Simple Motion Evasion Differential Game of One Pursuer and One Evader

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Abstract We study an evasion differential game of simple motion, involving one pursuer and one evader in the plane $\mathbb{R}^2$. The control functions of players are subjected to geometric constraints. Maximum speed of the pursuer is equal to 1, and maximal speed of the evader is $\alpha > 1$. Control set of the evader is a sector **S** whose radius is greater than 1. We say that evasion is possible if the state of the evader does not coincide with that of the pursuer at all times. The problem is to find the conditions of evasion. We obtained conditions that guarantee the evasion regardless of the location of initial position of players.

Keywords Differential game · Geometric constraint · Evasion · Control · Strategy

1 Introduction

The multiple players pursuit–evasion differential games of simple motion with geometric or integral constraints are widely studied in many works (see eg. [1–12]). For the case of many pursuers against one evader, the game is described by the following differential equations:

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad |u_i| \le \rho_i, \quad i = 1, \ldots, m,$$
$$\dot{y} = v, \quad y(0) = y_0, \quad |v| \le \sigma,$$

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where $x_i, y, u_i, v \in \mathbb{R}^n$, $\rho_i \geq 0$, $\sigma > 0$, $x_{i0} \neq y_0$, and $i = 1, \ldots, m$. Here, pursuit is said to be completed if $x_i(\tau) = y(\tau)$ for some $i \in \{1, \ldots, m\}$ and $\tau \geq 0$, and evasion is said to be possible if $x_i(t) \neq y(t)$ for all $i = 1, \ldots, m$, and $t \geq 0$.

One of the earliest work of evasion differential game in the case of many pursuers versus one evader was by Pshenichnii [11]. In his work, the maximum speed of each player is equal to 1. The necessary and sufficient condition of evasion was obtained where evasion is possible if and only if $y_0 \notin intconv\{x_{10}, \ldots, x_{m0}\}$.

The current paper is closely related to the following work by Chernous'ko [3]. The author examined the evasion differential game of one evader versus many pursuers which is described by the following simple differential equations:

$$\dot{x}_i = u_i, \quad |u_i| \leq kv, \quad 0 < k < 1,$$
$$\dot{y} = v, \quad |v| \leq v.$$

Trajectory of one evader against each pursuer is pictured in the following diagram in Fig. 1 where Q is the position of point P at time t_A; α is the angle between the ray x and segment AQ where $0 \leq \alpha \leq \pi$; R is the current distance QE; φ is the current angle between the segments QE and QA, and s is the arc length of curve AE. The author's strategy is to ensure $EP \geq L$ for all $t \geq t_0$ before the evader moving back to the horizontal line and continue to move away from pursuer horizontally. By the strategy, he proved that evasion is possible.

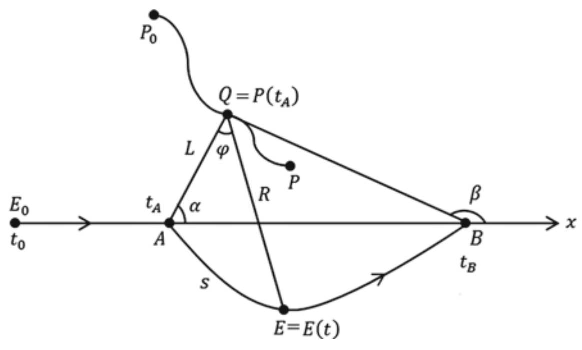
Also, the work by Ibragimov et al. [5] was devoted on differential game of evasion. The work deals with the evasion differential game of m pursuers $x_1, \ldots, x_m$ and one evader y described by the equations

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0},$$
$$\dot{y} = v, \quad y(0) = y_0, \quad x_{i0} \neq y_0, \quad i = 1, \ldots, m.$$

The game occurs in the plane and the following conditions:

$$\int_0^{\infty} |u_i(s)|^2 ds \leq \rho_i^2, \quad i = 1, \ldots, m, \quad \int_0^{\infty} |v(s)|^2 ds \leq \sigma^2,$$

Fig. 1 Trajectory of Evader



were imposed on control functions of the players. The main result of the paper is as follows: if the total resource of pursuers does not exceed that of evader, that is $\rho_1^2 + \rho_2^2 + \ldots \rho_m^2 \leq \sigma^2$, then evasion is possible.

We are to construct a different approach and strategy for evasion problem which is similar to Chernous'ko's [3], for the case of one evader against one pursuer. Like Chernous'ko's problem, evader is moving on a straight line as pursuer is approaching the evader. When the distance between them reaches a particular distance which is greater than zero, the evader will maneuver to avoid the pursuer. The strategy of the maneuver is different from Chernous'ko's where the control set of the evader in our problem is a sector. We also assume the maximal speed of the pursuer is equal to 1 and the maximal speed of the evader is α where $\alpha > 1$. We find a sufficient condition for the evader to escape from the pursuer.

2 Statement of the Problem

We study an evasion differential game of one pursuer P and one evader E with geometric constraints on controls of players described by equations:

$$P : \dot{x} = u, \ x(0) = x_0, \ |u| \leq 1, \tag{1}$$

$$E : \dot{y} = v, \ y(0) = y_0, \ v \in \mathbf{S}, \tag{2}$$

where $x, x_0, u, y, y_0, v \in \mathbb{R}^2$, $x_0 \neq y_0$, u is the control parameter of the pursuer P, v is that of the evader E,

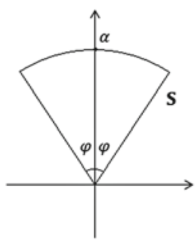
$$\mathbf{S} = \{(v_1, v_2) |\ v_1^2 + v_2^2 \leq \alpha^2, \ |v_1| \leq v_2 \tan\varphi, \ v_2 \geq 0\},$$

$\alpha > 1$ and φ, $0 < \varphi < \frac{\pi}{2}$, is a given angle. Note that $\mathbf{S}$ is a sector with the radius α and central angle 2φ (Fig. 2).

Definition 1 A Borel measurable function $u(t) = (u_1(t), u_2(t))$, $|u(t)| \leq 1$, $t \geq 0$, is called admissible control of the pursuer x.

Definition 2 A Borel measurable function $v(t) = (v_1(t), v_2(t))$, $v(t) \in \mathbf{S}$, $t \geq 0$, is called admissible control of the evader y.

Fig. 2 Sector **S**



Definition 3 A function

$$V(t, y, x, u),\ V : [0, \infty) \times R^2 \times H(0, \rho) \to \mathbf{S},$$

is called strategy of the evader, if for any admissible control $u(t)$ the following initial value problem

$$\dot{x} = u(t),\ x(0) = x_0,$$

$$\dot{y} = V(t, y, x, u(t)),\ y(0) = y_0,$$

has a unique solution $(x(t), y(t))$, $t \geq 0$, with absolutely continuous components $x(t)$ and $y(t)$.

Definition 4 We say that evasion is possible in the game (1)–(2) if there exists a strategy V of the evader E such that for any admissible control of the pursuer $x(t) \neq y(t)$ for all $t \geq 0$.

Problem
Find sufficient conditions of evasion in the game (1)–(2).

The condition $v(t) \in \mathbf{S}$, $t \geq 0$, implies that state of the evader $y(t)$ belongs to the sector $\mathbf{S}_1 = \{y_0 + ta \mid a \in \mathbf{S}, t \geq 0\}$. Initial position of the pursuer may be in $\mathbf{S}_1$ as well as outside $\mathbf{S}_1$. In process of pursuit, the pursuer can move throughout the plane. In this regard, the pursuer have advantage. However, the evader has advantage in speed, since $\alpha > 1$.

3 Main Result

The main result of the paper is the following statement.

Theorem 1 *If* $\alpha \cos \varphi_0 \geq 1$ *and* $\alpha \sin \varphi_0 > 1$ *at some* $0 < \varphi_0 \leq \varphi$, *then evasion is possible in the game* (1)–(2).

Proof Choose any number a, $0 < a < \min |x_0 - y_0|$. We construct a strategy for the evader as follows:

$$v(t) = \begin{cases} (0, \alpha),\ if\ 0 \leq t < \tau, \\ \left(\pm(c + |u_1(t)|), \sqrt{\alpha^2 - (c + |u_1(t)|)^2}\right), \\ \qquad\qquad\qquad if\ \tau \leq t \leq t_1, \\ (0, \alpha),\ if\ t > t_1, \end{cases} \tag{3}$$

where τ is the first time when $|y(\tau) - x(\tau)| = a$,

$$t_1 = \tau + \frac{a}{\sqrt{(\alpha - 1)^2 - c^2}}, \quad c = \alpha \sin \varphi_0 - 1.$$

Note that such a time τ may not exist. If so, then we let $v(t) = (0, \alpha)$ for all $t \geq 0$. In (3), $\pm$ means $v_1(t) = c + |u_1(t)|$, if $x_1(\tau) \leq y_1(\tau)$ and $v_1(t) = -(c + |u_1(t)|)$, if $x_1(\tau) \geq y_1(\tau)$, where $x(t) = (x_1(t), x_2(t))$, $y(t) = (y_1(t), y_2(t))$.

Estimate $|y(t) - x(t)|$, $\tau \leq t \leq t_1$. We have

$$\begin{aligned} |y(t) - x(t)| &= \left| y(\tau) + \int_\tau^t v(s)ds - \left(x(\tau) + \int_\tau^t u(s)ds \right) \right| \\ &\geq |y(\tau) - x(\tau)| - \left| \int_\tau^t v(s)ds \right| - \left| \int_\tau^t u(s)ds \right| \\ &\geq a - (t - \tau)(\alpha + 1). \end{aligned}$$

Without loss of generality, we assume that $y_1(\tau) \geq x_1(\tau)$, and hence $v_1(t) = c + |u_1(t)|$, $\tau \leq t \leq t_1$. Then, on the other hand, for the points $x(t)$ and $y(t)$ we have

$$\begin{aligned} |y(t) - x(t)| &\geq y_1(t) - x_1(t) \\ &= y_1(\tau) + \int_\tau^t v_1(s)ds - \left(x_1(\tau) + \int_\tau^t u_1(s)ds \right) \\ &= y_1(\tau) - x_1(\tau) + \int_\tau^t (v_1(s) - u_1(s))\, ds \\ &= y_1(\tau) - x_1(\tau) + \int_\tau^t (c + |u_1(s)| - u_1(s))\, ds \\ &\geq c \cdot (t - \tau) \end{aligned}$$

Thus, $|y(t) - x(t)| \geq f(t)$, where

$$f(t) = \max\{a - (t - \tau)(\alpha + 1),\ c(t - \tau)\}.$$

Note that the function $f(t)$ has only one minimum on $[\tau, t_1]$, since the first function in the max decreases, whereas the second function increases. The function $f(t)$ takes its minimum at

$$t_* = \tau + \frac{a}{\alpha(1 + \sin\varphi_0)} \in [\tau, t_1]\,.$$

We have

$$\begin{aligned} |y(t) - x(t)| &\geq c \cdot (t_* - \tau) \\ &= c \cdot \frac{a}{\alpha(1 + \sin\varphi_0)} \\ &\geq \frac{c}{2\alpha} \cdot a, \qquad \tau \leq t \leq t_1. \end{aligned}$$

In particular, at the time t_1

$$|y(t_1) - x(t_1)| \geq \frac{c}{2\alpha} \cdot a. \tag{4}$$

Moreover, at the time $t = t_1$ the pursuer cannot be above the horizontal line $y = y_2(t_1)$ of the xy-plane. Indeed,

$$\begin{aligned} y_2(t_1) - x_2(t_1) &= y_2(\tau) + \int_\tau^{t_1} v_2(t)dt - \left(x_2(\tau) + \int_\tau^{t_1} u_2(t)dt\right) \\ &= y_2(\tau) - x_2(\tau) + \int_\tau^{t_1} (v_2(t) - u_2(t))\, dt \\ &\geq -a + \int_\tau^{t_1} \left(\sqrt{\alpha^2 - (c + |u_1(t)|)^2} - \sqrt{1 - |u_1(t)|^2}\right) dt \end{aligned} \tag{5}$$

It is not difficult to show that

$$\sqrt{\alpha^2 - (c + |u_1(t)|)^2} - \sqrt{1 - |u_1(t)|^2} \geq \sqrt{(\alpha - 1)^2 - c^2}.$$

Then right-hand side of (5) can be estimated from below by

$$-a + \int_\tau^{t_1} \sqrt{(\alpha - 1)^2 - c^2} dt = -a + (t_1 - \tau)\sqrt{(\alpha - 1)^2 - c^2} = 0$$

Thus, $y_2(t_1) \geq x_2(t_1)$.

Next, according to (3) $v(t) = (0, \alpha)$, $t > t_1$. We estimate $|x(t) - y(t)|$ at $t > t_1$. Since

$$y_1(t) - x_1(t) = y_1(t_1) - x_1(t_1) - \int_{t_1}^t u_1(s)ds$$

and

$$y_2(t) - x_2(t) \geq y_2(t_1) - x_2(t_1) + \alpha(t - t_1) - \int_{t_1}^t |u_2(s)|ds \geq 0.$$

Last inequality because of $y_2(t_1) \geq x_2(t_1)$ and $|u_2(t)| \leq 1$.
Therefore

$$\begin{aligned} |x(t) - y(t)| \geq \Bigg(&\left(y_1(t_1) - x_1(t_1) - \int_{t_1}^t u_1(s)ds\right)^2 \\ &+ \left(y_2(t_1) - x_2(t_1) + \alpha(t - t_1) - \int_{t_1}^t |u_2(s)|ds\right)^2\Bigg)^{1/2}. \end{aligned}$$

Using the triangle inequality, we can estimate right-hand side of this inequality from below by

$$\sqrt{|y_1(t_1) - x_1(t_1)|^2 + (y_2(t_1) - x_2(t_1) + \alpha(t - t_1))^2}$$
$$-\sqrt{\left(\int_{t_1}^{t} u_1(s)ds\right)^2 + \left(\int_{t_1}^{t} |u_2(s)|ds\right)^2}.$$

By the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\left(\int_{t_1}^{t} u_1(s)ds\right)^2 + \left(\int_{t_1}^{t} |u_2(s)|ds\right)^2 &\le \left(\int_{t_1}^{t} 1\cdot|u_1(s)|ds\right)^2 + \left(\int_{t_1}^{t} 1\cdot|u_2(s)|ds\right)^2 \\ &\le \int_{t_1}^{t} 1^2 ds \int_{t_1}^{t} u_1^2(s)ds + \int_{t_1}^{t} 1^2 ds \int_{t_1}^{t} u_2^2(s)ds \\ &= (t - t_1)\int_{t_1}^{t} \left(u_1^2(s) + u_2^2(s)\right) ds \\ &\le (t - t_1)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}|x(t) - y(t)| &\ge \left((y_1(t_1) - x_1(t_1))^2 + (y_2(t_1) - x_2(t_1))^2 + (\alpha(t - t_1))^2\right)^{1/2} - (t - t_1) \\ &\ge \sqrt{\left(\frac{ac}{2\alpha}\right)^2 + \alpha^2(t - t_1)^2} - (t - t_1) \qquad (6)\end{aligned}$$

since $|y(t_1) - x(t_1)|^2 \ge (\frac{ac}{2\alpha})^2$.

To estimate the expression (6) from below, we let $\dfrac{ac}{2\alpha} = b$, $(t - t_1) = x$ and consider the function

$$g(x) = \sqrt{b^2 + \alpha^2 x^2} - x, \quad x \ge 0.$$

This function takes its minimum at $x = x_* = \dfrac{b}{\alpha\sqrt{\alpha^2 - 1}}$. Thus

$$\begin{aligned}|x(t) - y(t)| &\ge g(x_*) \\ &= \frac{\sqrt{\alpha^2 - 1}}{\alpha} b = \frac{\sqrt{\alpha^2 - 1}}{2\alpha^2} ac, \quad t \ge t_1.\end{aligned}$$

Next, we estimate $y_2(t) - x_2(t)$ for $t \ge \tau' = \tau + ra$ where

$$r = \frac{1}{\sqrt{(\alpha - 1)^2 - c^2}} + \frac{c}{2\alpha^2\sqrt{\alpha^2 - 1}}.$$

Since $y_2(t_1) \geq x_2(t_1)$,

$$\begin{aligned} y_2(t) - x_2(t) &= y_2(t_1) + \int_{t_1}^{t} v_2(s)ds - x_2(t_1) - \int_{t_1}^{t} u_2(s)ds \\ &\geq y_2(t_1) - x_2(t_1) + \int_{t_1}^{t} \alpha\, ds - \int_{t_1}^{t} 1\, ds \\ &\geq (\alpha - 1)(t - t_1) \\ &\geq (\alpha - 1)(\tau' - t_1) \\ &= \frac{(\alpha - 1)c}{2\alpha^2\sqrt{\alpha^2 - 1}} \cdot a \\ &= ap, \end{aligned}$$

where $p = \dfrac{(\alpha - 1)c}{2\alpha^2\sqrt{\alpha^2 - 1}}$. □

Thus, we can conclude that

(1) $|x(t) - y(t)| \geq a$ for $0 \leq t \leq \tau$,
(2) $|x(t) - y(t)| \geq \frac{c}{2\alpha} \cdot a$ for $\tau \leq t \leq t_1$,
(3) $y_2(t) > x_2(t)$, for $t \geq t_1$,
(4) $|x(t) - y(t)| \geq \dfrac{\sqrt{\alpha^2 - 1}}{2\alpha^2} ac$ for $t \geq t_1$,
(5) $y_2(t) - x_2(t) \geq ap$ for $t \geq \tau'$.

In particular, $y(t) \neq x(t),\ t > 0$.

Hence, at the time t_1 the evader will be above the horizontal line where is the pursuer x. Thus, at the time t_1 the pursuer x became "behind" the evader. Since $v_2(t) \geq \alpha \cos\varphi_0 \geq 1 \geq u_2(t)$, then $y_2(t) > x_2(t)$, for all $t \geq t_1$.

If a-approach occurs with the pursuer x at a time τ, then the evader uses maneuver on $[\tau, t_1]$ which ensures the inequality $y_2(t_1) \geq x_2(t_1)$. Further, the strategy of the evader guarantees him the inequality $y_2(t) > x_2(t)$ for all $t \geq t_1$.

4 Conclusion

A simple motion evasion differential game of one pursuer and one evader whose control set is a sector has been considered in the plane. If $\alpha \cos\varphi_0 \geq 1$ and $\alpha \sin\varphi_0 > 1$, then evasion from one pursuer has been presented. The strategy for the evader was constructed as well. Moreover, distances between the pursuer and evader have been estimated.

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