - \delta_{ki}\delta_{jh}\right),\ S = \left(\langle S(e_k, e_j)e_i, e_h \rangle\right) \in M_{n^2}(\mathbb{R})$, we obtain

$$(2\rho)^2 = \langle F, S \rangle^2 \leq \|F\|^2 \|S\|^2 = 2n(n-1)\|S\|^2,$$

which completes the proof.

Remark that even if we know $S(X, Y)Z = \frac{\rho}{n(n-1)}\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\}$, we cannot conclude that it is of constant sectional curvature. In fact, we can construct counterexamples from Hessian manifolds of constant Hessian curvature by using (7.17).

7.3 Holomorphic Statistical Manifolds

We continue to study the intrinsic properties of statistical manifolds. In this section, we proceed to their complex geometry.

Definition 5 Let M be an almost complex manifold with almost complex structure $J \in \Gamma(TM^{(1,1)})$. A quadruplet (M, ∇, g, J) is called a *holomorphic statistical manifold* if (1) (∇, g) is a statistical structure on M, and (2) ω is a ∇-parallel 2-form on M, where ω is defined by $\omega(X, Y) = g(X, JY)$.

A holomorphic statistical manifold is considered as a Kähler manifold with a certain connection. In fact, the skew-symmetricity of ω means that (g, J) is an almost Hermitian structure, and the condition $\nabla\omega = 0$ implies that ω is closed since ∇ is of torsion free.

A holomorphic statistical manifold (M, ∇, g, J) is nothing but a *special Kähler manifold* if ∇ is flat. See [1] for example.

Lemma 2 *The following hold for a holomorphic statistical manifold* (M, ∇, g, J)*:*

$$\nabla^J = \nabla^*, \quad \text{where} \quad \nabla^J_X Y = J^{-1}\nabla_X(JY), \tag{7.11}$$

$$\nabla_X(JY) = J\nabla^*_X Y, \tag{7.12}$$

$$R(X, Y)JZ = JR^*(X, Y)Z. \tag{7.13}$$

Proof The formula (7.11) is given in [8], for example. The properties (7.12) and (7.13) follow directly from (7.11). ∎

Lemma 3 *Let* (M, ∇, g, J) *be a holomorphic statistical manifold. The following holds for* $X, Y, Z, W \in \Gamma(TM)$*:*

$$\langle S(Z, W)JY, JX\rangle = \langle S(JZ, JW)Y, X\rangle = \langle S(Z, W)Y, X\rangle. \tag{7.14}$$

Proof The formula (7.13) implies (7.14) by (7.10) immediately. ∎

Definition 6 A holomorphic statistical manifold (M, ∇, g, J) is said to be *of constant holomorphic sectional curvature* $k(\in \mathbb{R})$ if the sectional curvature of (∇, g) is constant k for any $x \in M$ and for any J-invariant 2-dimensional subspace Π of T_xM.

Then, the formula

$$S(X, Y)Z = \frac{k}{4}\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\} \tag{7.15}$$

holds for $X, Y, Z \in \Gamma(TM)$.

If ∇ is the Levi–Civita connection of g, the notion of holomorphic sectional curvature coincides with the standard one in Kähler geometry.

Lemma 4 *Let* (M, ∇, g, J) *be a holomorphic statistical manifold, and* φ *a function on* M*. Define an affine connection* $\nabla^\varphi = \nabla + \varphi K$*, where* $K \in \Gamma(TM^{(1,2)})$ *is given as* $\nabla - \widehat{\nabla}$*.*

(1) $(M, \nabla^{\varphi}, g, J)$ *is also a holomorphic statistical manifold.*
(2) *The following hold:*

$$R^{\nabla^{\varphi}}(X,Y)Z = R(X,Y)Z + \varphi\{(\nabla_X K)(Y,Z) - (\nabla_Y K)(X,Z)\} \\ +\varphi^2[K_X, K_Y]Z + (X\varphi)K_Y Z - (Y\varphi)K_X Z, \quad (7.16)$$

$$S^{(\nabla^{\varphi},g)}(X,Y)Z = S(X,Y)Z + (\varphi^2 + 2\varphi)[K_X, K_Y]Z \\ = (\varphi+1)^2 S(X,Y)Z - \varphi(\varphi+2)\widehat{R}(X,Y)Z, \quad (7.17)$$

where $\widehat{R}$ *is the curvature tensor field of the Lev–Civita connection* $\widehat{\nabla}$.

This is obtained by direct calculation. Remark that the formula $\widehat{R}(X,Y)Z = S(X,Y)Z - [K_X, K_Y]Z$ holds for $X, Y, Z \in \Gamma(TM)$. We also remark that there is a misprint in the formula corresponding to (7.16) in [8, Remark 2.7].

Example 4 For $k \in \mathbb{R}$, let I be an interval in $\{t > 0 \mid 1 - 2kt^3 > 0\}$, and set a domain $\Omega = I \times \mathbb{R}$ in the (u^1, u^2)-plane $\mathbb{R}^2$. J denotes the standard complex structure on Ω, determined by $J\partial_1 = \partial_2$, where $\partial_j = \dfrac{\partial}{\partial u^j}$. Define a Riemannian metric g and an affine connection $\widetilde{\nabla}$ on Ω by

$$g = u^1\{(du^1)^2 + (du^2)^2\}, \\ \widetilde{\nabla}_{\partial_1}\partial_1 = -\frac{1}{2}\varphi(u^1)^{-1}\partial_1, \quad (7.18)$$

$$\widetilde{\nabla}_{\partial_1}\partial_2 = \widetilde{\nabla}_{\partial_2}\partial_1 = (u^1)^{-1}\left(1 + \frac{1}{2}\varphi(u^1)\right)\partial_2, \\ \widetilde{\nabla}_{\partial_2}\partial_2 = \frac{1}{2}\varphi(u^1)^{-1}\partial_2, \quad (7.19)$$

where

$$\varphi(t) = -1 \pm \sqrt{1 - 2kt^3}. \quad (7.20)$$

Then $(\Omega, \widetilde{\nabla}, g, J)$ is a holomorphic statistical manifold of constant holomorphic sectional curvature k.

We obtain the above example as follows: As the first step, we construct a special Kähler domain $(\overline{\Omega}(= (0,\infty) \times \mathbb{R}), \nabla, g)$, using Cortés and his collaborators' idea (see [6], for example). Take a complex coordinate $z = u^1 + \sqrt{-1}u^2$, and a holomorphic function $F(z) = \frac{\sqrt{-1}}{6}z^3$. Then a Riemannian metric $g = \mathrm{Im}F_{zz}dzd\bar{z}$ and an affine connection ∇ such that $x = u^1$ and $y = \mathrm{Re}F_z$ become affine coordinates give a special Kähler structure. Namely, we calculate g as in (7.18) and ∇ as

$$\nabla_{\partial_1}\partial_1 = \nabla_{\partial_2}\partial_2 = 0, \\ \nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = (u^1)^{-1}\partial_2. \quad (7.21)$$

Remark that it is a holomorphic statistical manifold with $S = 0$. Due to Lemma 4, $(\overline{\Omega}, \nabla^{\varphi}, g)$ for a function φ is still a holomorphic statistical manifold. As the second step, for given $K = \nabla - \widehat{\nabla}$ and k, find φ such that (7.15) and (7.17) hold. Thus we obtain $\widetilde{\nabla} = \nabla^{\varphi}$ in (7.19) and (7.20).

According to the sign in (7.20), we have two holomorphic statistical structures, which are corresponding to the pair of one and its dual. Since $(\Omega, \widetilde{\nabla}, g)$ is of complex one dimension, it is of constant sectional curvature k in the sense of Definition 3. However, we remark that it is not of constant curvature in Kurose's sense.

7.4 Statistical Submanifolds

In this section, we study the extrinsic theory of statistical manifolds. We give fundamental equations for statistical submanifolds.

Let $(\widetilde{M}, \widetilde{\nabla}, \tilde{g})$ be a statistical manifold. Let M be a submanifold of $\widetilde{M}$ and g the induced metric on M. We define an affine connection ∇ on M by

$$\nabla_X Y = (\widetilde{\nabla}_X Y)^{\top},$$

where $(\ )^{\top}$ denotes the orthogonal projection of $(\ )$ on the tangent space of M with respect to $\tilde{g}$, that is, $\langle \nabla_X Y, Z\rangle = \langle \widetilde{\nabla}_X Y, Z\rangle$ for $X, Y, Z \in \Gamma(TM)$. Then (M, ∇, g) becomes a statistical manifold, and this (∇, g) is called the *induced statistical structure* on M. We say that (M, ∇, g) is a *statistical submanifold* in $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ if (∇, g) is the induced statistical structure on M.

In this paper, we basically follow the notation of [22]. Let $T^{\perp}M$ be the normal bundle of M in $\widetilde{M}$, and $(\ )^{\perp}$ the orthogonal projection of $(\ )$ on the normal space of M with respect to $\widetilde{g}$. We define the *second fundamental form* of M for $\widetilde{\nabla}$ by

$$B(X, Y) = (\widetilde{\nabla}_X Y)^{\perp}$$

for $X, Y \in \Gamma(TM)$. Since $\widetilde{\nabla}$ is of torsion free, $B \in \Gamma(T^{\perp}M \otimes TM^{(0,2)})$ is symmetric. We define the *shape operator* and the *normal connection* for $\widetilde{\nabla}$, respectively, by

$$A_{\xi}X = -(\widetilde{\nabla}_X \xi)^{\top}, \quad D_X \xi = (\widetilde{\nabla}_X \xi)^{\perp}$$

for $\xi \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$. As in the Riemannian submanifold theory, $A \in \Gamma((T^{\perp}M)^* \otimes TM^{(1,1)})$ and D is a connection of a vector bundle $T^{\perp}M$. In the same fashion, we define these objects for the dual connection $\widetilde{\nabla}^*$. Summing up, we have

Proposition 1 *Let (M, ∇, g) be a statistical submanifold in $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$.*

(1) *The Gauss and Weingarten formulas are written by*

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \widetilde{\nabla}_X \xi = -A_{\xi}X + D_X \xi, \tag{7.22}$$

$$\widetilde{\nabla}^*_X Y = \nabla^*_X Y + B^*(X, Y), \quad \widetilde{\nabla}^*_X \xi = -A^*_{\xi}X + D^*_X \xi \tag{7.23}$$

for $X, Y \in \Gamma(TM)$ *and* $\xi \in \Gamma(T^{\perp}M)$.

(2) *The following hold for* $X, Y, Z \in \Gamma(TM)$ *and* $\xi, \eta \in \Gamma(T^{\perp}M)$*:*

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla^*_X Z\rangle, \tag{7.24}$$

$$\langle B(X, Y), \xi\rangle = \langle A^*_{\xi} X, Y\rangle, \quad \langle B^*(X, Y), \xi\rangle = \langle A_{\xi} X, Y\rangle, \tag{7.25}$$

$$X\langle \xi, \eta\rangle = \langle D_X \xi, \eta\rangle + \langle \xi, D^*_X \eta\rangle. \tag{7.26}$$

Let us prove (7.25) in order to check the difference from the Riemannian submanifold theory; $0 = X\langle Y, \xi\rangle = \langle \widetilde{\nabla}_X Y, \xi\rangle + \langle Y, \widetilde{\nabla}^*_X \xi\rangle = \langle B(X, Y), \xi\rangle - \langle Y, A^*_{\xi} X\rangle$.

Definition 7 Let (M, ∇, g) be a statistical submanifold of dimension n in $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$. We define the *mean curvature vector field* of M for $\widetilde{\nabla}$ by

$$H = \frac{1}{n}\mathrm{tr}_g B,$$

where tr_g is the trace with respect to g. M is said to be *totally geodesic with respect to* $\widetilde{\nabla}$ if the second fundamental form B of M for $\widetilde{\nabla}$ vanishes identically. M is said to be *totally umbilical with respect to* $\widetilde{\nabla}$ if $B = H \otimes g$ holds.

Proposition 2 *Let* (M, ∇, g) *be a statistical submanifold in* $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$*. The following equations of Gauss, Codazzi, and Ricci hold for* $X, Y, Z \in \Gamma(TM)$ *and* $\xi \in \Gamma(T^{\perp}M)$*:*

$$(\widetilde{R}(X, Y)Z)^{\top} = R(X, Y)Z - A_{B(Y,Z)}X + A_{B(X,Z)}Y, \tag{7.27}$$

$$(\widetilde{R}(X, Y)Z)^{\perp} = (\overline{\nabla}_X B)(Y, Z) - (\overline{\nabla}_Y B)(X, Z), \tag{7.28}$$

$$(\widetilde{R}(X, Y)\xi)^{\top} = (\overline{\nabla}_Y A)_{\xi} X - (\overline{\nabla}_X A)_{\xi} Y, \tag{7.29}$$

$$(\widetilde{R}(X, Y)\xi)^{\perp} = R^{\perp}(X, Y)\xi - B(X, A_{\xi} Y) + B(Y, A_{\xi} X), \tag{7.30}$$

where

$$(\overline{\nabla}_X B)(Y, Z) := D_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z), \tag{7.31}$$

$$(\overline{\nabla}_X A)_{\xi} Y := \nabla_X (A_{\xi} Y) - A_{D_X \xi} Y - A_{\xi} \nabla_X Y,$$

and $R^{\perp}$ *is the curvature tensor field with respect to the normal connection* D *for* $\widetilde{\nabla}$.

Remark 3 In the same setting in Proposition 2, we have equations for the dual connection $\widetilde{\nabla}^*$, using the Gauss and Weingarten formulas (7.23). For example, Eq. (7.27) of Gauss for $\widetilde{\nabla}^*$ is given as

$$(\widetilde{R}^*(X, Y)Z)^{\top} = R^*(X, Y)Z - A^*_{B^*(Y,Z)}X + A^*_{B^*(X,Z)}Y.$$

We refer to this equation as (7.27)* for short if there is no danger of confusion.

Proposition 3 *For a statistical submanifold* (M, ∇, g) *in* $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$*, we have*

$$2(\widetilde{S}(X,Y)Z)^{\top} = 2S(X,Y)Z - A_{B(Y,Z)}X + A_{B(X,Z)}Y - A^{*}_{B^{*}(Y,Z)}X + A^{*}_{B^{*}(X,Z)}Y, \quad (7.32)$$

$$2(\widetilde{S}(X,Y)Z)^{\perp} = (\overline{\nabla}_X B)(Y,Z) - (\overline{\nabla}_Y B)(X,Z) + (\overline{\nabla}^{*}_X B^{*})(Y,Z) - (\overline{\nabla}^{*}_Y B^{*})(X,Z), \quad (7.33)$$

$$2(\widetilde{S}(X,Y)\xi)^{\top} = (\overline{\nabla}_Y A)_{\xi}X - (\overline{\nabla}_X A)_{\xi}Y + (\overline{\nabla}^{*}_Y A^{*})_{\xi}X - (\overline{\nabla}^{*}_X A^{*})_{\xi}Y, \quad (7.34)$$

$$2(\widetilde{S}(X,Y)\xi)^{\perp} = 2S^{\perp}(X,Y)\xi - B(X, A_{\xi}Y) + B(Y, A_{\xi}X) - B^{*}(X, A^{*}_{\xi}Y) + B^{*}(Y, A^{*}_{\xi}X) \quad (7.35)$$

for $X, Y, Z \in \Gamma(TM)$ *and* $\xi \in \Gamma(T^{\perp}M)$*, where*

$$S^{\perp}(X,Y)\xi = \frac{1}{2}\{R^{\perp}(X,Y)\xi + R^{\perp *}(X,Y)\xi\}$$

and $R^{\perp *}$ *is the curvature tensor field of* D^{*}.

Example 5 Let $((\mathbb{R}^{+})^{n+1}, D, g_0)$ and $(\Delta^n, \nabla^{(e)}, g^F)$ be statistical manifolds in Examples 2 and 3. Define $\iota : \Delta^n \to (\mathbb{R}^{+})^{n+1}$ by

$$\Delta^n \ni \eta \mapsto \begin{bmatrix} 2\sqrt{p(1,\eta)} \\ \vdots \\ 2\sqrt{p(n,\eta)} \\ 2\sqrt{p(n+1,\eta)} \end{bmatrix} \in (\mathbb{R}^{+})^{n+1}.$$

Then,
(1) $\iota(\Delta^n) = S^n(2) \cap (\mathbb{R}^{+})^{n+1} = \{y \in \mathbb{R}^{n+1} \mid y^{\alpha} > 0, \sum_{\alpha=1}^{n+1}(y^{\alpha})^2 = 4\}$.
(2) The induced statistical structure by ι from $((\mathbb{R}^{+})^{n+1}, D, g_0)$ coincides with $(\nabla^{(e)}, g^F)$. In fact, the following hold:

$$\iota^{*}g_0 = g^F, \quad \begin{cases} D_X Y = \nabla^{(e)}_X Y - g^F(X,Y)\xi, \\ D_X \xi = 0, \end{cases} \quad \xi = \frac{1}{2}\iota.$$

7.5 CR-Submanifolds in Holomorphic Statistical Manifolds

In this section, we would like to build the statistical submanifold theory in holomorphic statistical manifolds. In the first place, we briefly review the submanifold theory in Kähler manifolds for later use following [22].

Let $(\widetilde{M}, \widetilde{g}, J)$ be a Kähler manifold, and M a submanifold in $\widetilde{M}$. Define $P \in \Gamma(TM^{(1,1)})$, $F \in \Gamma(T^{\perp}M \otimes TM^{(0,1)})$, $t \in \Gamma(TM \otimes (T^{\perp}M)^{*})$, and $f \in \Gamma(\mathrm{End}(T^{\perp}M))$ by

$$PX = (JX)^{\top}, \; FX = (JX)^{\perp}, \quad X \in \Gamma(TM),$$
$$t\xi = (J\xi)^{\top}, \; f\xi = (J\xi)^{\perp}, \quad \xi \in \Gamma(T^{\perp}M), \quad \text{i.e.,}$$
$$JX = PX + FX, \quad J\xi = t\xi + f\xi.$$

It is easy to have

$$\langle PX, Y\rangle = -\langle X, PY\rangle, \quad \langle f\xi, \eta\rangle = -\langle \xi, f\eta\rangle,$$
$$\langle FX, \xi\rangle = -\langle X, t\xi\rangle,$$
$$P^2 = -I - tF, \quad FP + fF = 0, \tag{7.36}$$
$$Pt + tf = 0, \quad f^2 = -I - Ft \tag{7.37}$$

for $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^{\perp}M)$.

A Riemannian submanifold M is called a *CR-submanifold* in $(\widetilde{M}, \widetilde{g}, J)$ if there exists a differentiable distribution $\mathfrak{D} : M \ni x \longmapsto \mathfrak{D}_x \subset T_xM$ on M satisfying the following two conditions:

(i) $\mathfrak{D}$ is holomorphic, i.e., $J\mathfrak{D}_x = \mathfrak{D}_x \subset T_xM$ for each $x \in M$,

(ii) the orthogonal complementary distribution $\mathfrak{D}^{\perp}$ is totally real, i.e., $J\mathfrak{D}^{\perp}_x \subset T^{\perp}_xM$ for each $x \in M$.

In this case, $\mathfrak{D}$ is called the *holomorphic distribution* and $\mathfrak{D}^{\perp}$ the *totally real distribution* of CR-submanifold M in $\widetilde{M}$. Let N be a subbundle of $T^{\perp}M$ defined as $N_x = \{\xi \in T^{\perp}_xM \mid \xi \perp J\mathfrak{D}^{\perp}_x\}$ for each $x \in M$. Accordingly, we have decompositions as vector bundles as follows:

$$T\widetilde{M} = TM \oplus T^{\perp}M = (\mathfrak{D} \oplus \mathfrak{D}^{\perp}) \oplus (J\mathfrak{D}^{\perp} \oplus N).$$

We then get

$$\begin{aligned} FP = 0, \quad fF = 0, \quad tf = 0, \quad Pt = 0, \\ P^3 = -P, \quad f^3 = -f, \end{aligned} \tag{7.38}$$

and that CR-submanifolds are characterized by the condition $FP = 0$ (see [22, p. 87]).

The class of CR-submanifolds contains various well-known classes of submanifolds. In fact, if $\mathfrak{D} = TM$, M is a *holomorphic* submanifold ($F = 0$ and $t = 0$), and if $\mathfrak{D}^{\perp} = TM$, M is a *totally real* submanifold ($P = 0$). If $J\mathfrak{D}^{\perp} = T^{\perp}M$ and $\mathfrak{D} \neq 0$, then M is called a *generic* submanifold ($f = 0$). If $\mathfrak{D}^{\perp} = TM$ and $J\mathfrak{D}^{\perp} = T^{\perp}M$, then M is called a *Lagrangian* submanifold ($P = 0$ and $f = 0$). If $\mathfrak{D} \neq 0$ and $\mathfrak{D}^{\perp} \neq 0$, then M is said to be *proper*.

We remark that the totally real distribution $\mathfrak{D}^{\perp}$ of a proper CR-submanifold in a Kähler manifold is completely integrable. In fact, we calculate $0 = d\omega(X, V, W) = -g([V, W], JX)$ for $X \in \mathfrak{D}$ and $V, W \in \mathfrak{D}^{\perp}$, where $\omega = g(\cdot, J\cdot)$. For short we denoted $X \in \Gamma(TM)$ such that $X_x \in \mathfrak{D}_x$ for each $x \in M$ by $X \in \mathfrak{D}$.

Now let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ be a holomorphic statistical manifold of dimension $2m \geq 4$. We say that (M, ∇, g) is a *CR-statistical submanifold* in $\widetilde{M}$ if (M, ∇, g) a statistical submanifold in $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ and (M, g) is a CR-submanifold in $(\widetilde{M}, \widetilde{g}, J)$. We call it just a CR-submanifold for short if there is no danger of confusion.

Lemma 5 *Let (M, ∇, g) be a statistical submanifold in $\widetilde{M}$. We have*

$$\nabla_X(PY) - A_{FY}X = P\nabla^*_X Y + tB^*(X, Y), \tag{7.39}$$

$$B(X, PY) + D_X(FY) = fB^*(X, Y) + F\nabla^*_X Y, \tag{7.40}$$

$$\nabla_X(t\xi) - A_{f\xi}X = -PA^*_\xi X + tD^*_X\xi, \tag{7.41}$$

$$B(X, t\xi) + D_X(f\xi) = -FA^*_\xi X + fD^*_X\xi \tag{7.42}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$.

Proof For $X, Y \in \Gamma(TM)$, we have

$$\begin{aligned} \widetilde{\nabla}_X(JY) &= \widetilde{\nabla}_X(PY) + \widetilde{\nabla}_X(FY) \\ &= \nabla_X(PY) + B(X, PY) - A_{FY}X + D_X(FY), \end{aligned} \tag{7.43}$$

$$\begin{aligned} J\widetilde{\nabla}^*_X Y &= J(\nabla^*_X Y + B^*(X, Y)) \\ &= P\nabla^*_X Y + F\nabla^*_X Y + tB^*(X, Y) + fB^*(X, Y). \end{aligned} \tag{7.44}$$

Comparing the tangent components of (7.43) and (7.44), we have (7.39). Comparing the normal components of (7.43) and (7.44), we have (7.40). In the same way, calculate $\widetilde{\nabla}_X(J\xi) = J\widetilde{\nabla}^*_X\xi$ to get (7.41) and (7.42). ∎

Remark 4 Let (M, ∇, g) be a holomorphic statistical submanifold in $\widetilde{M}$. From Lemma 5 we have

$$\nabla_X(JY) = J\nabla^*_X Y, \tag{7.45}$$

$$B(X, JY) = JB^*(X, Y), \tag{7.46}$$

$$A_{J\xi}X = JA^*_\xi X,$$

$$D_X(J\xi) = JD^*_X\xi$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$. In particular, (7.45) implies that (M, ∇, g, J) is a holomorphic statistical manifold, and (7.46) implies that

$$B(X, JY) = B(JX, Y), \quad B^*(X, JY) = B^*(JX, Y). \tag{7.47}$$

Accordingly, the mean curvature vector fields for $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ vanish.

Lemma 6 *Let (M, ∇, g) be a CR-submanifold in $\widetilde{M}$. We have*

$$A_{JV}W = A_{JW}V \quad \textit{and} \quad A^*_{JV}W = A^*_{JW}V \quad \textit{for} \quad V, W \in \mathfrak{D}^\perp.$$

Proof For $V, W \in \mathfrak{D}^\perp$, (7.39) implies

$$-A_{JW}V = P\nabla^*_V W + tB^*(V, W),$$

from which

$$A_{JV}W - A_{JW}V = P(\nabla^*_V W - \nabla^*_W V) = P[V, W] = 0,$$

since ∇^* is of torsion free and $\mathfrak{D}^\perp$ is completely integrable. ■

Example 6 Let us consider the following holomorphic statistical manifold, which is constructed as the product of the special Kähler domains given in (7.21):

$$\begin{aligned}
&\widetilde{M} = \{{}^t(u^1, \ldots, u^{2n}) \in \mathbb{R}^{2n} \mid u^1 > 0, \cdots, u^n > 0\},\\
&\widetilde{g} = \sum_{i=1}^{n} u^i\{(du^i)^2 + (du^{n+i})^2\},\\
&\widetilde{\nabla}_{\partial_i}\partial_{n+i} = \widetilde{\nabla}_{\partial_{n+i}}\partial_i = (u^i)^{-1}\partial_{n+i}, \quad \text{and otherwise,} \quad \widetilde{\nabla}_{\partial_\alpha}\partial_\beta = 0,\\
&J\partial_i = \partial_{n+i}, \quad J\partial_{n+i} = -\partial_i,
\end{aligned}$$

where the indices α, β run from 1 to $2n$, the index i from 1 to n, and $\partial_\alpha = \partial/\partial u^\alpha$. We can check that $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ is a special Kähler manifold. For positive constants $r_1, \ldots, r_n$, we set a statistical submanifold by

$$\iota : M \ni {}^t(x^1, \ldots, x^n) \mapsto {}^t(r_1 \cos x^1, \ldots, r_n \cos x^n, r_1 \sin x^1, \ldots, r_n \sin x^n) \in \widetilde{M},$$

where $M = (-\frac{\pi}{2}, \frac{\pi}{2})^n$. Then M is a Lagrangian submanifold in $\widetilde{M}$ and the statistical structure induced by ι is given as

$$\begin{aligned}
g &= \sum_{i=1}^{n} r_i^3 \cos x^i (dx^i)^2,\\
\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} &= -\delta_{ij} \sin(2x^j) \frac{\partial}{\partial x^j}.
\end{aligned}$$

We remark that ∇ is also flat. A basis of the normal space is given as $\{\xi_j = J\iota_* \dfrac{\partial}{\partial x^j} = -r_j(\cos x^j \partial_j + \sin x^j \partial_{n+j}) \mid j = 1, \ldots, n\}$, and the statistical shape operators are calculated as

$$\begin{aligned}
A_{\xi_j} \frac{\partial}{\partial x^i} &= 2\delta_{ij} \cos^2 x^i \frac{\partial}{\partial x^i},\\
A^*_{\xi_j} \frac{\partial}{\partial x^i} &= \delta_{ij}(1 + 2\sin^2 x^i) \frac{\partial}{\partial x^i}.
\end{aligned}$$

Theorem 7.1 *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ be a holomorphic statistical manifold and (M, ∇, g) a generic submanifold in $\widetilde{M}$ of codimension grater than one. If M is totally umbilical with respect to $\widetilde{\nabla}$, then M is totally geodesic with respect to $\widetilde{\nabla}$.*

This theorem is a statistical submanifold version of [5, Theorem 7.2]. Since $T^{\perp}M = J\mathfrak{D}^{\perp}$, the following lemma implies the theorem.

Lemma 7 *Let (M, ∇, g) be a CR-submanifold in $\widetilde{M}$. If M is totally umbilical with respect to $\widetilde{\nabla}$ and $\dim \mathfrak{D}^{\perp} \geq 2$, then the mean curvature vector field H for $\widetilde{\nabla}$ is perpendicular to $J\mathfrak{D}^{\perp}$.*

Proof For any $V \in \mathfrak{D}^{\perp}$, there exists a unit vector $W \in \mathfrak{D}^{\perp}$ which is perpendicular to V. Therefore, using Lemma 6, we obtain

$$\begin{aligned}\langle H, JV\rangle &= \langle B(W, W), JV\rangle = \langle A^{*}_{JV} W, W\rangle = \langle A^{*}_{JW} V, W\rangle = \langle B(V, W), JW\rangle \\ &= \langle V, W\rangle\langle H, JW\rangle = 0.\end{aligned}$$

■

Corollary 1 *Let $\widetilde{M}(\widetilde{c})$ be a holomorphic statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ of constant holomorphic sectional curvature $\widetilde{c}$ and (M, ∇, g) a generic submanifold in $\widetilde{M}(\widetilde{c})$. If M is totally umbilical with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^{*}$, then $\widetilde{c} = 0$.*

Proof In the case that M is a hypersurface, the proof will be given in Theorem 7.4. When M is not a hypersurface, Theorem 7.1 implies $B = B^{*} = 0$. Using Eq. (7.34) of Codazzi, we obtain

$$0 = (\widetilde{S}(X, Y)JV)^{\top} = \frac{\widetilde{c}}{2}\langle JX, Y\rangle V$$

for any $X, Y \in \mathfrak{D}$ and $V \in \mathfrak{D}^{\perp}$. ■

Hereafter, we will use the symbol $\widetilde{M}(\widetilde{c})$ for a $2m(\geq 4)$-dimensional holomorphic statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ of constant holomorphic sectional curvature $\widetilde{c}$ for short as in the previous corollary. At present we have little information about such spaces. It is very important to construct their standard models and to get a kind of uniformization for such spaces.

Proposition 4 *Let (M, ∇, g) be a statistical submanifold in $\widetilde{M}(\widetilde{c})$. Suppose that $\widetilde{c} \neq 0$. Then M is a proper CR-submanifold in $\widetilde{M}(\widetilde{c})$ if and only if the maximal holomorphic subspace $\mathfrak{D}_x = T_xM \cap JT_xM$, $x \in M$, defines a nontrivial differentiable distribution $\mathfrak{D}$ on M such that*

$$\langle \widetilde{S}(X, Y)V, W\rangle = 0 \quad \textit{for} \quad X, Y \in \mathfrak{D}, \;\; V, W \in \mathfrak{D}^{\perp},$$

where $\mathfrak{D}^{\perp}$ denotes the orthogonal complementary distribution of $\mathfrak{D}$ in M.

Proof From (7.15) we have

$$\langle \widetilde{S}(X, Y)V, W\rangle = \frac{\widetilde{c}}{2}\langle X, JY\rangle\langle JV, W\rangle$$

for $X, Y \in \mathfrak{D},\ V, W \in \mathfrak{D}^\perp$. If M is a CR-submanifold, $\langle JV, W\rangle = 0$. Conversely, suppose that $\langle X, JY\rangle\langle JV, W\rangle = 0$ for $X, Y \in \mathfrak{D},\ V, W \in \mathfrak{D}^\perp$. By setting $Y = JX (\neq 0)$, we can get $\langle JV, Z\rangle = 0$ for $Z \in \Gamma(TM)$. ∎

The following is a statistical submanifold version of [5, Theorem 7.3].

Theorem 7.2 *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ be a holomorphic statistical manifold and (M, ∇, g) a proper CR-submanifold in $\widetilde{M}$. If M is totally umbilical with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, then the sectional curvature of $\widetilde{M}$ for a CR-section of M vanishes, where a plane section $X \wedge V$ with $X \in \mathfrak{D}$ and $V \in \mathfrak{D}^\perp$ is called a CR-section of M.*

Proof Since M is totally umbilical with respect to $\widetilde{\nabla}$, it follows that

$$(\bar{\nabla}_X B)(Y, Z) = (\nabla_X g)(Y, Z)H + \langle Y, Z\rangle D_X H, \quad X, Y, Z \in \Gamma(TM).$$

Using Eq. (7.28) of Codazzi, we have

$$\begin{aligned}(\widetilde{R}(X, Y)Z)^\perp &= \{(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z)\}H + \langle Y, Z\rangle D_X H - \langle X, Z\rangle D_Y H \\ &= \langle Y, Z\rangle D_X H - \langle X, Z\rangle D_Y H\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$, from which we see

$$\langle \widetilde{R}(X, V)JX, JV\rangle = 0$$

for any $X \in \mathfrak{D}$ and $V \in \mathfrak{D}^\perp$. Similarly, we have

$$\langle \widetilde{R}^*(X, V)JX, JV\rangle = 0$$

and therefore

$$\begin{aligned}\langle \widetilde{S}(X, V)V, X\rangle &= -\langle \widetilde{S}(X, V)X, V\rangle = -\langle J\widetilde{S}(X, V)X, JV\rangle \\ &= -\langle \widetilde{S}(X, V)JX, JV\rangle = 0\end{aligned}$$

for any $X \in \mathfrak{D}$ and $V \in \mathfrak{D}^\perp$. ∎

Definition 8 Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ be a holomorphic statistical manifold and (M, ∇, g) a CR-submanifold in $\widetilde{M}$. M is said to be *mixed totally geodesic with respect to* $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$) if $B(X, V) = 0$ (resp. $B^*(X, V) = 0$) for $X \in \mathfrak{D}$ and $V \in \mathfrak{D}^\perp$. M is said to be $\mathfrak{D}$*-totally geodesic with respect to* $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$) if $B(X, Y) = 0$ (resp. $B^*(X, Y) = 0$) for all $X, Y \in \mathfrak{D}$.

Proposition 5 *Let (M, ∇, g) be a CR-submanifold in $\widetilde{M}$.*

(1) *If M is mixed totally geodesic with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$), then each leaf of $\mathfrak{D}^\perp$ is totally geodesic in M with respect to ∇ (resp. ∇^*).*

(2) *Suppose that (M, ∇, g) is a generic submanifold in $\widetilde{M}$. Then M is mixed totally geodesic with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$) if and only if each leaf of $\mathfrak{D}^\perp$ is totally geodesic in M with respect to ∇ (resp. ∇^*).*

(3) *If M is $\mathfrak{D}$-totally geodesic with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$), then $\mathfrak{D}$ is completely integrable and each leaf of $\mathfrak{D}$ is totally geodesic in M with respect to ∇^* (resp. ∇).*

These are direct conclusions from the following lemma:

Lemma 8 *Let (M, ∇, g) be a proper CR-submanifold in $\widetilde{M}$.*

(1) *An integral manifold L of $\mathfrak{D}^\perp$ is totally geodesic in M with respect to ∇ (resp. ∇^*) if and only if*

$$B(X, V) \in \Gamma(N) \quad (\text{resp. } B^*(X, V) \in \Gamma(N)) \quad \text{for } X \in \mathfrak{D},\ V \in \mathfrak{D}^\perp.$$

(2) *$\mathfrak{D}$ is completely integrable and each leaf of $\mathfrak{D}$ is totally geodesic in M with respect to ∇^* (resp. ∇) if*

$$B(X, Y) \in \Gamma(N) \quad (\text{resp. } B^*(X, Y) \in \Gamma(N)) \quad \text{for } X, Y \in \mathfrak{D}.$$

Proof For any $X, Y \in \mathfrak{D}$ and $V, W \in \mathfrak{D}^\perp$, we obtain

$$\begin{aligned} \langle B(X, V), JW \rangle &= \langle A^*_{JW} V, X \rangle = -\langle \widetilde{\nabla}^*_V JW, X \rangle \\ &= -\langle J\widetilde{\nabla}_V W, X \rangle = \langle \widetilde{\nabla}_V W, JX \rangle \\ &= \langle \nabla_V W, JX \rangle, \end{aligned} \tag{7.48}$$

$$\begin{aligned} \langle B(X, JY), JW \rangle &= \langle \widetilde{\nabla}_X JY, JW \rangle - \langle B^*(X, Y), W \rangle \\ &= \langle J(\widetilde{\nabla}^*_X Y - B^*(X, Y)), JW \rangle = \langle J\nabla^*_X Y, JW \rangle \\ &= \langle \nabla^*_X Y, W \rangle. \end{aligned} \tag{7.49}$$

The formulas (7.48) and (7.49) imply (1) and (2), respectively. ∎

Concerning on the integrability of the holomorphic distribution $\mathfrak{D}$ for a CR-statistical submanifold, we have the following:

Lemma 9 *Let (M, ∇, g) be a CR-submanifold in $\widetilde{M}$. Then the following conditions* (1), (2), (2)*, (3), (3)*, (4), (4)* *are equivalent:*

(1) *$\mathfrak{D}$ is completely integrable.*

(2) $\langle B(JX, Y), JV \rangle = \langle B(X, JY), JV \rangle$ *for $X, Y \in \mathfrak{D}$, $V \in \mathfrak{D}^\perp$.*

(3) $\langle JA_{JV} X, Y \rangle = -\langle A_{JV} JX, Y \rangle$ *for $X, Y \in \mathfrak{D}$, $V \in \mathfrak{D}^\perp$.*

(4) $B(JX, Y) = B(X, JY)$ *for $X, Y \in \mathfrak{D}$.*

As mentioned in Remark 3, for example, (2)* means the dual formula for (2), that is, $\langle B^*(JX, Y), JV \rangle = \langle B^*(X, JY), JV \rangle$ for $X, Y \in \mathfrak{D}$, $V \in \mathfrak{D}^\perp$.

Proof From (7.49), we have

$$\langle B(X, JY) - B(JX, Y), JV\rangle = \langle \nabla^*_X Y - \nabla^*_Y X, V\rangle = \langle [X, Y], V\rangle,$$

which implies (1) and (2) are equivalent, as well as (1) and (2)*. It is easy to check that the conditions (2) and (3)*, (3) and (2)* are equivalent, respectively.

To prove that (1) implies (4), we should remark (7.47) and that an integral manifold of $\mathfrak{D}$ is a holomorphic submanifold in $\widetilde{M}$. ■

Definition 9 Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ be a holomorphic statistical manifold and (M, ∇, g) a CR-submanifold in $\widetilde{M}$. M is said to be *mixed foliate* with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$) if M is mixed totally geodesic with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$) and $\mathfrak{D}$ is completely integrable.

Lemma 10 *Let (M, ∇, g) be a CR-submanifold in $\widetilde{M}$. If M is mixed foliate with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$), then we have*

$$PA^*_\xi = -A^*_\xi P \quad (\textit{resp.} \quad PA_\xi = -A_\xi P)$$

for any $\xi \in \Gamma(T^\perp M)$.

Proof Since M is mixed totally geodesic with respect to $\widetilde{\nabla}$, we have

$$\langle PA^*_\xi(X + V) + A^*_\xi P(X + V), Y + W\rangle = \langle B(X, JY) + B(JX, Y), \xi\rangle$$

for $X, Y \in \mathfrak{D}$, $V, W \in \mathfrak{D}^\perp$ and $\xi \in \Gamma(T^\perp M)$. ■

The following is a statistical submanifold version of [4, Proposition 3].

Theorem 7.3 *Let $\widetilde{M}(\widetilde{c})$ denote a holomorphic statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ of constant holomorphic sectional curvature $\widetilde{c}$ and (M, ∇, g) a proper CR-submanifold in $\widetilde{M}(\widetilde{c})$. If M is mixed foliate with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, then $\widetilde{c} \leq 0$.*

Proof For $X \in \mathfrak{D}$ and $V \in \mathfrak{D}^\perp$, using Lemma 10, we have

$$\begin{aligned}\nabla_X V &= (\widetilde{\nabla}_X V)^\top = -(J\widetilde{\nabla}^*_X JV)^\top = PA^*_{JV}X - t\,D^*_X JV \\ &= -A^*_{JV}JX - t\,D^*_X JV \end{aligned} \tag{7.50}$$

and similarly,

$$\nabla^*_X V = -A_{JV}JX - t\,D_X JV. \tag{7.51}$$

For $X, Y \in \mathfrak{D}$ and $V \in \mathfrak{D}^\perp$, using Eq. (7.33) of Codazzi, (7.50) and (7.51), we have

$$\begin{aligned}\widetilde{c}\langle X, JY\rangle JV &= 2\widetilde{S}(X, Y)V = 2(\widetilde{S}(X, Y)V)^\perp \\ &= (\bar{\nabla}_X B)(Y, V) - (\bar{\nabla}_Y B)(X, V) + (\bar{\nabla}^*_X B^*)(Y, V) - (\bar{\nabla}^*_Y B^*)(X, V) \\ &= -B(Y, \nabla_X V) + B(X, \nabla_Y V) - B^*(Y, \nabla^*_X V) + B^*(X, \nabla^*_Y V) \\ &= B(Y, A^*_{JV}JX) - B(X, A^*_{JV}JY) + B^*(Y, A_{JV}JX) - B^*(X, A_{JV}JY).\end{aligned}$$

Putting $X = JY$ in this equation, we see that

$$\begin{aligned}\widetilde{c}\|Y\|^2\|V\|^2 &= -\langle B(Y, A^*_{JV}Y), JV\rangle - B(JY, A^*_{JV}JY), JV\rangle \\ &\quad -\langle B^*(Y, A_{JV}Y), JV\rangle - \langle B^*(JY, A_{JV}JY), JV\rangle \\ &= -\left(\|A^*_{JV}Y\|^2 + \|A^*_{JV}JY\|^2 + \|A_{JV}Y\|^2 + \|A_{JV}JY\|^2\right).\end{aligned}$$

This proves our assertion. ■

Corollary 2 *Let $\widetilde{M}(\widetilde{c})$ be a holomorphic statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ of constant holomorphic sectional curvature $\widetilde{c} > 0$ and (M, ∇, g) a CR-submanifold in $\widetilde{M}(\widetilde{c})$. If M is mixed foliate with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, then M is a holomorphic submanifold or a totally real submanifold in $\widetilde{M}(\widetilde{c})$.*

7.6 Statistical Real Hypersurfaces

Real hypersurfaces in a Kähler manifold form an important class of CR-submanifolds. In this section, we study statistical real hypersurfaces. Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ be a holomorphic statistical manifold of dimension $2m \geq 4$ and (M, ∇, g) a statistical real hypersurface (for simplicity, statistical hypersurface) with unit normal vector field ξ in $\widetilde{M}$.

We define

$$U := -J\xi \in \Gamma(TM),$$

and $u \in \Gamma(TM^{(0,1)})$ by

$$u(X) := \langle U, X\rangle.$$

We set $h, h^* \in \Gamma(TM^{(0,2)})$ by

$$h(X, Y) := \langle B(X, Y), \xi\rangle, \quad h^*(X, Y) := \langle B^*(X, Y), \xi\rangle,$$

and

$$A := A_\xi,\ A^* := A^*_\xi \in \Gamma(TM^{(1,1)}),$$

and set $\tau, \tau^* \in \Gamma(TM^{(0,1)})$ by

$$\tau(X) := \langle D_X\xi, \xi\rangle, \quad \tau^*(X) := \langle D^*_X\xi, \xi\rangle.$$

That is, we have

$$\mathfrak{D}^{\perp} = \operatorname{span}\{U\}, \quad N = 0,$$
$$FX = u(X)\xi, \qquad t\xi = -U, \quad f = 0,$$
$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi, \quad \widetilde{\nabla}_X \xi = -AX + \tau(X)\xi.$$

We should remark that the definition of A and τ is different from the one in [8].

Lemma 11 *Let (M, ∇, g) be a statistical hypersurface in $\widetilde{M}$. The following formulas hold for $X, Y \in \Gamma(TM)$:*

$$\|U\| = 1, \quad PU = 0, \quad u \circ P = 0, \tag{7.52}$$
$$P^2 X = -X + u(X)U, \qquad \langle PX, PY\rangle = \langle X, Y\rangle - u(X)u(Y), \tag{7.53}$$
$$\tau(X) = u(\nabla^*_X U), \quad \tau^*(X) = -\tau(X), \tag{7.54}$$
$$h(X, Y) = \langle A^* X, Y\rangle, \tag{7.55}$$
$$\nabla^*_X Y = -P\nabla_X(PY) + u(Y)PAX + u(\nabla^*_X Y)U, \tag{7.56}$$
$$A^* X = -P\nabla_X U + u(A^* X)U, \tag{7.57}$$
$$\nabla_X U = PA^* X - \tau(X)U. \tag{7.58}$$

Proof The formulas (7.52) and (7.53) are obtained easily from (7.36) and (7.37). Equations (7.54) and (7.55) follow from (7.25) and (7.26). We get (7.56) from (7.39), (7.57) from (7.39)*, and (7.58) from (7.41). ■

As in the previous section, let $\widetilde{M}(\widetilde{c})$ denote a $2m(\geq 4)$-dimensional holomorphic statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ of constant holomorphic sectional curvature $\widetilde{c}$. Let (M, ∇, g) be a statistical hypersurface in $\widetilde{M}(\widetilde{c})$. In this case, equations of Gauss (7.32) and Codazzi (7.34) are written as follows:

Lemma 12 *Let (M, ∇, g) a statistical hypersurface in $\widetilde{M}(\widetilde{c})$. The following hold for $X, Y, Z \in \Gamma(TM)$:*

$$\begin{aligned}
&\frac{\widetilde{c}}{2}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle PY, Z\rangle PX - \langle PX, Z\rangle PY + 2\langle X, PY\rangle PZ\} \\
&\quad = 2(\widetilde{S}(X, Y)Z)^{\top} = 2S(X, Y)Z - \langle A^* Y, Z\rangle AX + \langle A^* X, Z\rangle AY \\
&\qquad - \langle AY, Z\rangle A^* X + \langle AX, Z\rangle A^* Y,
\end{aligned} \tag{7.59}$$

$$\begin{aligned}
&\frac{\widetilde{c}}{2}\{u(Y)PX - u(X)PY + 2\langle PX, Y\rangle U\} \\
&\quad = 2(\widetilde{S}(X, Y)\xi)^{\top} = -(\nabla_X A)Y + (\nabla_Y A)X - (\nabla^*_X A^*)Y \\
&\qquad + (\nabla^*_Y A^*)X - \tau(X)(A^* - A)Y + \tau(Y)(A^* - A)X.
\end{aligned} \tag{7.60}$$

Theorem 7.4 *Let $\widetilde{M}(\widetilde{c})$ denote a holomorphic statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ of constant holomorphic sectional curvature $\widetilde{c}$ and (M, ∇, g) a statistical hypersurface in $\widetilde{M}(\widetilde{c})$. If M is totally umbilical with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, then*

$$\widetilde{c} = 0, \tag{7.61}$$

and

$$X(\lambda + \lambda^*) = (\lambda - \lambda^*)\tau(X), \tag{7.62}$$
$$S(X, Y)Z = \lambda\lambda^*\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} \tag{7.63}$$

for any $X, Y, Z \in \Gamma(TM)$*, where* λ *and* λ^* *denote the eigenvalue of* A *and* A^**, respectively.*

Moreover, if $\lambda = \lambda^*$*, then* λ *is constant and* M *is of constant sectional curvature* λ^2.

Proof For any $X, Y \in \Gamma(TM)$, we have $(\nabla_Y A)X = (Y\lambda)X$ and $(\nabla_Y A^*)X = (Y\lambda^*)X$. Using Eq. (7.60) of Codazzi, we have

$$\begin{aligned}\frac{\widetilde{c}}{2}\{u(Y)PX - u(X)PY + 2\langle PX, Y\rangle U\} &= \{Y(\lambda + \lambda^*) - (\lambda - \lambda^*)\tau(Y)\}X \\ &\quad - \{X(\lambda + \lambda^*) - (\lambda - \lambda^*)\tau(X)\}Y.\end{aligned}$$

Putting $Y = U$, we get by (7.52) that

$$\frac{\widetilde{c}}{2}PX = \{U(\lambda + \lambda^*) - (\lambda - \lambda^*)\tau(U)\}X - \{X(\lambda + \lambda^*) - (\lambda - \lambda^*)\tau(X)\}U. \tag{7.64}$$

There exists a nonzero $X \in \Gamma(TM)$ such that $X \perp U$ because $\dim M \geq 3$. Since $X, PX = JX, U$ are linearly independent, we obtain (7.61) and (7.62) from (7.64). The conclusion (7.63) is driven from (7.61) and Eq. (7.59) of Gauss. ■

This is a statistical submanifold version of the Tashiro and Tachibana theorem [20, Theorem 3], which shows that if a complex space form admits a totally umbilical real hypersurface, then its holomorphic sectional curvature vanishes. See Milijević [15] for a generalization of Theorem 7.4.

Lemma 13 *Let* (M, ∇, g) *be a statistical hypersurface in* $\widetilde{M}(\widetilde{c})$*. If* U *is an eigenvector of* A *and* A^**, then it follows that*

$$\begin{aligned}\widetilde{c}\langle PX, Y\rangle = &\ Y(\lambda + \lambda^*)u(X) - X(\lambda + \lambda^*)u(Y) + \lambda\langle X, (PA + AP)Y\rangle \\ &+ \lambda^*\langle X, (PA^* + A^*P)Y\rangle - 2\langle X, (APA + A^*PA^*)Y\rangle \\ &+ (\lambda - \lambda^*)\{\tau(X)u(Y) - \tau(Y)u(X)\}\end{aligned}$$

for any $X, Y \in \Gamma(TM)$*, where* $\lambda := u(AU)$ *and* $\lambda^* := u(A^*U)$.

Proof From $AU = \lambda U$ and (7.58)*, we have

$$\begin{aligned}\langle(\nabla_X A)Y, U\rangle &= X\langle AY, U\rangle - \langle AY, \nabla^*_X U\rangle - \langle AU, \nabla_X Y\rangle \\ &= (X\lambda)u(Y) - \lambda\langle X, APY\rangle + \langle X, APAY\rangle,\end{aligned}$$

and similarly,

$$\langle (\nabla^*_X A^*)Y, U\rangle = (X\lambda^*)u(Y) - \lambda^*\langle X, A^*PY\rangle + \langle X, A^*PA^*Y\rangle$$

for any $X, Y \in \Gamma(TM)$. The assertion is driven from Eq. (7.60) of Codazzi. ∎

Proposition 6 *Let (M, ∇, g) be a statistical hypersurface in $\widetilde{M}(\widetilde{c})$. If M satisfies $PA + AP = PA^* + A^*P = 0$, then $\widetilde{c} \le 0$. Furthermore, if $\widetilde{c} = 0$, then* rank $A \le 1$ *and* rank $A^* \le 1$ *at each point in M.*

Proof Since $PA + AP = 0$, the formulas (7.52) and (7.53) imply $PAU = 0$ and hence $AU = \lambda U$ where $\lambda := u(AU)$. Similarly, we have $A^*U = \lambda^* U$. Then, using Lemma 13, we have

$$\widetilde{c}\|X\|^2 = -2(\|APX\|^2 + \|A^*PX\|^2) \le 0$$

for any $X \in \Gamma(TM)$ perpendicular to U.

If $\widetilde{c} = 0$, we have $\|APX\| = \|A^*PX\| = 0$. Therefore, we obtain $AX = \lambda\, u(X)U$ and $A^*X = \lambda^*\, u(X)U$ for any $X \in \Gamma(TM)$. Thus we have rank $A \le 1$ and rank $A^* \le 1$ at each point in M. ∎

Proposition 7 *Let (M, ∇, g) be a statistical hypersurface in $\widetilde{M}(\widetilde{c})$. Suppose that $\widetilde{c} > 0$. If M satisfies $AU = \lambda U$, $A^*U = \lambda^* U$ and $\tau(X) = 0$ for any $X \in \Gamma(TM)$ perpendicular to U, then $\lambda + \lambda^*$ is constant.*

Proof Since $\tau(Y) = 0$ for any $Y \in \Gamma(TM)$ perpendicular to U, we have $\tau(Y) = \tau(U)u(Y)$ for any $Y \in \Gamma(TM)$. Putting $X = U$ in the formula of Lemma 13, we get $Y(\lambda + \lambda^*) = (U(\lambda + \lambda^*))u(Y)$. Thus we have grad $(\lambda + \lambda^*) = \gamma U$, where $\gamma := U(\lambda + \lambda^*)$, from which (7.58) implies

$$\nabla_X \mathrm{grad}\,(\lambda + \lambda^*) = (X\gamma)U + \gamma(PA^*X - \tau(X)U).$$

Using this equation, we obtain

$$\begin{aligned} 0 &= X(Y(\lambda + \lambda^*)) - Y(X(\lambda + \lambda^*)) - [X, Y](\lambda + \lambda^*) \\ &= X\langle Y, \mathrm{grad}\,(\lambda + \lambda^*)\rangle - Y\langle X, \mathrm{grad}\,(\lambda + \lambda^*)\rangle - (\nabla^*_X Y - \nabla^*_Y X)(\lambda + \lambda^*) \\ &= \langle Y, \nabla_X \mathrm{grad}\,(\lambda + \lambda^*)\rangle - \langle X, \nabla_Y \mathrm{grad}\,(\lambda + \lambda^*)\rangle \\ &= (X\gamma)u(Y) + \gamma\langle PA^*X, Y\rangle - (Y\gamma)u(X) - \gamma\langle PA^*Y, X\rangle. \end{aligned}$$

Putting $Y = U$, we get $X\gamma = (U\gamma)u(X)$, and hence $\gamma(PA^* + A^*P) = 0$. Similarly, we have $\gamma(PA + AP) = 0$. Therefore, using Proposition 6, we obtain $\gamma = 0$ and hence we have the conclusion. ∎

Definition 10 Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ be a holomorphic statistical manifold and (M, ∇, g) a statistical hypersurface in $\widetilde{M}$. M is said to be *totally η-umbilical with respect to* $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$), if

$$AX = \mu X + (\lambda - \mu)\, u(X)U \quad (\text{resp. } A^*X = \mu^*X + (\lambda^* - \mu^*)\, u(X)U)$$

for some scalar functions λ and μ (resp. λ^* and μ^*) on M.

In this case we have

$$AP = \mu P = PA, \quad AU = \lambda\, U$$

and $\lambda = u(AU)$ (resp. $A^*P = \mu^*P = PA^*$, $A^*U = \lambda^*\, U$ and $\lambda^* = u(A^*U)$). If $\lambda = \mu$, then M is totally umbilical with respect to $\widetilde{\nabla}$. If $\mu = 0$, then $AX = \lambda\, u(X)U$ for any $X \in \Gamma(TM)$, and therefore $AX = 0$ for any X perpendicular to U. If $\lambda = 0$, then $AX = \mu(X - u(X)U)$ for any $X \in \Gamma(TM)$.

Proposition 8 *Let (M, ∇, g) be a statistical hypersurface in $\widetilde{M}(\widetilde{c})$. If M is totally η-umbilical with respect $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ with $\lambda = \lambda^* = 0$, then $\widetilde{c} \geq 0$. Furthermore, if $\widetilde{c} = 0$, then M is totally geodesic with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$.*

It follows directly from the following:

Lemma 14 *Let (M, ∇, g) be a statistical hypersurface in $\widetilde{M}(\widetilde{c})$. If M is totally η-umbilical with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, then it follows that*

$$\widetilde{c} = 2(\mu^2 + \mu^{*2} - \mu\lambda - \mu^*\lambda^*) = 2(2\mu\mu^* - \mu\lambda^* - \mu^*\lambda), \tag{7.65}$$

$$U(\mu + \mu^*) = (\mu - \mu^*)\tau(U) \quad \textit{and} \tag{7.66}$$

$$X(\lambda + \lambda^*) - U(\lambda + \lambda^*)u(X) = (\lambda - \lambda^*)\,(\tau(X) - \tau(U)u(X)) \tag{7.67}$$

for any $X \in \Gamma(TM)$.

Proof By Lemma 13, we have

$$\begin{aligned}\widetilde{c}\langle PX, Y\rangle &= ZY(\lambda + \lambda^*)u(X) - X(\lambda + \lambda^*)u(Y) - 2(\mu^2 + \mu^{*2} - \mu\lambda - \mu^*\lambda^*)\langle X, PY\rangle \\ &\quad + (\lambda - \lambda^*)\{\tau(X)u(Y) - \tau(Y)u(X)\},\end{aligned}$$

from which we get $(7.65)_1$ by taking $Y = PX$ so that $X \perp U$ and $\|X\| = 1$.
Using (7.58) and $(7.58)^*$, we calculate

$$\begin{aligned}(\nabla_X A)Y &= \nabla_X\{\mu Y + (\lambda - \mu)\langle U, Y\rangle U\} + \{\mu\nabla_X Y + (\lambda - \mu)u(\nabla_X Y)U\} \\ &= (X\mu)(Y - u(Y)U) + (X\lambda)u(Y)U + (\lambda - \mu)\{\mu^* u(Y)PX + \mu\langle JX, Y\rangle U\}\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, from which

$$\begin{aligned}(\nabla_X A)U &= (X\lambda)U + \mu^*(\lambda - \mu)PX, \\ (\nabla_U A)X &= (U\mu)(X - u(X)U) + (U\lambda)u(X)U\end{aligned}$$

for any $X \in \Gamma(TM)$. Similarly, we get

$$(\nabla^*_X A^*)U = (X\lambda^*)U + \mu(\lambda^* - \mu^*)PX,$$
$$(\nabla^*_U A^*)X = (U\mu^*)(X - u(X)U) + (U\lambda^*)u(X)U$$

for any $X \in \Gamma(TM)$. Substituting these equations in (7.60), we obtain

$$\begin{aligned}\frac{\widetilde{c}}{2}PX &= 2(\widetilde{S}(X,U)\xi)^\top \\ &= (2\mu\mu^* - \mu\lambda^* - \mu^*\lambda)PX + \{U(\mu+\mu^*) - (\mu-\mu^*)\tau(U)\}X \\ &\quad -\{X(\lambda+\lambda^*) - U(\lambda+\lambda^*)u(X) - (\lambda-\lambda^*)\left(\tau(X) - \tau(U)u(X)\right) \\ &\quad + \left(U(\mu+\mu^*) - (\mu-\mu^*)\tau(U)\right)u(X)\}U\end{aligned}$$

for any $X \in \Gamma(TM)$. Comparing the coefficients of PX, we have $(7.65)_2$. The X-component and the U-component imply (7.66) and (7.67), respectively. ■

7.7 CR-Statistical Submanifolds of Maximal CR-Dimension

As an appendix of the previous section, we briefly treat CR-submanifolds of maximal CR-dimension, which are generalizations of real hypersurfaces. Following [7], we will first remind you the definition. Let M be an $n(\geq 3)$-dimensional submanifold in a $2m$-dimensional Kähler manifold $\widetilde{M}$. If the maximal holomorphic subspace $\mathfrak{D}_x = T_xM \cap JT_xM$, $x \in M$, defines an $(n-1)$-dimensional differentiable distribution $\mathfrak{D}$, then $J\mathfrak{D}^\perp_x \subset T^\perp_x M$, where $\mathfrak{D}^\perp$ denotes the orthogonal complementary distribution of $\mathfrak{D}$ in M. Therefore, M is a CR-submanifold in $\widetilde{M}$. We call such an M a CR-submanifold *of maximal CR-dimension*. For $U \in \mathfrak{D}^\perp$ such that $\|U\| = 1$, we set $\xi := JU$ and call it the *distinguished normal* for M. A real hypersurface is a typical CR-submanifold of maximal CR-dimension.

Hereafter in this section, let (M, ∇, g) be an $n(\geq 3)$-dimensional CR-submanifold of maximal CR-dimension with distinguished normal ξ in a $2m$-dimensional holomorphic statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$. For the normal space of M, we take a local orthonormal frame $\{\xi, \xi_1, \ldots, \xi_{2q}\} = \{\xi, \xi_1, \ldots, \xi_q, \xi_{\bar{1}}, \ldots, \xi_{\bar{q}}\}$ such that $\xi_{q+a} = \xi_{\bar{a}} = J\xi_a$ where $2q + 1 = 2m - n$, and the indices α, β run from 1 to $2q$, and a, b from 1 to q. We then put $u \in \Gamma(TM^{(0,1)})$ by $u(X) := \langle U, X\rangle$. We set $A := A_\xi$, $A^* := A^*_\xi$, $A_\alpha := A_{\xi_\alpha}$, and $A^*_\alpha := A^*_{\xi_\alpha} \in \Gamma(TM^{(1,1)})$. We define $\tau, \tau_\alpha, \tau^*_\alpha, \tau_{\alpha\beta} \in \Gamma(TM^{(0,1)})$ by $\tau(X) := \langle D_X\xi, \xi\rangle$, $\tau_\alpha(X) := \langle D_X\xi, \xi_\alpha\rangle$, $\tau^*_\alpha(X) := \langle D^*_X\xi, \xi_\alpha\rangle$, $\tau_{\alpha\beta}(X) := \langle D_X\xi_\alpha, \xi_\beta\rangle$ for any $X \in \Gamma(TM)$. Summing up, we have

$$\mathfrak{D}^\perp = \mathrm{span}\{U\}, \quad N = \mathrm{span}\{\xi_1, \ldots, \xi_{2q}\},$$
$$FX = u(X)\xi, \quad t\xi = -U, \quad t\xi_\alpha = 0, \quad f\xi = 0, \quad f\xi_a = \xi_{\bar{a}},$$

$$\widetilde{\nabla}_X\xi = -AX + D_X\xi, \quad D_X\xi = \tau(X)\xi + \sum_{\alpha=1}^{2q}\tau_\alpha(X)\xi_\alpha,$$

$$\widetilde{\nabla}^*_X\xi = -A^*X + D^*_X\xi, \quad D^*_X\xi = -\tau(X)\xi + \sum_{\alpha=1}^{2q}\tau^*_\alpha(X)\xi_\alpha,$$

$$\widetilde{\nabla}_X\xi_\alpha = -A_\alpha X + D_X\xi_\alpha, \quad D_X\xi_\alpha = -\tau^*_\alpha(X)\xi + \sum_{\beta=1}^{2q}\tau_{\alpha\beta}(X)\xi_\beta,$$

$$\widetilde{\nabla}^*_X\xi_\alpha = -A^*_\alpha X + D^*_X\xi_\alpha, \quad D^*_X\xi_\alpha = -\tau_\alpha(X)\xi - \sum_{\beta=1}^{2q}\tau_{\beta\alpha}(X)\xi_\beta$$

for any $X \in \Gamma(TM)$.

Lemma 15 *Let (M, ∇, g) be a CR-submanifold of maximal CR-dimension in $\widetilde{M}$. The formulas (7.52)–(7.58) and the following hold:*

$$\tau_{ab} = -\tau_{\bar{b}\bar{a}}, \quad \tau_{a\bar{b}} = \tau_{b\bar{a}}, \quad \tau_{\bar{a}b} = \tau_{\bar{b}a}, \tag{7.68}$$

$$\tau_a(X) = -u(A_{\bar{a}}X), \quad \tau_{\bar{a}}(X) = u(A_aX), \quad X \in \Gamma(TM), \tag{7.69}$$

$$PA_aX = A^*_{\bar{a}}X + \tau^*_a(X)U, \tag{7.70}$$

$$PA_{\bar{a}}X = -A^*_aX + \tau^*_{\bar{a}}(X)U, \quad X \in \Gamma(TM),$$

$$\langle A_\alpha X, X\rangle + \langle A_\alpha JX, JX\rangle = 0, \quad X \in \mathfrak{D}. \tag{7.71}$$

Proof The formula $(7.68)_1$ is obtained by

$$\begin{aligned} 0 = X\langle\xi_a, \xi_b\rangle &= \langle D_X\xi_a, \xi_b\rangle + \langle J\xi_a, JD^*_X\xi_b\rangle \\ &= \langle D_X\xi_a, \xi_b\rangle + \langle J\xi_a, D_XJ\xi_b\rangle \\ &= \langle D_X\xi_a, \xi_b\rangle + \langle \xi_{\bar{a}}, D_X\xi_{\bar{b}}\rangle. \end{aligned}$$

We can get the others in (7.68) and the formulas (7.69) in the same way. The formulas (7.70) follow from (7.41). By (7.70), we have that $\langle A_aX, X\rangle = \langle A^*_{\bar{a}}X, PX\rangle$ for $X \in \mathfrak{D}$, which implies (7.71). ■

Now we shall generalize Theorem 7.4 for this setting.

Lemma 16 *Let (M, ∇, g) be a CR-submanifold of maximal CR-dimension with distinguished normal ξ in $\widetilde{M}(\widetilde{c})$. If there exist functions λ and λ^* on M satisfying $A_\xi X = \lambda X$ and $A^*_\xi X = \lambda^* X$ for any $X \in \Gamma(TM)$, then the following hold:*

$$
\begin{aligned}
\frac{\widetilde{c}}{2}\|PX\|^2 = 2\sum_{\alpha=1}^{2q}\tau_\alpha(X)\tau^*_\alpha(X) \\
&+ \sum_{a=1}^{q}\{\tau_a(U)\langle A^*_{\bar{a}}X, X\rangle + \tau^*_a(U)\langle A_{\bar{a}}X, X\rangle \\
&- \tau_{\bar{a}}(U)\langle A^*_a X, X\rangle - \tau^*_{\bar{a}}(U)\langle A_a X, X\rangle\}, \qquad (7.72)
\end{aligned}
$$

$$
\begin{aligned}
X(\lambda+\lambda^*) - (\lambda-\lambda^*)\tau(X) = 2\sum_{a=1}^{q}\{\tau_a(X)\tau_{\bar{a}}(U) + \tau^*_a(X)\tau^*_{\bar{a}}(U) \\
&- \tau_a(U)\tau_{\bar{a}}(X) - \tau^*_a(U)\tau^*_{\bar{a}}(X)\} \qquad (7.73)
\end{aligned}
$$

for any $X \in \Gamma(TM)$*, and*

$$
\widetilde{c}\|X\|^2 = 2\sum_{\alpha=1}^{2q}\left\{\tau_\alpha(X)\tau^*_\alpha(X) + \tau_\alpha(JX)\tau^*_\alpha(JX)\right\} \qquad (7.74)
$$

for any $X \in \mathfrak{D}$.

Proof Using (7.69), (7.69)*, (7.70), and (7.70)*, we have

$$
\begin{aligned}
\sum_{\alpha=1}^{2q}\tau_\alpha(U)\langle A_\alpha X, PX\rangle = -\sum_{a=1}^{q}\{\tau_a(U)\langle A^*_{\bar{a}}X, X\rangle - \tau_{\bar{a}}(U)\langle A^*_a X, X\rangle\} \\
&- u(X)\sum_{\alpha=1}^{2q}\tau_\alpha(U)\tau^*_\alpha(X), \qquad (7.75)
\end{aligned}
$$

$$
\sum_{\alpha=1}^{2q}\tau_\alpha(X)\langle A_\alpha U, PX\rangle = \sum_{\alpha=1}^{2q}\tau_\alpha(X)\tau^*_\alpha(X) - u(X)\sum_{\alpha=1}^{2q}\tau_\alpha(X)\tau^*_\alpha(U). \qquad (7.76)
$$

Equation (7.34) of Codazzi implies

$$
\begin{aligned}
&\frac{\widetilde{c}}{2}\{u(Y)PX - u(X)PY + 2\langle PX, Y\rangle U\} \\
&= (\nabla_Y A)X - (\nabla_X A)Y + (\nabla^*_Y A^*)X - (\nabla^*_X A^*)Y \\
&\quad - \tau(Y)(AX - A^*X) + \tau(X)(AY - A^*Y) \\
&\quad - \sum_{\alpha=1}^{2q}\left\{\tau_\alpha(Y)A_\alpha X - \tau_\alpha(X)A_\alpha Y + \tau^*_\alpha(Y)A^*_\alpha X - \tau^*_\alpha(X)A^*_\alpha Y\right\}
\end{aligned}
$$

$$= \left\{Y(\lambda+\lambda^*) - (\lambda-\lambda^*)\tau(Y)\right\}X - \left\{X(\lambda+\lambda^*) - (\lambda-\lambda^*)\tau(X)\right\}Y$$
$$- \sum_{\alpha=1}^{2q} \left\{\tau_\alpha(Y)A_\alpha X - \tau_\alpha(X)A_\alpha Y + \tau_\alpha^*(Y)A_\alpha^* X - \tau_\alpha^*(X)A_\alpha^* Y\right\}$$

for any $X, Y \in \Gamma(TM)$. Putting $Y = U$ in this equation, we get

$$\frac{\widetilde{c}}{2}PX = \left\{U(\lambda+\lambda^*) - (\lambda-\lambda^*)\tau(U)\right\}X - \left\{X(\lambda+\lambda^*) - (\lambda-\lambda^*)\tau(X)\right\}U$$
$$- \sum_{\alpha=1}^{2q} \left\{\tau_\alpha(U)A_\alpha X - \tau_\alpha(X)A_\alpha U + \tau_\alpha^*(U)A_\alpha^* X - \tau_\alpha^*(X)A_\alpha^* U\right\} \tag{7.77}$$

for any $X \in \Gamma(TM)$. Since $\langle PX, X\rangle = \langle PX, U\rangle = 0$, we have

$$\frac{\widetilde{c}}{2}\|PX\|^2 = \sum_{\alpha=1}^{2q}\{-\tau_\alpha(U)\langle A_\alpha X, PX\rangle + \tau_\alpha(X)\langle A_\alpha U, PX\rangle$$
$$-\tau_\alpha^*(U)\langle A_\alpha^* X, PX\rangle + \tau_\alpha^*(X)\langle A_\alpha^* U, PX\rangle\}$$

and by (7.75), (7.76) and their duals, we get (7.72).

Since $\widetilde{c}\|X\|^2 = \frac{\widetilde{c}}{2}(\|PX\|^2 + \|X\|^2)$ for any $X \in \mathfrak{D}$, using (7.72) and (7.71), we calculate (7.74).

On the other hand, since $\langle PX, X\rangle = 0$ again with (7.77), we have for any $X \in \mathfrak{D}$,

$$\left\{U(\lambda+\lambda^*) - (\lambda-\lambda^*)\tau(U)\right\}\|X\|^2$$
$$= \sum_{\alpha=1}^{2q}\left\{\tau_\alpha(U)\langle A_\alpha X, X\rangle + \tau_\alpha^*(U)\langle A_\alpha^* X, X\rangle\right\},$$

from which the similar calculation with (7.71) implies that the right-hand side of this equation vanishes, and that

$$U(\lambda+\lambda^*) = (\lambda-\lambda^*)\tau(U).$$

Substituting this equation in (7.77), we have

$$\frac{\widetilde{c}}{2}PX = -\left\{X(\lambda+\lambda^*) - (\lambda-\lambda^*)\tau(X)\right\}U$$
$$- \sum_{\alpha=1}^{2q} \left\{\tau_\alpha(U)A_\alpha X - \tau_\alpha(X)A_\alpha U + \tau_\alpha^*(U)A_\alpha^* X - \tau_\alpha^*(X)A_\alpha^* U\right\}$$

for any $X \in \Gamma(TM)$, from which we have (7.73) by (7.69) and $(7.69)^*$. ■

The formulas in Lemma 16 directly imply the following:

Theorem 7.5 *Let $\widetilde{M}(\widetilde{c})$ denote a holomorphic statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ of constant holomorphic sectional curvature $\widetilde{c}$, and (M, ∇, g) an $n(\geq 3)$-dimensional CR-submanifold of maximal CR-dimension with distinguished normal ξ in $\widetilde{M}(\widetilde{c})$. Suppose that there exist functions λ and λ^* on M satisfying $AX = \lambda X$ and $A^*X = \lambda^* X$ for any $X \in \Gamma(TM)$.*

(1) *If there exists $x \in M$ such that $D_X\xi = \tau(X)\xi$ for any $X \in \mathfrak{D}_x$, then $\widetilde{c} = 0$.*
(2) *If $\lambda = \lambda^*$ and $D_U\xi = -D^*_U\xi = \tau(U)\xi$, then $\lambda(= \lambda^*)$ is constant.*

Lemma 16 will give various observations in addition. For example, we have the following under the same setting. If there exists $x \in M$ such that $D_X\xi - D^*_X\xi = 2\tau(X)\xi$ for any $X \in \mathfrak{D}_x$, then $\widetilde{c} \geq 0$. If $D_U\xi = \tau(U)\xi$ and $D^*_X\xi = 0$ for any $X \in \mathfrak{D}$, then $\lambda + \lambda^*$ is constant.

We will proceed to properties for CR-submanifold of maximal CR-dimension corresponding to Propositions 6 and 7. The following lemma is a generalization of Lemma 13, which is obtained in the similar fashion.

Lemma 17 *Let (M, ∇, g) be a CR-submanifold of maximal CR-dimension with distinguished normal ξ in $\widetilde{M}(\widetilde{c})$. If $U = -J\xi$ is an eigenvector of $A := A_\xi$ and $A^* := A^*_\xi$, then it follows that*

$$\begin{aligned}\widetilde{c}\langle PX, Y\rangle = {} & Y(\lambda + \lambda^*)u(X) - X(\lambda + \lambda^*)u(Y) \\ & + \lambda\langle X, (PA + AP)Y\rangle + \lambda^*\langle X, (PA^* + A^*P)Y\rangle \\ & - 2\langle X, (APA + A^*PA^*)Y\rangle + (\lambda - \lambda^*)\{\tau(X)u(Y) - \tau(Y)u(X)\} \\ & + 2\sum_{a=1}^{q}\{\tau_a(X)\tau_{\bar{a}}(Y) - \tau_{\bar{a}}(X)\tau_a(Y) + \tau^*_a(X)\tau^*_{\bar{a}}(Y) - \tau^*_{\bar{a}}(X)\tau^*_a(Y)\}\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, where $\lambda := u(AU)$ and $\lambda^* := u(A^*U)$.

Proposition 9 *Let (M, ∇, g) be a CR-submanifold of maximal CR-dimension with distinguished normal ξ in $\widetilde{M}(\widetilde{c})$. If M satisfies $PA_\xi + A_\xi P = PA^*_\xi + A^*_\xi P = 0$ and if there exist $x \in M$ and $X \in \mathfrak{D}_x$ such that $D_X\xi = -D^*_X\xi = \tau(X)\xi$, then $\widetilde{c} \leq 0$.*

Proposition 10 *Let (M, ∇, g) be a CR-submanifold of maximal CR-dimension with distinguished normal ξ in $\widetilde{M}(\widetilde{c})$. Suppose that there exist functions λ and λ^* on M satisfying $A_\xi U = \lambda U$ and $A^*_\xi U = \lambda^* U$. If $\widetilde{c} > 0$ and $D_X\xi = D^*_X\xi = 0$ for any $X \in \mathfrak{D}$, then $\lambda + \lambda^*$ is constant.*

7.8 Totally Real Statistical Submanifolds

We finally proceed to generalizations of the totally real submanifold theory. Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ be a $2m$-dimensional holomorphic statistical manifold and (M, ∇, g) an n-dimensional statistical submanifold in $\widetilde{M}$.

Lemma 18 (1) *Let* (M, ∇, g) *be a totally real statistical submanifold in* $\widetilde{M}$. *Then we have the following fundamental formulas:*

$$P = 0, \quad \mathfrak{D} = 0, \quad \mathfrak{D}^{\perp} = TM,$$

$$A_{JX}Y = A_{JY}X, \tag{7.78}$$

$$A_{JY}X = -tB^{*}(X, Y), \tag{7.79}$$

$$D_X(JY) = J\nabla^{*}_X Y + fB^{*}(X, Y), \tag{7.80}$$

$$\nabla_X Y = -t\, D^{*}_X(JY), \tag{7.81}$$

$$B(X, Y) = JA^{*}_{JX}Y - fD^{*}_X(JY), \tag{7.82}$$

$$A_{J\zeta}X = -t\, D^{*}_X\zeta, \tag{7.83}$$

$$D_X(f\zeta) = -JA^{*}_{\zeta}X + f\, D^{*}_X\zeta \tag{7.84}$$

for any $X, Y \in \Gamma(TM)$ *and* $\zeta \in \Gamma(N)$.

(2) *Furthermore, let* (M, ∇, g) *be a Lagrangian submanifold in* $\widetilde{M}$. *Then we have the following fundamental formulas:*

$$f = 0, \quad N = 0,$$

$$D_X(JY) = J\nabla^{*}_X Y, \tag{7.85}$$

$$R^{\perp}(X, Y)JZ = JR^{*}(X, Y)Z, \quad S^{\perp}(X, Y)JZ = JS(X, Y)Z, \tag{7.86}$$

$$B(X, Y) = JA^{*}_{JX}Y \tag{7.87}$$

for each $X, Y, Z \in \Gamma(TM)$.

Proof (1) The formula (7.78) follows from Lemma 6. The formulas (7.39), (7.40), (7.41), and (7.42) imply (7.79), (7.80), (7.81), and (7.82), respectively. We get (7.83) and (7.84) by $\widetilde{\nabla}_X(J\zeta) = J\widetilde{\nabla}^{*}_X\zeta$. (2) The formula (7.80) implies (7.85), from which (7.86) follows. The formula (7.82) directly implies (7.87). ■

Proposition 11 *Let* (M, ∇, g) *be a totally real submanifold in* $\widetilde{M}(\widetilde{c})$. *Suppose that the mean curvature vector field H for* $\widetilde{\nabla}$ *is parallel with respect to D and M is totally umbilical with respect to* $\widetilde{\nabla}$ *and* $\widetilde{\nabla}^{*}$. *Then M is of constant sectional curvature.*

It immediately follows from the following:

Lemma 19 *Let* (M, ∇, g) *be a totally real submanifold in* $\widetilde{M}(\widetilde{c})$. *If M is totally umbilical with respect to* $\widetilde{\nabla}$ *and* $\widetilde{\nabla}^{*}$, *then*

$$S(X, Y)Z = \left(\frac{\widetilde{c}}{4} + \langle H, H^{*}\rangle\right)\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} \tag{7.88}$$

for any $X, Y, Z \in \Gamma(TM)$, *and*

$$D_X H + D^{*}_X H^{*} = 0, \tag{7.89}$$

where H and H^{*} *denote the mean curvature vector fields for* $\widetilde{\nabla}$ *and* $\widetilde{\nabla}^{*}$, *respectively.*

Proof Since $B(X,Y) = \langle X, Y\rangle H$, $A_\xi X = \langle H, \xi\rangle X$ and their duals hold, Eq. (7.32) of Gauss drives (7.88). Furthermore, we have

$$(\bar{\nabla}_X B)(Y,Z) = \langle Y, Z\rangle D_X H + \langle \nabla^*_X Y - \nabla_X Y, Z\rangle H, \quad X, Y, Z \in \Gamma(TM),$$

from which

$$(\bar{\nabla}_X B)(Y,Z) - (\bar{\nabla}_Y B)(X,Z) = \langle Y, Z\rangle D_X H - \langle X, Z\rangle D_Y H.$$

Using Eq. (7.33) of Codazzi, we obtain

$$0 = \langle Y, Z\rangle (D_X H + D^*_X H^*) - \langle X, Z\rangle (D_Y H + D^*_Y H^*). \tag{7.90}$$

For any $X \in \Gamma(TM)$, putting nonzero $Y = Z \in \Gamma(TM)$ perpendicular to X in (7.90), we have (7.89). ■

Proposition 12 *Let (M, ∇, g) be a totally real submanifold in $\widetilde{M}(\widetilde{c})$. Suppose that M is totally umbilical with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$. If $H = H^*$, then $\|H\|$ is constant and M is of constant sectional curvature $\dfrac{\widetilde{c}}{4} + \|H\|^2$.*

Proof Using Lemma 19, we have

$$X\|H\|^2 = X\langle H, H^*\rangle = \langle D_X H + D^*_X H^*, H\rangle = 0$$

for any $X \in \Gamma(TM)$. ■

Theorem 7.6 *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ be a holomorphic statistical manifold and (M, ∇, g) a Lagrangian submanifold in $\widetilde{M}$. If $A_{JX}A^*_{JY} = A^*_{JY}A_{JX}$ for each $X, Y \in \Gamma(TM)$, then $(\widetilde{S}(X,Y)Z)^\top = S(X,Y)Z$ for each $X, Y, Z \in \Gamma(TM)$. In particular, if $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ is of constant holomorphic sectional curvature $\widetilde{c}$ additionally, then M is of constant sectional curvature $\widetilde{c}/4$.*

Proof For $X, Y \in \Gamma(TM)$, using (7.78) and (7.87), we obtain

$$A_{B(Y,Z)}X = A_{JA^*_{JY}Z}X = A_{JX}A^*_{JY}Z.$$

Similarly, we have

$$A^*_{B^*(Y,Z)}X = A^*_{JX}A_{JY}Z,$$

from which, using Eq. (7.32) of Gauss, we get

$$\begin{aligned} 2S(X,Y)Z - 2(\widetilde{S}(X,Y)Z)^\top &= A_{B(Y,Z)}X - A_{B(X,Z)}Y + A^*_{B^*(Y,Z)}X - A^*_{B^*(X,Z)}Y \\ &= A_{JX}A^*_{JY}Z - A_{JY}A^*_{JX}Z + A^*_{JX}A_{JY}Z - A^*_{JY}A_{JX}Z \\ &= 0. \end{aligned}$$

■

We remark that the shape operators and their duals of the Lagrangian submanifold in Example 6 commute and the induced statistical structure is of constant sectional curvature zero.

For a normal bundle valued (0, 2)-tensor field σ on M, we denote

$$\overline{R}(X,Y)\cdot\sigma := \overline{\nabla}_X\overline{\nabla}_Y\sigma - \overline{\nabla}_Y\overline{\nabla}_X\sigma - \overline{\nabla}_{[X,Y]}\sigma, \quad X, Y \in \Gamma(TM),$$

where $\overline{\nabla}$ is defined as in (7.31) for $\widetilde{\nabla}$. Then we have

$$(\overline{R}(X,Y)\cdot\sigma)(Z,W) = R^{\perp}(X,Y)\sigma(Z,W) - \sigma(R(X,Y)Z,W) \\ - \sigma(Z,R(X,Y)W),$$

from which we obtain

$$(\overline{S}(X,Y)\cdot\sigma)(Z,W) := \frac{1}{2}\{(\overline{R}(X,Y)\cdot\sigma)(Z,W) + (\overline{R}^*(X,Y)\cdot\sigma)(Z,W)\} \\ = S^{\perp}(X,Y)\sigma(Z,W) - \sigma(S(X,Y)Z,W) - \sigma(Z,S(X,Y)W) \quad (7.91)$$

for any $X, Y, Z, W \in \Gamma(TM)$.

Proposition 13 *Let* (M, ∇, g) *be a Lagrangian submanifold of constant sectional curvature* c *in* $\widetilde{M}$. *If* M *satisfies* $S^{\perp}(X,Y)H = 0$ (*resp.* $S^{\perp}(X,Y)H^* = 0$), *then either* $c = 0$ *or* $H = 0$ (*resp.* $H^* = 0$).

Proof Using (7.86), we have

$$0 = S^{\perp}(X,Y)H = -J(S(X,Y)JH) \\ = c\{\langle X, JH\rangle JY - \langle Y, JH\rangle JX\}$$

for any $X, Y \in \Gamma(TM)$. Putting $X = JH$ and $0 \neq Y \perp JH$ in this equation, we have the conclusion. ■

Proposition 14 *Let* (M, ∇, g) *be a Lagrangian submanifold of constant sectional curvature* c *in* $\widetilde{M}$. *If* M *satisfies* $\overline{S}(X,Y)\cdot B = 0$ (*resp.* $\overline{S}(X,Y)\cdot B^* = 0$), *then either* $c = 0$ *or* $B = 0$ (*resp.* $B^* = 0$).

Proof Using (7.91), we have

$$S^{\perp}(X,Y)B(Z,W) = B(S(X,Y)Z,W) + B(Z,S(X,Y)W) \\ = c\{\langle Y,Z\rangle B(X,W) - \langle X,Z\rangle B(Y,W) \\ + \langle Y,W\rangle B(X,Z) - \langle X,W\rangle B(Y,Z)\}, \quad (7.92)$$

from which

$$n\,S^{\perp}(X,Y)H = S^{\perp}(X,Y)\sum_{i=1}^{n} B(e_i,e_i) = 2\sum_{i=1}^{n} B(S(X,Y)e_i,e_i) = 0,$$

where $\{e_1,\dots,e_n\}$ denotes a local orthonormal frame on M. Therefore, using Proposition 13, we have either $c = 0$ or $H = 0$.

On the other hand, using (7.87), $(7.86)_2$, we have

$$S^{\perp}(X,Y)B(Z,W) = c\{\langle B(Y,W), JZ\rangle JX - \langle B(X,W), JZ\rangle JY\},$$

which implies with (7.92) that

$$c\{(n+1)B(Y,Z) - n\langle Y,Z\rangle H - n\langle H, JZ\rangle JY\} = 0.$$

Therefore, if $c \neq 0$, we have $B = 0$. ■

Theorem 7.7 *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ be a holomorphic statistical manifold and (M, ∇, g) a totally real submanifold in $\widetilde{M}$. Suppose that*

(1) $D_X(J\zeta) = JD^*_X\zeta$ *for any $X \in \Gamma(TM)$, $\zeta \in \Gamma(N)$,*
(2) $\overline{S}(X,Y)\cdot B = 0$ *(resp. $\overline{S}(X,Y)\cdot B^* = 0$) for any $X, Y \in \Gamma(TM)$, and*
(3) *M is of constant sectional curvature c.*

Then either $c = 0$ or $B = 0$ (resp. $B^* = 0$).

Since the condition (1) in the theorem holds when M is a Lagrangian submanifold, Theorem 7.7 is a generalization of Proposition 14.

Proof We will prove as Lemmas 20 and 21 that the condition (1) derives formulas (7.85), $(7.85)^*$ (7.86), $(7.86)^*$ (7.87), and $(7.87)^*$. It shows that the proof of Proposition 14 works in this case as well. ■

Lemma 20 *Let (M, ∇, g) be a totally real submanifold in $\widetilde{M}$. Then the formulas $(7.85)^*$, (7.87) and the following four conditions are equivalent to each other:*

$$A^*_\zeta = 0, \tag{7.93}$$

$$D_X(f\zeta) = fD^*_X\zeta, \tag{7.94}$$

$$fB(X,Y) = 0, \tag{7.95}$$

$$fD^*_X(JY) = 0, \tag{7.96}$$

where $X, Y \in \Gamma(TM)$ and $\zeta \in \Gamma(N)$.

Moreover, these conditions imply that $(7.86)^*$ holds and $JH \in \Gamma(TM)$.

Proof The formula (7.93) implies (7.94) by (7.84). The formula (7.94) implies (7.87) by (7.82). The formula (7.87) implies (7.95) because $-A^*_{JX}Y = JB(X,Y) = tB(X,Y) + fB(X,Y)$. The formula (7.95) implies $(7.85)^*$ by $(7.80)^*$. The formula

(7.85)* implies (7.96) by (7.38). The formula (7.96) implies (7.93) by $\langle A^*_\zeta X, Y\rangle = \langle B(X,Y),\zeta\rangle = \langle JA^*_{JX}Y - fD^*_X(JY),\zeta\rangle = 0$. ■

Lemma 21 *Let* (M,∇,g) *be a totally real submanifold in* $\widetilde{M}$. *Then the condition* (1) *of Theorem 7.7 is equivalent to that* $A_\zeta = A^*_\zeta = 0$ *for* $\zeta \in \Gamma(N)$.

Proof We have

$$\begin{aligned} D_X(J\zeta) - JD^*_X\zeta &= \widetilde{\nabla}_X(J\zeta) + A_{J\zeta}X - JD^*_X\zeta = J\widetilde{\nabla}^*_X\zeta + A_{J\zeta}X - JD^*_X\zeta \\ &= -JA^*_\zeta X + A_{J\zeta}X. \end{aligned}$$ ■

These kinds of properties for totally real submanifolds in Kähler manifolds are obtained by Kassabov [10]. A generalization for the case that the ambient space is a holomorphic statistical manifold is due to Milijević [14], in which she used the notion of constant curvature in Kurose's sense.

Let M be an n-dimensional totally real submanifold in a holomorphic statistical manifold $\widetilde{M}$. We set functions on M by

$$\sigma_1 = \frac{1}{4}\sum_{h,k,j,i=1}^{n} \langle [A_k, A^*_j]e_i - [A_j, A^*_k]e_i, e_h\rangle^2 \tag{7.97}$$

$$= \sum_{j,i=1}^{n} \mathrm{tr}\,\left((A_i)^2(A^*_j)^2 - (A_iA^*_j)^2 + A_iA^*_iA_jA^*_j - A_iA_jA^*_iA^*_j\right), \tag{7.98}$$

$$\sigma_2 = \frac{1}{4}\sum_{j,i=1}^{n}\{\mathrm{tr}\,(A_jA^*_i + A_iA^*_j)\}^2 \tag{7.99}$$

$$= \frac{1}{2}\sum_{j,i=1}^{n}\mathrm{tr}\,(A_iA^*_iA_jA^*_j + A_iA_jA^*_jA^*_i), \tag{7.100}$$

where $A_i := A_{Je_i},\ A^*_i := A^*_{Je_i}$, and $\{e_1,\ldots,e_n\}$ is a local orthonormal frame on M. To obtain (7.98) from (7.97) and (7.100) from (7.99), respectively, we need (7.78) and easy long calculation.

Lemma 22 *Let* (M,∇,g) *be an n-dimensional totally real submanifold in* $\widetilde{M}(\widetilde{c})$. *If* $H = H^* = 0$ *and* $D_X(J\zeta) = JD^*_X\zeta$ *for any* $X \in \Gamma(TM),\ \zeta\in\Gamma(N)$, *then*

$$\sigma_1 \geq \frac{2}{n(n-1)}\left(\sum_{i=1}^{n}\mathrm{tr}\,(A_iA^*_i)\right)^2 \quad \textit{and} \quad \sigma_2 \geq \frac{1}{n}\left(\sum_{i=1}^{n}\mathrm{tr}\,(A_iA^*_i)\right)^2.$$

In the first inequality, the equality holds if and only if $S(X,Y)Z = \frac{\rho}{n(n-1)}\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}$ *for any* $X,Y,Z\in\Gamma(TM)$. *In the second inequality, the equality holds if and only if* $L(X,Y) = \frac{\rho}{n}\langle X,Y\rangle$ *for any* $X,Y\in\Gamma(TM)$.

Proof Since $D_X(J\zeta) = JD^*_X\zeta$ for any $X \in \Gamma(TM)$, $\zeta \in \Gamma(N)$, by Lemmas 20 and 21, we have (7.87) and (7.87)*. Using them with (7.15) and (7.32), we have

$$S(X,Y)Z = \frac{\widetilde{c}}{4}\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} + \frac{1}{2}([A_{JX}, A^*_{JY}]Z - [A_{JY}, A^*_{JX}]Z) \tag{7.101}$$

for any $X, Y, Z \in \Gamma(TM)$. The condition $H = H^* = 0$ implies that $\sum_{i=1}^n A^*_i e_i = \sum_{i=1}^n A_i e_i = 0$. Therefore, L and ρ are given by

$$L(X,Y) = \frac{1}{4}(n-1)\widetilde{c}\langle X,Y\rangle - \frac{1}{2}\mathrm{tr}\,(A_{JX}A^*_{JY} + A_{JY}A^*_{JX}), \tag{7.102}$$

$$\rho = \frac{1}{4}n(n-1)\widetilde{c} - \sum_{i=1}^n \mathrm{tr}\,(A_iA^*_i). \tag{7.103}$$

From (7.101), (7.102), and (7.103), we have

$$\begin{aligned}
\|S\|^2 &= \frac{1}{8}n(n-1)\widetilde{c}^2 - \widetilde{c}\sum_{i=1}^n \mathrm{tr}\,(A_iA^*_i) + \frac{1}{4}\sum_{k,j,i=1}^n \|([A_k, A^*_j] - [A_j, A^*_k])e_i\|^2 \\
&= \frac{1}{8}n(n-1)\widetilde{c}^2 - \widetilde{c}\sum_{i=1}^n \mathrm{tr}\,(A_iA^*_i) + \sigma_1 \\
&= \frac{2}{n(n-1)}\rho^2 + \sigma_1 - \frac{2}{n(n-1)}\left(\sum_{i=1}^n \mathrm{tr}\,(A_iA^*_i)\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
\|L\|^2 &= \frac{1}{16}n(n-1)^2\widetilde{c}^2 - \frac{1}{2}(n-1)\widetilde{c}\sum_{i=1}^n \mathrm{tr}\,(A_iA^*_i) \\
&\quad + \frac{1}{4}\sum_{j,i=1}^n\{\mathrm{tr}\,(A_jA^*_i) + \mathrm{tr}\,(A_iA^*_j)\}^2 \\
&= \frac{1}{16}n(n-1)^2\widetilde{c}^2 - \frac{1}{2}(n-1)\widetilde{c}\sum_{i=1}^n \mathrm{tr}\,(A_iA^*_i) + \sigma_2 \\
&= \frac{1}{n}\rho^2 + \sigma_2 - \frac{1}{n}\left(\sum_{i=1}^n \mathrm{tr}\,(A_iA^*_i)\right)^2.
\end{aligned}$$

Using Remark 2, we have the conclusion. ■

Theorem 7.8 *Let $\widetilde{M}(\widetilde{c})$ be a holomorphic statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, J)$ of constant holomorphic sectional curvature $\widetilde{c}$, and (M, ∇, g) an n-dimensional totally real submanifold in $\widetilde{M}(\widetilde{c})$. Suppose that*

(1) $D_X(J\zeta) = JD^*_X\zeta$ *for any* $X \in \Gamma(TM)$, $\zeta \in \Gamma(N)$,
(2) $\overline{S}(X, Y) \cdot B = 0$ *for any* $X, Y \in \Gamma(TM)$ *and*
(3) $H = H^* = 0$.

Then

$$\rho \sum_{i=1}^{n} \mathrm{tr}\,(A_i A_i^*) \geq 0. \tag{7.104}$$

Moreover, the left-hand side of (7.104) *vanishes if and only if*

$$S(X, Y)Z = \frac{\rho}{n(n-1)}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}$$

holds for any $X, Y, Z \in \Gamma(TM)$. *Especially,* $\rho = 0$ *if and only if* $S = 0$.

Proof By the condition (1) and Lemma 20, we have (7.86) and (7.87). Using these with the condition (2), we calculate

$$\begin{aligned} 0 &= \langle -J(\overline{S}(X, Y) \cdot B)(Z, W_1), W_2\rangle \\ &= \langle S(X, Y)A^*_{JZ}W_1 - A^*_{JW_1}S(X, Y)Z - A^*_{JZ}S(X, Y)W_1, W_2\rangle \end{aligned}$$

and putting $W_1 = A_{JU}V$ and $W_2 = W$, by (7.101) we obtain

$$\begin{aligned} 0 = \frac{1}{4}\widetilde{c}\,\big\{&\langle Y, A^*_{JZ}A_{JU}V\rangle\langle X, W\rangle - \langle X, A^*_{JZ}A_{JU}V\rangle\langle Y, W\rangle \\ &-\langle Y, A_{JU}V\rangle\langle A^*_{JZ}X, W\rangle + \langle X, A_{JU}V\rangle\langle A^*_{JZ}Y, W\rangle \\ &-\langle Y, Z\rangle\langle A^*_{JX}A_{JU}V, W\rangle + \langle X, Z\rangle\langle A^*_{JY}A_{JU}V, W\rangle\big\} \\ +\frac{1}{2}\big\{&\langle([A_{JX}, A^*_{JY}] - [A_{JY}, A^*_{JX}])A^*_{JZ}A_{JU}V, W\rangle \\ &-\langle([A_{JX}, A^*_{JY}] - [A_{JY}, A^*_{JX}])A_{JU}V, A^*_{JZ}W\rangle \\ &-\langle([A_{JX}, A^*_{JY}] - [A_{JY}, A^*_{JX}])Z, A^*_{JW}A_{JU}V\rangle\big\} \end{aligned}$$

for any $U, V, W, X, Y, Z \in \Gamma(TM)$. Putting $X = W = e_i$, $Y = U = e_j$, $Z = V = e_k$ and summing up i, j, k from 1 to n in this equation, we have

$$0 = \frac{1}{4}(n+1)\widetilde{c}\sum_{i=1}^{n} \mathrm{tr}\,(A_i A_i^*) - (\sigma_1 + \sigma_2)\,.$$

We remark that the condition (3) implies (7.103). Therefore, using it with the above equation, we get

$$\frac{n+1}{n(n-1)}\rho\sum_{i=1}^{n} \mathrm{tr}\,(A_i A_i^*) = \frac{1}{4}(n+1)\widetilde{c}\sum_{i=1}^{n} \mathrm{tr}\,(A_i A_i^*) - \frac{n+1}{n(n-1)}\left(\sum_{i=1}^{n} \mathrm{tr}\,(A_i A_i^*)\right)^2$$

$$= \left\{ \sigma_1 - \frac{2}{n(n-1)} \left(\sum_{i=1}^{n} \mathrm{tr}\,(A_i A_i^*) \right)^2 \right\} + \left\{ \sigma_2 - \frac{1}{n} \left(\sum_{i=1}^{n} \mathrm{tr}\,(A_i A_i^*) \right)^2 \right\},$$

from which Lemma 22 implies the conclusion. ■

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Chapter 8
CR-Submanifolds in Complex and Sasakian Space Forms

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8.1 Introduction

In Riemannian geometry the manifolds endowed with certain endomorphisms of their tangent bundles play an important role. Among these, the most important are the almost complex structures (on even-dimensional manifolds) and almost contact structures (on odd-dimensional manifolds). In particular the Kähler manifolds and the Sasakian manifolds, respectively, are the most studied such manifolds, because they have the most interesting properties and applications.

In order to have the highest degree of homogeneity (i.e., the group of isometries has the maximum dimension), the spaces of constant sectional curvatures are the most investigated. It is known that a Kähler manifold with constant sectional curvature is flat. For this reason the notion of complex space form (a Kähler manifold with constant holomorphic sectional curvature) was introduced. Analogously, the Sasakian space forms were defined.

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On the other hand, starting from the classical theory of curves and surfaces in Euclidean spaces, the theory of submanifolds is an important field of research in Riemannian geometry.

There are certain important specific classes of submanifolds in Kähler manifolds and Sasakian manifolds, respectively. For example, complex and Lagrangian submanifolds in Kähler manifolds and invariant and Legendrian submanifolds in Sasakian manifolds. A notion which generalizes the above-mentioned submanifolds is the class of *CR*-submanifolds in a Kähler manifold and, respectively, the contact *CR*-submanifolds in Sasakian manifolds.

The purpose of this chapter is to present recent topics of research and recent results on *CR*-submanifolds in complex space forms and contact *CR*-submanifolds in Sasakian space forms, respectively. First, we recall basic results on Kähler manifolds and Sasakian manifolds and their submanifolds. In particular, the *CR*-submanifolds in Kähler manifolds and contact *CR*-submanifolds in Sasakian manifold are defined. The next section is devoted to *CR*-submanifolds in complex space forms. The Ricci curvature and the k-Ricci curvature of such submanifolds are estimated in terms of the squared mean curvature. The generalized Wintgen inequality conjecture, also known as the DDVV conjecture, was recently solved in its general settings, i.e., for submanifolds in Riemannian space forms of arbitrary dimensions and codimensions. We mention some contributions of the present authors in this respect. A Wintgen-type inequality for totally real surfaces in complex space forms is proved. The equality case holds identically if and only if the ellipse of curvature is a circle at every point of the surface. An interesting example of a totally real surface in $\mathbf{C}^2$ satisfying the equality case identically is given. Afterwards, a generalized Wintgen inequality for Legendrian submanifolds in complex space forms was established. A Wintgen-type inequality for *CR*-submanifolds in complex space forms is stated. The warped product manifolds play an important role in Riemannian geometry as well as in physics. Recently B.Y. Chen investigated warped product submanifolds in Riemannian space forms. Also he introduced and studied the notion of a *CR*-warped product in a Kähler manifold. After recalling some important results of B.Y. Chen, we present some contributions of one of the present authors. A geometric inequality for warped product manifolds satisfying a certain condition (in particular *CR*-warped product submanifolds) in complex space forms is proved, the equality case is characterized and examples of the equality case are given. Also some obstructions to the minimality of warped product *CR*-submanifolds in complex space forms are derived. Finally, the scalar curvature of such submanifolds is estimated and classifications of submanifolds in complex space forms satisfying the equality case, identically, are given. The last section deals with certain results on contact *CR*-submanifolds in Sasakian space forms. We state geometric inequalities for the Ricci curvature and k-Ricci curvature of contact *CR*-submanifolds in Sasakian space forms. Recently, one of the present authors proved a generalized Wintgen inequality for C-totally real submanifolds in Sasakian space forms. We extend this result to contact *CR*-submanifolds in Sasakian space forms. The last subsection contains certain results on contact *CR*-warped products in Sasakian space forms.

8.2 Submanifolds in Kähler and Sasakian Manifolds

8.2.1 Submanifolds in Kähler Manifolds

Let $\tilde{M}$ be a complex manifold of dimension m and J its standard almost complex structure. A *Hermitian metric* on $\tilde{M}$ is a Riemannian metric g invariant with respect to J, i.e.,

$$g(JX, JY) = g(X, Y),\ \forall X, Y \in \Gamma(T\tilde{M}).$$

The pairing $(\tilde{M}, g)$ is called a *Hermitian* manifold.

Any complex manifold admits a Hermitian metric.

A Hermitian metric g on a complex manifold $\tilde{M}$ defines a nondegenerate 2-form $\omega(X, Y) = g(JX, Y)$, $X, Y \in \Gamma(T\tilde{M})$, which is called the *fundamental* 2-form. Clearly, $\omega(JX, JY) = \omega(X, Y)$.

Definition A Hermitian manifold is called a *Kähler* manifold if the fundamental 2-form ω is closed.

Necessary and sufficient conditions for a Hermitian manifold to be a Kähler manifold are given by the following:

Theorem 8.2.1.1 *Let $(\tilde{M}, g)$ be an m-dimensional Hermitian manifold and $\tilde{\nabla}$ the Levi-Civita connection associated to g. The following statements are equivalent to each others:*

(i) *M is a Kähler manifold;*

(ii) *the standard almost complex structure J on $\tilde{M}$ is parallel with respect to $\tilde{\nabla}$, i.e., $\tilde{\nabla} J = 0$;*

(iii) *For any $z_0 \in \tilde{M}$, there exists a holomorphic coordinate system in a neighborhood of z_0 such that*

$$g = (\delta_{kj} + h_{kj}) dz^k d\bar{z}^j,$$

where $h_{kj}(z_0) = \frac{\partial h_{kj}}{\partial z^l}(z_0) = 0$, for any $k, j, l = 1, \dots, m$;

(iv) *locally, there exists a real differentiable function F such that the fundamental 2-form is given by $\omega = i\partial\bar{\partial}F$, where the exterior differentiation d is decomposed in $d\alpha = \partial\alpha + \bar{\partial}\alpha$.*

Remark If locally the fundamental 2-form is given by

$$\omega = i\partial\bar{\partial}F = i\frac{\partial^2 F}{\partial z^j \partial \bar{z}^k} dz^j \wedge d\bar{z}^k,$$

then the Hermitian metric can be expressed by

$$g = \frac{\partial^2 F}{\partial z^j \partial \bar{z}^k} dz^j d\bar{z}^k.$$

Examples of Kähler Manifolds [35]

1. $\mathbf{C}^n$ with the Euclidean metric $g = \sum_{k=1}^n dz^k d\bar{z}^k$.
The fundamental 2-form is given by $\omega = i \sum_{k=1}^n dz^k \wedge d\bar{z}^k$.

2. The *complex torus* $T^n = \mathbf{C}^n/_G$ with the Hermitian structure induced by the Euclidean metric of $\mathbf{C}^n$.

3. The *complex projective space* $P^n(\mathbf{C})$.
Let $(z_j^0, \dots, z_j^{j-1}, z_j^{j+1}, \dots, z_j^n)$ be the local coordinates on $U_j \subset P^n(\mathbf{C})$. One defines $f_j(z) = \sum_{k=0}^n |z_j^k|^2$, where $z_j^j = 1$.

On $U_j \cap U_k$, we have $f_k(z) = |z_k^j|^2 f_j(z)$.

The 2-form ω defined on U_j by $\omega = i\partial\bar{\partial} \ln f_j$, is globally defined on $P^n(\mathbf{C})$.

For $j = 0$, $f_0 = 1 + \sum_{k=1}^n |z^k|^2$ and

$$\omega = i \frac{(1 + z^s \bar{z}^s) dz^k \wedge d\bar{z}^k - \bar{z}^k dz^k \wedge z^j d\bar{z}^j}{(1 + z^s \bar{z}^s)^2}.$$

The Hermitian metric g has the coefficients:

$$g_{j\bar{k}} = \frac{(1 + z^s \bar{z}^s)\delta_{jk} - z^k \bar{z}^j}{(1 + z^s \bar{z}^s)^2}.$$

This metric is called the *Fubini–Study metric*.

4. The *complex Grassmann manifold* $G_p(\mathbf{C}^{p+q})$.

We will define a Kählerian metric on $G_p(\mathbf{C}^{p+q})$. Let $M^*(p+q, p, \mathbf{C})$ be the space of $(p+q, p)$-matrices with complex coefficients, of rank p. The canonical projection $M^*(p+q, p, \mathbf{C}) \to G_p(\mathbf{C}^{p+q})$ defines a principal fiber bundle with structural group $GL(p, \mathbf{C})$.

An element $Z \in M^*(p+q, p, \mathbf{C})$ can be written as $Z =^t (Z_0, Z_1)$, where Z_0 is a quadratic matrix of order p and Z_1 a matrix of order (q, p).

We consider the open subset of $G_p(\mathbf{C}^{p+q})$ defined by $\det Z_0 \neq 0$ and denoted by $T = Z_1 Z_0^{-1}$. A Kählerian structure is defined by the 2-form $\omega = i\partial\bar{\partial} \log \det(I +^t \bar{T} T)$.

The associated Kählerian metric is called the *generalized Fubini–Study metric* on $G_p(\mathbf{C}^{p+q})$.

5. Let $D^n = \mathrm{Int}\, S^{2n-1}$ be the unit disk in $\mathbf{C}^n$, i.e., $D^n = \left\{ z \in \mathbf{C}^n \mid \sum_{j=1}^n |z^j|^2 < 1 \right\}$.

We put $\omega = -i\partial\bar{\partial} \ln(1 - \sum_{j=1}^n |z^j|^2)$. The associated Hermitian metric has the coefficients

$$g_{j\bar{k}} = \frac{(1 - z^s \bar{z}^s)\delta_{jk} + \bar{z}^j z^k}{(1 - z^s \bar{z}^s)^2}.$$

This metric is called the *Bergman metric*.

6. Any orientable surface is a Kähler manifold.

We state obstructions to the existence of Kählerian metrics on a compact complex manifold (see [25, 35, 38]).

Theorem 8.2.1.2 *On a compact Kähler manifold the Betti numbers of even order are nonzero.*

As an application, we see that the Calabi manifolds $S^{2m+1} \times S^{2n+1}$ do not admit any Kähler metric if $(m, n) \neq (0, 0)$. In particular, Hopf manifolds are not Kähler manifolds.

Theorem 8.2.1.3 *On a compact Kähler manifold the Betti numbers of odd order are even.*

We can construct an almost Kähler manifold which does not admit any Kählerian metric.

Let $H \subset \mathrm{GL}(3, \mathbb{R})$ be the *Heisenberg group*, i.e.,

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x, y, z \in \mathbb{R} \right\}.$$

Let Γ be the maximal discrete subgroup of H, defined as the set of matrices of H with integer components. Because Γ is closed, it follows that $H/_\Gamma$ is a homogeneous space.

Let $S^1 = \{e^{2\pi i t} | t \in \mathbb{R}\}$ be the unit circle.

We consider the compact homogeneous space $M = H/_\Gamma \times S^1$.

A basis in the Lie algebra of H is given by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z}.$$

These fields are invariant under the action of Γ, inducing the fields e_1, e_2, e_3 linearly independent on H/Γ. We denote by e_4 the standard vector field $\frac{d}{dt}$ on S^1.

The dual 1-forms of the vector fields X_1, X_2, X_3 are

$$\theta_1 = dx,\ \theta_2 = dy,\ \theta_3 = dz - x dy.$$

They together with dt induce 1-forms $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ linearly independent on M.

We define the 2-form ω of maximum rank on M, by $\omega = \alpha_4 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3$. It is closed.

We consider the invariant Riemannian metric $\tilde{g}$ on $H \times S^1$, defined by $\tilde{g} = dx^2 + dy^2 + (dz - xdy)^2 + dt^2$. The metric g induced on M is $g = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2$.

An almost complex structure J on M is defined by $Je_j = (-1)^j e_{5-j}$, $j \in \{1, 2, 3, 4\}$.

Then ω is the fundamental 2-form associated to the almost Hermitian manifold (M, J, g). Since ω is closed, M is an almost Kähler manifold.

But M does not admit Kählerian metrics, because the following result holds good.

Proposition 8.2.1.4 ([38]) *The first Betti number* $\beta_1(M) = 3$.

We recall the geometric interpretation of the *sectional curvature* determinated by the linearly independent vectors $u, v \in T_p\tilde{M}$, $p \in \tilde{M}$. It represents the Gauss curvature of the surface

$$(\lambda, \mu) \mapsto \exp_p(\lambda u + \mu v).$$

If a Kähler manifold of dimension $n > 1$ has constant sectional curvature, then it is flat (see [49]). It follows that the notion of constant sectional curvature for a Kähler manifold is not significant. One introduces the notion of *holomorphic sectional curvature* and one states a Schur-like theorem.

Let $\tilde{M}$ be a Kähler manifold and J its standard almost complex structure.

The sectional curvature of $\tilde{M}$ in direction of an invariant 2-plane section by J is called the *holomorphic sectional curvature* of $\tilde{M}$.

For the 2-plane section π invariant by J, we take an orthonormal basis $\{X, JX\}$, with unit X. Then the holomorphic sectional curvature is given by $K(\pi) = \tilde{R}(X, JX, X, JX)$.

The curvature tensor of a Kähler manifold satisfies:

(i) $\tilde{R}(X, Y, Z, W) = -\tilde{R}(Y, X, Z, W) = -\tilde{R}(X, Y, W, Z)$;
(ii) $\tilde{R}(X, Y, Z, W) = \tilde{R}(Z, W, X, Y)$;
(iii) $\tilde{R}(X, Y, Z, W) + \tilde{R}(X, Z, W, Y) + \tilde{R}(X, W, Y, Z) = 0$;
(iv) $\tilde{R}(JX, JY, Z, W) = \tilde{R}(X, Y, JZ, JW) = \tilde{R}(X, Y, Z, W)$,
for any vector fields X, Y, Z, W on $\tilde{M}$.

Definition Let $\tilde{M}$ be a Kähler manifold. If the function holomorphic sectional curvature K is constant for all 2-plane sections π of $T_p\tilde{M}$ invariant by J for any $p \in \tilde{M}$, then $\tilde{M}$ is called a *space with constant holomorphic sectional curvature* (or *complex space form*).

Using the above properties, the following **Schur-like theorem** can be proved.

Theorem ([49]) *Let $\tilde{M}$ be a connected Kähler manifold of complex dimension $n \geq 2$. If the holomorphic sectional curvature depends only on $p \in \tilde{M}$ (and does not depend of the 2-plane sections π of $T_p\tilde{M}$ invariant by J), then $\tilde{M}$ is a complex space form.*

It follows that the curvature tensor of a complex space form of constant holomorphic sectional curvature $4c$, denoted by $\tilde{M}(4c)$, has the expression

$$\begin{aligned}\tilde{R}(X, Y, Z, W) = c[&g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\\ &+ g(X, JZ)g(Y, JW) - g(X, JW)g(Y, JZ)\\ &+ 2g(X, JY)g(Z, JW)].\end{aligned}$$

Recall that the Riemannian manifold (M, g) is an *Einstein manifold* if the Ricci tensor S is proportional to the Riemannian metric g, i.e., $S = \lambda g$, where λ is a real number.

Corollary 8.2.1.5 *Each complex space form is an Einstein manifold.*

Examples of Complex Space Forms

1. $\mathbf{C}^n$ with the Euclidean metric is a flat complex space form.

2. $P^n(\mathbf{C})$ with the Fubini–Study metric has holomorphic sectional curvature equal to 4.

3. D^n with the Bergman metric has holomorphic sectional curvature equal to -4.

Conversely, the following result holds good [25].

Theorem 8.2.1.6 *Let $\tilde{M}$ be a connected, simply connected, and complete complex space form. Then $\tilde{M}$ is isometric either to $\mathbf{C}^n$, $P^n(\mathbf{C})$ or D^n.*

Let $(\tilde{M}, J, g)$ be an m-dimensional Kähler manifold and M an n-dimensional submanifold of $\tilde{M}$. The induced Riemannian metric on M is also denoted by g. We denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connections on $\tilde{M}$ and M, respectively. We recall the fundamental formulae and equations for a submanifold.

Let h be the second fundamental form of the submanifold M. Then the *Gauss formula* is written as

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

for any $X, Y \in \Gamma(TM)$.

Denoting by $\nabla^{\perp}$ the connection in the normal bundle and by A the shape operator, one has the *Weingarten formula*

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi,$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$.

Let $\tilde{R}$, R, and $R^\perp$ be the curvature tensors with respect to $\tilde{\nabla}$, ∇, and $\nabla^\perp$, respectively.

For any $X, Y, Z, W \in \Gamma(TM)$, the *Gauss equation* is expressed by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, Z), h(Y, W)) + g(h(X, W), h(Y, Z)).$$

We put

$$(\nabla_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(X, \nabla_Y Z);$$

then the normal component of $\tilde{R}(X, Y)Z$ is given by

$$(\widetilde{R}(X, Y)Z)^\perp = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z).$$

The above relation represents the *Codazzi equation*.

Using the Weingarten formula, one obtains the *Ricci equation*.

$$\begin{aligned}\widetilde{R}(X, Y, \xi, \eta) &= R^\perp(X, Y, \xi, \eta) - g(A\eta A_\xi X, Y) + g(A_\xi A\eta X, Y)\\ &= R^\perp(X, Y, \xi, \eta) + g([A_\xi, A\eta]X, Y),\end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^\perp M)$.

If the second fundamental form h vanishes identically, M is a *totally geodesic* submanifold.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM, $p \in M$, and H be the *mean curvature vector*, i.e.,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

The submanifold M is said to be *minimal* if $H(p) = 0, \forall p \in M$.

There are no compact minimal submanifolds of $\mathbb{R}^m$.

For a normal section V on M, if A_V is everywhere proportional to the identity transformation I, i.e., $A_V = aI$, for some function a, then V is called an *umbilical section* on M, or M is said to be *umbilical with respect to* V. If the submanifold M is umbilical with respect to every local normal section of M, then M is said to be *totally umbilical*.

An equivalent definition is the following: M is *totally umbilical* if $h(X, Y) = g(X, Y)H$, for any vector fields X, Y tangent to M.

Any submanifold M which is both minimal and totally umbilical is totally geodesic.

If the second fundamental form and the mean curvature of M in $\widetilde{M}$ satisfy $g(h(X, Y), H) = fg(X, Y)$ for some function f on M, then M is called *pseudo-umbilical*.

The submanifold M is a *parallel* submanifold if the second fundamental form h is parallel, that is $\nabla h = 0$, identically.

We denote also by

$$h_{ij}^r = g(h(e_i, e_j), e_r),\ i, j = 1, \ldots, n;\ r = n + 1, \ldots, 2m,$$

where $\{e_{n+1}, \ldots, e_{2m}\}$ is an orthonormal basis of $T_p^{\perp}M$, the components of the second fundamental form, and by

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

According to the behavior of the tangent spaces of a submanifold M under the action of the almost complex structure J of the ambient space $\tilde{M}$, we distinguish two special classes of submanifolds:

(i) *Complex submanifolds*, if $J(T_pM) = T_pM, \forall p \in M$.
(ii) *Totally real submanifolds*, if $J(T_pM) \subset T_p^{\perp}M, \forall p \in M$.

Any complex submanifold of a Kähler manifold is a Kähler manifold and a minimal submanifold.

If the real dimension of the totally real submanifold M is equal to the complex dimension of the Kähler manifold $\tilde{M}$, then M is called a *Lagrangian submanifold*. In other words, a *Lagrangian submanifold* is a totally real submanifold of maximum dimension.

The notion of a generic submanifold of a Kähler manifold was introduced by B.Y. Chen [4]. It is a natural generalization of both complex and totally real submanifolds.

Let $\tilde{M}$ be a Kähler manifold with complex structure J and Kähler metric g. Let M be a real submanifold of $\widetilde{M}$. For each point $p \in M$, denote by $\mathcal{D}_p$ the maximal holomorphic subspace of the tangent space T_pM, i.e., $\mathcal{D}_p = T_pM \cap J(T_pM)$.

Definition ([4]) If the dimension of $\mathcal{D}_p$ is constant along M and $\mathcal{D}_p$ defines a differentiable distribution $\mathcal{D}$ over M, then M is called a *generic submanifold* of $\tilde{M}$.

The distribution $\mathcal{D}$ is called the *holomorphic distribution* of the generic submanifold M.

For each point $p \in M$, denote by $\mathcal{D}_p^{\perp}$ the orthogonal complementary subspace of $\mathcal{D}_p$ in T_pM. On a generic submanifold, $\mathcal{D}_p^{\perp}(p \in M)$ define a differentiable distribution $\mathcal{D}^{\perp}$ over M, called the *purely real distribution*. M is said to be a *proper generic submanifold* if both $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are nontrivial.

Remark It is known (see [4]) that every submanifold of $\tilde{M}$ is the closure of the union of some open generic submanifolds of $\tilde{M}$.

Let M be a generic submanifold of the Kähler manifold $\tilde{M}$. For a vector field X tangent to M, we put $JX = PX + FX$, where PX and FX are the tangential and normal components of JX, respectively. Then P is an endomorphism of TM and F is a normal bundle-valued 1-form on TM.

Put $\alpha = \dim_{\mathbf{C}} \mathcal{D}$, $\beta = \dim_{\mathbb{R}} \mathcal{D}^{\perp}$. Then we have $\dim M = 2\alpha + \beta$.

A *CR-submanifold* is a particular case of a generic submanifold.

Definition A generic submanifold M of a Kähler manifold $\tilde{M}$ is said to be a *CR-submanifold* if its purely real distribution $\mathcal{D}^{\perp}$ is totally real, i.e., $J(\mathcal{D}^{\perp}_p) \subset T^{\perp}_p M,\ p \in\ M.$

The *CR*-submanifolds were studied by B.Y. Chen [5], A. Bejancu [1], K. Yano and M. Kon [48], etc. Both complex and totally real submanifolds are improper *CR*-submanifolds. It is easily seen that a real hypersurface of a Kähler manifold is a proper *CR*-submanifold. The first main result on *CR*-submanifolds was obtained by B.Y. Chen.

Theorem 8.2.1.7 ([5]) *The totally real distribution $\mathcal{D}^{\perp}$ on a CR-submanifold M of a Kähler manifold $\tilde{M}$ is completely integrable.*

For a differentiable distribution $\mathcal{H}$ on a Riemannian manifold M, we set $h^0(X, Y) = (\nabla_X Y)^{\perp}$, for any vector fields X, Y in $\mathcal{H}$, where $(\nabla_X Y)^{\perp}$ denotes the component of $\nabla_X Y$ in the complementary orthogonal subbundle $\mathcal{H}^{\perp}$ to $\mathcal{H}$ in TM. Let $\{e_1, \ldots, e_r\}$ be an orthonormal basis of $\mathcal{H}$, $r = \dim \mathcal{H}$. If we put $H^0 = \frac{1}{r}\sum_{i=1}^{r} h^0(e_i, e_i)$, then H^0 is a well-defined $\mathcal{H}^{\perp}$-valued vector field on M. It is called *the mean curvature vector field of the distribution* $\mathcal{H}$. A distribution $\mathcal{H}$ on a Riemannian manifold M is said to be *minimal* if its mean curvature vector vanishes identically. For the holomorphic distribution $\mathcal{D}$ on a *CR*-submanifold, the following general result was proved (see [4]).

Theorem 8.2.1.8 *The holomorphic distribution $\mathcal{D}$ on a CR-submanifold M of a Kähler manifold $\tilde{M}$ is a minimal distribution.*

The integrability of the holomorphic distribution $\mathcal{D}$ is characterized as follows:

Theorem 8.2.1.9 *Let M be a CR-submanifold of a Kähler manifold $\tilde{M}$. Then the holomorphic distribution $\mathcal{D}$ is completely integrable if and only if*

$$h(X, JY) = h(JX, Y),\ \forall X, Y \in \Gamma(\mathcal{D}).$$

In contrast with the integrability of $\mathcal{D}^{\perp}$ and the minimality of $\mathcal{D}$ on a *CR*-submanifold, we have the following:

Theorem 8.2.1.10 ([4]) *Let M be a compact CR-submanifold of a Kaehler manifold $\tilde{M}$. If its de Rham cohomology group $H^{2k}(M;\mathbb{R})=0$, for some $k \leq \dim_{\mathbf{C}} \mathcal{D}$, then either $\mathcal{D}$ is not integrable or $\mathcal{D}^{\perp}$ is not minimal.*

In order to justify the denomination of a *CR*-submanifold, D.E. Blair and B.Y. Chen proved the following result (in [3]).

Theorem 8.2.1.11 *Each CR-submanifold of a Kähler manifold is a Cauchy–Riemann (abbreviated as CR-) manifold in the sense of S. Greenfield.*

We recall the definition of a *CR-structure* on a differentiable manifold following S. Greenfield (see [1]). Let M be a differentiable manifold and $T_{\mathbf{C}}M = TM \otimes_{\mathbb{R}} \mathbf{C}$ its complexified tangent bundle. A subbundle of $T_{\mathbf{C}}M$ is called a *complex distribution* on M.

Definition A differentiable manifold is said to have a *CR-structure* if it admits a complex distribution $\mathcal{B}$ satisfying the following conditions:

(i) $\mathcal{B} \cap \bar{\mathcal{B}} = \{0\}$, where $\bar{\mathcal{B}}$ means the complex conjugated distribution of $\mathcal{B}$;

(ii) $\mathcal{B}$ is involutive, i.e., for any $A, B \in \Gamma(\mathcal{B})$, their Lie bracket $[A, B]$ belongs to $\Gamma(\mathcal{B})$.

Each *CR*-submanifold of a complex manifold $\widetilde{M}$ is endowed with a *CR*-structure, defined by $\mathcal{B} = \{X - iJX | X \in \Gamma(\mathcal{D})\}$.

8.2.2 Sasakian Manifolds and Their Submanifolds

Roughly speaking, a Sasakian manifold is the odd-dimensional correspondent of a Kähler manifold. A $(2m+1)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ is said to be a *Sasakian manifold* if it admits an endomorphism ϕ of its tangent bundle $T\widetilde{M}$, a vector field ξ, and a 1-form η, satisfying

$$\begin{cases} \phi^2 = -\mathrm{Id} + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \\ (\tilde{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \ \tilde{\nabla}_X \xi = \phi X, \end{cases}$$

for any vector fields X, Y on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

A plane section π in $T_p\tilde{M}$ is called a *ϕ-section* if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ-*sectional curvature*. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\widetilde{M}(c)$.

The curvature tensor of $\widetilde{R}$ of a Sasakian space form $\widetilde{M}(c)$ is given by [49]

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ & + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ & + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\}, \end{aligned} \tag{8.2.2.1}$$

for any tangent vector fields X, Y, Z on $\widetilde{M}(c)$.

As examples of Sasakian space forms we mention $\mathbb{R}^{2m+1}$ and S^{2m+1}, with standard Sasakian structures (see [2, 49]).

Let M be an n-dimensional submanifold in a Sasakian manifold $\widetilde{M}$. We denote by ∇ and h the Riemannian connection of M and the second fundamental form, respectively. Let R be the Riemann curvature tensor of M.

Then the equation of Gauss is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,Z),h(Y,W)) + g(h(X,W),h(Y,Z)), \tag{8.2.2.2}$$

for any vectors X, Y, Z, W tangent to M.

By analogy with the submanifolds of a Kähler manifold, we distinguish special classes of submanifolds of Sasakian manifolds.

A submanifold M normal to ξ in a Sasakian manifold $\widetilde{M}$ is said to be a *C-totally real* submanifold. In this case, it follows that ϕ maps any tangent space of M into the normal space, that is, $\phi(T_pM) \subset T_p^{\perp}M$, for every $p \in M$.

In particular, if $\dim \widetilde{M} = 2\dim M + 1$, then M is called a *Legendrian* submanifold.

For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. We mention the following:

(i) A submanifold M tangent to ξ is called an *invariant* submanifold if ϕ preserves any tangent space of M, that is, $\phi(T_pM) \subset T_pM$, for every $p \in M$.

(ii) A submanifold M tangent to ξ is called an *anti-invariant* submanifold if ϕ maps any tangent space of M into the normal space, that is, $\phi(T_pM) \subset T_p^{\perp}M$, for every $p \in M$.

An important class of submanifolds tangent to ξ in Sasakian manifolds are the contact *CR*-submanifolds. Both invariant and anti-invariant submanifolds are contact *CR*-submanifolds.

Definition A submanifold M tangent to ξ is called a *contact CR-submanifold* if it admits an invariant differentiable distribution $\mathcal{D}$ with respect to ϕ whose orthogonal complementary orthogonal distribution $\mathcal{D}^{\perp}$ is anti-invariant, that is, $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$, with $\phi(\mathcal{D}_p) \subset \mathcal{D}_p$ and $\phi(\mathcal{D}_p^{\perp}) \subset T_p^{\perp}M$, for every $p \in M$.

If, in particular, $\phi(\mathcal{D}_p^{\perp}) = T_p^{\perp}M$, for every $p \in M$, then M is called an *anti-ϕ-holomorphic* contact *CR*-submanifold.

The integrability of structure distributions is characterized by the following:

Theorem 8.2.2.1 ([49]) *Let M be an $(n+1)$-dimensional contact CR-submanifold of a $(2m+1)$-dimensional Sasakian manifold $\widetilde{M}$. Then the distribution $\mathcal{D}^{\perp}$ is always completely integrable and its maximal integral submanifold is either a C-totally real submanifolds of $\widetilde{M}$ or an anti-invariant submanifold tangent to ξ.*

Theorem 8.2.2.2 ([49]) *Let M be an $(n+1)$-dimensional contact CR-submanifold of a $(2m+1)$-dimensional Sasakian manifold $\widetilde{M}$. The distribution $\mathcal{D}$ is completely integrable if and only if its second fundamental form h satisfies*

$$h(PX, Y) = h(X, PY),$$

for any vector fields $X, Y \in \Gamma(\mathcal{D})$, where PX is the tangential component of ϕX. Moreover, the maximal integral submanifold of $\mathcal{D}$ is an invariant submanifold tangent to ξ.

Next we provide an **example** ([49]) of a contact *CR*-submanifold.

Let S^{2m+1} be a $(2m+1)$-dimensional unit sphere with standard Sasakian structure. We denote by $S^k(r)$ a k-dimensional sphere with radius r. We consider the following immersion:

$$M = S^{k_1}(r_1) \times \cdots \times S^{k_q}(r_q) \to S^{n+q} \subset S^{2m+1}, \ n+1 = \sum_{i=1}^{q} k_i,$$

where $k_1, \ldots, k_q$ are odd numbers, $r_1^2 + \cdots + r_q^2 = 1$ and $n+q$ is also odd.

Then M is an anti-ϕ-holomorphic contact *CR*-submanifold of S^{n+q} and a contact *CR*-submanifold of S^{2m+1}.

If $r_i = \left(\frac{k_i}{n+1}\right)^{1/2}$, for any $i \in \{1, \ldots, q\}$, then M is a minimal submanifold both in S^{n+q} and S^{2m+1}.

Moreover, if all $k_i = 1$, then M is a Legendrian submanifold in S^{n+q} and a C-totally real submanifold in S^{2m+1}.

8.3 CR-Submanifolds in Complex Space Forms

8.3.1 Ricci and k-Ricci Curvatures of CR-Submanifolds in Complex Space Forms

Let M be an n-dimensional submanifold of an m-dimensional complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$.

For any $p \in M$ and any unit vector X tangent to M at p, we consider an orthonormal basis $\{e_1 = X, e_2, \ldots, e_n\}$ of T_pM. Then the *Ricci curvature* of X is defined by

$$\mathrm{Ric}(X) = \sum_{i=2}^{n} K(X \wedge e_i),$$

where $K(X \wedge e_i)$ is the sectional curvature of the 2-plane section spanned by X and e_i, $i = 2, \ldots, n$.

If J is the standard almost complex structure on $\widetilde{M}(4c)$, we put $JX = PX + FX$, where PX and FX are the tangential and normal components, respectively, of JX.

B.Y. Chen established a sharp relationship between Ricci and k-Ricci curvatures, respectively, and the squared mean curvature for submanifolds in real space forms (see [8]).

In [27], we proved a similar inequality for an n-dimensional Riemannian submanifold M of an m-dimensional complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$ (see also [30]).

Theorem 8.3.1.1 *Let M be an n-dimensional submanifold in an m-dimensional complex space form $\widetilde{M}(4c)$ with constant holomorphic sectional curvature $4c$. Then*

(i) *For each unit vector $X \in T_pM$, we have*

$$\mathrm{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + (n-1)c + \frac{3}{2}c \|PX\|^2 . \tag{8.3.1.1}$$

(ii) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (8.3.1.1) if and only if $X \in \ker h_p$.*

(iii) *The equality case of (8.3.1.1) holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof Let $X \in T_pM$ be a unit tangent vector X at p. We choose an orthonormal basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}$ such that $e_1, \ldots, e_n$ are tangent to M at p, with $e_1 = X$.

Then, from the equation of Gauss, we have

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - [n(n-1) + 3\|P\|^2]c. \tag{8.3.1.2}$$

From (8.3.1.2), we get

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau + \sum_{r=n+1}^{2m} \left[(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r)^2 + 2\sum_{i<j} (h_{ij}^r)^2 \right] \\ &\quad - 2 \sum_{r=n+1}^{2m} \sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - [n(n-1) + 3\|P\|^2]c \\ &= 2\tau + \frac{1}{2} \sum_{r=n+1}^{2m} \left[(h_{11}^r + h_{22}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2 \right] \\ &\quad + 2 \sum_{r=n+1}^{2m} \sum_{i<j} (h_{ij}^r)^2 - 2 \sum_{r=n+1}^{2m} \sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - [n(n-1) + 3\|P\|^2]c. \end{aligned} \tag{8.3.1.3}$$

From the equation of Gauss, we find

$$K(e_i \wedge e_j) = \sum_{r=n+1}^{2m} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + [1 + 3g^2(e_i, Je_j)]c, \; 2 \le i < j \le n, \tag{8.3.1.4}$$

and consequently

$$\begin{aligned} \sum_{2\le i<j\le n} K(e_i \wedge e_j) &= \sum_{r=n+1}^{2m} \sum_{2\le i<j\le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{(n-1)(n-2)}{2} c \\ &\quad + (\|P\|^2 - \|Pe_1\|^2) \frac{3c}{2}. \end{aligned} \tag{8.3.1.5}$$

Substituting (8.3.1.5) in (8.3.1.3), one gets

$$\begin{aligned} n^2 \|H\|^2 &\ge 2\tau + \frac{1}{2} n^2 \|H\|^2 + 2 \sum_{r=n+1}^{2m} \sum_{j=2}^{n} (h_{1j}^r)^2 - 2 \sum_{2\le i<j\le n} K(e_i \wedge e_j) \\ &\quad + [(n-1)(n-2) - n(n-1)]c - 3\|Pe_1\|^2 c. \end{aligned} \tag{8.3.1.6}$$

Therefore,

$$\frac{1}{2} n^2 \|H\|^2 \ge 2\mathrm{Ric}(X) - 2(n-1)c - 3\|PX\|^2 c, \tag{8.3.1.7}$$

or equivalently (8.3.1.1).

(i) Assume $H(p) = 0$. Equality holds in (8.3.1.1) if and only if

$$\begin{cases} h^r_{12} = \cdots = h^r_{1n} = 0, \\ h^r_{11} = h^r_{22} + \cdots + h^r_{nn}, r \in \{n+1, \ldots, 2m\}. \end{cases} \tag{8.3.1.8}$$

Then $h^r_{1j} = 0, \forall j \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\}$, i.e., $X \in \ker h_p$.

(ii) The equality case of (8.3.1.1) holds for all unit tangent vectors at p if and only if

$$\begin{cases} h^r_{ij} = 0, i \neq j, r \in \{n+1, \ldots, 2m\}, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\}. \end{cases} \tag{8.3.1.9}$$

We distinguish two cases:

(a) $n \neq 2$; then p is a totally geodesic point;

(b) $n = 2$; it follows that p is a totally umbilical point.

The converse is trivial.

In particular, for CR-submanifolds in complex space forms we have

Corollary 8.3.1.2 *Let M be an n-dimensional CR-submanifold of an m-dimensional complex space form $\widetilde{M}(4c)$. Then*

(i) *For any unit vector $X \in \mathcal{D}_p$,*

$$\operatorname{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + \left(n + \frac{1}{2}\right) c.$$

(ii) *For any unit vector $X \in \mathcal{D}^\perp_p$,*

$$\operatorname{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + (n-1)c.$$

Next we prove a relationship between the k-Ricci curvature and the squared mean curvature for submanifolds in complex space forms.

First, we state a relationship between the sectional curvature and the squared mean curvature.

Theorem 8.3.1.3 *Let M be an n-dimensional submanifold in a complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - c - \frac{3c\,\|P\|^2}{n(n-1)}.$$

Proof Let $p \in M$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis of T_pM.

From the equation of Gauss for $X = Z = e_i, Y = W = e_j$, by summing, we obtain

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - [n(n-1) + 3\|P\|^2]c. \tag{8.3.1.10}$$

We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \ldots, e_n$ diagonalize the shape operator A_{n+1}. Then the shape operators take the forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix} \tag{8.3.1.11}$$

$$A_r = (h_{ij}^r),\ i, j = 1, \ldots, n; r = n+2, \ldots, 2m, \text{trace } A_r = 0.$$

From (8.3.1.11), we get

$$n^2 \|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - [n(n-1) + 3\|P\|^2]c. \tag{8.3.1.12}$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_i a_i^2 - 2 \sum_{i<j} a_i a_j,$$

we obtain

$$n^2 \|H\|^2 = \left(\sum_{i=1}^{n} a_i \right)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i<j} a_i a_j \leq n \sum_{i=1}^{n} a_i^2, \tag{8.3.1.13}$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2 .$$

We have from (8.3.1.12)

$$n^2 \|H\|^2 \geq 2\tau + n\|H\|^2 - [n(n-1) + 3\|P\|^2]c, \tag{8.3.1.14}$$

or, equivalently,

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - c - \frac{3c\|P\|^2}{n(n-1)},$$

which represents the inequality to prove.

We recall the definitions of the k-Ricci curvature and sectional curvature of a k-plane section, respectively ([8]).

Let $p \in M$, L a k-plane section at p, $2 \leq k \leq n$, and $X \in L$ a unit vector. Take an orthonormal basis $\{X = v_1, v_2, \ldots, v_k\}$ of L. Then the Ricci curvature $\mathrm{Ric}_L(X)$ of L at X is given by

$$\mathrm{Ric}_L(X) = \sum_{2 \leq r \leq k} K(X \wedge v_r).$$

We simply call it a *k-Ricci curvature*.

The *scalar curvature* of L is $\tau(L) = \sum_{1 \leq i < j \leq k} K(v_i \wedge v_j)$.

The Riemannian invariant Θ_k is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} \mathrm{Ric}_L(X),$$

where L runs over all k-plane sections in $T_p M$ and X runs over all unit vectors in L.

Using Theorem 8.3.1.3, we obtain the following:

Theorem 8.3.1.4 ([30]) *Let M be an n-dimensional submanifold in a complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2 (p) \geq \Theta_k(p) - c - \frac{3c\,\|P\|^2}{n(n-1)}. \tag{8.3.1.15}$$

Proof Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_p M$. Denote by $L_{i_1 \ldots i_k}$ the k-plane section spanned by $e_{i_1}, \ldots, e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} \mathrm{Ric}_{L_{i_1 \ldots i_k}}(e_i),$$

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \tau(L_{i_1 \ldots i_k}).$$

From the above equations, one derives

$$\tau(p) \geq \frac{n(n-1)}{2} \Theta_k(p),$$

which implies (8.3.1.15).

Finally, on a CR-submanifold of a complex space form, we establish a corresponding inequality.

Corollary 8.3.1.5 *Let M be an n-dimensional CR-submanifold in a complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - c - \frac{6\alpha c}{n(n-1)},$$

where $2\alpha = \dim \mathcal{D}$.

8.3.2 Wintgen-Type Inequalities for CR-Submanifolds in Complex Space Forms

For surfaces M in the Euclidean space $\mathbf{E}^3$, the Euler inequality $K \leq ||H||^2$ is fulfilled, where K is the (intrinsic) Gauss curvature of M and $||H||^2$ is the (extrinsic) squared mean curvature.

Furthermore, $K = ||H||^2$ everywhere on M if and only if M is totally umbilical, or still, by a theorem of Meusnier, if and only if M is an open portion of a plane $\mathbf{E}^2$ or it is an open portion of a round sphere S^2 in $\mathbf{E}^3$.

In 1979, P. Wintgen [47] proved that the Gauss curvature K, the squared mean curvature $\|H\|^2$, and the normal curvature $K^\perp$ of any surface M^2 in $\mathbf{E}^4$ always satisfy the inequality

$$K \leq \|H\|^2 - |K^\perp|;$$

the equality holds if and only if the ellipse of curvature of M^2 in $\mathbf{E}^4$ is a circle.

The Whitney 2-sphere satisfies the equality case of the Wintgen inequality identically.

A survey containing recent results on surfaces satisfying identically the equality case of Wintgen inequality can be read in [16].

Later, the Wintgen inequality was extended by B. Rouxel [44] and by I.V. Guadalupe and L. Rodriguez [23] independently, for surfaces M of arbitrary codimension in real space forms $\widetilde{M}(c)$; namely

$$K \leq \|H\|^2 - |K^\perp| + c.$$

The equality case was also investigated.

A corresponding inequality for totally real surfaces in n-dimensional complex space forms was obtained by one of the present authors in [28]. The equality case was studied and a nontrivial example of a totally real surface satisfying the equality case identically was given.

More recently, Wintgen-type inequalities for affine surfaces in $\mathbb{R}^4$ and $\mathbb{R}^5$ were discussed in [33].

Let M be a totally real surface of the complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$ and of complex dimension m. Recall that the curvature tensor $\widetilde{R}$ is given by

$$\widetilde{R}(X,Y,Z,W) = c[g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + g(JX,Z)g(JY,W) - g(JX,W)g(JY,Z) + 2g(X,JY)g(Z,JW)]. \tag{8.3.2.1}$$

For a point $p \in M$, let $\{e_1, e_2\}$ be an orthonormal basis of the tangent plane T_pM and $\{e_3, \ldots, e_{2m}\}$ an orthonormal basis of the normal space $T_p^{\perp}M$.

The *ellipse of curvature* at a point $p \in M$ is the subspace E_p of the normal space given by

$$E_p = \{h_p(X,X) \mid X \in T_pM, \|X\| = 1\}.$$

For any vector $X = (\cos\theta)e_1 + (\sin\theta)e_2$, $\theta \in [0, 2\pi)$, we have

$$h_p(X,X) = H(p) + \frac{1}{2}(\cos 2\theta)(h_{11} - h_{22}) + (\sin 2\theta)h_{12},$$

where $h_{ij} = h(e_i, e_j)$, for $i, j = 1, 2$.

The following result [23] holds good.

Proposition 8.3.2.1 *If the ellipse of curvature is nondegenerated, then the vectors $h_{11} - h_{22}$ and h_{12} are linearly independent.*

Using a similar method as in [23] and the above proposition, we can define a 2-plane subbundle P of the normal bundle, with the induced connection.

Then we will define the *scalar normal curvature* by the formula

$$K_N = -g([A_{e_3}, A_{e_4}]e_1, e_2),$$

where $\{e_1, e_2\}$, $\{e_3, e_4\}$ are orthonormal-oriented bases of T_pM and P_p, respectively, and A is the shape operator.

Remark This definition of the scalar normal curvature coincides with the definition of the *normal curvature* (used by Wintgen [47] and also by Guadalupe and Rodriguez [23] by the formula $K^{\perp} = g(R^{\perp}(e_1, e_2)e_4, e_3))$, if the ambient space $\tilde{M}(c)$ is a real space form.

We can choose $\{e_1, e_2\}$ such that the vectors $u = \frac{1}{2}(h_{11} - h_{22})$ and $v = h_{12}$ are perpendicular, in which case they coincide with the half-axes of the ellipse. Then we will take $e_3 = \frac{u}{\|u\|}$ and $e_4 = \pm\frac{v}{\|v\|}$.

From the equation of Ricci and the definition of K_N, we have

$$|K_N| = \|h_{11} - h_{22}\| \cdot \|h_{12}\|. \tag{8.3.2.2}$$

Also, from the Gauss equation we obtain the formula of the Gauss curvature K of the totally real surface M of the complex space form $\widetilde{M}(c)$:

$$K = g(h_{11}, h_{22}) - \|h_{12}\|^2 + c. \tag{8.3.2.3}$$

By the definition of the mean curvature vector, (8.3.2.3) and the relation $\|h\|^2 = \|h_{11}\|^2 + \|h_{22}\|^2 + 2\|h_{12}\|^2$, we have

$$4\|H\|^2 = \|h\|^2 + 2(K - c). \tag{8.3.2.4}$$

Then

$$\begin{aligned} 0 &\le (\|h_{11} - h_{22}\| - 2\|h_{12}\|)^2 = \|h\|^2 - 2(K - c) - 4|K_N| \\ &= 4\|H\|^2 - 4(K - c) - 4|K_N|, \end{aligned} \tag{8.3.2.5}$$

which is equivalent to

$$\|H\|^2 \ge K + |K_N| - c. \tag{8.3.2.6}$$

The equality sign is realized if and only if $\|h_{11} - h_{22}\| = 2\|h_{12}\|$, i.e., $\|u\| = \|v\|$, so the ellipse of curvature is a circle.

Thus, we proved the following Wintgen-type inequality (see [28]).

Theorem 8.3.2.2 *Let M be a totally real surface of the complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$ and of complex dimension m. Then, at any point $p \in M$ we have*

$$\|H\|^2 \ge K + |K_N| - c.$$

Moreover, the equality sign is realized if and only if the ellipse of curvature is a circle.

We will give an **example** of a Lagrangian surface in $\mathbf{C}^2$ with the standard almost complex structure J_0, for which the equality sign is realized (which we call an *ideal surface*).

Let M be the *rotation surface* of Vrănceanu [46], given by

$$X(u, v) = r(u)(\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v),$$

where r is a positive C^∞-differentiable function.

Let $\{e_1, e_2\}$ be an orthonormal basis of the tangent plane and $\{e_3, e_4\}$ an orthonormal basis of the normal plane such that $\{e_1, e_1, e_3, e_4\}$ is positively oriented.

Then it is easy to find the following expressions for $e_i, i \in \{1, 2, 3, 4\}$ (see also [45]):

$$e_1 = (-\cos u \sin v, \cos u \cos v, -\sin u \sin v, \sin u \cos v),$$

$$e_2 = \frac{1}{A}(B\cos v, B\sin v, C\cos v, C\sin v),$$

$$e_3 = \frac{1}{A}(-C\cos v, -C\sin v, B\cos v, B\sin v),$$

$$e_4 = (-\sin u \sin v, \sin u \cos v, \cos u \sin v, -\cos u \cos v),$$

where $A = [r^2 + (r')^2]^{\frac{1}{2}}$, $B = r'\cos u - r\sin u$, $C = r'\sin u + r\cos u$.

Also, after technical calculations, we find

$$h^3_{11} = \frac{1}{[r^2 + (r')^2]^{\frac{1}{2}}}, \quad h^3_{12} = 0, \quad h^3_{22} = \frac{-rr'' + 2(r')^2 + r^2}{[r^2 + (r')^2]^{\frac{3}{2}}},$$

$$h^4_{11} = 0, \quad h^4_{12} = -\frac{1}{[r^2 + (r')^2]^{\frac{1}{2}}}, \quad h^4_{22} = 0.$$

M is a totally real surface of maximum dimension, so is a *Lagrangian surface* of $\mathbf{C}^2$. Also, M verifies the equality sign of the inequality in Theorem 8.3.2.2 (it is an *ideal surface*) if and only if

$$r(u) = \frac{1}{(|\cos 2u|)^{\frac{1}{2}}}$$

(the ellipse of curvature at every point of M is a circle).

Moreover, M is a minimal surface (see [41]) and $X = c_1 \otimes c_2$ is the tensor product immersion of $c_1(u) = \frac{1}{(|\cos 2u|)^{\frac{1}{2}}}(\cos u, \cos v)$ (an orthogonal hyperbola) and $c_2(u) = (\cos v, \sin v)$ (a circle centered at the origin).

In 1999, P.J. De Smet et al. [18] formulated the conjecture on Wintgen inequality for submanifolds of real space forms, which is also known as the *DDVV conjecture*.

Let M be an n-dimensional submanifold of a real space form $\widetilde{M}(c)$. We denote by K and $R^\perp$ the sectional curvature function and the normal curvature tensor on M, respectively.

Then the normalized scalar curvature is given by

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} K(e_i \wedge e_j),$$

where τ is the scalar curvature, and the normalized normal scalar curvature by

$$\rho^\perp = \frac{2\tau^\perp}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1 \le i < j \le n} \sum_{1 \le \alpha < \beta \le m} (R^\perp(e_i, e_j, \xi_\alpha, \xi_\beta))^2},$$

where $\{e_1, \ldots, e_n\}$ and $\{\xi_1, \ldots, \xi_m\}$ are orthonormal frames tangent and normal to M, respectively.

DDDV Conjecture. *Let $f : M \to \widetilde{M}(c)$ be an isometric immersion, where $\widetilde{M}(c)$ is a real space form of constant sectional curvature c and M an n-dimensional submanifold of codimension m. Then*

$$\rho \leq \|H\|^2 - \rho^{\perp} + c.$$

This conjecture was proven by the authors for submanifolds M of arbitrary dimension $n \geq 2$ and codimension 2 in real space forms $\widetilde{M}(c)$ of constant sectional curvature c.

Also, a detailed characterization of the equality case in terms of the shape operators of M in $\widetilde{M}(c)$ was given.

T. Choi and Z. Lu [17] proved that this conjecture is true for all 3-dimensional submanifolds M of arbitrary codimension $m \geq 2$ in $\widetilde{M}(c)$. The characterization of the equality case gives the specific forms of shape operators of M in $\widetilde{M}(c)$.

For normally flat submanifolds, i.e., $R^{\perp} = 0$, the normal scalar curvature vanishes; B.Y. Chen [7] established the inequality

$$\rho \leq \|H\|^2 + c.$$

Hence, the conjecture is true for hypersurfaces of real space forms.

Other extensions of Wintgen inequality have been studied by P.J. De Smet, F. Dillen, J. Fastenakels, J. Van der Veken, L. Verstraelen, L. Vrancken and the present authors for certain submanifolds in Kähler, nearly Kähler, and Sasakian spaces (see [20, 21, 31, 39], etc.).

Recently, the DDVV conjecture was finally settled for the general case by Z. Lu [26] and independently by J. Ge and Z. Tang [22].

Theorem 8.3.2.3 *The generalized Wintgen inequality*

$$\rho \leq \|H\|^2 - \rho^{\perp} + c,$$

holds for every submanifold M in any real space form $\widetilde{M}(c)$ (dim $M = n \geq 2$, codim $M = m \geq 2$).

The equality case holds identically if and only if, with respect to suitable orthonormal frames $\{e_i | i = \Gamma, n\}$ and $\{\xi_\alpha | \alpha = \Gamma, m\}$, the shape operators of M in $\widetilde{M}(c)$ take the forms

$$A_{\xi_1} = \begin{pmatrix} \lambda_1 & \mu & 0 & \cdots & 0 \\ \mu & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{pmatrix},$$

$$A_{\xi_2} = \begin{pmatrix} \lambda_2 + \mu & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{pmatrix},$$

$$A_{\xi_3} = \begin{pmatrix} \lambda_3 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_3 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3$*, and* μ *are real functions on* M*,*

$$A_{\xi_4} = \cdots = A_{\xi_m} = 0.$$

Recently, one of the present authors [39] proved a generalized Wintgen inequality for Lagrangian submanifolds in complex space forms.

Theorem 8.3.2.4 *Let* M *be an* n*-dimensional Lagrangian submanifold of a complex space form* $\widetilde{M}(4c)$*. Then*

$$(\rho^{\perp})^2 \leq (\|H\|^2 - \rho + c)^2 + \frac{4}{n(n-1)}(\rho - c)c + \frac{2c^2}{n(n-1)}.$$

Corollary 8.3.2.5 *Let* M *be a minimal Lagrangian submanifold of* $\mathbf{C}^n$*. Then*

$$\rho \leq -\rho^{\perp}.$$

Remark (i) The inequality in the above theorem for $n = 3$ was established by A. Mihai [31].

(ii) The inequality in the above corollary for $n = 3, 4$ was given by F. Dillen et al. [21].

Another Wintgen-type inequality for totally real submanifolds in complex space forms was established in [39].

Let M be an n-dimensional totally real submanifold of an m-dimensional complex space form $\tilde{M}(4c)$ and $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2m}\}$ orthonormal frames on M tangent and normal to M, respectively.

By analogy with the two-dimensional case we introduced a *scalar normal curvature* K_N defined by

$$K_N = \frac{1}{2} \sum_{1 \le r < s \le n} (\mathrm{Trace}[A_r, A_s])^2 = \sum_{1 \le r < s \le n} \sum_{1 \le i < j \le n} (g([A_r, A_s]e_i, e_j)^2.$$

The *normalized* scalar normal curvature is

$$\rho_N = \frac{2}{n(n-1)} \sqrt{K_N}.$$

For this extrinsic invariant we proved the following inequality.

Proposition 8.3.2.6 *Let M be an n-dimensional totally real submanifold of an m-dimensional complex space form $\widetilde{M}(4c)$. Then we have*

$$\|H\|^2 \ge \rho + \rho_N - c.$$

We can generalize the above inequality to *CR*-submanifolds in complex space forms.

Let M be an n-dimensional *CR*-submanifold of an m-dimensional complex space form $\tilde{M}(4c)$. If $\{e_1, \ldots, e_n\}$ and $\{\xi_1, \ldots, \xi_{2m-n}\}$ are orthonormal frames tangent and normal to M, respectively, we denote as usual by $h^r_{ij} = g(h(e_i, e_j), \xi_r)$, $1 \le i, j \le n$, $1 \le r \le 2m - n$, the components of the second fundamental form.

As in [39], we can prove that

$$\sum_{r=1}^{2m-n} \sum_{1 \le i < j \le n} (h^r_{ii} - h^r_{jj})^2 + 2n \sum_{r=1}^{2m-n} \sum_{1 \le i < j \le n} (h^r_{ij})^2$$
$$\ge 2n \left[\sum_{1 \le r < s \le 2m-n} \sum_{1 \le i < j \le n} \left(\sum_{k=1}^{n} (h^r_{jk} h^s_{ik} - h^r_{ik} h^s_{jk}) \right)^2 \right]^{\frac{1}{2}}. \qquad (8.3.2.7)$$

We use Gauss equation, which implies

$$\tau = \sum_{1 \le i < j \le n} R(e_i, e_j, e_i, e_j) = \frac{n(n-1)}{2} c + 3\alpha c + \sum_{r=1}^{2m-n} \sum_{1 \le i < j \le n} \left[h^r_{ii} h^r_{jj} - (h^r_{ij})^2 \right],$$

where $2\alpha = \dim \mathcal{D}$.

By substituting the last equation in (8.3.2.7), we obtain the following:

Theorem 8.3.2.7 ([34]) *Let M be an n-dimensional CR-submanifold of an m-dimensional complex space form $\tilde{M}(4c)$. Then, we have*

$$||H||^2 \ge \rho + \rho_N - c - \frac{6\alpha c}{n(n-1)}.$$

8.3.3 General Inequalities for CR-Warped Products in Complex Space Forms

The notion of *warped product* plays some important role in differential geometry as well as in Physics [12]. For instance, the best relativistic model of the Schwarzschild spacetime that describes the out space around a massive star or a black hole is given as a warped product [43].

One of the most fundamental problems in the theory of submanifolds is the immersibility (or nonimmersibility) of a Riemannian manifold in a Euclidean space (or, more generally, in a space form). According to a well-known theorem on Nash, every Riemannian manifold can be isometrically immersed in some Euclidean spaces with sufficiently high codimension.

Nash's theorem implies, in particular, that every warped product $M_1 \times_f M_2$ can be immersed as a Riemannian submanifold in some Euclidean space. Moreover, many important submanifolds in real and complex space forms are expressed as a warped product submanifold.

Isometric immersions of warped products were studied by Nölker [42]. Every Riemannian manifold of constant curvature c can be locally expressed as a warped product whose warping function satisfies $\Delta f = cf$. For example,

(i) $S^n(1)$ is locally isometric to $(0, \pi) \times_{\cos t} S^{n-1}(1)$,
(ii) $\mathbf{E}^n$ is locally isometric to $(0, \infty) \times_x S^{n-1}(1)$,
(iii) $H^n(-1)$ is locally isometric to $\mathbb{R} \times_{e^x} \mathbf{E}^{n-1}$
(see [12]).

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. The *warped product* of M_1 and M_2 is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$.

Let ∇, ∇^1, and ∇^2 be the Levi-Civita connections on $M_1 \times_f M_2$, M_1, and M_2, respectively.

We recall the following formulae for ∇ on a warped product manifold.

(i) $\nabla_X Y = \nabla^1_X Y,\ \forall X, Y \in \Gamma(TM_1)$;

(ii) $\nabla_X Z = \nabla_Z X = (X \ln f) Z,\ \forall X \in \Gamma(TM_1),\ Z \in \Gamma(TM_2)$;

(iii) $\nabla_Z W = \nabla^2_Z W - \frac{1}{f} g(Z, W) \nabla f,\ \forall Z, W \in \Gamma(TM_2)$.

The sectional curvature K of a warped product can be determined in terms of the sectional curvatures K_1 and K_2 of its factors. We have

(a) $K(X \wedge Y) = K_1(X \wedge Y),\ \forall X, Y \in \Gamma(TM_1)$ linearly independent;

(b) $K(X \wedge Z) = \frac{1}{f}[-X^2 f + (\nabla_X X) f],\ \forall X \in \Gamma(TM_1),\ Z \in \Gamma(TM_2)$ unit vector fields;

(c) $K(Z \wedge W) = \frac{1}{f^2}[K_2(Z \wedge W) - ||\nabla f||^2],\ \forall Z, W \in \Gamma(TM_2)$ orthonormal vector fields.

B.Y. Chen [11] established a sharp inequality for the warping function of a warped product submanifold in a Riemannian space form in terms of the squared mean curvature.

Theorem 8.3.3.1 *Let $\phi : M_1 \times_f M_2 \to \tilde{M}(c)$ be an isometric immersion of a warped product into a Riemannian m-manifold of constant sectional curvature c. Then we have*

$$\frac{\Delta f}{f} \leq \frac{(n_1+n_2)^2}{4n_2}||H||^2 + n_1 c,$$

where $n_i = \dim M_i$, $i = 1, 2$, $||H||^2$ is the squared mean curvature of $M_1 \times_f M_2$ and Δ is the Laplacian operator of M_1.

The equality sign holds identically if and only if ϕ is mixed totally geodesic and trace $h_1 =$ trace h_2.

As applications, the author derived obstructions to the minimality of warped product submanifolds in Riemannian space forms.

Later, in [13], the same author studied warped product submanifolds in complex hyperbolic spaces.

We recall in this section some results given by B.Y. Chen in [14], more precisely a general inequality for a *CR*-warped product $M_1 \times_f M_2$ in a complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$ and the complete classification of *CR*-warped products in complex space forms satisfying the equality case of the inequality.

We point out that in [9, 10] it was proved that there do not exist warped products of the form $M_2 \times_f M_1$ in a Kähler manifold beside *CR*-products, with M_2 a totally real submanifold and M_1 a complex submanifold. There exist many *CR*-submanifolds which are warped products of the form $M_1 \times_f M_2$, by reversing the two factors, called *CR-warped products*.

In [9, 10] B.Y. Chen proved that every *CR*-warped product $M_1 \times_f M_2$ in a Kähler manifold $\widetilde{M}$ satisfies the inequality

$$\|h\|^2 \geq 2p\,\|\nabla \ln f\|^2, \tag{8.3.3.1}$$

where h is the second fundamental form, $\nabla \ln f$ is the gradient of $\ln f$ and $p = \dim M_2$. In the same articles classifications of *CR*-warped products in complex space forms satisfying the equality case of the inequality (8.3.3.1) was given.

In [14] another stronger general inequality has been proved.

$$\|h\|^2 \geq 2p[\|\nabla \ln f\|^2 + \Delta(\ln f)] + 4qpc, \tag{8.3.3.2}$$

where Δ is the Laplacian operator on M_1, $q = \dim_{\mathbf{C}} M_1$, $p = \dim M_2$.

We would like to point out that an interesting map was defined in Chen's paper, playing a central role. More precisely, let $\mathbf{C}^q_* = \mathbf{C}^q - \{0\}$ and $j : S^p \to \mathbf{E}^{p+1}$ be the inclusion of the unit hypersphere S^p centered at the origin into $\mathbf{E}^{p+1}$. For a natural number $\alpha \le q$ and a vector X tangent to $\mathbf{C}_*^\alpha$ at a point $z \in \mathbf{C}_*^\alpha$, one decomposes $X = X_z^{||} + X_z^\perp$, where $X_z^{||}$ is parallel to z and $X_z^\perp$ is perpendicular to z.

For any given three natural numbers q, p, α satisfying $\alpha \le q$, the map $\phi_\alpha^{qp} : \mathbf{C}_*^q \times S^p \to \mathbf{C}^{\alpha p+q}$ is defined by

$$\phi_\alpha^{qp}(z, w) = (w_0 z_1, \dots, w_p z_1, \dots, w_0 z_\alpha, \dots, w_p z_\alpha, z_{\alpha+1}, \dots, z_q), \tag{8.3.3.3}$$

for $z = (z_1, \dots, z_q) \in \mathbf{C}_*^q$ and $w = (w_0, \dots, w_p) \in S^p \subset \mathbf{E}^{p+1}$ with $\sum_{i=0}^p w_i^2 = 1$.

Theorem 8.3.3.2 ([14]) *For* $1 \le \alpha \le q$ *and* $p \ge 1$, *the map* $\phi_\alpha^{qp:} : \mathbf{C}_*^q \times S^p \to \mathbf{C}^{\alpha p+q}$ *defined by (8.3.3.3) satisfies the following properties:*

(i) $\phi_\alpha^{qp} : \mathbf{C}_*^q \times S^p \to \mathbf{C}^{\alpha p+q}$ *is an isometric immersion with warping function* $f = \sqrt{\sum_{j=1}^\alpha z_j \bar{z}_j}$.

(ii) ϕ_α^{qp} *is a CR-warped product.*

(iii) *The second fundamental form h of* ϕ_α^{qp} *satisfies the equality*

$$\|h\|^2 = 2p[\|\nabla \ln f\|^2 + \Delta(\ln f)]. \tag{8.3.3.4}$$

The equality case of the inequality (8.3.3.2) is studied, when the ambient space is the complex Euclidean space, complex projective space, and complex hyperbolic space, respectively.

In this section we recall only the result for complex Euclidean space, the other cases being similar, with adequate constructions of the isometric immersions of the *CR*-warped products, by composing them with projections.

Theorem 8.3.3.3 ([14]) *Let* $\phi : M_1 \times_f M_2 \to \mathbf{C}^m$ *be a CR-warped product in complex Euclidean space* $\mathbf{C}^m$. *Then we have*

(i) *The squared norm of the second fundamental form h of ϕ satisfies*

$$\|h\|^2 \ge 2p[\|\nabla \ln f\|^2 + \Delta(\ln f)]. \tag{8.3.3.5}$$

(ii) *If the CR-warped product satisfies the equality case of (8.3.3.5), then we have*

(ii.a) M_1 *is an open portion of* $\mathbf{C}_*^q$.

(ii.b) M_2 *is an open portion of* S^p;

(ii.c) *There exists a natural number* $\alpha \le q$ *and a complex coordinate system* $\{z_1, \dots, z_q\}$ *on* $\mathbf{C}_*^q$ *such that the warping function is given by* $f = \sqrt{\sum_{j=1}^\alpha z_j \bar{z}_j}$;

(ii.d) *Up to rigid motions of* $\mathbf{C}^m$, *the immersion ϕ is given by* ϕ_α^{qp} *in a natural way; namely, we have*

$$\phi(z, w) = (w_0 z_1, \dots, w_p z_1, \dots, w_0 z_\alpha, \dots, w_p z_\alpha, z_{\alpha+1}, \dots, z_q, 0, \dots, 0), \tag{8.3.3.6}$$

for $z = (z_1, \dots, z_q) \in \mathbf{C}_*^q$ *and* $w = (w_0, \dots, w_p) \in S^p \subset \mathbf{E}^{p+1}$ *with* $\sum_{i=0}^p w_i^2 = 1$.

8.3.4 Warped Product Submanifolds in Complex Space Forms

In the present subsection, we establish an inequality between the warping function f (intrinsic structure), the squared mean curvature $\|H\|^2$, and the holomorphic sectional curvature $4c$ (extrinsic structures) for warped product submanifolds $M_1 \times_f M_2$ with $J\mathcal{D}_1 \perp \mathcal{D}_2$ (in particular, CR-warped product submanifolds and CR-Riemannian products) in any complex space form $\widetilde{M}(4c)$. Examples of such submanifolds which satisfy the equality case are given.

Let M be an n-dimensional submanifold in a complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$, endowed with the almost complex structure J. We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$, and ∇ the Riemannian connection of M, respectively. Also, let h be the second fundamental form and R the Riemann curvature tensor of M.

Recall that the equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

for any vectors X, Y, Z, W tangent to M.

Let $p \in M$ and $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}\}$ an orthonormal basis of the tangent space $T_p\widetilde{M}(c)$, such that $e_1, \dots, e_n$ are tangent to M at p. We denote by H the mean curvature vector, that is,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \dots, n\}, r \in \{n+1, \dots, 2m\}.$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For any tangent vector field X tangent to M, we put $JX = PX + FX$, where PX and FX are the tangential and normal components of JX, respectively. We denote by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j).$$

If M is a Riemannian n-manifold, $\{e_1, \dots, e_n\}$ an orthonormal frame field on M and f a differentiable function on M, the Laplacian Δf of f is defined by

$$\Delta f = \sum_{j=1}^{n} [(\nabla_{e_j} e_j) f - e_j e_j f].$$

Let $x : M_1 \times_f M_2 \to \widetilde{M}(4c)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a complex space form $\widetilde{M}(4c)$. We denote by $H_i = \frac{1}{n_i} \operatorname{trace} h_i$ the *partial mean curvatures*, where trace h_i is the trace of h restricted to M_i and $n_i = \dim M_i$ $(i = 1, 2)$.

For a warped product $M_1 \times_f M_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors tangent to leaves and fibers, respectively. Thus, $\mathcal{D}_1$ is obtained from the tangent vectors of M_1 via the horizontal lift and $\mathcal{D}_2$ by tangent vectors of M_2 via the vertical lift.

In this section, we investigate warped product submanifolds with $J\mathcal{D}_1 \perp \mathcal{D}_2$ in a complex space form $\widetilde{M}(4c)$. We mention that CR-submanifolds have this property.

As applications we will give some nonimmersion theorems [29].

We recall an algebraic lemma of Chen.

Lemma ([6]) *Let $n \geq 2$ and $a_1, \dots, a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i \right)^2 = (n-1) \left(\sum_{i=1}^{n} a_i^2 + b \right).$$

Then $2a_1 a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

Using it, we can prove the following:

Theorem 8.3.4.1 *Let $x : M_1 \times_f M_2 \to \widetilde{M}(4c)$ be an isometric immersion of an n-dimensional warped product with $J\mathcal{D}_1 \perp \mathcal{D}_2$ into a $2m$-dimensional complex space form $\widetilde{M}(4c)$. Then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 c, \tag{8.3.4.1}$$

where $n_i = \dim M_i$, $i = 1, 2$, and Δ is the Laplacian operator of M_1.

Moreover, the equality case of (8.3.4.1) holds identically if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i, $i = 1, 2$, are the partial mean curvature vectors.

Proof Let $M_1 \times_f M_2$ be a warped product submanifold into a complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$.

Since $M_1 \times_f M_2$ is a warped product, from the formula of the Riemannian connection ∇ it follows that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z, \tag{8.3.4.2}$$

for any vector fields X, Z tangent to M_1, M_2, respectively.

Then, if X and Z are unit vector fields, the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}[(\nabla_X X)f - X^2 f]. \tag{8.3.4.3}$$

We choose a local orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$, such that $e_1, \ldots, e_{n_1}$ are tangent to M_1, $e_{n_1+1}, \ldots, e_n$ are tangent to M_2, e_{n+1} is parallel to the mean curvature vector H.

Then, using (8.3.4.3), we get

$$\frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \tag{8.3.4.4}$$

for each $s \in \{n_1 + 1, \ldots, n\}$.

From the equation of Gauss, we have

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - n(n-1)c - 3c\|P\|^2. \tag{8.3.4.5}$$

We set

$$\delta = 2\tau - n(n-1)c - 3\|P\|^2 c - \frac{n^2}{2}\|H\|^2. \tag{8.3.4.6}$$

Then (8.3.4.5) can be written as

$$n^2 \|H\|^2 = 2(\delta + \|h\|^2). \tag{8.3.4.7}$$

With respect to the above orthonormal frame, (8.3.4.7) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i \neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right\}.$$

If we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\le i\ne j\le n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n} (h_{ij}^r)^2 - \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1}h_{tt}^{n+1}\right\}.$$

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for $n = 3$), i.e.,

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right).$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3$.
In the case under consideration, this means

$$\sum_{1\le j<k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1} \ge \frac{\delta}{2} + \sum_{1\le\alpha<\beta\le n} (h_{\alpha\beta}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n} (h_{\alpha\beta}^r)^2. \tag{8.3.4.8}$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{8.3.4.9}$$

Using again the Gauss equation, we have

$$\begin{aligned} n_2\frac{\Delta f}{f} &= \tau - \sum_{1\le j<k\le n_1} K(e_j\wedge e_k) - \sum_{n_1+1\le s<t\le n} K(e_s\wedge e_t) \\ &= \tau - \frac{n_1(n_1-1)c}{2} - \sum_{r=n+1}^{2m}\sum_{1\le j<k\le n_1} [h_{jj}^r h_{kk}^r - (h_{jk}^r)^2] - 3c\sum_{1\le j<k\le n_1} g^2(Je_j, e_k) \\ &\quad - \frac{n_2(n_2-1)c}{2} - \sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n} [h_{ss}^r h_{tt}^r - (h_{st}^r)^2] - 3c\sum_{n_1+1\le s<t\le n} g^2(Je_s, e_t). \end{aligned} \tag{8.3.4.10}$$

Combining (8.3.4.8) and (8.3.4.9) and taking account of (8.3.4.4), we obtain

$$\begin{aligned}
n_2\frac{\Delta f}{f} &\le \tau - \frac{n(n-1)c}{2} + n_1 n_2 c - \frac{\delta}{2} - 3c\sum_{1\le j<k\le n_1} g^2(Je_j, e_k) \\
&\quad - 3c\sum_{n_1+1\le s<t\le n} g^2(Je_s, e_t) - \sum_{1\le j\le n_1;\, n_1+1\le t\le n} (h_{jt}^{n+1})^2 - \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n}(h_{\alpha\beta}^r)^2 \\
&\quad + \sum_{r=n+2}^{2m}\sum_{1\le j<k\le n_1}[(h_{jk}^r)^2 - h_{jj}^r h_{kk}^r] + \sum_{r=n+2}^{2m}\sum_{n_1+1\le s<t\le n}[(h_{st}^r)^2 - h_{ss}^r h_{tt}^r] \\
&= \tau - \frac{n(n-1)c}{2} + n_1 n_2 c - \frac{\delta}{2} - \sum_{r=n+1}^{2m}\sum_{1\le j\le n_1;\, n_1+1\le t\le n}(h_{jt}^r)^2 - 3c\sum_{1\le j<k\le n_1} g^2(Je_j, e_k) \\
&\quad - 3c\sum_{n_1+1\le s<t\le n} g^2(Je_s, e_t) - \frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{j=1}^{n_1} h_{jj}^r\right)^2 - \frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{t=n_1+1}^{n} h_{tt}^r\right)^2 \\
&\le \tau - \frac{n(n-1)c}{2} + n_1 n_2 c - \frac{\delta}{2} - 3c\sum_{1\le j<k\le n_1} g^2(Je_j, e_k) - 3c\sum_{n_1+1\le s<t\le n} g^2(Je_s, e_t).
\end{aligned} \tag{8.3.4.11}$$

Since we assume that $J\mathcal{D}_1 \perp \mathcal{D}_2$, the last relation implies the inequality (8.3.4.1).

We see that the equality sign of (8.3.4.11) holds if and only if

$$h_{jt}^r = 0, \quad 1 \le j \le n_1, n_1+1 \le t \le n, n+1 \le r \le 2m, \tag{8.3.4.12}$$

and

$$\sum_{i=1}^{n_1} h_{ii}^r = \sum_{t=n_1+1}^{n} h_{tt}^r = 0, \quad n+2 \le r \le 2m. \tag{8.3.4.13}$$

Obviously (8.3.4.12) is equivalent to the mixed totally geodesic of the warped product $M_1 \times_f M_2$ (i.e., $h(X, Z) = 0$, for any X in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$) and (8.3.4.9) and (8.3.4.13) imply $n_1 H_1 = n_2 H_2$.

The converse statement is straightforward.

Remark For $c \le 0$ the inequality is true without the condition $J\mathcal{D}_1 \perp D_2$ (see [13]).

As applications, we derive certain obstructions to the existence of minimal warped product submanifolds in complex hyperbolic spaces.

Let $x : M_1 \times_f M_2 \to \widetilde{M}(4c)$ be a minimal isometric immersion. Then the above theorem implies

$$\frac{\Delta f}{f} \le n_1 c.$$

Thus, if $c < 0$, f cannot be a harmonic function or an eigenfunction of Laplacian with positive eigenvalue.

We resume this remark into the following:

Proposition 8.3.4.2 ([13]) *If f is a harmonic function, then $M_1 \times_f M_2$ does not admit any isometric minimal immersion into a complex hyperbolic space.*

Proposition 8.3.4.3 ([13]) *If f is an eigenfunction of Laplacian on M_1 with corresponding eigenvalue $\lambda > 0$, then $M_1 \times_f M_2$ does not admit any isometric minimal immersion into a complex hyperbolic space or a complex Euclidean space.*

Next, we will give some **examples** which satisfy the equality case of the inequality (8.3.4.1).

Recall that the *Hopf submersion* is the canonical projection of $\mathbf{C}^{n+1} - \{0\} \to P^n(\mathbf{C})$, restricted to S^{2n+1} (where S^{2n+1} is regarded as the set $\{z \in \mathbf{C}^{n+1}; \sum_{j=1}^{n+1} |z^j|^2 = 1\}$).

Examples

1. Let us consider the immersion $\psi : M \to S^7$, where $M = (-\pi/2, \pi/2) \times_{\cos t} N^2$, with N^2 is a minimal C-totally real submanifold in S^7, defined by $\psi(t, p) = (\cos t)p + (\sin t)v$, where v is a vector tangent to S^7, but normal to S^5.

Let $\pi : S^7 \to P^3(\mathbf{C})$ be the Hopf submersion. Then

$$\pi \circ \psi : M \to P^3(\mathbf{C})$$

is a Lagrangian minimal immersion which satisfies the equality case.

2. Let $\psi : S^n \to S^{2n+1}$ be an immersion defined by

$$\psi(x^1, \dots, x^{n+1}) = (x^1, 0, x^2, 0, \dots, x^{n+1}, 0),$$

and $\pi : S^{2n+1} \to P^n(\mathbf{C})$ the Hopf submersion.

Then $\pi \circ \psi : S^n \to P^n(\mathbf{C})$ satisfies the equality case.

3. On $S^{n_1+n_2}$ let us consider the spherical coordinates $u_1, \dots, u_{n_1+n_2}$ and on S^{n_1} the function $f(u_1, \dots u_n) = \cos u_1 \dots \cos u_{n_1}$ (f is an eigenfunction of Δ).

Then $S^{n_1+n_2} = S^{n_1} \times_f S^{n_2}$.

Let $i : S^{n_1+n_2} \to S^{2(n_1+n_2)+1}$ be the standard immersion and $\pi : S^{2(n_1+n_2)+1} \to P^{n_1+n_2}(\mathbf{C})$ the Hopf submersion.

Then $\pi \circ i : S^{n_1+n_2} \to P^{n_1+n_2}(\mathbf{C})$ satisfies the equality case.

Moreover, the examples given by B.Y. Chen in [11] for $c = 0$ in the real case are true in the complex case too, for $c = 0$.

8.3.5 Scalar Curvature of Warped Product Submanifolds in Complex Space Forms

Let $M_1 \times_f M_2$ be a warped product submanifold into a complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$.

Since $M_1 \times_f M_2$ is a warped product, it is known that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z, \tag{8.3.5.1}$$

for any vector fields X, Z tangent to M_1 and M_2, respectively.

Let X and Z be unit vector fields tangent to M_1 and M_2, respectively. It follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}[(\nabla_X X)f - X^2 f]. \tag{8.3.5.2}$$

We choose a local orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$, such that $e_1 = X, e_2, \ldots, e_{n_1}$ are tangent to M_1, $e_{n_1+1} = Z, e_{n_1+2}, \ldots, e_n$ are tangent to M_2 and e_{n+1} is parallel to the mean curvature vector H.

Then, using (8.3.5.2), we get

$$\frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \tag{8.3.5.3}$$

for each $s \in \{n_1 + 1, \ldots, n\}$.

From the equation of Gauss, we have

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - n(n-1)c - 3\|P\|^2 c. \tag{8.3.5.4}$$

We set

$$\delta = 2\tau - [n(n-1) + 3\|P\|^2 - 2]c - \frac{n^2(n-2)}{n-1}\|H\|^2. \tag{8.3.5.5}$$

Combining the above formulae, we obtain

$$n^2 \|H\|^2 = (n-1)(\|h\|^2 + \delta - 2c). \tag{8.3.5.6}$$

With respect to the above orthonormal frame, (8.3.5.6) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left\{\delta - 2c + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right\}.$$

If we put $a_1 = h_{11}^{n+1}$, $a_2 = h_{22}^{n+1}$,..., $a_n = h_{nn}^{n+1}$,

$$b = \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 + \delta - 2c,$$

the above equation becomes

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Thus $a_1, a_2, \ldots, a_n, b$ satisfy the Lemma of Chen. Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

In the case under consideration, this means

$$2h_{11}^{n+1}h_{(n_1+1)(n_1+1)}^{n+1} \geq \sum_{1\leq i\neq j\leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 + \delta - 2c. \qquad (8.3.5.7)$$

Applying the Gauss equation, it follows that

$$\begin{aligned} K(X\wedge Z) &= K(e_1\wedge e_{n_1+1}) = R(e_1, e_{n_1+1}, e_1, e_{n_1+1}) \\ &= [1 + 3g^2(Je_1, e_{n_1+1})]c + \sum_{r=n+1}^{2m}[(h_{11}^{r}h_{(n_1+1)(n_1+1)}^{r} - (h_{1(n_1+1)}^{r})^2] \\ &\geq 3cg^2(Je_1, e_{n_1+1}) + \frac{1}{2}\sum_{r=n+1}^{2m}\sum_{j\in\Omega_{1(n_1+1)}}[(h_{1j}^{r})^2 + (h_{(n_1+1)j}^{r})^2] \\ &\quad + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{i,j\in\Omega_{1(n_1+1)}}(h_{ij}^{r})^2 + \frac{1}{2}\sum_{r=n+2}^{2m}(h_{11}^{r} + h_{(n_1+1)(n_1+1)}^{r})^2 + \frac{\delta}{2} \\ &\geq 3cg^2(Je_1, e_{n_1+1}) + \frac{\delta}{2}, \end{aligned} \qquad (8.3.5.8)$$

where $\Omega_{1(n_1+1)} = \{1, \ldots, n\} \setminus \{1, n_1+1\}$.

From (8.3.5.8), using (8.3.5.2) and (8.3.5.5), one obtains

$$\frac{1}{f}\{(\nabla_{e_1}e_1)f - e_1^2 f\} \geq 3cg^2(Je_1, e_{n_1+1}) + \tau - [n(n-1) + 3\|P\|^2 - 2]\frac{c}{2} - \frac{n^2(n-2)}{2(n-1)}\|H\|^2. \quad (8.3.5.9)$$

Then

$$\tau \leq \frac{1}{f}[(\nabla_{e_1}e_1)f - e_1^2 f] + \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + [n(n-1) + 3\|P\|^2 - 2 - 6g^2(Je_1, e_{n_1+1})]\frac{c}{2}. \quad (8.3.5.10)$$

The equality sign in (8.3.5.10) holds if and only if

$$A_{n+1} = \begin{pmatrix} a & 0 & . & . & . & . & . & 0 \\ 0 & \mu & 0 & . & . & . & . & 0 \\ 0 & . & . & 0 & . & . & . & 0 \\ 0 & . & 0 & \mu & 0 & . & . & 0 \\ 0 & . & . & 0 & b & 0 & . & 0 \\ 0 & . & . & . & 0 & \mu & . & 0 \\ 0 & . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & 0 & \mu \end{pmatrix},$$

with $a + b = \mu$, and

$$A_r = \begin{pmatrix} h_{11}^r & 0 \; . \; 0 & h_{1(n_1+1)}^r & 0 \; . \; 0 \\ 0 & . \; . \; . & . & . \; . \; 0 \\ . & . \; . \; . & . & . \; . \; . \\ 0 & . \; . \; . & 0 & . \; . \; 0 \\ h_{1(n_1+1)}^r & 0 \; . \; 0 & -h_{11}^r & 0 \; . \; 0 \\ 0 & . \; . \; . & 0 & . \; . \; 0 \\ . & . \; . \; . & . & . \; . \; . \\ 0 & . \; . \; . & . & . \; . \; 0 \end{pmatrix},$$

for $r = n+2, \ldots, 2m$.

For $X = e_2, e_3, \ldots, e_{n_1}$, respectively, summing in (8.3.5.10), we obtain

$$n_1\tau \leq \frac{\Delta f}{f} + n_1\frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \{n_1[n(n-1) + 3\|P\|^2 - 2] - 6\sum_{j=1}^{n} g^2(Je_j, e_{n_1+1})\}\frac{c}{2}. \quad (8.3.5.11)$$

The equality sign holds good in (8.3.5.11) if and only if the shape operators have the following form:

$$A_{n+1} = \begin{pmatrix} 0 & 0 & . & . & . & . & . & 0 \\ 0 & \mu & 0 & . & . & . & . & 0 \\ 0 & . & . & 0 & . & . & . & 0 \\ 0 & . & 0 & \mu & 0 & . & . & 0 \\ 0 & . & . & 0 & \mu & 0 & . & 0 \\ 0 & . & . & . & 0 & \mu & . & 0 \\ 0 & . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & 0 & \mu \end{pmatrix} \tag{8.3.5.12}$$

and $A_r = 0$, for any $r = n+2, \dots, 2m$.

Summing up, we proved the following:

Theorem 8.3.5.1 [32] *Let $M_1 \times_f M_2$ be a warped product submanifold of $\mathbf{C}^m$. Then*

$$n_1 \tau \leq \frac{\Delta f}{f} + n_1 \frac{n^2(n-2)}{2(n-1)} \|H\|^2,$$

with equality holding if and only if the shape operator A_{n+1} is given by (8.3.5.12) and $A_r = 0$, for any $r = n+2, \dots, 2m$.

Theorem 8.3.5.2 [32] *Let $M_1 \times_f M_2$ be a warped product submanifold into a complex space form $\widetilde{M}(4c)$ with $c > 0$. Then*

$$n_1 \tau \leq \frac{\Delta f}{f} + n_1 \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + n_1[(n+1)(n-2) + 3\|P\|^2]\frac{c}{2},$$

with equality holding if and only if the shape operator A_{n+1} is given by (8.3.5.12) and $A_r = 0$, for any $r = n+2, \dots, 2m$ and $J(TM_1)$ is orthogonal to TM_2.

Theorem 8.3.5.3 [32] *Let $M_1 \times_f M_2$ be a warped product submanifold into a complex space form $\widetilde{M}(4c)$ with $c < 0$ and $J(TM_1)$ orthogonal to TM_2. Then*

$$n_1 \tau \leq \frac{\Delta f}{f} + n_1 \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + n_1 (n+1)(n-2)\frac{c}{2},$$

with equality holding if and only if the shape operator A_{n+1} is given by (8.3.5.12) and $A_r = 0$, for any $r = n+2, \dots, 2m$.

There are two cases:

(I) If $\mu = 0$, then $M_1 \times_f M_2$ is totally geodesic and, by Gauss equation, the warped product submanifold can be either a complex space form $M(4c)$ or a real space form $M(c)$.

(II) If $\mu \neq 0$, then, by reference to Chen [15] and Dillen [19], $M_1 \times_f M_2$ is a rotational hypersurface with a geodesic as a profile curve, i.e., $I \times N^{n-1}(\delta)$, where I is a real interval and $N^{n-1}(\delta)$ a real space form, and $\Delta f = 4cf$.

8.4 Contact CR-Submanifolds in Sasakian Manifolds

8.4.1 Ricci and k-Ricci Curvatures of Contact CR-Submanifolds in Sasakian Space Forms

B.Y. Chen [8] established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms. We proved (see [36]) such inequalities for certain submanifolds of a Sasakian space form.

Theorem 8.4.1.1 *Let M be an n-dimensional C-totally real submanifold of a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then*

(i) *For each unit vector $X \in T_pM$, we have*

$$\mathrm{Ric}(X) \leq \frac{1}{4}[(n-1)(c+3) + n^2||H||^2]. \tag{8.4.1.1}$$

(ii) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (8.4.1.1) if and only if $X \in \ker h_p$.*

(iii) *The equality case of (8.4.1.1) holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Theorem 8.4.1.2 *Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form and M an n-dimensional submanifold tangent to ξ. Then*

(i) *For each unit vector $X \in T_pM$ orthogonal to ξ, we have*

$$\mathrm{Ric}(X) \leq \frac{1}{4}[(n-1)(c+3) + (3||PX||^2 - 2)(c-1)/2 + n^2||H||^2], \tag{8.4.1.2}$$

where PX is the tangential component of ϕX.

(ii) *If $H(p) = 0$, then a unit tangent vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of (8.4.1.2) if and only if $\in \ker h_p$.*

(iii) *The equality case of (8.4.1.2) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

In particular, for contact CR-submanifolds in Sasakian space forms, we derive the following:

Corollary 8.4.1.3 *Let M be an n-dimensional contact CR-submanifold of a Sasakian space form $\tilde{M}(c)$. Then*

(i) *For each unit vector $X \in \mathcal{D}_p$ orthogonal to ξ, we have*

$$\mathrm{Ric}(X) \leq \frac{1}{4}[(n-1)(c+3) + (c-1)/2 + n^2||H||^2].$$

(ii) *For each unit vector $X \in \mathcal{D}_p^{\perp}$, we have*

$$\mathrm{Ric}(X) \leq \frac{1}{4}[(n-1)(c+3) - c + 1 + n^2||H||^2].$$

We also state an inequality for the k-Ricci curvature of a contact CR-submanifold in a Sasakian space form.

Proposition 8.4.1.4 ([36]) *Let M be an n-dimensional contact CR-submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer k, $2 < k < n$, and any point $p \in M$, we have*

$$||H(p)||^2 \geq \Theta_k(p) - \frac{c+3}{4} + \frac{(3\alpha - n + 1)(c-1)}{2n(n-1)},$$

where $2\alpha = \dim T_pM \cap \phi(T_pM)$.

8.4.2 A Generalized Wintgen Inequality for Contact CR-Submanifolds in Sasakian Space Forms

Recently, in [40], we established certain generalized Wintgen inequalities for C-totally real submanifolds, in particular Legendrian submanifolds, in Sasakian space forms. We use the notations from Sect. 8.3.2.

First, we state an inequality involving the normalized scalar curvature ρ, the normalized scalar normal curvature ρ_N, and the squared mean curvature $||H||^2$ for C-totally real submanifolds in Sasakian space forms.

Proposition 8.4.2.1 *Let M be an n-dimensional C-totally real submanifold of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then we have*

$$||H||^2 + \frac{c+3}{4} \geq \rho + \rho_N.$$

The equality case holds identically if and only if, with respect to suitable orthonormal frames $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2m}, e_{2m+1} = \xi\}$, the shape operators of M^n in $\widetilde{M}^{2m+1}(c)$ take the forms

$$A_{e_{n+1}} = \begin{pmatrix} \lambda_1 & \mu & 0 & \cdots & 0 \\ \mu & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{pmatrix},$$

$$A_{e_{n+2}} = \begin{pmatrix} \lambda_2 + \mu & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{pmatrix},$$

$$A_{e_{n+3}} = \begin{pmatrix} \lambda_3 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_3 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3$ *and* μ *are real functions on* M^n*,*

$$A_{e_{n+4}} = \cdots = A_{e_{2m}} = A_{e_{2m+1}} = 0.$$

Next we are able to derive a generalized Wintgen inequality for Legendrian submanifolds in Sasakian space forms.

Theorem 8.4.2.2 *Let M be an n-dimensional Legendrian submanifold of a Sasakian space form $\widetilde{M}(c)$. Then*

$$(\rho^{\perp})^2 \leq \left(\|H\|^2 - \rho + \frac{c+3}{4}\right)^2 + \frac{4}{n(n-1)}\left(\rho - \frac{c+3}{4}\right) \cdot \frac{c-1}{4} + \frac{(c-1)^2}{8n(n-1)}.$$

Corollary 8.4.2.3 *Let M be a minimal Legendrian submanifold of S^{2n+1}. Then*

$$\rho \leq 1 - \rho^{\perp}.$$

Using similar methods we can state a generalized Wintgen inequality for contact CR-submanifolds in Sasakian space forms.

Let M be an $(n+1)$-dimensional contact CR-submanifold of an $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$.

The difference is made by the Gauss equation. It implies

$$\tau = n + \frac{n(n-1)(c+3)}{8} + \frac{3\alpha(c-1)}{4} + \sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2].$$

Finally, we obtain the following:

Theorem 8.4.2.4 [34] *Let M be an $(n+1)$-dimensional contact CR-submanifold of a Sasakian space form $\tilde{M}(c)$. Then one has*

$$\rho + \rho_N \leq ||H||^2 + \frac{c+3}{4} - \frac{2}{n-1} + \frac{3\alpha(c-1)}{2n(n-1)}.$$

8.4.3 Geometric Inequalities for Contact CR-Warped Product Submanifolds in Sasakian Space Forms

B.Y. Chen established a sharp relationship between the warping function f of a warped product CR-submanifold $M_1 \times_f M_2$ of a Kaehler manifold $\widetilde{M}$ and the squared norm of the second fundamental form $\|h\|^2$ (see [9]).

In [24] we proved a similar inequality for contact CR-warped product submanifolds in a Sasakian manifold.

In this subsection, we investigate warped products $M = M_1 \times_f M_2$ which are contact CR-submanifolds of a Sasakian manifold $\widetilde{M}$. Such submanifolds are always tangent to the structure vector field ξ.

We distinguish two cases:
(a) ξ is tangent to M_1;
(b) ξ is tangent to M_2.

In case (a), one has two subcases:
(1) M_1 is an anti-invariant submanifold and M_2 is an invariant submanifold of $\widetilde{M}$;
(2) M_1 is an invariant submanifold and M_2 is an anti-invariant submanifold of $\widetilde{M}$.

We start with the subcase (1).

Theorem 8.4.3.1 *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold. Then there do not exist warped product submanifolds $M = M_1 \times_f M_2$ such that M_1 is an anti-invariant submanifold tangent to ξ and M_2 an invariant submanifold of $\widetilde{M}$.*

Proof Assume $M = M_1 \times_f M_2$ is a warped product submanifold of a Sasakian manifold $\widetilde{M}$, such that M_1 is an anti-invariant submanifold tangent to ξ and M_2 an invariant submanifold of $\widetilde{M}$.

Gauss formula is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM). \tag{8.4.3.1}$$

Recall that we have

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X, \tag{8.4.3.2}$$

for any vector fields Z and X tangent to M_1 and M_2, respectively.

In particular, for $Z = \xi$, we derive $\xi f = 0$.

Using the last structure equation of a Sasakian manifold, (8.4.3.1) and (8.4.3.2), it follows that

$$\phi X = \widetilde{\nabla}_X \xi = \nabla_X \xi = (\xi \ln f) X = 0. \tag{8.4.3.3}$$

Thus M_2 cannot exist, which achieves the proof.

Consider now the subcase (2).

Theorem 8.4.3.2 *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold and $M = M_1 \times_f M_2$ an n-dimensional warped product submanifold, such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional C-totally real submanifold of $\widetilde{M}$. Then*

(i) *The squared norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq 2\beta[\|\nabla(\ln f)\|^2 + 1], \tag{8.4.3.4}$$

where $\nabla(\ln f)$ is the gradient of $\ln f$.

(ii) *If the equality sign of (8.4.3.4) holds identically, then M_1 is a totally geodesic submanifold and M_2 is a totally umbilical submanifold of $\widetilde{M}$. Moreover, M is a minimal submanifold of $\widetilde{M}$.*

Proof Let $M = M_1 \times_f M_2$ be a warped product submanifold of a Sasakian manifold $\widetilde{M}$, such that M_1 is an invariant submanifold tangent to ξ and M_2 a C-totally real submanifold of $\widetilde{M}$.

For any unit vector fields X tangent to M_1 and Z, W tangent to M_2, respectively, we have

$$\begin{aligned} g(h(\phi X, Z), \phi Z) &= g(\widetilde{\nabla}_Z \phi X, \phi Z) = g(\phi \widetilde{\nabla}_Z X, \phi Z) \\ &= g(\widetilde{\nabla}_Z X, Z) = g(\nabla_Z X, Z) = X \ \ln f. \end{aligned} \tag{8.4.3.5}$$

On the other hand, since the ambient manifold $\widetilde{M}$ is Sasakian, it is easily seen that

$$h(\xi, Z) = \phi Z. \tag{8.4.3.6}$$

Therefore, by (8.4.3.5) and (8.4.3.6) the inequality (8.4.3.4) follows immediately.

Denote by h'' the second fundamental form of M_2 in M. Then, we get

$$g(h''(Z, W), X) = g(\nabla_Z W, X) = -(X \ \ln f) g(Z, W),$$

or equivalently

$$h''(Z, W) = -g(Z, W)\nabla(\ln f). \tag{8.4.3.7}$$

If the equality sign of (8.4.3.4) holds identically, then we obtain

$$h(\mathcal{D}, \mathcal{D}) = 0, \quad h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0, \quad h(\mathcal{D}, \mathcal{D}^\perp) \subset \phi\mathcal{D}^\perp. \tag{8.4.3.8}$$

The first condition (8.4.3.8) implies that M_1 is totally geodesic in M. On the other hand, one has

$$g(h(X, \phi Y), \phi Z) = g(\widetilde{\nabla}_X \phi Y, \phi Z) = g(\nabla_X Y, Z) = 0.$$

Thus M_1 is totally geodesic in $\widetilde{M}$.

The second condition (8.4.3.8) and (8.4.3.7) implies that M_2 is totally umbilical in $\widetilde{M}$. Moreover, by (8.4.3.8), it follows that M is a minimal submanifold of $\widetilde{M}$.

Corollary 8.4.3.3 *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form of constant ϕ-sectional curvature c and $M = M_1 \times_f M_2$ an n-dimensional nontrivial warped product submanifold, such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional C-totally real submanifold of $\widetilde{M}$ satisfying*

$$\|h\|^2 = 2\beta[\|\nabla(\ln f)\|^2 + 1].$$

Then, we have

(i) *M_1 is a totally geodesic invariant submanifold of $\widetilde{M}(c)$. Hence M_1 is a Sasakian space form of constant ϕ-sectional curvature c.*

(ii) *M_2 is a totally umbilical C-totally real submanifold of $\widetilde{M}(c)$. Hence M_2 is a real space form of sectional curvature $\varepsilon > \frac{c+3}{4}$.*

(iii) *If $\beta > 1$, then the warping function f satisfies $\|\nabla f\|^2 = (\varepsilon - \frac{c+3}{4}) f^2$.*

Proof See [24].

Assume now that $M_1 \times_f M_2$ is a warped product submanifold of a Sasakian manifold $\widetilde{M}$ such that ξ is tangent to M_2.

If we put $Z = \xi$ in (8.4.3.2), it follows that $Xf = 0$, for all vector fields X tangent to M_1. Thus f is constant and the warped product becomes a Riemannian product. Also, it follows that M_1 is a C-totally real submanifold of $\widetilde{M}(c)$.

Proposition 8.4.3.4 *Any warped product submanifold $M_1 \times_f M_2$ of a Sasakian manifold $\widetilde{M}$ such that ξ is tangent to M_2 is a Riemannian product. Moreover, M_1 is a C-totally real submanifold.*

Theorem 8.4.3.1 and Proposition 8.4.3.4 show that the only warped products with nonconstant warping function which are contact *CR*-submanifolds in a Sasakian manifold $\widetilde{M}$ have the form $M = M_1 \times_f M_2$, with M_1 an invariant submanifold tangent to ξ and M_2 a C-totally real submanifold of $\widetilde{M}$. We simply call such submanifolds *contact CR-warped products*.

Inspired by [14], we determined the minimum codimension of a contact *CR*-warped product in an odd-dimensional sphere endowed with the standard Sasakian structure.

Theorem 8.4.3.5 ([24]) *Let* $M = M_1 \times_f M_2$ *be a contact CR-warped product in the* $(2m+1)$*-dimensional sphere* S^{2m+1}*. If* M_1 *is compact, then we have*

$$m \geq \alpha + \beta + \alpha\beta,$$

where $\dim M_1 = 2\alpha + 1$ *and* $\dim M_2 = \beta$.

Proof Let $M = M_1 \times_f M_2$ be a warped product submanifold of the $(2m+1)$-dimensional sphere S^{2m+1}, such that M_1 is an invariant submanifold tangent to ξ and M_2 a C-totally real submanifold of S^{2m+1}.

We denote by ν the normal subbundle orthogonal to $\phi\mathcal{D}^\perp$. Obviously, we have

$$T^\perp M = \phi\mathcal{D}^\perp \oplus \nu, \quad \phi\nu = \nu.$$

For any vector fields X tangent to M_1 and orthogonal to ξ and Z tangent to M_2, (8.2.2.1) gives

$$\widetilde{R}(X, \phi X, Z, \phi Z) = 0.$$

On the other hand, by Codazzi equation, we have

$$\begin{aligned}\widetilde{R}(X, \phi X, Z, \phi Z) = &-g(\nabla^\perp_X h(\phi X, Z) - h(\nabla_X \phi X, Z) - h(\phi X, \nabla_X Z), \phi Z) \\ &+ g(\nabla^\perp_{\phi X} h(X, Z) - h(\nabla_{\phi X} X, Z) - h(X, \nabla_{\phi X} Z), \phi Z).\end{aligned} \tag{8.4.3.9}$$

Using (8.4.3.2) and structure equations of a Sasakian manifold, we get

$$\begin{aligned}g(\nabla^\perp_X h(\phi X, Z), \phi Z) &= Xg(h(\phi X, Z), \phi Z) - g(h(\phi X, Z), \nabla^\perp_X \phi Z) \\ &= Xg(\nabla_Z X, Z) - g(h(\phi X, Z), \phi\widetilde{\nabla}_X Z) \\ &= X((X \ln f)g(Z, Z)) - (X \ln f)g(h(\phi X, Z), \phi Z) \\ &\quad - g(h(\phi X, Z), \phi h_\nu(X, Z)) \\ &= (X^2 \ln f)g(Z, Z) + (X \ln f)^2 g(Z, Z) - \|h_\nu(X, Z)\|^2,\end{aligned}$$

where we denote by $h_\nu(X, Z)$ the ν-component of $h(X, Z)$.

Also, we have

$$\begin{aligned}
&g(h(\nabla_X\phi X, Z), \phi Z) = g(\widetilde{\nabla}_Z\nabla_X\phi X, \phi Z)\\
&= g(\widetilde{\nabla}_Z\widetilde{\nabla}_X\phi X, \phi Z) - g(\widetilde{\nabla}_Z h(X, \phi X), \phi Z)\\
&= -g(X, X)g(Z, Z) + ((\nabla_X X)\ln f)g(Z, Z),\\
&g(h(\phi X, \nabla_X Z), \phi Z) = (X\ln f)g(h(\phi X, Z), \phi Z) = (X\ln f)^2 g(Z, Z).
\end{aligned}$$

Substituting the above relations in (8.4.3.9), we find

$$\begin{aligned}
\widetilde{R}(X, \phi X, Z, \phi Z) = \|h_\nu(X, Z)\|^2 &- (X^2\ln f)g(Z, Z) - (X\ln f)^2 g(Z, Z)\\
&+ ((\nabla_X X)\ln f)g(Z, Z) - g(X, X)g(Z, Z) + (X\ln f)^2 g(Z, Z)\\
&+ \|h_\nu(X, Z)\|^2 - ((\phi X)^2\ln f)g(Z, Z) - ((\phi X)\ln f)^2 g(Z, Z)\\
&+ ((\nabla_{\phi X}\phi X)\ln f)g(Z, Z) - g(X, X)g(Z, Z) + ((\phi X)\ln f)^2 g(Z, Z).
\end{aligned} \tag{8.4.3.10}$$

We recall that the Hessian of f is defined by $H^f(X, Y) = XYf - (\nabla_X Y)f$.

Then (8.4.3.10) becomes

$$\|h_\nu(X, Z)\|^2 = [g(X, X) + \frac{1}{2}(H^{\ln f}(X, X) + H^{\ln f}(\phi X, \phi X))]g(Z, Z). \tag{8.4.3.11}$$

Let $\{X_0 = \xi, X_1, \ldots, X_{2\alpha}, Z_1, \ldots, Z_\beta\}$ be a local orthonormal frame on M such that $X_0, \ldots, X_{2\alpha}$ are tangent to M_1 and $Z_1, \ldots, Z_\beta$ are tangent to M_2.

Since the Hessian and the second fundamental form are bilinear, by polarization we obtain

$$\begin{aligned}
&g(h_\nu(X_i, Z_s), h_\nu(X_j, Z_t))\\
&\quad= [1 + \frac{1}{2}(H^{\ln f}(X_i, X_j) + H^{\ln f}(\phi X_i, \phi X_j))]\delta_{ij}\delta_{st},
\end{aligned} \tag{8.4.3.12}$$

which implies that $\{h_\nu(X_i, Z_t) | i = 1, \ldots, 2\alpha; t = 1, \ldots, \beta\}$ are mutually orthogonal vector fields.

Let us assume that M_1 is compact. Then the function $\ln f$ has an absolute minimum at some point $u \in M_1$. At this critical point, the Hessian $H^{\ln f}$ is nonnegative definite. Then, by (8.4.3.4), each $h_\nu(X_i, Z_t) \neq 0$. Therefore, the rank of ν is at least $2\alpha\beta$.

It follows that $m \geq \alpha + \beta + \alpha\beta$.

Remark Same result holds for contact *CR*-warped products in Sasakian space forms $\widetilde{M}(c)$, with $c > -3$.

In this case, we have

$$\|h_\nu(X,Z)\|^2 = \left[\frac{c+3}{4}g(X,X) + \frac{1}{2}(H^{\ln f}(X,X) + H^{\ln f}(\phi X,\phi X))\right]g(Z,Z). \tag{8.4.3.13}$$

Then, using similar arguments as in the proof of Theorem 8.4.3.5, we obtain the same result.

Corollary 8.4.3.6 *Let $\widetilde{M}(c)$ be a Sasakian space form, with $c < -3$. Then there do not exist contact CR-warped products $M_1 \times_f M_2$, with M_1 a compact invariant submanifold tangent to ξ and M_2 a C-totally real submanifold of $\widetilde{M}(c)$.*

Proof Assume there exists a contact *CR*-warped product $M_1 \times_f M_2$ in a Sasakian space form $\widetilde{M}(c)$, with $c < -3$, such that M_1 is compact. Then the function $\ln f$ has an absolute maximum at some point $u \in M_1$. At this critical point, the Hessian $H^{\ln f}$ is nonpositive definite. Thus (8.4.3.13) leads to a contradiction.

In [37] we have improved the Theorem 8.4.3.2 for contact *CR*-warped product submanifolds in Sasakian space forms.

Theorem 8.4.3.7 *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold and $M = M_1 \times_f M_2$ an n-dimensional warped product submanifold, such that M_1 is a $(2\alpha + 1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional C-totally real submanifold of $\widetilde{M}$. Then*

(i) *The squared norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq 2\beta[\|\nabla(\ln f)\|^2 - \Delta(\ln f) + 1] + \alpha\beta(c+3), \tag{8.4.3.14}$$

where Δ is the Laplacian operator on M_1.

(ii) *The equality sign of (8.4.3.14) holds identically if and only if we have*

(a) *M_1 is a totally geodesic submanifold of $\widetilde{M}(c)$. Hence M_1 is a Sasakian space form of constant ϕ-sectional curvature c.*

(b) *M_2 is a totally umbilical submanifold of $\widetilde{M}$. Hence M_2 is a real space form of constant sectional curvature $\varepsilon \geq \frac{c+3}{4}$.*

If M_1 is compact, by integrating the above inequality we obtain the following:

Corollary 8.4.3.8 *Let $M = M_1 \times_f M_2$ be an n-dimensional contact CR-warped product submanifold with compact M_1 in the unit sphere S^{2m+1}. Then for any $q \in M_2$, we have*

$$\int_{M_1\times\{q\}} ||h||^2 \mathrm{dvol}_1 \geq 2\beta(2\alpha+1)\mathrm{vol}(M_1),$$

where dvol$_1$ *and* vol(M_1) *are the volume element and the volume of* M_1*, respectively.*

The equality sign holds identically if and only if we have

(i) *The warping function f is constant.*

(ii) (M_1, g_1) *is isometric to* $S^{2\alpha+1}$ *and it is isometrically immersed in* S^{2m+1} *as a totally geodesic invariant submanifold.*

(iii) (M_2, f^2g_2) *is isometric to an open portion of the sphere* S^β *of constant sectional curvature* 1 *and it is isometrically immersed in* S^{2m+1} *as a totally geodesic C-totally real submanifold.*

If the warping function f is nonconstant, we can improve the above inequality.

Corollary 8.4.3.9 ([37]) *Let* $M = M_1 \times_f M_2$ *be an n-dimensional contact CR-warped product submanifold, with compact* M_1 *and nonconstant warping function* f*, in the unit sphere* S^{2m+1}*. Then for any* $q \in M_2$*, we have*

$$\int_{M_1\times\{q\}} ||h||^2 \mathrm{dvol}_1 \geq 2\beta\lambda_1 \int_{M_1\times\{q\}} (\ln f)^2 \mathrm{dvol}_1 + 2\beta(2\alpha+1)\mathrm{vol}(M_1),$$

where λ_1 *is the first positive eigenvalue of the Laplacian* Δ *on* M_1.

The equality sign holds identically if and only if we have

(i) *The warping function is an eigenfunction of* Δ *corresponding to the eigenvalue* λ_1 *of* Δ.

(ii) *The contact CR-warped product* $M_1 \times_f M_2$ *is both* M_1*-totally geodesic and* M_2*-totally geodesic.*

In [37] we completely classified the contact *CR*-warped products in the unit sphere S^{2m+1} which satisfy identically the equality case of the inequality (8.4.3.14).

Theorem 8.4.3.10 *Let* $x : M = M_1 \times_f M_2 \to S^{2m+1}$ *be an isometric immersion of an n-dimensional contact CR-warped product such that* M_1 *is a* $(2\alpha+1)$*-dimensional invariant submanifold tangent to* ξ *and* M_2 *a* β*-dimensional C-totally real submanifold into the unit sphere* S^{2m+1}*. Then*

(i) *The squared norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq 2\beta[\|\nabla(\ln f)\|^2 - \Delta(\ln f) + 1] + 4\alpha\beta, \quad (8.4.3.15)$$

where Δ *is the Laplacian operator on* M_1.

(ii) *The contact warped product M satisfies the sign of (8.4.3.15) identically if and only if*

(a) M_1 *is an open portion of the unit sphere* $S^{2\alpha+1}$.

(b) M_2 *is an open portion of the unit sphere* S^β.

(c) *There exists a natural number* $h \leq \alpha$ *such that, up to rigid motions, x is given by*

$$x(z, w) = (w^0 z^0, \dots, w^0 z^h, \dots, w^\beta z^0, \dots, w^\beta z^h, z^{h+1}, \dots, z^\alpha, 0, \dots, 0),$$

where $z = (z^0, \dots, z^\alpha) \in S^{2\alpha+1} \subset \mathbf{C}^{\alpha+1}$, $w = (w^0, \dots, w^\beta) \in S^\beta \in \mathbf{E}^{\beta+1}$.

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Chapter 9
CR-Doubly Warped Product Submanifolds

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9.1 Introduction

In 1978, A. Bejancu introduced the notion of a CR-submanifold which is a generalization of holomorphic and totally real submanifolds in an almost Hermitian manifold [4].

Afterwards many papers and books were written in this field. The first main result on CR-submanifolds was obtained by Chen [13]: any CR-submanifold of a Kähler manifold is foliated by totally real submanifolds. As nontrivial examples of CR-submanifolds, we can mention the (real) hypersurfaces of Hermitian manifolds.

In [15], Chen introduced the notion of a CR-warped product submanifold in a Kähler manifold and proved a lot of interesting results on these submanifolds. In particular, he established a sharp relationship between the warping function f of a warped product *CR*-submanifold $M_1 \times_f M_2$ of a Kähler manifold $\widetilde{M}$ and the squared norm of the second fundamental form $||h||^2$.

Later, I. Hasegawa and I. Mihai established a sharp inequality for the squared norm of the second fundamental form (an extrinsec invariant) in terms of the warping function for contact *CR*-warped products isometrically immersed in Sasakian

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manifolds in [20]. Moreover, I. Mihai [25] improved the same inequality for contact *CR*-warped products in Sasakian space forms and he gave some applications. A classification of contact *CR*-warped products in spheres, which satisfy the equality case, identically, was also given.

Furthermore, in [2], K. Arslan, R. Ezentaş, I. Mihai, and C. Murathan considered contact *CR*-warped product submanifolds in Kenmotsu space forms and they obtained sharp estimates for the squared norm of the second fundamental form in terms of the warping function for contact *CR*-warped products isometrically immersed in Kenmotsu space forms.

In [1], R. Al-Ghefari, F. Al-Solamy and M.H. Shahid studied contact *CR*-warped product submanifolds in generalized Sasakian space forms.

Recently, in [3], M. Atçeken studied contact *CR*-warped product submanifolds of a cosymplectic space form and obtained a necessary and sufficient condition for such a submanifold to be a contact *CR*-product.

In [37], S. Sular and C. Özgür considered contact *CR*-warped product submanifolds of a trans-Sasakian generalized Sasakian space forms and obtained a necessary and sufficient condition for a contact *CR*-warped product submanifold of a trans-Sasakian generalized Sasakian space form to be a contact *CR*-product.

Singly warped products or simply warped products were first defined by Bishop and O'Neill in [6]. They used this concept to construct Riemannian manifolds with negative sectional curvature. In general, doubly warped products can be considered as a generalization of singly warped products.

In [27], M.I. Munteanu and then in [23] (see, also, [11]), K. Matsumoto and V. Bonanzinga studied doubly warped product *CR*-submanifolds in locally conformal Kähler manifolds.

In [29], the author established general inequalities for *CR*-doubly warped products isometrically immersed in Sasakian space forms. Later, in [30], the author obtained sharp estimates for the squared norm of the second fundamental form (an extrinsic invariant) in terms of the warping functions (intrinsic invariants) for contact *CR*-doubly warped products isometrically immersed in Kenmotsu space forms. The equality case is considered. Some applications are derived.

The paper presents the results obtained by the author on *CR*-doubly warped products and is organized as follows:

In Sect. 9.2 we give a brief introduction to submanifolds, providing some basic notations, formulas and definitions for later use.

Section 9.3 contains some necessary background on doubly warped products.

In Sect. 9.4 we survey results from [29, 32] for *CR*-doubly warped product submanifolds in Sasakian space forms.

Then, in Sect. 9.5 we survey results from [30, 32] for *CR*-doubly warped product submanifolds in Kenmotsu space forms.

9.2 Preliminaries

9.2.1 Basic Notations and Formulas

In this section, we recall some definitions and basic formulas which we will use later.

Let $\widetilde{M}$ be an m-dimensional Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(u, v)$ the sectional curvature of $\widetilde{M}$ associated with a plane section $\pi \subset T_p\widetilde{M}$, where $\{u, v\}$ is an orthonormal basis of π. For any n-dimensional subspace $L \subseteq T_p\widetilde{M}$, $2 \leq n \leq m$, its scalar curvature $\tau(L)$ is given by

$$\tau(L) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j), \tag{9.1}$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of L. If $L = T_p\widetilde{M}$, then $\tau(L)$ is just the scalar curvature $\tau(p)$ of $\widetilde{M}$ at p.

For an n-dimensional submanifold M in a Riemannian m-manifold $\widetilde{M}$, we denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M and $\widetilde{M}$, respectively. The Gauss and Weingarten formulas are:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h\,(X, Y) \text{ and } \widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{9.2}$$

respectively, for vector fields X, Y tangent to M and ξ normal to M, where h denotes the second fundamental form, $\nabla^\perp$ the normal connection and A the shape operator of M [12]. Denote by $\widetilde{R}$, R, $R^\perp$, the curvature tensors with respect to $\widetilde{\nabla}$, ∇ and $\nabla^\perp$, respectively. Then the *Gauss equation* is expressed by

$$\begin{aligned} \widetilde{R}\,(X, Y, Z, W) &= R\,(X, Y, Z, W) \\ &\quad - g\,(h\,(X, Z)\,, h\,(Y, W)) + g\,(h\,(X, W)\,, h\,(Y, Z))\,, \end{aligned} \tag{9.3}$$

for all vector fields X, Y, Z, W tangent to M [12]. We put

$$(\nabla_X h)\,(Y, Z) = \nabla^\perp_X h\,(Y, Z) - h\,(\nabla_X Y, Z) - h\,(X, \nabla_Y Z)\,; \tag{9.4}$$

then the normal component of $\widetilde{R}\,(X, Y)\,Z$ is given by

$$\left(\widetilde{R}\,(X, Y)\,Z\right)^\perp = (\nabla_X h)\,(Y, Z) - (\nabla_Y h)\,(X, Z)\,. \tag{9.5}$$

The above relation represents the *Codazzi equation*. Using the Weingarten formula, one obtains the *Ricci equation*

$$\begin{aligned} \widetilde{R}\,(X, Y, \xi, \eta) &= R^\perp\,(X, Y, \xi, \eta) \\ &\quad - g\left(A_\eta A_\xi X, Y\right) + g\left(A_\xi A_\eta X, Y\right) \end{aligned}$$

$$= R^{\perp}(X, Y, \xi, \eta) + g\left([A_{\xi}, A_{\eta}]X, Y\right), \tag{9.6}$$

for vector fields X, Y tangent to M and ξ, η normal to M. If the second fundamental form h vanishes identically, M is a *totally geodesic* submanifold.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM, $p \in M$. Let H be the mean curvature vector, i.e.,

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{9.7}$$

The submanifold M is said to be *minimal* if $H = 0$. We denote by

$$h_{ij}^{r} = g\left(h\left(e_i, e_j\right), e_r\right), i, j \in \{1, \ldots, n\}, r \in \{n+1, \ldots, m\}$$

the coefficients of the second fundamental form h with respect to $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$, and

$$||h||^2 = \sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right). \tag{9.8}$$

Let M be an n-dimensional Riemannian manifold and $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on M. For a differentiable function f on M, the *Laplacian* Δf of f is defined by

$$\Delta f = \sum_{j=1}^{n} \left\{\left(\nabla_{e_j} e_j\right) f - e_j e_j f\right\}. \tag{9.9}$$

We recall the following result of Chen for later use.

Lemma 1 *[14] Let $n \geq 2$ and $a_1, a_2, \ldots, a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

9.2.2 Almost Contact Metric Manifolds

A $(2m+1)$-dimensional Riemannian manifold $\left(\widetilde{M}, g\right)$ is said to be an *almost contact metric manifold* if it admits an endomorphism ϕ, a vector field ξ (called the *structure vector field or Reeb vector field*), a 1-form η and a Riemannian metric g satisfying the following properties:

$$\phi^2 = -Id + \eta \otimes \xi, \eta(\xi) = 1, \phi\xi = 0, \eta \circ \phi = 0, \tag{9.10}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi), \tag{9.11}$$

$$g(X, \phi Y) = -g(\phi X, Y) \tag{9.12}$$

for any vector fields X, Y on $\widetilde{M}$.

An almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$ is said to be a *contact metric manifold* if $d\eta = \Phi$, where

$$\Phi(X, Y) = g(\phi X, Y) \tag{9.13}$$

is called the *fundamental* 2-*form* of $\widetilde{M}$.

If, in addition, ξ is a Killing vector field, then $\widetilde{M}$ is said to be a K-*contact manifold*. It is well known that a contact metric manifold is a K-contact manifold if and only if $\widetilde{\nabla}_X \xi = -\phi X$, for any vector field X on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. In a K-contact manifold, we have

$$\left(\widetilde{\nabla}_X \phi\right) Y = \widetilde{R}(X, \xi)Y, \tag{9.14}$$

for any vector fields X, Y.

On the other hand, the almost contact metric structure of $\widetilde{M}$ is said to be *normal* if

$$[\phi, \phi](X, Y) = -2d\eta(X, Y)\xi, \tag{9.15}$$

for any X, Y, where $[\phi, \phi]$ denotes the Nijenhuis torsion of ϕ, given by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]. \tag{9.16}$$

A *Sasakian manifold* is a normal contact metric manifold.

It can be proved that an almost contact metric manifold is Sasakian if and only if the Riemann curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{9.17}$$

for any vector fields X, Y on $\widetilde{M}$.

An almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$ is called *Kenmotsu manifold* if

$$\left(\widetilde{\nabla}_X \phi\right) Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{9.18}$$

$$\widetilde{\nabla}_X \xi = X - \eta(X)\xi, \tag{9.19}$$

for any vector fields X, Y on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

In this case, it is well known that

$$\widetilde{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X. \tag{9.20}$$

Remark 1 A Kenmotsu manifold is normal, but not Sasakian.

Given an almost contact metric manifold $\left(\widetilde{M}, \phi, \xi, \eta, g\right)$, a *$\phi$-section* of $\widetilde{M}$ at $p \in \widetilde{M}$ is a section $\pi \subseteq T_p\widetilde{M}$ spanned by X_p and ϕX_p, where X_p is a unit tangent vector orthogonal to ξ_p. The sectional curvature of a ϕ-section is called a *ϕ-sectional curvature*. A Sasakian (resp. Kenmotsu) manifold with constant ϕ-sectional curvature is a *Sasakian* (resp. *Kenmotsu*) *space form* and is denoted by $\widetilde{M}(c)$. As examples of Sasakian space forms, we mention R^{2m+1} and S^{2m+1} with standard Sasakian structures (see [7]). For a Sasakian space form the Riemann curvature tensor is given by [7]

$$\begin{aligned}\widetilde{R}(X,Y)Z = {}& \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\}\\ &+ \frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}\\ &+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},\end{aligned} \tag{9.21}$$

for any vector fields X, Y, Z on $\widetilde{M}(c)$. The Riemann curvature tensor $\widetilde{R}$ of a Kenmotsu space form is given by [21]

$$\begin{aligned}\widetilde{R}(X,Y)Z = {}& \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\}\\ &+ \frac{c+1}{4}\{[\eta(X)Y - \eta(Y)X]\eta(Z) + [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi\\ &+ \omega(Y,Z)\phi X - \omega(X,Z)\phi Y - 2\omega(X,Y)\phi Z\}.\end{aligned} \tag{9.22}$$

9.2.3 Submanifolds of Almost Contact Metric Manifolds

Let $\left(\widetilde{M}, \phi, \xi, \eta, g\right)$ be an almost contact manifold. A submanifold M normal to ξ in $\widetilde{M}$ is said to be a *C-totally real* submanifold. If $\widetilde{M}$ is a K-contact manifold, it follows that ϕ maps any tangent space of M into the normal space, that is, $\phi\left(T_pM\right) \subset T_p^{\perp}M$, for every $p \in M$. For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. We mention the following:

(1) A submanifold M tangent to ξ is said to be *invariant* submanifold if ϕ preserves any tangent space of M, that is $\phi\left(T_pM\right) \subset T_pM, \forall p \in M$.
(2) A submanifold M tangent to ξ is said to be *anti-invariant* submanifold if ϕ maps any tangent space of M into the normal space, that is, $\phi\left(T_pM\right) \subset T_p^{\perp}M$, $\forall p \in M$.
(3) A submanifold M tangent to ξ is called a *contact CR-submanifold* if there exists a pair of orthogonal differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on M, such that:

   (a) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \{\xi\}$, where $\{\xi\}$ is the 1-dimensional distribution spanned by ξ;
   (b) $\mathcal{D}$ is invariant by ϕ, i.e., $\phi\left(\mathcal{D}_p\right) = \mathcal{D}_p, \forall p \in M$;
   (c) $\mathcal{D}^{\perp}$ is invariant by ϕ, i.e., $\phi\left(\mathcal{D}_p^{\perp}\right) \subset \mathcal{D}_p^{\perp}, \forall p \in M$.
   In particular, if $\mathcal{D}^{\perp} = \{0\}$ (resp. $\mathcal{D} = \{0\}$), M is an invariant (resp. anti-invariant) submanifold.

9.3 Doubly Warped Products

Doubly warped products can be considered as a generalization of singly warped products.

Definition 1 Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and let $\sigma_1 : M_1 \to (0, \infty)$ and $\sigma_2 : M_2 \to (0, \infty)$ be differentiable functions.

The doubly warped product $M = {}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ is the product manifold $M_1 \times M_2$ endowed with the metric

$$g = \sigma_2^2 g_1 + \sigma_1^2 g_2. \tag{9.23}$$

More precisely, if $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are natural projections, the metric g is defined by

$$g = (\sigma_2 \circ \pi_2)^2\, \pi_1^* g_1 + (\sigma_1 \circ \pi_1)^2\, \pi_2^* g_2. \tag{9.24}$$

The functions σ_1 and σ_2 are called warping functions.

Remark 2 If either $\sigma_1 \equiv 1$ or $\sigma_2 \equiv 1$, but not both, then we obtain a warped product. If both $\sigma_1 \equiv 1$ and $\sigma_2 \equiv 1$, then we have a Riemannian product manifold. If neither σ_1 nor σ_2 is constant, then we have a nontrivial doubly warped product.

Examples ([19])

(1) Assume that M_1 ($\dim M_1 \geq 2$) is an open subset of $R^r \backslash \{(0, \ldots, 0)\}$, $g_{1,ab} = \delta_{ab}$, $a, b \in \{1, \ldots, r\}$,

$$\sigma_1 = \sigma_1\left(x^1, \ldots, x^r\right) = \frac{B}{2}\left(\sum_{a=1}^{r} \left(x^a\right)^2\right),$$

and M_2 ($\dim M_2 \geq 2$) is an open subset of $\mathbb{R}^{n-r} \setminus \{(0, \ldots, 0)\}$, $g_{2,\alpha\beta} = \delta_{\alpha\beta}$, $\alpha, \beta \in \{r+1, \ldots, n\}$,

$$\sigma_2 = \sigma_2\left(x^{r+1}, \ldots, x^n\right) = \frac{A}{2}\left(\sum_{\alpha=r+1}^{n} (x^\alpha)^2\right),$$

with positive constants A and B. Thus, ${}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ is a doubly warped product.

(2) Let $M_1 = \{(x, y) \in \mathbb{R}^2 | \ y > 0\}$ with the metric tensor g_1 defined by

$$g_1\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = uv, \ g_1\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \frac{1}{v}, \ g_1\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0,$$

where $u = u(x)$, $v = v(y)$ are smooth functions not vanishing at any point on M_1. Next let $\sigma_1 = \sigma_1(y) = y$ and $M_2 = \{(z, t) \in \mathbb{R}^2 | t > 0\}$ with g_2 defined by

$$g_2\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = pq, \ g_2\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \frac{1}{q}, \ g_2\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right) = 0,$$

where $p = p(z)$, $q = q(t)$ are smooth functions not vanishing at any point on M_2, and $\sigma_2 = \sigma_2(t) = t$.

${}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ is also a doubly warped product.

(3) Let M_1 ($\dim M_1 = 2$) and σ_1 be as in Example 2, and M_2 ($\dim M_2 = n-2 \geq 2$) and σ_2 be as in Example 1. Then, ${}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ is a doubly warped product.

In a doubly warped product ${}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ we have

$$\begin{cases} \nabla_X Y = \nabla^1_X Y - \frac{\sigma_2^2}{\sigma_1^2} g_1(X, Y)\, \nabla^2(\ln \sigma_2), \\ \nabla_X Z = Z(\ln \sigma_2)\, X + X(\ln \sigma_1)\, Z, \end{cases} \tag{9.25}$$

for any vector fields X, Z tangent to M_1 and M_2, respectively, where ∇^1 and ∇^2 are the Levi-Civita connections of the Riemannian metrics g_1 and g_2, respectively (see [22, 38]). Here, $\nabla^2(\ln \sigma_2)$ denotes the gradient of $\ln \sigma_2$ with respect to the metric g_2.

If X and Z are unit vector fields, the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = \frac{1}{\sigma_1}\left\{\left(\nabla^1_X X\right)\sigma_1 - X^2\sigma_1\right\} + \frac{1}{\sigma_2}\left\{\left(\nabla^2_Z Z\right)\sigma_2 - Z^2\sigma_2\right\}, \tag{9.26}$$

where ∇^1 and ∇^2 are the Riemannian connections of the Riemannian metrics g_1 and g_2, respectively.

9.4 CR-doubly Warped Products in Sasakian Space Forms

9.4.1 A General Inequality

In this section, we present the following results from [29, 32] for contact *CR*-doubly warped product submanifolds in Sasakian space forms.

First, we investigat doubly warped products $M =_{\sigma_2} M_1 \times_{\sigma_1} M_2$ which are *CR*-submanifolds of a Sasakian manifold $\widetilde{M}$. Such manifolds are always tangent to the structure vector field ξ. We distinguish 2 cases

(1) ξ is tangent to M_1;
(2) ξ is tangent to M_2.

In case 1, one has two subcases:

(a) M_1 is an anti-invariant submanifold and M_2 is an invariant submanifold of $\widetilde{M}$;

(b) M_1 is an invariant submanifold and M_2 is an anti-invariant submanifold of $\widetilde{M}$.

We start with the subcase (a).

Theorem 9.1 *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold. Then there do not exist doubly warped product submanifolds $M =_{\sigma_2} M_1 \times_{\sigma_1} M_2$ such that M_1 is an anti-invariant submanifold tangent to ξ and M_2 is an invariant submanifold of $\widetilde{M}$.*

Proof Assume $M =_{\sigma_2} M_1 \times_{\sigma_1} M_2$ is a doubly warped product submanifold of a Sasakian manifold $\widetilde{M}$, such that M_1 is an anti-invariant submanifold tangent to ξ and M_2 is an invariant submanifold of $\widetilde{M}$.

By Eq. (9.25), we have

$$\nabla_X Z = \nabla_Z X = Z(\ln \sigma_2)\, X + X(\ln \sigma_1)\, Z, \tag{9.27}$$

for any vector fields X and Z tangent to M_1 and M_2, respectively.

In particular, for $X = \xi$, we derive $\xi\sigma_1 = 0$.

Using the last structure equation of a Sasakian manifold, Gauss formula and (9.27) we get

$$\phi Z = \widetilde{\nabla}_Z \xi = (Z \ln \sigma_2)\, \xi + (\xi \ln \sigma_1)\, Z. \tag{9.28}$$

It follows that $Z \ln \sigma_2 = 0$ for any vector field Z tangent to M_2.

Thus σ_2 is a constant and the doubly warped product becomes a warped product.

So, there do not exist doubly warped product submanifolds $M =_{\sigma_2} M_1 \times_{\sigma_1} M_2$ such that M_1 is an anti-invariant submanifold tangent to ξ and M_2 is an invariant submanifold of $\widetilde{M}$.

Consider now the subcase (b).

Theorem 9.2 *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold and $M =_{\sigma_2} M_1 \times_{\sigma_1} M_2$ an n-dimensional doubly warped product such that M_1 is a $(2\alpha+1)$ -dimensional invariant submanifold tangent to ξ and M_2 is a β-dimensional C-totally real submanifold of $\widetilde{M}$.Then:*

(i) The squared norm of the second fundamental form of M satisfies:

$$||h||^2 \geq 2\beta \left[||\nabla (\ln \sigma_1)||^2 + 1 \right], \tag{9.29}$$

where $\nabla (\ln \sigma_1)$ is the gradient of $\ln \sigma_1$.

(ii) If the equality sign of (9.29) holds identically, then both M_1 and M_2 are totally umbilical submanifolds of $\widetilde{M}$. Moreover, M is a minimal submanifold of $\widetilde{M}$.

Proof Let $M =_{\sigma_2} M_1 \times_{\sigma_1} M_2$ be a doubly warped product submanifold of a Sasakian manifold $\widetilde{M}$, such that M_1 is an invariant submanifold tangent to ξ and M_2 is a C-totally real submanifold of $\widetilde{M}$.

For any unit vector fields X tangent to M_1 and Z, W tangent to M_2, respectively, we have:

$$\begin{aligned} g(h(\phi X, Z), \phi Z) &= g\left(\widetilde{\nabla}_Z \phi X, \phi Z\right) = g\left(\phi \widetilde{\nabla}_Z X, \phi Z\right) \\ &= g\left(\widetilde{\nabla}_Z X, Z\right) = g(\nabla_Z X, Z) = X \ln \sigma_1. \end{aligned} \tag{9.30}$$

On the other hand, since the ambient manifold $\widetilde{M}$ is Sasakian, it is easily seen that

$$h(\xi, Z) = \phi Z. \tag{9.31}$$

Therefore, by (9.30) and (9.31) the inequality (9.29) is immediately obtained.

Denote by h'' the second fundamental form of M_2 in M. Then we get

$$g\left(h''(Z, W), X\right) = g(\nabla_Z W, X) = -(X \ln \sigma_1) g(Z, W,)$$

or equivalently

$$h''(Z, W) = -g(Z, W) \nabla (\ln \sigma_1). \tag{9.32}$$

If the equality sign of (9.29) holds identically, then we obtain

$$h(\mathcal{D},\mathcal{D}) = 0, h\left(\mathcal{D}^{\perp},\mathcal{D}^{\perp}\right) = 0, h\left(\mathcal{D},\mathcal{D}^{\perp}\right) \subset \phi\mathcal{D}^{\perp}. \tag{9.33}$$

The first condition (9.33) implies that M_1 is totally geodesic in M. On the other hand, one has

$$g(h(X, \phi Y), \phi Z) = g\left(\widetilde{\nabla}_X \phi Y, \phi Z\right) = g(\nabla_X Y, Z) = -Z(\ln \sigma_2) g(X, Y).$$

Thus M_1 is totally umbilical in $\widetilde{M}$.

The second condition (9.33) and (9.32) imply that M_2 is totally umbilical submanifold in $\widetilde{M}$.

Moreover, by (9.33), it follows that M is a minimal submanifold of $\widetilde{M}$.

Corollary 1 *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form of constant ϕ-sectional curvature c and $M = {}_{\sigma_2} M_1 \times_{\sigma_1} M_2$ an n-dimensional nontrivial doubly warped product submanifold, such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 is a β-dimensional C-totally real submanifold of $\widetilde{M}(c)$ satisfying*

$$||h||^2 = 2\beta \left[||\nabla (\ln \sigma_1)||^2 + 1 \right].$$

Then we have

(a) M_1 is a totally umbilical invariant submanifold of $\widetilde{M}(c)$. Hence M_1 is a Sasakian space form of constant ϕ-sectional curvature $< c$.

(b) M_2 is a totally umbilical C-totally real submanifold of $\widetilde{M}(c)$. Hence M_2 is a real space form of sectional curvature $\varepsilon > \frac{c+3}{4}$.

Proof Statement (a) follows from Theorem 9.2 and the Gauss equation.

Also, we know that M_2 is a totally umbilical C-totally real submanifold of $\widetilde{M}(c)$. Gauss equation implies that M_2 is a real space form of sectional curvature $\varepsilon \geq \frac{c+3}{4}$.

Moreover, by (9.25), we see that $\varepsilon = \frac{c+3}{4}$ if and only if the warping function σ_1 is constant.

Assume now that ${}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ is a doubly warped product submanifold of a Sasakian manifold $\widetilde{M}$ such that ξ is tangent to M_2 (case *2*).

Proposition 1 *Any doubly warped product submanifold ${}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ of a Sasakian manifold $\widetilde{M}$ such that ξ is tangent to M_2 is a warped product. Moreover, M_1 is a C -totally real submanifold.*

Proof Let ${}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ be a doubly warped product submanifold of a Sasakian manifold $\widetilde{M}$ such that ξ is tangent to M_2.

If we put $Z = \xi$ in (9.25), we get

$$\nabla_X \xi = \xi(\ln \sigma_2)X + X(\ln \sigma_1)\xi.$$

It follows that $X \ln \sigma_1 = 0$, for any vector field X tangent to M_1. Thus σ_1 is constant and the doubly warped product becomes a warped product.

Also, it follows that M_1 is a C-totally real submanifold of $\widetilde{M}$.

Theorem 9.2 and Proposition 1 show that the only nontrivial doubly warped products which are contact CR-submanifolds in a Sasakian manifold $\widetilde{M}$ have the form $M = {}_{\sigma_2} M_1 \times_{\sigma_1} M_2$, with M_1 is an invariant submanifold tangent to ξ and M_2 a C-totally real submanifold submanifold of $\widetilde{M}$. We simply call such submanifolds as *contact CR-doubly warped products*.

9.4.2 Another Inequality

Next we improve the inequality (9.29) from Theorem 9.2.

Theorem 9.3 ([29]) *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form of constant ϕ-sectional curvature c and $M = _{\sigma_2} M_1 \times_{\sigma_1} M_2$ an n-dimensional contact CR-doubly warped product submanifold, such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 is a β-dimensional C-totally real submanifold of $\widetilde{M}(c)$. Then:*

(i) The squared norm of the second fundamental form of M satisfies

$$||h||^2 \geq 2\beta\left[||\nabla(\ln\sigma_1)||^2 - \Delta_1(\ln\sigma_1) + 1\right] + \alpha\beta(c+3), \tag{9.34}$$

where Δ_1 denotes the Laplace operator on M_1.

(ii) The equality sign of (9.34) holds identically if and only if we have:

(a) M_1 is a totally umbilical invariant submanifold of $\widetilde{M}(c)$. Hence M_1 is a Sasakian space form of constant ϕ-sectional curvature $< c$.

(b) M_2 is a totally umbilical C-totally real submanifold of $\widetilde{M}(c)$. Hence M_2 is a real space form of sectional curvature $\varepsilon > \frac{c+3}{4}$.

Proof Let $M = _{\sigma_2} M_1 \times_{\sigma_1} M_2$ be a contact CR-doubly warped submanifold in a Sasakian space form $\widetilde{M}(c)$, such that $\dim M_1 = 2\alpha+1$ and $\dim M_2 = \beta$.

Let $\{X_0 = \xi, X_1, \ldots, X_\alpha, X_{\alpha+1} = \phi X_1, \ldots, X_{2\alpha} = \phi X_\alpha, Z_1, \ldots, Z_\beta\}$ be a local orthonormal frame on M such that $X_0, \ldots, X_{2\alpha}$ are tangent to M_1 and $Z_1, \ldots, Z_\beta$ are tangent to M_2.

For any unit vector fields X tangent to M_1 and Z, W tangent to M_2, respectively, we have:

$$\begin{aligned} g(h(\phi X, Z), \phi Z) &= g\left(\widetilde{\nabla}_Z \phi X, \phi Z\right) = g\left(\phi\widetilde{\nabla}_Z X, \phi Z\right) \\ &= g\left(\widetilde{\nabla}_Z X, Z\right) = g(\nabla_Z X, Z) = X \ln\sigma_1. \end{aligned} \tag{9.35}$$

On the other hand, since the ambient manifold $\widetilde{M}$ is Sasakian, it is easily seen that

$$h(\xi, Z) = \phi Z. \tag{9.36}$$

We denote by $h_{\phi\mathcal{D}^\perp}(X, Z)$ the component of $h(X, Z)$ in $\phi\mathcal{D}^\perp$. Therefore, by (9.35) and (9.36), it follows that

$$\sum_{i=0}^{2\alpha}\sum_{t=1}^{\beta} ||h_{\phi\mathcal{D}^\perp}(X_i, Z_t)||^2 = \beta\left[||\nabla(\ln\sigma_1)||^2 + 1\right]. \tag{9.37}$$

Let ν be the normal subbundle orthogonal to $\phi\mathcal{D}^\perp$. Obviously, we have

$$T^\perp M = \phi\mathcal{D}^\perp \oplus \nu, \; \phi\nu = \nu.$$

For any unit vector fields X tangent to M_1 and orthogonal to ξ and Z tangent to M_2, Eq. (9.21) gives

$$\widetilde{R}(X, \phi X, Z, \phi Z) = \frac{c-1}{2}.$$

On the other hand, by Codazzi equation (9.5), we have

$$\widetilde{R}(X, \phi X, Z, \phi Z) = -g\left(\nabla^{\perp}_{X} h(\phi X, Z) - h(\nabla_X \phi X, Z) - h(\phi X, \nabla_X Z), \phi Z\right) + g\left(\nabla^{\perp}_{\phi X} h(X, Z) - h\left(\nabla_{\phi X} X, Z\right) - h\left(\phi X, \nabla_{\phi X} Z\right), \phi Z\right). \tag{9.38}$$

Using Eq. (9.25) and structure equations of a Sasakian manifold, we get

$$\begin{aligned} g\left(\nabla^{\perp}_{X} h(\phi X, Z), \phi Z\right) &= Xg(h(\phi X, Z), \phi Z) - g\left(h(\phi X, Z), \nabla^{\perp}_{X} \phi Z\right) \\ &= Xg(\nabla_Z X, Z) - g\left(h(\phi X, Z), \phi \widetilde{\nabla}_X Z\right) \\ &= X((X \ln \sigma_1) g(Z, Z)) - (X \ln \sigma_1) g(h(\phi X, Z), \phi Z) \\ &\quad - g(h(\phi X, Z), \phi h_{\nu}(X, Z)) \\ &= \left(X^2 \ln \sigma_1\right) g(Z, Z) + (X \ln \sigma_1)^2 g(Z, Z) - ||h_{\nu}(X, Z)||^2, \end{aligned}$$

where we denote by $h_{\nu}(X, Z)$ the ν-component of $h(X, Z)$. Also, we have

$$\begin{aligned} g(h(\nabla_X \phi X, Z), \phi Z) &= g\left(\widetilde{\nabla}_Z \nabla_X \phi X, \phi Z\right) \\ &= g\left(\widetilde{\nabla}_Z \widetilde{\nabla}_X \phi X, \phi Z\right) - g\left(\widetilde{\nabla}_Z h(X, \phi X), \phi Z\right) \\ &= -g(X, X) g(Z, Z) + ((\nabla_X X) \ln \sigma_1) g(Z, Z), \\ &\quad g(h(\phi X, \nabla_X Z), \phi Z) \\ &= (X \ln \sigma_1) g(h(\phi X, Z), \phi Z) = (X \ln \sigma_1)^2 g(Z, Z). \end{aligned}$$

Substituting the above relations in (9.38), we find

$$\begin{aligned} &\widetilde{R}(X, \phi X, Z, \phi Z) \\ &= 2||h_{\nu}(X, Z)||^2 - \left(X^2 \ln \sigma_1\right) g(Z, Z) \\ &\quad + ((\nabla_X X) \ln \sigma_1) g(Z, Z) - 2g(X, X) g(Z, Z) \\ &\quad - \left((\phi X)^2 \ln \sigma_1\right) g(Z, Z) + \left(\left(\nabla_{\phi X} \phi X\right) \ln \sigma_1\right) g(Z, Z). \end{aligned} \tag{9.39}$$

By summing the Eq. (9.39), one finds

$$\sum_{i=0}^{2\alpha} \sum_{t=1}^{\beta} ||h_{\nu}(X_i, Z_t)||^2 = \frac{c+3}{2} \alpha\beta - \beta \Delta_1(\ln \sigma_1). \tag{9.40}$$

Combining (9.37) and (9.40), we obtain the inequality (9.34). Denote by h'' the second fundamental form of M_2 in M . Then we get

$$g\left(h''(Z,W),X\right)=g\left(\nabla_Z W,X\right)=-\left(X\ln\sigma_1\right)g(Z,W)$$

or equivalently

$$h''(Z,W)=-g(Z,W)\,\nabla\left(\ln\sigma_1\right). \tag{9.41}$$

If the equality sign of (9.34) holds identically, then we obtain

$$h(\mathcal{D},\mathcal{D})=0,\ h\left(\mathcal{D}^{\perp},\mathcal{D}^{\perp}\right)=0. \tag{9.42}$$

The first condition (9.42) implies that M_1 is totally geodesic in M. On the other hand, one has

$$\begin{aligned} g\left(h(X,\phi Y),\phi Z\right) &= g\left(\widetilde{\nabla}_X\phi Y,\phi Z\right) \\ &= g\left(\nabla_X Y,Z\right)=-Z(\ln\sigma_2)g(X,Y). \end{aligned}$$

Thus M_1 is totally umbilical in $\widetilde{M}(c)$, and hence is a Sasakian space form with constant ϕ-sectional curvature $< c$. The second condition (9.42) and (9.41) imply that M_2 is totally umbilical submanifold in $\widetilde{M}(c)$. Moreover, by (9.42), it follows that M is a minimal submanifold of $\widetilde{M}(c)$. Gauss equation (9.3) implies that M_2 is a real space form of sectional curvature $\varepsilon\geq\frac{c+3}{4}$. Moreover, by (9.25), we see that $\varepsilon=\frac{c+3}{4}$ if and only if the warping function σ_1 is constant.

9.4.3 Minimum Codimension of a Contact CR-doubly Warped Product

Now we determine the minimum codimension of a contact CR-doubly warped product in a Sasakian space form $\widetilde{M}(c)$ with $c>-3$.

Theorem 9.4 *Let $M={}_{\sigma_2}M_1\times_{\sigma_1}M_2$ be a contact CR-doubly warped product into a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ with $c>-3$. If M_1 is compact, then we have*

$$m\geq\alpha+\beta+\alpha\beta, \tag{9.43}$$

where $\dim M_1=2\alpha+1$ *and* $\dim M_2=\beta$.

Proof Let $M={}_{\sigma_2}M_1\times_{\sigma_1}M_2$ be a doubly warped product submanifold of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$ with $c>-3$, such that M_1 is an invariant submanifold tangent to ξ and M_2 a C-totally real submanifold of $\widetilde{M}(c)$. We denote by ν the normal subbundle orthogonal to $\phi\mathcal{D}^{\perp}$. Obviously, we have

$$T^{\perp}M=\phi\mathcal{D}^{\perp}\oplus\nu,\ \phi\nu=\nu.$$

For any unit vector fields X tangent to M_1 and orthogonal to ξ and Z tangent to M_2, Eq. (9.21) gives

$$\widetilde{R}(X, \phi X, Z, \phi Z) = \frac{c-1}{2}.$$

On the other hand, by Codazzi equation (9.5), we have

$$\begin{aligned}\widetilde{R}(X, \phi X, Z, \phi Z) = &-g\left(\nabla^{\perp}_{X} h(\phi X, Z) - h(\nabla_X \phi X, Z) - h(\phi X, \nabla_X Z), \phi Z\right)\\ &+ g\left(\nabla^{\perp}_{\phi X} h(X, Z) - h\left(\nabla_{\phi X} X, Z\right) - h\left(\phi X, \nabla_{\phi X} Z\right), \phi Z\right). \quad (9.44)\end{aligned}$$

Using the second equation from (9.25) and structure equations of a Sasakian manifold, we get

$$\begin{aligned} g\left(\nabla^{\perp}_{X} h(\phi X, Z), \phi Z\right) &= Xg(h(\phi X, Z), \phi Z) - g\left(h(\phi X, Z), \nabla^{\perp}_{X} \phi Z\right)\\ &= Xg(\nabla_Z X, Z) - g\left(h(\phi X, Z), \phi \widetilde{\nabla}_X Z\right)\\ &= X((X \ln \sigma_1) g(Z, Z)) - (X \ln \sigma_1) g(h(\phi X, Z), \phi Z)\\ &\quad - g(h(\phi X, Z), \phi h_{\nu}(X, Z))\\ &= \left(X^2 \ln \sigma_1\right) g(Z, Z) + (X \ln \sigma_1)^2 g(Z, Z) - ||h_{\nu}(X, Z)||^2, \end{aligned}$$

where we denote by $h_{\nu}(X, Z)$ the ν-component of $h(X, Z)$. Also, we have

$$\begin{aligned} g(h(\nabla_X \phi X, Z), \phi Z) &= g\left(\widetilde{\nabla}_Z \nabla_X \phi X, \phi Z\right)\\ &= g\left(\widetilde{\nabla}_Z \widetilde{\nabla}_X \phi X, \phi Z\right) - g\left(\widetilde{\nabla}_Z h(X, \phi X), \phi Z\right)\\ &= -g(X, X) g(Z, Z) + ((\nabla_X X) \ln \sigma_1) g(Z, Z),\\ & g(h(\phi X, \nabla_X Z), \phi Z)\\ &= (X \ln \sigma_1) g(h(\phi X, Z), \phi Z) = (X \ln \sigma_1)^2 g(Z, Z). \end{aligned}$$

Substituting the above relations in (9.44), we find

$$\begin{aligned} &\widetilde{R}(X, \phi X, Z, \phi Z)\\ &\quad = ||h_{\nu}(X, Z)||^2 - \left(X^2 \ln \sigma_1\right) g(Z, Z) - (X \ln \sigma_1)^2 g(Z, Z)\\ &\qquad + ((\nabla_X X) \ln \sigma_1) g(Z, Z) - g(X, X) g(Z, Z) + (X \ln \sigma_1)^2 g(Z, Z)\\ &\qquad + ||h_{\nu}(X, Z)||^2 - \left((\phi X)^2 \ln \sigma_1\right) g(Z, Z) - ((\phi X) \ln \sigma_1)^2 g(Z, Z)\\ &\qquad + \left(\left(\nabla_{\phi X} \phi X\right) \ln \sigma_1\right) g(Z, Z)\\ &\qquad - g(X, X) g(Z, Z) + ((\phi X) \ln \sigma_1)^2 g(Z, Z). \end{aligned} \quad (9.45)$$

We recall that the Hessian of σ_1 is defined by

$$H^{\sigma_1}(X, Y) = XY\sigma_1 - (\nabla_X Y)\sigma_1.$$

We will use the complex Hessian given by

$$H_C^{\sigma_1}(X, Y) = \frac{1}{2}H^{\sigma_1}(X, Y) + \frac{1}{2}H^{\sigma_1}(\phi X, \phi Y).$$

Then the Eq. (9.45) becomes

$$||h_\nu (X, Z)||^2 = \left[\frac{c+3}{4}g(X, X) + H_C^{\ln \sigma_1}(X, X)\right] g(Z, Z). \tag{9.46}$$

Let $\{X_0 = \xi, X_1, \ldots, X_{2\alpha}, Z_1, \ldots, Z_\beta\}$ be a local orthonormal frame on M such that $X_0, \ldots, X_{2\alpha}$ are tangent to M_1 and $Z_1, \ldots, Z_\beta$ are tangent to M_2. Since the Hessian and the second fundamental form are bilinear, by polarization we obtain

$$g(h_\nu (X_i, Z_s), h_\nu (X_j, Z_t)) = \left[\frac{c+3}{2}\delta_{ij} + H_C^{\ln \sigma_1}(X_i, X_j)\right]\delta_{st}. \tag{9.47}$$

Let assume that M_1 is compact. Then the function $\ln \sigma_1$ has an absolute minimum at some point $u \in M_1$. At this critical point, the complex Hessian $H_C^{\ln \sigma_1}$ is non-negative definite. Then by (9.46), each $h_\nu (X_i, Z_t) \neq 0$ at u. Since $H_C^{\ln \sigma_1}$ is self-adjoint, we can choose an orthonormal basis X_0, ..., $X_{2\alpha}$ at $u \in M_1$ which diagonalizes $H_C^{\ln \sigma_1}$. Then by (9.47), it follows that the vectors

$$h_\nu (X_i, Z_t);\ i = 1, ..., 2\alpha, t = 1, ..., \beta,$$

at u are mutually orthogonal nonzero vectors. Therefore, the rank of ν is at least $2\alpha\beta$. It follows that $m \geq \alpha + \beta + \alpha\beta$.

Corollary 2 *Let $\widetilde{M}(c)$ be a Sasakian space form with $c < -3$. Then there do not exist contact CR-doubly warped products ${}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ with M_1 a compact invariant submanifold tangent to ξ and M_2 a C-totally real submanifold of $\widetilde{M}(c)$.*

Proof Assume there exists a contact CR-doubly warped product ${}_{\sigma_2}M_1 \times_{\sigma_1} M_2$ in a Sasakian space form with $c < -3$, such that M_1 is compact. Then the function $\ln \sigma_1$ has an absolute maximum at some point $u \in M_1$. At this critical point, the Hessian $H^{\ln \sigma_1}$ is non-positive definite. Thus (9.46) leads to a contradiction.

9.5 CR-doubly Warped Products in Kenmotsu Space Forms

9.5.1 A General Inequality

In this section, we present the following results from [30] and [32] for contact *CR*-doubly warped product submanifolds in Kenmotsu space forms.

We prove estimates of the squared norm of the second fundamental form in terms of the warping function. Equality cases are investigated and obstructions to the existence of contact *CR*-doubly warped product submanifolds in Kenmotsu space forms are derived.

A doubly warped product submanifold $M = {}_{f_2} M_1 \times_{f_1} M_2$ of a Kenmotsu manifold $\widetilde{M}$, with M_1 a $(2\alpha + 1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional anti-invariant submanifold of $\widetilde{M}$ is said to be a *contact CR-doubly warped product submanifolfd*. We state the following estimate of the squared norm of the second fundamental form for contact *CR*-doubly warped products in Kenmotsu manifolds.

Theorem 9.5 *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Kenmotsu manifold and $M = {}_{f_2} M_1 \times_{f_1} M_2$ an n-dimensional contact CR-doubly warped product submanifold, such that M_1 is a $(2\alpha + 1)$-dimensional invariant submanifold tangent to ξ and M_2 is a β-dimensional anti-invariant submanifold of $\widetilde{M}(c)$. Then:*

(i) The squared norm of the second fundamental form of M satisfies

$$||h||^2 \geq 2\beta||\nabla(\ln f_1)||^2 - 1], \tag{9.48}$$

where $\nabla(\ln f_1)$ *is the gradient of* $\ln f_1$.

(ii) If the equality sign of (9.48) holds identically, then both M_1 and M_2 are totally umbilical submanifolds of $\widetilde{M}$. Moreover, M is a minimal submanifold of $\widetilde{M}$.

Proof Let $M = {}_{f_2} M_1 \times_{f_1} M_2$ be a doubly warped product submanifold of a Kenmotsu manifold $\widetilde{M}$, such that M_1 is an invariant submanifold tangent to ξ and M_2 is an anti-invariant submanifold of $\widetilde{M}$. For any unit vector fields X tangent to M_1 and Z, W tangent to M_2, respectively, we have:

$$\begin{aligned} g(h(\phi X, Z), \phi Z) &= g\left(\widetilde{\nabla}_Z \phi X, \phi Z\right) = g\left(\phi \widetilde{\nabla}_Z X, \phi Z\right) \\ &= g\left(\widetilde{\nabla}_Z X, Z\right) = g(\nabla_Z X, Z) = X \ln f_1, \\ g(h(\phi X, Z), \phi W) &= (X \ln f_1)\, g(Z, W). \end{aligned} \tag{9.49}$$

On the other hand, since the ambient manifold $\widetilde{M}$ is a Kenmotsu manifold, it is easily seen that

$$h(\xi, Z) = 0. \tag{9.50}$$

Obviously, (9.25) implies $\xi \ln f_1 = 1$. Therefore, by (9.49) and (9.50), the inequality (9.48) is immediately obtained. Denote by h'' the second fundamental form of M_2 in M. Then we get

$$g\left(h''(Z, W), X\right) = g(\nabla_Z W, X) = -(X \ln f_1)\, g(Z, W)$$

or equivalently

$$h''(Z, W) = -g(Z, W)\nabla(\ln f_1). \tag{9.51}$$

Thus M_2 is a totally umbilical submanifold of M. Analogously, M_1 is totally umbilical in M. If the equality sign of (9.48) holds identically, then we obtain

$$h(\mathcal{D},\mathcal{D}) = 0,\ h\left(\mathcal{D}^{\perp},\mathcal{D}^{\perp}\right) = 0,\ h\left(\mathcal{D},\mathcal{D}^{\perp}\right) \subset \phi\mathcal{D}^{\perp}. \tag{9.52}$$

The first condition (9.52) implies that M_1 is totally geodesic in M. On the other hand, one has

$$\begin{aligned} g(h(X,\phi Y),\phi Z) &= g\left(\widetilde{\nabla}_X\phi Y,\phi Z\right) \\ &= g(\nabla_X Y, Z) = -Z(\ln\sigma_2)g(X,Y). \end{aligned}$$

It follows that M_1 is totally umbilical in $\widetilde{M}$. The second condition (9.52) and (9.51) imply that M_2 is totally umbilical submanifold in $\widetilde{M}$. Moreover, by (9.52), it follows that M is a minimal submanifold of $\widetilde{M}$.

In particular, if the ambient space is a Kenmotsu space form, one has the following.

Corollary 3 *Let $\widetilde{M}(c)$ be a $(2m+1)$ -dimensional Kenmotsu space form of constant ϕ-sectional curvature c and $M = {}_{f_2} M_1 \times_{f_1} M_2$ an n-dimensional nontrivial contact CR-doubly warped product submanifold, satisfying*

$$||h||^2 = 2\beta\left[||\nabla(\ln f_1)||^2 - 1\right].$$

Then we have

(a) M_1 is a totally umbilical invariant submanifold of $\widetilde{M}(c)$. Hence M_1 is a Kenmotsu space form of constant ϕ-sectional curvature $< c$.

(b) M_2 is a totally umbilical anti-invariant submanifold of $\widetilde{M}(c)$. Hence M_2 is a real space form of sectional curvature $\varepsilon > \frac{c-3}{4}$.

Proof Statement (a) follows from Theorem 9.5. Also, we know that M_2 is a totally umbilical submanifold of $\widetilde{M}(c)$. Gauss equation implies that M_2 is a real space form of sectional curvature $\varepsilon \geq \frac{c-3}{4}$. Moreover, by (9.25), we see that $\varepsilon = \frac{c-3}{4}$ if and only if the warping function f_1 is constant.

9.5.2 Another Inequality

In [30], we improved the inequality (9.48) for contact *CR*-doubly warped product submanifolds in Kenmotsu space forms. Equality case was characterized.

Theorem 9.6 *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Kenmotsu space form of constant ϕ-sectional curvature c and $M = {}_{f_2} M_1 \times_{f_1} M_2$ an n-dimensional contact CR-doubly warped product submanifold, such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 is a β-dimensional anti-invariant submanifold of $\widetilde{M}(c)$. Then* *(i) The squared norm of the second fundamental form of M satisfies*

$$||h||^2 \geq 2\beta\left[||\nabla(\ln f_1)||^2 - \Delta_1(\ln f_1) - 1\right] + \alpha\beta(c+1), \tag{9.53}$$

where Δ_1 denotes the Laplace operator on M_1.

(ii) The equality sign of (9.53) holds identically if and only if we have:

(a) M_1 is a totally umbilical invariant submanifold of $\widetilde{M}(c)$. Hence M_1 is a Kenmotsu space form of constant ϕ-sectional curvature $< c$.

(b) M_2 is a totally umbilical anti-invariant submanifold of $\widetilde{M}(c)$. Hence M_2 is a real space form of sectional curvature $\varepsilon \geq \frac{c-3}{4}$.

Proof Let $M = {}_{f_2}M_1 \times_{f_1} M_2$ be a contact CR-doubly warped product submanifold of a $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$, such that M_1 is an invariant submanifold tangent to ξ and M_2 is an anti-invariant submanifold of $\widetilde{M}(c)$. We denote by ν be the normal subbundle orthogonal to $\phi(TM_2)$. Obviously, we have

$$T^{\perp}M = \phi(TM_2) \oplus \nu, \phi\nu = \nu.$$

For any vector fields X tangent to M_1 and orthogonal to ξ and Z tangent to M_2, Eq. (9.22) gives

$$\widetilde{R}(X, \phi X, Z, \phi Z) = \frac{c+1}{2} g(X, X)g(Z, Z).$$

On the other hand, by Codazzi equation (9.5), we have

$$\begin{aligned}
&\widetilde{R}(X, \phi X, Z, \phi Z) \\
&\quad = -g\left(\nabla^{\perp}_X h(\phi X, Z) - h(\nabla_X \phi X, Z) - h(\phi X, \nabla_X Z), \phi Z\right) \\
&\qquad + g\left(\nabla^{\perp}_{\phi X} h(X, Z) - h\left(\nabla_{\phi X} X, Z\right) - h\left(X, \nabla_{\phi X} Z\right), \phi Z\right).
\end{aligned} \tag{9.54}$$

Using the Eq. (9.25) and structure equations of a Kenmotsu manifold, we get

$$\begin{aligned}
&g\left(\nabla^{\perp}_X h(\phi X, Z), \phi Z\right) \\
&\quad = Xg(h(\phi X, Z), \phi Z) - g\left(h(\phi X, Z), \nabla^{\perp}_X \phi Z\right) \\
&\quad = Xg(\nabla_Z X, Z) - g\left(h(\phi X, Z), \phi\widetilde{\nabla}_X Z\right) \\
&\quad = X((X\ln f_1)g(Z, Z)) \\
&\qquad - (X\ln f_1)g(h(\phi X, Z), \phi Z) - g(h(\phi X, Z), \phi h_\nu(X, Z)) \\
&\quad = \left(X^2 \ln f_1\right) g(Z, Z) \\
&\qquad + (X\ln f_1)^2 g(Z, Z) - ||h_\nu(X, Z)||^2,
\end{aligned}$$

where we denote by $h_\nu(X, Z)$ the ν-component of $h(X, Z)$. Also, by (9.49) and (9.25), we obtain, respectively,

$$g(h(\nabla_X \phi X, Z), \phi Z) = ((\nabla_X X) \ln f_1)\, g(Z, Z),$$
$$g(h(\phi X, \nabla_X Z), \phi Z) = (X \ln f_1) g(h(\phi X, Z), \phi Z) = (X \ln f_1)^2\, g(Z, Z).$$

Substituting the above relations in (9.54), we find

$$\begin{aligned}\widetilde{R}(X, \phi X, Z, \phi Z) = 2||h_\nu(X, Z)||^2 &- \left(X^2 \ln f_1\right) g(Z, Z) \\ &+ ((\nabla_X X) \ln f_1)\, g(Z, Z) \\ &- \left((\phi X)^2 \ln f_1\right) g(Z, Z) + \left(\left(\nabla_{\phi X} \phi X\right) \ln f_1\right) g(Z, Z).\end{aligned} \tag{9.55}$$

Then the Eq. (9.55) becomes

$$\begin{aligned}2||h_\nu(X, Z)||^2 = \Bigg[\frac{c+1}{2} g(X, X) &+ \left(X^2 \ln f_1\right) - ((\nabla_X X) \ln f_1) \\ &+ \left((\phi X)^2 \ln f_1\right) - \left(\left(\nabla_{\phi X} \phi X\right) \ln f_1\right)\Bigg] g(Z, Z).\end{aligned} \tag{9.56}$$

Let $\{X_0 = \xi, X_1, \ldots, X_\alpha, X_{\alpha+1} = \phi X_1, \ldots, X_{2\alpha} = \phi X_\alpha, Z_1, \ldots, Z_\beta\}$ be a local orthonormal frame on M such that $X_0, \ldots, X_{2\alpha}$ are tangent to M_1 and $Z_1, \ldots, Z_\beta$ are tangent to M_2. Therefore

$$2 \sum_{j=1}^{2\alpha} \sum_{t=1}^{\beta} ||h_\nu\left(X_j, Z_t\right)||^2 = \alpha\beta(c+1) - 2\beta\Delta_1(\ln f_1). \tag{9.57}$$

Combining (9.48) and (9.57), we obtain the inequality (9.53). The equality case can be solved similarly to the Corollary 3.

Corollary 4 *Let $\widetilde{M}(c)$ be a Kenmotsu space form with $c < -1$. Then there do not exist contact CR-doubly warped product submanifolds ${}_{f_2}M_1 \times_{f_1} M_2$ in $\widetilde{M}(c)$ such that* $\ln f_1$ *is a harmonic function on M_1.*

Proof Assume there exists a contact *CR*-doubly warped product submanifold ${}_{f_2}M_1 \times_{f_1} M_2$ in a Kenmotsu space form $\widetilde{M}(c)$ such that $\ln f_1$ is a harmonic function on M_1 .Then (9.57) implies $c \geq -1$.

Corollary 5 *Let $\widetilde{M}(c)$ be a Kenmotsu space form with $c \leq -1$. Then there do not exist contact CR-doubly warped product submanifolds ${}_{f_2}M_1 \times_{f_1} M_2$ in $\widetilde{M}(c)$ such that* $\ln f_1$ *is a nonnegative eigenfunction of the Laplacian on M_1 corresponding to an eigenvalue $\lambda > 0$.*

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Chapter 10
Ideal CR Submanifolds

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10.1 Introduction

In submanifold theory, it is important to establish relations between extrinsic and intrinsic invariants of submanifolds.

In the early 1990s, the notion of δ-invariants was introduced by Chen (see [11, 13, 14]). These invariants are obtained by subtracting a certain amount of sectional curvatures from the scalar curvature. Furthermore, he established pointwise optimal inequalities involving δ-invariants and the squared mean curvature of arbitrary submanifolds in real and complex space forms. A submanifold is said to be δ-*ideal* if it satisfies an equality case of the inequalities everywhere. During the past two decades, many interesting results on δ-ideal submanifolds have been obtained.

The main purpose of this chapter is to survey some of the known results on δ-ideal CR submanifolds in complex space forms, the nearly Kähler 6-sphere and odd dimensional unit spheres. For a given CR manifold M equipped with a compatible metric, the δ-ideal CR immersions of M minimize the λ-bienergy functional among all isometric CR immersions of M. In view of this fact, some topics on variational problem for the λ-bienergy functional are also presented.

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10.2 Preliminaries

Let M be an n-dimensional submanifold of a Riemannian manifold $\tilde{M}$. Let us denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M and $\tilde{M}$, respectively. The Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y),$$
$$\tilde{\nabla}_X V = -A_V X + D_X V$$

for tangent vector fields X, Y and normal vector field V, where B, A and D are the second fundamental form, the shape operator and the normal connection.

The mean curvature vector field H is defined by $H = (1/n)\mathrm{trace}\, B$. The function $|H|$ is called the *mean curvature*. If it vanishes identically, then M is called a *minimal submanifold*. In particular, if B vanishes identically, then M is called a *totally geodesic submanifold*.

Definition 22 ([6]) Let M be a Riemannian submanifold of an almost Hermitian manifold $\tilde{M}$ and let J be the complex structure of $\tilde{M}$. A submanifold M is called a *CR submanifold* if there exist differentiable distributions $\mathcal{H}$ and $\mathcal{H}^{\perp}$ such that

$$TM = \mathcal{H} \oplus \mathcal{H}^{\perp}, \quad J\mathcal{H} = \mathcal{H}, \quad J\mathcal{H}^{\perp} \subset T^{\perp}M,$$

where $T^{\perp}M$ denotes the normal bundle of M. A CR submanifold is called a *Kähler submanifold* (resp. *totally real submanifold*) if rank $\mathcal{H}^{\perp} = 0$ (resp. rank $\mathcal{H} = 0$). A totally real submanifold is called a *Lagraingian submanifold* if $J(TM) = T^{\perp}M$. A CR submanifold is said to be *proper* if rank $\mathcal{H} \neq 0$ and rank $\mathcal{H}^{\perp} \neq 0$.

10.3 δ-Invariants

Let M be an n-dimensional Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$. For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let L be a subset of T_pM of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of the r-plane section L by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

For an integer $k \geq 0$, denote by $\mathcal{S}(n, k)$ the finite set which consists of unordered k-tuples $(n_1, \ldots, n_k)$ of integers satisfying $2 \leq n_1 \ldots, n_k < n$ and $n_1 + \cdots + n_k \leq n$. We denote by $\mathcal{S}(n)$ the set of k-tuples with $k \geq 0$ for a fixed n.

For each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, the notion of *$\delta$-invariant* $\delta(n_1, \ldots, n_k)$ was introduced by Chen [13] as follows:

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

where L_1, ..., L_k run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j, j = 1, \ldots, k$.

Let $\overline{Ric}$ denote the maximum Ricci curvature function on M defined by

$$\overline{Ric}(p) = \max\{S(X, X)|X \in U_pM\},$$

where S is the Ricci tensor and U_pM is the unit tangent vector space of M at p. Then, we have $\delta(n-1)(p) = \overline{Ric}(p)$.

Let M be a Kähler manifold with real dimension $2n$. For each k-tuple $(2n_1, \ldots, 2n_k) \in \mathcal{S}(2n)$, Chen [13] also introduced the notion of *complex δ-invariant* $\delta^c(2n_1, \ldots, 2n_k)$, which is defined by

$$\delta^c(2n_1, \ldots, 2n_k)(p) = \tau(p) - \inf\{\tau(L_1^c) + \cdots + \tau(L_k^c)\},$$

where L_1^c, ..., L_k^c run over all k mutually orthogonal complex subspaces of T_pM such that $\dim L_j = 2n_j, j = 1, \ldots, k$.

For simplicity, we denote $\delta(\lambda, \ldots, \lambda)$ and $\delta^c(\lambda, \ldots, \lambda)$ by $\delta_k(\lambda)$ and $\delta_k^c(\lambda)$, respectively, where λ appears k times.

10.4 Inequalities involving δ-Invariants on Submanifolds

For each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, let $c(n_1, \ldots, n_k)$ and $b(n_1, \ldots, n_k)$ be the constants given by

$$c(n_1, \ldots, n_k) = \frac{n^2\left(n + k - 1 - \sum_{j=1}^k n_j\right)}{2(n + k - \sum_{j=1}^k n_j)},$$
$$b(n_1, \ldots, n_k) = \frac{1}{2}\left(n(n-1) - \sum_{j=1}^k n_j(n_j - 1)\right).$$

Chen obtained the following inequality for an arbitrary submanifold in a real space form.

Theorem 10.1 ([14]) *Given an n-dimensional submanifold M in an m-dimensional real space form $R^m(\epsilon)$ of constant sectional curvature ϵ, we have*

$$\delta(n_1,\ldots,n_k) \le c(n_1,\ldots,n_k)|H|^2 + b(n_1,\ldots,n_k)\epsilon. \tag{10.1}$$

Equality sign of (10.1) holds at a point $p \in M$ for some $(n_1,\ldots,n_k) \in \mathcal{S}(n)$ if and only if there exists an orthonormal basis $\{e_1,\ldots,e_{2m}\}$ at p such that $e_1,\ldots,e_n$ are tangent to M and the shape operators of M in $R^m(\epsilon)$ at p take the following forms:

$$A_{e_r} = \begin{pmatrix} \begin{matrix} A_1^r & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & A_k^r \end{matrix} & 0 \\ 0 & \mu_r I \end{pmatrix}, \tag{10.2}$$

$$r = n+1,\ldots,2m,$$

where each A_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$\text{trace}(A_1^r) = \cdots = \text{trace}(A_k^r) = \mu_r. \tag{10.3}$$

Let $\tilde{M}^m(4\epsilon)$ be a complex space form of complex dimension m and constant holomorphic sectional curvature 4ϵ and let J be the complex structure of $\tilde{M}^m(4\epsilon)$.

Let M be an n-dimensional submanifold in $\tilde{M}^m(4\epsilon)$. For any vector X tangent to M, we put $JX = PX + FX$, where PX and FX are tangential and normal components of JX, respectively. For a subspace $L \subset T_pM$ of dimension r, we set

$$\Psi(L) = \sum_{1\le i<j\le r} \langle Pu_i, u_j\rangle^2,$$

where $\{u_1,\ldots,u_r\}$ is an orthonormal basis of L.

For an arbitrary submanifold in a complex space form, we have

Proposition 26 ([14]) *Let M be an n-dimensional submanifold in a complex space form $\tilde{M}^m(4\epsilon)$. Then, for mutually orthogonal subspaces $L_1,\ldots, L_k$ of T_pM such that* $\dim L_j = n_j$, *we have*

$$\tau - \sum_{i=1}^k \tau(L_i) \le c(n_1,\ldots,n_k)|H|^2 + b(n_1,\ldots,n_k)\epsilon + \frac{3}{2}|P|^2\epsilon - 3\epsilon\sum_{i=1}^k \Psi(L_i). \tag{10.4}$$

The equality case of inequality (10.4) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1,\ldots,e_{2m}\}$ at p such that

(a) $L_j = \text{Span}\{e_{n_1+\cdots+n_{j-1}+1},\ldots,e_{n_1+\cdots+n_j}\}$, $\quad j = 1,\ldots,k$,

(b) *the shape operators of M in $\tilde{M}^m(4\epsilon)$ at p satisfy* (10.2) *and* (10.3).

Using Proposition 26, we obtain the following inequalities.

Proposition 27 ([14]) *Let M be a Kähler submanifold with real dimension $2n$ in a complex space form $\tilde{M}^m(4\epsilon)$. Then, we have*

$$\delta^c(2n_1, \ldots, 2n_k) \leq 2\left(n(n+1) - \sum_{j=1}^{k} n_j(n_j+1)\right)\epsilon. \tag{10.5}$$

The equality case of inequality (10.5) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_{2m}\}$ at p such that $e_1, \ldots, e_{2n}$ are tangent to M and $e_{2l} = Je_{2l-1}$ $(1 \leq l \leq k)$, and moreover, the shape operators of M in $\tilde{M}^m(4\epsilon)$ at p take the following forms:

$$A_{e_r} = \begin{pmatrix} \begin{matrix} A_1^r & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & A_k^r \end{matrix} & 0 \\ 0 & 0 \end{pmatrix},$$

$$r = 2n+1, \ldots, 2m,$$

where each A_j^r is a symmetric $(2n_j) \times (2n_j)$ submatrix satisfying $\operatorname{trace}(A_j^r) = 0$.

Proposition 28 ([42]) *Let M be an n-dimensional CR submanifold with* $\operatorname{rank} \mathcal{H} = 2h$ *in $\mathbb{C}H^m(-4)$. Then, we have*

$$\delta(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k)|H|^2 - b(n_1, \ldots, n_k) - 3h + \frac{3}{2}\sum_{j=1}^{k} n_j. \tag{10.6}$$

Equality sign of (10.6) holds at a point $p \in M$ for some $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_{2m}\}$ at p such that

(a) *each $L_j = \operatorname{Span}\{e_{n_1+\cdots+n_{j-1}+1}, \ldots, e_{n_1+\cdots+n_j}\}$ satisfies $\Psi(L_j) = n_j/2$ for $1 \leq j \leq k$,*

(b) *the shape operators of M in $\mathbb{C}H^m(-4)$ at p satisfy* (10.2) *and* (10.3).

Proposition 29 ([46]) *Let M be an n-dimensional CR submanifold with* $\operatorname{rank}\mathcal{H} = 2h$ *in $\mathbb{C}P^m(4)$. Then, we have*

$$\delta(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k)|H|^2 + b(n_1, \ldots, n_k) + 3h. \tag{10.7}$$

Equality sign of (10.7) holds at a point $p \in M$ for some $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_{2m}\}$ at p such that

(a) *each $L_j = \operatorname{Span}\{e_{n_1+\cdots+n_{j-1}+1}, \ldots, e_{n_1+\cdots+n_j}\}$ satisfies $\Psi(L_j) = 0$ for $1 \leq i \leq k$,*

(b) *the shape operators of M in $\mathbb{C}P^m(4)$ at p satisfy (10.2) and (10.3).*

Definition 23 A submanifold is said to be $\delta(n_1, \ldots, n_k)$-*ideal* if it satisfies the equality case of (10.1), (10.6) or (10.7) identically for a k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$. Similarly, a Kähler submanifold is said to be $\delta^c(2n_1, \ldots, 2n_k)$-*ideal* if it satisfies the equality case of (10.5) identically for a k-tuple $(2n_1, \ldots, 2n_k) \in \mathcal{S}(2n)$.

For more information on δ-invariants and δ-ideal submanifolds, we refer the reader to [16].

Definition 24 A submanifold is said to be *linearly full* in $\tilde{M}^m(4\epsilon)$ if it does not lie in any totally geodesic Kähler hypersurfaces of $\tilde{M}^m(4\epsilon)$.

10.5 Ideal CR Submanifolds in Complex Hyperbolic Space

We first recall some basic definitions on hypersurfaces.

Definition 25 Let N be a submanifold in a Riemannian manifold $\tilde{M}$ and $UN^\perp$ the unit normal bundle of N. Then, for a sufficiently small $r > 0$, the following mapping is an immersion:

$$f_r : UN^\perp \to \tilde{M}, \quad f_r(p, V) = \exp_p(rV),$$

where exp denotes the exponential mapping of $\tilde{M}$. The hypersurface $f_r(UN^\perp)$ of $\tilde{M}$ is called the *tubular hypersurface* over N with radius r. If N is a point x in $\tilde{M}$, then the tubular hypersurface over x is a geodesic hypersphere centered at x.

Definition 26 For a given point $p \in \mathbb{C}H^m(-4)$, let $\gamma(t)$ be a geodesic with $\gamma(0) = p$, which is parametrized by arch length. Denote by $S_t(\gamma(t))$ the geodesic hypersphere centered at $\gamma(t)$ with radius t. The limit of $S_t(\gamma(t))$ when t tends to infinity is called a *horosphere*.

Definition 27 Let M be a real hypersurface in an almost Hermitian manifold and V be a unit normal vector. A hypersurface M is called a *Hopf hypersurface* if JV is a principal curvature vector.

A real hypersurface in an almost Hermitian manifold is a proper CR submanifold with rank $\mathcal{H}^\perp = 1$. The following theorem characterizes the horosphere of $\mathbb{C}H^m(-4)$ in terms of $\delta_k(2)$.

Theorem 10.2 ([14]) *Let M be a $\delta_k(2)$-ideal real hypersurface of $\mathbb{C}H^m(-4)$. Then $k = m - 1$ and M is an open portion of the horosphere in $\mathbb{C}H^m(-4)$.*

Remark 15 The third case of (9.5) in [14] does not occur, because $L_1 \ldots, L_k$ are complex planes. Therefore, case (1) of Theorem 9.1 in [14] shall be removed from the list of $\delta_k(2)$-ideal real hypersurfaces in $\mathbb{C}H^m(-4)$.

For $\delta(2m - 2)$-ideal real hypersurfaces in $\mathbb{C}H^m(-4)$, Chen proved the following.

Theorem 10.3 ([15]) *Let M be a real hypersurface of $\mathbb{C}H^m(-4)$. Then M is $\delta(2m-2)$-ideal if and only if M is a Hopf hypersurface with constant mean curvature given by $2\alpha/(2m-1)$, where $A_V JV = \alpha JV$ for a unit normal vector V. If M has constant principal curvatures, then M is an open portion of one of the following real hypersurfaces:*

(1) *the horosphere of $\mathbb{C}H^2(-4)$;*

(2) *the tubular hypersurface over totally geodesic $\mathbb{C}H^{m-1}(-4)$ in $\mathbb{C}H^m(-4)$ with radius $r = \tanh^{-1}(1/\sqrt{2m-3})$, where $m \geq 3$.*

It was proved in [15] that if $m = 2$ in Theorem 10.3, then the assumption of the constancy of principal curvatures is satisfied. That is to say, we have

Corollary 22 ([15]) *Let M be a $\delta(2)$-ideal real hypersurface of $\mathbb{C}H^2(-4)$. Then M is an open portion of the horosphere.*

Let $\mathbb{C}_1^{m+1}$ be the complex number $(m+1)$-space endowed with the complex coordinates $(z_0, \ldots, z_m)$, the pseudo-Euclidean metric given by $\tilde{g} = -dz_0 d\bar{w}_0 + \sum_{i=1}^m dz_i d\bar{w}_i$ and the standard complex structure. For $\epsilon < 0$, we put $H_1^{2m+1}(\epsilon) = \{z \in \mathbb{C}_1^{m+1} | \langle z, z \rangle = 1/\epsilon\}$, where $\langle , \rangle$ denote the inner product on $\mathbb{C}_1^{m+1}$ induced from $\tilde{g}$. For a given $z \in H_1^{2m+1}(\epsilon)$, we put $[z] = \{\lambda z | \lambda \in \mathbb{C}, \lambda\bar{\lambda} = 1\}$. The Hopf fibration is given by

$$\varpi_{\{m,\epsilon\}} : H_1^{2m+1}(\epsilon) \to \mathbb{C}H^m(4\epsilon) : z \mapsto [z].$$

For $\delta_k(2)$-ideal proper CR submanifolds in $\mathbb{C}H^m(-4)$ whose codimensions are greater than one, we have the following representation formula.

Theorem 10.4 ([40]) *Let M be a linearly full $(2n+1)$-dimensional $\delta_k(2)$-ideal CR submanifold in $\mathbb{C}H^m(-4)$ such that* rank $\mathcal{H}^\perp = 1$, $k \geq 1$ *and* $m > n+1$. *Then, up to holomorphic isometries of $\mathbb{C}H^m(-4)$, the immersion of M into $\mathbb{C}H^m(-4)$ is given by the composition $\varpi_{\{m,-1\}} \circ z$, where*

$$z = \left(-1 - \frac{1}{2}|\Psi|^2 + iu, -\frac{1}{2}|\Psi|^2 + iu, \Psi\right)e^{it}, \tag{10.8}$$

and Ψ is a $2n$-dimensional $\delta_n^c(2)$-ideal Kähler submanifold in $\mathbb{C}^{m-1}$.

Up to holomorphic isometries of $\mathbb{C}H^m(-4)$, the horosphere in $\mathbb{C}H^m(-4)$ is a real hypersurface defined by $\{[z] : z \in H_1^{2m+1}(-1), \quad |z_0 - z_1| = 1\}$ (see, for example, [48]). Hence, Theorem 10.4 can be considered as an extension of Theorem 10.2.

As an immediate corollary of Theorem 10.4, we obtain

Corollary 23 ([19]) *Let M be a linearly full 3-dimensional $\delta(2)$-ideal CR submanifold in $\mathbb{C}H^m(-4)$ such that* rank $\mathcal{H}^\perp = 1$ *and $m > 2$. Then, up to holomorphic isometries of $\mathbb{C}H^m(-4)$, the immersion of M into $\mathbb{C}H^m(-4)$ is given by the composition $\varpi_{\{m,-1\}} \circ z$, where z is given by (10.8) and $\Psi(w)$ is a holomorphic curve in $\mathbb{C}^{m-1}$ with $\Psi'(w) \neq 0$.*

For general $\delta(n_1, \ldots, n_k)$-ideal proper CR submanifolds in $\mathbb{C}H^m(-4)$ whose codimensions are greater than one, the following classification result has been obtained.

Theorem 10.5 ([42, 46]) *Let M be a linearly full $(2n+1)$-dimensional $\delta(n_1, \ldots, n_k)$-ideal CR submanifold in $\mathbb{C}H^m(-4)$ such that* rank $\mathcal{H}^{\perp} = 1$*, $k \geq 1$ and $m > n+1$. Then, we have $JH \in \mathcal{H}^{\perp}$, $A_V JV = (2n/\sqrt{k(2n-k)})JV$ for $V = H/|H|$, $DH = 0$, and moreover, the mean curvature is given by*

$$\frac{2n(k+1)}{(2n+1)\sqrt{k(2n-k)}}.$$

If all principal curvatures of M with respect to $H/|H|$ are constant, then one of the following two cases occurs:

(1) *M is locally congruent with the immersion described in Theorem 10.4.*

(2) *$n/k \in \mathbb{Z} - \{1\}$, $n_1 = \cdots = n_k = 2n/k$, and M is locally congruent with the immersion*

$$\varpi_{\{m,-1\}}\left(\varpi^{-1}_{\{m-1, \frac{2k-2n}{2n-k}\}}(\Psi), \sqrt{\frac{k}{2n-2k}}e^{it}\right),$$

where Ψ is a $2n$-dimensional $\delta^c_k(2n/k)$-ideal Kähler submanifold in $\mathbb{C}H^{m-1}\left(\frac{8k-8n}{2n-k}\right)$.

If $n > 1$, $k = 1$ and $n_1 = 2n$ in Theorem 10.5, then we have

Corollary 24 ([41, 46]) *Let M be a linearly full $(2n+1)$-dimensional CR submanifold in $\mathbb{C}H^m(-4)$ such that* rank $\mathcal{H}^{\perp} = 1$*, $n > 1$ and $m > n+1$. Then M is $\delta(2n)$-ideal if and only if $JH \in \mathcal{H}^{\perp}$, $DH = 0$, $A_V JV = (2n/\sqrt{(2n-1)})JV$ for $V = H/|H|$, and moreover, the mean curvature is given by*

$$\frac{4n}{(2n+1)\sqrt{2n-1}}.$$

If all principal curvatures of M with respect to $H/|H|$ are constant, then, up to holomorphic isometries of $\mathbb{C}H^m(-4)$, the immersion of M into $\mathbb{C}H^m(-4)$ is given by

$$\varpi_{\{m,-1\}}\left(\varpi^{-1}_{\{m-1, \frac{2-2n}{2n-1}\}}(\Psi), \sqrt{\frac{1}{2n-2}}e^{it}\right),$$

where Ψ is a $2n$-dimensional Kähler submanifold in $\mathbb{C}H^{m-1}\left(\frac{8-8n}{2n-1}\right)$.

A hypersurface given by (2) in Theorem 10.3 can be rewritten as follows (see, for example, [37, Example 6.1]):

$$\varpi_{\{m,-1\}}\left(H_1^{2m-1}\left(\frac{4-2m}{2m-3}\right) \times S^1\left(\frac{1}{\sqrt{2m-4}}\right)\right),$$

where $S^1(r) = \{z \in \mathbb{C} | z\bar{z} = r^2\}$. Thus, Corollary 24 can be regarded as an extension of Theorem 10.3.

Let N be a Kähler hypersurface with real dimension $2n$ in a complex space form. Let V and JV be normal vector fields of N. Since $A_{JV} = JA_V$ and $JA_V = -A_V J$ holds (cf. [34, p. 175]), there exists an orthonormal basis $\{e_1, Je_1, \ldots, e_n, Je_n\}$ of T_pN with respect to which the shape operators A_V and A_{JV} take the following forms:

$$A_V = \begin{pmatrix} \lambda_1 & & & & 0 \\ & -\lambda_1 & & & \\ & & \ddots & & \\ & & & \lambda_n & \\ 0 & & & & -\lambda_n \end{pmatrix}, \quad A_{JV} = \begin{pmatrix} 0 & \lambda_1 & & & & 0 \\ \lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & & 0 & \lambda_n \\ 0 & & & & \lambda_n & 0 \end{pmatrix}.$$

Hence, it follows from Proposition 27 that every Kähler hypersurface with real dimension $2n$ in a complex space form is $\delta_k^c(2n/k)$-ideal for any natural number k such that $n/k \in \mathbb{Z}$. Accordingly, applying Theorem 10.4 yields the following.

Corollary 25 ([40]) *Let M be a linearly full $(2n+1)$-dimensional $\delta_k(2)$-ideal CR submanifold in $\mathbb{C}H^{n+2}(-4)$ such that* rank $\mathcal{H}^\perp = 1$ *and $k \geq 1$. Then, up to holomorphic isometries of $\mathbb{C}H^{n+2}(-4)$, the immersion of M into $\mathbb{C}H^{n+2}(-4)$ is given by the composition $\varpi_{\{n+2,-1\}} \circ z$, where z is given by (10.8), and Ψ is a Kähler hypersurface in $\mathbb{C}^{n+1}$.*

Similarly, we obtain the following corollary of Theorem 10.5.

Corollary 26 *Let M be a linearly full $(2n+1)$-dimensional $\delta(n_1, \ldots, n_k)$-ideal CR submanifold in $\mathbb{C}H^{n+2}(-4)$ such that* rank $\mathcal{H}^\perp = 1$ *and $k \geq 1$. If all principal curvatures of M with respect to $H/|H|$ are constant, then one of the following two cases occurs*:

(1) *M is locally congruent with the immersion described in Corollary 25.*

(2) *$n/k \in \mathbb{Z} - \{1\}$, $n_1 = \cdots = n_k = 2n/k$, and M is locally congruent with the immersion*

$$\varpi_{\{n+2,-1\}}\left(\varpi^{-1}_{\{n+1,\frac{2k-2n}{2n-k}\}}(\Psi), \sqrt{\frac{k}{2n-2k}}e^{it}\right),$$

where Ψ is a Kähler hypersurface in $\mathbb{C}H^{n+1}(\frac{8k-8n}{2n-k})$.

It is natural to ask the following problem.

Problem 1 Find $\delta(n_1, \ldots, n_k)$-ideal CR submanifolds with rank $\mathcal{H}^\perp = 1$ in $\mathbb{C}H^m(-4)$ such that the principal curvatures with respect to $H/|H|$ are not all constant.

Generally, $\delta(n_1, \ldots, n_k)$-ideal proper CR submanifolds in $\mathbb{C}H^m(-4)$ have the following properties.

Theorem 10.6 ([42]) *Let M be a linearly full $(2n+q)$-dimensional $\delta(n_1, \ldots, n_k)$-ideal CR submanifold in $\mathbb{C}H^m(-4)$ such that* rank$\mathcal{H}^\perp = q$. *If $q > 1$, then M is minimal. If $q = 1$ and $m > n+1$, then M is non-minimal and satisfies $DH = 0$.*

A differentiable manifold M is called an *almost contact manifold* if it admits a unit vector field ξ, a one-form η and a (1, 1)-tensor field ϕ satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi.$$

Every almost contact manifold admits a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

The quadruplet (ϕ, ξ, η, g) is called an *almost contact metric structure*.

An almost contact metric structure is called a *contact metric structure* if it satisfies

$$d\eta(X, Y) = \frac{1}{2}\Big(X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])\Big) = g(X, \phi Y).$$

A contact metric structure is said to be *Sasakian* if the tensor field S defined by

$$S(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X, Y)\xi$$

vanishes identically. A manifold equipped with a Sasakian structure is called a *Sasakian manifold*. We refer the reader to [8] for more information on Sasakian manifolds.

Let M be a CR submanifold with rank $\mathcal{H}^\perp = 1$ in a complex space form. We define a one-form η by $\eta(X) = g(U, X)$, where U is a unit tangent vector field lying in $\mathcal{H}^\perp$, and g is an induced metric on M. We put $\bar{U} = (1/\sqrt{r})U$, $\bar{\eta} = \sqrt{r}\eta$ and $\bar{g} = rg$ for a positive constant r. Then, the quadruplet $(P, \bar{U}, \bar{\eta}, \bar{g})$ defines an almost contact structure on M (cf. [21, p. 96]).

For the almost contact structure $(P, \bar{U}, \bar{\eta}, \bar{g})$ on a CR submanifold described in Theorem 10.5, we have the following.

Proposition 30 ([46]) *An almost contact structure $(P, \bar{U}, \bar{\eta}, \bar{g})$ with $r = \sqrt{\frac{k}{2n-k}}$ on a CR submanifold in Theorem 10.5. becomes a Sasakian structure. In particular, in the case of* (1)*, the structure is Sasakian with respect to the induced metric.*

10.6 Ideal CR Submanifolds in Complex Projective Space

All $\delta_k(2)$-ideal Hopf hypersurfaces of $\mathbb{C}P^m(4)$ have been determined as follows:

Theorem 10.7 ([14]) *Let M be a $\delta_k(2)$-ideal Hopf hypersurface of $\mathbb{C}P^m(4)$. Then, one of the following three cases occurs*:

(1) *$k = 1$ and M is an open portion of a geodesic sphere with radius $\pi/4$;*

(2) *m is odd, $k = m - 1$, and M is an open portion of a tubular hypersurface with radius $r \in (0, \pi/2)$ over a totally geodesic $\mathbb{C}P^{(m-2)/2}(4)$;*

(3) *$m = 2$, $k = 1$, and M is an open portion of a tubular hypersurface over the complex quadric curve* $Q_1 := \{[z_0, z_1, z_2] \in \mathbb{C}P^2 : z_0^2 + z_1^2 + z_2^2 = 0\}$, *with radius* $r = \tan^{-1}((1 + \sqrt{5} - \sqrt{2 + 2\sqrt{5}})/2)$. *Here,* $[z_0, z_1, z_2]$ *is a homogeneous coordinate of* $\mathbb{C}P^2$.

A real hypersurface of $\mathbb{C}P^m(4)$ is called a *ruled real hypersurface* if $\mathcal{H}$ is integrable and each leaf of its maximal integral manifolds is locally congruent to $\mathbb{C}P^{m-1}(4)$. For a unit normal vector V of a ruled real hypersurface M, the shape operator A_V satisfies

$$A_V JV = \mu JV + \nu U \ \ (\nu \neq 0), \quad AU = \nu JV, \quad AX = 0 \tag{10.9}$$

for all X orthogonal to both JV and U, where U is a unit vector orthogonal to JV, and μ and ν are smooth functions on M. Thus, all ruled real hypersurfaces of $\mathbb{C}P^m(4)$ are non-Hopf (see [33]).

Using Proposition 29 and (10.9), we find that every minimal ruled real hypersurface in $\mathbb{C}P^m(4)$ is $\delta_k(2)$-ideal for $1 \le k \le m-1$. Such a hypersurface can be represented as follows:

Theorem 10.8 ([1]) *A minimal ruled hypersurface of* $\mathbb{C}P^m(4)$ *is congruent to* $\varpi \circ z$, *where* $\varpi : S^{2m+1}(1) \to \mathbb{C}P^m(4)$ *is the Hopf fibration and*

$$z(s, t, \theta, w) = e^{\sqrt{-1}\theta}\Big(\cos s \cos t, \cos s \sin t, (\sin s) w\Big)$$

for $w \in \mathbb{C}^{m-1}$, $|w|^2 = 1$, $-\pi/2 < s < \pi/2$, $0 \le t, \theta < 2\pi$.

It seems interesting to consider the following problem.

Problem 2 Classify $\delta(n_1, \ldots, n_k)$-ideal non-Hopf real hypersurfaces in $\mathbb{C}P^m(4)$.

Let M be an n-dimensional $\delta(n_1, \ldots, n_k)$-ideal CR submanifold in $\mathbb{C}P^m(4)$. Let L_j be subspaces of T_pM defined in (a) of Proposition 29. Define the subspace L_{k+1} by $L_{k+1} = \mathrm{Span}\{e_{n_1+\cdots+n_k+1}, \ldots, e_n\}$. It is clear that $T_pM = L_1 \oplus \cdots \oplus L_{k+1}$. We denote by $\mathcal{L}_i$ the distribution which is generated by L_i. Then, we have the following codimension reduction theorem.

Theorem 10.9 ([46]) *Let M be an n-dimensional $\delta(n_1, \ldots, n_k)$-ideal CR submanifold with* rank $\mathcal{H}^\perp = 1$ *in* $\mathbb{C}P^m(4)$. *If* $\mathcal{H}^\perp \subset \mathcal{L}_i$ *for some* $i \in \{1, \ldots, k+1\}$, *then M is contained in a totally geodesic Kähler submanifold* $\mathbb{C}P^{\frac{n+1}{2}}(4)$ *in* $\mathbb{C}P^m(4)$.

It was proved in [46] that if $\dim M = 3$, then the assumption on $\mathcal{H}^\perp$ in Theorem 10.9 holds. That is to say, we have the following.

Corollary 27 ([46]) *Let M be a 3-dimensional $\delta(2)$-ideal proper CR submanifold in* $\mathbb{C}P^m(4)$. *Then, M is contained in* $\mathbb{C}P^2(4)$.

The following problem arises naturally.

Problem 3 Find $\delta(n_1, \ldots, n_k)$-ideal CR submanifolds with rank $\mathcal{H}^\perp = 1$ in $\mathbb{C}P^m(4)$ such that the codimensions are greater than one.

10.7 Ideal CR Submanifolds in the Nearly Kähler 6-sphere

Let $\mathcal{O}$ be the Cayley algebra, and denote by Im $\mathcal{O}$ the purely imaginary part of $\mathcal{O}$. We identify Im $\mathcal{O}$ with $\mathbb{R}^7$ and define the exterior product $u \times v$ on it by

$$u \times v = \frac{1}{2}(uv - vu).$$

The canonical inner product on $\mathbb{R}^7$ is given by $\langle u, v\rangle = -(uv + vu)/2$.

We define the tensor field J of type $(1, 1)$ on $S^6(1) = \{p \in \text{Im}\,\mathcal{O} | \langle , \rangle = 1\}$ by

$$JX = p \times X$$

for any $p \in S^6(1)$, $X \in T_pS^6(1)$. Let g be the standard metric on $S^6(1)$. Then $(S^6(1), J, g)$ is a nearly Kähler manifold, i.e., an almost Hermitian manifold satisfying $(\nabla_X J)X = 0$ for any $X \in TS^6(1)$, where ∇ is the Levi-Civita connection with respect to g (cf. [34, pp. 139–140]).

For 3-dimensional $\delta(2)$-ideal proper CR submanifolds in the nearly Kähler $S^6(1)$, we have the following result.

Theorem 10.10 ([22, 23]) *Let M be a 3-dimensional $\delta(2)$-ideal proper CR submanifold in the nearly Kähler $S^6(1)$. Then, M is minimal and locally congruent with the following immersion*:

$$\begin{aligned} f(t, u, v) = (&\cos t \cos u \cos v, \sin t, \cos t \sin u \cos v,\\ &\cos t \cos u \sin v, 0, -\cos t \sin u \sin v, 0). \end{aligned} \tag{10.10}$$

Remark 16 A CR submanifold (10.10) can be rewritten as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_6^2 = 1, \quad x_5 = x_7 = 0, \quad x_3x_4 + x_1x_6 = 0,$$

which implies that it lies in $S^4(1)$.

The following theorem determines 4-dimensional $\delta(2)$-ideal proper CR submanifolds in the nearly Kähler $S^6(1)$.

Theorem 10.11 ([2, 3]) *Let M be a 4-dimensional $\delta(2)$-ideal proper CR submanifold in the nearly Kähler $S^6(1)$. Then, M is minimal and locally congruent with the following immersion*:

$$\begin{aligned} f(t, u, v, w) = (&\cos w \cos t \cos u \cos v, \sin w \sin t \cos u \cos v,\\ &\sin 2w \sin v \cos u + \cos 2w \sin u, 0, \sin w \cos t \cos u \cos v,\\ &\cos w \sin t \cos u \cos v, \cos 2w \sin v \cos u - \sin 2w \sin u). \end{aligned} \tag{10.11}$$

Remark 17 A CR submanifold given by (10.11) lies in $S^5(1)$.

Definition 28 A 2-dimensional submanifold N of the nearly Kähler $S^6(1)$ is called an *almost complex curve* if $J(T_pN) = T_pN$ for any $p \in N$.

Chen has classified $\delta_k(n_1, \ldots, n_k)$-ideal Hopf hypersurfaces of the nearly Kähler $S^6(1)$ as follows:

Theorem 10.12 ([16, p. 415]) *A Hopf hypersurface of the nearly Kähler $S^6(1)$ is $\delta(n_1, \ldots, n_k)$-ideal if and only if it is either*

(1) *a totally geodesic hypersurface, or*

(2) *an open part of a tubular hypersurface with radius $\pi/2$ over a non-totally geodesic almost complex curve of $S^6(1)$.*

Remark 18 A tubular hypersurface described in (2) of Theorem 10.12 is a minimal $\delta(\lambda)$-ideal hypersurface for $\lambda \in \{2, 3, 4\}$.

It is natural to consider the following problem.

Problem 4 Classify 4-dimensional $\delta(2, 2)$-ideal and $\delta(3)$-ideal CR submanifolds in the nearly Kähler $S^6(1)$.

10.8 Ideal Contact CR Submanifolds in Odd Dimensional Unit Spheres

For any point $x \in S^{2n+1}(1) \subset \mathbb{C}^{n+1}$, we set $\xi = Jx$, where J denotes the canonical complex structure of $\mathbb{C}^{n+1}$. Let g be the standard metric on $S^{2n+1}(1)$ and η be the one-form given by $\eta(X) = g(X, \xi)$. We consider the orthogonal projection $P : T_x\mathbb{C}^{n+1} \to T_xS^{2n+1}(1)$. We define a $(1, 1)$-tensor field ϕ on $S^{2n+1}(1)$ by $\phi = P \circ J$. Then, the quadruplet (ϕ, ξ, η, g) is a Sasakian structure (see, for example, [7]).

Definition 29 ([35]) Let M be a Riemannian submanifold tangent to ξ of a Sasakian manifold. A submanifold M is called a *contact CR submanifold* if there exist differentiable distributions $\mathcal{H}$ and $\mathcal{H}^\perp$ such that

$$TM = \mathbb{R}\xi \oplus \mathcal{H} \oplus \mathcal{H}^\perp, \quad \phi\mathcal{H} = \mathcal{H}, \quad \phi\mathcal{H}^\perp \subset T^\perp M.$$

A contact CR submanifold is said to be *proper* if rank $\mathcal{H} \neq 0$ and rank $\mathcal{H}^\perp \neq 0$.

Non-minimal $\delta(2)$-ideal submanifolds in a sphere have been completely described in [20]. For minimal $\delta(2)$-ideal proper contact CR submanifolds in $S^{2m+1}(1)$, we have the following codimension reduction theorem.

Theorem 10.13 ([38]) *Let M^n be a minimal $\delta(2)$-ideal proper contact CR submanifold in $S^{2m+1}(1)$. Then n is even and there exits a totally geodesic Sasakian $S^{2n+1}(1)$ in $S^{2m+1}(1)$ containing M^n as a hypersurface.*

Therefore, it is sufficient to investigate the case of hypersurfaces. Let N be a minimal surface in $S^n(1)$ and let $UN^{\perp}$ be its unit normal bundle. Then, a map

$$F : UN^{\perp} \to S^n(1) : V_p \mapsto V_p$$

is a minimal $\delta(2)$-ideal codimension one immersion (see [13, Example 9.8]).

Munteanu and Vrancken proved the following.

Theorem 10.14 ([38]) *Let M^{2n} be a minimal $\delta(2)$-ideal proper contact CR hypersurface in $S^{2n+1}(1)$. Then M^{2n} can be locally considered as the unit normal bundle of the Clifford torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3(1) \subset S^{2n+1}(1)$.*

10.9 Related Topics

This section gives an account of the relationship between $\delta(n_1, \ldots, n_k)$-ideal immersions and critical points of the λ-bienergy functional $E_{2,\lambda}$. Some topics about variational problems for $E_{2,\lambda}$ are also presented.

10.9.1 λ-Bienergy Functional

Let $f : M \to N$ be a smooth map of an n-dimensional Riemannian manifold into another Riemannian manifold. The *tension field* $\tau(f)$ of f is a section of the induced vector bundle f^*TN defined by

$$\tau(f) = \sum_{i=1}^{n} \{\nabla^f_{e_i} df(e_i) - df(\nabla_{e_i} e_i)\}$$

for a local orthonormal frame $\{e_i\}$ on M, where ∇^f and ∇ denote the induced connection and the Levi-Civita connection of M, respectively. If f is an isometric immersion, then we have

$$\tau(f) = nH. \tag{10.12}$$

A smooth map f is called a *harmonic map* if it is a critical point of the energy functional

$$E(f) = \int_{\Omega} |df|^2 dv_g$$

over every compact domain Ω of M, where dv_g is the volume form of M. A smooth map f is harmonic if and only if $\tau(f)$ vanishes identically on M.

Definition 30 For each smooth map f of a compact domain Ω of M into N, the λ-bienergy functional is defined by

$$E_{2,\lambda}(f) = \int_{\Omega} |\tau(f)|^2 dv_g + \lambda E(f).$$

For simplicity, we denote $E_{2,0}(f)$ by $E_2(f)$, which is called the bienergy functional.

Eliasson [24] proved that $E_{2,\lambda}$ satisfies Condition (C) of Palais-Smale if the dimension of the domain is 2 or 3 and the target is non-positively curved. In general, $E_{2,\lambda}$ does not satisfy Condition (C) (see [35]).

10.9.2 Ideal CR Immersions as Critical Points of λ-Bienergy Functional

Let (M, HM, J_H, g) be a compact Riemannian almost CR manifold (with or without boundary) whose CR dimension is h, i.e., a compact smooth manifold equipped with a subbundle HM of TM of rank $2h$ together with a bundle isomorphism $J_H : HM \to HM$ such that $(J_H)^2 = -I$, and a compatible Riemannian metric g such that $g(X, Y) = g(J_H X, J_H Y)$ for all $X, Y \in HM$.

An immersion f of (M, HM, J_H, g) into $\tilde{M}^m(4\epsilon)$ is called a *CR immersion* if $J(df(X)) = df(J_H(X))$ for any $X \in HM$. If f is an isometric immersion, then $f(M)$ is a CR submanifold of $\tilde{M}^m(4\epsilon)$. We denote by $\mathcal{ICR}(M, \tilde{M}^m(4\epsilon))$ the family of isometric CR immersions of M into $\tilde{M}^m(4\epsilon)$. By Propositions 28 and 29, we see that a $\delta(n_1, \ldots, n_k)$-ideal CR immersion of M into $\tilde{M}^m(4\epsilon)$ is a stable critical point of $E_{2,\lambda}$ within the class of $\mathcal{ICR}(M, \tilde{M}^m(4\epsilon))$.

10.9.3 λ-Biharmonic Submanifolds and Their Extensions

Definition 31 ([25]) A smooth map $f : M \to N$ is called a *λ-biharmonic map* if it is a critical point of the λ-bienergy functional with respect to all variations with compact support. If f is a λ-biharmonic isometric immersion, then M is called a *λ-biharmonic submanifold* in N. In the case of $\lambda = 0$, we simply call it a *biharmonic submanifold*.

The Euler–Lagrange equation for $E_{2,\lambda}$ is given by (see [30] and [25, p. 515])

$$\tau_{2,\lambda} := -\Delta_f(\tau(f)) + \mathrm{trace} R^N(\tau(f), df)df - \lambda\tau(f) = 0, \tag{10.13}$$

where $\Delta_f = -\sum_{i=1}^{n}(\nabla^f_{e_i}\nabla^f_{e_i} - \nabla^f_{\nabla_{e_i}e_i})$ and R^N is the curvature tensor of N, which is defined by

$$R^N(X,Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z$$

for the Levi-Civita connection ∇^N of N. For simplicity, we denote $\tau_{2,0}(f)$ by $\tau_2(f)$.

By decomposing the left-hand side of (10.13) into its tangential and normal components, we have

Proposition 31 ([4]) *Let M be an n-dimensional submanifold of $\tilde{R}^m(\epsilon)$. Then M is λ-biharmonic if and only if*

$$\begin{cases} \Delta^D H + \text{trace}\, B(\cdot, A_H(\cdot)) + (\lambda - \epsilon n)H = 0, \\ 4\text{trace} A_{D_{(\cdot)}H}(\cdot) + n\text{grad}(|H|^2) = 0, \end{cases}$$

where $\Delta^D = -\sum_{i=1}^n \{D_{e_i}D_{e_i} - D_{\nabla_{e_i}e_i}\}$.

Proposition 32 ([25]) *Let M be an n-dimensional submanifold of $\tilde{M}^m(4\epsilon)$ such that JH is tangent to M. Then M is λ-biharmonic if and only if*

$$\begin{cases} \Delta^D H + \text{trace}\, B(\cdot, A_H(\cdot)) + \{\lambda - \epsilon(n+3)\}H = 0, \\ 4\text{trace}\, A_{D_{(\cdot)}H}(\cdot) + n\text{grad}(|H|^2) = 0. \end{cases}$$

Remark 19 By Proposition 32, we see that all hypersurfaces with constant principal curvatures in $R^m(\epsilon)$ and $\tilde{M}^m(4\epsilon)$ are $\{-|B|^2 + \epsilon(m-1)\}$-biharmonic and $\{-|B|^2 + 2\epsilon(m+1)\}$-biharmonic, respectively.

It follows from (10.12) and (10.13) that any minimal submanifold is λ-biharmonic. Thus, it is interesting to investigate non-minimal λ-biharmonic submanifolds.

Remark 20 Let $f : M \to \mathbb{R}^n$ be an isometric immersion. We denote the mean curvature vector field of M by $H = (H_1, \ldots, H_n)$. Then, it follows from (10.12) and (10.13) that M is λ-biharmonic if and only if it satisfies

$$\Delta_M H_i = -\lambda H_i, \quad 1 \leq i \leq n, \tag{10.14}$$

where Δ_M is the Laplace operator acting on $C^\infty(M)$. Hence, the notion of biharmonic submanifolds in Definition 31 is same as one defined by B.Y. Chen (cf. [17]). It was proved in [10] that a submanifold M satisfies (10.14) if and only if one of the following three cases occurs:

(1) f satisfies $\Delta_M f = -\lambda f$;

(2) f can be written as $f = f_0 + f_1$, $\quad \Delta_M f_0 = 0, \quad \Delta_M f_1 = -\lambda f_1$;

(3) M is a biharmonic submanifold.

An immersion described in (1) (resp. (2)) is said to be of 1-*type* (resp. *null* 2-*type*). An immersion $f : M \to \mathbb{R}^n$ is of 1-type if and only if either M is a minimal submanifold of $\mathbb{R}^n$ or M is a minimal submanifold of a hypersphere in $\mathbb{R}^n$ (cf. [12, Theorem 3.2]). The classification of null 2-type immersions is not yet complete.

There exist many non-minimal biharmonic submanifolds in a sphere or a complex projective space (see, for example, [5, 25]). On the other hand, the following conjecture proposed by Chen [12] is still open.

Conjecture 1 Any biharmonic submanifold in Euclidean space is minimal.

Several partial positive answers to this conjecture have been obtained (see [17]). For example, Chen and Munteanu [18] proved that Conjecture 1 is true for hypersurfaces which are $\delta(2)$-ideal or $\delta(3)$-ideal in Euclidean space of arbitrary dimension.

As an extension of the notion of biharmonic submanifolds, the following notion was introduced by Loubeau and Montaldo in [36].

Definition 32 An isometric immersion $f : M \to N$ is called a λ-*biminimal* if it is a critical point of the λ-bienergy functional with respect to all *normal variations* with compact support. Here, a normal variation means a variation f_t through $f = f_0$ such that the variational vector field $V = df_t/dt|_{t=0}$ is normal to $f(M)$. In this case, M is called a λ-*biminimal submanifold* in N. In the case of $\lambda = 0$, we simply call it *biminimal* submanifold.

An isometric immersion f is λ-biminimal if and only if

$$[\tau_{2,\lambda}(f)]^{\perp} = 0,$$

where $[\cdot]^{\perp}$ denotes the normal component of $[\cdot]$ (see [36]). It is known that there exist ample examples of λ-biminimal submanifolds in real and complex space forms, which are not λ-biharmonic (see, for example, [36, 43, 45, 47]).

In [44], the notion of tangentially biharmonicity for submanifolds was introduced as follows:

Definition 33 Let $f : M \to N$ be an isometric immersion. Then M is called a *tangentially biharmonic submanifold* in N if it satisfies

$$[\tau_2(f)]^{\top} = 0, \tag{10.15}$$

where $[\cdot]^{\top}$ denotes the tangential part of $[\cdot]$.

Example 20 Let $x : M^{n-1} \to \mathbb{R}^n$ be an isometric immersion. The normal bundle $T^{\perp}M^{n-1}$ of M^{n-1} is naturally immersed in $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ by the immersion $f(\xi_x) := (x, \xi_x)$, which is expressed as

$$f(x, s) = (x, sV) \tag{10.16}$$

for the unit normal vector field V along x. We equip $T^{\perp}M^{n-1}$ with the metric induced by f. If we define the complex structure J on $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ by $J(X, Y) := (-Y, X)$, then $T^{\perp}M^{n-1}$ is a Lagrangian submanifold in $\mathbb{C}^n$ (see [26, III.3.C]). It was proved in [44] that $T^{\perp}M^2$ is a tangentially biharmonic Lagrangian submanifold in $\mathbb{C}^3$ if and only if M^2 is either minimal, a part of a round sphere or a part of a circular cylinder in $\mathbb{R}^3$.

Remark 21 For any $\lambda \in \mathbb{R}$, we have $[\tau_{2,\lambda}(f)]^\top = [\tau_2(f)]^\top$.

Remark 22 By the first variation formula for E_2 obtained in [30], we see that an isometric immersion $f : M \to N$ is tangentially biharmonic if and only if it is a critical point of E_2 with respect to all *tangential variations* with compact support. Here, a tangential variation means a variation f_t through $f = f_0$ such that the variational vector field $V = df_t/dt|_{t=0}$ is tangent to $f(M)$.

Remark 23 As described by Hilbert [28], the stress-energy tensor associated to a variational problem is a symmetric 2-covariant tensor which is conservative, namely, divergence-free at critical points. The stress-energy tensor S_2 for $E_2(f)$ was introduced by Jiang [31] as follows:

$$S_2(X, Y) = \frac{1}{2}|\tau(f)|^2 \langle X, Y\rangle + \left\langle df, \nabla^f \tau(f)\right\rangle \\ - \left\langle df(X), \nabla^f_Y \tau(f)\right\rangle - \left\langle df(Y), \nabla^f_X \tau(f)\right\rangle.$$

It satisfies $\operatorname{div} S_2 = -\langle \tau_2(f), df\rangle$. Hence, an isometric immersion f is tangentially biharmonic if and only if $\operatorname{div} S_2 = 0$. Caddeo et al. [9] called these submanifolds satisfying such a condition as *biconservative submanifolds*, and moreover, classified biconservative surfaces in 3-dimensional real space forms.

Remark 24 Hasanis and Vlachos [27] classified hypersurfaces in $\mathbb{R}^4$ satisfying (10.15). They called such hypersurfaces as *H-hypersurfaces*. Afterwards, the biharmonic ones are picked out in the class. As a result, the non-existence of non-minimal biharmonic hypersurfaces in $\mathbb{R}^4$ was proved.

Remark 25 It follows from Definitions 32 and 33 that a map f is λ-biharmonic (resp. λ-biminimal) if and only if it is a critical point of $E_2(f)$ for all variations (resp. normal variations) with compact support and *fixed energy*. Here, λ is the Lagrange multiplier.

10.9.4 Biharmonic Ideal CR Submanifolds

For homogeneous real hypersurfaces in $\mathbb{C}P^m(4)$, namely, orbits under some subgroups of the projective unitary group $PU(m+1)$, we have

Theorem 10.15 ([29]) *Let M be a homogeneous hypersurface in $\mathbb{C}P^m(4)$. Then, M is non-minimal biharmonic if and only if it is congruent to an open portion of one of the following real hypersurfaces*:

(1) *a tubular hypersurface over $\mathbb{C}P^q(4)$ with radius*

$$r = \cot^{-1}\left(\sqrt{\frac{m+2 \pm \sqrt{(2q-m+1)^2 + 4(m+1)}}{2m-2q-1}}\right).$$

(2) *a tubular hypersurface over the Plücker imbedding of the complex Grassmann manifold* $Gr_2(\mathbb{C}^5) \subset \mathbb{C}P^9(4)$ *with radius r, where* $0 < r < \pi/4$ *and* $t = \cot r$ *is a unique solution of the equation*

$$41t^6 + 43t^4 + 41t^2 - 15 = 0.$$

(3) *a tubular hypersurface over the canonical imbedding of the Hermitian symmetric space* $SO(10)/U(5) \subset \mathbb{C}P^{15}(4)$ *with radius r, where* $0 < r < \pi/4$ *and* $t = \cot r$ *is a unique solution of the equation*

$$13t^6 - 107t^4 + 43t^2 - 9 = 0.$$

For details on the canonical imbedding of a compact Hermitian symmetric space into $\mathbb{C}P^m(4)$, we refer the reader to Sect. 4 of [39].

Remark 26 Let M be a real hypersurface in $\mathbb{C}P^m(4)$. Kimura [32] proved that M is a Hopf hypersurface with constant principal curvatures if and only if it is homogeneous.

Combining Theorem 10.7, Proposition 32 and Theorem 10.15, we obtain

Corollary 28 *Let M be a* $\delta_k(2)$*-ideal non-minimal biharmonic Hopf hypersurface in* $\mathbb{C}P^m(4)$. *Then, m is odd and M is an open portion of a tubular hypersurface over* $\mathbb{C}P^{(m-1)/2}(4)$ *with radius*

$$r = \cot^{-1}\left(\sqrt{\frac{m+2\pm 2\sqrt{m+1}}{m}}\right).$$

Example 21 On each CR submanifold described in Theorem 10.5, there exists an orthonormal frame $\{e_1, \dots, e_{2m}\}$ such that $e_{2r} = Je_{2r-1}$ for $r \in \{1, \dots, n\}$, $JH || e_{2n+1} \in \mathcal{H}^\perp$ and the second fundamental form B takes the following form:

$$\begin{aligned}
B(e_{2r-1}, e_{2r-1}) &= \sqrt{\frac{k}{2n-k}} Je_{2n+1} + \phi_r \xi_r, \\
B(e_{2r}, e_{2r}) &= \sqrt{\frac{k}{2n-k}} Je_{2n+1} - \phi_r \xi_r, \\
B(e_{2r-1}, e_{2r}) &= \phi_r J\xi_r, \\
B(e_{2n+1}, e_{2n+1}) &= \frac{2n}{\sqrt{k(2n-k)}} Je_{2n+1}, \\
B(u_i, u_j) &= h(u_i, e_{2n+1}) = 0 \quad (i \neq j),
\end{aligned}$$

where ϕ_r are functions, $\xi_r \in \nu$ and $u_j \in L_j$ (see Lemma 7 of [42]). Here, ν denotes an orthogonal complement of $J\mathcal{H}^\perp$ in $T^\perp M$. Therefore, by using Proposition 32, we find that all ideal CR submanifolds given in Theorem 10.5 are non-minimal λ-biharmonic submanifolds with

$$\lambda = -\frac{2n(2n + k^2)}{k(2n - k)} - 2n - 4 \ \ (\neq 0).$$

The following problem seems interesting.

Problem 5 Classify $\delta(n_1, \ldots, n_k)$-ideal proper CR submanifolds in $\mathbb{C}P^m(4)$ which are non-minimal biharmonic.

Example 22 The standard product $S^{2r+1}(1/\sqrt{2}) \times S^{2s+1}(1/\sqrt{2})$ in $S^{2(r+s)+3}(1)$ is a biharmonic contact CR hypersurface (see [49, Example 5.1] and [5, p. 92]). Its principal curvatures are $\{1, -1\}$ with multiplicities $\{2r + 1, 2s + 1\}$. We may assume that $r \geq s$. By Theorem 10.1, we see that the biharmonic hypersurface $S^{2r+1}(1/\sqrt{2}) \times S^{2s+1}(1/\sqrt{2})$ is minimal and $\delta_{2s+1}(2)$-ideal if $r = s$; otherwise it is non-minimal and $\delta(4s + 3)$-ideal.

Incidentally, the following conjectures proposed in [4] remains open.

Conjecture 2 The only non-minimal biharmonic hypersurfaces in S^{m+1} are the open parts of hyperspheres $S^m(1/\sqrt{2})$ or of the standard products $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, where $m_1 + m_2 = m$ and $m_1 \neq m_2$.

Conjecture 3 Any non-minimal biharmonic submanifold in $S^n(1)$ has constant mean curvature.

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Chapter 11
Submersions of CR Submanifolds

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11.1 Introduction

The study of Riemannian submersions $\pi : M \longrightarrow N$ of a Riemannian manifold M onto a Riemannian manifold N was initiated by O'Neill (cf. [28]) and later such submersions have been studied widely between manifolds endowed with additional geometric structures. The simplest example of Riemannian submersion is the projection of Riemannian product manifold on one of its factors. We note that a submersion gives two distributions on total manifold called horizontal and vertical distributions. It is also important to mention that the vertical distribution of a Riemannian submersion is always integrable.

Bejancu introduced a remarkable class of submanifolds of a Kaehler manifold that are known as CR submanifolds (see [3, 4]). A CR submanifold M of an almost Hermitian manifold $\bar{M}$ with an almost complex structure J requires two orthogonal complementary distributions D and $D^{\perp}$ defined on M such that D is invariant under

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J and $D^{\perp}$ is totally real. Kobayashi (cf. [21]) observed this similarity between the total space of the submersion $\pi : M \longrightarrow N$ of a CR submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N such that the distributions D and $D^{\perp}$ of M become, respectively, the horizontal and vertical distributions required by the submersion and π restricted to D becomes a complex isometry.

Deshmukh et al. studied (cf. [13]) submersions of CR submanifolds of a Kaehler manifold and in [14], Deshmukh, Tehseen, and Hashem considered similar problem to CR submanifolds of manifolds in different classes of almost Hermitian manifolds, viz., quasi-Kaehler manifold, nearly Kaehler manifold. Submersions of CR submanifolds of locally conformal Kaehler ($l.c.k.$), quaternionic Kaehler manifold, and para-quaternionic Kaehler manifold were studied in [26, 31] and [20], respectively, while Mangione studied (cf. [22]) submersions of CR hypersurfaces of Kaehler–Einstein manifold. B. Sahin studied (cf. [30]) horizontally conformal submersions of CR submanifolds of Kaehler manifold. On the other hand, submersions of contact CR submanifolds were studied by Massamba and Matamba (cf. [24]).

The aim of this article is to survey the contributions on submersions of CR submanifolds of different classes of almost Hermitian manifolds, viz., Kaehler manifolds, quasi-Kaehler manifolds, nearly Kaehler manifolds, and $l.c.k.$ manifolds, and some almost contact metric manifolds, viz., quasi-K-cosymplectic manifold and quasi-Kenmotsu manifold.

11.2 Preliminaries

Let $(\bar{M}, g)$ be an almost Hermitian manifold. This means that $\bar{M}$ admits a tensor field J of type (1, 1) on $\bar{M}$ such that for any $X, Y \in T\bar{M}$ we have

$$J^2 = -Id, \qquad g(JX, JY) = g(X, Y).$$

Let M be an m-dimensional submanifold of $\bar{M}$. The Riemannian connection $\bar{\nabla}$ on $\bar{M}$ induces the Riemannian connections ∇ and $\nabla^{\perp}$ on M and in the normal bundle of M in $\bar{M}$, respectively. These connections are related by *Gauss* and *Weingarten* formulae:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{11.1}$$

$$\bar{\nabla}_X N' = -\bar{A}_{N'} X + \nabla^{\perp}_X N' \tag{11.2}$$

for any $X, Y \in TM$ and $N' \in TM^{\perp}$. h and $\bar{A}$ are the second fundamental form and Weingarten map and it is easy to see that $g(h(X, Y), N') = g(\bar{A}_{N'} X, Y)$.

Let $\bar{R}$ and R be the curvature tensors of $\bar{M}$ and M. The equations of *Gauss*, *Codazzi*, and *Ricci* are given by [13]

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W)), \tag{11.3}$$

$$[\bar{R}(X, Y)Z]^{\perp} = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \tag{11.4}$$

$$R(X, Y, N_1, N_2) = R^{\perp}(X, Y, N_1, N_2) - g([\overline{A}_{N_1}, \overline{A}_{N_2}]X, Y) \tag{11.5}$$

for any $X, Y, Z, W \in TM$ and $N_1, N_2 \in T^{\perp}M$where $[\ \ ]^{\perp}$ denotes the normal component.

Definition 1 Let M be a CR submanifold of an almost Hermitian manifold $\bar{M}$ with distributions D and $D^{\perp}$ and the normal bundle $T^{\perp}M$. By a submersion $\pi : M \longrightarrow N$ of M onto an almost Hermitian manifold N we mean a *Riemannian submersion* $\pi : M \to N$ together with the following conditions (cf. [21]):

(i) $D^{\perp}$ is the kernel of π_*, that is, $\pi_* D^{\perp} = \{0\}$,

(ii) $\pi_* D_p = T_{\pi(p)}N$ is complex isometry, where $p \in M$ and $T_{\pi(p)}N$ is the tangent space of N at $\pi(p)$,

(iii) J interchanges $D^{\perp}$ and $T^{\perp}M$.

Thus using the metric of M we decompose the tangent bundle TM into a direct sum

$$TM = \mathcal{H} \oplus \mathcal{V} \tag{11.6}$$

where $\mathcal{H}$ is called horizontal distribution and $\mathcal{V}$ is called vertical distribution. We use the same letters to denote the orthogonal projections onto these distributions. Therefore if $X \in TM$, we may write

$$X = \mathcal{H}X + \mathcal{V}X$$

where $\mathcal{H}$ and $\mathcal{V}$ are the projections of X on D and $D^{\perp}$ and are called horizontal and vertical part of X, respectively.

We recall that a vector field $X \in TM$ for the submersion $\pi : M \longrightarrow N$ is said to be *basic* vector field if $X \in D$ and X is π-related to a vector field on N, that is, there exists a vector X_* on N such that $(\pi_* X)_p = X_{*\pi(p)}$ for each $p \in M$ (cf. [28]).

The following lemma is known for basic vector fields (cf. [28]).

Lemma 1 *Let X and Y be basic vector fields of M. Then*

(a) $g(X, Y) = g'(X_*, Y_*) \circ \pi$, *$g$ is the metric on M and g' is the Riemannian metric on N.*

(b) *the horizontal part $\mathcal{H}[X, Y]$ of $[X, Y]$ is a basic vector and corresponds to $[X_*, Y_*]$, i.e., $\pi_*(\mathcal{H}[X, Y]) = [X_*, Y_*] \circ \pi$.*

(c) $[V, X] \in D^{\perp}$ *for any $V \in D^{\perp}$.*

(d) *$\mathcal{H}(\nabla_X Y)$ is the basic vector field corresponding to $\nabla'_{X_*} Y_*$, where ∇' is the Riemannian connection on N.*

Now we put

$$\tilde{\nabla}'_X Y = \mathcal{H}(\nabla_X Y) \text{ for any } X, Y \in D \tag{11.7}$$

then $\tilde{\nabla}'_X Y$ is basic vector field and from Lemma 1, we have

$$\pi_*(\tilde{\nabla}'_X Y) = \nabla'_{X_*} Y_*. \tag{11.8}$$

A tensor field C on M is defined by setting (cf. [13])

$$\nabla_X Y = \tilde{\nabla}'_X Y + C(X, Y) \quad \text{for any } X, Y \in D \tag{11.9}$$

where $C(X, Y)$ denotes the vertical part of $\nabla_X Y$ and is denoted by $\mathcal{V}(\nabla_X Y)$ for any $X, Y \in D$. It is easy to check that C is skew symmetric and satisfies (cf. [21])

$$C(X, Y) = \frac{1}{2}\mathcal{V}[X, Y] \quad \text{for any } X, Y \in D. \tag{11.10}$$

Furthermore, for any $X \in D$ and $V \in D^\perp$, A is defined by the following equation (cf. [13]):

$$\nabla_X V = \mathcal{V}(\nabla_X V) + A_X V\ .$$

Thus $A_X V$ is the horizontal component of $\nabla_X V$. Since by Lemma 1, $[V, X] \in D^\perp$ for any $X \in D$ and $V \in D^\perp$, we have

$$\mathcal{H}(\nabla_X V) = \mathcal{H}(\nabla_V X) = A_X V.$$

The operators A and C are related by

$$g(A_X V, Y) = -g(V, C(X, Y) \tag{11.11}$$

for any $X, Y \in D$, $V \in D^\perp$. The curvature tensor R of M and R' of N are related by (cf. [13])

$$\begin{aligned} R(X, Y, Z, W) = R'(X_*, Y_*, Z_*, W_*) + g(C(X, Z), C(Y, W)) \\ - g(C(Y, Z), C(X, W)) + 2g(C(X, Y), C(Z, W)) \end{aligned} \tag{11.12}$$

for any $X, Y, Z, W \in D$.

11.3 Submersions of CR Submanifolds

An almost Hermitian manifold $\bar{M}$ with almost complex structure J and Hermitian metric g is said to be a *Kaehler* manifold if

$$\bar{\nabla}_X JY = J\bar{\nabla}_X Y \quad \text{for any} \quad X, Y \in T\bar{M}. \tag{11.13}$$

First, we prove the following theorem (cf. [21]).

Theorem 11.1 *Let $\pi : M \longrightarrow N$ be a submersion of a CR submanifold of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N. Then*

(i) *N is a Kaehler manifold.*

(ii) *If $\bar{H}$ and H' are the holomorphic sectional curvatures of $\bar{M}$ and that of N, respectively, then for any horizontal unit vector $X \in D$,*

$$\bar{H}(X) = H^{'}(X_*) - 4\,\|h(X, X)\|^2 .$$

Proof For any basic vector fields X, Y, we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y).$$

Using (11.9) in the last equation, we get

$$\bar{\nabla}_X Y = \tilde{\nabla}'_X Y + C(X, Y) + h(X, Y).$$

Hence

$$J\bar{\nabla}_X Y = J\tilde{\nabla}'_X Y + JC(X, Y) + Jh(X, Y). \tag{11.14}$$

Similarly, we have

$$\bar{\nabla}_X JY = \tilde{\nabla}'_X JY + C(X, JY) + h(X, JY). \tag{11.15}$$

Since $\bar{M}$ is Kaehler, we have from (11.14) and (11.15)

$$J\tilde{\nabla}'_X Y = \tilde{\nabla}'_X JY, \tag{11.16}$$

$$JC(X, Y) = h(X, JY), \tag{11.17}$$

$$C(X, JY) = Jh(X, Y) \tag{11.18}$$

from which it follows easily that

$$h(X, JY) + h(JX, Y) = 0. \tag{11.19}$$

Equation (11.16) shows that N is Kaehler. Further using (11.3), (11.12), (11.17), and (11.18), for any unit basic vector field X, we derive

$$\bar{R}(X, JX, X, JX) = R'(X_*, J'X_*, X_*, J'X_*) - 4\,\|h(X, X)\|^2$$

or

$$\bar{H}(X) = H'(X_*) - 4\,\|h(X,X)\|^2\,. \qquad \square$$

Definition 2 The invariant distribution D (respectively the totally real distribution $D^\perp$) is said to be *parallel* if $\nabla_X Y \in D$ for any $X, Y \in D$ (respectively if $\nabla_X Y \in D^\perp$ for any $X, Y \in D^\perp$) .

Then we have the following (cf. [13]).

Proposition 1 *Let $\pi : M \longrightarrow N$ be a submersion of a CR submanifold of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N. If D is integrable and $D^\perp$ is parallel, then M is the product $M_1 \times M_2$ where M_1 is a complex submanifold and M_2 is a totally real submanifold of $\bar{M}$.*

Proof Integrability of distribution D yields (cf. [3])

$$h(X, JY) = h(Y, JX) \quad \text{for any} \quad X, Y \in D. \tag{11.20}$$

Combining (11.19) and (11.20) we find

$$h(X, JY) = 0 \quad \text{for any} \quad X, Y \in D.$$

Therefore from (11.17) we obtain

$$C(X, Y) = 0,$$

and thus from (11.9), $\nabla_X Y \in D$ for any $X, Y \in D$ which proves D is parallel. This completes the proof. $\square$

Combining Theorem 11.1 and Proposition 1, we have the following (cf. [13]).

Corollary 1 *Let $\pi : M \longrightarrow N$ be a submersion of a CR submanifold of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N with integrable distribution D. Then*

$$\bar{H}(X) = H'(X_*)$$

for any $X \in D$.

Definition 3 A CR submanifold is said to be *mixed foliate* if D is integrable and $h(X, Y) = 0$ for any $X \in D, Y \in D^\perp$, whereas it is called *mixed totally geodesic* if $h(X, Y) = 0$ for any $X \in D, Y \in D^\perp$ [13].

Thus we have the following (cf. [13]).

Proposition 2 *Let $\pi : M \longrightarrow N$ be a submersion of a mixed foliate CR submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N. Then M is the product $M_1 \times M_2$, where M_1 is a complex submanifold and M_2 is a totally real submanifold of $\bar{M}$.*

Proof From Proposition 1, we have

$$h(X, Y) = 0$$

for any $X, Y \in D$. Now M being mixed foliate, we have

$$h(X, Y) = 0$$

for any $X \in D$, $Y \in D^\perp$. Further for any $X, Y \in D^\perp$, we have from Kaehlerian property

$$(\bar{\nabla}_X J)Y = 0$$

which gives

$$\bar{\nabla}_X JY = J\bar{\nabla}_X Y.$$

Using Gauss and Weingarten formulas, we get

$$-\bar{A}_X JY + \nabla_X^\perp JY = J\nabla_X Y + Jh(X, Y).$$

Comparing normal components on both sides, we find

$$\nabla_X^\perp JY = J\nabla_X Y$$

proving that $\nabla_X Y \in D^\perp$ for any $X, Y \in D^\perp$, i.e., $D^\perp$ is parallel and thus the proof follows from Proposition 1. □

Now for any $U, V \in D^\perp$, L is defined by (cf. [13])

$$\nabla_U V = \hat{\nabla}_U V + L(U, V)$$

where $\hat{\nabla}_U V = \mathcal{V}(\nabla_U V)$ and $L(U, V) = \mathcal{H}(\nabla_U V)$. The sectional curvatures $\bar{K}$ of $\bar{M}$ and $\hat{K}$ of the fibers are related by (cf. [13]).

Proposition 3 *Let $\pi : M \longrightarrow N$ be a submersion of a CR submanifold of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N. Then*

$$\bar{K}(U \wedge V) = \hat{K}(U \wedge V) - g([\bar{A}_{JU}, \bar{A}_{JV}]U, V)$$

for any orthonormal vector fields $U, V \in D^\perp$.

Proof By a simple calculation and using (11.3) we have

$$\begin{aligned}\bar{K}(U \wedge V) = \bar{R}(U, V, U, V) = \hat{K}(U \wedge V) - g(L(U, V), L(U, V))\\ + g(L(U, U), L(V, V)) - g(h(U, V), h(U, V)) + g(h(U, U), h(V, V))\end{aligned}$$

for any orthonormal vectors $U, V \in D^{\perp}$. Now using $\mathcal{H}(\bar{A}_{JU}V) = -JL(U, V)$ and $\mathcal{V}(\bar{A}_{JU}V) = -Jh(U, V)$ [13] in the above equation, we obtain

$$\begin{aligned}\bar{K}(U \wedge V) &= \hat{K}(U \wedge V) - g(\bar{A}_{JU}\bar{A}_{JV}U, V) + g(\bar{A}_{JV}\bar{A}_{JU}U, V)\\ &= \hat{K}(U \wedge V) - g([\bar{A}_{JU}, \bar{A}_{JV}]U, V).\end{aligned}$$ □

For a mixed totally geodesic CR submanifold, we have the following (cf. [13]).

Proposition 4 *Let $\pi : M \longrightarrow N$ be a submersion of a mixed totally geodesic CR submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N then*

$$\bar{R}(X, V, Y, W) = -g((\nabla_V C)(X, Y), W) - g(A_X V, A_Y W) + g(h(X, Y), h(V, W)) \quad (11.21)$$

for any $X, Y \in D$ and $V, W \in D^{\perp}$.

As an application of Proposition 4 we prove the following (cf. [13]).

Proposition 5 *Let $\pi : M \longrightarrow N$ be a submersion of a mixed totally geodesic CR submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N. Then for the unit vectors $X \in D$ and $V \in D^{\perp}$*

$$\bar{K}(X \wedge V) = -\left\|\bar{A}_{JV}X\right\|^2 + g(h(X, X), h(V, V)). \quad (11.22)$$

Proof Taking $X = Y$, $W = V$ in Proposition 4 and noting $C(X, X) = 0$, we get

$$\bar{K}(X \wedge V) = g(h(X, X), h(V, V)) - \|A_X V\|^2. \quad (11.23)$$

Also using $(\bar{\nabla}_X J)V = 0$, we get

$$-\bar{A}_{JV}X + \nabla^{\perp}_X JV = JA_X V + J\mathcal{V}\nabla_X V + Jh(X, V).$$

Since M is mixed totally geodesic, we have $h(X, V) = 0$ for any $X \in D$, $V \in D^{\perp}$ and this implies $\bar{A}_{JV}X \in D$ for any $X \in D$, $\mathcal{V} \in D^{\perp}$. Thus equating horizontal component in the above equation we get

$$A_X V = J\bar{A}_{JV}X. \quad (11.24)$$

Therefore, Eqs. (11.23) and (11.24) imply (11.22). □

The following proposition is an easy consequence of Propositions 1 and 4 (cf. [13]).

Proposition 6 *Let $\pi : M \longrightarrow N$ be a submersion of a mixed foliate CR submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N, then the curvature tensor $\bar{R}$ of $\bar{M}$ satisfies*

$$\bar{R}(X, V, Y, W) = 0$$

for any $X, Y \in D$ *and any* $V, W \in D^{\perp}$.

Definition 4 If the almost complex structure J on almost Hermitian manifold $\bar{M}$ satisfies

$$(\bar{\nabla}_X J)(Y) + (\bar{\nabla}_{JX} J)(JY) = 0 \quad \text{for any} \quad X, Y \in T\bar{M} \tag{11.25}$$

then $\bar{M}$ is called a *quasi-Kaehler* manifold and if it satisfies

$$(\bar{\nabla}_X J)(Y) + (\bar{\nabla}_Y J)(X) = 0 \quad \text{for any} \quad X, Y \in T\bar{M} \text{ or } ((\bar{\nabla}_X J)(X) = 0) \tag{11.26}$$

then it is called a *nearly Kaehler* manifold [14].

Equation (11.26) yields

$$(\bar{\nabla}_X J)(JY) = -J(\bar{\nabla}_X J)(Y)$$

and

$$(\bar{\nabla}_{JX} J)(Y) = -J(\bar{\nabla}_X J)(Y)$$

for any $X, Y \in T\bar{M}$. Combining this with equation (11.26) gives

$$(\bar{\nabla}_{JX} J)(JY) + (\bar{\nabla}_X J)(Y) = -(\bar{\nabla}_X J)(Y) + (\bar{\nabla}_X J)(Y) = 0$$

which shows that a nearly Kaehler manifold is a quasi-Kaehler manifold. On the other hand a nearly Kaehler manifold with vanishing Nijenhuis torsion is a Kaehler manifold. Now we have the following theorem (cf. [14]).

Theorem 11.2 *Let* $\pi : M \longrightarrow N$ *be a submersion of a CR submanifold of a quasi-Kaehler manifold* $\bar{M}$ *onto an almost Hermitian manifold* N. *Then*

(i) N *is a quasi-Kaehler manifold.*

(ii) *If* $\bar{H}$ *and* H' *are the holomorphic sectional curvatures of* $\bar{M}$ *and that of* N, *respectively, then for any horizontal unit vector* $X \in D$

$$\bar{H}(X) = H'(X_*) + \|h(X, JX)\|^2 - g(h(JX, JX), h(X, X)) - 3\,\|C(X, JX)\|^2 .$$

Proof Let $X, Y \in D$ be the basic vector fields. Then using (11.1) and (11.9) in (11.25) we get

$$\begin{aligned}
&\tilde{\nabla}'_X JY - \tilde{\nabla}'_{JX} Y - J\tilde{\nabla}'_X Y - J\tilde{\nabla}'_{JX} JY \\
&\quad + C(X, JY) - C(JX, Y) + h(X, JY) - h(JX, Y) \\
&\quad - J(C(X, Y) + C(JX, JY)) - J(h(X, Y) + h(JX, JY)) = 0.
\end{aligned}$$

Equating horizontal, vertical, and normal components in the above equation, we get

$$\tilde{\nabla}'_X JY - \tilde{\nabla}'_{JX} Y - J\tilde{\nabla}'_X Y - J\tilde{\nabla}'_{JX} JY = 0, \quad (11.27)$$

$$C(X, JY) - C(JX, Y) = J(h(X, Y) + h(JX, JY)), \quad (11.28)$$

$$J(C(X, Y) + C(JX, JY)) = h(X, JY) - h(JX, Y). \quad (11.29)$$

Equation (11.27) shows that N is a quasi-Kaehler manifold. Next using (11.8)–(11.11), we obtain

$$R(X, Y, Z, W) = R'(X_*, Y_*, Z_*, W_*) - g(C(Y, Z), C(X, W)) \\ + g(C(X, Z), C(Y, W)) + 2g(C(X, Y), C(Z, W))$$

for any $X, Y, Z, W \in D$. Now using equation of Gauss (11.3) and the fact that C is skew symmetric, the above equation yields

$$\bar{R}(X, JX, JX, X) = R'(X_*, J'X_*, J'X_*, X_*) + \|h(X, JX)\|^2 \\ - g(h(JX, JX), h(X, X)) - 3\|C(X, JX)\|^2$$

or

$$\bar{H}(X) = H'(X_*) + \|h(X, JX)\|^2 - g(h(JX, JX), h(X, X)) \\ - 3\|C(X, JX)\|^2 . \qquad \square$$

In [1] one can find analogous results for submersions of a CR submanifold of a non-Kaehler, nearly Kaehler manifold $\overline{M}$. In fact by adding (11.15) with the corresponding value of $\bar{\nabla}_Y JX$, using (11.14) and the skew symmetric property of C we get

$$(\tilde{\nabla}'_X J)(Y) + (\tilde{\nabla}'_Y J)(X) + C(X, JY) + C(Y, JX) \\ + h(X, JY) + h(Y, JX) = 2Jh(X, Y).$$

for any basic vector fields X, Y. Thus we have the following theorem (cf. [1]).

Theorem 11.3 *Let $\pi : M \longrightarrow N$ be a submersion of a CR submanifold of a nearly Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N. Then*

(i) *N is a nearly Kaehler manifold.*

(ii) *If $\bar{H}$ and H' are the holomorphic sectional curvatures of $\bar{M}$ and N, then for any horizontal unit vector $X \in D$*

$$\bar{H}(X) = H'(X_*) - 4\|h(X, X)\|^2 .$$

Definition 5 The normal connection of M in $\bar{M}$ is called *D-flat* if $R^{\perp}(X, Y)N' = 0$ for any $X, Y \in D$ and $N' \in T^{\perp}M$ [1].

We now have the following theorem (cf. [1]).

Theorem 11.4 *Let $\pi : M \longrightarrow N$ be a submersion of a mixed foliate CR submanifold of a nearly Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N. Then the normal connection of M in $\bar{M}$ is D-flat.*

Proof It is known from [1] that for nearly Kaehler manifolds we have

$$\bar{R}(X, Y, JZ, JW) = \bar{R}(X, Y, Z, W) - g((\bar{\nabla}_X J)(Y), (\bar{\nabla}_Z J(W) \tag{11.30}$$

for any $X, Y, Z, W \in T\bar{M}$. Combining (11.30) and Ricci equation (11.5), for any $X, Y \in D$ and $V, W \in D^{\perp}$, we obtain

$$\begin{aligned} R^{\perp}(X, Y, JV, JW) = \bar{R}(X, Y, V, W) - g((\bar{\nabla}_X J)(Y), (\bar{\nabla}_V J)(W)) \\ + g([\bar{A}_{JV}, \bar{A}_{JW}](X), Y). \end{aligned} \tag{11.31}$$

Again in the Bianchi identity

$$\bar{R}(X, Y, V, W) + \bar{R}(Y, V, X, W) + \bar{R}(V, X, Y, W) = 0$$

the last two terms vanish from Proposition 2.4 of [1] and hence we have

$$\bar{R}(X, Y, V, W) = 0. \tag{11.32}$$

Thus for any $X, Y \in D$ and $V, W \in D^{\perp}$ we get

$$\begin{aligned} g([\bar{A}_{JV}, \bar{A}_{JW}](X), Y) &= g(A_X W, A_Y V) - g(A_X V, A_Y W) \\ &= -g(C(X, A_Y V), W) + g(C(X, A_Y W), V) = 0 \end{aligned} \tag{11.33}$$

where we have used (11.24), (11.11) and the fact that for a foliate CR submanifold $C(X, Y) = 0$. Therefore, we get $R^{\perp}(X, Y, JV, JW) = 0$, for any $X, Y \in D$, $V, W \in D^{\perp}$ from which theorem follows. □

Finally, in this section, we discuss how the submersion $\pi : M \to N$ of a CR submanifold M with integrable invariant distribution D affects the topology of M. Let M be a CR submanifold of a Hermitian manifold $\bar{M}$ with almost complex structure J. Let

$$\{E_1,\ E_2, \ldots, E_p,\ JE_1,\ JE_2, \ldots,\ JE_p,\ E_{2p+1},\ E_{2p+2}, \ldots,\ E_m\}$$

be the local orthonormal frame on M such that $\{E_j,\ JE_j : 1 \leq j \leq p\}$ is a local orthonormal frame of D and $\{E_{2p+1},\ E_{2p+2}, \cdots,\ E_m\}$ is a local orthonormal frame of $D^{\perp}$. Let

$$\{\omega^1,\ \omega^2, \ldots, \omega^{2p},\ \omega^{2p+1}, \ldots,\ \omega^m\}$$

be the dual frame of 1-forms to the above local orthonormal frame. We write a global $2p$-form Ω on M, independent of the choice of $\{E_j\ ,\ JE_j : 1 \leq j \leq p\}$, as

$$\Omega = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^{2p}. \tag{11.34}$$

Definition 6 Let $\mathcal{D}$ be a s-dimensional distribution on a Riemannian manifold M.If $\sum_{i=1}^{s} \nabla_{E_i} E_i \in \mathcal{D}$ then the distribution is said to be *minimal*, where ∇ is the Riemannian connection on M and $\{E_1, E_2, ..., E_s\}$ is a local orthonormal frame of $\mathcal{D}$.

First, we state the following (cf. [14]).

Proposition 7 *Let $\pi : M \to N$ be submersion of a complete, simply connected CR submanifold M of a Hermitian manifold $\bar{M}$ onto an almost Hermitian manifold N. If D is integrable and $D^\perp$ is parallel then M is the Riemannian product $M_1 \times M_2$, where M_1 is an invariant submanifold, and M_2 is a totally real submanifold of $\bar{M}$.*

Now we prove the following theorem (cf. [14]).

Theorem 11.5 *Let $\pi : M \to N$ be submersion of a closed CR submanifold M of a Hermitian manifold $\bar{M}$ with integrable distribution D onto an Hermitian manifold N. Then the $2p$-form Ω is closed which defines a canonical de Rham cohomology class $[\Omega] \in H^{2p}(M, R)$, where $2p = \dim D$. Moreover, the cohomology group $H^{2p}(M, R)$ is nontrivial if $D^\perp$ is minimal.*

Theorem 11.5 is an immediate consequence of the following lemma in [9].

Lemma 2 *Let M be a closed CR submanifold of a Hermitian manifold $\overline{M}$.*

(i) *If D is minimal and $D^\perp$ is integrable then Ω is closed* (*i.e., $d\Omega = 0$*).

(ii) *If D is integrable and $D^\perp$ is minimal then Ω is co-closed* (*i.e., $\delta\Omega = 0$*).

In particular if both D and $D^\perp$ are integrable and minimal, $H^{2p}(M, \mathbb{R}) \neq 0$, where p is the complex rank of the holomorphic distribution D.

The result was stated in [9] under the assumption that the ambient space $\overline{M}$ is a Kaehlerian manifold (and if this is the case the hypothesis in (i) of Lemma 2 holds true to start with). The proof is, however, more general (and will be outlined below, for the reader's convenience). Equation (11.34) of Ω gives

$$d\Omega = \sum_{i=1}^{2p} (-1)^{i-1} \omega^1 \wedge \cdots \wedge d\omega^i \wedge ..... \wedge \omega^{2p}.$$

From the above equation, it follows that $d\Omega = 0$ if and only if

$$d\Omega(Z, W, E_1, E_2, \ldots, E_{2p-1}) = d\Omega(Z, E_1, ..., E_{2p}) = 0 \tag{11.35}$$

for any $Z, W \in D^\perp$ and $E_1, E_2, \ldots, E_{2p} \in D$. Choosing $E_a \in D$, $1 \leq a \leq 2p$, as a local orthonormal frame $\{E_j\ ,\ JE_j : 1 \leq j \leq p\}$ of D to which $\{\omega^1, \omega^2, ..., \omega^{2p}\}$

works as a dual frame of 1-forms, we can say that the first equation in (11.35) holds if and only if $D^{\perp}$ is integrable, whereas the second part in (11.35) is satisfied if and only if D is minimal. In particular, under the assumptions in (i) of Chen's Lemma 2 the form Ω is closed and thus gives rise to a well-defined de Rham cohomology class $[\Omega] \in H^{2p}(M, R)$. Next, let us consider the $(m-2p)$-form $\Omega^{\perp}$ on M given by

$$\Omega^{\perp} = \omega^{2p+1} \wedge \cdots \wedge \omega^{m}$$

where $\{\omega^{2p+1}, \ldots, \omega^{m}\}$ is a dual frame to the local orthonormal frame $\{E_{2p+1}, \ldots, E_m\}$ of $D^{\perp}$. Then with the similar argument for Ω, it follows that $d\Omega^{\perp} = 0$ if D is integrable and $D^{\perp}$ is minimal. It may then be shown (by expressing the Hodge operator in terms of the chosen local frames) that $d\Omega^{\perp} = 0$ is equivalent to $\delta\Omega = 0$. Consequently, Ω is a harmonic form on M so that (as M is closed) the corresponding de Rham cohomology class is nontrivial ($[\Omega] \neq 0$). □

11.4 Horizontally Conformal Submersions

In this section, we discuss the results on horizontally conformal submersions of CR submanifolds of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N obtained by B. Sahin, [30].

Definition 7 Let (M^m, g) and (N^n, g') be Riemannian manifolds. Suppose that $\pi : (M^m, g) \longrightarrow (N^n, g')$ is a map between Riemannian manifolds and $p \in M$. Then π is called a *horizontally weakly conformal* map at p if either

(i) $\pi_{*p} = 0$ or

(ii) π_{*p} maps the horizontal space $\mathcal{H} = \{\ker(\pi_{*p})\}^{\perp}$ conformally onto $T_{\pi(p)}N$, i.e., π_{*p} is surjective and there exists a number $\lambda(p) \neq 0$ such that

$$g'(\pi_{*p}(X), \pi_{*p}(Y)) = \lambda(p) g(X, Y) \tag{11.36}$$

for any $X, Y \in \mathcal{H}$. If a point p is of type (i), then it is called *critical point* of π. A point p of type (ii) is called *regular*. The number $\lambda(p)$ is called *square dilation*. The map π is called *horizontally conformal submersion* if π has no critical point.

Thus a Riemannian submersion is a horizontally conformal submersion with square dilation identically one. A horizontally conformal submersion $\pi : M \longrightarrow N$ is said to be *horizontally homothetic* if the gradient of λ is vertical, i.e.,

$$\mathcal{H}(grad\lambda) = 0. \tag{11.37}$$

The unsymmetrized second fundamental form $A^{\mathcal{V}}$ of $\mathcal{V}$ is defined by

$$A^{\mathcal{V}}_E F = \mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F) \tag{11.38}$$

and the symmetrized second fundamental form $B^{\mathcal{V}}$ of V is given by

$$B^{\mathcal{V}}(E, F) = \frac{1}{2}\{A^{\mathcal{V}}_E F + A^{\mathcal{V}}_F E\} = \frac{1}{2}\{\mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F) + \mathcal{H}(\nabla_{\mathcal{V}F}\mathcal{V}E)\} \tag{11.39}$$

for any $E, F \in TM$ [30]. The integrability tensor of $\mathcal{V}$ is the tensor field $I^{\mathcal{V}}$ given by

$$I^{\mathcal{V}}(E, F) = A^{\mathcal{V}}_E F - A^{\mathcal{V}}_F E - \mathcal{H}([\mathcal{V}E, \mathcal{V}F]). \tag{11.40}$$

Moreover, the mean curvature of $\mathcal{V}$ is defined by [30]

$$\mu^{\mathcal{V}} = \frac{1}{q}\sum_{r=1}^{q}\mathcal{H}(\nabla_{E_r}E_r) \tag{11.41}$$

where $\{E_1, E_2, \ldots, E_q\}$ is a local frame of $\mathcal{V}$. By reversing the roles of $\mathcal{V}$ and $\mathcal{H}$, we can define $B^{\mathcal{H}}$, $A^{\mathcal{H}}$ and $I^{\mathcal{H}}$. For example

$$B^{\mathcal{H}}(E, F) = \frac{1}{2}\{\mathcal{V}(\nabla_{\mathcal{H}E}\mathcal{H}F) + \mathcal{V}(\nabla_{\mathcal{H}F}\mathcal{H}E)\} \tag{11.42}$$

Hence, we have the mean curvature of $\mathcal{H}$ as

$$\mu^{\mathcal{H}} = \frac{1}{m-q}\sum_{s=1}^{m-q}\mathcal{V}(\nabla_{E_s}E_s)$$

where $\{E_1, E_2, \ldots, E_{m-q}\}$ is a local frame of $\mathcal{H}$.

It is known that if a horizontally conformal submersion $\pi : M \to N$ is horizontally homothetic, then (cf. e.g., [2])

$$\mu^{\mathcal{H}} = grad \ln \lambda = \frac{1}{2}(grad \ln \|\pi_*\|^2). \tag{11.43}$$

The tension field $\tau(\pi)$ for horizontally conformal submersion is given by [2]

$$\tau(\pi) = -(n-2)\pi_*(grad \ln \lambda) - (m-n)\pi_*(\mu^{\mathcal{V}}). \tag{11.44}$$

Let M be a CR submanifold of a Kaehler manifold $(\bar{M}, J)$ and (N, J') be an almost Hermitian manifold. Let $\pi : M \to N$ be a horizontally conformal submersion such that

$$D^{\perp} = \ker(\pi_*), \quad D = \{\ker(\pi_*)\}^{\perp} = \mathcal{H}, \tag{11.45}$$

$$J(D^{\perp}) = T^{\perp}M, \quad J' \circ \pi_* = \pi_* \circ J.$$

Now we have the following theorem (cf. [30]).

Theorem 11.6 *Let $\pi : M \longrightarrow N$ is a horizontally homothetic submersion of a CR submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N under the assumptions (11.45). Then π is a Riemannian submersion up to a scale. Moreover, π is a harmonic map if M is mixed geodesic.*

Proof We know from (11.37) that $\mathcal{H}(grad \ln \lambda) = 0$. On the other hand, from (11.42) we have

$$g(B^{\mathcal{H}}(X, X), V) = g(\nabla_X X, V)$$

for any $X \in D$ and $V \in D^{\perp}$. Using Gauss formula (11.1), we find

$$g(B^{\mathcal{H}}(X, X), V) = g(\bar{\nabla}_X X, V)$$

which, after considering that $\bar{M}$ is Kaehler, gives

$$g(B^{\mathcal{H}}(X, X), V) = g(\bar{\nabla}_X JY, JV).$$

It implies that

$$g(B^{\mathcal{H}}(X, X), V) = g([X, JX] + \bar{\nabla}_{JX} X, JV).$$

From this, we derive

$$g(B^{\mathcal{H}}(X, X), V) = g(\bar{\nabla}_{JX} X, JV).$$

Using (11.13) and (11.1) we have

$$g(B^{\mathcal{H}}(X, X), V) = -g(\nabla_{JX} JX, V) \quad \text{for any} \quad X \in D, \; V \in D^{\perp}$$

Hence we have

$$g(B^{\mathcal{H}}(X, X), V) = -g(B^{\mathcal{H}}(JX, JX), V). \tag{11.46}$$

Thus we have

$$g(\mu^{\mathcal{H}}, V) = \frac{1}{2p} \sum_{i=1}^{p} \{g(B^{\mathcal{H}}(E_i, E_i), V) + g(B^{\mathcal{H}}(JE_i, JE_i), V)\}.$$

Using (11.46) in the above equation yields

$$g(\mu^{\mathcal{H}}, V) = \frac{1}{2p}\sum_{i=1}^{p}\{g(B^{\mathcal{H}}(E_i, E_i), V) - g(B^{\mathcal{H}}(E_i, E_i), V)\} = 0$$

which implies that

$$\mu^{\mathcal{H}} = 0. \tag{11.47}$$

Combining (11.43) and (11.47) we find *grad* $\ln\lambda = 0$. Hence, λ is a constant on M. Thus π is a Riemannian submersion up to scale. On the other hand from (11.38) and (11.39) we have

$$g(B^{\mathcal{V}}(Z, Z), X) = g(\nabla_Z Z, X)$$

for any $X \in D$ and $Z \in D^{\perp}$. Then using (11.13) we derive

$$g(B^{\mathcal{V}}(Z, Z), X) = g(\bar{\nabla}_Z JZ, JX).$$

Thus from (11.2) we get

$$g(B^{\mathcal{V}}(Z, Z), X) = -g(h(Z, JX), JZ). \tag{11.48}$$

Then since M is mixed totally geodesic, last equation along with the definition of $\mu^{\mathcal{V}}$ gives

$$g(\mu^{\mathcal{V}}, X) = 0. \tag{11.49}$$

The harmonicity of π now follows from (11.44), (11.49) and *grad* $\ln\lambda = 0$. □

For horizontally conformal submersion and N, a Kaehler manifold, we have (cf. [30])

Theorem 11.7 *Let $\pi : M \longrightarrow N$ be a horizontally conformal submersion of a CR submanifold M of a Kaehler manifold $\bar{M}$ onto a Kaehler manifold N under the assumptions* (11.45)*. Then π is a Riemannian submersion up to a scale. Moreover, π is a harmonic map if M is mixed totally geodesic.*

11.5 Submersions of Totally Umbilical CR Submanifolds

A submanifold M of a Kaehler manifold $\bar{M}$ is said to be *totally umbilical* if $h(X, Y) = g(X, Y)H$, where $H = \frac{1}{m}(trace\ h)$ is called the mean curvature and *extrinsic hyperspheres* are defined to be totally umbilical hypersurfaces having nonzero parallel mean curvature vector field (cf. e.g., [27]). Many of the basic results concerning extrinsic spheres in Riemannian and Kaehlerian geometry were studied by Chen [8]. First, we discuss the submersions of totally umbilical CR submanifolds M of a Kaehler manifold $\bar{M}$ (cf. [13]) and in this case the Gauss and Weingarten formulae become

$$\bar{\nabla}_X Y = \nabla_X Y + g(X, Y)H, \tag{11.50}$$

$$\bar{\nabla}_X N' = -g(N', H)X + \nabla^{\perp}_X N' \tag{11.51}$$

for any $X, Y \in TM$ and $N' \in T^{\perp}M$. Further in this case, the Codazzi equation becomes (11.4)

$$[\bar{R}(X, Y)Z]^{\perp} = g(Y, Z)\nabla^{\perp}_X H - g(X, Z)\nabla^{\perp}_Y H. \tag{11.52}$$

If $\bar{M}(c)$ is a complex space form of constant holomorphic sectional curvature c, the curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by (cf. [13])

$$\begin{aligned}\bar{R}(X, Y, Z, W) = \frac{c}{4}[&g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + g(JY, Z)g(JX, W)\\ &- g(JX, Z)g(JY, W) + 2g(X, JY)g(JZ, W)].\end{aligned} \tag{11.53}$$

Theorem 11.8 (cf. [13]) *Let $\pi : M \longrightarrow N$ be a submersion of a totally umbilical CR submanifold M ($\dim M \geq 5$) of a complex space form $\bar{M}(c)$ onto an almost Hermitian manifold N. Then N is also a complex space form.*

Proof Since in case of submersion $\pi : M \longrightarrow N$, $JD^{\perp} = T^{\perp}M$, from a theorem [5] it follows that either $H = 0$ or $\dim D^{\perp} = 1$. In case $H = 0$, it follows that N is also a complex space form [21] . Now suppose $\dim D^{\perp} = 1$. We easily get the following expression for the curvature tensor R' of N:

$$\begin{aligned}R'(X_*, Y_*, Z_*, W_*) = \left(\frac{c}{4} + \|H\|^2\right)\{&g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\\ &+ g(JY, Z)g(JX, W) - g(JX, Z)g(JY, W)\\ &+ 2g(X, JY)g(JZ, W)\}\end{aligned}$$

for any $X, Y, Z, W \in D$. Since $\dim M \geq 5$ we can choose vectors $X, Y \in D$ such that

$$g(X, Y) = g(X, JY) = 0.$$

From (11.53) it follows that $\overline{R}(JY, X, JY, N') = 0$. Thus (11.52) gives

$$g(\nabla^{\perp}_X H, N') = 0 \text{ for } N' \in T^{\perp}M \tag{11.54}$$

which shows that

$$\nabla^{\perp}_X H = 0 \quad \text{for any} \quad X \in D.$$

Next let $X \in D^{\perp}$. Then using some curvature properties of $\bar{M}$ given in [13] and (11.53), we obtain

$$\bar{R}(X, Y, Y, X) = \bar{R}(X, Y, JY, N') = 0, \ N' = JX.$$

Furthermore, using linearity of $\bar{R}$ in $\bar{R}(X, Y, Y, X) = 0$, we get

$$\bar{R}(X, Y, Y, N') = 0.$$

Using this in (11.52), we find

$$g(\nabla_X^{\perp} H, N') = 0.$$

Since dim $D^{\perp} = 1$ we get $\nabla_X^{\perp} H = 0$ for any $X \in D^{\perp}$. Therefore, for any X on M, we have $X \|H\|^2 = Xg(H, H) = 2g(\nabla_X^{\perp} H, H) = 0$ showing $\|H\|^2 =$ constant and hence the theorem. □

Next we have the following theorem (cf. [13]).

Theorem 11.9 *Let $\pi : M \longrightarrow N$ be a submersion of a totally umbilical CR submanifold M of a Kaehler manifold $\bar{M}$ with parallel distribution D onto an almost Hermitian manifold N. Then M is the product $M_1 \times M_2$, where M_1 is a complex submanifold and M_2 is a totally real submanifold of $\bar{M}$.*

Proof Since M is totally umbilical CR submanifold, using Gauss and Weingarten formulae (11.1) and (11.2) respectively in $(\bar{\nabla}_X J)(JH) = 0$, we get

$$g(H, H)X - \nabla_X^{\perp} H = J\nabla_X JH + h(X, JH)JH \tag{11.55}$$

which, after taking inner product with $X \neq 0 \in D$, gives

$$\|H\|^2 \|X\|^2 = g(JH, \nabla_X JX). \tag{11.56}$$

As D is parallel, $\nabla_X JX \in D$ for any $X \in D$, which implies that $g(\nabla_X JX, JH) = 0$, the above relation (11.56) gives $\|H\|^2 = 0$, i.e., M is totally geodesic and hence the result. □

Now we shall study some curvature results for submersions of CR submanifolds of Kaehler manifold. First, we quote the following theorem (cf. [13]).

Theorem 11.10 *Let $\pi : M \longrightarrow N$ be a submersion of a mixed foliate CR submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N. Then the Ricci tensors $\bar{S}$ and S' of $\bar{M}$ and N, respectively, satisfy the relation*

$$\bar{S}(X, Y) = S'(X, Y) \tag{11.57}$$

for any $X, Y \in D$.

Definition 8 The Kaehler manifold $\bar{M}$ is said to be an *Einstein space* if there exist a constant σ such that the Ricci tensor $\bar{S}$ of $\bar{M}$ satisfies

$$\bar{S}(X, Y) = \sigma g(X, Y) \tag{11.58}$$

for all tangent vectors X, Y on $\bar{M}$.

As a direct consequence of (11.58), Theorem 11.10 gives the following theorem (cf. [13]).

Theorem 11.11 *Let $\pi : M \longrightarrow N$ be a submersion of a mixed foliate CR submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N. Then N is Einstein space if and only if $\bar{M}$ is Einstein.*

Now, simple calculations starting from (11.3), (11.12), and (11.53) yield

$$\begin{aligned} R'(Z_*, X_*, Y_*, W_*) = \frac{c}{4}\{&g(X, Y)g(Z, W) - g(Z, Y)g(X, W) \\ &+ g(JX, Y)g(JZ, W) - g(JZ, Y)g(JX, W) + 2g(Z, JX)g(JY, W)\} \\ &+ g(h(Z, W), h(X, Y)) - g(h(Z, Y), h(X, W)) \\ &+ g(C(X, Y), C(Z, W)) - g(C(Z, Y), C(X, W)) - 2g(C(Z, X), C(Y, W)) \end{aligned} \tag{11.59}$$

for any basic vector fields X, Y, Z, W on M. The above equation is simplified to the following form:

$$\begin{aligned} S'(X_*, Y_*) = \frac{(p+1)c}{2} g(X, Y) &+ \sum_{i=1}^{2p}\{g(h(X, Y), h(E_i, E_i)) - g(h(E_i, Y), h(E_i, X))\} \\ &+ 3\sum_{i=1}^{2p}\{g(h(E_i, JY), h(E_i, JX))\}. \end{aligned} \tag{11.60}$$

From (11.60), the scalar curvature ρ' of N is given in the following form:

$$\begin{aligned} \rho' = p(p+1)c & \\ &+ \sum_{i,j=1}^{2p}\{g(h(E_i, E_i), h(E_j, E_j)) - g(h(E_i, E_j), h(E_i, E_j))\} \\ &+ 3\sum_{i,j=1}^{2p}\{g(h(E_i, JE_j), h(E_i, JE_j))\}. \end{aligned} \tag{11.61}$$

Thus we have the following theorem (cf. [13]).

Theorem 11.12 *Let $\pi : M \longrightarrow N$ be a submersion of a mixed foliate CR submanifold M of a complex space form $\bar{M}(c)$ of constant holomorphic sectional curvature c onto an almost Hermitian manifold N. Then the Ricci tensor S' of N satisfies*

$$S'(X_*, Y_*) = \frac{(p+1)c}{2} g(X, Y)$$

for any horizontal vector fields $X, Y \in D$.

In particular for scalar curvature ρ' of N, we have the following theorem (cf. [13]).

Theorem 11.13 *Let $\pi : M \longrightarrow N$ be a submersion of a foliate CR submanifold M of a complex space form $\bar{M}(c)$ of constant holomorphic sectional curvature c onto an almost Hermitian manifold N. Then M is D-totally geodesic if and only if the scalar curvature ρ' of N satisfies*

$$\rho' = p(p+1)c.$$

The following corollary follows from Theorem 11.12 (cf. [13]).

Corollary 2 *Let $\pi : M \longrightarrow N$ be a submersion of a mixed foliate CR submanifold M of a complex space form $\bar{M}(c)$ of constant holomorphic sectional curvature c, onto an almost Hermitian manifold N. Then N is an Einstein space.*

Before we close this section, we study submersions of CR hypersurfaces of Kaehler manifold. For this we denote by N' the global unit normal vector field to M. Then $\zeta = -JN'$. It should be noted that M is a CR hypersurface of $\bar{M}$ such that $TM = D \oplus D^{\perp}$, where $D^{\perp}$ is one-dimensional anti-invariant distribution generated by the vector field ζ on M. Thus we prove a theorem via curvatures for extrinsic hyperspheres as follows (cf. [22]).

Theorem 11.14 *Let M be an orientable extrinsic hypersphere of a Kaehler–Einstein manifold $\bar{M}$. If $\pi : M \longrightarrow N$ is a CR submersion of M onto an almost Hermitian manifold N, then N is a Kaehler–Einstein manifold.*

Proof From Gauss formula (11.1) and the umbilicality of M, we get $\bar{\nabla}_X \zeta = \nabla_X \zeta$ for any vector field $X \in TM$. Then we have (cf. [22])

$$g(\bar{\nabla}_X JN', Y) = -g(\nabla_X \zeta, Y) = -g(\mathcal{H}\nabla_X \zeta, Y) = -g(A_X \zeta, Y). \tag{11.62}$$

If we put $\|H\| = k$, then k is a nonzero constant function on the extrinsic hypersphere M. Since $\bar{M}$ is Kaehler and M is totally umbilical, for any horizontal vector field X and Y, we have

$$\begin{aligned} g(\bar{\nabla}_X JN', Y) &= g(J\bar{\nabla}_X N', Y) = -g(\bar{\nabla}_X N', JY) \\ &= g(h(X, JY), N') = g(X, JY)g(H, N') = kg(X, JY). \end{aligned} \tag{11.63}$$

Combining (11.62) and (11.63) we conclude that

$$g(A_X \zeta, Y) = -kg(X, JY).$$

Consequently, from the last equation we have

$$g(A_X\zeta, A_Y\zeta) = k^2 g(X, Y). \tag{11.64}$$

It is known that for any horizontal vector fields X and Y, $A_X Y$ is a vertical vector field and hence (cf. [22])

$$A_X Y = g(A_X Y, \zeta)\zeta.$$

Therefore, from the last equation we derive

$$g(A_X Y, A_Z W) = k^2 g(X, JY)g(Z, JW) \tag{11.65}$$

for any $X, Y, Z, W \in D$. Also, using the Gauss equation (11.3) and umblicality of M, we get

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) + k^2\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\} \tag{11.66}$$

for any $X, Y, Z, W \in D$. Now it is known that the anti-invariant distribution $D^\perp$ is integrable and its leaves are totally geodesic in M and for Riemannian submersions with totally geodesic fibers, the following formula is known (cf. [22]):

$$\bar{R}(X, V, Y, U) = g((\nabla_V A)(X, Y), U) + g(A_X V, A_Y U) \tag{11.67}$$

for any $X, Y \in D$ and $U, V \in D^\perp$. It is noted that the first term on the right part of the above equation is skew symmetric with respect to the vertical vector fields V and U. Combining (11.66) and (11.67) we arrive at

$$\bar{R}(\zeta, X, Y, \zeta) = 0, \quad \bar{R}(\zeta, JX, JY, \zeta) = 0. \tag{11.68}$$

On the other hand, for any $X, Y, Z, W \in D$, we have (cf. [28])

$$\begin{aligned} R(X, Y, Z, W) = R'(X_*, Y_*, Z_*, W_*) &- 2g(A_X Y, A_Z W) + g(A_Y Z, A_X W) \\ &- g(A_X Z, A_Y W). \end{aligned} \tag{11.69}$$

From (11.66), (11.68), and (11.69), we get

$$\begin{aligned} \bar{R}(X, Y, Z, W) = R'(X_*, Y_*, Z_*, W_*) &- k^2\{g(X, JZ)g(Y, JW) \\ &- g(X, JW)g(Y, JZ) + 2g(X, JY)g(Z, JW)\} \\ &- k^2\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\}. \end{aligned} \tag{11.70}$$

Using the above facts in (11.70), it follows that N is a Kaehler–Einstein manifold. □

The following corollary is an easy consequence of the last theorem (cf. [22]).

Corollary 3 *Let $\pi : M \longrightarrow N$ be a submersion of an orientable CR hypersurface of a complex space form $\bar{M}(c)$ onto an almost Hermitian manifold N. Then the base space N is also a complex space form.*

It is noted that Shahid and Solamy [31] studied submersions of quaternion CR submanifolds of quaternion Kaehler manifold onto an almost quaternion manifold while in [20], Ianus, Marchiafava, and Vilcu defined the para-quaternionic CR submersions as semi-Riemannian submersions from quaternionic CR submanifold onto an almost para-quaternionic Hermitian manifold and obtained some properties concerning their geometry. They also discussed curvature properties of fibers and base manifold for para-quaternionic CR submersions.

11.6 Submersions of CR Submanifolds of l.c.K. Manifolds

CR submanifolds of locally conformal Kaehler manifolds were studied by Dragomir [15], Matsumoto [25], Narita [26], and others. In this section, we give some results on submersions of CR submanifolds of locally conformal Kaehler manifolds [26].

Let $\bar{M}$ be an almost Hermitian manifold with fundamental 2-form Ω. Then the manifold $\bar{M}$ is characterized to be a *locally conformal Kaehler* manifold if the following holds:

$$d\Omega = \omega \wedge \Omega, \quad d\omega = 0 \tag{11.71}$$

where ω is a globally defined 1-form on $\bar{M}$. We call ω the *Lee form*. We define the *Lee* vector field $\mathcal{L}$ by

$$g(X, \mathcal{L}) = \omega(X). \tag{11.72}$$

The *Weyl connection* $\bar{\nabla}^W$ is the linear connection defined by

$$\bar{\nabla}^W_X Y = \bar{\nabla}_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)\mathcal{L}. \tag{11.73}$$

It is known from [32] that an almost Hermitian manifold $\bar{M}$ is a locally conformal Kaehler manifold if and only if there is a closed 1-form ω on $\bar{M}$ such that

$$\bar{\nabla}^W_X J = 0. \tag{11.74}$$

Furthermore, (11.74) is equivalent to

$$\bar{\nabla}_X JY - \frac{1}{2}\omega(JY)X + \frac{1}{2}g(X, JY)\mathcal{L} = J\nabla_X Y - \frac{1}{2}\omega(Y)JX + \frac{1}{2}g(X, Y)J\mathcal{L} \tag{11.75}$$

for any vector fields X and Y on $\bar{M}$. If $\bar{R}^W$ is the curvature tensor field of the Weyl connection ∇^W, then we have

$$\begin{aligned}\bar{R}^W(X,Y)Z = \bar{R}(X,Y)Z \\ &- \frac{1}{2}\{[(\nabla_X\omega)Z + \frac{1}{2}\omega(X)\omega(Z)]Y \\ &- [(\nabla_Y\omega)Z + \frac{1}{2}\omega(Y)\omega(Z)]X - g(Y,Z)(\nabla_X\mathcal{L} + \frac{1}{2}\omega(X)\mathcal{L}) \\ &+ g(X,Z)(\nabla_Y\mathcal{L} + \frac{1}{2}\omega(Y)\mathcal{L})\} - \frac{1}{4}\|\omega\|^2(g(Y,Z)X - g(X,Z)Y)\end{aligned} \tag{11.76}$$

for any vector fields X and Y on $\bar{M}$.

Definition 9 A locally conformal Kaehler manifold $(\bar{M}, g, J)$ is said to be a *generalized Hopf* manifold if the Lee form is parallel, i.e., $\bar{\nabla}\omega = 0 (\omega \neq 0)$. A generalized Hopf manifold is called a $\mathcal{P}_o K$-manifold if the Weyl curvature tensor is zero.

First, we have the following theorem (cf. [26]).

Theorem 11.15 *Let $\pi : M \to N$ be a submersion of a CR submanifold M of a locally conformal Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold N with Lee vector field $\mathcal{L} \in T^\perp M$. Then N is a Kaehler manifold.*

Proof For any vector field X tangent to M and $\mathcal{L} \in T^\perp M$, we have $\omega(X) = 0$. Since M is a CR submanifold of $\bar{M}$, (11.75) implies

$$\bar{\nabla}_X JY + \frac{1}{2}g(X, JY)\mathcal{L} = J\nabla_X Y + \frac{1}{2}g(X,Y)J\mathcal{L} \tag{11.77}$$

where X and Y are any horizontal vector fields. It is easy to see that

$$\bar{\nabla}_X Y = \mathcal{H}\nabla_X Y + A_X Y + h(X,Y). \tag{11.78}$$

From (11.77) and (11.78), we get

$$\mathcal{H}\nabla_X JY = J\mathcal{H}\nabla_X Y \in D, \tag{11.79}$$

$$A_X JY = Jh(X,Y) + \frac{1}{2}g(X,Y)J\mathcal{L} \in D^\perp, \tag{11.80}$$

$$h(X, JY) + \frac{1}{2}g(X,JY)\mathcal{L} = JA_X Y \in T^\perp M \tag{11.81}$$

where X and Y are any horizontal vector fields on M. Since π_* is a complex isometry, we have $\pi_* \circ J = J' \circ \pi_*$. Therefore, if X is a basic vector field, JX is also a basic vector field. Using Lemma 1, (11.7) and (11.79), we have

$$\nabla_{X_*}{}' J' Y_* = J' \nabla_{X_*}{}' Y_*.$$

Hence, N is a Kaehler manifold. □

Next we have the following theorem (cf. [26]).

Theorem 11.16 *Let $\pi : M \to N$ be a submersion of a CR submanifold M of a $\mathcal{P}_o K$-manifold $\bar{M}$ onto an almost Hermitian manifold N. If M is a totally umbilical submanifold whose mean curvature vector is parallel and $\mathcal{L} \in T^{\perp}M$, then N is a locally symmetric Kaehler manifold and $H'(X_*) > 0$ where H' is the holomorphic sectional curvature of N and X_* is any unit tangent vector on N.*

Proof Since $\bar{M}$ is a $\mathcal{P}_o K$-manifold, we have $\bar{R}^W = 0$ and $\bar{\nabla}\omega = 0$. We set $c = \frac{\|\omega\|}{2}$. Since $\bar{\nabla}\Omega = 0$, we have $\bar{\nabla}\mathcal{L} = 0$ and c = constant (cf. [33]). From (11.76) we have

$$\bar{R}(X, Y)Z = \frac{1}{4}\{[\omega(X)Y - \omega(Y)X]\omega(Z) + [g(X, Z)\omega(Y) - g(Y, Z)\omega(X)]\mathcal{L}\} + c^2(g(Y, Z)X - g(X, Z)Y). \tag{11.82}$$

Using $\bar{\nabla}\omega = 0$ and $\bar{\nabla}\mathcal{L} = 0$, we obtain $\bar{\nabla}\bar{R} = 0$ (cf. [15]). Since $\mathcal{L} \in T^{\perp}M$, using (11.3) and (11.82), for any vector fields X, Y, Z, and W tangent to M we have

$$\bar{R}(W, Z, X, Y) = c^2(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) + g(h(Y, Z), h(X, W)) - g(h(X, Z), h(Y, W)). \tag{11.83}$$

Since M is a totally umbilical submanifold of $\bar{M}$ and the mean curvature vector is parallel, the second fundamental form is parallel. Thus M is a locally symmetric space. Using (11.83) and $h(X, Y) = g(X, Y)H$, then for any $X, Y, Z \in D$ and $V \in D^{\perp}$ we obtain $\bar{R}(X, Y, Z, V) = 0$. Moreover, since $h(X, Y) = g(X, Y)H$ and $\mathcal{L} \in T^{\perp}M$, the fibers of π are totally geodesic (cf. [15]). Hence, reflections $\varphi_{\pi^{-1}(x)}$ with respect to the fibers are isometries (cf. [10]). Therefore, N is a locally symmetric space [10, 26]. From Theorem 11.15, N is a Kaehler manifold. Using (11.82), for any horizontal unit vector X, we get $\bar{H}(X) = c^2$. Also, it is easy to see that

$$\bar{H}(X) = H'(X_*) - 3\,\|A_X JX\|^2 - \|h(X, X)\|^2.$$

Therefore, we have $H'(X_*) > 0$, where X_* is any unit tangent vector on N. □

11.7 Submersions of Contact CR Submanifolds

In this section, we shall discuss almost contact metric submersions of contact CR submanifolds M of quasi-K-cosymplectic and quasi-Kenmotsu manifolds $\bar{M}$ onto an almost contact metric manifold N (cf. [24]). It should be noted that the theory of

CR submersion was extended to the case where the total space is a semi-invariant submanifold (as meant in Sasakian geometry) by Papaghiuc (cf. [29]). He obtained the basic properties of CR submersions of a semi-invariant submanifold of a Sasakian manifold onto an almost contact metric manifold.

Let $\bar{M}$ be a differentiable manifold. An *almost contact structure* on $\bar{M}$ is a triplet (ϕ, ξ, η), where ϕ is a tensor field of type (1,1), ξ is a vector field, and η is a 1-form satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1 \tag{11.84}$$

for any vector field X tangent to $\bar{M}$. If $\bar{M}$ is equipped with a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

then (g, ϕ, ξ, η) is called an *almost contact metric structure*. So the quintuple $(\bar{M}, g, \phi, \xi, \eta)$ is an *almost contact metric manifold*.

It is known that, similar to almost Hermitian manifolds, any almost contact metric manifold admits a fundamental 2-form Φ defined by

$$\Phi(X, Y) = g(X, \phi Y).$$

We shall consider the following structures (for further study see [24]) referred to as *quasi-K-cosymplectic* if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_{\phi X}\phi)\phi Y - \eta(Y)(\bar{\nabla}_{\phi X}\xi) = 0, \tag{11.85}$$

[where $\bar{\nabla}$ is the Levi-Civita connection on $(\bar{M}, g)$] and *quasi-Kenmotsu* if $d\eta = 0$ and

$$(\bar{\nabla}_X \Phi)(Y, Z) + (\bar{\nabla}_{\phi X}\Phi)(\phi Y, Z) = \eta(Y)\Phi(Z, X) + 2\eta(Z)\Phi(X, Y) \tag{11.86}$$

for any $X, Y, Z \in T\overline{M}$.

Definition 10 Let the Riemannian manifold M be isometrically immersed in $\bar{M}$ with tangential structure vector field ξ. Then the submanifold M is called a *contact* CR submanifold if it is endowed with the pair of distribution $(D, D^\perp)$ satisfying the following conditions:

(i) $TM = D \oplus D^\perp \oplus \{\xi\}$ and D, $D^\perp$, and ξ are orthogonal to each other.
(ii) the distribution D is invariant, i.e., $\phi(D) = D$.
(iii) the distribution $D^\perp$ is anti-invariant, i.e., $\phi(D^\perp) \subset T^\perp M$.

The projections of TM to D and $D^\perp$ are denoted by $\mathcal{H}$ and $\mathcal{V}$, respectively, i.e., for any $X \in TM$, we have

$$X = \mathcal{H}X + \mathcal{V}X + \eta(X)\xi. \tag{11.87}$$

Applying ϕ on both sides of the above equation, we have

$$\phi X = FX + TX, \quad \text{for any } X \in TM \tag{11.88}$$

where $FX = \phi\mathcal{H}X$ and $TX = \phi\mathcal{V}X$ are tangential and normal components of ϕX, respectively. Similarly, from the decomposition $T^{\perp}M = \phi D^{\perp} \oplus \nu$, we have

$$U = uU + tU \tag{11.89}$$

for any $U \in T^{\perp}M$. Applying ϕ on both sides of (11.89), we get

$$\phi U = fU + mU \tag{11.90}$$

where $fU = \phi uU \in D^{\perp}$ and $mU = \phi tU \in \nu$. Let M be a contact CR submanifold of a quasi-K-cosymplectic (respectively quasi-Kenmotsu) manifold $\bar{M}$ and N be an almost contact metric manifold with the almost contact metric structure (ϕ', ξ', η', g'). Let there be a submersion $\pi : M \longrightarrow N$ such that [24]

(i) $D^{\perp} = \ker \pi_*$,

(ii) $\pi_* : D \oplus \{\xi\} \longrightarrow TN$ is an isometry which satisfies $\pi_* \circ \phi = \phi' \circ \pi_*$; $\eta = \eta' \circ \pi_*$; $\pi_* \circ \xi = \xi'$.

We recall the following result (cf. [24]).

Lemma 3 *Let X, Y be basic vector fields on M. Then*
(i) $g(X, Y) = g'(X_*, Y_*) \circ \pi$;
(ii) *the component $\mathcal{H}([X, Y]) + \eta([X, Y])\xi$ of $[X, Y]$ is basic vector field and corresponds to $[X_*, Y_*]$*;
(iii) $[U, X] \in D^{\perp}$ *for any $U \in D^{\perp}$*;
(iv) $\mathcal{H}(\nabla_X Y) + \eta(\nabla_X Y)\xi$ *is a basic vector field corresponding to $\nabla'_{X_*} Y_*$ where ∇' denotes the Levi-Civita connection on N.*

First, we discuss some preliminary results on contact CR submanifolds of quasi-K-cosymplectic and quasi-Kenmotsu manifolds. The following result (cf. [24]) is an easy consequence of (11.85) (resp (11.86)).

Lemma 4 *For a contact CR submanifold M of a quasi-K-cosymplectic* (*resp. quasi-Kenmotsu*) *manifold $\bar{M}$, the following equations hold*

$$F^2 + fT = -I + \eta \otimes \xi, \quad TF + mT = 0, \tag{11.91}$$

$$Ff + fm = 0, \quad m^2 + Tf = -I. \tag{11.92}$$

Moreover, we have the following result (cf. [24]).

Lemma 5 *For a contact CR submanifold M of a quasi-K-cosymplectic* (*respectively quasi-Kenmotsu*) *manifold $\bar{M}$, the following equations hold:*

$$\ker F = D^{\perp} \oplus \{\xi\}, \quad \ker T = D \oplus \{\xi\}, \tag{11.93}$$

$$\ker m = TD^{\perp}, \qquad \ker f = \nu. \tag{11.94}$$

Next, the covariant derivative of the structure vector field ξ is given, for quasi-K-cosymplectic, by

$$\bar{\nabla}_X \xi = \phi(\bar{\nabla}_{\phi X} \xi) \tag{11.95}$$

and for a quasi-Kenmotsu manifold by

$$\bar{\nabla}_X \xi = -2\phi^2 X + \phi(\bar{\nabla}_{\phi X} \xi), \tag{11.96}$$

for any $X \in T\overline{M}$ [24]. □

Then we have the following (cf. [24]).

Lemma 6 *Let M be a contact CR submanifold of an almost contact manifold $\bar{M}$. Then, if $\bar{M}$ is quasi-K-cosymplectic, we have the following identities:*

$$\nabla_X \xi = F(\nabla_{FX} \xi) + fh(FX, \xi), \tag{11.97}$$

$$h(X, \xi) = T(\nabla_{FX} \xi) + mh(FX, \xi) \tag{11.98}$$

for any $X \in D$. Moreover, if $\bar{M}$ is quasi-Kenmotsu, we have

$$\nabla_X \xi = 2\{X - \eta(X)\xi\} + F(\nabla_{FX} \xi) + fh(FX, \xi), \tag{11.99}$$

$$h(X, \xi) = T(\nabla_{FX} \xi) + mh(FX, \xi), \tag{11.100}$$

for any $X \in D$.

Proof Let $\bar{M}$ be a quasi-K-cosymplectic manifold. Then for any $X \in D$, (11.95) gives

$$\begin{aligned}\nabla_X \xi + h(X, \xi) &= \phi(\nabla_{FX} \xi) + \phi h(FX, \xi)\\ &= F(\nabla_{FX} \xi) + T(\nabla_{FX} \xi) + mh(FX, \xi) + fh(FX, \xi).\end{aligned}$$

Similarly for a quasi-Kenmotsu manifold, from (11.96), we get

$$\begin{aligned}\nabla_X \xi + h(X, \xi) &= -2\phi^2 X + \phi(\nabla_{FX} \xi) + \phi h(FX, \xi)\\ &= -2\phi^2 X + F(\nabla_{FX} \xi) + T(\nabla_{FX} \xi) + fh(FX, \xi) + mh(FX, \xi).\end{aligned}$$

Then comparing tangential and normal components on both sides of these equations completes the proof. □

The differential of the fundamental 2-form Φ gives, for any $X, Y, Z \in T\overline{M}$,

$$3\,d\Phi(X, Y, Z) = X(\Phi(Y, Z)) + Y(\Phi(Z, X)) + Z(\Phi(X, Y)) \\ - \Phi([X, Y], Z) - \Phi([Z, X], Y) - \Phi([Y, Z], X).$$

Now for any $Y, Z \in D^{\perp}$ we find

$$3\,d\Phi(X, Y, Z) = g(\phi[Y, Z], X). \tag{11.101}$$

Hence, $d\Phi(X, Y, Z) = 0$ if and only if $[Y, Z] \in \ker(F)$, i.e., $[Y, Z] = \mathcal{V}[Y, Z] + \eta([Y, Z])\xi$. Also we have

$$\eta([Y, Z]) = g(\bar{\nabla}_Z\xi, Y) - g(\bar{\nabla}_Y\xi, Z).$$

For a quasi-K-cosymplectic manifold, we have $\bar{\nabla}_Z\xi = \phi(\bar{\nabla}_{\phi Z}\xi)$ and which gives for any $Y, Z \in D^{\perp}$

$$g(\bar{\nabla}_Z\xi, Y) = -g(\overline{A}_{\phi Y}\xi, \phi Z) = 0.$$

Consequently, $\eta([Y, Z]) = 0$ and $[Y, Z] \in D^{\perp}$ for any $Y, Z \in D^{\perp}$. Similar is the case if $\bar{M}$ is a quasi-Kenmotsu manifold. Therefore, we have the following theorem (cf. [24]).

Theorem 11.17 *Let M be a contact CR submanifold of a quasi-K-cosymplectic (or quasi-Kenmotsu) manifold $\bar{M}$. The distribution $D^{\perp}$ is integrable if and only if $d\Phi(X, Y, Z) = 0$, for any X tangent to M and $Y, Z \in D^{\perp}$.*

We now discuss submersions of contact CR submanifolds of a quasi-K-cosymplectic (quasi-Kenmotsu) manifold $\bar{M}$ onto an almost contact metric manifold N. For any $X, Y \in D \oplus \{\xi\}$ we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

or

$$\bar{\nabla}_X Y = \tilde{\nabla}'_X Y + C(X, Y) + uh(X, Y) + th(X, Y). \tag{11.102}$$

If $\bar{M}$ is a quasi-K-cosymplectic manifold, we know

$$(\bar{\nabla}_X\phi)Y = \bar{\nabla}_X\phi Y - \phi(\bar{\nabla}_X Y) = -(\bar{\nabla}_{\phi X}\phi)\phi Y + \eta(Y)(\bar{\nabla}_{\phi X}\xi). \tag{11.103}$$

Now combining (11.102) and (11.103) we find

$$\tilde{\nabla}'_X\phi Y + C(X, \phi Y) + uh(X, \phi Y) + th(X, \phi Y) \\ - \phi\tilde{\nabla}'_X Y - \phi C(X, Y) - \phi uh(X, Y) - \phi th(X, Y) \\ = -(\bar{\nabla}_{\phi X}\phi)\phi Y + \eta(Y)(\bar{\nabla}_{\phi X}\xi). \tag{11.104}$$

Similarly for quasi-Kenmotsu manifold we have

$$\begin{aligned} &\tilde{\nabla}'_X \phi Y + C(X, \phi Y) + uh(X, \phi Y) + th(X, \phi Y) \\ &\quad - \phi \tilde{\nabla}'_X Y - \phi C(X, Y) - \phi uh(X, Y) - \phi th(X, Y) \\ &= \phi((\bar{\nabla}_{\phi X} \phi) Y) + g(\phi X, Y)\xi - 2\eta(Y)\phi X. \end{aligned} \tag{11.105}$$

We now prove the following theorem (cf. [24]).

Theorem 11.18 *Let $\pi : M \longrightarrow N$ be a submersion of a contact CR submanifold of a manifold $\bar{M}$ onto an almost contact metric manifold N. Then*

(i) *If $\bar{M}$ is quasi-K-cosymplectic, for any $X, Y \in D \oplus \{\xi\}$*

$$(\tilde{\nabla}'_X \phi) Y + (\tilde{\nabla}'_{\phi X} \phi)\phi Y = \eta(Y) \tilde{\nabla}'_{\phi X} \xi, \tag{11.106}$$

$$C(X, \phi Y) - C(\phi X, Y) = f\{h(X, Y) + h(\phi X, \phi Y)\}, \tag{11.107}$$

$$t\{h(X, \phi Y) - h(\phi X, Y)\} = m\{h(X, Y) + h(\phi X, \phi Y)\}, \tag{11.108}$$

$$u\{h(X, \phi Y) - h(\phi X, Y)\} = \phi\{C(X, Y) + C(\phi X, \phi Y)\}. \tag{11.109}$$

(ii) *If $\bar{M}$ is quasi-Kenmotsu, for any $X, Y \in D \oplus \{\xi\}$*

$$(\tilde{\nabla}'_X \phi) Y - \phi((\tilde{\nabla}'_{\phi X} \phi) Y) = g(\phi X, Y)\xi - 2\eta(Y)\phi X, \tag{11.110}$$

$$C(X, \phi Y) - C(\phi X, Y) = fh(X, Y), \tag{11.111}$$

$$C(X, Y) = -C(\phi X, \phi Y), \tag{11.112}$$

$$uh(X, \phi Y) = \phi th(X, Y). \tag{11.113}$$

Proof If $\bar{M}$ is a quasi-K-cosymplectic manifold, we have

$$\bar{\nabla}_{\phi X} \xi = \tilde{\nabla}'_{\phi X} \xi + C(\phi X, \xi) + h(\phi X, \xi) \tag{11.114}$$

and

$$\begin{aligned} (\bar{\nabla}_{\phi X} \phi)\phi Y = (\tilde{\nabla}'_{\phi X} \phi)\phi Y - -C(\phi X, Y) + \eta(Y) C(\phi X, \xi) - h(\phi X, Y) \\ + \eta(Y) h(\phi X, \xi) - \phi C(\phi X, \phi Y) - \phi h(\phi X, \phi Y) \end{aligned} \tag{11.115}$$

for any $X, Y \in D \oplus \{\xi\}$. Combining (11.104), (11.114), and (11.115) and comparing the components of $D \oplus \{\xi\}$, $D^\perp$, $\phi D^\perp$, and ν, respectively, on both sides, we get (i) of the theorem. Similarly, we can prove (ii) in case $\bar{M}$ is a quasi-Kenmotsu manifold.
□

Next we give the following result (cf. [24]).

Theorem 11.19 *Let $\pi : M \longrightarrow N$ be a submersion of a contact CR submanifold of a manifold $\bar{M}$ onto an almost contact metric manifold N.*

(i) *If $\bar{M}$ is quasi-K-cosymplectic, then N is also a quasi-K-cosymplectic manifold.*

(ii) *If $\bar{M}$ is quasi-Kenmotsu, then M is $D \oplus \{\xi\}$-totally geodesic and N is also a quasi-Kenmotsu manifold.*

Definition 11 Let $(N^{2r'}, g', J')$ be an almost Hermitian manifold, then the Riemannian submersion

$$\pi : M^{2r+1} \longrightarrow N^{2r'}$$

is called an *almost contact metric submersion of type II* if

$$\pi_* \circ \phi = J' \circ \pi_*.$$

First, we state the following theorem (cf. [24]).

Theorem 11.20 *Let $\pi : M^{2r+1} \longrightarrow N^{2r'}$ be an almost contact metric submersion of type II of contact CR submanifold M of a quasi-K-cosymplectic (or quasi-Kenmotsu) manifold $\bar{M}$ onto an almost contact metric manifold N. Then the base space is a quasi-Kaehler manifold.*

Now we prove the following theorem (cf. [24]).

Theorem 11.21 *Let $\pi : M \longrightarrow N$ be a submersion of a contact CR submanifold of a quasi-K-cosymplectic (or quasi-Kenmotsu) manifold $\bar{M}$ onto an almost contact metric manifold N. If the horizontal distribution $D \oplus \{\xi\}$ is integrable and the vertical distribution $D^\perp$ is parallel, then M is CR product.*

Proof Since $D \oplus \{\xi\}$ is integrable we have $\mathcal{V}[X, Y] = 0$ for any $X, Y \in D \oplus \{\xi\}$. This implies that $C(X, Y) = 0$ which gives $\nabla_X Y = \tilde{\nabla}'_X Y \in D \oplus \{\xi\}$ for any $X, Y \in D \oplus \{\xi\}$. This equation shows that $D \oplus \{\xi\}$ is parallel. Thus using de Rham's theorem, it follows that M is the product $M_1 \times M_2$, where M_1 is the invariant submanifold of $\bar{M}$ and M_2 is the totally real submanifold of $\bar{M}$. Hence, M is a CR product. □

Finally, we discuss holomorphic sectional curvature of the submersion of contact CR submanifold of quasi-K-cosymplectic (resp. quasi-Kenmotsu manifold) $\bar{M}$ onto an almost contact metric manifold N. From Gauss Eq. (11.12), we find (cf. [24])

$$\begin{aligned}\bar{R}(X, \phi X, \phi Y, Y) = R'(X_*, \phi_* X_*, \phi_* Y_*, Y_*) \\ - g(C(X, Y), C(\phi X, \phi Y)) + g(C(X, \phi Y), C(\phi X, Y)) \\ + 2g(C(X, \phi X), C(\phi Y, Y)) \\ - g(uh(X, Y), uh(\phi X, \phi Y)) - g(th(X, Y), th(\phi X, \phi Y)) \\ + g(uh(X, \phi Y), uh(\phi X, Y)) + g(th(X, \phi Y), th(\phi X, Y)).\end{aligned} \tag{11.116}$$

Suppose that the distribution $D \oplus \{\xi\}$ is integrable. Then we have $C(X, Y) = 0$ and $h(\phi X, \phi Y) = -h(X, Y)$ for any $X, Y \in D \oplus \{\xi\}$. Using these relations, (11.116) becomes

$$\bar{R}(X, \phi X, \phi Y, Y) = R'(X_*, \phi' X_*, \phi' Y_*, Y_*) + \|h(X, Y)\|^2 + \|h(X, \phi Y)\|^2$$

which implies that

$$\bar{H}(X) = H'(X_*) + \|h(X, Y)\|^2 + \|h(X, \phi Y)\|^2$$

where $\bar{H}$ and H' are the holomorphic sectional curvatures of $\bar{M}$ and N, respectively. □

From the above discussion we have the following results (cf. [24]).

Theorem 11.22 *Let $\pi : M \longrightarrow N$ be a submersion of a contact CR submanifold of a quasi-K-cosymplectic manifold $\bar{M}$ onto an almost contact metric manifold N with integrable $D \oplus \{\xi\}$. Then the holomorphic sectional curvatures $\bar{H}$ and H' of $\bar{M}$ and N respectively satisfy*

$$\bar{H}(X) \geq H'(X_*),$$

for any $X \in D \oplus \{\xi\}$, $\|X\| = 1$, and the equality holds if and only if M is $D \oplus \{\xi\}$-totally geodesic.

Theorem 11.23 *Let $\pi : M \longrightarrow N$ be a submersion of a contact CR submanifold of a quasi-Kenmotsu manifold $\bar{M}$ onto an almost contact metric manifold N. Then the holomorphic sectional curvatures $\bar{H}$ and H' of $\bar{M}$ and N, respectively, satisfy*

$$\bar{H}(X) \geq H'(X_*),$$

for any $X \in D \oplus \{\xi\}$, $\|X\| = 1$, and the equality holds if and only if the distribution $D \oplus \{\xi\}$ is integrable.

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Chapter 12
CR-Submanifolds of Semi-Riemannian Kaehler Manifolds

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Mathematics Subject Classification 53C50 · 53C55 · 83C50

12.1 Introduction

An almost complex structure on a smooth manifold M is a (1, 1)-tensor field J satisfying the condition

$$J^2 = -I \tag{12.1}$$

where I is the identity operator on the tangent space at each point. M furnished with an almost complex structure is known as an almost complex manifold and is even-dimensional and orientable. An almost complex structure that comes from a complex structure is called integrable, and when one wishes to specify a complex structure as opposed to an almost complex structure, one calls it an integrable complex structure. This integrability condition is equivalent to the vanishing of the Nijenhuis' tensor $[J, J]$ defined by

$$[J, J](X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]$$

for arbitrary vector fields X and Y on M. An almost complex manifold is called an almost Hermitian manifold if there exists a Riemannian metric g such that $g(JX, JY) = g(X, Y)$. An almost Hermitian manifold is said to be Hermitian if the underlying almost complex structure is integrable.

One may note that, if g is semi-Riemannian, then its signature has even number (including 0) of positive signs and even number (including 0) of negative signs.

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A Kaehler manifold is a Hermitian manifold whose complex structure J is parallel with respect to the Levi-Civita connection ∇ of g. A holomorphic section of a Kaehler manifold is section obtained by a plane element spanned by a non-null tangent vector X (i.e. $g(X, X) \neq 0$) at a point and JX. The sectional curvature of M with respect to a holomorphic section is called the holomorphic sectional curvature. A Kaehler manifold is said to be a complex space-form if its holomorphic sectional curvature is independent of the choice of a holomorphic section at each point. A complex space-form with constant holomorphic sectional curvature c is denoted by $M(c)$ whose curvature tensor is given by

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ] \tag{12.2}$$

One can easily check that c is constant on M.

At this point, we denote an almost Hermitian manifold by $(\bar{M}, g)$ with the almost complex structure J. Generalizing the cases of invariant and anti-invariant (i.e., totally real) submanifolds, Bejancu [2] introduced the notion of a CR-submanifold as follows.

Definition 1 A CR-submanifold of a semi-Riemannian almost Hermitian manifold $(\bar{M}, g)$ is a non-degenerate submanifold (M, g) of $(\bar{M}, g)$ admitting a smooth distribution $D : p \to D_p \subset T_pM$ such that

(1) D is invariant, i.e., $JD_p = D_p$ at each point $p \in M$ and
(2) the orthogonal complementary distribution $D^\perp$ is anti-invariant, i.e. $JD_p^\perp \subset (T_pM)^\perp$ for each $p \in M$.

Remark 1 The above definition would not be feasible if D was degenerate with respect to g, because D and $D^\perp$ are not necessarily complementary in TM for degenerate D. Henceforth, we will assume D and $D^\perp$ both non-degenerate with respect to g.

Recall that a CR-structure on a smooth manifold M is a complex subbundle H of the complexified tangent bundle $C(TM)$ of M such that $(H \cap \bar{H})_p = 0$ at each $p \in M$ and H is involutive, i.e., $X, Y \in H \Rightarrow [X, Y] \in H$. It is known that, on a CR-manifold, there exist a real distribution D and a field of endomorphisms $P : D \to D$ such that $P^2 = -I$, $D = Re(H \oplus \bar{H})$ and $H_p = \{X - iPX : X \in D_p\}$. Blair and Chen [7] proved that a proper CR-submanifold M of a Hermitian manifold is a CR-manifold. This justifies the term "CR-submanifold."

If the holomorphic distribution D is equal to the tangent bundle TM, then M reduces to an invariant submanifold of $\bar{M}$, and if the totally real distribution $D^\perp$ equals TM then M reduces to a totally real submanifold of $\bar{M}$. When the dimensions of $D^\perp$ and $(TM)^\perp$ are equal, M is said to be a generic submanifold of $\bar{M}$.

For a tangential vector field X and a normal vector field V on a CR-submanifold of an almost Hermitian manifold $\bar{M}$, we have the following decomposition formulas:

$$JX = PX + FX \tag{12.3}$$

$$JV = tV + fV \tag{12.4}$$

where PX and tV are the tangential parts of JX and JV, respectively, and FX and fV are the normal parts of JX and JV, respectively. It is easy to verify from the preceding two equations that

$$g(FX, V) + g(X, tV) = 0 \tag{12.5}$$

and that $g(PX, Y)$ is skew-symmetric in X, Y, and $g(fU, V)$ is skew-symmetric in U, V. Operation of J on Eqs. (12.3) and (12.4) yields the following relations:

$$P^2 = -I - tF, \quad FP + fF = 0 \tag{12.6}$$

$$Pt + tf = 0, \quad f^2 = -I - Ft \tag{12.7}$$

Let us denote the projection operator on D by l and that on $D^\perp$ bt $l^\perp$. Then, evidently

$$l + l^\perp = I, \quad l^2 = l, \quad (l^\perp)^2 = l^\perp, \quad ll^\perp = l^\perp l = 0$$

$$l^\perp Pl = 0, \quad Fl = 0, \quad Pl = P$$

Using this in the second equation of (12.6) one gets

$$FP = 0 \tag{12.8}$$

Thus, the second equation of (12.6) reduces to

$$fF = 0$$

Taking fV for V in Eq. (12.5) we find

$$tf = 0 \tag{12.9}$$

Using this in the first equation of (12.7) gives

$$Pt = 0 \tag{12.10}$$

Consequently, the first equation of (12.6) implies

$$P^3 + P = 0 \tag{12.11}$$

and the second equation of (12.7) implies

$$f^3 + f = 0 \tag{12.12}$$

Equations (12.11) and (12.12) show that P and f define f-structures (for details on an f-structure, we refer to Yano [18]) on the tangent and normal bundles of M, respectively. Setting $l = P^2$ and $l^{\perp} = I - l$, one can easily verify the following result of Yano and Kon [21].

Theorem 12.1 *A submanifold M of an almost Hermitian manifold $\bar{M}$ is a CR-submanifold if and only if $FP = 0$.*

12.2 Basic Equations and Results

According to a theorem of Flaherty [12], we know that the signature of a Hermitian metric g on an almost complex manifold has even number of positive signs and even number of negative signs. Thus, g cannot be Lorentzian which is essential for a physical space-time of relativity. For a four dimensional space-time, we can choose a coordinate system comprising two real coordinates x, y and complex null coordinates $z + it$ and $z - it$. The aforementioned facts suggest that a complex structure can be defined only on its two dimensional submanifold defined by $x =$ constant and $y =$ constant. With this motivation and the purpose of applying our results in relativity theory, we consider a class of submanifolds of a semi-Riemannian Kaehler manifold such that there may be complementary complex and real distributions. One of the settings for such a distribution can be provided by singling out holomorphic distributions of the *CR*-submanifolds (see for example, Penrose [15]).

As pointed out in the previous section, a *CR*-submanifold (M, g) has an induced f-structure defined by the (1, 1) tensor field P on M, and hence the metric g can be Lorentzian. Our study is not only applicable within the framework of general relativity, but also in the theory of semi-Riemannian manifolds whose metrics have signatures compatible with the induced f-structure. We also note in our context that indefinite Kaehler manifolds (in particular, complex space-forms) were studied by Barros and Romero in [1].

We denote the Levi-Civita connection of the induced metric g on the *CR*-submanifold (M, g) of a semi-Riemannian Kaehler manifold $(\bar{M}, g, J)$ by ∇ and that of $(\bar{M}, g)$ by $\bar{\nabla}$. The second fundamental form of M is denoted by B and the Weingarten operator by A_V for an arbitrary normal vector field V on M. They are related by $g(A_V X, Y) = g(B(X, Y), V)$. The Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)$$

$$\bar{\nabla}_X V = -A_V X + D_X V$$

The Gauss and Codazzi equations are

$$g(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) - g(B(Y, Z), B(X, W)) + g(B(X, Z), B(Y, W))$$

$$[\bar{R}(X, Y)Z]^{\perp} = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)$$

where

$$(\nabla_X B)(Y, Z) = D_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) \tag{12.13}$$

X, Y, Z, W denote arbitrary vector fields tangent to M, and D denotes the normal connection of M. Also, R and $\bar{R}$ denote the curvature tensors of ∇ and $\bar{\nabla}$, respectively.

The covariant derivatives of the operators P, F, t, f are defined along M as

$$(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y) \tag{12.14}$$

$$(\nabla_X F)Y = D_X(FY) - F(\nabla_X Y) \tag{12.15}$$

$$(\nabla_X t)V = \nabla_X(tV) - t(D_X V) \tag{12.16}$$

$$(\nabla_X f)V = D_X(fV) - f(D_X V) \tag{12.17}$$

At this point, we use the Kaehlerian condition $\bar{\nabla} J = 0$. The Gauss and Weingarten formulas provide the following equations

$$(\nabla_X P)Y = A_{FY}X + tB(X, Y) \tag{12.18}$$

$$(\nabla_X F)Y = -B(X, PY) + fB(X, Y) \tag{12.19}$$

$$(\nabla_X t)V = A_{fV}X - PA_V X \tag{12.20}$$

$$(\nabla_X f)V = -FA_V X - B(X, tV) \tag{12.21}$$

We now recall the following results and definitions from Yano and Kon [20] and Yano and Ishihara [19], that will be used later.

Lemma 1 *Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then, for any vector fields X and Y in $D^{\perp}$ we have*

$$A_{FX}Y = A_{FY}X. \tag{12.22}$$

Theorem 12.2 *Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then, the totally real distribution $D^{\perp}$ is integrable and its maximal integral submanifold $M^{\perp}$ is an anti-invariant (totally real) submanifold of M.*

Definition 2 The f-structure induced on the CR-submanifold of a Kaehler manifold is said to be partially integrable if D is integrable and the almost complex structure induced on each leaf of D is integrable.

Theorem 12.3 *Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then the f-structure induced on M is partially integrable if and only if*

$$B(PX, Y) = B(X, PY) \tag{12.23}$$

for all X and Y in D.

Definition 3 The f-structure induced on the CR-submanifold M of a Kaehler manifold is said to be normal if the (1, 2)-tensor field S defined by

$$S(X, Y) = [P, P](X, Y) - t((\nabla_X F)Y - (\nabla_Y F)X)$$

vanishes identically on M.

This normality condition is equivalent (see [20]) to $A_{FX} = PA_{FX}$ for any X tangent to M.

Definition 4 A CR-submanifold of a Kaehler manifold is said to be mixed totally geodesic if $B(X, Y) = 0$ for any vector field $X \in D$ and $Y \in D^{\perp}$.

12.3 Mixed Foliate *CR*-Submanifolds

In this section, we will deal with a subclass of mixed totally geodesic CR-submanifolds characterized by the partial integrability of f-structure induced on them.

Definition 5 A CR-submanifold of a Kaehler manifold is called mixed foliate if it is mixed totally geodesic and the f-structure induced on it is partially integrable.

Next, we recall the following lemma (see Yano and Kon [20]).

Lemma 2 *Let M be a mixed foliate CR-submanifold of a Kaehler manifold $\bar{M}$. Then, for all $V \in (TM)^{\perp}$ we have*

$$A_V P + PA_V = 0 \tag{12.24}$$

Now, we recall the following theorem of Bejancu et al. [4], which holds for a positive definite Kaehler metric.

Theorem 12.4 *If M is a mixed foliate proper CR-submanifold of a complex space-form $\bar{M}(c)$, then $c \leq 0$.*

So, the following question arises "what sort of constraint is imposed on the possible values of c when the metric of $\bar{M}(c)$ is indefinite?" Sharma and Duggal [17] provided an answer to this question in a special case in the form of the following result.

Theorem 12.5 *If the mixed foliate proper CR-submanifold of a semi-Riemannian complex space-form $\bar{M}(c)$ is such that the metric g restricted to D is definite and g restricted to $D^{\perp}$ is indefinite, then $c = 0$.*

Proof The curvature tensor of $\bar{M}(c)$ is given by Eq. (12.2). Restricting the vector fields X, Y to D and Z to $D^{\perp}$ we find that

$$[\bar{R}(X, Y)Z]^{\perp} = \frac{c}{2} g(PY, X)JZ \tag{12.25}$$

Equation (12.13) provides

$$(\nabla_X B)(Y, Z) = -B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

since M is mixed totally geodesic. Anti-symmetrizing the last equation with respect to X and Y we get

$$\begin{aligned}(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = &-B([X, Y], Z)\\ &- B(Y, \nabla_X Z) + B(X, \nabla_Y Z)\end{aligned}$$

As per our hypothesis, M is mixed foliate and hence, by the integrability of D, the above equation reduces to

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = -B(Y, \nabla_X Z) + B(X, \nabla_Y Z) \tag{12.26}$$

As Z is any vector field in $D^{\perp}$, there is a normal vector field V such that $Z = JV$. Therefore, $Z = tV$ and $fV = 0$. Consequently, we have

$$\nabla_Y Z = (\nabla_Y t)V + tD_Y V = tD_Y V - PA_V Y$$

where we used Eq. (12.20). The Eq. (12.10) shows that $tD_Y V \in D^{\perp}$. The use of Lemma 2 transforms equation (12.26) into

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(X, A_V PY) + B(PY, A_V X) \tag{12.27}$$

Taking $X = PY$, substituting the value JV for Z, and taking inner product with V provides

$$g(A_V PY, A_V PY) = -\frac{c}{4} g(PY, PY)g(V, V) \tag{12.28}$$

By hypothesis, g restricted to D is definite. By Lemma 2, $A_V PY = -PA_V Y$. Thus, the above equation implies the inequality

$$cg(V, V) = cg(Z, Z) \leq 0$$

for any $Z \in D^{\perp}$. Again, by hypothesis, g restricted to $D^{\perp}$ is indefinite, and therefore, $D^{\perp}$ does contain at least one space-like vector field Z_1 (i.e., $g(Z_1, Z_1) > 0$) and a time-like vector field Z_2 (i.e., $g(Z_2, Z_2) < 0$). Consequently, we get $c \geq 0$ and $c \leq 0$. Thus, we conclude that $c = 0$, completing the proof.

Corollary 1 *Under the hypothesis of the preceding theorem, we have*

$$A_V P = 0 \tag{12.29}$$

for every $V \in JD^{\perp}$.

Proof It follows from (12.28) and the conclusion $c = 0$ of Theorem 12.5, that $g(A_V PY, A_V PY) = 0$. Since $A_V P = -PA_V$, the vector field $A_V PY \in D$. The hypothesis that g is definite on D, implies that (12.28) holds.

Remark 2 The following result of Chen [8] "A CR-submanifold of C^n is mixed foliate if and only if it is a CR-product, i.e., the product of the leaves of D and $D^{\perp}$" can be shown to be valid for both definite and indefinite metrics.

Employing it for the mixed foliate CR-submanifold M under the hypothesis of the Theorem 12.5, it follows straightaway that M is a CR-product. This can be proved independently (without using Chen's theorem) in another way to gain more insight into the structure of M. First, let us establish the following lemma.

Lemma 3 *A necessary and sufficient condition for the integrability of the f-structure induced on a mixed foliate proper CR-submanifold M of a Kaehler manifold $\bar{M}$ is that $A_{FY} P = 0$ for any vector field Y tangent to M.*

Proof We know that a proper mixed foliate CR-submanifold M of a Kaehler manifold $\bar{M}$ has a partially integrable f-structure and integrable $D^{\perp}$. The f-structure would be completely integrable if its Nijenhuis tensor $[P, P]$ vanishes identically, i.e.,

$$[PX, PY] + P^2[X, Y] - P[PX, Y] - P[X, PY] = 0.$$

For $X, Y \in D$, the integrability of D implies $[P, P](X, Y) = 0$. For $X, Y \in D^{\perp}$, the integrability of $D^{\perp}$ implies $[P, P](X, Y) = 0$. For $X \in D$ and $Y \in D^{\perp}$ we observe that

$$\begin{aligned} [P, P](X, Y) &= (\nabla_{PX} P)Y - (\nabla_{PY} P)X - P[(\nabla_X P)Y - (\nabla_Y P)X] \\ &= A_{FY} PX - PA_{FY} X = 2A_{FY} PX \end{aligned}$$

where Eqs. (12.18) and (12.24) have been used. Hence, the f-structure induced on M is integrable if and only if $A_{FY} P = 0$ for any V in $JD^{\perp}$.

Definition 6 The f-structure induced on the CR-submanifold of a Kaehler manifold is said to be normal if the (1, 2)-tensor field S defined by

$$S(X, Y) = [P, P](X, Y) - t[(\nabla_X F)Y - (\nabla_Y F)X]$$

vanishes identically.

It has been shown in [20] that the normality of the f-structure induced on the CR-submanifold of a Kaehler manifold is equivalent to $A_{FX}P = PA_{FX}$, for any vector field X tangent to M. This holds for a definite as well as an indefinite metric. The following result characterizes the intrinsic structure of M hypothesized as in Theorem 12.5.

Theorem 12.6 *Under the hypothesis of Theorem 12.5, the f-structure induced on M is integrable and normal. Moreover, if $D^{\perp}$ is parallel, then M is locally a CR-product $M^T \times M^{\perp}$, where M^T is flat and $M^{\perp}$ is a totally geodesic real submanifold of M.*

Proof We conclude from Eq. (12.28) and the conclusion $c = 0$ of Theorem 12.5 that $A_V P = 0$. Hence Lemma 3 asserts that the f-structure on M is integrable. Now, the necessary and sufficient condition for the normality of the f-structure on M is $PA_V = A_V P$ for any $V \in JD^{\perp}$. This is automatically satisfied since we have $A_V P = 0$ (corollary to Theorem 12.5 and Lemma 2). Thus, the structure is also normal. It can be shown with the aid of Eq. (12.18) that the fundamental 2-form Ω of the f-structure is closed. It therefore follows from a result of Goldberg [13] that $(\nabla_X P)Y = 0$ for $X \in D$. The expression (12.18) for $(\nabla_X P)Y$ ensures that it lies in $D^{\perp}$, which is clear from the result $PA_V X = -A_V PX = 0$ so that $A_V X \in D^{\perp}$. Next, from Eq. (12.18) and Lemma 1 we have $(\nabla_X P)Y = (\nabla_Y P)X$ for all $X, Y \in D^{\perp}$. Therefore, $g((\nabla_X P)Y, Z) = g((\nabla_Y P)X, Z)$ for all $X, Y \in D^{\perp}$ and $Z \in T(M)$. This means, $(\nabla_X \Omega)(Y, Z) = -(\nabla_Y \Omega)(Z, X)$ whence we find $(\nabla_Z \Omega)(X, Y) = 0$. This shows that $(\nabla_Z P)X \in D$, but as shown earlier, $(\nabla_Z P)X \in D^{\perp}$ for any Z and X tangent to M. We had also proved that $(\nabla_Z P)X = 0$ whenever $X \in D$ and Z is tangent to M. Consequently, we obtain $(\nabla_Z P)X = 0$ for any Z and X tangent to M, i.e. $\nabla P = 0$. Applying Chen's result [8] "A CR-submanifold of a Kaehler manifold is a CR-product if and only if $\nabla P = 0$," we conclude that M is $M^T \times M^{\perp}$, where M^T is a leaf of D totally geodesic in M and $M^{\perp}$ is a leaf of $D^{\perp}$ totally geodesic in M. This shows that M^T is flat, and hence completes the proof.

Proposition 1 *Under the hypothesis of Theorem 12.5, if $D^{\perp}$ is parallelizable and the normal connection is flat, then M is locally flat.*

Proof Since $D^{\perp}$ is parallelizable, we can choose an orthonormal base (ξ_a) of $D^{\perp}$. If (η^a) denotes its dual, then one can show that $FX = \eta^a(X)J\xi_a$ and $tJ\xi_a = -\xi_a$. Hence, we have

$$S(X, Y) = N_P(X, Y) - t[(\nabla_X F)Y - (\nabla_Y F)X]$$

Therefore, as the f-structure is integrable and normal, we obtain

$$(d\eta^a)(X, Y)\xi_a - \eta^a(Y)tD_X J\xi_a + \eta^a(X)tD_Y J\xi_a = 0$$

But the normal connection is flat, and so $d\eta^a = 0$. For such a structure we know from Blair [5] that $L_{\xi_a} g = 0$. Consequently, $\nabla\xi_a = 0$ and hence $R(X, Y)\xi_a = 0$, i.e., $M^\perp$ is locally flat. Hence, M is locally flat. This completes the proof.

Remark 3 If M of the foregoing proposition was of dimension 4 and complete, and the f-structure globally framed, then M would be a quotient of the Minkowski space-time of special relativity.

12.4 Normal Mixed Totally Geodesic *CR*-Submanifolds

Let us consider a class of *CR*-submanifolds of a Kaehler manifold, which are mixed totally geodesic with distribution D not necessarily integrable (unlike that of a mixed foliate *CR*-submanifold) and the f-structure induced on M is normal.

Definition 7 A *CR*-submanifold M of a Kaehler manifold $\bar{M}$ is said to be normal mixed totally geodesic if it is mixed totally geodesic and the f-structure induced on M is normal.

We state and prove the following result.

Theorem 12.7 *Let M be a normal mixed totally geodesic CR-submanifold of a complex space-form $\bar{M}(c)$. Then,*

(1) if g and $W = A_V^2 + A_{FA_V Z}$ ($V \in JD^\perp$ and $Z = JV$) are positive definite on D, then $c \geq 0$ and
(2) if g is positive definite on D and indefinite on $D^\perp$, then W cannot be definite on D. Also, $c = 0$ if and only if $W = 0$ on D.

Proof Supposing $X, Y \in D$, using Codazzi equation and the expression (12.2) for the curvature tensor of $\bar{M}(c)$ we can show that

$$B(Y, PA_V X) - B(X, PA_V Y) - B([X, Y], Z) = \frac{c}{2} g(PY, X)JZ$$

where $Z = JV \in D^\perp$. Taking its inner product with V we get

$$g(A_V Y, PA_V X) - g(A_V X, PA_V Y) - g(A_V Z, [X, Y]) + \frac{c}{2} g(PY, X)g(V, V) = 0 \tag{12.30}$$

It can be shown by a straightforward computation that

$$B(PX, Y) - B(X, PY) = F[X, Y]$$

which, in turn, implies that

$$g(A_V PX + PA_V X, Y) = g([X, Y], Z)$$

Substituting PY for X in Eq. (12.30) gives

$$g(WY, Y) = \frac{c}{2} g(V, V) g(Y, Y) \tag{12.31}$$

where we have used the normality condition $A_V P = PA_V$. If g and W (as defined in Theorem 12.7) are positive definite on D, then (12.31) implies that $c \geq 0$, which proves part (1). Let W be definite on D. If g is positive definite on D and indefinite on $D^\perp$, then (12.31) implies $cg(V, V) = cg(Z, Z) \geq 0$. Now, Z being in $D^\perp$ could be space-like or time-like. Hence $c = 0$, and therefore the operator W vanishes on D, which contradicts our hypothesis that W is definite. The last part of (2) follows from Eq. (12.31). This completes the proof.

Remark 4 For the case when g is definite on D and indefinite on $D^\perp$, we compare Theorem 12.5 and part (2) of the Theorem 12.7. As a consequence of Theorem 12.5, M reduces to a CR-product provided the f-structure on it is globally framed. On the other hand Theorem 12.7 involves the operator W on D. The condition that c may vanish, is that W must vanish identically on D. This is quite compatible with the consequence of Theorem 12.5 in that if we assume that M of Theorem 12.7 (part (2)) is a CR-product then we must have A_V vanish on D and hence the operator W vanishes on D, thus reducing c to 0. Hence, we claim to have gotten a wider class of CR-submanifolds, as hypothesized in part (2) of Theorem 12.7, which can be embedded in C^n.

12.5 Totally Umbilical *CR*-Submanifolds

This section is devoted to totally umbilical CR-submanifolds of a Kaehler manifold. We denote the dimension of the totally real distribution $D^\perp$ by q. First, we state and prove

Proposition 2 *Let M be a CR-submanifold of a Kaehler manifold. Then both the distributions D and $D^\perp$ are non-degenerate.*

Proof Let D be degenerate. Then there exists a nonzero vector field $X \in D$ such that $g(X, Y) = 0$ for all $Y \in D$. As D and $D^\perp$ are complementary and orthogonal to each other, we conclude that $g(X, Y) = 0$ for all $Y \in TM$. Hence $X = 0$ because TM is nondegenerate. But X is nonzero. Hence, we arrive at a contradiction. This proves that D is nondegenerate. Similarly, one can show that $D^{\perp}$ is non-degenerate.

Next, we state and prove

Proposition 3 *The mean curvature vector μ of a totally umbilical CR-submanifold of a Kaehler manifold belongs to $JD^{\perp}$.*

Proof Total umbilicity of M means $B(X, Y) = g(X, Y)\mu$. Consider any $X \in D$ and V in the complementary orthogonal subbundle to $JD^{\perp}$ in $TM^{\perp}$. Then we have

$$\begin{aligned} g(J(\bar{\nabla}_X X), JV) &= g(\bar{\nabla}_X JX, JV) \\ &= g(\nabla_X JX + g(X, JX)\mu, JV) = 0 \end{aligned}$$

$$\begin{aligned} g(J(\bar{\nabla}_X X), JV) &= g(\bar{\nabla}_X X, V) \\ &= g(\nabla_X X + g(X, X)\mu, V) = g(X, X)g(\mu, V). \end{aligned}$$

Thus we find $g(X, X)g(\mu, V) = 0$. By the preceding proposition, we conclude that $g(\mu, V) = 0$, i.e. $f\mu = 0$. Hence $\mu \in JD^{\perp}$, completing the proof.

Let us recall the following result of Bejancu [3] for a positive definite metric.

Theorem 12.8 *Let M be a totally umbilical proper CR-submanifold of a Kaehler manifold M. For $q > 1$, M reduces to a totally geodesic submanifold and is locally a Riemannian product of an invariant and an anti-invariant submanifold of M.*

That this theorem holds for an indefinite metric, was proved by Duggal and Sharma [11]. The proof is slightly longer than that for the positive definite case, and is given below.

Proof By virtue of Lemma 1, we have $A_{FX}Y = A_{FY}X$ for all $X, Y \in D^{\perp}$. As $t\mu \in D^{\perp}$, for any $X \in D^{\perp}$ we have $A_{FX}t\mu = A_{Ft\mu}X$. As M is totally umbilical, we have $B(X, Y) = g(X, Y)\mu$ and $A_V X = g(\mu, V)X$. Hence we obtain

$$g(t\mu, X)t\mu = g(t\mu, t\mu)X \tag{12.32}$$

for all $X \in D^{\perp}$. Since $q > 1$, it follows, upon contraction of Eq. (12.32) at X with respect to a local orthonormal basis of $D^{\perp}$, that $g(t\mu, t\mu) = 0$. Hence $t\mu = 0$. Now, let X be a vector field tangent to M. Then

$$(\nabla_X t)\mu = \nabla_X t\mu - tD_X\mu = -tD_X\mu$$

Using Eq. (12.20) in the above equation provides

$$-tD_X\mu = A_{f\mu}X - PA_{\mu}X = -g(\mu, \mu)PX$$

Using $Pt = 0$, we get $g(\mu, \mu)P^2X = 0$. As M is proper CR-submanifold, we conclude that $g(\mu, \mu) = 0$. Further, $(\nabla_X P)Y = A_{FY}X + tB(X, Y) = -g(Y, t\mu) = 0$. Thus we obtain $\nabla_X P = 0$, which implies through Chen's result [8] mentioned earlier, that M is locally a product of an invariant submanifold M^T and an anti-invariant submanifold $M^\perp$ of M. What remains to be proved is that $\mu = 0$. Suppose that $Y \in D$ so that $FY = 0$. As D is parallel, $\nabla_X Y \in D$ and hence $F(\nabla_X Y) = 0$. Consequently, $(\nabla_X F)Y = 0$ and using Eq. (12.19) we have $g(X, PY)\mu = g(X, Y)f\mu$ for every $Y \in D$. Substituting $X = PY$ and noting the skew-symmetry $g(PX, Y) = -g(PY, X)$ we find that $\mu = 0$. Hence $B = 0$, i.e., M is totally geodesic and locally a CR-product of the leaves of D and $D^\perp$. This completes the proof.

The case $q = 1$ was not covered in the preceding theorem. Chen [9] proved the following result.

Theorem 12.9 *Let M be a totally umbilical CR-submanifold of a Kaehler manifold $\bar{M}$. Then (i) M is totally geodesic, or (ii) $q = 1$, or (iii) M is totally real.*

Note that if M was a proper CR-submanifold in the above theorem, then the possibility (iii) would be ruled out. Also, note that (i) and (ii) are not mutually exclusive. The case (ii) has been investigated by Chen [9], in the context of a locally Hermitian symmetric space $\bar{M}$ with dim. $\bar{M} \geq 5$. In [11], Duggal and Sharma studied the case (ii) by relaxing these conditions and assuming M to be proper, and proved the following result.

Theorem 12.10 *Let M be a proper totally umbilical CR-submanifold of a semi-Riemannian Kaehler manifold $\bar{M}$ with $q = 1$ and g positive definite on $D^\perp$. Suppose that the mean curvature vector μ vanishes nowhere on M. Then the following statements are equivalent: (1) M has an α-Sasakian structure, (2) μ has a constant norm, (3) μ is parallel in the normal bundle, (4) second fundamental form of M is parallel.*

For α-Sasakian structures, we refer to [6, 14].

Proof As $\mu \neq 0$ and $\mu \in JD^\perp$ by Proposition 3, it follows that $t\mu \neq 0$ and lies in $D^\perp$. Now since $q = 1$, any vector field in $D^\perp$ is a scalar multiple of $t\mu$. For any X tangent to M we can show, using Eq. (12.20), that

$$g(\mu, \mu)P^2X = g(t\mu, X)t\mu - g(t\mu, t\mu)X$$

Operating P on this gives $g(\mu, \mu) = g(t\mu, t\mu)$. Hence we get

$$g(t\mu, t\mu)(P^2X + X) = g(t\mu, X)t\mu \tag{12.33}$$

In this case too, Eq. (12.32) holds, which shows ($q = 1$) that $g(t\mu, t\mu) \neq 0$ and hence $g(\mu, \mu) \neq 0$. Hence the Eq. (12.33) assumes the form

$$P^2X = -X + [g(t\mu, t\mu)]^{-1}g(t\mu, X)t\mu \tag{12.34}$$

As g is positive definite on $D^{\perp}$, and μ vanishes nowhere on M, we have $g(t\mu, t\mu) = \alpha^2$. Hence Eq. (12.34) becomes

$$P^2X = -X + \eta(X)\xi \tag{12.35}$$

where $\xi = \frac{1}{\alpha}t\mu$ is a unit vector field, and η is a 1-form on M given by $\eta(X) = g(X, \xi)$. One can easily verify that $P\xi = 0$, $\eta(PX) = 0$, rank$(P) = n - 1$, and

$$g(PX, PY) = g(X, Y) - \eta(X)\eta(Y) \tag{12.36}$$

Use of Eq. (12.18) and total umbilicity shows that

$$(\nabla_X P)Y = \alpha[g(X, Y)\xi - g(\xi, Y)X] \tag{12.37}$$

Equations (12.35)–(12.37) show that (M, g) is an α-Sasakian manifold if and only if $g(\mu, \mu)$ is constant. This proves the equivalence of (1) to (2). In virtue of the equality (whose proof is easy)

$$tD_X\mu = (Xln|g(t\mu, t\mu)|)t\mu$$

the statement (2) is equivalent to $tD_X\mu = 0$. Differentiating the result $f\mu = 0$ obtained earlier, and operating f^2 on the derived equation provides $f(D_X\mu) = 0$. Hence (2) is equivalent to $D_X\mu = 0$, i.e. the statement (3). The statement (4) means

$$D_X(B(Y, Z)) = B(\nabla_X Y, Z) + B(Y, \nabla_X Z)$$

Substituting totally umbilical condition (hypothesis) $B(X, Y) = g(X, Y)\mu$ in the preceding equation shows that (4) is equivalent to (3). This completes the proof.

Remark 5 In particular, if M was a real hypersurface of $\bar{M}$ (as hypothesized in the foregoing theorem, second case), then the statement (3) would have been automatically true, as apparent from the fact that $D_X\mu$ does not belong to $JD^{\perp}$.

12.6 Application to General Relativity

The CR-submanifolds under the hypothesis of Theorem 12.8 are locally decomposable as $M^T \times M^{\perp}$. Recall that these submanifolds carry a parallel f-structure: $P^3 + P = 0$, rank $(P) = 2p$, and $\nabla P = 0$, where $2p$ is the dimension of D. M has a pair of complementary orthogonal distributions $D^{\perp}$ (of dimension q) and D defined respectively by the projection operators $-P^2$ and $P^2 + I$ acting on the tangent space of M at every point. For simplicity, we assume that D and $D^{\perp}$ are each of dimension 2 (i.e. $2p = 2$, $q = 2$). Let $D^{\perp}$ be parallelizable so that there exist vector fields ξ_1, ξ_2 spanning $D^{\perp}$ and their duals η^1, η^2 such that

$$P^2(X) = -X + \eta^1(X)\xi_1 + \eta^2(X)\xi_2$$

$$P\xi_1 = P\xi_2 = 0, \quad P\xi_3 = \xi_4, P\xi_4 = -\xi_3$$

where ξ_3, ξ_4 is a basis of D such that $(\xi_1, \xi_2, \xi_3, \xi_4)$ is an orthonormal basis of TM. Thus, if the metric g on M is indefinite on $D^\perp$ and positive definite on D, then it can be expressed canonically as

$$g = -\eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2 + \eta^3 \otimes \eta^3 + \eta^4 \otimes \eta^4.$$

Using the condition $\nabla P = 0$, we can show that

$$\nabla_X \xi_1 = h(X)\xi_2, \nabla_X \xi_2 = h(X)\xi_1$$
$$\nabla_X \xi_3 = w(X)\xi_4, \nabla_X \xi_4 = -w(X)\xi_3$$

where h and w are smooth 1-forms on M. A straightforward computation gives the curvature tensor R, Ricci tensor Ric and scalar curvature r as follows:

$$R(X, Y)Z = 2H(\eta^2 \wedge \eta^1)(X, Y)[\eta^1(Z)\xi_2 + \eta^2(Z)\xi_1]$$
$$+ 2W(\eta^4 \wedge \eta^3)(X, Y)[\eta^3(Z)\xi_4 - \eta^4(Z)\xi_3]$$

$$Ric = H(-\eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2) + W(\eta^3 \otimes \eta^3 + \eta^4 \otimes \eta^4)$$

$$r = 2(H + W)$$

where $H = (dh)(\xi_2, \xi_1)$ and $W = (dw)(\xi_4, \xi_3)$. Let us call such a manifold (M, g) a Lorentzian Framed (LF)-manifold.

Evidently, LF-manifolds are Ricci-flat if and only if h and w are closed. Also, LF-manifolds are Einstein if and only if $4H = 4W = r$. By a straightforward calculation one can verify that LF-manifolds are conformally flat if and only if $r = 0$. Let us consider a coordinate frame $(\frac{\partial}{\partial x^i})$ (abbreviated ∂_i) compatible with the LF-structure, for a local coordinate system $(t, x, y, z) = (x^i)$ such that

$$\partial_1 = \sigma\xi_1, \partial_2 = \sigma\xi_2, \partial_3 = \tau\xi_3, \partial_4 = \tau\xi_4$$

where σ and τ are nonzero smooth functions. Under such a coordinate system, the metric g takes the form

$$ds^2 = \sigma^2(-dt^2 + dx^2) + \tau^2(dy^2 + dz^2)$$

where $\sigma = \sigma(t, x)$ and $\tau = \tau(y, z)$ and are related to H and W by partial differential equations

$$(ln\sigma)_{,tt} - (ln\sigma)_{,xx} = H\sigma^2, (ln\tau)_{,yy} + (ln\tau)_{,zz} = -W\tau^2$$

The Ricci tensor is expressed in terms of the coordinates (t, x, y, z) as

$$Ric = H(-dt \otimes dt + dx \otimes dx) + W(dy \otimes dy + dz \otimes dz).$$

Exact solutions of the Einstein's field equations

$$Ric - \frac{r}{2}g = 8\pi T$$

with a given energy-momentum tensor T, have been obtained by Duggal and Sharma in [10] under various cases such as flat (Minkowski), Einstein, Conformally flat, Scalar field and nonsingular simple electromagnetic field. For details we refer to [10] and the Ph.D. dissertation of Sharma [16].

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Chapter 13
Paraquaternionic CR-Submanifolds

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13.1 Introduction

The paraquaternionic structures, previously called quaternionic structures of second kind, have been introduced in geometry by P. Libermann [50], in 1952. The theory of paraquaternionic manifolds parallels the theory of quaternionic manifolds, but uses the algebra of paraquaternionic numbers, in which two generators have square 1 and one generator has square -1. Accordingly, such manifolds are equipped with a subbundle of rank 3 in the bundle of the endomorphisms, locally spanned by two almost product structures and one almost complex structure. From the metric point of view, the almost paraquaternionic Hermitian manifolds have neutral signature. An example of such kind of structure has been considered on the tangent bundle of a manifold endowed with an almost complex structure and a linear connection in [35]. The differential geometry of manifolds equipped with paraquaternionic structures was developed in the last two decades by papers of Blažić [14], Vukmirović [67], García-Río, Matsushita, Vázquez-Lorenzo [32], Bonome, Castro, García-Río, Hervella, Vázquez-Lorenzo [16], Ivanov, Zamkovoy [39, 68], Alekseevsky, Cortes, Kamishima [2, 3], Marchiafava [51, 52], Dancer, Jorgensen, Swann [28], David [29], etc. Almost all important constructions from quaternionic geometry have been

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adapted (for example, the twistor space has been studied in [4, 12, 13, 45]), but the majority has remained at the stage of definition and first properties. Recently, there has been a lot of work in this direction (in Germany, at Augsburg, in Italy, at Roma II and Bari, several PhD thesis on this topic have been defended [19, 49, 59]) and also there are some constructions which were not tackled.

The counterpart in odd dimension of paraquaternionic geometry was introduced in [40]. It is called mixed 3-structure, which appears in a natural way on lightlike hypersurfaces in paraquaternionic manifolds (there are two paracontact structures and one contact structure, that satisfy analogous conditions to those satisfied by 3-Sasakian structures). The first properties were obtained in [21] and [36]. A compatible metric with a mixed 3-structures is necessarily semi-Riemann and mixed 3-Sasakian manifolds are Einstein, hence there is a possible importance of these structures in theoretical physics [43].

This work is organized as follows: in Sects. 13.2 and 13.3, one recalls basic definitions and fundamental properties of manifolds endowed with paraquaternionic and mixed 3-structures.

In Sect. 13.4, the geometry of the semi-Riemannian hypersurfaces of co-index both 0 and 1 in a manifold endowed with a mixed 3-structure and a compatible metric is presented [38].

Section 13.5 deals with the class of paraquaternionic CR-submanifolds in the paraquaternionic Kähler manifolds. This is a natural extension in paraquaternionic setting of CR-submanifolds, first introduced in Kählerian geometry by Bejancu [6]. We define the paraquaternionic CR-submersions (in the sense of Kobayashi [47]) as semi-Riemannian submersions from paraquaternionic CR-submanifolds onto an almost paraquaternionic hermitian manifold and obtain some properties concerning their geometry [44]. We also discuss curvature properties of fibers and base manifold for paraquaternionic CR-submersions.

In Sect. 13.6, we introduce a new class of semi-Riemannian submersions from a manifold endowed with a metric mixed 3-structure onto an almost paraquaternionic hermitian manifold [64]. We obtain some fundamental properties which discuss the transference of structures and the geometry of the fibers. In particular we obtain that such a submersion is a harmonic map provided that the total space is mixed 3-cosymplectic or 3-Sasakian. Moreover, some nontrivial examples are given.

13.2 Paraquaternionic Structures on Manifolds

Let $\widetilde{\mathbb{H}}$ be the algebra of paraquaternions and identify $\widetilde{\mathbb{H}}^n = \mathbb{R}^{4n}$. It is well known that $\widetilde{\mathbb{H}}$ is a real Clifford algebras with $\widetilde{\mathbb{H}} = C(1,1) \cong C(0,2)$. In fact $\widetilde{\mathbb{H}}$ is generated by the unity 1 and generators J_1^0, J_2^0, J_3^0 satisfying the paraquaternionic identities

$$(J_1^0)^2 = (J_2^0)^2 = -(J_3^0)^2 = 1, \ J_1^0 J_2^0 = -J_2^0 J_1^0 = J_3^0.$$

We assume that $\widetilde{\mathbb{H}}$ acts on $\widetilde{\mathbb{H}}^n$ by right multiplication and use the convention that $SO(2n, 2n)$ acts on $\widetilde{\mathbb{H}}^n$ on the left.

An *almost product structure* on a smooth manifold M is a tensor field P of type $(1, 1)$ on M, $P \neq \pm Id$, such that $P^2 = Id$, where Id is the identity tensor field of type $(1, 1)$ on M. The pair (M, P) is called an *almost product manifold*. An *almost para-complex manifold* is an almost product manifold (M, P) such that the two eigenbundles T^+M and T^-M associated with the two eigenvalues $+1$ and -1 of P, respectively, have the same rank. Equivalently, a splitting of the tangent bundle TM of a differentiable manifold M, into the Whitney sum of two subbundles $T^{\pm}M$ of the same fiber dimension is called an *almost para-complex structure* on M.

An *almost para-Hermitian structure* on a smooth manifold M is a pair (g, P), where g is a pseudo-Riemannian metric on M and P is an almost product structure on M, which is compatible with g, i.e., $P^*g = -g$. In this case, the triple (M, g, P) is called an *almost para-Hermitian manifold*. Moreover, (M, g, P) is said to be a *para-Hermitian manifold* if P is integrable, i.e., if the Nijenhuis N_P defined by

$$N_P(X, Y) = [PX, PY] - P[X, PY] - P[PX, Y] + [X, Y]$$

vanishes. An *almost complex structure* on a smooth manifold M is a tensor field J of type $(1, 1)$ on M such that $J^2 = -Id$. The pair (M, J) is called an *almost complex manifold*. We note that the dimension of an almost (para-)complex manifold is necessarily even (see [27, 48]).

An *almost pseudo-Hermitian structure* on a smooth manifold M is a pair (g, J), where g is a pseudo-Riemannian metric on M and J is an almost complex structure on M, which is compatible with g, i.e., $J^*g = g$. In this case, the triple (M, g, J) is called an *almost pseudo-Hermitian manifold*. Moreover, (M, g, J) is said to be *pseudo-Hermitian* if J is integrable, i.e., if the Nijenhuis N_J defined by

$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$$

vanishes.

An *almost para-hypercomplex structure* on a smooth manifold M is a triple $H = (J_1, J_2, J_3)$ of $(1, 1)$-type tensor fields on M satisfying

$$J_\alpha^2 = -\tau_\alpha Id,\ J_\alpha J_\beta = -J_\beta J_\alpha = \tau_\gamma J_\gamma, \tag{13.1}$$

for any $\alpha \in \{1, 2, 3\}$ and for any even permutation (α, β, γ) of $(1, 2, 3)$, where $\tau_1 = \tau_2 = -1 = -\tau_3$. In this case, (M, H) is said to be an *almost para-hypercomplex manifold*. We remark that from (13.1) it follows that J_1 and J_2 are almost product structures on M, while $J_3 = J_1 J_2$ is an almost complex structure on M. We note that the almost para-hypercomplex structures have been introduced in geometry by P. Libermann [50] under the name of quaternionic structures of second kind (*structures presque quaternioniennes de deuxième espèce*).

A semi-Riemannian metric g on (M, H) is said to be *compatible* or *adapted* to the almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=1,2,3}$ if it satisfies:

$$g(J_\alpha X, J_\alpha Y) = \tau_\alpha g(X, Y)$$

for all vector fields X,Y, on M and $\alpha \in \{1, 2, 3\}$. Moreover, the pair (g, H) is called an *almost para-hyperhermitian structure* on M and the triple (M, g, H) is said to be an *almost para-hyperhermitian manifold*. We note that any almost para-hyperhermitian manifold is of dimension $4m,\ m \geq 1$, and any adapted metric is necessarily of neutral signature $(2m, 2m)$. If $\{J_1, J_2, J_3\}$ is parallel with respect to the Levi-Civita connection of g, then the manifold is called *para-hyper-Kähler*. We note that the simplest example of para-hyper- Kähler manifold is the paraquaternionic vector space $\widetilde{\mathbb{H}}^n$.

An almost para-hypercomplex manifold (M, H) is called a *para-hypercomplex manifold* if each J_α, $\alpha = 1, 2, 3$, is integrable. In this case H is said to be a *para-hypercomplex structure* on M. Moreover, if g is a semi-Riemannian metric adapted to the para-hypercomplex structure H, then the pair (g, H) is said to be a *para-hyperhermitian structure* on M and (M, g, H) is called a *para-hyperhermitian manifold*.

An *almost paraquaternionic Hermitian manifold* is a triple (M, σ, g) where M is a smooth manifold, σ is an almost paraquaternionic structure on M, i.e., a rank 3-subbundle of $End(TM)$ which is locally spanned by an almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=1,2,3}$ and g is a compatible metric with respect to H. We remark that, if $\{J_1, J_2, J_3\}$ and $\{J'_1, J'_2, J'_3\}$ are two canonical local bases of σ in U and in another coordinate neighborhood U' of M, then for all $x \in U \cap U'$

$$\left(J'_\alpha\right)_x = \sum_{\beta=1}^{3} s_{\alpha\beta}(x) \left(J'_\beta\right)_x, \alpha = 1, 2, 3,$$

where $S_{UU'}(x) = \left(s_{\alpha\beta}(x)\right)_{\alpha,\beta=1,2,3} \in \mathrm{SO}(2, 1)$, because $\{J_1, J_2, J_3\}$ and $\{J'_1, J'_2, J'_3\}$ satisfy the paraquaternionic identities (13.1).

If (M, σ, g) is an almost paraquaternionic Hermitian manifold such that the bundle σ is preserved by the Levi-Civita connection ∇ of g, then (M, σ, g) is said to be a *paraquaternionic Kähler manifold* [32]. Equivalently, we can write

$$\nabla J_\alpha = \tau_\beta \omega_\gamma \otimes J_\beta - \tau_\gamma \omega_\beta \otimes J_\gamma, \tag{13.2}$$

where (α, β, γ) is an even permutation of $(1, 2, 3)$ and $\omega_1, \omega_2, \omega_3$ are locally defined 1-forms. We note that the prototype of paraquaternionic Kähler manifold is the paraquaternionic projective space $P^n(\widetilde{\mathbb{H}})$ as described by Blažić [14].

If the Riemannian curvature tensor R is taken with the sign convention

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all vector fields X, Y, Z on M, then a consequence of (13.2) is that R satisfies

$$[R, J_\alpha] = \tau_\beta A_\gamma \otimes J_\beta - \tau_\gamma A_\beta \otimes J_\gamma,$$

for any even permutation (α, β, γ) of $(1, 2, 3)$, where

$$A_\alpha = d\omega_\alpha + \tau_\alpha \omega_\beta \wedge \omega_\gamma.$$

If we consider $\Omega_\alpha := g(J_\alpha \cdot, \cdot)$ the fundamental form associated with J_α, $\alpha = 1, 2, 3$, then we have the following structure equations (see [4, 13]):

$$d\omega_\alpha + \tau_\alpha \omega_\beta \wedge \omega_\gamma = \tau_\alpha \nu \Omega_\alpha,$$

for any even permutation (α, β, γ) of $(1, 2, 3)$, where $\nu = \frac{Sc}{4n(n+2)}$ is the reduced scalar curvature, Sc being the scalar curvature defined as the trace of the Ricci tensor ρ.

We recall that the main property of manifolds endowed with this kind of structure is as follows:

Theorem 13.1 ([32]) *Any paraquaternionic Kähler manifold* (M, σ, g) *is an Einstein space, provided that* $\dim M > 4$.

Let (M, σ, g) be a $4n$-dimensional paraquaternionic Kähler manifold. Then the Ricci 2-forms of the Levi-Civita connection of g are defined as (see [45]):

$$\rho_\alpha(X, Y) = -\frac{\tau_\alpha}{2} \mathrm{Trace}(Z \to J_\alpha R(X, Y)Z), \ \alpha = 1, 2, 3,$$

and for $n > 1$ it follows:

$$\rho(X, Y) = \frac{n+2}{n} \rho_\alpha(X, J_\alpha Y), \ \alpha = 1, 2, 3.$$

Using Theorem 13.1, we obtain the following relations

$$\rho_\alpha(X, Y) = -\tau_\alpha \frac{Sc}{4(n+2)} g(X, J_\alpha Y), \ \alpha = 1, 2, 3.$$

We consider now the general case of a $4n$-dimensional smooth manifold M endowed with an almost paraquaternionic structure σ and with a *paraquaternionic connection* ∇, i.e., a linear connection which preserves σ, and following [45] we recall next some basic facts concerning the twistor and reflector spaces of M.

Let $p \in M$, any linear frame u on T_pM can be considered as an isomorphism $u : \mathbb{R}^{4n} \to T_pM$. Taking such a frame u, we can define a subspace of the space of the all endomorphisms of T_pM by $u(\mathrm{sp}(1, \mathbb{R}))u^{-1}$. This subset is a paraquaternionic structure and we define $P(M)$ to be the set of all linear frames u which satisfy $u(\mathrm{sp}(1, \mathbb{R}))u^{-1} = \sigma$, where $\mathrm{sp}(1, \mathbb{R}) = \mathrm{Span}\{\mathrm{J}_1^0, \mathrm{J}_2^0, \mathrm{J}_3^0\}$ is the Lie algebra of $\mathrm{Sp}(1, \mathbb{R})$. It is clear that $P(M)$ is the principal frame bundle of M with structure group $\mathrm{GL}(n, \widetilde{\mathbb{H}})\mathrm{Sp}(1, \mathbb{R})$, where

$$\mathrm{GL}(n, \widetilde{\mathbb{H}}) = \{A \in \mathrm{GL}(4n, \mathbb{R}) | A(\mathrm{sp}(1, \mathbb{R}))A^{-1} = \mathrm{sp}(1, \mathbb{R})\}.$$

We denote by $\pi : P(M) \to M$ the natural projection and remark that the Lie algebra of $\mathrm{GL}(n, \widetilde{\mathbb{H}})$ is

$$\mathrm{gl}(n, \widetilde{\mathbb{H}}) = \{A \in \mathrm{gl}(4n, \mathbb{R}) | AB = BA \text{ for all } B \in \mathrm{sp}(1, \mathbb{R})\}.$$

We also denote by (,) the inner product in $\mathrm{gl}(4n, \mathbb{R})$ given by $(A, B) = \mathrm{Trace}(AB^t)$, for $A, B \in \mathrm{gl}(4n, \mathbb{R})$.

We split now the curvature of ∇ into $\mathrm{gl}(n, \widetilde{\mathbb{H}})$-valued part R' and $\mathrm{sp}(1, \mathbb{R})$-valued part R'' following the classical scheme (see e.g., [10]). We denote the splitting of the $\mathrm{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathrm{sp}(1, \mathbb{R})$-valued curvature 2-form Ω on $P(M)$ according to the splitting of the curvature R by

$$\Omega = \Omega' + \Omega'',$$

where Ω' is a $\mathrm{gl}(n, \widetilde{\mathbb{H}})$-valued 2-form and Ω'' is a $\mathrm{sp}(1, \mathbb{R})$-valued form. Explicitly,

$$\Omega'' = \Omega''_1 J_1^0 + \Omega''_2 J_2^0 + \Omega''_3 J_3^0,$$

where $\Omega''_\alpha, \ \alpha = 1, 2, 3$ are 2-forms. If $\xi, \eta \in \mathbb{R}^{4n}$, then the 2-forms $\Omega''_\alpha, \ \alpha = 1, 2, 3$, are given by

$$\Omega''_\alpha(B(\xi), B(\eta)) = \frac{1}{2n}\rho_\alpha(X, Y) = -\tau_\alpha \frac{Sc}{8n(n+2)} g(X, J_\alpha Y),$$

where $X = u(\xi), \ Y = u(\eta)$ (see [45]).

For each $u \in P(M)$ we consider two linear isomorphism $j^+(u)$ and $j^-(u)$ on $T_{\pi(u)}M$ defined by:

$$j^+(u) = uJ_1^0 u^{-1}, \ \ j^-(u) = uJ_3^0 u^{-1}.$$

It is easy to see that

$$(j^-(u))^2 = -Id, \ \ (j^+(u))^2 = Id$$

and

$$g(j^-(u)X, j^-(u)Y) = g(X, Y), \ \ g(j^+(u)X, j^+(u)Y) = -g(X, Y),$$

for all $X, Y \in T_{\pi(u)}M$.

As in [45], for each $p \in M$ we consider

$$Z_p^{\pm}(M) = \{j^{\pm}(u) | u \in P(M), \pi(u) = p\}$$

and we define the *twistor space* Z^- of M and the *reflector space* Z^+ of M, by setting

$$Z^{\pm} = Z^{\pm}(M) = \bigcup_{p \in M} Z_p^{\pm}(M).$$

Then the twistor space $Z^-(M)$ is the unit pseudo-sphere bundle with fiber the 2-sheeted hyperboloid $S_1^2(-1) = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = -1\}$ and the reflector space $Z^+(M)$ is the unit pseudo-sphere bundle with fiber the 1-sheeted hyperboloid $S_1^2(1) = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = 1\}$.

We denote by A^* (resp. $B(\xi)$) the fundamental vector field (resp. the standard horizontal vector field) on $P(M)$, the principal frame bundle of M, corresponding to $A \in \mathrm{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathrm{sp}(1, \mathbb{R})$ (resp. $\xi \in \mathbb{R}^{4n}$). Let $u \in P(M)$ and Q_u be the horizontal subspace of the tangent space $T_u P(M)$ induced by the connection ∇ on M (see [48]). As in [45] we have the decompositions

$$T_u P(M) = (h_i)_u^* \oplus (m_i)_u^* \oplus Q_u, \; i = 1, 3$$

and the following isomorphisms

$$j_{*u}^-|_{(m_3)_u^* \oplus Q_u} : (m_3)_u^* \oplus Q_u \to T_{j^-(u)} Z^-,$$

$$j_{*u}^+|_{(m_1)_u^* \oplus Q_u} : (m_1)_u^* \oplus Q_u \to T_{j^+(u)} Z^+$$

where

$$h_3 = \{A \in \mathrm{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathrm{sp}(1, \mathbb{R}) | A J_3^0 = J_3^0 A\},$$

$$h_1 = \{A \in \mathrm{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathrm{sp}(1, \mathbb{R}) | A J_1^0 = J_1^0 A\},$$

$$m_3 = \{A \in \mathrm{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathrm{sp}(1, \mathbb{R}) | A J_3^0 = -J_3^0 A\} = \mathrm{Span}\{J_1^0, J_2^0\},$$

$$m_1 = \{A \in \mathrm{gl}(n, \widetilde{\mathbb{H}}) \oplus \mathrm{sp}(1, \mathbb{R}) | A J_1^0 = -J_1^0 A\} = \mathrm{Span}\{J_2^0, J_3^0\},$$

and

$$(h_i)_u^* = \{A_u^* | A \in h_i\}, \; (m_i)_u^* = \{A_u^* | A \in m_i\}, \; i = 1, 3.$$

Now, we can define two almost complex structures I_1 and I_2 on Z^- by (see [45])

$$\begin{aligned} I_1(j_{*u}^- A^*) &= j_{*u}^- (J_3^0 A)^*, \\ I_2(j_{*u}^- A^*) &= -j_{*u}^- (J_3^0 A)^* \\ I_i(j_{*u}^- B(\xi)) &= j_{*u}^- B(J_3^0 \xi), \; i = 1, 2, \end{aligned}$$

for $u \in P(M), \; A \in m_3, \; \xi \in \mathbb{R}^{4n}$.

Similarly, it can be defined two almost para-complex structures P_1 and P_2 on Z^+ by (see [45])

$$P_1(j^+_{*u}A^*) = j^+_{*u}(J^0_1 A)^*,$$
$$P_2(j^+_{*u}A^*) = -j^+_{*u}(J^0_1 A)^*$$
$$P_i(j^+_{*u}B(\xi)) = j^+_{*u}B(J^0_1\xi),\ i = 1, 2,$$

for $u \in P(M),\ A \in m_1,\ \xi \in \mathbb{R}^{4n}$.

We remark that the almost complex structures defined above were also defined and investigated in [13] on paraquaternionic Kähler manifolds, the authors proving that the almost complex structure I_2 is never integrable while I_1 is always integrable. Moreover, we note that in [45] the authors found that the para-complex structures P_2 is never integrable on reflector space, while P_1 is always integrable.

We recall now that the *-Ricci tensor of a $2n$-dimensional almost pseudo-Hermitian manifold (M, g, J) is defined by

$$\rho^*(X, Y) = \sum_{i=1}^{2n} \varepsilon_i R(X, E_i, JY, JE_i),$$

where R denotes the curvature of the metric g, $\{E_1, \ldots, E_{2n}\}$ is a pseudo-orthonormal basis at an arbitrary point p, and X, Y are tangent vectors at p. If the *-Ricci tensor is scalar multiple of the metric then the manifold is said to be **-Einstein*.

On the other hand, we note that A. Gray introduced [33] three basic classes $AH1,\ AH2,\ AH3$ of almost Hermitian manifolds, whose curvature tensors resemble that of a Kähler manifold. They are defined by the following curvature identities:

$$AH1 : R(X, Y, Z, T) = R(X, Y, JZ, JT),$$
$$AH2 : R(X, Y, Z, T) = R(JX, JY, Z, T) + R(JX, Y, JZ, T) + R(JX, Y, Z, JT),$$
$$AH3 : R(X, Y, Z, T) = R(JX, JY, JZ, JT),$$

where R is the curvature tensor of the manifold. It is easy to see that

$$AH1 \subset AH2 \subset AH3.$$

By analogy, one says that an almost para-Hermitian manifold satisfies the para-Gray identities if

$$APH1 : R(X, Y, Z, T) = -R(X, Y, JZ, JT),$$
$$APH2 : R(X, Y, Z, T) = -R(JX, JY, Z, T) - R(JX, Y, JZ, T) - R(JX, Y, Z, JT),$$
$$APH3 : R(X, Y, Z, T) = R(JX, JY, JZ, JT),$$

where R is the curvature tensor of the manifold. We note that para-Gray-like identities were considered in [18, 24] and it can be easily checked that

$$APH1 \subset APH2 \subset APH3.$$

The main curvature properties of (Z^-, I_i, h_t) and (Z^+, P_i, h_t), $i = 1, 2$, are given in the following two theorems.

Theorem 13.2 ([66]) *Let* (M, σ, g) *be a* $4n$*-dimensional paraquaternionic Kähler manifold and* (Z^-, I_i, h_t)*,* $i = 1, 2$ *the twistor spaces associated. Then:*
(i) The manifolds (Z^-, I_i, h_t)*,* $i = 1, 2$*, belong always to* $AH2$ *and* $AH3$ *and are with pseudo-Hermitian Ricci tensor and with pseudo-Hermitian *-Ricci tensor;*
(ii) The manifolds (Z^-, I_1, h_t) *belong to* $AH1$ *iff* $Sc = 0$ *or* $Sc = \frac{4(n+2)}{t}$*;*
(iii) The manifolds (Z^-, I_2, h_t) *belong to* $AH1$ *iff* $Sc = 0$*;*
(iv) The manifolds (Z^-, I_i, h_t)*,* $i = 1, 2$ *are Einstein iff*

$$Sc = \frac{4(n+2)}{t} \text{ or } Sc = \frac{4(n+2)}{(n+1)t};$$

(v) The manifolds (Z^-, I_1, h_t) *are *-Einstein iff*

$$Sc = \frac{4(n+2)}{t} \text{ or } Sc = -\frac{4(n+2)}{nt};$$

(vi) The manifolds (Z^-, I_2, h_t) *are *-Einstein iff*

$$Sc = \frac{2(n+2)}{(n-1)t}\left[3n - 1 - \sqrt{9n^2 - 10n + 5}\right]$$

or

$$Sc = \frac{2(n+2)}{(n-1)t}\left[3n - 1 + \sqrt{9n^2 - 10n + 5}\right].$$

Theorem 13.3 ([66]) *Let* (M, σ, g) *be a* $4n$*-dimensional paraquaternionic Kähler manifold and* (Z^+, P_i, h_t)*,* $i = 1, 2$*, the reflector spaces associated. Then:*
(i) The manifolds (Z^+, P_i, h_t)*,* $i = 1, 2$*, belong always to* $APH2$ *and* $APH3$ *and are with para-Hermitian Ricci tensor and with para-Hermitian *-Ricci tensor;*
(ii) The manifolds (Z^+, P_1, h_t) *belong to* $APH1$ *iff* $Sc = 0$ *or* $Sc = -\frac{4(n+2)}{t}$*;*
(iii) The manifolds (Z^+, P_2, h_t) *belong to* $APH1$ *iff* $Sc = 0$*;*
(iv) The manifolds (Z^+, P_i, h_t)*,* $i = 1, 2$ *are Einstein iff*

$$Sc = -\frac{4(n+2)}{t} \text{ or } Sc = -\frac{4(n+2)}{(n+1)t};$$

(v) The manifolds (Z^+, P_1, h_t) *are *-Einstein iff*

$$Sc = -\frac{4(n+2)}{t} \text{ or } Sc = \frac{4(n+2)}{nt};$$

(vi) The manifolds (Z^+, P_2, h_t) *are *-Einstein iff*

$$Sc = \frac{2(n+2)}{(n-1)t}\left[1 - 3n - \sqrt{9n^2 - 10n + 5}\right]$$

or

$$Sc = \frac{2(n+2)}{(n-1)t}\left[1 - 3n + \sqrt{9n^2 - 10n + 5}\right].$$

13.3 Mixed 3-Structures on Manifolds

Definition 1 Let $\bar{M}$ be a differentiable manifold equipped with a triple (φ, ξ, η), where φ is a field of endomorphisms of the tangent spaces, ξ is a vector field and η is a 1-form on $\bar{M}$. If we have:

$$\varphi^2 = \tau(-I + \eta \otimes \xi), \quad \eta(\xi) = 1 \tag{13.3}$$

then we say that:
(i) (φ, ξ, η) is an almost contact structure on $\bar{M}$, if $\tau = 1$ [56].
(ii) (φ, ξ, η) is an almost paracontact structure on $\bar{M}$, if $\tau = -1$ [57].

We note that from (13.3) we can easily obtain $\varphi\xi = 0, \ \eta \circ \varphi = 0$ (see [11]).

Definition 2 ([21, 38]) A mixed 3-structure on a smooth manifold $\bar{M}$ is a triple of structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1, 2, 3\}$, which are almost paracontact structures for $\alpha = 1, 2$ and almost contact structure for $\alpha = 3$, satisfying the following conditions:

$$\eta_\alpha(\xi_\beta) = 0, \tag{13.4}$$

$$\varphi_\alpha(\xi_\beta) = \tau_\beta \xi_\gamma, \ \ \varphi_\beta(\xi_\alpha) = -\tau_\alpha \xi_\gamma, \tag{13.5}$$

$$\eta_\alpha \circ \varphi_\beta = -\eta_\beta \circ \varphi_\alpha = \tau_\gamma \eta_\gamma\,, \tag{13.6}$$

$$\varphi_\alpha \varphi_\beta - \tau_\alpha \eta_\beta \otimes \xi_\alpha = -\varphi_\beta \varphi_\alpha + \tau_\beta \eta_\alpha \otimes \xi_\beta = \tau_\gamma \varphi_\gamma\,, \tag{13.7}$$

where (α, β, γ) is an even permutation of $(1, 2, 3)$ and $\tau_1 = \tau_2 = -\tau_3 = -1$.

If a manifold $\bar{M}$ equipped with a mixed 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ admits a semi-Riemannian metric $\bar{g}$ such that:

$$\bar{g}(\varphi_\alpha X, \varphi_\alpha Y) = \tau_\alpha[\bar{g}(X, Y) - \varepsilon_\alpha \eta_\alpha(X)\eta_\alpha(Y)], \tag{13.8}$$

for any X, Y tangent to $\bar{M}$ and $\alpha = 1, 2, 3$, where $\varepsilon_\alpha = \bar{g}(\xi_\alpha, \xi_\alpha) = \pm 1$, then we say that $\bar{M}$ has a metric mixed 3-structure and $\bar{g}$ is called a compatible metric.

Remark 1 From (13.8) we obtain

$$\eta_\alpha(X) = \varepsilon_\alpha \bar{g}(X, \xi_\alpha), \ \bar{g}(\varphi_\alpha X, Y) = -\bar{g}(X, \varphi_\alpha Y) \tag{13.9}$$

for any X, Y tangent to $\bar{M}$ and $\alpha = 1, 2, 3$.

We remark that if $(\bar{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \bar{g})$ is a manifold endowed with a metric mixed 3-structure then from (13.9) it follows

$$\bar{g}(\xi_1, \xi_1) = \bar{g}(\xi_2, \xi_2) = -\bar{g}(\xi_3, \xi_3).$$

Hence the vector fields ξ_1 and ξ_2 are either spacelike or timelike and these force the causal character of the third vector field ξ_3. We may therefore distinguish between positive and negative metric mixed 3-structures, according as ξ_1 and ξ_2 are both spacelike or both timelike vector fields. Because at each point of M there always exists a pseudo-orthonormal frame field given by

$$\{(E_i, \varphi_1 E_i, \varphi_2 E_i, \varphi_3 E_i)_{i=\overline{1,n}}, \xi_1, \xi_2, \xi_3\}$$

We conclude that the dimension of the manifold is $4n+3$ and the signature of g is $(2n+1, 2n+2)$, if the metric mixed 3-structure is positive (i.e., $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$), or the signature of g is $(2n+2, 2n+1)$, if the metric mixed 3-structure is negative (i.e., $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -1$).

In what follows, we denote by $\Gamma(V)$ the module of all differentiable sections on a vector bundle V over $\bar{M}$.

Definition 3 ([21, 38]) Let $(\bar{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \bar{g})$ be a manifold with a metric mixed 3-structure.

(*i*) If $(\varphi_1, \xi_1, \eta_1, \bar{g})$ and $(\varphi_2, \xi_2, \eta_2, \bar{g})$ are para-cosymplectic structures and $(\varphi_3, \xi_3, \eta_3, \bar{g})$ is a cosymplectic structure, i.e., the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ satisfies

$$\bar{\nabla}\varphi_\alpha = 0 \tag{13.10}$$

for all $\alpha \in \{1, 2, 3\}$, then $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \bar{g})$ is said to be a mixed 3-cosymplectic structure on $\bar{M}$.

(*ii*) If $(\varphi_1, \xi_1, \eta_1, \bar{g}), (\varphi_2, \xi_2, \eta_2, \bar{g})$ are para-Sasakian structures and $(\varphi_3, \xi_3, \eta_3, \bar{g})$ is a Sasakian structure, i.e.,

$$(\bar{\nabla}_X \varphi_\alpha)Y = \tau_\alpha[g(X, Y)\xi_\alpha - \varepsilon_\alpha \eta_\alpha(Y)X] \tag{13.11}$$

for all $X, Y \in \Gamma(T\bar{M})$ and $\alpha \in \{1, 2, 3\}$, then $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \bar{g})$ is said to be a mixed 3-Sasakian structure on $\bar{M}$.

We remark that from (13.10) we obtain

$$\bar{\nabla}\xi_\alpha = 0$$

and from (13.11) it follows

$$\bar{\nabla}_X \xi_\alpha = -\varepsilon_\alpha \varphi_\alpha X,$$

for all $\alpha \in \{1, 2, 3\}$ and $X \in \Gamma(T\bar{M})$.

We note that the main property of a manifold endowed with a mixed 3-Sasakian structure is given by the following theorem.

Theorem 13.4 ([21, 36]) *Any* $(4n+3)-$*dimensional manifold endowed with a mixed* 3*-Sasakian structure is an Einstein space with Einstein constant* $\lambda = (4n+2)\varepsilon$*, with* $\varepsilon = \mp 1$*, according as the metric mixed 3-structure is positive or negative, respectively.*

Several examples of manifolds endowed with mixed 3-cosymplectic and mixed 3-Sasakian structures can be found in [38, 42]. We note that $\mathbb{R}^{4n+3}_{2n+1}$ admits a positive mixed 3-cosymplectic structure and $\mathbb{R}^{4n+3}_{2n+2}$ admits a negative mixed 3-cosymplectic structure. On the other hand, the unit pseudo-sphere $S^{4n+3}_{2n+2} \subset \mathbb{R}^{4n+4}_{2n+2}$ is the canonical example of manifold with negative mixed 3-Sasakian structure, while the pseudo-hyperbolic space $H^{4n+3}_{2n+1} \subset \mathbb{R}^{4n+4}_{2n+2}$ can be endowed with a canonical positive mixed 3-Sasakian structure. We note that other interesting examples were recently given in [5, 22, 23].

Using the cones over pseudo-Riemannian manifolds (see [4]) and the techniques from [17], we obtain the following characterization of manifolds endowed with mixed 3-Sasakian structures (see [42] for the negative mixed 3-Sasakian case).

Theorem 13.5 *Let* $(\overline{M}, \overline{g})$ *be a semi-Riemannian manifold. Then the following five assertions are mutually equivalent:*

(i) There is a positive mixed 3-Sasakian structure, a negative mixed 3-Sasakian structure, respectively, on $(\overline{M}, \overline{g})$*.*

(ii) There is a para-hyper-Kähler structure on the timelike cone

$$(C(\overline{M}), h) = (\mathbb{R}_+ \times \overline{M}, -dr^2 + r^2\overline{g}),$$

respectively, on the spacelike cone

$$(C(\overline{M}), h) = (\mathbb{R}_+ \times \overline{M}, dr^2 + r^2\overline{g}).$$

(iii) There exists three orthogonal Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$ *on* $\overline{M}$*, with* ξ_1, ξ_2 *unit spacelike vector fields and* ξ_3 *unit timelike vector field, respectively,* ξ_1, ξ_2 *unit timelike vector fields and* ξ_3 *unit spacelike vector field, satisfying*

$$[\xi_\alpha, \xi_\beta] = (\epsilon_\beta \tau_\alpha + \epsilon_\alpha \tau_\beta)\xi_\gamma, \tag{13.12}$$

where (α, β, γ) *is an even permutation of (1, 2, 3), such that the tensor fields* ϕ_α *of type (1, 1), defined by:*

$$\phi_\alpha X = -\epsilon_\alpha \overline{\nabla}_X \xi_\alpha, \ \alpha = 1, 2, 3,$$

satisfy (13.11).
(iv) There exists three orthogonal Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$ *on* $\overline{M}$, *with* ξ_1, ξ_2 *unit spacelike vector fields and* ξ_3 *unit timelike vector field, respectively,* ξ_1, ξ_2 *unit timelike vector fields and* ξ_3, *unit spacelike vector field, satisfying (13.12), such that:*

$$\overline{R}(X, \xi_\alpha)Y = \overline{g}(\xi_\alpha, Y)X - \overline{g}(X, Y)\xi_\alpha,\ \alpha = 1, 2, 3,$$

where $\overline{R}$ *is the Riemannian curvature tensor of the Levi-Civita connection* $\overline{\nabla}$ *of* $\overline{g}$.
(v) There exists three orthogonal Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$ *on* $\overline{M}$, *with* ξ_1, ξ_2 *unit spacelike vector fields and* ξ_3 *unit timelike vector field, respectively,* ξ_1, ξ_2 *unit timelike vector fields and* ξ_3 *unit spacelike vector field, satisfying (13.12), such that the sectional curvature of every section containing* ξ_1, ξ_2 *or* ξ_3 *equals 1.*

Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold and let M be an immersed submanifold of $\bar{M}$. Then M is said to be *non-degenerate* if the restriction of the semi-Riemannian metric $\bar{g}$ to TM is non-degenerate at each point of M [9, 54]. We denote by g the semi-Riemannian metric induced by $\bar{g}$ on M and by $TM^\perp$ the normal bundle to M. Then we have the following orthogonal decomposition:

$$T\bar{M} = TM \oplus TM^\perp.$$

We denote by $\bar{\nabla}$ and ∇ the Levi-Civita connection on $\bar{M}$ and M, respectively. Then the Gauss formula is given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

for all $X, Y \in \Gamma(TM)$, where $h : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$ is the second fundamental form of M in $\bar{M}$.

On the other hand, the Weingarten formula is given by

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, where $-A_N X$ is the tangential part of $\bar{\nabla}_X N$ and $\nabla^\perp_X N$ is the normal part of $\bar{\nabla}_X N$; A_N and $\nabla^\perp$ are called the shape operator of M with respect to N and the normal connection, respectively. Moreover, it is well known that h and A_N are related by

$$\bar{g}(h(X, Y), N) = g(A_N X, Y)$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$.

For the rest of this paper, we shall assume that the induced metric is non-degenerate.

13.4 Semi-Riemannian Hypersurfaces

13.4.1 Semi-Riemannian Hypersurfaces of Manifolds Endowed with Metric Mixed 3-Structures, Tangent to the Structure Vector Fields

Let $(\overline{M}, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ be a manifold with a metric mixed 3-structure and let (M, g) be a semi-Riemannian hypersurface of $\overline{M}$ with $g = \overline{g}_{|M}$, i.e., a semi-Riemannian submanifold of codimension 1, isometrically immersed in $\overline{M}$; thus the co-index of M must be 0 or 1. The sign of the semi-Riemannian hypersurface M, denoted by ε_M, is defined as +1 if the co-index of M is 0, and -1 if the co-index of M is 1 (see [54]). We note that, unlike the positive definite case where every hypersurface is Riemannian with sign +1, in the indefinite case, sign -1 is as natural as +1.

Next we suppose that M is tangent to the structure vector fields ξ_1, ξ_2, ξ_3 and denote by T_pM and $T_pM^\perp$ the tangent space and the normal space, respectively, to M at $p \in M$. Let N be the unit spacelike or timelike vector field normal to M, correspondent to the co-index 0, respectively. Since

$$g(\phi_\alpha N, N) = 0,\ \ \alpha = 1, 2, 3$$

we can define three one-dimensional distributions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ on M, spanned by $\phi_1 N$, $\phi_2 N$ and $\phi_3 N$, respectively, i.e.,

$$\mathcal{D}_\alpha : p \longrightarrow \mathcal{D}_{\alpha p} = \phi_\alpha(T_pM^\perp) \subset T_pM,\ \ \alpha = 1, 2, 3.$$

We denote by

$$\mathcal{D}^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$$

and

$$\xi = \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\},$$

where $\{\xi_\alpha\}$, $\alpha = 1, 2, 3$, are the one-dimensional distributions spanned by structure vector fields ξ_α on M. Concerning these distributions, we have the following result.

Proposition 1 ([38]) *Let (M, g) be a semi-Riemannian hypersurface of a manifold $\overline{M}$ endowed with a metric mixed 3-structure $((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$, such that ξ_1, ξ_2, ξ_3 are tangent to M. Then we have the following assertions:*
(i) The distributions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are mutually orthogonal.
(ii) The distribution $\mathcal{D}^\perp$ is orthogonal to the distribution ξ.

We denote now by $\mathcal{D}$ the orthogonal complementary distribution to $\mathcal{D}^{\perp} \oplus \xi$ in TM. Thus we have the decomposition

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \xi$$

and we can easily remark that

$$\phi_{\alpha}(\mathcal{D}_{\alpha}) = TM^{\perp}, \ \alpha = 1, 2, 3$$

and

$$\phi_{\alpha}(\mathcal{D}_{\beta}) = \mathcal{D}_{\gamma},$$

for any cyclic permutation (α, β, γ) of $(1, 2, 3)$. Moreover, the distribution $\mathcal{D}$ is invariant with respect to each ϕ_{α}, i.e., $\phi_{\alpha}(\mathcal{D}) = \mathcal{D}$, $\alpha = 1, 2, 3$ (see [38]).

Concerning the integrability of the above distributions and the geometry of the induced foliations, we have the following theorems which extends in semi-Riemannian setting some results obtained in [7, 58] for hypersurfaces of manifolds endowed with Sasakian or hypercosymplectic 3-structures.

Theorem 13.6 ([38]) *Let (M, g) be a semi-Riemannian hypersurface of a mixed 3-Sasakian or mixed 3-cosymplectic manifold $(\overline{M}, (\phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha})_{\alpha=\overline{1,3}}, \overline{g})$, such that ξ_1, ξ_2, ξ_3 are tangent to M. Then the distribution ξ is integrable.*

Theorem 13.7 ([38]) *Let (M, g) be a semi-Riemannian hypersurface of a mixed 3-Sasakian or mixed 3-cosymplectic manifold $(\overline{M}, (\phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha})_{\alpha=\overline{1,3}}, \overline{g})$, such that ξ_1, ξ_2, ξ_3 are tangent to M. Then the following statements are equivalent.*
(i) M is $\mathcal{D}$-geodesic, i.e., $h(X, Y) = 0, \ \forall X, Y \in \Gamma(\mathcal{D})$.
(ii) The distribution $\mathcal{D} \oplus \xi$ is integrable.
(iii) The second fundamental form h of M satisfies

$$h(X, \phi_{\alpha} Y) = h(\phi_{\alpha} X, Y), \ \forall X, Y \in \Gamma(\mathcal{D}), \ \alpha = 1, 2, 3.$$

Theorem 13.8 ([38]) *Let (M, g) be a semi-Riemannian hypersurface of a mixed 3-cosymplectic manifold $(\overline{M}, (\phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha})_{\alpha=\overline{1,3}}, \overline{g})$, such that ξ_1, ξ_2, ξ_3 are tangent to M. Then we have:*
(i) The distribution $\mathcal{D} \oplus \mathcal{D}^{\perp}$ is integrable.
(ii) The distribution $\mathcal{D}$ is integrable if and only if:

$$g([X, Y], Z) = 0, \ \forall X, Y \in \Gamma(\mathcal{D}), \ Z \in \Gamma(\mathcal{D}^{\perp}).$$

(iii) The distribution $\mathcal{D}^{\perp}$ is integrable if and only if:

$$g([X, Y], Z) = 0, \ \forall X, Y \in \Gamma(\mathcal{D}^{\perp}), \ Z \in \Gamma(\mathcal{D}).$$

Theorem 13.9 ([38]) *If* (M, g) *is a semi-Riemannian hypersurface of a mixed 3-Sasakian manifold* $(\overline{M}, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$*, such that* ξ_1, ξ_2, ξ_3 *are tangent to* M*, then we have:*
(i) The distributions $\mathcal{D}_\alpha \oplus \xi_\alpha$*,* $\alpha = 1, 2, 3$*, are integrable.*
(ii) The distribution $\mathcal{D}^\perp$ *is not integrable.*
(iii) The distribution $\mathcal{D}$ *is not integrable, provided that dimension of* $\overline{M}$ *is strictly greater than 7.*

Theorem 13.10 ([38]) *Let* (M, g) *be a semi-Riemannian hypersurface of a mixed 3-Sasakian or mixed 3-cosymplectic manifold* $(\overline{M}, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$*, such that* ξ_1, ξ_2, ξ_3 *are tangent to* M*. Then the following statements are equivalent:*
(i) The distribution $\mathcal{D}^\perp \oplus \xi$ *is integrable.*
(ii) M *is* $(\mathcal{D}, \mathcal{D}^\perp)$*-geodesic, i.e.,* $h(X, Y) = 0$*,* $\forall X \in \Gamma(\mathcal{D})$*,* $Y \in \Gamma(\mathcal{D}^\perp)$*.*

Theorem 13.11 ([38]) *Let* (M, g) *be a semi-Riemannian hypersurface of a mixed 3-Sasakian or mixed 3-cosymplectic manifold* $(\overline{M}, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$*, such that* ξ_1, ξ_2, ξ_3 *are tangent to* M*. If* $\mathcal{D} \oplus \xi$ *is integrable, then its leaves are totally geodesic immersed in* $\overline{M}$*.*

Theorem 13.12 ([38]) *Let* (M, g) *be a semi-Riemannian hypersurface of a mixed 3-Sasakian or mixed 3-cosymplectic manifold* $(\overline{M}, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$*, such that* ξ_1, ξ_2, ξ_3 *are tangent to* M*. If* $\mathcal{D}^\perp \oplus \xi$ *is integrable, then the induced foliation* $\mathfrak{F}(\mathcal{D}^\perp \oplus \xi)$ *is totally geodesic, i.e., each leaf of the foliation is totally geodesic immersed in* M*. Moreover, if* $\overline{M}$ *is a mixed 3-Sasakian manifold, then any leaf is never totally geodesic immersed in* $\overline{M}$*.*

13.4.2 Semi-Riemannian Hypersurfaces of Manifolds Endowed with Metric Mixed 3-Structures, Normal to the Structure Vector Fields

Let $(\overline{M}, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ be a manifold with a metric mixed 3-structure and let (M, g) be a semi-Riemannian hypersurface of $\overline{M}$, such that ξ_i is normal to M, where i is settled in $\{1, 2, 3\}$. Then it is clear that $TM^\perp = \{\xi_i\}$ and ξ_j, ξ_k are tangent to M, provided that $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$. Moreover, if the metric mixed 3-structure is positive, then the sign of the hypersurface is $\varepsilon_M = +1$, provided that $i = 1$ or $i = 2$, and $\varepsilon_M = -1$, provided that $i = 3$. Analogous, if the metric mixed 3-structure is negative, then the sign of the hypersurface is $\varepsilon_M = +1$, provided that $i = 3$, and $\varepsilon_M = -1$, provided that $i = 1$ or $i = 2$.

Next, we settle j and k such that (i, j, k) to become an even permutation of $(1, 2, 3)$. We consider the distribution $\xi_{jk} = \{\xi_j\} \oplus \{\xi_k\}$ and let $\xi_{jk}^\perp$ be the orthogonal complementary distribution to ξ_{jk} in TM. We have the decomposition $TM = \xi_{jk} \oplus \xi_{jk}^\perp$ and we can easily remark that $\phi_i(\xi_{jk}) = \xi_{jk}$. Moreover, the distribution $\xi_{jk}^\perp$ is invariant by each ϕ_α, $\alpha = 1, 2, 3$ (see [38]) and we can prove the following results.

Proposition 2 ([38]) *If (M, g) is a semi-Riemannian hypersurface of a mixed 3-Sasakian or mixed 3-cosymplectic manifold $(\overline{M}, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ such that ξ_i is normal to M, then the distribution ξ_{jk} is integrable.*

Theorem 13.13 *Let (M, g) be a semi-Riemannian hypersurface of a manifold $\overline{M}$ endowed with a metric mixed 3-structure $((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ such that ξ_i is normal to M.*
(i) If $(\overline{M}, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ is a mixed 3-cosymplectic manifold then the distribution $\xi_{jk}^\perp$ is integrable. Moreover each leaf of $\xi_{jk}^\perp$ is totally geodesic immersed in $\overline{M}$.
(ii) If $(\overline{M}, (\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ is a mixed 3-Sasakian manifold then the distribution $\xi_{jk}^\perp$ is never integrable, provided that dimension of $\overline{M}$ is strictly greater than 3.

Remark 2 We note that non-degenerate invariant and anti-invariant submanifolds in manifolds endowed with mixed 3-structures were recently studied in [46], the relevant ambient being mixed 3-Sasakian and mixed 3-cosymplectic.

13.5 Paraquaternionic CR-submanifolds

Let N be an n-dimensional non-degenerate submanifold of an almost paraquaternionic hermitian manifold (M, σ, g). We say that (N, g) is a *paraquaternionic CR-submanifold* of M if there exists a non-degenerate distribution $\mathcal{D} : x \to \mathcal{D}_x \subseteq T_xN$ such that on any $U \cap N$ we have (see [44])
(i) $\mathcal{D}$ is a *paraquaternionic* distribution, i.e.,

$$J_\alpha \mathcal{D}_x = \mathcal{D}_x, \ \alpha \in \{1, 2, 3\}$$

and
(ii) $\mathcal{D}^\perp$ is a *totally real* distribution, i.e.,

$$J_\alpha \mathcal{D}_x^\perp \subset T_x^\perp N, \ \alpha \in \{1, 2, 3\}$$

for any local basis $\{J_1, J_2, J_3\}$ of σ on U and $x \in U \cap N$, where $\mathcal{D}^\perp$ is the orthogonal complementary distribution to $\mathcal{D}$ in TN.

A non-degenerate submanifold N of an almost paraquaternionic hermitian manifold (M, σ, g) is called a *paraquaternionic* (respectively, *totally real*) *submanifold* if $\mathcal{D}^\perp = 0$ (respectively, $\mathcal{D} = 0$). A paraquaternionic CR-submanifold is said to be *proper* if it is neither paraquaternionic nor totally real.

We remark that other important classes of submanifolds in paraquaternionic Kähler manifolds have been recently investigated: Kähler and para-Kähler submanifolds in [4, 49, 60], normal semi-invariant submanifolds in [1, 8, 62], anti-invariant submanifolds in [26], and lightlike submanifolds in [37, 40].

Example 1 (i) The canonical immersion of $P^n\widetilde{\mathbb{H}}(c)$ into $P^m\widetilde{\mathbb{H}}(c)$, where $n \leq m$, provides us a very natural example of paraquaternionic submanifold (see [51]).

(ii) The real projective space $P^n_s\mathbb{R}(\frac{c}{4})$ is a totally real submanifold of the paraquaternionic projective space $P^n\widetilde{\mathbb{H}}(c)$, where $s \in \{0, \ldots, n\}$ denotes the index of the manifold, defined as the dimension of the largest negative definite vector subspace of the tangent space.
(iii) Let (M_1, g_1, σ_1) and (M_2, g_2, σ_2) be two paraquaternionic Kähler manifolds. If U_1 and U_2 are open subsets of M_1 and M_2, respectively, on which local basis $\{J_1^{(1)}, J_2^{(1)}, J_3^{(1)}\}$ and $\{J_1^{(2)}, J_2^{(2)}, J_3^{(2)}\}$ for σ_1 and σ_2, respectively, are defined, then the product manifold $U = U_1 \times U_2$ can be endowed with an almost paraquaternionic hermitian non-Kähler structure (g, σ) (see [61]). Now, if N_1 is a paraquaternionic submanifold of U_1 and N_2 is a totally real submanifold of U_2, then $N = N_1 \times N_2$ is a proper paraquaternionic *CR*-submanifold of the almost paraquaternionic hermitian manifold (U, g, σ).
(iv) A large class of examples of proper paraquaternionic CR-submanifolds can be constructed using the paraquaternionic momentum map [67] and the technique from [55]. Suppose that a Lie group G acts freely and isometrically on the paraquaternionic Kähler manifold (M, σ, g), preserving the fundamental 4-form Ω of the manifold. We denote by $\mathfrak{g}$ the Lie algebra of G, by $\mathfrak{g}^*$ its dual and by V the unique Killing vector field corresponding to a vector $V^* \in \mathfrak{g}$. Then there exists a unique section f of bundle $\mathfrak{g}^* \otimes \sigma$ such that (see [67])

$$\nabla f_{V^*} = \theta_{V^*},$$

for all $V^* \in \mathfrak{g}$, where the section θ_{V^*} of the bundle $\Omega^1(\sigma)$ with values in σ is well defined globally by

$$\theta_{V^*}(X) = \sum_{\alpha=1}^{3} \omega_\alpha(V, X) J_\alpha, \ \forall X \in TM.$$

Moreover, the group G acts by isometries on the pre-image $f^{-1}(0)$ of the zero-section $0 \in \mathfrak{g}^* \otimes \sigma$. Similarly as in [55], we have the decomposition

$$T_x(f^{-1}(0)) = T_x(G \cdot x) \oplus H_x, \forall x \in M,$$

where $G \cdot x$ represents the orbit of G through x, supposed to be non-degenerate, and H_x is the orthogonal complementary subspace of $T_x(G \cdot x)$ in $T_x(f^{-1}(0))$. Because H_x is invariant under the action of σ and $T_x(G \cdot x)$ is totally real, we can state now the following result.

Proposition 3 ([44]) *If $f^{-1}(0)$ is a smooth submanifold of a paraquaternionic Kähler manifold (M, σ, g), then $f^{-1}(0)$ is a proper paraquaternionic CR-submanifold of M.*

We remark that, in general, $f^{-1}(0)$ is not a differentiable submanifold of M, but always we can take a subset $N \subset f^{-1}(0)$ which is invariant under the action of G and which is a submanifold of M. A particular example is given in [15], as a paraquaternionic version of the example constructed by Galicki and Lawson in [31]: if p and q are distinct and relatively prime natural numbers, then we have the action of the Lie group $G = \{e^{jt} | t \in \mathbb{R}\}$ on $P^2\widetilde{\mathbb{H}}$ defined by

$$\phi_{p,q}(t) \cdot [u_0, u_1, u_2] := [e^{jqt}u_0, e^{jpt}u_1, e^{jpt}u_2],$$

where $e^{jt} = \cosh t + j\sinh t$ and $[u_0, u_1, u_2]$ are homogenous coordinates on $P^2\widetilde{\mathbb{H}}$. We can see that this action is free, isometric, and preserves the paraquaternionic structure on $P^2\widetilde{\mathbb{H}}$ and, moreover, we have that the pre-image by the momentum map $f_{p,q} : P^2\widetilde{\mathbb{H}} \to \mathrm{Im}\widetilde{\mathbb{H}}$ of the zero-section $0 \in \mathrm{Im}\widetilde{\mathbb{H}}$ is (see also [67]):

$$f_{p,q}^{-1}(0) = \{[u_0, u_1, u_2] \in P^2\widetilde{\mathbb{H}} | q\bar{u}_0 j u_0 + p\bar{u}_1 j u_1 + p\bar{u}_2 j u_2 = 0\}.$$

Finally, we conclude that the subset N of the regular points of $f_{p,q}^{-1}(0)$, given by

$$N = \{[u_0, u_1, u_2] \in f_{p,q}^{-1}(0) | q^2|u_0|^2 + p^2|u_1|^2 + p^2|u_2|^2 \neq 0\}$$

is a proper paraquaternionic CR-submanifold of $P^2\widetilde{\mathbb{H}}$.

Let now N be a paraquaternionic CR-submanifold of an almost paraquaternionic hermitian manifold (M, σ, g). Then we say that:
(i) N is $\mathcal{D}$-geodesic if $h(X, Y) = 0,\ \forall X, Y \in \Gamma(\mathcal{D})$;
(ii) N is $\mathcal{D}^\perp$-geodesic if $h(X, Y) = 0,\ \forall X, Y \in \Gamma(\mathcal{D}^\perp)$;
(iii) N is mixed geodesic if $h(X, Y) = 0,\ \forall X \in \Gamma(\mathcal{D}), Y \in \Gamma(\mathcal{D}^\perp)$;
(iv) N is mixed foliated if N is mixed geodesic and $\mathcal{D}$ is integrable.

Since any paraquaternionic submanifold of a paraquaternionic Kähler manifold is a totally geodesic paraquaternionic Kähler submanifold, we deduce the next property.

Proposition 4 ([44])

(1) Let (N, g) be a paraquaternionic submanifold of a paraquaternionic Kähler manifold (M, σ, g). Then:
(i) dim$N = 4n, n \geq 1$ and the signature of $g|_{TN}$ is (2n, 2n);
(ii) N is an Einstein manifold, provided that dim$N > 4$.
(2) The paraquaternionic submanifolds of $\mathbb{R}^{4m}_{2m}$ and of paraquaternionic projective space $P^m\widetilde{\mathbb{H}}$ are locally isometric with $\mathbb{R}^{4n}_{2n}$ and $P^n\widetilde{\mathbb{H}}$, respectively, where $n \leq m$.

Next, let (N, g) be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M, σ, g). We put $\nu_{\alpha x} = J_\alpha(D_x^\perp),\ \alpha \in \{1, 2, 3\}$ and $\nu_x^\perp = \nu_{1x} \oplus \nu_{2x} \oplus \nu_{3x}$, and remark that $\nu_{1x},\ \nu_{2x},\ \nu_{3x}$ are mutually orthogonal non-degenerate vector subspaces of $T_xN^\perp$, for any $x \in U \cap N$. We also note that the subspaces $\nu_{\alpha x}$ depend on the choice of the local base $(J_\alpha)_\alpha$, while $\nu_x^\perp$ does not depend on it. Moreover, we can prove the following result:

Proposition 5 ([44]) *Let (N, g) be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M, σ, g). Then we have:*

(i) $J_\alpha(\nu_{\alpha x}) = \mathcal{D}_x^\perp, \ \forall x \in U \cap N, \ \alpha \in \{1, 2, 3\}$*;*
(ii) $J_\alpha(\nu_{\beta x}) = \nu_{\gamma x}$*, for any even permutation* (α, β, γ) *of* $(1, 2, 3)$ *and* $x \in U \cap N$*;*
(iii) The mapping

$$\nu^\perp : x \in N \to \nu_x^\perp$$

defines a non-degenerate distribution of dimension $3s$*, where* $s = dim D_x^\perp$*;*
(iv) $J_\alpha(\nu_x) = \nu_x, \ \forall x \in U \cap N, \ \alpha \in \{1, 2, 3\}$*, where* ν *is the complementary orthogonal distribution to* $\nu^\perp$ *in* $TN^\perp$*.*

Concerning the integrability of the distributions involved in the definition of the paraquaternionic CR-submanifolds, we have the following results.

Theorem 13.14 ([44]) *The distribution* $\mathcal{D}^\perp$ *is integrable.*

Theorem 13.15 ([44]) *The paraquaternionic distribution* $\mathcal{D}$ *is integrable if and only if* N *is* $\mathcal{D}$*-geodesic.*

Moreover, we can prove the following.

Theorem 13.16 ([44]) *The paraquaternionic distribution* $\mathcal{D}$ *is minimal.*

Theorem 13.17 ([44]) *Let* N *be a closed paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold* (M, σ, g)*. Then the* $4r$*-form* ω *is closed and defines a canonical de Rham cohomology class* $[\omega]$ *in* $H^{4r}(M, \mathbb{R})$*. Moreover, this cohomology class is nontrivial if* $\mathcal{D}$ *is integrable and* $\mathcal{D}^\perp$ *is minimal.*

Theorem 13.18 ([44]) *Let* N *be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold* (M, σ, g)*. The next assertions are equivalent:*
(i) The foliation $\mathfrak{F}^\perp$ *induced by* $\mathcal{D}^\perp$ *is totally geodesic;*
(ii) $h(X, Y) \in \Gamma(\nu), \forall X \in \Gamma(\mathcal{D}), Y \in \Gamma(\mathcal{D}^\perp)$*;*
(iii) $A_N X \in \Gamma(\mathcal{D}^\perp), \forall X \in \Gamma(\mathcal{D}^\perp), N \in \Gamma(\nu^\perp)$*;*
(iv) $A_N Y \in \Gamma(\mathcal{D}), \forall Y \in \Gamma(\mathcal{D}), N \in \Gamma(\nu^\perp)$*.*

A submanifold N of a semi-Riemannian manifold (M, g) is said to be a *ruled submanifold* if it admits a foliation whose leaves are totally geodesic submanifolds immersed in (M, g) [9, 63].

A paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold which is a ruled submanifold with respect to the foliation $\mathfrak{F}^\perp$ is called *totally real ruled paraquaternionic CR-submanifold*. We have next characterization of totally real ruled paraquaternionic CR-submanifolds which extends in paraquaternionic setting a previous result obtained in [41] for quaternionic CR-submanifolds.

Theorem 13.19 ([44]) *Let* N *be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold* (M, σ, g)*. Then the following assertions are mutually equivalent:*
(i) N *is a totally real ruled paraquaternionic CR-submanifold.*
(ii) N *is* $\mathcal{D}^\perp$*-geodesic and:*

$$B(X, Y) \in \Gamma(\nu), \ \forall X \in \Gamma(\mathcal{D}), \ Y \in \Gamma(\mathcal{D}^{\perp}).$$

(iii) The subbundle $\nu^{\perp}$ is $\mathcal{D}^{\perp}$-parallel, i.e.,

$$\nabla^{\perp}_X J_\alpha Z \in \Gamma(\nu^{\perp}), \ \forall X, Z \in \Gamma(\mathcal{D}^{\perp}), \ \alpha \in \{1, 2, 3\}$$

and the second fundamental form satisfies:

$$B(X, Y) \in \Gamma(\nu),$$

for all $X \in \Gamma(\mathcal{D}^{\perp}), \ Y \in \Gamma(TN)$.
(iv) The shape operator satisfies:

$$A_{J_\alpha Z} X = 0, \ \forall X, Z \in \Gamma(\mathcal{D}^{\perp}), \ \alpha \in \{1, 2, 3\}$$

and

$$A_N X \in \Gamma(\mathcal{D}), \ \forall X \in \Gamma(\mathcal{D}^{\perp}), \ N \in \Gamma(\nu).$$

Remark 3 It is known that semi-Riemannian submersions were introduced by O'Neill [54]. Let (M, g) and (M', g') be two connected semi-Riemannian manifolds of index s $(0 \leq s \leq dimM)$ and s' $(0 \leq s' \leq dimM')$, respectively, with $s' \leq s$. A semi-Riemannian submersion is a smooth map $\pi : M \to M'$ which is onto and satisfies the following conditions:

(i) $\pi_{*|p} : T_pM \to T_{\pi(p)}M'$ is onto for all $p \in M$;
(ii) The fibers $\pi^{-1}(p'), \ p' \in M'$, are semi-Riemannian submanifolds of M;
(iii) π_* preserves scalar products of vectors normal to fibers.

The vectors tangent to fibers are called *vertical* and those normal to fibers are called *horizontal*. We denote by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution, and by v and h the vertical and horizontal projection. An horizontal vector field X on M is said to be *basic* if X is π-related to a vector field X' on M'. It is clear that every vector field X' on M' has a unique horizontal lift X to M and X is basic.

Remark 4 A semi-Riemannian submersion $\pi : M \to M'$ determines, as well as in the Riemannian case (see [30]), two $(1, 2)$ tensor fields T and A on M, by the formulas:

$$T(E, F) = h\nabla_{vE} vF + v\nabla_{vE} hF$$

and

$$A(E, F) = v\nabla_{hE} hF + h\nabla_{hE} vF$$

respectively, for any $E, F \in \Gamma(TM)$. We remark that for $U, V \in \Gamma(\mathcal{V})$, $T(U, V)$ coincides with the second fundamental form of the immersion of the fiber submanifolds and for $X, Y \in \Gamma(\mathcal{H})$, $A(X, Y) = \frac{1}{2}v[X, Y]$ characterizes the complete integrability of the horizontal distribution $\mathcal{H}$.

Remark 5 In [47], S. Kobayashi observed the next similarity between a Riemannian submersion and a CR-submanifold of a Kähler manifold: both involve two distributions (the vertical and horizontal distribution), one of them being integrable. Then he introduced the concept of CR-submersion, as a Riemannian submersion from a CR-submanifold to an almost hermitian manifold. Next, we'll consider CR-submersions from paraquaternionic CR-submanifolds of a paraquaternionic Kähler manifold.

Let N be a paraquaternionic CR-submanifold of an almost paraquaternionic hermitian manifold (M, σ, g) and (M', σ', g') be an almost hermitian manifold. A semi-Riemannian submersion $\pi : N \to M'$ is said to be a paraquaternionic CR-submersion if the following conditions are satisfied [44]

(i) $\mathcal{V} = \mathcal{D}^{\perp}$;

(ii) For each $p \in N$, $\pi_* : \mathcal{D}_p \to T_{\pi(p)}M'$ is an isometry with respect to each complex and product structure of $\mathcal{D}_p$ and $T_{\pi(p)}M'$, where $T_{\pi(p)}M'$ denotes the tangent space to M' at $\pi(p)$.

Example 2 We remarked below that

$$N = \{[u_0, u_1, u_2] \in f_{p,q}^{-1}(0) | q^2|u_0|^2 + p^2|u_1|^2 + p^2|u_2|^2 \neq 0\}$$

is a proper paraquaternionic *CR*-submanifold of $P^2\widetilde{\mathbb{H}}$. Moreover, the Lie group $G = \{e^{jt} | t \in \mathbb{R}\}$ acts freely and isometrically on N. Now using the paraquaternionic Kähler reduction (see Theorem 5.2 from [67]), we obtain that the manifold $M' = N/G$ equipped with the submersed metric (i.e., the one g' which makes the projection $\pi : (N, g) \to (M', g')$ a semi-Riemannian submersion) is again a paraquaternionic Kähler manifold with respect to the structure σ' induced on M' from the structure σ by the projection π. Moreover, $\pi : N \to N/G$ is a paraquaternionic CR-submersion.

The main properties of the paraquaternionic CR-submersions are as follows.

Theorem 13.20 ([44]) *Let N be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M, σ, g) and (M', σ', g') be an almost paraquaternionic hermitian manifold. If $\pi : N \to M'$ is a paraquaternionic CR-submersion, then (M', σ', g') is a paraquaternionic Kähler manifold.*

Theorem 13.21 ([44]) *Let N be a mixed foliated paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M, σ, g) and (M', σ', g') be an almost paraquaternionic hermitian manifold. If $\pi : N \to M'$ is a paraquaternionic CR-submersion, then N is locally a semi-Riemannian product of a paraquaternionic submanifold and a totally real submanifold of M. In particular, if N is complete and simply connected then it is a global semi-Riemannian product.*

Remark 6 Let (M, g) be a semi-Riemannian manifold. The sectional curvature K of a 2-plane in T_pM, $p \in M$, spanned by $\{X, Y\}$, is defined by:

$$K(X,Y)=\frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2}. \tag{13.13}$$

It is clear that the above definition makes sense only for non-degenerate planes, i.e., those satisfying $Q(X,Y)=g(X,X)g(Y,Y)-g(X,Y)^2\neq 0$.

The main curvature properties of the paraquaternionic CR-submersions are the following.

Theorem 13.22 ([44]) *Let N be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M,σ,g) and (M',σ',g') be an almost paraquaternionic hermitian manifold. If $\pi:N\to M'$ is a paraquaternionic CR-submersion, then the sectional curvatures of M and the fibers are related by:*

$$\begin{aligned}\overline{K}(U,V)&=\hat{K}(U,V)-\epsilon_\alpha\theta_U\theta_V[g(A_{J_\alpha U}U,A_{J_\alpha V}V)-g(A_{J_\alpha V}U,A_{J_\alpha V}U)]\\&-\epsilon_\alpha\theta_U\theta_V[g(\overline{h}\nabla^\perp_U J_\alpha U,\overline{h}\nabla^\perp_V J_\alpha V)-g(\overline{h}\nabla^\perp_U J_\alpha V,\overline{h}\nabla^\perp_U J_\alpha V)]\end{aligned}$$

for any unit spacelike or timelike orthogonally vector fields $U,V\in\Gamma(\mathcal{D}^\perp)$ and $\alpha\in\{1,2,3\}$, where $\theta_U=g(U,U)\in\{-1,1\}$ and $\theta_V=g(V,V)\in\{-1,1\}$.

Theorem 13.23 ([44]) *Let N be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M,σ,g) and (M',σ',g') be an almost paraquaternionic hermitian manifold. If $\pi:N\to M'$ is a paraquaternionic CR-submersion, then for any unit spacelike or timelike horizontal vector field X one has:*

$$\overline{H}_\alpha(X)=H'_\alpha(\pi_*X)-4g(\overline{v}B(X,X),\overline{v}B(X,X))+2g(\overline{h}B(X,X),\overline{h}B(X,X)),$$

for $\alpha\in\{1,2,3\}$, where $\overline{H}_\alpha$ and H'_α are the holomorphic sectional curvatures of M and M', defined by $\overline{H}_\alpha(X)=\overline{K}(X,J_\alpha X)$ and $H'_\alpha(X)=K'(X,J_\alpha X)$, respectively.

Corollary 1 ([44]) *Let N be a totally geodesic paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold (M,σ,g) and (M',σ',g') be an almost paraquaternionic hermitian manifold. If $\pi:N\to M'$ is a paraquaternionic CR-submersion one has:*

$$\overline{H}_\alpha(X)=H'_\alpha(\pi_*X),$$

for any unit spacelike or timelike horizontal vector field X.

Remark 7 It is known that the natural product of two paraquaternionic Kähler manifolds does not become a paraquaternionic Kähler manifold, but it is an almost paraquaternionic Kähler product manifold (see [61]). Motivated by this result, the concept of paraquaternionic CR-submanifolds was introduced recently in a more general setting, namely for a paraquaternionic Kähler product manifold in [65].

13.6 Mixed Paraquaternionic 3-Submersions

Let $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ be a manifold with a metric mixed 3-structure and (M', σ, g') be an almost paraquaternionic hermitian manifold. Then a semi-Riemannian submersion $\pi : M \to M'$ is called a *mixed paraquaternionic 3-submersion* (see [64]) if for any $p \in M$ there exists a canonical local basis $\{J'_\alpha\}_{\alpha=\overline{1,3}}$ of σ on an open neighborhood U of $\pi(p)$ such that

$$A^{\alpha\beta}\left(J'_\beta\right)_{\pi(y)} \circ \pi_{*,y} \circ (\varphi_\alpha)_y = \pi_{*,y}$$

for any $y \in \pi^{-1}(U)$, where $A^{\alpha\beta}$ are the entries of a matrix A in SO(2, 1).

Remark 8 Since SO(2, 1) preserves the identities (13.1) and (13.4)–(13.7), we can choose in the above definition the local basis $\{J'_\alpha\}_{\alpha=\overline{1,3}}$ such that $A^{\alpha\beta} = -\tau_\alpha\delta_{\alpha\beta}$. In the sequel we shall assume this. Therefore, we can suppose that for each $p \in M$, $\sigma_{\pi(p)}$ admits a canonical local basis $\{J'_\alpha\}_{\alpha=\overline{1,3}}$ such that:

$$\pi_*\varphi_\alpha = J'_\alpha\pi_*, \quad \alpha = 1, 2, 3.$$

Concerning the transference of structures and the geometry of the fibers, we have the following results.

Theorem 13.24 ([64]) *Let $\pi : (M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g) \to (M', \sigma, g')$ be a mixed paraquaternionic 3-submersion. If the total space of the submersion is mixed 3-cosymplectic or mixed 3-Sasakian then*

(i) the base space is a locally para-hyper-Kähler manifold;
(ii) the fibers inherit the structure of the total space;
(iii) the fibers are totally geodesic immersed.

Theorem 13.25 ([64]) *Let $\pi : (M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g) \to (M', \sigma, g')$ be a mixed paraquaternionic 3-submersion. If the total space of the submersion is mixed 3-cosymplectic then the horizontal distribution is integrable.*

Taking account of Vilms Theorem (which states that a semi-Riemannian submersion is a harmonic map if and only if its fibers are minimal submanifolds [25]), we deduce from Theorem 13.24 the following result.

Theorem 13.26 ([64]) *Any mixed paraquaternionic 3-submersion from a mixed 3-cosymplectic or mixed 3-Sasakian manifold is a harmonic map.*

Remark 9 Let $\pi : (M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g) \to (M', \sigma, g')$ be a mixed paraquaternionic 3-submersion. We denote by B_{φ_α} the φ_α-(para-)holomorphic bisectional curvature, defined for any pair of nonzero non-lightlike orthogonal vector fields E and G on M by

$$B_{\varphi_\alpha}(E, G) = \frac{R(E, \varphi_\alpha E, G, \varphi_\alpha G)}{||E||^2||G||^2}.$$

On the other hand, the φ_α-(para-)holomorphic sectional curvature is defined for any nonzero non-lightlike vector field E on M by $H_{\varphi_\alpha}(E) = K(E, \varphi_\alpha E)$.

We denote by $\hat{B}_{\varphi_\alpha}$ and $\hat{H}_{\varphi_\alpha}$ the bisectional and the sectional (para-) holomorphic curvatures of a fiber. On the other hand, for each of the three-structures $\{J_1, J_2, J_3\}$ on the almost paraquaternionic hermitian manifold M', we consider the (para)-holomorphic bisectional curvature tensor $B'_{J'_\alpha}$ and (para-)holomorphic sectional curvature tensor $H'_{J'_\alpha}$ defined in the usual way for almost (para-)hermitian manifolds.

The following theorem is a translation of the results of Gray [34] and O'Neill [53] to the present situation.

Theorem 13.27 ([64]) *Let* $\pi : (M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g) \to (M', \sigma, g')$ *be a mixed paraquaternionic 3-submersion. Let X and Y be unit spacelike or timelike horizontal vector fields, and U and V be unit spacelike or timelike vertical vector fields, orthogonal to* ξ_α. *Then, we have:*

(i) $B_{\varphi_\alpha}(U, V) = \hat{B}_{\hat{\varphi}_\alpha}(U, V) - \theta_U\theta_V[g(T_U V, T_{\varphi_\alpha U}\varphi_\alpha V) - g(T_U\varphi_\alpha V, T_{\varphi_\alpha U} V)]$;
(ii)

$$\begin{aligned} B_{\varphi_\alpha}(X, U) = \theta_X\theta_U[&g((\nabla_U A)_X(\varphi_\alpha X), \varphi_\alpha U) - g((\nabla_{\varphi_\alpha U} A)_X(\varphi_\alpha X), U) \\ &+g(A_X U, A_{\varphi_\alpha X}\varphi_\alpha U) - g(A_X\varphi_\alpha U, A_{\varphi_\alpha X} U) \\ &-g(T_U X, T_{\varphi_\alpha U}\varphi_\alpha X) + g(T_U\varphi_\alpha X, T_{\varphi_\alpha U} X)]; \end{aligned}$$

(iii)

$$\begin{aligned} B_{\varphi_\alpha}(X, Y) = B'_{J'_\alpha}(\pi_* X, \pi_* Y) - \theta_X\theta_Y[&2g(A_X\varphi_\alpha X, A_Y\varphi_\alpha Y) \\ &-g(A_{\varphi_\alpha X} Y, A_X\varphi_\alpha Y) + g(A_X Y, A_{\varphi_\alpha X}\varphi_\alpha Y)], \end{aligned}$$

for $\alpha = 1, 2, 3,$ *where* $\theta_U = g(U, U) \in \{\pm 1\}$, $\theta_V = g(V, V) \in \{\pm 1\}$, $\theta_X = g(X, X) \in \{\pm 1\}$ *and* $\theta_Y = g(Y, Y) \in \{\pm 1\}$.

Using Theorem 13.27, we obtain the following curvature properties of mixed paraquaternionic 3-submersions.

Theorem 13.28 ([64]) *Let* $\pi : (M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g) \to (M', \sigma, g')$ *be a mixed paraquaternionic 3-submersion. If X is a unit spacelike or timelike horizontal vector field and U is a unit spacelike or timelike vertical vector field orthogonal to* ξ_α, *then we have for* $\alpha = 1, 2, 3$:

(i) $H_{\varphi_\alpha}(U) = \hat{H}_{\hat{\varphi}_\alpha}(U) - \tau_\alpha[g(T_U U, T_{\varphi_\alpha U}\varphi_\alpha U) - g(T_U\varphi_\alpha U, T_{\varphi_\alpha U} U)]$;
(ii) $H_{\varphi_\alpha}(X) = H'_{J'_\alpha}(\pi_* X) - 3\tau_\alpha||A_X\varphi_\alpha X||^2$.

Theorem 13.29 ([64]) *Let* $\pi : (M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g) \to (M', \sigma, g')$ *be a mixed paraquaternionic 3-submersion. If M is mixed 3-cosymplectic, then we have*

$$H_{\varphi_\alpha}(U) = \hat{H}_{\hat{\varphi}_\alpha}(U),$$
$$H_{\varphi_\alpha}(X) = H'_{J'_\alpha}(\pi_* X),$$

for any unit spacelike or timelike horizontal vector field X and any unit spacelike or timelike vertical vector field U orthogonal to ξ_α*, where* $\alpha = 1, 2, 3$.

Theorem 13.30 ([64]) *Let* $\pi : (M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g) \to (M', \sigma, g')$ *be a mixed paraquaternionic 3-submersion. If M is mixed 3-Sasakian, then we have for any unit spacelike or timelike horizontal vector field X and any unit spacelike or timelike vertical vector field U orthogonal to* ξ_α*:*

$$H_{\varphi_\alpha}(U) = \hat{H}_{\hat{\varphi}_\alpha}(U),$$
$$H_{\varphi_\alpha}(X) = H'_{J'_\alpha}(\pi_* X) - 3\varepsilon,$$

with $\varepsilon = \mp 1$*, according as the metric mixed 3-structure is positive or negative, respectively, and* $\alpha = 1, 2, 3$.

Finally, we give some examples of mixed paraquaternionic 3-submersions.

Example 3 ([64]) We consider $M' = \mathbb{R}^4_2 \cong \widetilde{\mathbb{H}}$ equipped with the canonical para-hyper Kähler structure $(H = (J_\alpha)_{\alpha=1,2,3}, g')$ and we take $M = \mathbb{R}^{11}_s$ ($s = 5$, respectively, 6) endowed with the standard positive (negative) mixed 3-cosymplectic structure $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}, g)$. Then the map $\pi : M \to M'$ defined by

$$\pi(x_1, x_2, x_3, \dots, x_{11}) = \left(\frac{x_4 + x_8}{\sqrt{2}}, \frac{x_6 + x_{10}}{\sqrt{2}}, \frac{x_7 + x_{11}}{\sqrt{2}}, \frac{x_5 + x_9}{\sqrt{2}}\right),$$

is a mixed paraquaternionic 3-submersion.

Example 4 ([64]) Let $(M', (J'_\alpha)_{\alpha=1,2,3}, g')$ be an almost para-hyperhermitian manifold of dimension $4n$ and let $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}, g)$ be a manifold endowed with a metric mixed 3-structure having dimension $4m + 3$. Then we can obtain a mixed 3-structure on the product manifold $\bar{M} = M' \times M$ by setting

$$\bar{\varphi}_\alpha = \begin{pmatrix} J'_\alpha & 0 \\ 0 & \varphi_\alpha \end{pmatrix}, \; \bar{\xi}_\alpha = \frac{m+n}{m}\begin{pmatrix} 0 \\ \xi_\alpha \end{pmatrix}, \; \bar{\eta}_\alpha = \frac{m}{m+n}(0, \; \eta_\alpha),$$

for $\alpha = 1, 2, 3$.

If we define now a semi-Riemannian metric $\bar{g}$ on $\bar{M}$ by

$$\bar{g}((E', E), (G', G)) := g'(E', G') + \frac{(m+n)^2}{m^2} g(E, G),$$

for all $E', G' \in \Gamma(TM')$ and $E, G \in \Gamma(TM)$, then a direct computation show that $\bar{g}$ is compatible with the mixed 3-structure $(\bar{\varphi}_\alpha, \bar{\xi}_\alpha, \bar{\eta}_\alpha)_{\alpha=1,2,3}$ on $\bar{M} = M' \times M$.

Moreover, it can be easily proved that $\bar{M}$ is mixed 3-cosymplectic if and only if M is mixed 3-cosymplectic and M' is para-hyper-Kähler.

We consider now the canonical projection $\pi_1 : M' \times M \to M'$ and we remark that we have

$$\pi_{1*}\bar{\varphi}_\alpha(E', E) = \pi_{1*}(J'_\alpha E', \varphi_\alpha E) = J'_\alpha E'.$$

On the other hand, it is obvious that

$$J'_\alpha \pi_{1*}(E', E) = J'_\alpha E'.$$

Thus, we have $\pi_{1*}\bar{\varphi}_\alpha = J'_\alpha \pi_{1*}$, for $\alpha = 1, 2, 3$, and therefore we deduce that π_1 is a mixed paraquaternionic 3-submersion.

Example 5 ([64]) Let $(M', H' = (J'_\alpha)_{\alpha=1,2,3}, g')$ be an almost para-hyperhermitian manifold and let (TM', π, M') its tangent bundle. Then it is known from [20] that the almost para-hypercomplex structure H' determines an almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=1,2,3}$ on TM' such that (TM', H, G) is an almost para-hyperhermitian manifold, where G is the Sasaki metric on TM'. Moreover, the canonical projection $\pi : TM' \to M'$ is a semi-Riemannian submersion which commutes with the structure tensor fields of type (1, 1), i.e.,

$$\pi_* J_\alpha = J'_\alpha \pi_*, \ \alpha = 1, 2, 3 \tag{13.14}$$

and if we suppose that M' is para-hyper-Kähler with g' flat, then it follows that TM' is also a para-hyper-Kähler manifold.

Next, let $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}, g)$ be a manifold endowed with a mixed 3-cosymplectic structure, then we can construct, as in the previous example, a mixed 3-cosymplectic structure $((\bar{\varphi}_\alpha, \bar{\xi}_\alpha, \bar{\eta}_\alpha)_{\alpha=1,2,3}, \bar{g})$ on the product manifold $\bar{M} = TM' \times M$. If we define the map $F : \bar{M} \to M'$ by

$$F(E, E') = \pi(E),$$

for all $E \in \Gamma(TTM')$ and $E' \in \Gamma(TM)$, then using (13.14) it follows that

$$F_*\bar{\varphi}_\alpha = J'_\alpha F_*, \ \alpha = 1, 2, 3.$$

Therefore F is a mixed paraquaternionic 3-submersion.

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