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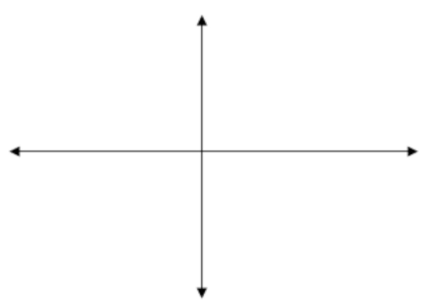
5. MAKING COMPLEX ARITHMETIC REAL!

With $i = \sqrt{-1}$ being called an imaginary number, it's easy to see why many students don't view complex numbers as actual numbers and/or take any interest in them. Such complex analysis is an important tool in engineering, and students need real-world examples that will motivate them to study the topic of complex numbers. We might humor our students by telling them that the real reason they need to know about complex numbers is to pass the next test or class. Instead, we should offer some simple exercises that give insight into the value of complex numbers.

We can associate with each complex number, z = a + bi, the point (a, b), in the plane. The real number a is called the real part of z and the real number b is called the imaginary part of z. Adding two complex numbers z = a + bi and w = c + di is done by adding the real parts and imaginary parts: z+w = (a+c) + (b+d)i. Combining this definition of arithmetic and the geometry of the numbers themselves leads to the following activities.

Plot points associated with the following complex numbers: $z_1 = 1 + i$, $z_2 = -1 + i$, $z_3 = -1 - i$, and $z_4 = 1 - i$. Add the complex number w = 2+3i to each of the numbers z_1 to z_4, plotting the sums as well. To complete the picture, connect the numbers z_1 to z_4 as vertices of a square. Connect the sums in the same way. The resulting picture gives us our first clue as to the power of complex arithmetic: adding a complex number translates another complex number or set of numbers. This is a way of doing geometry with arithmetic.

z	z+(2+3i)
1+i	
–1+i	
–1–i	
1–i	



What about complex multiplication? Multiplying two complex numbers z = a + bi and w = c + di is performed the same way as multiplication of any two real binomials, while also using the identity, $i^2 = -1$. We can verify that zw = (ac–bd) + (ad+bc)i.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 17–19.
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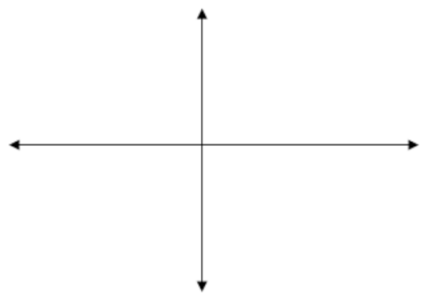
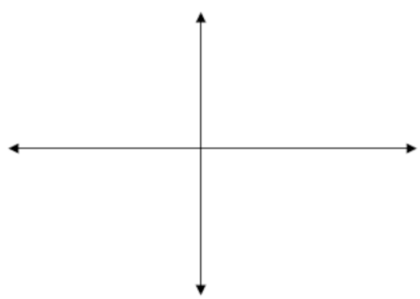
Start with a clean plot of the points z_1 to z_4 again. Multiply each of them by the complex number $w = \sqrt{2}/2 + \sqrt{2}/2i$ Again, connect the numbers z_1 to z_4 as vertices of a square, and do the same for the resulting products. The resulting picture gives us more evidence of the power of complex arithmetic: multiplying by a complex number can rotate another complex number or set of numbers. We have extended the reach of arithmetic into geometry.

z	z*w
1+i	
–1+i	
–1–i	
1–i	



If we repeat the multiplication exercise with w = 1 + 1i = 1+i instead, we get a slightly different result. Instead of a simple rotation, the original square is stretched as well. Yes, there is a connection between the multiplication and, $\sqrt{2}\left(\sqrt{2}/2 + \sqrt{2}/2i\right) = 1 + i$ and the stretch. This gives us evidence that multiplying by a real number dilates (stretches/contracts) another complex number or set of complex numbers.

z	z*(1+i)
1+i	
–1+i	
–1–i	
1–i	



Who knew that it would be possible to translate, rotate or stretch a figure just by arithmetic? And it doesn't end there. We might wonder if it is possible to do reflections, inversions or other geometric transformations with complex numbers. Why, of course.

Here are some exercises to work on. In them, S refers to the square in the plane, with vertices z_1 to z_4, as above.

Suppose there is a translation of S so that z_1 has image, –3 + 2i. What complex number, w, could be added to the vertices of S for that translation?

Multiplying the vertices of S by $w = (\sqrt{2}/2 + \sqrt{2}/2i)$ rotated the square counterclockwise by 45 degrees. What would happen if you multiplied the vertices in S by w^2? (Try guessing the transformation and then confirming your guess by calculating w^2 and all the products.) What would happen if you multiplied the vertices in S by w^3? What about higher powers?

By what angle does multiplication by w = 0 + 1i = i rotate the square S?

What complex arithmetic would be needed to rotate the square S by 45 degrees counterclockwise, stretch it by a factor of 2, and then move it up 2 units?

If you have completed the exercises and want some background on the rotations and dilations, here it is.

For any purely complex number z = a + bi (that is, neither a nor b is zero), we can draw the right triangle with one vertex at the origin, (0, 0), and another at the associated point (a, b). The final vertex, at the right angle, will be the point (a, 0). The length of the hypotenuse of this triangle is $r = \sqrt{(a^2 + b^2)}$. In that triangle, we can see that a = r cos(t) and b = r sin(t), where t is the angle at the origin. In complex analysis, t is called the argument of z. Factoring out the common r, we can rewrite z = r(cos(t) + sin(t)i).

What would we get if we multiply two complex numbers in this form? Suppose $z_1 = r_1(\cos(t_1)+\sin(t_2)i)$ and $z_2 = r_2(\cos(t_2)+\sin(t_2)i)$. Then $z_1*z_2= r_1*r_2*(\cos(t_1)+\sin(t_1)i)*(\cos(t_2)+\sin(t_2)i)$. Applying what we saw for multiplication of complex numbers, this would be rewritten as

$$z_1*z_2 = r_1*r_2*((\cos(t_1)\cos(t_2) - \sin(t_1)\sin(t_2)) + (\cos(t_1)\sin(t_2)+\sin(t_1)\cos(t_2)i).$$

Look familiar? The real and imaginary parts (not including the factor of r_1*r_2) are the angle addition formulas for cosine and sine, respectively. That is

$$z1*z_2 = r1*r_2(\cos(t1+t_2)+\sin(t1+t_2)i)$$

So, the result of multiplying two complex numbers can be done by multiplying their lengths and adding their angles.