Pedagogy and Content in Middle and High School Mathematics

G. Donald Allen and
Amanda Ross (Eds.)

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TABLE OF CONTENTS

PREFACE

This book provides articles, included in *Focus on Mathematics Pedagogy and Content*, a newsletter for middle and high school teachers, published by Texas A&M University. The book covers all five NCTM content strands, focusing only on grades 6–12. The articles may be used as a reference for teachers, on both effective ways to teach mathematics, as well as mathematics content knowledge.

High level mathematics content and problem solving processes are presented in different ways, including via historical information and creative real-world contexts. This book offers historical perspectives and connections, which are not typically found in other books that examine instructional strategies for various mathematics topics. The book will benefit those readers, who desire to learn more about the history of mathematics and its connection to teaching in the mathematics classroom. As related to problem solving, many articles present different ways of representing mathematics content, ways of connecting these representations, and different ways to approach the same type of problem. In addition, student misconceptions are interspersed throughout the book.

The book also briefly delves into assessments, looking at an amalgamation of topics, related to formative and summative assessments. These articles focus on test construction, viewpoints, background, and types of assessments. Finally, a whole section on “Teaching Tips” is included in the book, in addition to a section on games and technology integration.

We would like to thank all of the contributors to this book. All of the contributions were guided, based upon personal interest. Articles were not submitted, in response to a particular call for articles or particular content domain request. Authors were given complete autonomy in deciding on article content. This lack of structure resulted in a wide variety of articles on many different mathematics content and pedagogy topics, which certainly added to the uniqueness of this book.

We hope you enjoy the book!

PART I

CONTENT AND PEDAGOGY

SECTION 1

NUMBER

G. DONALD ALLEN

1. A BRIEF HISTORY OF ZERO

Zero is a relatively recent addition to mathematics. Indeed, entire civilizations lasting longer than our entire western era (i.e. more than two thousand years) flourished and perished having built the pyramids and the wonders of the world, without any notion of zero. Zero, which is taught to youngsters, is such an important concept, and like much of mathematics, was invented out of necessity.

One of our principal uses of zero is as a placeholder in our system of enumeration. How else could we write 2005 without the zero? The ancient Egyptians, Babylonians, Greeks, and Romans all knew how to do so. Placeholders are a mere convenience of our enumeration, not an essential part of enumeration. Systems of enumeration are shown in Figure 1.

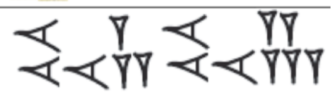
Egyptian (hieroglyphs)	
Babylonian (cuneiform)	
Greek (alpha-numerical)	‘βε
Roman (alpha-numerical)	**MMV**

Figure 1. Systems of enumeration

However, they do help with our algorithms of calculation such as addition, subtraction, multiplication, and division. For example, our modern division algorithm is about five hundred years old, and once was an advanced subject taught only in Italy. These algorithms take keen advantage of our zero placeholder, and make rapid hand calculations possible. More important, they make the realization of truly large numbers such as a google, 10^{100}, possible.

The other use of zero is as a number itself. You can be the judge of which is more important. But for basic business type mathematics calculations, it would be the place holding value that is probably greatest. In higher math, the actual value of zero is extremely important.

While the Babylonians and ancient Greeks did finally evolve to a symbolic placeholder for zero, it was not really a number. What we do know is that by around

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 5–6.

650 AD, the use of zero as a number came into Indian mathematics. Its original form is very much like our own zero, 0, only a little smaller, though there is evidence also that a single dot, . , was used to denote an empty space. Links: History of zero: http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Zero.html Enumeration: chapters 3 and 4 of http://www.math.tamu.edu/%7Edallen/masters/hist_frame.htm

G. DONALD ALLEN

2. APPROXIMATING PI

Ever since mankind began surveying areas and building, the need to measure circles has been important. For this task, we need π. As we know today, π is one of the most peculiar of numbers. It is not rational, but instead, irrational, and is a special kind of irrational number, called a transcendental number, meaning that it cannot be the solution of a polynomial equation with integer coefficients. Indeed, it was only in 1840 that such numbers were even found, and this was thousands of years after the ancients first mused on what π might be. Well, everyone knew π was a little bit larger than three, but to achieve an accurate approximation was an elusive task. Let's look at a couple of methods and approximations from various civilizations.

ANCIENT EGYPTIANS

"A square of side 8 has the area of a circle of diameter 9." The area of any circle was then approximated using proportion using the formula and what was understood to be an area to square of the radius formula.

$$\frac{A}{r^2} = \frac{64}{\left(\frac{9}{2}\right)^2} = \frac{256}{81}$$

The form we use today is shown below.

$$A = \frac{256}{81} r^2$$

Cut off each corner of the square of side length 9 divided horizontally and vertically in thirds, as shown, and add the resulting five squares of radius three and four triangles of half that size to get $5(3)^2 + 4\left(\frac{1}{2} \cdot 3^2\right) = 63 \approx 64$. See Figure 1.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 7–10.

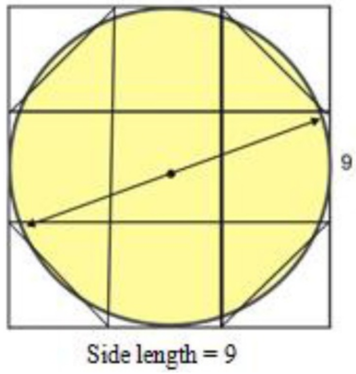


Figure 1. Approximation by Ancient Egyptians

ANCIENT GREEKS

$\pi \approx \frac{22}{7}$. This incredibly remarkable formula was determined by Archimedes, the greatest of the ancient mathematicians. He was able to determine the areas of inscribed and circumscribed regular polygons of 6, 12, 24, 48, and finally 96 sides. In this way, he found lower and upper estimates of π, the lower estimate being $3\frac{10}{71}$. See Figure 2, where polygons of up to 24 sides are shown. What Archimedes did was discover a very clever relation between the areas of these figures, as the number of sides increased. This allowed the prodigious computational feat. The formulas of Archimedes were used even until modern times to compute ever more accurate approximations to π.

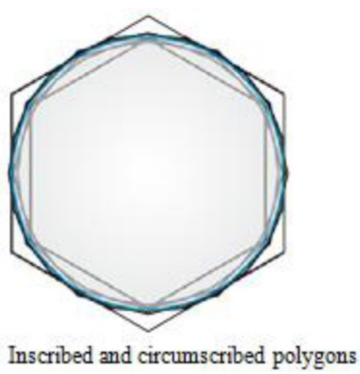


Figure 2. Approximation by Ancient Greeks

ANCIENT CHINESE

While it is uncertain how the computation was made, the Chinese of the 5th century gave us the purely elementary fraction approximation to π given by

$$\frac{355}{113} = 3.14159292$$

Now to ten places π = 3.141592654. So you can see that the approximation, $\pi - \frac{355}{113} = -0.0000002667$, is very, very accurate, almost beyond any current needs. Note the pattern of the number, which uses the digits 1, 1, 3, 3, 5, 5, stacked to make the fraction. Now, how good are these? Can we do better? Well, $3\frac{1}{6} = \frac{19}{6}$ is not nearly as accurate as $\frac{22}{7}$. Yet, $\frac{22}{7}$, is the most accurate fraction approximation up to $\frac{179}{57}$, but the improved accuracy of the latter fraction is slight. (Ask your students to compute these differences. They will begin to use very small numbers.) On the other hand, the next better approximation than $\frac{355}{113}$ is the whopping big fraction $\frac{53228}{16943}$, and as before the improvement is only slight. This should give the idea that these two revered fractions, $\frac{22}{7}$ and $\frac{355}{113}$, have a special place in the world of approximations.

MODERN TIMES

The current, best approximation to π is accurate to 1,240,000,000,000 places. (That's more than a trillion digits.) To give an idea how many digits these are, typing them all out at 10 digits to the inch, the entire approximation would run 1,957,070 miles. This is almost 78 times around the earth, or four round trips to the moon! The method uses a complex formula involving the arc tangent function. It was computed a HITACHI SR8000/MP supercomputer under the project direction of Yasumasa Kanada. See http://www.super-computing.org/ for more information.

ACTIVITIES

1. Make the decimal approximations of all the fractions above. Compare the decimals to one another and then to π itself.
2. Ask your students to recreate the clever diagram of the ancient Egyptians.

3. In the 6th century, Indian mathematicians used this description, "Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle of which the diameter is 20,000. What is their effective π? Answer: Computing, we have $\frac{(100+4)\times 8+62,000}{20,000}=3\frac{177}{1250}=3.1416$

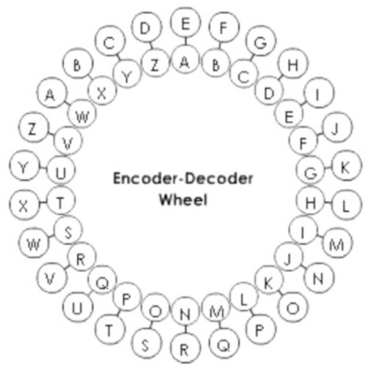
READINGS

"A History of Pi";
http://www-groups.dcs.stand.ac.uk/history/HistTopics/Pithroughtheages.html

G. DONALD ALLEN

3. THE CAESAR CYPHER

A **cypher** (or encryption) is a method of transforming a message into a set of alternate characters that conceals the contents of the message. The **Caesar cypher** (or **shift**) was one of the earliest cyphers ever used. More than two thousand years ago, Julius Caesar was able to convey secret messages to his generals and colleagues. It is simple and effective. Each character is shifted a specified number of places to the right, with the provision that at the end of the alphabet, the characters “wrap around” to the beginning of the alphabet. This is shown below for a shift of four places. So, “A” is shifted to “E”; “B” is shifted to “F”, “W” is shifted to “A”, and so on. For example, the message “Send more money” is encrypted to “Wirh qsvi qsric.”



wheel

Shift by four characters

An alternative to the Encoder-Decoder wheel is a linear representation of the shift.

Original	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
New	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D

However, the cypher is now easy to decipher using frequency analysis. That is, the number of each letter occurrence is counted and compared with standard frequency counts for normal text. For example, “E” is the most commonly occurring letter, occurring 12.702% of the time. So, it is natural to guess that the letter in the encrypted message occurring most often is an “E.” This can help the cryptographer determine or guess the true letters. In the version of the Caesar cypher below, numbers are

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 11–12.

shifted as numbers, capital letters are shifted as capitals, and lower case letters are shifted as lower case. For each shift, the Encoder-Decoder wheel is shown. (Press "Show encoder wheel.") Frequency counts of letters for the English language can be found at http://en.wikipedia.org/wiki/Letter_frequencies

Example. With the four character shift, the message "I love American Idol," is encoded as "L oryh Dphulfdq Lgro."

Where's the Math? It lies in what is call **modular** arithmetic. Modular arithmetic is based on a specific modulus. We define

$$a = b \pmod{c}$$

to be the remainder of *b* divided by *c*. The **modulus** is **c**. For example 5 = 12 (mod 7) because 5 is the remainder of 12 divided by 7. For our present situation we are working with the twenty six letters of the alphabet; so our modulus *c* = 26. **Encode** the letters as the numbers 0, 1, 2, ... , 25 as shown in the chart below.

Alphabet	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
Encoding	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

So with a shift of four, each encoded letter is increased by four units. The letter "A" is encoded as "0" and this is shifted to "4" which is "E". Written in modular arithmetic this is 4 = 4 (mod 26). On the other hand, the letter "Y" is encoded as "24" and this is shifted by "4" which is 28. Now dividing by 26 gives the remainder 2 and this is decoded to "C". Written in modular arithmetic this is 2 = 28 (mod 26).

In summary, letters are **encoded** as numbers. To encrypt the letters, we perform modular arithmetic on the numbers, in this case add 4, and then the numbers are **decoded** back to letters. Essentially every modern encryption scheme uses the encoding of letters to numbers.

TERMINOLOGY:

- Cypher (also spelled as *Cipher*) - Encryption
- Caesar cypher (cipher)
- Encode - Encoding
- Decode - Decoding
- Modular arithmetic
- Modulus

G. DONALD ALLEN

4. HOW BIG IS INFINITY?

Infinity is a relatively recent term in mathematics, having officially been around only since the nineteenth century. It was paradoxically created in an attempt to solve a complicated mathematical problem, but its presence was felt for centuries prior to that. It was felt in philosophy, religion, and in mathematics, partly due to the seeming need for infinite processes, infinite time span, infinitesimal divisibility of matter, and so on. In the months ahead, we can consider all of these if there becomes a need, but today we want to discover how big it is. Historically, infinity was a number, concept, or idea that could be approached but never reached or achieved. Today, it has attained its rightful place as a number, with precise rules on how to use it and what it means.

Everyone knows infinity is the biggest "thing" there is, no matter what the context. So, it matters not to discuss this aspect, the bigness. What is more fun is to consider some of the anomalies it creates.

ARITHMETIC

To work numerically with infinity, mathematicians have created the *extended number system*, which consists of all real numbers plus infinity (∞) and minus infinity ($-\infty$). For real numbers, the rules are the same as usual, but for the arithmetic involving ∞, we have for every real a

$$a + \infty = \infty$$

$$a * \infty = \pm\infty \text{ for } a \neq 0 \text{ with the } \pm \text{ generated by the sign of } a$$

$$\frac{a}{\infty} = 0$$

$$0 * \infty = 0$$

$$\infty + \infty = \infty$$

$$\infty - \infty = \text{ is undefined and } \frac{\infty}{\infty} \text{ is undefined}$$

With this set of rules, the extended real numbers form a consistent number system that obeys the laws of closure, commutativity, associativity, and distibutivity. The reason $\infty - \infty$ and $\frac{\infty}{\infty}$ are undefined is because we can't make any consistent sense

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 13–15.

of what it should be to maintain a consistent number system. For example, suppose we do what seems natural and define $\infty - \infty = 0$. Then, adding 1 to both sides of the equation gives $1 + \infty - \infty = 1 + 0$. By associativity, we have $(1 + \infty) = \infty$. So, $1 + \infty - \infty = (1 + \infty) - \infty = \infty - \infty = 0$, but $1 + 0 = 1$. This implies $0 = 1$, which we know is not so. Selecting $\infty - \infty$ to be anything else similarly will result in a contradiction. Ditto for $\frac{\infty}{\infty}$.

MAGNITUDE

We know ∞ is really, really big. But, it's bigger than that. One way of comparing the size of a basket of apples and a basket of pears is to put them in correspondence. In other words, you would place each apple next to a pear until one of the baskets is exhausted of fruit. Then, we can say that the basket with fruit remaining in it is the larger in size. Basically, we also do this by counting. However, there are records of some tribes of Indians, not knowing counting, who used this correspondence idea.

So, now let's suppose we have our baskets filled with an infinite quantity of objects. To make things easier, let's work with just the real numbers in the interval [0,1] and [0,2]. Clearly, both contain an infinite number of numbers, and just as clearly, the larger interval has "twice" the size. But, does it? We can't do the arithmetic $2\infty - \infty$ because this quantity is not defined. (Note. $2\infty - \infty = \infty - \infty$, and we've shown there can be no meaning to this.) Using the correspondence idea, we take any number x in [0,1] and multiply it by 2 to get $2x$. Now, all the numbers in [0,1] have been put in correspondence with those in [0,2]. Not only is the correspondence perfect but, it is one-to-one. This remarkable observation shows that these two sets with one obviously "twice" the other have exactly the same number of points in them. (Note. Use rational numbers, and the argument is the same.)

Definition: An infinite number of numbers is called *countable* if it can be put into one-to-one correspondence with the natural numbers.

Problem: Show that there is exactly the same number of even positive integers as there are positive integers.

Problem: The famous Hotel Infinity, with an infinite number of rooms, is completely filled, and a new guest arrives requesting lodging for the night. The clerk says, "Of course, sir", and makes a room available. How did he do it?

HIGHER ORDERS OF INFINITY

Infinity gets more bizarre when we consider all the real numbers in [0,1] and all the natural numbers, i.e., 1, 2, 3…. Both sets of numbers have an infinite number of members. This time, there is not a possible way to put them in a perfect correspondence. This means the infinity of [0,1] is *fundamentally larger* than that of the natural numbers. So there are magnitudes of infinity, just like there are magnitudes of numbers.

Let's show that the number of numbers in [0,1] cannot be put into one-to-one correspondence with the natural numbers. Each number in [0,1] has a decimal expansion. Assume we can put all of them in correspondence with the natural numbers. We express this assumption by writing $d_1, d_2, d_3, \ldots$ as all of the decimals written with their correspondent integer. Each decimal number has a full decimal expansion. So,

$$d_1 = 0.d_{11}d_{12}d_{13}\cdots$$
$$d_2 = 0.d_{21}d_{22}d_{23}\cdots$$

and so on, where all the digits d_{ij} are integers 0, 1, 2, …, 9. Now, construct a brand-new decimal $f = f_1, f_2, f_3 \ldots$ by the rule that $f_1 \neq d_{11}$, $f_2 \neq d_{22}$, $f_3 \neq d_{33}$, and so on. In this way, f cannot equal any of the $d_1, d_2, d_3, \ldots$, and this "proves" we cannot make the correspondence we assumed we could.

It was hardly 140 years ago that these ideas of orders of infinity turned mathematics on its axis. Infinity is now quite tamed; it is no longer the mystery it once was.

REFERENCE

http://www.math.tamu.edu/~dallen/masters/infinity/content2.htm

GEORGE TINTERA

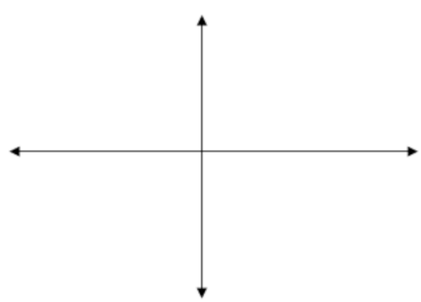
5. MAKING COMPLEX ARITHMETIC REAL!

With $i = \sqrt{-1}$ being called an imaginary number, it's easy to see why many students don't view complex numbers as actual numbers and/or take any interest in them. Such complex analysis is an important tool in engineering, and students need real-world examples that will motivate them to study the topic of complex numbers. We might humor our students by telling them that the real reason they need to know about complex numbers is to pass the next test or class. Instead, we should offer some simple exercises that give insight into the value of complex numbers.

We can associate with each complex number, $z = a + bi$, the point (a, b), in the plane. The real number a is called the real part of z and the real number b is called the imaginary part of z. Adding two complex numbers $z = a + bi$ and $w = c + di$ is done by adding the real parts and imaginary parts: $z+w = (a+c) + (b+d)i$. Combining this definition of arithmetic and the geometry of the numbers themselves leads to the following activities.

Plot points associated with the following complex numbers: $z_1 = 1 + i$, $z_2 = -1 + i$, $z_3 = -1 - i$, and $z_4 = 1 - i$. Add the complex number $w = 2+3i$ to each of the numbers z_1 to z_4, plotting the sums as well. To complete the picture, connect the numbers z_1 to z_4 as vertices of a square. Connect the sums in the same way. The resulting picture gives us our first clue as to the power of complex arithmetic: adding a complex number translates another complex number or set of numbers. This is a way of doing geometry with arithmetic.

z	z+(2+3i)
1+i	
–1+i	
–1–i	
1–i	

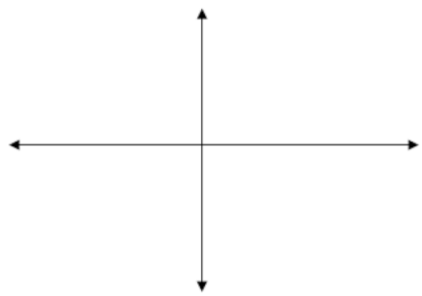


What about complex multiplication? Multiplying two complex numbers $z = a + bi$ and $w = c + di$ is performed the same way as multiplication of any two real binomials, while also using the identity, $i^2 = -1$. We can verify that $zw = (ac-bd) + (ad+bc)i$.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 17–19.

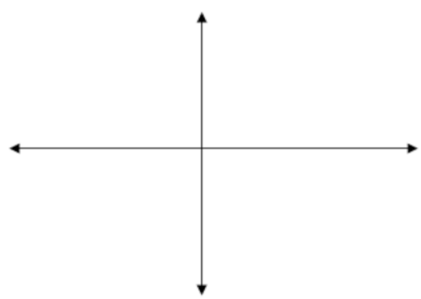
Start with a clean plot of the points z_1 to z_4 again. Multiply each of them by the complex number $w = \sqrt{2}/2 + \sqrt{2}/2i$ Again, connect the numbers z_1 to z_4 as vertices of a square, and do the same for the resulting products. The resulting picture gives us more evidence of the power of complex arithmetic: multiplying by a complex number can rotate another complex number or set of numbers. We have extended the reach of arithmetic into geometry.

z	z*w
1+i	
–1+i	
–1–i	
1–i	



If we repeat the multiplication exercise with w = 1 + 1i = 1+i instead, we get a slightly different result. Instead of a simple rotation, the original square is stretched as well. Yes, there is a connection between the multiplication and, $\sqrt{2}\left(\sqrt{2}/2 + \sqrt{2}/2i\right) = 1 + i$ and the stretch. This gives us evidence that multiplying by a real number dilates (stretches/contracts) another complex number or set of complex numbers.

z	z*(1+i)
1+i	
–1+i	
–1–i	
1–i	



Who knew that it would be possible to translate, rotate or stretch a figure just by arithmetic? And it doesn't end there. We might wonder if it is possible to do reflections, inversions or other geometric transformations with complex numbers. Why, of course.

Here are some exercises to work on. In them, S refers to the square in the plane, with vertices z_1 to z_4, as above.

Suppose there is a translation of S so that z_1 has image, –3 + 2i. What complex number, w, could be added to the vertices of S for that translation?

Multiplying the vertices of S by $w = (\sqrt{2}/2 + \sqrt{2}/2i)$ rotated the square counterclockwise by 45 degrees. What would happen if you multiplied the vertices in S by w^2? (Try guessing the transformation and then confirming your guess by calculating w^2 and all the products.) What would happen if you multiplied the vertices in S by w^3? What about higher powers?

By what angle does multiplication by w = 0 + 1i = i rotate the square S?

What complex arithmetic would be needed to rotate the square S by 45 degrees counterclockwise, stretch it by a factor of 2, and then move it up 2 units?

If you have completed the exercises and want some background on the rotations and dilations, here it is.

For any purely complex number z = a + bi (that is, neither a nor b is zero), we can draw the right triangle with one vertex at the origin, (0, 0), and another at the associated point (a, b). The final vertex, at the right angle, will be the point (a, 0). The length of the hypotenuse of this triangle is $r = \sqrt{(a^2 + b^2)}$. In that triangle, we can see that a = r cos(t) and b = r sin(t), where t is the angle at the origin. In complex analysis, t is called the argument of z. Factoring out the common r, we can rewrite z = r(cos(t) + sin(t)i).

What would we get if we multiply two complex numbers in this form? Suppose $z_1 = r_1(\cos(t_1)+\sin(t_2)i)$ and $z_2 = r_2(\cos(t_2)+\sin(t_2)i)$. Then $z_1*z_2= r_1*r_2*(\cos(t_1)+\sin(t_1)i)*(\cos(t_2)+\sin(t_2)i)$. Applying what we saw for multiplication of complex numbers, this would be rewritten as

$$z_1*z_2 = r_1*r_2*((\cos(t_1)\cos(t_2) - \sin(t_1)\sin(t_2)) + (\cos(t_1)\sin(t_2)+\sin(t_1)\cos(t_2))i).$$

Look familiar? The real and imaginary parts (not including the factor of r_1*r_2) are the angle addition formulas for cosine and sine, respectively. That is

$$z1*z_2 = r1*r_2(\cos(t1+t_2)+\sin(t1+t_2)i)$$

So, the result of multiplying two complex numbers can be done by multiplying their lengths and adding their angles.

G. DONALD ALLEN

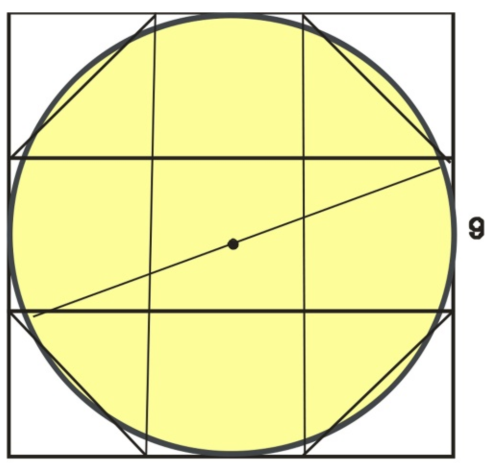
6. EARLY PI

Egypt. In 1856, a Scotsman antiquarian, named Henry Rhind, purchased a papyrus in Egypt. Currently, it is in the Bristish Museum listed as manuscript or artifact 10057. This is a copy dating to 1,650 B.C. It was copied by the scribe, Ahmes, from a lost text from the reign of king Amenemhat III (12th dynasty). It was written in the hieratic script, one of the forms of Egyptian at the time, and as a scroll, it measures about 15 feet long by and about 13 inches wide. It was first literally translated in the nineteenth century, though it has not been fully translated. It consisted of a collection of mathematics problems prepared for a typical mathematics student of the day. There was no text explanation of anything.

One of the important problems of the day, and to this day, was to compute the area of a circle. Of course we know the familiar formula $A = \pi r^2$ where r is the radius of the circle. But at the time, there was no understanding, really, of general formulae, nor an understanding of π. So, how did the Egyptians compute the area of a circle? It was achieved by an example. Students or practitioners were then to use proportionality to compute the area of any circle. With no exact value of π, just like as today, areas were only approximate.

Problem 50. A circular field of diameter 9 has the same area as a square of side 8.

Problem 48 gives a hint of how this formula is constructed.



Side length = 9

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 21–23.

Trisect each side. Remove the corner triangles. The resulting octagonal figure approximates the circle. The area of the octagonal figure is:

$$9\times 9-4(\frac{1}{2}\cdot 3\cdot 3)=63\approx 64=8^2$$

Thus the number

$$4(\frac{8}{9})^2=3\frac{13}{81}$$

plays the role of π. That this octagonal figure, whose area is easily calculated, so accurately approximates the area of the circle is just plain good luck. Obtaining a better approximation to the area using finer divisions of a square and a similar argument is not easy. Try it. From this statement, the "student" was expected to compute the area of any circle. Fortunately, the concept of proportion was well understood, and thus by scaling, any circular area was easily determined.

Effectively, using the fact that the true area of a circle of diameter 9 is

$$A=\pi r^2=\pi(4.5)^2\approx 63.62$$

and the area of a square of side 8 is 64, we easily see that the *effective* value for π is

$$\pi\approx 3.1605 \quad \text{or} \quad 3\frac{13}{81}$$

Most modern histories give this value to be

$$3\frac{1}{6}.$$

From the Bible:

> Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about. (Old Testament, 1 Kings 7:23)

Thus, biblically speaking, π is 3. Again, proportionality was the tool to get the area of any circle.

Babylon The Baylonians enjoyed a sophisticated society. Located in the Mesopotamian River Valley, they created large monuments to their gods and kings while engaging in commerce throughout the Mediterranian. To do this, they required some considerable mathematical skills. Excellent records of their whole society, including their mathematics, have been preserved in the tens of thousands of clay tablets.

At Susa (about 200–300 miles from Babylon), a new set of tablets were unearthed in 1936. These tablets were on whole concerned with the ratios of areas and perimeters of regular polygons to their respective side lengths. For example, we find

> *The ratio of the perimeter of a regular inscribed hexagon to the circumference of the circle in sexagesimal (base 60) numbers is*

$$0;57,36 = 0 + \frac{57}{60} + \frac{35}{60^2} \approx .9600$$

The correct value of this ratio is $\frac{6}{2\pi}$ and this gives an *effective* value of π to be $3\frac{1}{8}$.

G. DONALD ALLEN

7. PRIMES, PERFECT NUMBERS AND MAGIC NUMBERS (JUST FOR FUN)

We know very well that primes provide one of the most important types of numbers. Their opposites consist of composite numbers, or those that have two factors not equal to one or itself. Many other types of numbers exist. There are the triangular numbers, square numbers, Pythagorean triples, even or odd numbers, rational numbers, and so on. In this article, we take up a few types – just a bit off the well-worn path. For the record, we have:

Definition. A prime number is a natural number that has exactly two distinct natural number divisors: 1 and itself. A composite number is any number not prime.

Theorem. There are an infinite number of primes.

This fact was known to the ancient Greeks. Their proof is very elegant and most students can follow it, but the method of proof is indirect. That is, we suppose there are only a finite number of primes, and then show something goes logically wrong. So, let $p_1, p_2, \&, p_n$ be all the primes. Then, of course, the product $N = p_1\ p_2 \& p_n$ is composite and is divisible by all of the primes. But since $N > 2$ (why?) we know that $N - 1 \neq 2$ is also divisible by at least one of the primes, say p_j. This fact means that both N and $N - 1$ are both divisible by p_j and hence so is their difference. However, the difference is 1, which is impossible. We conclude there are infinitely many primes.

Theorem. (Fundamental Theorem of Arithmetic) Every composite number can be written as the product of two or more (not necessarily distinct) primes; furthermore, this representation is unique, due to the order of the factors.

Primes are a lot of fun and have attracted students and mathematicians for at least two thousand years. Prime numbers today are most useful in public-encryption systems. In earlier encryption systems, an algorithm (e.g. take each letter to the next letter) is used to encrypt the message and the same algorithm, applied backwards (e.g. take each letter to the previous letter) is used to decrypt the message. With a public key system, there are two completely different keys related to large prime (128 digits or more) numbers. The first to encrypt the message is the public one. The second is used to decrypt (the private one). So, with my public key, you can encrypt any message and send to me, but only I can decrypt it. Just too cool!

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 25–28.

MAGIC NUMBERS

Definition. A number p is called magic if, when we sum its digits and then sum the digits of that and so, the result is one.

Example. The following numbers are magic: 28, 7992, 100 for

$$
\begin{aligned}
28 &\to 2+8=10\to 1\\
7993 &\to 7+9+9+3=28\to 2+8=10\to 1\\
100 &\to 1
\end{aligned}
$$

Questions one can ask are whether there are an infinite number of magic numbers, but this is trivial. Why? The reason is that all powers of 10, namely 10^m for m = 1,2,…1, are all magic. In some sense, they form the root class of magic numbers in the sense that every other magic number must have its divisors eventually add up to a power of ten, since the final sum is to be one. As you can see, 28 is at the next level. Let's call it level two. From the calculation above, we see 7993 is one level above 28, on what we will call level three. In this way, we can create higher levels of magic numbers. In sum, we have created a hierarchy of magic numbers.

In consequence, we can ask a myriad of questions concerning hierarchy. For example, how many levels are there? One? Two? ... Infinity? We can also ask for the minimum number in some level. We can also question the rate at which this minimum increases. It is possible, in fact, to pose hundreds of completely useless questions about the levels of magic number – which you may note have no actual value themselves.

PERFECT NUMBERS

Recall the definition a perfect number.

Definition. A number p is *perfect* if its divisors add up to twice itself.

The first four perfect numbers, 6, 28, 496, and 8128, were known to the ancient Greeks. Examples: The divisors of 6 are 1, 2, 3, and 6, the sum of which is 12. The divisors of 28 are 1, 2, 4, 7, 14, 28, the sum of which is $2 \times 28 = 56$. In this definition, we take every number to be a divisor of itself. Changing the word "add" to "multiply" gives the idea of the next definition. Perfect numbers date back to antiquity. Euclid also gives a characterization of how to find them. Later, much later, more than 2000 years later, Euler showed that Euclid's characterization what complete for all *even* perfect numbers. The Theorem is this

Theorem. (Euclid-Euler) An even number is perfect if and only if it has the form $2^{p-1}(2^p-1)$, where (2^p-1) is prime.

Prime numbers of the form 2^p-1 are called *Mersenne primes* and can only occur when p is prime. By 1947, it was established that $p =$ 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107 and 127 generate Mercenne primes. We have the following abbreviated list:

Number	Prime	Mercenne prime	Perfect Number
1	$p = 2$	$(2^2–1)$	$2^1(2^2–1) = 6$
2	$p = 3$	$(2^3–1) = 7$	$2^2(2^3–1) = 28$
3	$p = 5$	$(2^5–1) = 31$	$2^4(2^5–1) = 496$
4	$p = 7$	$(2^7–1) = 127$	$2^6(2^7–1) = 8128$
5	$p = 13$	$(2^{13}–1) = 8191$	$2^{12}(2^{13}–1) = 33,550,336$
6	$p = 17$	$(2^{17}–1) = 1310712$	$2^{16}(2^{17}–1) = 8,589,869,056$

Mersenne primes and their correspondent perfect numbers get very large quickly. To date, only about 47 Mersenne primes have been discovered. The most recent were discovered last in 2008; they are $2^{43,112,609} – 1$ and $2^{37,156,667} – 1$, which are respectively 12,978,189 and 11,185,272 digits long. Are these numbers really big? Well, suppose we type on a page of paper 80 digits per line and 50 lines per page. Then, each page would contain 4,000 digits. This gives a mighty tome of 3,245 pages to hold the new prime number. Mercenne primes with more than a million digits are called *mega-primes*. There are a lot of websites about Mercenne primes. Just Google "Mercenne primes." Is there an infinite number of perfect numbers? This is unknown. It is also unknown if there is an infinite number of Mersenne primes.

Are there *odd* perfect numbers? So far, none have been found, but many have considered the problem. Using complex computer algorithms, it was discovered in 1991 that an odd perfect number, if one exists, must exceed 10^{300}. See, R. P. Brent, G. L. Cohen and H. J. J. te Riele, "Improved techniques for lower bounds for odd perfect numbers," Math. Comp., 57:196 (1991) 857–868. MR 92c:11004. In addition, it must be a perfect square of an odd power of a single prime and be divisible by at least eight primes. There is even more...

Definition. A number p is *multiplicatively perfect* if it equals the product of its divisors other than itself.

Example. Examples of multiplicatively perfect numbers are 6, 8, 27, 125, 134.

Naturally, we may ask if there are an infinite number of multiplicatively perfect numbers, but this is easily answered in the affirmative by the following theorem.

Theorem. Every number that is the product of exactly two distinct primes is multiplicatively perfect.

Proof. Let $p = p_1p_2$ where p_1 and p_2 are primes. Then the divisors of p are exactly (and only) p_1 and p_2

But are there other multiplicatively perfect numbers?

PROBLEMS

1. Find all magic numbers in the interval [1,100]
2. Is the *number of levels* of magic numbers finite?
3. What is the minimum and maximum number of nonzero digits of a magic number on level two?
4. Characterize all magic numbers on level two. (Hint. You will not be able write them all down, but you can give an economical description.)
5. Find the smallest magic number on level three.
6. Show that there are no "twin" magic numbers, namely a pair of magic numbers with (arithmetic) difference two.
7. Let d be any of the integers 1,2,…,9. Let's define the magic-d numbers analogously to magic numbers where the final sum is d. So, for instance, with $d = 4$, the number 85 is magic-4. Note: $85 \to 8 + 5 = 13 \to 4$. Also, magic-1 numbers are the same as the original magic numbers.
8. Show that every number is in one of the magic-d classes. Can you find a relationship between the classes? Why don't we pick $d = 0$?
9. For a given value, d, define the analogous levels of magic-$d$numbers.
10. Prove that every magic-9 number on level 2 (of magic-9 numbers) is divisible by 9.
11. Let p be prime. Then p^3 is multiplicatively perfect.
12. Let p be prime. Then p^k is not multiplicatively perfect if $k \neq 3$.
13. Show that if given, $n = pqr$, or the product of three distinct primes, then n is **not** multiplicatively perfect.

Good luck! These are great problems for students.

G. DONALD ALLEN

8. HOW DO WE DEFINE THE NUMBER "1?"

Preamble. In human history, the origin of the numbers came from definite practical needs.The famous statement by the German mathematician, Leopold Kronecker, (1823–1891), "God made the integers; all else is the work of man," has spawned a lively modern philosophical discussion, and this discussion begins by trying to get a philosophical handle on "1." This approach remains under heavy discussion.

In this short note, we look more toward the historical origins of that most important, number "1." Its history is curiously interesting but its story needs to be told in multiple chapters, as "1," having at least *five* forms or types, is a little more complicated than you might think. It reveals how important "1" is, how naturally it arose, and how abstract it can become.

The story of "1." The first two are the regular types, the ordinal and the cardinal. The *ordinal*, as in first, second, third, etc probably originated first. It was likely used as the order in which certain people or things were presented, perhaps in ceremonies. It is also an aspect of social behavior in animal groups, as we see with the term "alpha" (i.e. first) male or female. In many animal groups (e.g. chimpanzees), there is a ranking of individuals for priority for feeding, grooming, and other animal activities. Most folks understand this as the "pecking order."

The *cardinal* use of "1" puts it as the first counting number. This is the number "1" we think of. It is the "1" we use with all the other natural numbers for arithmetic, to create the fractions, the reals, and the entire world of mathematics. It is also the "1" considered in philosophical discussions, which posits the single unit upon which all the integers are defined, usually as successors. This is the use in most mathematical foundational works. See, for example, the works of Friedrich Ludwig Gottlob Frege (1848–1925).

By Bertrand Russell (1872–1970), there is another concept of "1" that might we call a *state*. Let's refer to this state as "oneness." This abstract idea or comprehension of "oneness" came long after the number one itself. Examples of one stone, one rabbit, and one child preceded our current idea of "one" as "oneness." That is, consider collections of all objects of which the *number state* is "oneness." Actually, Russell explains all this in terms of "twoness." For comparison, other states are color, density, magnitude, and many more.

The idea of "1" as a *unit* is a rather important aspect of this basic number. As a unit, it forms the basis of all kinds of measurement from money to miles. The natural evolution of this idea of "1" most probably arrived only when people had a

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 29–30.

specific need to measure (as opposed to count) quantities. Naturally, the difficulties of teaching the concept of unit for purposes of teaching fractions are legend.

Finally, there is the *relational* use of "one" as in one-to-one. For example, the Vedda tribesmen in their idea of counting cattle and the like had no number words, but did have the concept of *one-to-one* comparing, for example, the size of their herd as equivalent to a number of sticks to which there was made a one-to-one relation.

Conclusion. Most of us use these forms transparently and interchangeably without difficulty. For the newcomer to math, they are all different and all need some explanation at one time or another. Nonetheless, in all said here, there is no arithmetic, little math, and next to no philosophy. The number "1" is foundational in nearly every aspect of our lives – for five different reasons. The hope is you agree that indeed there are five forms of "1," all of which evolved naturally over a great period. The origins of numbers and counting are fascinating. See for example, *The Universal History of Numbers: From Prehistory to the Invention of the Computer* by Georges Ifrah and David Bello, Wiley (2000).

Epilogue. What about *zero*? For this late arrival on the number scene, there is another story here – full books even. See *The Nothing that Is: A Natural History of Zero,* Robert Kaplan, Oxford University Press (1999). And *Zero, The Biography of a Dangerous Idea*, Charles Seife, Penguin Books (2000). http://disted6.math.tamu.edu/newsletter/newsletters/mathstar-tamu-volume1number4.pdf.

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9. PERCENTAGE PROBLEMS

Problems concerning percentages. These are not trick problems. This kind of data and/or situations actually occur. Maybe they are not so simple, however they do require some clarity of thought to resolve them. These problems may be new to your students. The usual methods are used for solving them, but they are posed in a different way.

Problem 1. Jim makes 75% of what Sally makes, and Mary makes 62% of what Jim makes. However, it is known that Jim and Mary together make 1.5 times what Sally makes. How much does each make?

Answer: We do know that Mary makes 62% of 75% of what Sally makes, or in decimals 0.62 * 0.75 of Sally's salary. But… This is an impossible problem as together Jim and Mary make (0.62 * 0.75+0.75) =1.215 of what Sally makes. So, together they cannot make 1.5 what Sally makes. In addition, we can never know how much each makes unless we know how much at least one of them makes. There is the certain confusion of the percentages (62% and 75%) given and the factor 1.5. Unless the student knows the 1.5 means 150%, he/she has little hope of solving the problem.

Answer alternative: What one student could say is that because we don't know what any of the three makes we cannot possibly know what all make. This is a little superficial because the problem is missed completely – although the answer is correct. If there is a tricky part to the problem, it must be that there is *no solution*. Students, at all levels of mathematical learning, are simply not accustomed to confronting problems that have no solution.

Curiously, when multiplying percentages (of the same thing), the result is a percentage, not percentage-squared. This is unlike the product of length measurements, where you do indeed get length * length = length-squared, or area. You can also multiply area by length and get volume, as you know. Yet, multiplying areas doesn't give much of anything, including areas-squared, which has no meaning. No wonder kids have math problems with incredibly basic problems. They don't teach this, I think.

Problem 2. This problem is based on actual data. In 1961, 71% of American men were married, when just last year, 51% of American men were married. So, the teacher said we have a drop of married men over this period of 21%. Sally Ann, from the last row, said she thought is was more like 30%. Who is correct?

Answer: In fact, there is a 21% drop in the number of men actually married – in absolute quantities, but the drop in percentage is just like the drop in any quantities. You take

(previous – now)/previous * 100%

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 31–32.

to get the result, which is about 30%. The confusion is with percentages and what they mean, and what "drop" means.

Problem 3. At camp Wilderness, 45% of the campers got poison ivy, while 70% had mosquito bites. The teacher asked, "How many campers had mosquito bites and not poison ivy?" The teacher then asked, "How many had one or the other?" Bobby answered the first by subtracting to come of with 25%. Karen answered the second by summing to come up with 115%. Ken volunteered his ideas by saying both solutions are incorrect but gave no numbers. Which answers are correct?

Answer: The issue here is with "number-banging," that is combining numbers in some fashion. The tip off for Karen is that adding the percentages gives a result that is unreal, i.e. more than 100%. The problem with Bobby is that he doesn't know how the populations of the mosquito and poison ivy folks interact. Fundamentally, what we haven't considered is that the percentages are relative and not absolute. Ken is correct, and not giving the answers is correct – it cannot be done with the information at hand.

Alternative problem form… At camp Wilderness, 25% of the campers got poison ivy while 70% had mosquito bites. The teacher asked, "How many campers had mosquito bites and not poison ivy?" The teacher then asked, "How many had one or the other?" Bobby answered the first by subtracting to come of with 45%. Karen answered the second by summing to come up with 95%. Ken volunteered his ideas by saying both solutions are incorrect but gave no numbers. Whose is correct? *Answer:* The result is the same as above, but this time, the sum of 25% and 70%, giving 95% as the result, is less a tip-off for a problem with the math. Ken is again correct. We definitely need to know how many had both mosquito bites and poison ivy.

Problem 4. In Mason City, there are 1,000,000 TV sets. What is known is that at a given time, 44% of all TV's turned on were tuned to Channel 3. It is also known that 56,000 TVs turned on were not tuned to Channel 3. How many TV's were tuned to Channel 3?

Answer: The red herring here is the total number of TV's in Mason City. This value is not needed. So, we need to focus on the 56,000 TVs on but not turned to Channel 3. Since this number is 56% (coincidence) of the number of TVs (i.e. 100%–44%), we know there must have been 100,000 TVs turned on. Therefore, there were 44,000 TVs tuned to Channel 3. What is the meaning of the 1,000,000 TVs? It is to make the problem solvable. See below.

Alternate problem. In Mason City, there are 75,000 TV sets. What is known is that at a given time 44% of all TV's turned on were tuned to Channel 3. It is also known that 56,000 TVs turned on were not tuned to Channel 3. How many TV's were tuned to Channel 3?

Answer: This problem cannot be resolved because if there are 56,000 TVs turned to other channels, this represents a fraction of more than 56/75 TVs tuned to other channels, and this value exceeds 56% of the total. Therefore, it is not possible that 44% of TVs turned on are tuned to Channel 3. In this problem, we have inconsistent conditions given.

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10. MICROWAVE MATH

How does a microwave oven work? It works by infusing energy in the form of microwaves. Microwaves are another one of the waveforms in the electromagnetic spectrum of energy. To give a measure of where these fellows are, it should suffice to say their wavelength is between the ultraviolet and radio wave spectra. There are many, many microwaves, but the ones used in household microwave ovens vibrate at *exactly* the "natural" frequency of water molecules, H_2O. In so doing, this increases friction within the molecule and this friction converts to heat. So your microwave oven, sending this type of energy into water-based food, heats up.

[I guarantee not one in five students has even a clue about how these ovens work.]

Other uses of microwaves include spacecraft communication, data transfer, TV, cell phone and telephone communication, radar, and of course microwave ovens. If you've ever received a ticket for speeding, it was those microwaves used in the X, K, and Ka bands that were the likely markers used to measure your speed. (Laser detectors use quite a different technology.)

If the object contains little or no water, little heat is generated within it by microwaves in this region of the (microwave) spectrum. Here's an experiment. Take two identical blocks of wood. Place one of them in the oven at 200°F for three hours and the other submerged in a tub of water for the same time. Place each in your microwave for 2 minutes separately. You will notice the dry block heats up hardly at all, but the wet block becomes rather hot to touch. Thus, the magic of the microwave is revealed.

Time level. This short lesson will take you through timing and proportion. First on the agenda is how microwave ovens keep time. It uses both base 10 and base 60 scales. The rightmost two digits of what you enter will be the number of seconds and the last digits number of minutes. Put in a 45 into the timer and the oven will operate for 45 seconds. Put in 245, and it will operate for two minutes and 45 seconds. There is one small caveat to the base 60 rule, and that is one can enter in any two digits, and these are interpreted as seconds.

Question. Suppose Bill enters 80 into the digital time meter on his microwave and Sally puts in 110 on hers. Both start their ovens at the same time. Whose microwave turns off first?

Answer. Bill put in $80 = 60 + 20$ seconds, and this means one minute and 20 seconds. Sally put in 110, and this means her microwave will cook for one minute and 10 seconds. Therefore, Sally's microwave oven turns off first.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 33–35.

Energy level. The amount of energy you power your microwave at is featured in the power control. It is usually accomplished by entering a digit one through 10. Here, 10 is the highest and default power setting. In the problems below, we assume that by placing nine as the power level, it means the energy is 90% of a 10. The eight is 80% of a 10. The seven is 70% of it, and so on. Now when a lower power energy level is selected, you are naturally introducing less energy according to the level of the descriptor. So putting 9 as the energy level for six minutes introduces 90% of the total energy of putting a 10 for six minutes. So the total energy generated at level a for *b* minutes is *ab*.

We will ignore the amount of energy because that depends on the energy generating tube, the magnetron tube, which generates the microwaves. If you purchase a microwave oven, the power of the oven is determined by the maximum output of the magnetron tube in watts, the basic unit of power. You see this rating especially on light bulbs, where you know that a 100 watt light bulb consumes energy at double the rate of a 50 watt bulb.

Discussion question. On what other household devices do you see the wattage indicated as the measure of how "powerful" the devices may be? (Hint. Electric space heaters, stoves, conventional ovens, etc.)

Question. Suppose Bill puts an object in his microwave at an energy level of 10 for three minutes. Sally decides to use an energy level up six. How many minutes should Sally select so the same amount of energy is introduced into her microwave as for Bill's? This is clearly a problem of proportion. We must have the equality ten over three equals X over six. Solve for X. We obtain X = 6 × 10, divided by three. And of course, this means Sally must put 20 into her timer.

Discussion question. Why is the "defrost" level on a microwave always at a much lower energy level than the maximum level? (Hint. Be practical! Suppose you put into your microwave oven a frozen potato at full strength, say 10. What problem will result? Here we have a realistic concern in that the microwaves don't penetrate equally through any object. For a dense object, the microwaves act upon water molecules nearer the surface of the potato. This is similar to heating an object in a conventional oven. The outer layers or layer absorb the heat first with the interior getting heat only eventually. So, the defrost setting delivers less energy for a much longer period, allowing the microwaves to penetrate to the center of the potato.)

Sample Test for Bill and Sally.

1. Bill puts 220 into his microwave and Sally puts 180 into hers. Both begin at the same moment. Which stops first?
   a. Bill's
   b. Sally's
   c. There is no clear answer.
   d. Both end at the same time.*

2. Bill puts 2222 into his microwave and turns it on. After how many minutes does it turn off?
   a. Two hours, two minutes and 22 seconds.
   b. Twenty two minutes and 22 seconds.*
   c. Two thousand two hundred and 22 seconds.
   d. 22.22 minutes.
3. Sally puts her mac ‘n cheese dish into her microwave with a time setting 130 at a power control setting of 6. Bill wants to heat his same food with the same amount of energy for only 65 seconds. What power control setting should he select?
   a. 9
   b. 3
   c. 6
   d. There is no power level compatible with the 1...9 settings.*
4. Sally puts a bagel with no dish into her microwave oven for 55 seconds. If she puts the bagel on a dish, how much extra time should she add to heat it to the same degree?
   a. 5 seconds
   b. 2 seconds
   c. 1 second
   d. No extra seconds*

*Correct

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11. FANTASY PROBLEMS AND FLYING CARPET SOLUTIONS

One issue in teaching about big numbers is for students to understand just how big numbers can get. With modern computers and storage devices, we have become accustomed to some big numbers, notably megabytes, gigabytes, and even terabytes. In this brief article, we examine just how big numbers can get from innocent sounding problems. This is combined with new ideas of fantasy problems and flying carpet solutions. Fantasy problems are those arising in our minds from an array of possible disconnects with reality, and flying carpet solutions are those offered with disregard of possibility. Some sound good; but fantasy may be the best descriptor.

One particular example is given about tracking of all people, not dissimilar from information you may have about how your various devices can currently track you. It is a good teaching exercise of how numbers can get huge, and completely unmanageable. The tools are only simple facts about the multiplication of exponential powers. It also taxes the largest numbers for which there are actual names, e.g. billion, trillion, quadrillion….

FANTASY PROBLEMS

Given a problem for which there is essentially no verifiable information outside of emotions or beliefs, for which the meaning is unclear, or for which the understanding is at most vague, we call this a *fantasy problem*. On the one level, fantasy problems come from the formation of beliefs and using a pleasing imagination to the exclusion of evidence, reality, and rationality. On another level, fantasy problems arrive owing to ignorance, impracticality, misunderstanding, misconceptions, biased or faulty judgment. Indeed, though applied to making particular conclusions, Kunda (1990) has reported that considerable research has documented ways in which people evaluate evidence in a biased manner.

Some problems of religion may appear to be fantasy problems, and many are eager to dismiss them as exactly so. Yet, many of these are reflective of deep internal workings and yearnings of the mind combined with beliefs, faith, and a need to understand the universe.

Certainly fantasy problems are mixed with wishful thinking and self-deception, even at a state level. Surely, there are some people who believe humans can fly if

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 37–40.

only the wind was just right, the arms were flapped just so, and the person fully believed in the possibility. Some believe that there really is a utopia, and if mankind would only follow simple precepts, it can be ours for all time. Others may ask how many angels can dance on the head of a pin. Sometimes, the problem posed is beyond complex tending, toward grandiose or massive. Sometimes the question is itself not only fantasy-based but vaguely stated. (Remember some problems come in the form of statements.) When such a problem is posed, responses may exhibit a nonsensical nature.

EXAMPLES

1. Who will win the NSF Superbowl this year?
2. End all war. End crime in the streets. End bigotry.
3. Define the universe. Give three examples.
4. From the perennial "C" student… This year I will get all A's.
5. We want to track the location of every citizen of the world each second. This way of determining for interconnectedness of persons of interest will be searchable.

Except for the avid fan/statistician who studies the game intensely and can make a probabilistic estimate of the winner – not the winner, most agree the first is fantasy. If it wasn't, our interest in these kinds of sports would disappear. The second is more serious; many might consider one or more of them as actually possible. The third is clearly a fantasy, as the only precise definition of the universe must be "what is there." To say the least, this is a vague definition, as it is not actually known what is there. The fourth is established as possible only by very rare exceptions, and thus overall is student fantasy.

The last of these is of some interest, as some government operative may consider it a viable option for the actual tracking of people. The procedure is simple to understand. First, embed a transmitting chip in each person, not at all difficult, and then collect the information. Second, process the information. Numbers are involved, however. The main formula we use is familiar: for $a > 0$, and b,c real, $a^b \text{ x } a^c = a^{b+c}$. In the first table below, we show the components of how much data will be collected and stored in one year.

Assuming you haven't seen too many domegemegrottebytes in your daily experience, let's take a look at what this means in terms of more familiar big, actually huge numbers. In Table 2, each value 10^n shown represents a one followed by n zeros. For example, $10^6 = 1{,}000{,}000$.

The terabyte is perhaps the size of your backup storage drive, and it is at least 100 times the random access memory of your computer. This number is puny compared with research from the University of California, San Diego, which reported that in 2008, Americans consumed 3.6 zettabytes of information. Or by Mark Liberman, (2003), who calculated the storage requirements for all human speech ever spoken at 42 zettabytes if digitized as 16 kHz 16-bit audio. By the way, the largest single

Table 1. Tracking data

Tracking the locations of all people by second per year		
Items	**Scientific**	**Named**
Number of people on earth	4×10^9	4 billion
Identifier per person - in bytes	10^2	100
Seconds per year	3×10^7	30 million
Location data- in coordinates (assuming resolution to the square foot)*	6×10^{15}	6 petabytes
Total data per year collected	$\mathbf{7\times10^{34}}$	**70 domegemegrottebytes**

** This is the number of square foot area locations on earth, implying that it takes about 6×10^{15} digits of data to set coordinates to all such areas. Note we've combined traditional names with the more modern prefix-notation – using bytes. The formula used: $S=4\pi r^2$, for the surface area of a sphere (earth) of radius r.*

Table 2. Orders of magnitude

Value	Symbol	Name
10^3	kB	kilobyte
10^6	MB	megabyte
10^9	GB	gigabyte
10^{12}	TB	terabyte
10^{15}	PB	petabyte
10^{18}	EB	exabyte
10^{21}	ZB	zettabyte
10^{24}	YB	yottabyte
10^{27}	XB	xenottabyte
10^{30}	SB	shilentnobyte
10^{33}	DB	domegemegrottebyte

storage drive is about one exabyte, as of this writing. Clearly, the tracking problem is fantastical as realized by this back of the envelop calculation, though sounding realistic at first blush. The domegemegrottebyte is a million, million times more than a zettabyte, already the largest value commonly used to measure uncommonly large quantities.

FLYING CARPET SOLUTIONS

Given a problem for which an unrealistic, impractical, unfeasible, or naïve solution is offered, we call this a *flying carpet solution*. Such solutions are derived for essentially the same reasons as fantasy problems, though probably ignorance and wishful thinking dominate (Bastardi, 2011).

EXAMPLES

Problem. The use of steel bars in prisons is damaging to the self-esteem of the prisoners. Solve this problem. Solutions. Use force fields ala Star Trek. Use an honor system for those willing to sign a pledge not to escape.

Problem. Fix the educational mess. Solutions most commonly offered. (a) Change the curriculum. (b) Put education on a firm business model. What is remarkable is that these solutions have been offered time and again, and incredibly they have proponents. Unrealistically, many believe such simplistic solutions will actually work.

Problem. Find a completely green system of transportation within a community. Solutions offered. Bicycles for everyone. Everyone should use a flying carpet. Use only electric vehicles.

This example could be presented in a single class period, with students gaining an understanding of how easily big numbers arise in their lives.

REFERENCES

Bastardi, A., Uhlmann, E. L., & Ross, L. (2011). Wishful thinking: Belief, desire, and the motivated evaluation of scientific evidence. *Psychological Science, 22*(6), 731–732.

Kunda, Z. (1990). The case for motivated reasoning. *Psychological Bulletin, 108*, 480–498.

Lieberman, M. (2014). *Zettascale linguistics* (2003–2014). Retrieved from http://itre.cis.upenn.edu/~myl/languagelog/archives/000087.html

SECTION 2

ALGEBRA

G. DONALD ALLEN

12. THE MYSTERY OF THE NEW "PLANET" (OR HOW REGRESSION SAVED THE DAY)

In 1781, the new planet, Uranus, was discovered. It was found to satisfy the *Titius-Bode Law,* formulated by Johann Bode and Johann Titius around 1770, that the semi-major axis of a planet of the first seven planets from the sun are proportional to the numbers $0.4 + (0.3 \times 2^n)$, for $n = -\infty,0,1,2,4,5,6$. The planets are Mercury, Venus, Earth, Mars, mystery planet, Jupiter, Saturn, and Uranus. Now the contest was to find and identify this planet.

$-\infty$	Mercury
0	Venus
1	Earth
2	Mars
3	Asteroid
4	Jupiter
5	Saturn
6	Uranus

On the first of January in 1801, the Italian astronomer Joseph Piazzi discovered a planetoid. He named the planetoid, Ceres, and observed it 22 times over a period of 41 days until it disappeared behind the rays of the sun. The angle swept out by Ceres was only 9°, and the problem became to determine its orbit to see if it was planet 3 in the Titius-Bode Law.

The situation was that there was very little information from which to compute the orbit. There was no precedent for doing this. When Uranus was discovered in 1781, its discoverer, William Herschel, had data from many, many observations in the sky. Not only that the orbit was nearly circular (though we know it to actually be an ellipse), so the simplifying assumption allowed astronomers to know the complete orbit with excellent accuracy.

So what could be done with so little data over such a small angle? The orbit was known to be elliptical but the eccentricity of the ellipse was not known and could not be assumed to be one value or another. Recall the eccentricity, e, of an ellipse is its

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 43–46.

measure of flatness. A circle has eccentricity of 0, and eccentricity can range from 0 to one. An ellipse has the formula $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Assuming that $a > b$, the eccentricity of the ellipse is given by $e=\frac{\sqrt{a^2-b^2}}{a}$.

At the age of just 18, a budding young mathematician, Carl Frederich Gauss, heard of this problem and had just happened to have developed his new method of least squares. By this method, one can fit data to a model curve in such a way that the sum of the squared deviations of the data to the curve is minimized. It's sort of like a grand average that results in determining the best parameters *a* and *b*.

Gauss in about 1803

With least squares, he was able to avoid assumptions of eccentricity and proceed directly to the computation of the equation of the ellipse itself, computing the eccentricity as a by-product. Making corrections for the rotation of the earth's axis, the earth's rotation about the sun during that period, he was able to do so with over 100 hours of computation! At this time, all calculations were made by hand and by experts. So, while we may be impressed with this amount of time at computation, it was not uncommon at the time. The calculations were published in September of the same year, together with the calculations of others. When astronomers found

Ceres again toward the end of November of the same year, it was determined that its position was very nearly that predicted by Gauss. Gauss' determination of Ceres' orbit made him instantly world famous in academic circles and established his reputation in the scientific and mathematical communities. It won him a position as director at the Gottingen Observatory, where he remained his entire life. Remarkable as this story is, it was only one of many absolutely incredible results Gauss produced in his lifetime.

The end of the story is that Ceres, with a diameter of 1003 km, did satisfy the Titius-Bode law for $n = 3$. However, it is merely the largest of the known asteroids in the asteroidal belt between Mars and Jupiter. While nearly circular, the orbit eccentricity is about 0.0789. The equation of an ellipse with this eccentricity is given by $x^2 + \frac{y^2}{(0.99688)^2} = 1$, while the equation of the circle is $x^2 + y^2 = 1$.

You really can't see the difference between the graphs of these curves.

Remark: The astronomical methods described in Theoria Motus Corporum Coelestium, the book in which Gauss eventually published his results, are still in use today. Only a few modifications have been necessary to adapt them for modern computers.

ECCENTRICITY

As indicated above, an ellipse has the formula $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The values a and b are called the semi-major and semi-minor axes. Assuming that a > b , the eccentricity of the ellipse is given by

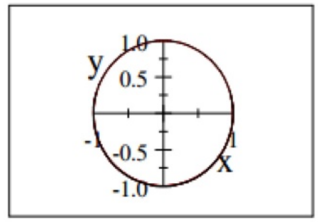
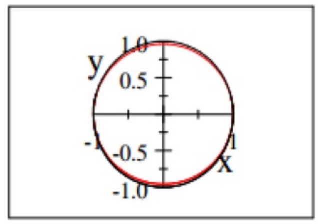
$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

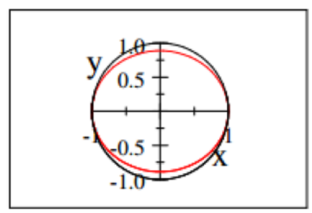
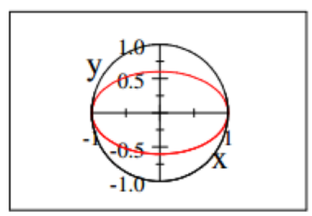
So, using this formula we can determine the values a and b for an ellipse with given eccentricity. To find the formula given above we took a = 1 and solved the equation

$$\frac{\sqrt{1 - b^2}}{1} = \sqrt{1 - b^2} = e$$

$$b = \sqrt{1 - e^2}$$

Thus, with e = 0.0789, we get naturally, by scaling, i.e. $b = \sqrt{1 - e^2} = 0.99688$. Multiplying a and b by a factor, we can make the ellipse to have any proportional semi-major axes with the same eccentricity e. Let's look at a few more eccentricities and compute the b in the table and graph the resulting ellipse as inscribed in a unit circle.

e	b	ellipse	Graph
0.1	0.994 99	$x^2 + \frac{y^2}{(0.99499)^2} = 1$	
0.25	0.968 25	$x^2 + \frac{y^2}{(0.968\,25)^2} = 1$	

e	b	Ellipse	Graph
0.50	0.88603	$x^2 + \frac{y^2}{(0.88603)^2} = 1$	
0.80	0.60000	$x^2 + \frac{y^2}{(0.88603)^2} = 1$	

REFERENCES

Hall, T. (1970). *Carl Friedrich Gauss: A biography*. Cambridge, MA: MIT Press.

Herrmann, D. (1984). *The history of astronomy from Herschel to Hertzsprung*. New York, NY: Cambridge University Press.

Schorn, R. (1998). *Planetary astronomy: From ancient times to the third millenium*. College Station, TX: Texas A&M University Press.

DIEM M. NGUYEN

13. GEOGEBRA IN THE CLASSROOM

One of the most effective uses of software technology is the seamless relationship between graphs and their functions. Algebraic functions essentially come alive with the use of the software, allowing students to gain a greater understanding between the algebraic symbols, their graphs, and the meanings of the graphs (Mackrell & Johnston-Wilder, 2005). Indeed, the use of mathematical software has drastically changed the way mathematics is presented to students. The visualization of abstract functions, the aspect that would take gigantic efforts and plenty of times with the use of just a chalkboard and chalk (Pimm, 2005), can be done fondly, quickly, and meaningfully with the use of a dynamic software. As technology becomes an integral aspect of mathematics teaching, delivering an algebraic lesson with dynamic applets is the goal of the teacher in the classroom.

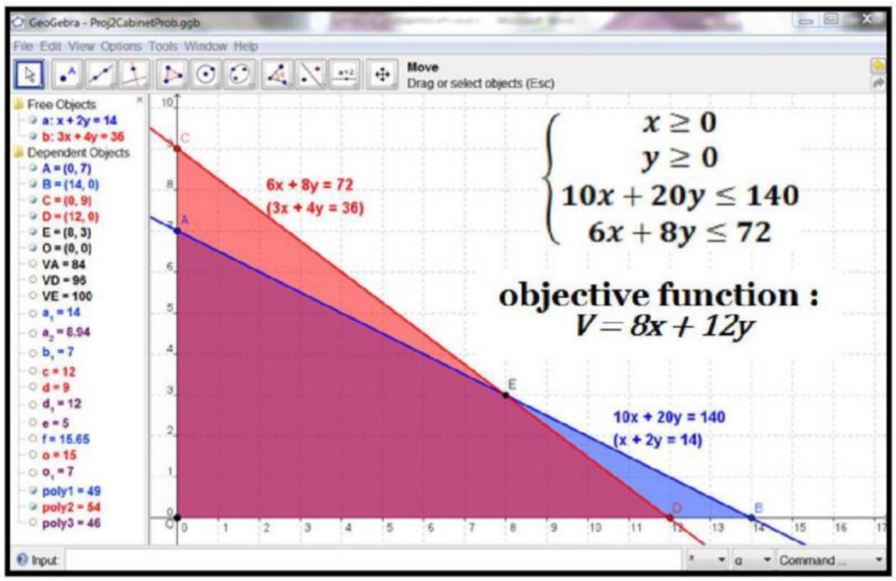
Indeed, recently developed *GeoGebra* software enables teachers to lively demonstrate abstract functions and allow students to explore concepts and make conjectures (Hohenwarter, 2009). *GeoGebra* is a free, dynamic web-based software package that is becoming popular in mathematics classrooms because of its accessibility and functionality. It can be used on any computer platform. To install *GeoGebra*, visit: http://www.geogebra.org/.

GeoGebra was developed on the KISS principle of "keep it short and simple." This simplicity allows users to use the program with limited computer skills. *GeoGebra* lends itself to being a student-centered program that requires active learning. The fact that students can freely manipulate work, investigate changes, and create algebraic functions, based on shapes, makes *GeoGebra* more valuable than any existing calculator or textbook resource.

According to Grandgenett (2007), *GeoGebra* effectively bridges algebra and geometry in an easy to use visual manner. It has unlimited uses in regards to polynomial and trigonometric functions and their graphs, which allows students to gain a greater understanding between graphs and functions through explorations. Many of the functions are akin to those found on Geometer's Sketchpad, but all objects are placed on a coordinate plane, easing the transition to algebraic interpretation of the geometric objects. The aforementioned objects can be created using tools from the toolbar interface or by manually defining them in the dependent objects – independent objects interface. The PowerPoint lesson, *GeoGebra* – A Simple Way to Solve Linear Programming Problems located at http://podcast.bgsu.edu/dnguyen/Courses/GeoGebra/GeoGebra%20

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 47–48.

Linear%20Programming%20Problems.ppt and complete with audio instruction will demonstrate:



How to use GeoGebra to solve a basic linear programming problem

- a sample linear programming problem to use in the classroom, along with its *GeoGebra* solution
- a few *GeoGebra* tips for beginners.

REFERENCES

Grandgenett, N. (2007). GeoGebra. *Mathematics and Computer Education*, *41*(3), 276–278.

Hohenwarter, J. (2009). Introducing dynamic mathematics software to secondary school teachers: The case of Geogebra. *Journal of Computers in Mathematics and Science Teaching, 28*(2), 135–146.

Mackrell, K., & Johnston-Wilder, P. (2005). Thinking geometrically. In S. Johnston-Wilder & D. Pimm (Eds.), *Teaching secondary mathematics with ICT* (pp. 81–100). Milton Keynes: Open University Press.

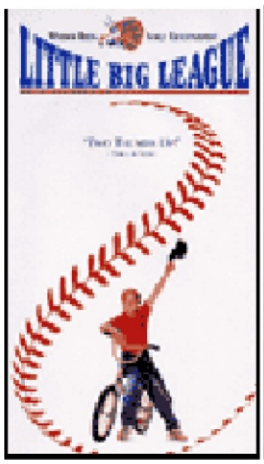
Pimm, D., & Johnston-Wilder, S. (2005). Technology, mathematics and secondary schools: A brief UK historical perspective. In S. Johnston-Wilder & D. Pimm (Eds.), *Teaching secondary mathematics with ITC* (pp. 3–17). Berkshire: Open University Press.

G. DONALD ALLEN

14. WORKING TOGETHER

The Math Part

In the movie, *Little Big League,* Billy Heywood needs to solve a homework question: Bill can paint a house in five hours, and Mary can paint a house in three hours. How many hours did it take to paint the house? Suppose there were other painters, as well.



This is a rate problem. You need to deal with rates!!

Answer. We need to find a common basis for comparing the painters. We know that Bill paints a house in five hours and Mary paints a house in three hours. This means that

Bill paints $\frac{1}{5}$ house in one hour.

Mary paints $\frac{1}{3}$ house in one hour.

Together, they paint $\frac{1}{5}+\frac{1}{3}$ house in one hour.

We need the *rates equation*:

$$\left(m\frac{\text{houses}}{\text{hour}}\right)\cdot\left(n\text{ hours}\right) = mn\text{ houses (painted)}$$

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 49–52.

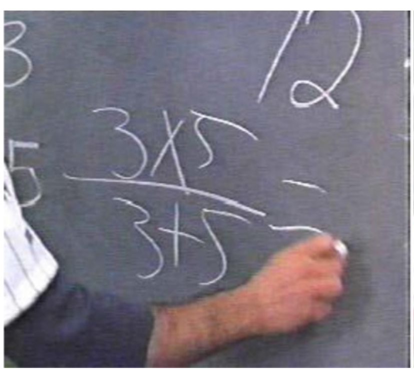
For our problem, we want just one house painted. So

$$\left(\frac{1}{5}+\frac{1}{3}\frac{\text{houses}}{\text{hour}}\right)\cdot(n \text{ hours}) = 1 \text{ houses (painted)}$$

$$\left(\frac{1}{5}+\frac{1}{3}\right)n = 1$$

$$\frac{3+5}{3\cdot 5}n = 1$$

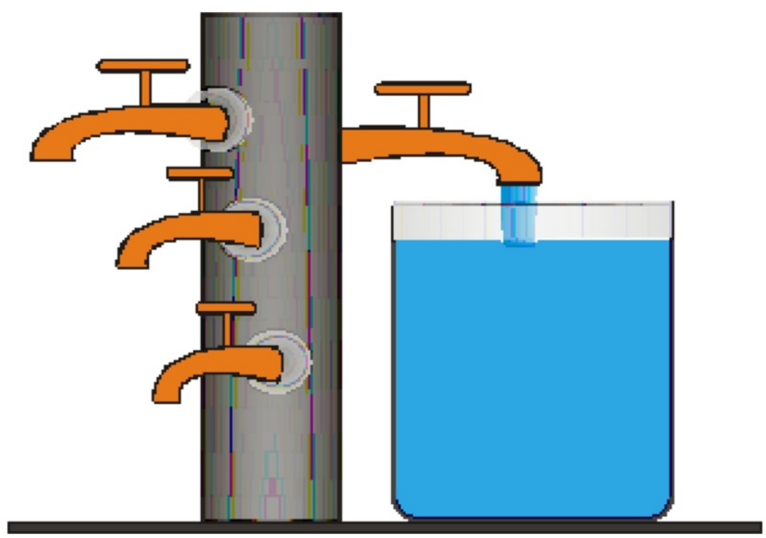
$$n = \frac{15}{8} = 1\frac{7}{8} \text{ hours}$$



What you see in the picture above is the formula, $\frac{3\times 5}{3+5}$, used for the correct answer. But, this doesn't explain how to solve the problem. The principle behind solving the problem must be understood. Indeed, in the problems below, all of which are similar rate problems, just knowing the answer to the problem above, will be of little help.

Now, it's your turn. Use the rates equation and/or ideas above.

1. Bob paints a house in 4 hours, and Ted paints a house in 3 hours. How long does it take them to paint two houses? (Hint. The wrong answer: 3 + 4 = 7 hours. Use the rate equation!)
2. Sean paints a house in 6 hours, Gary paints a house in 2 hours, and Susan paints a house in 3 hours. Working together, how long does it take them to paint a house?
3. Mary and Sally paint a house in 6 hours, when working together. Mary paints a house by herself in 2 hours. How long does it take Sally to paint a house by herself?
4. Bob takes two days longer than Ted, when painting a house. Together they can paint a house in two days. How long does it take each to paint a single house? (Taken from a 19th century algebra text)
5. Of the four spouts, one fills the whole tank in one day, one in two days, one in three days, and one in four days. How long will it take all four to fill the tank? (From the Greek *Anthology,* c. 500 CE)



AMANDA ROSS

15. A BRIEF LOOK AT CIRCLES

The study of circles spans pre-kindergarten through higher education. During the early grades, students are able to recognize circles and categorize them as plane, two-dimensional shapes that are not polygonal. Students later learn to create circles, explain the meaning of circles, and transform circles. In this brief article, an overview of some of the big topics, related to instruction of circles through high school, is presented.

HOW TO CONSTRUCT A CIRCLE

In the early grades, a circle may be constructed, using a piece of string, tied to a pencil. Students place the end of the string on a piece of paper and then rotate the string in a circular fashion, tracing the circle, with the fastened pencil. Students learn the meaning of the center of a circle and radius in a very concrete way, using this approach.

Later, students are able to construct a circle, using a compass. Students understand the point represents the center of the circle, while the width of the opening represents the radius.

In addition, students may be asked to construct a circle, using a table of values or a list of ordered pairs.

HOW TO CREATE A CIRCLE WITH A CENTER AT THE ORIGIN

In high school, students learn to create circles with a center at the origin, given the radius, or the equation of the circle. For example, students may be asked to create a circle with a center at the origin and radius of 4. The equation of the circle is $x^2 + y^2 = 16$. Students may manually construct the circle on grid paper by placing the center at (0, 0) and plotting the intercepts, (4, 0), (−4, 0), (0, 4), and (0, −4). Students may also construct the circle, using either a graphing calculator or geometry software, such as GeoGebra (www.geogebra.org). When using GeoGebra, a student may enter a point on the circle and choose the function that creates a circle, from a given center point and point on the circle, or directly enter the equation of the circle.

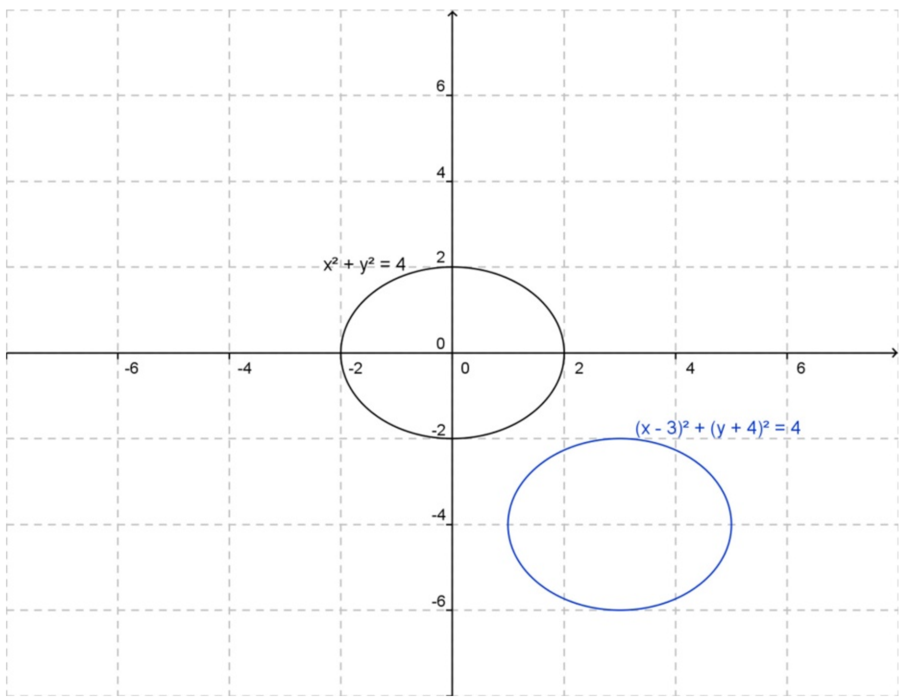
Note. The equation for a circle, with radius, r, is $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) represents the center of the circle. A circle, centered at the origin, has a

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 53–56.

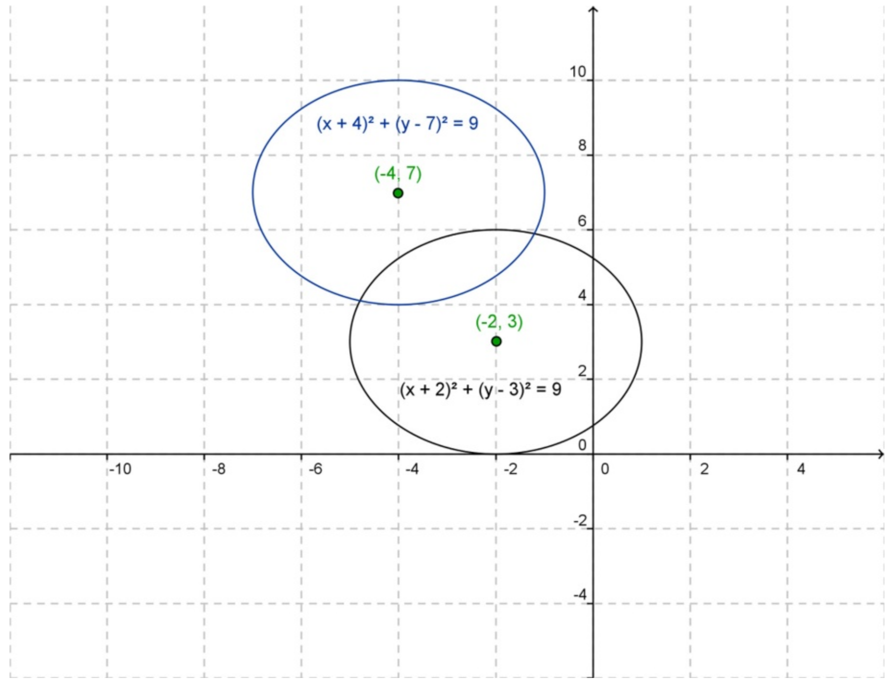
center at (0, 0). Therefore, the equation for a circle, centered at the origin is simply $x^2 + y^2 = r^2$.

HOW TO CREATE A TRANSLATED CIRCLE

A translated circle is any circle with a center not at the origin, or (0, 0). Suppose students are asked to graph the circle, given by the equation, $(x - 3)^2 + (y + 4)^2 = 4$. This circle is shifted 3 units to the right and 4 units down, from a circle, with a center at the origin and the same radius. Refer to the diagram below:



Now, suppose students are asked to translate a circle, with a center *not* at the origin. For example, suppose students are asked to shift the circle, given by the equation, $(x + 2)^2 + (y - 3)^2 = 9$, 2 units to the left and 4 units up. Students should be able to conceptually link the shift of 2 units to the left with a decrease of 2 units to the x-values. Likewise, students should be able to connect the shift of 4 units up to an increase of 4 units to the y-values. The center of the original equation is $(-2, 3)$. Thus, a shift of 2 units left will result in an x-value of -4. A shift of 4 units up will result in a y-value of 7. Therefore, the translated equation is $(x + 4)^2 + (y - 7)^2 = 9$. The translation is shown below:



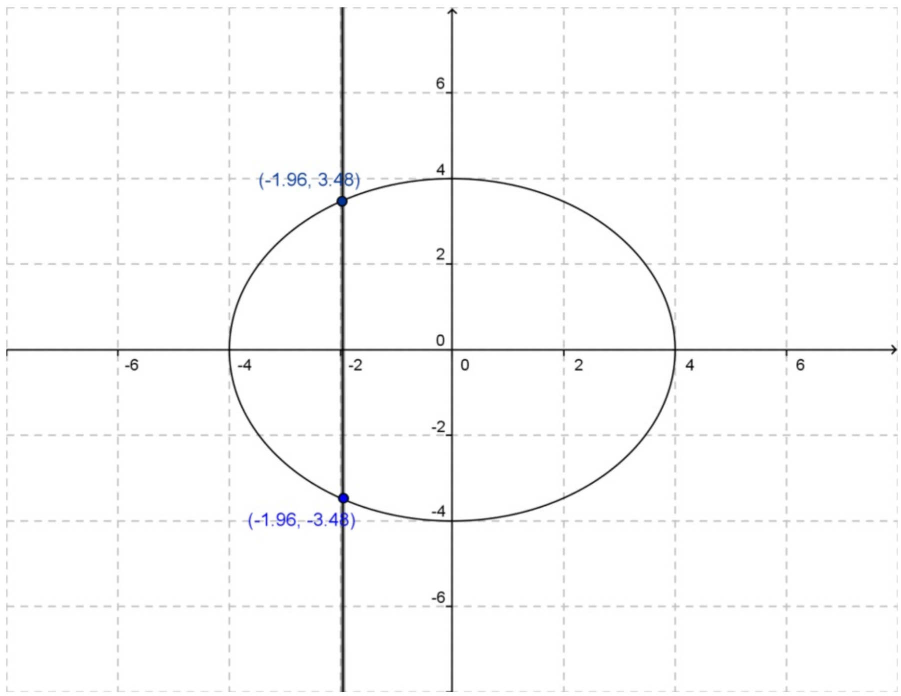
HOW TO CREATE A CIRCLE USING A GRAPHING CALCULATOR

When graphing a circle, using a graphing calculator, the positive square root may be entered in Y_1, with the negative square root entered in Y_2. The equation, $x^2 + y^2 = 16$, should first be solved for y. Doing so gives: $y^2 = -x^2 + 16$. Taking the square root of both sides gives: $y = \pm\sqrt{16 - x^2}$. The positive square root, or $\sqrt{16 - x^2}$, may be entered in Y_1. The negative square root, or $-\sqrt{16 - x^2}$, may be entered in Y_2. Note, you may also enter Y_1 in Y_2.

EXPLAINING WHY A CIRCLE IS NOT A FUNCTION

Students often incorrectly think that a circle is a function. For students who do realize a circle is not a function, they may be limited in their justification. Students may not be able to relate the requirements of a function to the vertical line test. In order to promote conceptual understanding, related to relations that are or are not functions, students must have a firm understanding of the definition of function. By definition, a function is a relation, whereby each element of the domain is mapped to one and only one element in the range. Therefore, given the ordered pair, (4, 0), found in a particular relation, the x-value of 4 may not be mapped to any other y-value. If it is, the relation is *not* a function.

The circle below highlights the mapping of the *x*-value of −1.96 to the *y*-values of 3.48 AND −3.48. Therefore, the circle is *not* a function.



Students need a conceptual understanding of the structure of circles. Students must be adept at manual construction of circles, as well as employment of various technologies. Students must also be facile at creating and using various representations of circles, including transformations of circles.

G. DONALD ALLEN

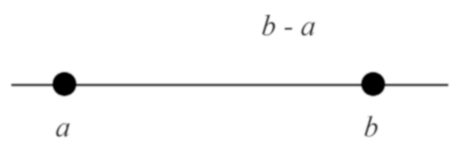
16. OPTIMIZATION – WITH AND WITHOUT CALCULUS

Calculus is full of *optimization* problems, some to minimize and some to maximize. This is a fact all calculus students come to know. But in fact there are lots of minimization problems that are solved from the early grades through high school. The central theme in many of these is geometry, or more specifically, the distance between two points. Many problems are understandable to students when they relate to something they know. This article is divided into a few short sections.

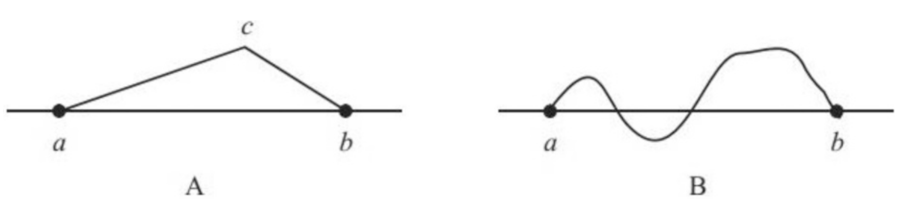
1. Distances between points in the plane
2. The first type of optimization problems – running, fencing
3. The second type of minimization problems – multi-media problems

1. DISTANCES BETWEEN POINTS IN THE PLANE

If we look at the real line and two points on it, say, and if we ask the question of the distance between *a* and *b,* everyone will say it is *b* – *a*

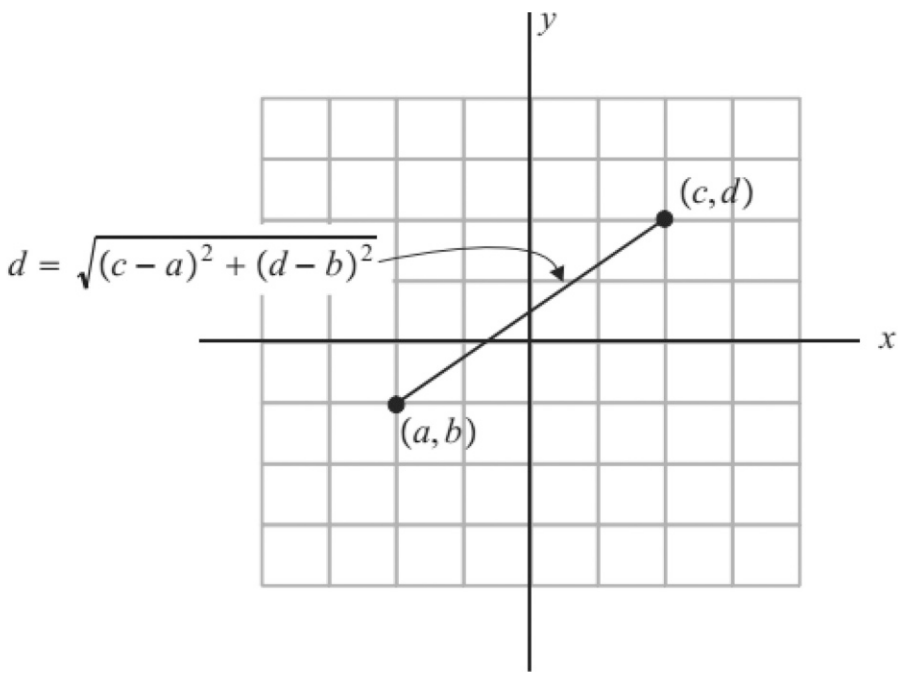


Of course, this is true because that is the definition of distance. However, if we ask the question, "What is the shortest distance between a and b, along some path?," this problem is suddenly a little puzzling. But then you come to your senses and realize that *b* – *a* is the minimum distance. But the problems still nags a little. What if there is another path over which the distance is shorter? Let's consider briefly. Two cases are shown below.



G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 57–66.

In the first case, A, if the path is from *a* to *c* to *b,* and the same distance is used, we can appeal to the fact from geometry that the sum of two legs of a triangle is greater than the third. However, how do we handle the second example, B? It's just not clear what can be done. Indeed, it is a problem of the calculus of variations, very advanced, that is required to prove there is no path that produces a shorter distance between the points. Throughout, this will not be a worry. *We assume the shortest distance between two points is the length of the straight line joining them.* So even in the plane, we have this. The distance between the two points, (*a,b*) and (*c,d*), is given by the value $d = \sqrt{(c-a)^2 + (d-b)^2}$, which is determined by the Pythagorean Theorem.



So much for the review.

2. The first type of optimization problems - running and fencing

Example 1. First we minimize cost. Suppose that we need to build a fence from (1,2) to (5,8) and fencing costs \$25/foot. What is the minimum cost to build this fence? The cost for the fence is

Cost of fence = \$25 x *length of fence* Since we are looking for the *minimum cost,* we use the minimum path, i.e. the straight line. Hence the minimum cost is

$$25 = \sqrt{(5-1)^2 + (8-2)^2}$$

or about \$180.28.

Example 2. Next we minimize time. Suppose Bob can run at 7 miles per hour, and needs to run from (1,2) to (5,8) . What is the minimum time he can run to get there? For this problem we need the relation between time *t,* distance *d,* and speed *v.* That formula is

$$d = vt$$

or t = d/v

or about 1.0302 hours.

In both these example, what to do is rather straight forward because we know the "minimum" whatever will be related to distance or length between points. We begin with a maximum area problem.

Example 3. A rancher needs to construct a rectangular stock pen, using the local creek as one of the sides. The other three sides will be constructed out of 800 m of fencing material (see Figure 1).

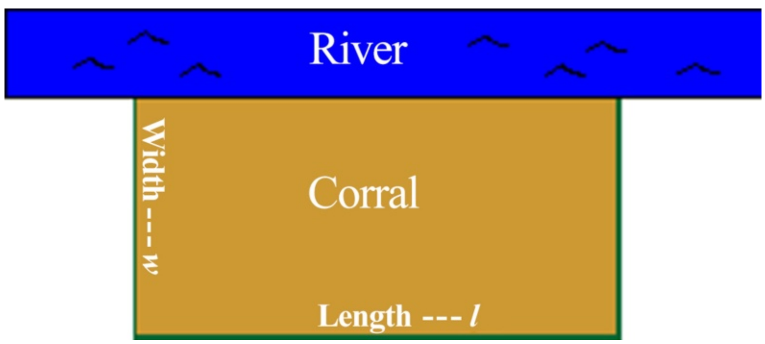


Figure 1. Rectangular stock pen diagram

He can make the stock pen a variety of shapes-long and thin, short and fat, square or any shape in between. We want to determine which rectangular shape will give him the maximum area. The area is $A = Iw$, and the perimeter for fencing is $P = 2w + l$. With 800 m of fencing material we know that

$$2w + l = 800$$
$$l = 800 - 2w$$

Now substitute into the area expression to obtain

$$A = w(800 - 2w)$$
$$= 800w - 2w^2$$

Here is a good case to use variables other than *x*, since students are already accustomed to using these symbols for the unknown quantities as considered. However, you may wish to use *x*, particularly if you will use a graphing calculator in this problem. Let's graph *A*, as in Figure 2.

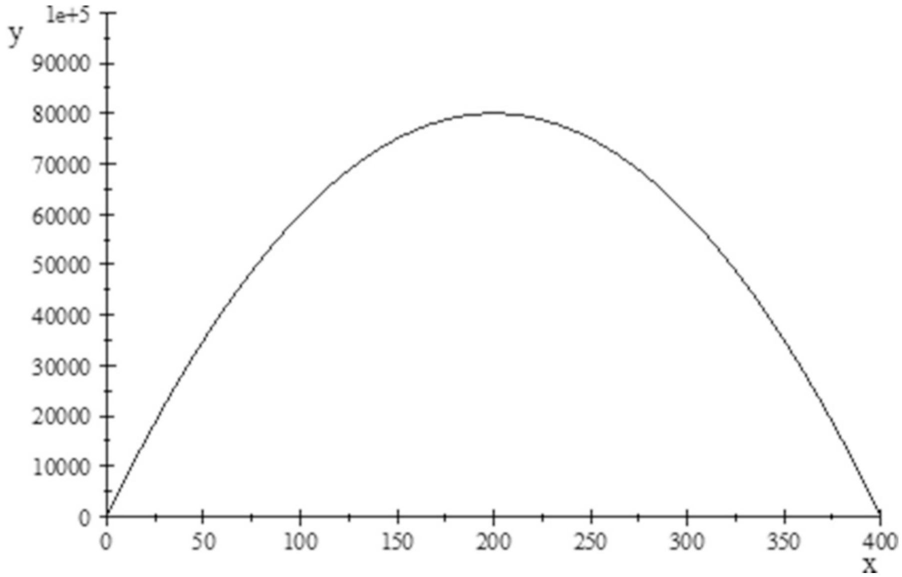


Figure 2. Area as a function of width

Graphing calculator solution: Graph A and trace to see that the maximum occurs when the width is 200 ft. This gives the length to be 400 ft. At this value, the area is $A = lw = 200 \cdot 400 = 80{,}000$ ft^2.

Algebra solution. We know the maximum of A (as a parabola) occurs at its vertex. This is easily computed to be when $w = 200$. The length is 400 and the area is the same. *Calculus solution.* We wish to maximize A. Find the critical points by differentiating the area A with respect to w and set it to zero.

$$\frac{dA}{dw} = \frac{d}{dw}\left(800w - 2w^2\right) = 800 - 4w = 0$$
$$w = 200$$

In all three, we get the same solution. In the first, the graphing solution, we get the correct answer only because tracing is particularly easy. Had 1821 ft of material or some other not very nice number been given, the tracing would have an error. However, the other two methods give exact solutions.

In the next set of problem, we upgrade the complexity just a bit. These are of the two media variety. For example, we may wish to minimize the cost of a fence where the construction is made of two different types of fencing, or the cost of construction differs because of some reason, such as proximity to a precipice. Or we may wish to minimize the time of transit when the path taken is through two media. For example, on part of the path, the speed of passage has one value and on another part of the path, the speed is something else.

3. The second type of minimization problems - multi-media problems
   Under this category, we need to find a path which minimizes some related quantity, something like the problems above. The rather important difference here is that the path is over two different terrains which are associated with two different rates. You can check out the pony problem that appeared in a recent issue of our Texas A&M University newsletter. See, Why Study Math when I Have Technology, in Volume 4, Number 2, April 2012 we considered the pony problem. It is similar to those below. (http://disted6.math.tamu.edu/newsletter/newsletters_new/2012_4_02.pdf)

Example 4. In the next problem, a lifeguard must rescue a swimmer. A lifeguard on a beach with a straight shoreline sees a swimmer needing help 70 yards down the beach and 40 yards out in the water. The lifeguard wishes to reach the swimmer in the fastest possible time. He can run down the beach at 7 yards/second and swim 2 yards/second. What path should he take to arrive in the minimum time?

Note. This is a classical calculus problem that students will usually see in their first semester of college calculus. What troubles the students most is how to set it up from the problem description. One reason is that there are so many possible paths available. The student must assign a variable to the problem and work exclusively from that variable and the given data in the problem.

Assumption. The lifeguard will run down the beach parallel to the shoreline a certain distance and then enter the water and swim in a straight line to the swimmer. Any other variation will intuitively make the trip take longer. This can be proved mathematically using an advanced subject called the calculus of variations, but doing so would take us far past the scope of this problem.

Make a diagram. It is key to make a diagram of the situation, as it is difficult to build a mental image of what is required. See the illustration below.

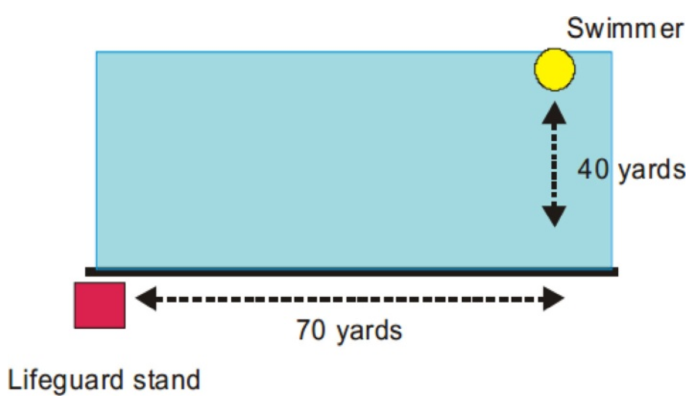


Figure 3. Diagram for the swim/race

Here is a possible path for the lifeguard. Run *x* yards down the beach and then swim. See figure below. Another path would be to swim directly from the lifeguard stand to the swimmer. As we will see, this is not good because swimming is so much slower and the trip-time would be longer, even though the distance is shortest.

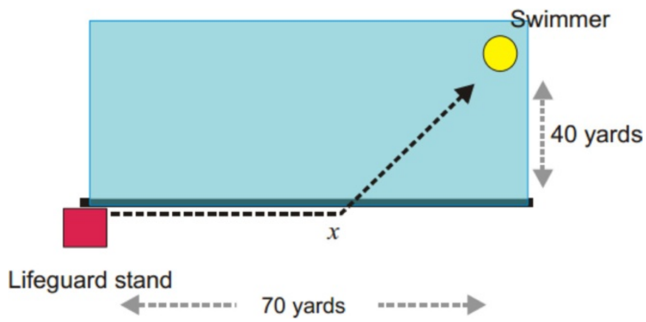


Figure 4. Diagram for swim/race with variable x

Information needed: From this diagram, it is clear that two distances are involved in the problem, the distance the lifeguard runs down the beach, and the distance from the water's edge to the swimmer. Thus we need the following information. (1) The distance between two points. (2) The formula relating distance and time given velocity. (3)

Given two points *(a,b)* and *(c,d),* the distance between them is given by

$$(c - a)^2 + (d - b)^2$$

For an object traveling x meters are a velocity of v meters/second, the distance traveled in t seconds is

$$d = vt$$

Since we will be seeking a "shortest time," we solve for t to get

$$\frac{dT}{dx} = \frac{d}{dx} = \left(\frac{x}{7} + \frac{\sqrt{(70-x)^2 + 40^2}}{2} \right) = \frac{1}{\sqrt{x^2 - 140x + 6500}} \left(\frac{1}{2}x - 35 \right) + \frac{1}{7} = 0$$

$$x = 70 - \frac{16}{3}\sqrt{5} = 58.074 \qquad t = \frac{d}{v}$$

$$\frac{1}{2}x - 35 = -\frac{1}{7}\sqrt{x^2 - 140x + 6500}$$

$$T = 500000x + 1000000\sqrt{3^2 + (6-x)^2}$$

Both formulas should be familiar to the student. Without such familiarity, the student will not know how to proceed. So, be sure the students have the proper mathematical tools before setting them on a problem-solving task. On the other hand, this problem could be rephrased solely in terms of what measures are needed to solve the problem. In this inquiry-based approach, students will be ready for the formulas, knowing they have a purpose to solve an interesting problem.

Setting up the problem. Set up a coordinate system. Place the origin at the lifeguard stand. The positive x-axis will be the beach at the water's edge. The positive y-axis will be perpendicular to the beach in the direction of the water. The swimmer is located at the point (70, 40), i.e. 70 yards down the beach and 40 yards out into the water.

As mentioned earlier, there must be a variable. Let the distance the lifeguard runs down the beach be denoted by x. Clearly, $x < 70$; the lifeguard would certainly not overshoot the swimmer laterally. Sometimes the dual use of the symbol x to label the axis and also to denote a variable point on it causes confusion. The alternative is adding another variable - this causes unneeded complexity. So, the two legs of the trip are from (0,0) to $(x,0)$ and then from $(x,0)$ to (70,40). The two distances are respectively

$$\sqrt{(70-x)^2+(40-0)^2}$$

We compute the times traveled from the given distances and velocities to be

$$\text{Time running on the beach} = \frac{x}{7}$$

$$\text{Time swimming} = \frac{\sqrt{(70-x)^2+40^2}}{2}$$

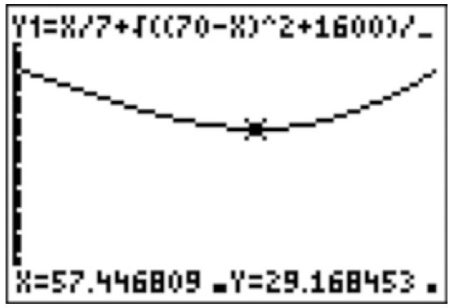
This gives the total time T to be

$$T = \frac{x}{7} + \frac{\sqrt{(70-x)^2+(40)^2}}{2}$$

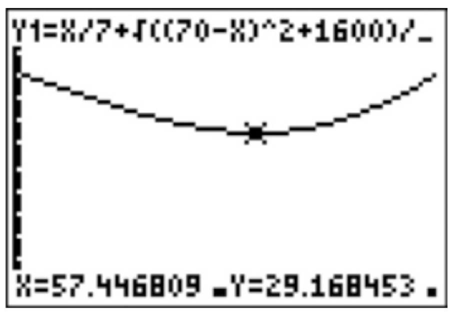
This expression gives the transit time as a function of how far the lifeguard runs down the beach before entering the water. It remains to minimize *T,* that is, to find x where the time function is minimized. To achieve this requires some calculus, and this cannot be assumed in a pre-calculus course.

Graphing calculator solution. A graphing calculator can be used to approximate the minimum. The graph of the function is shown below. Since the vertical axis is the time axis, we see the minimum time is in the vicinity of $x = 57$.

Using a TI-84 graphing calculator we see



We have turned on the trace function and placed the cursor at approximately the minimum value. It seems to be about 57.45. But moving the cursor every so slightly, we see this



Calculus Solution. As in the above problem, we differentiate the appropriate function to find critical points. So with

$$T = \frac{x}{7} + \frac{\sqrt{(70-x)^2 + 40^2}}{2}$$

We solve the following for *x*.

$$\frac{dT}{dx} = \frac{d}{dx} = \left(\frac{x}{7} + \frac{\sqrt{(70-x)^2 + 40^2}}{2} \right) = \frac{1}{\sqrt{x^2 - 140x + 6500}} \left(\frac{1}{2}x - 35 \right) + \frac{1}{7} = 0$$

$$x = 70 - \frac{16}{3}\sqrt{5} = 58.074$$

To solve this equation algebraically, it must first be rewritten as

$$\frac{1}{2}x - 35 = -\frac{1}{7}\sqrt{x^2 - 140x + 6500}$$

(Don't forget, you have assumed the radical term was nonzero when you multiplied across the equality.) Now square both sides and solve the resulting quadratic. The graph of

$$T = \frac{x}{7} + \frac{\sqrt{(70-x)^2 + 40^2}}{2}$$

is shown below. Note when $x = 0$, the T-value, just over 40, corresponds to the time swimming all the way – the slowest path of all even though the distance to the swimmer is smallest. Also, the proper domain of this function is [0, 70]. Why?

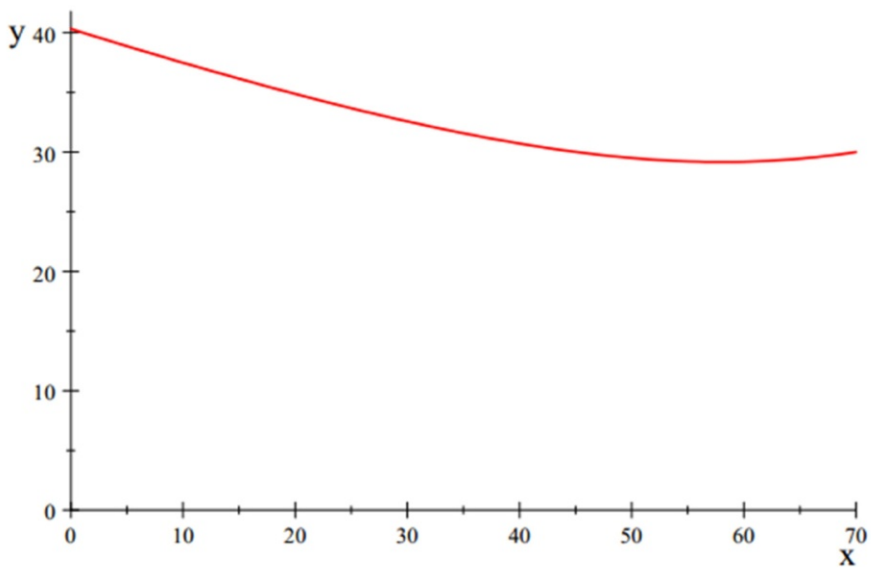


Figure 5. Finding the minimum

There are countless variations on this example. The next problem, another variation, involves a path through mixed media where the costs are different.

Example 5. Osega oil services company was hired to build a pipeline Point *A* on one side of a river to Point *B* on the other side of the river. Point *B* is eight miles down stream from Point A. The river is three miles across. It costs \$500,000 per mile to lay pipe on land and \$1,000,000 to lay pipe over the water. What path between the two points, on land and under water results in the least cost?

Solution. In the interest of brevity for this article, we note the setup is almost identical to the previous problem. The function to be optimized is

$$T = 500000x + 1000000\sqrt{3^2 + (6-x)^2}$$

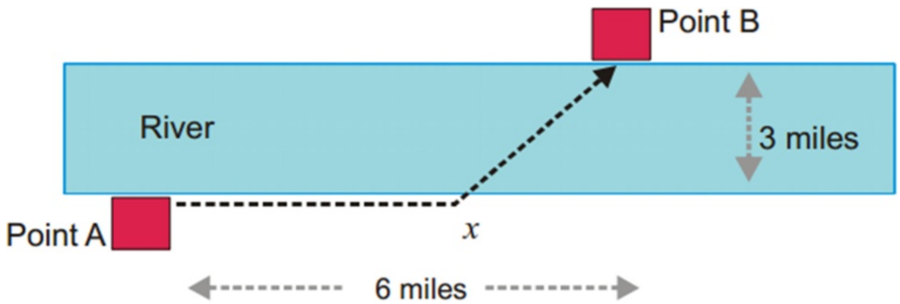


Figure 6. The Oil Services Diagram

REMARKS

- One point of these problems is that for the most part they can be approached from both graphical and from a theoretical viewpoints.
- These problems have many variations can crop up in any type of realistic problem.

They furnish a good introduction to optimization at an appropriate level. Algebra 1 and 2.

FACT. If your students will take calculus in college, it is a virtual certainty they will see both of these problems. They will be asked to use calculus to solve the problems.

G. DONALD ALLEN

17. PYTHAGOREAN TRIPLES PLUS ONE

Pythagorean triples. It is well known that for any two integers *m* and *n* we can take

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

and easily see that

$$\begin{aligned} a^2 + b^2 &= \left(m^2 - n^2\right)^2 + \left(2mn\right)^2 \\ &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \\ &= \left(m^2 + n^2\right)^2 = c^2 \end{aligned}$$

So $c^2 = a^2 + b^2$. This is the structure of Pythagorean triples. It is also possible to show that all Pythagorean triples have this form – almost. Indeed, any positive integer multiple will generate another, i.e. (*ka, kb, kc*). A short list of triples include: (3, 4, 5), (5, 12, 13), (8, 15, 17), (7, 24, 25), (20, 21, 29), (12, 35, 37), and (9, 40, 41). These are called *primitive* because *gcd* (*a, b*) = 1. Naturally, any triple can be multiplied (all values) by any positive integer to get another. So, the formula generates an infinite set of triples, and with all positive multiples, we get them all.

Pythagorean quadruples. The next step in our story is to consider Pythagorean quadruples, which is to say a set of four positive integers (*a,b,c,d*) such that

$$d^2 = a^2 + b^2 + c^2$$

There is a formula for these. Beginning with four positive integers, *m, n, p,* and *q*, define

$$\begin{aligned} a &= m^2 + n^2 - p^2 - q^2 \\ b &= 2\left(mq + np\right) \\ c &= 2\left(nq - mp\right) \\ d &= m^2 + n^2 + p^2 + q^2 \end{aligned}$$

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 67–71.

Then it is easy (mercifully omitted) to square and expand these terms to show that $a^2+b^2+c^2=\left(m^2+n^2+p^2+q^2\right)^2=d^2$. The numbers, *m, n, p,* and *q*, should be selected so that *a, b,* and *c* are positive. Here are a few of the smaller ones: (1,2,2,3), (2,3,6,7), (1,4,8,9), (4,4,7,9), (2,6,9,11), (6,6,7,11), (3,4,12,13), (2,5,14,15). A very important point is that this formulation generates *all* primitive Pythagorean quadruples, with *gcd* (*a, b, c*) = 1. This fact, not proved here, should be compared with the other formulations we will see presently.

The final step. We're almost done. Take $m = 1$, and let (p, q, n) be any Pythagorean triple, meaning $n^2 - p^2 - q^2 = 0$. This gives

$$a=1^2+n^2-p^2-q^2=1$$

Thus

$$d^2=1+b^2+c^2$$

And this is our *Pythagorean triple plus 1*. Of course, we've changed the lettering a bit. It is easy to see that we get one of these Pythagorean triples plus 1 for each Pythagorean triple. There are infinitely many of them. Here are a few of the smaller ones: (1, 38, 34, 51), (1, 172, 104, 201), (1, 154, 302, 339), (1, 526, 242, 579), (1, 512, 616, 801), (1, 398, 1186, 1251), (1, 1268, 472, 1353), (1, 1258, 1118, 1683). Well, maybe not so small. However, they are built from actual Pythagorean triples. There are others, for example, (1,2,2,3) as $3^2=1^2+2^2+2^2$, with no Pythagorean triple connected.

Other formulas for Pythagorean quadruples. The formula given above for quadruples is not unique. There are others. We give two other formulas. Neither are exhaustive in finding all such quads, but they are simpler to apply.

A. Let [3]

$$a=m^2, b=2mn, c=2n^2$$

Then

$$\begin{aligned} a^2+b^2+c^2 &= \left(m^2\right)^2+\left(2mn\right)^2+\left(2n^2\right)^2 \\ &= m^4+4m^2n^2+4n^4 \\ &= \left(m^2+n^2\right)^2=d^2 \end{aligned}$$

We can compute Pythagorean triples plus one by merely taking arbitrary values for $m=1$, and arbitrary values for *n*. The first few are (1, 2, 2, 3), (1, 4, 8, 9), (1, 6, 18, 19), (1, 8, 32, 33), (1, 10, 50, 51), (1, 12, 72, 73), (1, 14, 98, 99), (1, 16, 128, 129), (1, 18, 162, 163), (1, 20, 200, 201). These are somewhat smaller than the set listed above. However, both sets appear to have no members in common.

By restricting $m = 1$, we generate a set of Pythagorean triples plus one as

$$\left(1, 2n, 2n^2, 2n^2+1\right)$$

for each n = 1,2,3,….

An alternate derivation of the formula for Pythagorean triples plus one follows from those for which the difference of the third value from the fourth is one. So, consider the Pythagorean triple plus one to be (1, *a, b, b* + 1). We must have

$$(b+1)^2 = 1 + a^2 + b^2$$
$$2b = a^2$$
$$b = \frac{a^2}{2}$$

Let's select any value a so that $\frac{a^2}{2}$ is an integer. That is, let a be any even number, and $b = \frac{a^2}{2}$. Then

$$\left(\frac{a^2}{2} + 1\right)^2 = 1 + a^2 + \frac{a^2}{4}$$

This gives a direct method for computing such numbers, namely, for each the quadruple n = 1, 2, 3, …, is $\left(1, 2n, 2n^2, 2n^2 + 1\right)$, a Pythagorean triple plus one.

B. Let [4]

$$a = 2mp$$
$$b = 2np$$
$$c = p^2 - \left(m^2 + n^2\right)$$
$$d = p^2 + \left(m^2 + n^2\right)$$

Then

$$\begin{aligned} a^2 + b^2 + c^2 &= (2mp)^2 + (2np)^2 + \left(p^2 - \left(m^2 + n^2\right)\right)^2 \\ &= m^4 + 2m^2n^2 + 2m^2p^2 + n^4 + 2n^2p^2 + p^4 \\ &= \left(m^2 + n^2 + p^2\right)^2 \end{aligned}$$

So $d = p^2 + \left(m^2 + n^2\right)$. To compute Pythagorean triples plus 1 is not that easy, as we've seen. Of course, a and b are even. This means c = 1. And thus, $p^2 = 1 + \left(m^2 + n^2\right)$. Therefore, in order for this scheme to generate a Pythagorean triple plus one, it is necessary to begin with one from the start. Each written with the digits from smallest to largest, here are some: (1,2,2,3), (2,4,4,6), (6,6,7,11), (2,4,4,6), (1,4,8,9), (4,6,12,14), (2,6,9,11), (4,6,12,14), (1,6,18,19), (1,4,8,9), (4,6,12,14), (4,4,7,9), (4,8,8,12), (1,12,12,17). Several are multiples of others. Few of those with a "1" in them are very large. This implies the class of Pythagorean triples plus one is not completely generated from Pythagorean triples, per se. With this formula, here are the first 14 Pythagorean plus one triples as generated: (1,2,2,3), (1,4,8,9), (1,12,12,17), (1,6,18,19), (1,8,32,33), (1,10,50,51), (1,12,72,73), (1,14,98,99), (1,64,112,129), (1,72,144,161), (1,18,162,163), (1,20,200,201), (1,22,242,243), (1,24,288,289). Some of these have the pattern noted above, but not all.

Conclusion. We have studied a rather simple sounding problem, where to solve it we needed to examine the solutions to a more complicated problem, and then apply them particularly. Along the way, we've discovered an alternate way to generate Pythagorean triples plus one. However, if we asked for values, *a, b* and *c* for which $c^2 = a^2 + b^2 + 2$, i.e. Pythagorean triples plus two, all of the above offers us little help. This is all too typical in working with numbers and number theory, in general. Changing the problem only slightly can plunge it from the realm of being solvable to being really, really difficult.

EXERCISES

11. For the Pythagorean quadruple formula, show that $d^2 = a^2 + b^2 + c^2$. That is, make the calculation omitted above.
12. Can you find a Pythagorean quintuple, integers $a, b, c, d,$ and e so that $e^2 = a^2 + b^2 + c^2 + d^2$?
13. Find a formula for Pythagorean triples plus two, that is find values *a, b* and *c* for which $c^2 = a^2 + b^2 + 2$. Examples. $3^2 + 5^2 + 2 = 6^2, 7^2 + 25^2 + 2 = 26^2$. Remarkably, this problem has a very nice solution.
14. Suppose we want a Pythagorean quadruple with one of the values given as 23. How can it be determined? Are there an infinite number of them?

REFERENCES

Carmichael, R. D. (1915). *Diophantine analysis*, New York, NY: John Wiley & Sons, 1915.

Dickson, L.E. (1921). Some relations between the theory of numbers and other branches of mathematics. In H. Villat (Ed.), *Conférence générale, Comptes rendus du Congrès international des mathématiciens* (pp. 41–56). Strasbourg, Toulouse. (Reprint Nendeln/Liechtenstein: Kraus Reprint Limited, 1967; Collected Works 2, pp. 579–594).

Dutch, S. *Power page: Pythagorean quartets*. Retrieved from http://www.uwgb.edu/dutchs/RECMATH/rmpowers.htm#pythquart

Mordell, L. J. (1969). *Diophantine equations*. London: Academic Press.

Spira, R. (1962). The diophantine equation $x^2 + y^2 + z^2 = m^2$. *American Mathematical Monthly, 69*, 360–365.

Wikipedia. (n.d.). Retrieved from http://en.wikipedia.org/wiki/Pythagorean_quadruple

Solution to Problem 3. We note from actual Pythagorean triples (*a, b, c*) that by taking *a* and *c* for some of them, we do get a^2+c^2+2 is the square number $(c+1)^2$. This was just an observation. So, assuming the Pythagorean triple plus 2 of the form $(a,n,n+1)$, we must have

$$(n+1)^2-n^2=(n+1)^2-n^2=2n+1=a^2+2$$
$$n=\frac{1}{2}a^2+\frac{1}{2}$$

Checking, we have

$$\begin{aligned} a^2+n^2+2 &= a^2+\left(\frac{1}{2}a^2+\frac{1}{2}\right)^2+2 \\ &= \frac{1}{4}a^4+\frac{3}{2}a^2+\frac{9}{4} \\ &= \left(\frac{1}{2}a^2+\frac{1}{2}+1\right)^2 \neq (n+1)^2 \end{aligned}$$

This validates our model. Pythagorean triples plus 2 can have the form

$$\left(a,\frac{1}{2}a^2+\frac{1}{2},\frac{1}{2}a^2+\frac{3}{2}\right)$$

We have not shown these are the only such triples. Can you? The first few of these Pythagorean triples plus 2 taking as odd are: (1,1,2), (3,5,6), (5,13,14), (7,25,26), (9,41,42), (11,61,62), (13,85,86), (15,113,114), (17,145,146), (19,181,182), (21,221,222). Without the observation and then the guess, finding this form could have been difficult.

AMANDA ROSS

18. A BRIEF LOOK AT DERIVATIVES

The concept of a derivative is a mystery to many students. Oftentimes, students memorize a sequence of steps, in order to find a derivative, but never fully understand the meaning. On the same note, students often do not understand why there is more than one way to find a derivative. The varying notations of derivatives pose great confusion for many students. This article serves as a very brief introduction to the topic of derivatives. The information provided here may be used to segue into a more in-depth study of the teaching of derivatives.

First, let's explore the meaning of a derivative. A derivative is simply the rate of change at any point on a function. It is the understood limit of an instantaneous rate of change. Once a derivative is found, it can be used to find the instantaneous rate of change, for any value of x.

A derivative is stated as the ratio of "delta y" over "delta x." It is the ratio of the change in y to the change in x. The derivative can be represented as $\frac{\Delta y}{\Delta x}$. Typically, the derivative is represented by $f'(x)$, $\frac{d}{dx}(y)$, or $\frac{dy}{dx}$. These different equivalent notations are often a point of confusion for students. For this article, we will use $f'(x)$ (the Newton form); just know that it means the same as "d over dx" and "dy over dx."

Let's look at an example of finding a derivative. We can find it, according to a shortcut rule, according to a formal definition, or by using a TI graphing calculator. Consider the function below:

$$f(x) = 3x^2 + 6$$

We can use the shortcut rule shown below for finding a derivative. Recall that the derivative of a constant, c, is 0, while the derivative of the variable, x, is 1.

$$f'(x) = n \cdot ax^{n-1}$$

So, $f'(x) = 2 \cdot 3x^{2-1}$ or $f'(x) = 6x$

The formal definition states that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. Note. You can also substitute Δx, or "the change in x" for h.
So, we have the following:

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 73–76.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{3(x+h)^2+6-(3x^2+6)}{h} \\
&= \lim_{h\to 0}\frac{3(x^2+2xh+h^2)+6-3x^2-6}{h} \\
&= \lim_{h\to 0}\frac{3x^2+6xh+3h^2+6-3x^2-6}{h} \\
&= \lim_{h\to 0}\frac{6xh+3h^2}{h} \\
&= \lim_{h\to 0}\frac{h(6x+3h)}{h} \\
&= \lim_{h\to 0}(6x+3h) \\
&= \lim_{h\to 0}6x
\end{aligned}
$$

Again, we find the derivative to be $6x$.

Finally, we can also find the derivative, using the graphing calculator. The steps are shown below:

1. Enter the function into the y= screen.
2. Graph.
3. Choose Calculate (2nd Trace).
4. Select 6: *dy*/*dx*.
5. Type any value of *x*; the calculative gives the derivative for that *x*-value. It can do this for any point.
6. If we select an *x*-value of 4, we get a derivative of 24. If we select an *x*-value of 5, we get a derivative of 30. If we select an *x*-value of 6, we get a derivative of 36. If we select an *x*-value of 10, we get a derivative of 60. So, we can see that the derivative is 6*x*.

There is another way to find the derivative, using the graphing calculator. You can use the nDeriv function. These steps are shown below:

1. Choose Math.
2. Select 8: nDeriv(.
3. Type the expression, followed by the variable, and the value at which the derivative will be evaluated. It should look like nDeriv(function, *x*, *x*).

Let's evaluate our function, for the same *x*-values of 4, 5, 6, and 10.

We would enter $\text{nDeriv}\left(3x^2+6,x,4\right)$, $\text{nDeriv}\left(3x^2+6,x,5\right)$, $\text{nDeriv}\left(3x^2+6,x,6\right)$, and $\text{nDeriv}\left(3x^2+6,x,10\right)$. We get the same respective derivatives of 24, 30, 36, and 60. Again, we have shown the derivative to be $6x$.

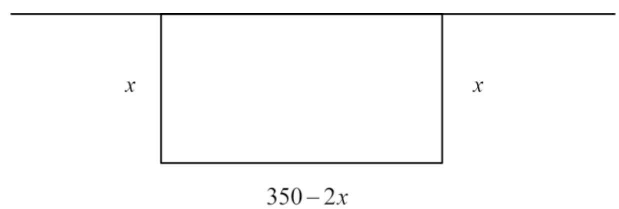
So, we have actually shown four ways to get the derivative of this function.

Now, that you know what a derivative is and how to find it, let's look at an example of how a derivative may be used to solve a real-world problem.

A derivative may be used to solve optimization problems. In other words, a derivative may be used to find the x-value at which a maximum area occurs. Consider the problem below.

Josiah wishes to build a dog pen that will be attached to the back wall of his house. He only has 350 feet of fencing. What dimensions will give the maximum area? What is the maximum area of the dog pen?

To solve, we will first illustrate the scenario:



We can represent the area as:

$$\begin{aligned} A(x) &= x\left(350-2x\right) \\ &= 350x-2x^2 \end{aligned}$$

The derivative may be used to find the x-value at which the maximum area occurs. The derivative is represented as:

$$A'(x) = 350-4x$$

The x-value that gives a derivative of 0 will be the x-value at which the maximum occurs. So, we will set the derivative equal to 0.

$$\begin{aligned} 0 &= 350-4x \\ 4x &= 350 \\ x &= 87.5 \end{aligned}$$

The maximum area will occur when x is 87.5. Thus, the dimensions that give the maximum area are 87.5 feet and $350-2(87.5)$ or 175 feet. So, the maximum area is $\left(87.5\right)\left(175\right)$ or 15,312.5 feet.

This article has briefly presented the meaning of a derivative, ways to represent and find a derivative, as well as a sample real-world application of a derivative. This information may be useful in helping students synthesize their understanding of derivatives. The skills presented in this article may also be used as a pretest measure of what students know. You may also present the real-world optimization problem and ask students to brainstorm other possible real-world derivative applications.

AMANDA ROSS

19. A BRIEF LOOK AT ARITHMETIC AND GEOMETRIC SEQUENCES

Some sequences are arithmetic or geometric, meaning they either change by some common difference or common ratio. An arithmetic sequence is a sequence with a common difference, d. All linear functions represent arithmetic sequences. A geometric sequence is a sequence with a common ratio, r. All exponential functions represent geometric sequences. Other sequences are neither arithmetic nor geometric, such as the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13,…). Notice there is not a constant rate of change or constant ratio, relating the terms, in this sequence. This brief article will touch upon arithmetic and geometric sequences.

ARITHMETIC SEQUENCES

Let's first look at an example of an arithmetic sequence.

2, 4, 6, 8, 10, 12,…

This is a simple sequence, which actually represents the set of even, positive integers. Recall that 0 is neither even nor odd. We know the sequence is an arithmetic sequence because there is a constant rate of change of 2. This is the common difference of 2. We can represent the sequence, using either a recursive or explicit formula. A recursive formula represents the value of a term, based upon the value of the preceding term. An explicit formula does not require knowledge of the value of any preceding term; it only requires knowledge of the nth term (or position) in question.

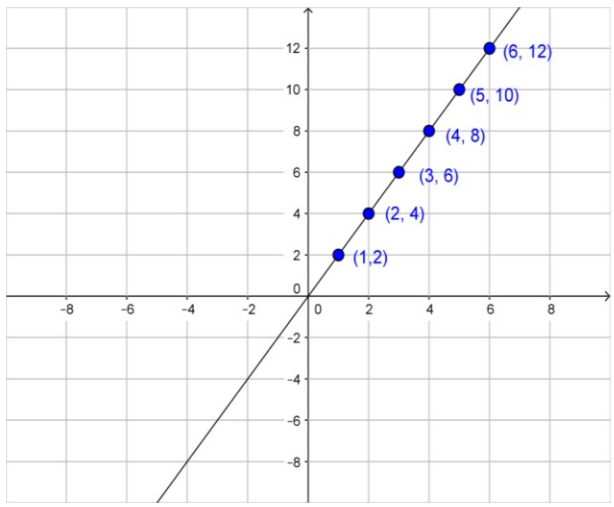
For this sequence, the recursive form would be $a_n = a_{(n-1)} + 2$, $n \geq 2$.

The explicit form for an arithmetic sequence can be found, by using the formula, $a_n = a_1 + (n - 1)d$, where a_1 represents the initial term value, n represents the nth term, and d represents the common difference. Substituting 2 for a_1 and 2 for d gives $a_n = 2 + (n - 1)2$ or $a_n = 2 + 2n - 2$, which simplifies as $a_n = 2n$.

The explicit form can also be found by substituting the common difference of 2 for the slope and the x-value of 1 and y-value of 2 into the slope-intercept form of an equation, or $y = mx + b$. (The position numbers or term numbers of the sequence represent the x-values.) Doing so gives $2 = 2(1) - b$, or $b = 0$. Since $b = 0$ and $m = 2$, the explicit form equation is $y = 2x$ or $a_n = 2n$.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 77–80.

This sequence can also be graphed by plotting ordered pairs, whereby the term numbers, *n*, represent the *x*-values, and the values of the terms represent the *y*-values. We would have the ordered pairs, (1, 2), (2, 4), (3, 6), (4, 8), (5, 10), and (6, 12). The graph of these points is shown below.



Notice the graph of the function representing the arithmetic sequence is linear.

GEOMETRIC SEQUENCES

Now, let's look at an example of a geometric sequence.

2, 4, 8, 16, 32, 64, …

We know the sequence is a geometric sequence because there is a constant ratio between terms. The constant ratio is 2. As with the arithmetic sequence, we can represent the sequence, using either a recursive or explicit formula.

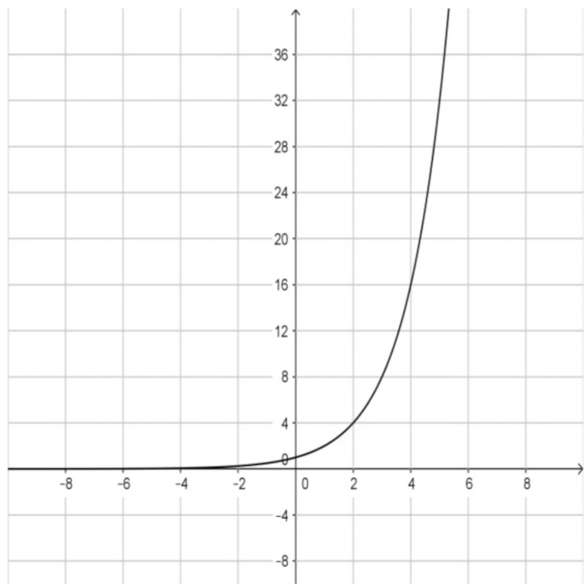
For this sequence, the recursive form would be $a_n = 2a_{n-1}$, $n \geq 2$.

The explicit form for an arithmetic sequence can be found, by using the formula, $a_n = a_1 \cdot r^{n-1}$, where a_1 represents the initial term value, *n* represents the nth term, and *r* represents the common ratio. Substituting 2 for a_1 and 2 for *r* gives $a_n = 2 \cdot 2^{n-1}$.

The explicit form can also be found by substituting any *x*- and corresponding *y*-value into the form, $y = a^x$. Substituting the *x*-value of 1 and *y*-value of 2, into the exponential equation form gives $2 = a^1$. Thus, $a = 2$. So the explicit form is $y = 2^x$ or $a_n = 2^n$.

If both of these equation forms were graphed, you would see that the graphs are the same. The functions lie perfectly on top of one another. So, the two different explicit forms above are equivalent.

The graph of this sequence is shown below.



Notice the graph of the function representing a geometric sequence is exponential.

FINDING THE NTH TERM

It is important to note that the explicit form of a sequence may be used to find the value of the nth term of the sequence. For example, with the first sequence of 2, 4, 6, 8, 10, 12,…, the value of the 100^{th} term can be found by evaluating the explicit form for an *n*-value of 100. Doing so gives $a_{100} = 2(100)$ or $a_{100} = 200$. We can also find the value of the 100^{th} term of the second sequence, or 2, 4, 8, 16, 32, 64,…, by again evaluating the explicit form for an *n*-value of 100. Doing so gives $a_{100} = 2 \cdot 2^{100-1}$. This would be a really big number, so we could leave it in this form or provide an approximation. The table feature of a graphing calculator can also be used to look at the terms of any arithmetic or geometric sequence. This is a very handy tool. The functions are simply entered into the y= screen.

SERIES

When adding the terms of a sequence, we call it a *series*. In other words, adding the terms of the sequence, $a_n = 2, 4, 6, 8, 10, 12,\ldots$, for the first six terms gives $S_6 = 2 + 4 + 6 + 8 + 10 + 12$. We can represent the sum of a series with sigma notation, which will be covered in more depth in the next article. For this sequence, we would write:

$$\sum_{n=1}^{6} 2n$$

This expression tells us to start at 1 and go to an *n*-value of 6, substituting each *n*-value into the expression, 2*n*, and summing the results. In expanded form, this would look like:

$$\sum_{n=1}^{6} 2n = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) + 2(6) = 42$$

AMANDA ROSS

20. A BRIEF LOOK AT SIGMA NOTATION

Sigma notation indicates that you must sum terms. Sigma means "to sum." Sigma notation indicates the starting and ending *n*-value (variable value), for which an expression must be evaluated. Then, each of these evaluations must be summed.

The table below shows some sigma examples.

Sigma notation	*Verbal translation*	*Evaluated expression*
$\sum_{n=1}^{4} n$	"the sum of 1 to 4 of *n* "	$\sum_{n=1}^{4} n = 1+2+3+4=10$
$\sum_{n=1}^{5} n^2$	"the sum of 1 to 5 of *n*-squared"	$\sum_{n=1}^{5} n^2 = (1)^2+(2)^2+(3)^2+(4)^2+(5)^2 = 55$
$\sum_{n=1}^{6} (n-3)$	"the sum of 1 to 6 of the quantity, *n* minus 3"	$\sum_{n=1}^{6} (n-3) = (1-3)^2+(2-3)^2+(3-3)^2$ $+(4-3)^2+(5-3)^2+(6-3)^2=19$
$\sum_{n=1}^{4} \sqrt{n}$	"the sum of 1 to 4 of the square root of *n*"	$\sum_{n=1}^{4} \sqrt{n} = \sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4} = 3+\sqrt{2}+\sqrt{3}$
$\sum_{n=1}^{6} \frac{1}{n}$	"the sum of 1 to 6 of the ratio of 1 to *n*"	$\sum_{n=1}^{6} \frac{1}{n} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6} = 2\frac{9}{20}$

Just as a sigma notation expression can be evaluated, you can also start with a sequence and determine the sigma notation which represents the series. Let's look at an example.

8, 10, 12, 14, 16

This sequence represents a linear function, with a slope or common difference of 2. Substituting the *a*-value of 8 and common difference of 2 into the explicit formula

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 81–82.

of $a_n = a_1 + (n-1)d$ gives $a_n = 8 + (n-1)2$, which simplifies as $a_n = 8 + 2n - 2$ or $a_n = 6 + 2n$. So, we can write

$$\sum_{n=1}^{5} 6 + 2n = 8 + 10 + 12 + 14 + 16 = 60$$

Note that the explicit formula may have also been derived by evaluating the slope-intercept form of a linear equation for the slope of 2 and any *x*-value and corresponding *y*-value. Doing so will give the *y*-intercept, which may then be used in place of *b*, in the equation, $y = mx + b$.

A graphing calculator can also be used to evaluate a summation. Let's evaluate the sigma notation, for the example we just finished.

Step 1: Press 2nd Stat.
Step 2: Select Math.
Step 3: Select 5: sum(
Step 4: Press Enter.
Step 5: Press 2nd Stat again.
Step 6: Select OPS.
Step 7: Select 5: seq(
Step 8: Press Enter.
Step 9: Enter sum(seq(6 + 2*x*, *x*, 1, 5, 1)
Step 10: Press Enter.

The output is 60, which is indeed the evaluation we got from our manual calculation.

Sigma notation can also be evaluated, using Excel. It would be good to have your students explore the different technologies available for evaluating series. Encourage students to work with many examples of sigma notation, noting that not all notation will indicate a starting value with "*n* = 1."

G. DONALD ALLEN

21. COMMUNICATING WITH LIFE IN THE UNIVERSE[1]

The title, which makes loads of suppositions, could also be called "One really big equation." It illustrates that real-life equations often involve multiple variables. From the very existence of life in the universe to how to communicate with it and points in between is a matter of pure conjecture. In this note, we will discuss the fundamental problem of determining if there is intelligent life in the universe and how we might go about communicating with it. Along the way, we will need an important and very speculative equation, nowadays called the Drake equation, what line of sight communication means and how we can compute it on the planet's surface, and finally a couple of methods of communications.

Certainly when discussing the possibility of *extraterrestrial intelligent species*, ETIS, some will view it as a journey into science fiction, while others feel it is an unproved certainty and its consideration is meritorious of our very best efforts.

THE DRAKE EQUATION

In this article, we consider the Drake equation for the determination of the number of extraterrestrial intelligent species, ETIS. In 1961, radio astronomer Frank Drake,

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 83–88.

working at the Greenbank observatory, proposed a method of estimating the number of civilizations in our Galaxy that could be detectable from Earth. It is written as an equation to determine the possible number of planets that have intelligent life. This equation is basically constructed as a product of two parts.

Number of planets with intelligent life = Number of possible planets × various probabilities

The number of possible planets involves counting a number of important factors, such as the rate of star formation and the mean lifetime (in years) of a communicative civilization. The various fractions are many, but all practically selected. They include the fraction of stars with planets, planets with suitable environment for life, inhabited planets where intelligent life-form evolves, planets with intelligent life that has developed radio communications, and others. This equation can be applied to our galaxy, the Milky Way, or to other galaxies, or even to the entire universe. The *Drake* (or Greenbank) equation is

$$N = R\, f_p\, n_e\, f_l\, f_i\, f_c\, f_l$$

where

- N is the number of *ETIS* in the galaxy
- R is mean rate of star formation
- f_p is the fraction of suitable suns with planetary systems
- n_e is the mean number of planets suitable for life (as we know it)
- f_l is the fraction of such planets on which life actually originates
- f_i is the fraction of such planets on which, after the origin of life, some form of intelligence arises
- f_c is the fraction of such intelligent species that develop the ability and desire to communicate with other civilizations
- L the average lifetime with a communicative civilization

Some astronomy authors are willing to publish their guesses for all of the terms in the Drake equation even though estimates of n_e and f_l are only rough and values quoted for the last three, f_i, f_c, L, are just wild guesses.

There is also the so-called the Fermi paradox, after Enrico Fermi who first publicized the subject. Fermi's response to a large N was that if there were very many advanced extraterrestrial civilizations in our galaxy, then, "Where are they?" and "Why haven't we seen any traces of intelligent extraterrestrial life?" This suggests that our understanding of what is a "conservative" value for some of the parameters may be overly optimistic or that some other factor is involved to suppress the development of intelligent space-faring life. Those that adhere to the premise behind the Fermi Paradox often refer to that premise as the Fermi Principle.

Here is a link to a ETIS calculator: http://www.as.utexas.edu/cgi-bin/drake.pl

Consider using numbers in the ranges below.

Variable	Value range
R	10–50
f_p	1/15–1
n_e	3–10
f_c	10^{-6}–1
f_i	10^{-3}–1
L	60–10^8

However, feel free to enter any appropriate numbers.

Finally, we note that there are various other forms of the Drake equation, some with more factors, some with fewer, and some with different factors. One version of the equation is called the Drake-Sagan equation, in honor of the popular astronomer, Carl Sagan.

SETI (SEARCH FOR EXTRA-TERRESTRIAL INTELLIGENCE)

In their own words, the SETI Institutes states, “The mission of the SETI Institute is to explore, understand and explain the origin, nature, prevalence and distribution of life in the universe.” The SETI Institute, founded in 1984, is a private, nonprofit organization which performs research on a number of fronts, while engaged in public outreach. Currently, it employs more than 100 scientists, educators, and staff. It is comprised of two centers, the center for SETI and the center for the Study of Life, both chaired by distinguished scientists. Without giving a long discussion and description of its history and highlights, the list below shows some of the current programs and projects in which SETI is currently engaged. As you will notice, some of them will produce interesting research in and of themselves and irrespective of ETIS.

CURRENT PROJECTS BY THE CENTER FOR SETI INCLUDE

Allen Telescope Array – The Allen Telescope Array will consist of approximately 350 6.1-meter offset Gregorian dishes arrayed at the Hat Creek Radio Observatory site.

Project Phoenix – is the world’s most sensitive and comprehensive search for extraterrestrial intelligence by listening for radio signals. .

Optical SETI – a new experiment to look for powerful light pulses beamed our way from other star systems.

New Search System Development – a $15 million campaign to build and deploy the real-time signal detection equipment it has recently designed and prototyped.

Interstellar – is the search in a multitude of forms for the search for interstellar messages.

Haughton Mars Project – an investigation of a bit of chilly real estate on Devon Island, located 450 miles above the Arctic Circle, just beyond better-known Baffin Island's northern tip. In some ways, it is very much like Mars.

What's Living in the World's Highest Lake? – the study of what lives in the highest lake in the world. This lake lies in the crater of a dormant volcano in the Andes, nearly 6,000 meters above sea level. Isolated and exposed to extreme doses of UV radiation.

Mars Exploration Rovers Landing Sites: First Cut – If you're going Mars … Where to land? What can machinery withstand?

Lakes on Mars? – high-resolution imagery from the Mars Global Surveyor, have recently claimed that Mars is most definitely host to sedimentary stone. The implications are, of course, exciting, namely lakes and oceans.

Rovers to Mars – recent research has investigated how humans and robot rovers could interact to explore the surface of Mars.

Clues to the Origin of Life – there have been created chemical compounds that may have been important for life's origin.

SETI@HOME

As we have learned, SETI is a rather large scaled scientific effort which seeks to determine if there is intelligent life beyond the Earth. SETI researchers use many methods of which one popular method, radio SETI, listens for artificial radio signals coming from other stars. SETI@home is a scientific experiment that uses Internet-connected computers in the Search for Extraterrestrial Intelligence (SETI). Anyone can participate by running a free program that downloads and analyzes radio telescope data.

For a little background, most of the SETI computer analysis programs today use very large computers to analyze data from the telescope in real time. None of these computers look very deeply at the data for weak signals nor do they look for a large class of signal types. The reason for this is because they are limited by the amount of computer power available for data analysis. To extract the presence of the weakest signals, extensive computing power is required. It would take a monstrous supercomputer to get the job done. SETI programs could never afford to build or buy that computing power. There is a trade-off that they can make. Rather than a huge computer to do the job, they could use a smaller computer, but just take longer to do it. But then there would be lots of data piling up. What if they used LOTS of small computers, all working simultaneously on different parts of the analysis? Where can the SETI team possibly find thousands of computers they'd need to analyze the data continuously streaming from Arecibo?

It was determined that there are literally thousands of substantially idle computers available for sorting through this massive amount of data. These computers are in the many private homes and public institutions around the world. If only there was a program that could be used. So, the UC-Berkeley SETI team devised a program, aptly called SETI@home, which would process data and look for patterns among the background of the radio telescope data. Since there is so much data, this team also partitioned it into bite-sized pieces for individual home computers.

You can participate in this project by downloading the program and just setting it to work when you have finished your own computing for the day. The remarkable thing about SETI@home is that it was written so that individual computational results could later be synthesized into a single data stream. This makes the multi-home computer system one massive distributed parallel computing network. You can learn more about SETI@home and even download the program by following this: http://setiathome.ssl.berkeley.edu/.

As a possible thought or discussion question, you might consider what would be the consequences if a discernible pattern was convincingly identified. It could definitely change some of our ideas about the universe. Of course, you will need to know how to discern a real message. This is discussed in full at the SETI@home site.

SPONSORS

SETI's list of sponsors is impressive. Projects have been sponsored by NASA Ames Research Center, NASA Headquarters, the National Science Foundation, the Department of Energy, the US Geological Survey, the Jet Propulsion Laboratory (JPL), the International Astronomical Union, Argonne National Laboratory, the Alfred P. Sloan Foundation, the David & Lucile Packard Foundation, the Paul G. Allen Foundation, the Moore Family Foundation, the Universities Space Research Association (USRA), the Pacific Science Center, the Foundation for Microbiology, Sun Microsystems, Hewlett Packard Company, other private industry, William and Rosemary Hewlett, Bernard M. Oliver and many other private donations.

NOTE

1 The author has borrowed substantially from SETI sites.

WEBSITES

Life in the Universe: http://www.personal.psu.edu/faculty/r/8/r81/055/life/life.html
Wikipedia: http://www.wikipedia.org/wiki/Drake_equation
Alien life. http://www.2think.org/aliens.shtml

Drake Equation for SETI: http://www.seti-inst.edu/seti/seti_science/Welcome.html
SETI: http://www.seti-inst.edu/Welcome.html
SETI@home: http://setiathome.ssl.berkeley.edu/

REFERENCES

Drake, F., & Sobel, D. (1992). *Is anyone out there? The scientific search for extraterrestrial intelligence*. New York, NY: Delacorte.

Lineweaver, C. H., & Davis, T. M. (2002). Does the rapid appearance of life on earth suggest that life is common in the universe? *Astrobiology, 2*(3), 293–304.

Rood, R. T., & Trefil, J. (1983). *Are we alone? The possibility of extraterrestrial civilizations*. New York, NY: Scribner.

Shermer, M. (2002). Why ET hasn't called. *Scientific American, 287*(2), 33.

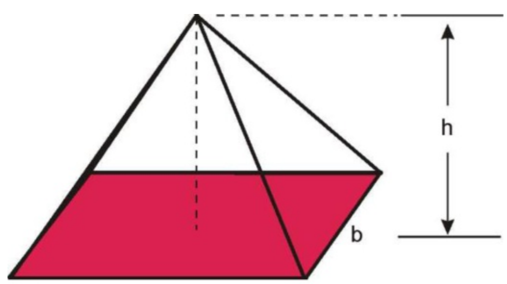
SECTION 3

GEOMETRY AND MEASUREMENT

G. DONALD ALLEN

22. THE MATHEMATICAL MYSTERY OF THE PYRAMID

It is not uncommon when studying ancient cultures that the only information available to historians and archaeologists leads to conclusions at odds with apparent facts. A remarkable example comes from ancient Egypt. Our knowledge of ancient Egyptian mathematics comes substantially from a collection of papyri, and most particularly the Rhind and Moscow papyri. The Rhind Mathematical Papyrus is named for A. H. Rhind (1833–1863) who purchased it at Luxor in 1858. It originates from about 1650 BCE but was written very much earlier. At 18 feet long and 13 inches wide, it is a collection of typical mathematics problems for students to learn. It is also called the Ahmes Papyrus after the scribe that last copied it. The Moscow Mathematical Papyrus was purchased by V. S. Golenishchev (d. 1947). It originates from about 1700 BC and is 15 ft long by 3 inches wide.



Pyramid

Both scrolls consist of very little more than applied arithmetic and simple geometry. Yet they possessed the exact formula for the volume of a pyramid, $V = \frac{1}{3}b^2h$, where b is the square base length and h is the height. Modern students prove this formula in their first calculus class. So, the question is how did the Egyptians come to know it when the only proofs available use infinite processes? Such types of processes would not be discovered for another thousand years in ancient Greece. See http://distance-ed.math.tamu.edu/historymath/egypt babylon/pyramid sand.pdf For a more extensive treatment on Egyptian mathematics and other problems in the papyri, see http://distance-ed.math.tamu.edu/historymath/egypt babylon/egypt.pdf

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 91.

G. DONALD ALLEN

23. THE SPHERE PACKING PROBLEM

The centuries old problem of the most efficient way to pack spheres has been solved – we think. You've seen how the grocer stacks oranges – in a kind of lattice where the next layer sits in the "holes" formed by the layer below and so on. The problem is to find the most efficient way to pack them. Let's make our definitions clear. We are talking about packing a huge, essentially infinite, volume with spheres (oranges, if you will). This problem was formally posed by Johannes Kepler, the astronomer and sometime cleric in 1611. Kepler posed the grocer's orange packing as the most efficient. Obviously, there are spaces between the spheres, but the overall *density* of spheres is about 0.74. Stacking layers of spheres, one directly atop the other gives the much lower density of about 0.52.

Closest packing of spheres

This is a difficult problem that was reduced by Lazlo Toth in 1953 to a huge calculation on many specific cases. University of Michigan mathematician, Thomas Hales, developed in 1994 a five step plan to make the calculations suggested by Toth, and over a period of several years carried them out. The proof involves 250 pages of text and about 3 gigabytes of computer programs and data.

But is it really solved? Hales' proof has been extremely difficult to verify. The published version of the proof will have the *unusual* editorial note stating parts of the paper have not been possible to check. Even still, a team of 12 reviewers worked hard for four years to verify the proof. The reviewers say they are almost certain it is correct.

LINKS

Sphere packing: http://mathworld.wolfram.com/SpherePackina.html
Kepler: http://scienceworld.wolfram.com/biography/Kepler.html

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 93.

G. DONALD ALLEN

24. BIG, BUT NOT REALLY BIG NUMBERS

Suppose we have a square that measures one centimeter on its side, and this square contains a set of 500,000 given points. We wish to divide this set into various subsets. Interestingly, if the number of points is smaller, most students don't have any trouble with these problems. It is when the number of points becomes unmanageably large that difficulties in seeing what to do emerge.

1. *Show that there is a straight line that passes through the square and touches none of these points.*
   Solution. Using our friend infinity and basic counting, we can solve this problem easily. Here's how. Select any point *P not* in the set. Now consider all the lines through this point. There is an infinity of them, of course. And they mutually intersect only at the selected point *P*. At most, 500,000 of them intersect one of the given points. So, there is clearly one that intersects none of the points.
2. *Show that there is a straight line that passes through the square and touches none of these points and divides the square into two parts each of which has exactly 250,000 points.*
   Solution. This solution of this problem is not so obvious. Simple counting may not be enough. We start the same way as above with a selected point *P* not in the set of given points. Now any line through this point divides the given set of points into three parts: the points on one side, the points on the other side, and the points on the line itself. Now rotate this line until the number of points to the left (say) is as close as possible to, but not exceeding, 250,000. Call this n. If $n = 250{,}000$, we are done. Otherwise, the slightest rotation shifts the number of point to be greater than 250,000 and therefore means that there are two or more points from the set on this line. Now find a point *Q* on the line where the number of points to one side plus n is exactly 250,000. Rotate the line through *Q* so that the one side has exactly 250,000 points. How can this be done? The answer is that the points are discrete, and this means there is some minimal distance between any pair of them. Thus, we can rotate the line about *Q* so that the distance of the new line from the original line is never greater than this minimal distance.
3. *Find two intersecting straight lines that divide the full set of points into four equal sets of points of 125,000 points each.*
   Solution. Curiously, this problem has no solution. For example, if all the points are co-linear, there is no possible subdivision into four equal parts.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 95–96.

ACTIVITIES

1. Ask your students to solve problems 1 or 2 when the number of points is just 4 or 6, or some small number. Most will clearly see the principle involved and solve it right away.
2. In our solution of both problems, we use the idea of rotation of the line. Can you instead use a sliding-the-line principle?
3. Find conditions for problem 3 to have a solution. (This may be tricky!)

SANDRA NITE

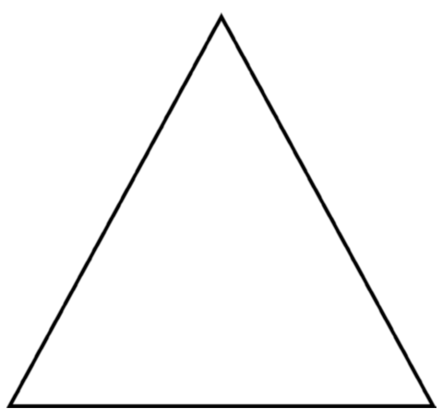
25. THE NET OF A CONE

One misconception that many high school students have is that the net of a cone is a triangle. Rather than simply telling or drawing for them the correct net, an activity for them to explore and/or discover the net of a cone could be more helpful to (1) engage their interest, and (2) increase the probability that they will remember the information. Two very simple activities are given to improve student understanding of the cone and its net. Neither of them should take more than 20–30 minutes of class time, and the teacher could add some questions appropriate for the lesson at the particular point in the curriculum in which the activity is used.

The first activity given (or an adaptation of it) could be used as a guided discovery or introductory activity. Note that the title does not give away the result, that the net of a cone will be discussed. The second activity given could be used if the students have already been exposed to the formula for the area of a sector of a circle and integrates review of several concepts and skills.

TRIANGLE EXPLORATION

1. Trace the isosceles triangle below and cut it out.



2. Gently fold the triangle so that the congruent sides meet, and tape them together.
3. Is the resulting figure a cone? Why or why not?
4. Trim the perimeter of the base of the figure so that it makes a cone (and sits flat on your desk with the apex point upward).

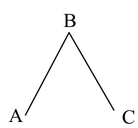
G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 97–98.

5. Carefully cut the taped edges to open up the net again. Draw the resulting figure in the space below.
6. What type of figure is the net of the cone?

The Texas Essential Knowledge and Skills (TEKS) addressed: 7.8.B The student uses geometry to model and describe the physical world. The student is expected to make a net (two-dimensional model) of the surface area of a three-dimensional figure.

THE TRIANGLE AND THE SECTOR

1. On a blank sheet of paper, use your protractor to draw two segments of length 5 cm with an included angle of 90 degrees. Label the three endpoints A, B, and C as shown below.



2. Draw a second figure exactly like the one above, but label the three endpoints D, E, and F.
3. Draw segment AC to form Δ ABC. Find the area of Δ ABC.
4. Find the area of a sector of a circle with angle 90° and radius 5 cm.
5. Does Δ ABC or the sector in #4 have a greater area? By how much?
6. Using the points D, E, and F in #2 and your compass, draw a sector as described in # 4.
7. Shade the area representing the answer to #5 above.
8. Which of the following is the net of a cone? If you are not sure, trace them, cut them out, and try to form a cone.
   A. triangle
   B. cone
   C. neither

The Texas Essential Knowledge and Skills (TEKS) addressed: 8.A. The student uses procedures to determine measures of three-dimensional figures. The student is expected to find lateral and total surface area of prisms, pyramids, and cylinders using concrete models and nets (two-dimensional models).

DIANNE GOLDSBY

26. AREA OF A CIRCLE

The formula for the area of a circle is one which can be derived by middle grades students with a hands-on activity. If you ask students for the formula, they will say $A = \pi r^2$. Asking students to explain where the formula comes from, one may be met with silence, statements that the derivation is not known, and/or shrugs. Students can generally explain the derivation of the formula for the area of a triangle from the halving of a rectangle. The formula for a circle is one that seems a little out there for them.

For the activity, you will need large circles cut from 8 ½ × 10 or larger sheets of paper and scissors.

Give each student a circle (or have them cut out one marked on the sheets).

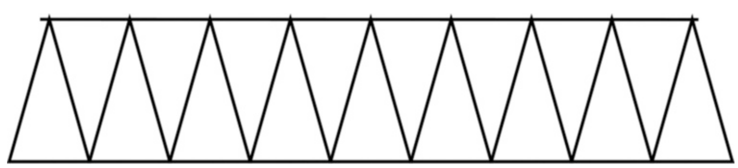
Ask them to fold the circle in half (lining up the edges and folding along a diameter. You can ask what this line is called.)

Then have them fold again being sure to press the folded edges.

Continue folding as many times as they can but at least a total of four times.

Have them open the circle and carefully cut the wedges formed.

Then, ask them to place the wedges, alternating point up and point down with sides touching.



When they do this, they form a figure that is almost a parallelogram. (Showing the figure with 4 sectors, then 8, then 16, etc. will show how the arcs "flatten" and approach a straight line.) They should know the formula for the area of a parallelogram: A = bh where b is the length of the base and h the height length.

The height of the figure is the radius of the circle and the base is ½ the circumference.

Substituting in the formula for the area of a parallelogram, we have $A = (½)(C) \times r$.

Since the circumference is 2π r, the area formula becomes

$A = (½)\,(2\,\pi\,r) \times r$
$A = \pi \times r \times r$
$A = \pi r^2$

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 99–100.

Having students explore and derive the formula is more likely to enable them to remember the formula, as they can see where it was derived.

This URL has an animation for the derivation of the formula:
http://curvebank.calstatela.edu/circle/circle.htm

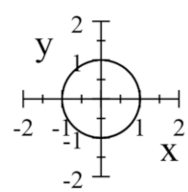
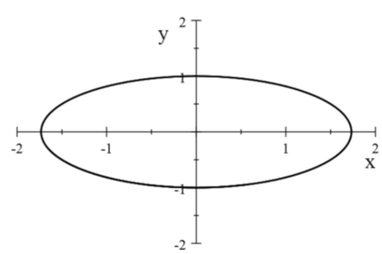
G. DONALD ALLEN

27. GEOMETRY MEETS ALGEBRA – SUPER-CONIC CONSTRUCTIONS, PART I

INTRODUCTION

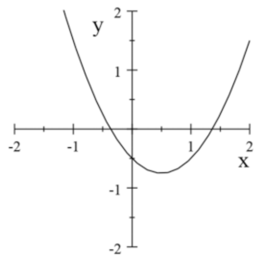
In this two part article, we highlight some of the ways we can do algebra geometrically. We will find square roots and cube roots and solve quadratics, cubics, and even quartics. These are all achieved by showing the intersection of various conics, which can yield the solutions to a great many problems. We will also show algebraic and geometric connections.

Conics. The basic conics we will use are the circle, ellipse, parabola, and hyperbola. Conics have wonderful geometric descriptions through the slicing of a double-napped cone. They also have descriptions in terms of relative distances from one or two foci and one or two directrices. We have the fundamental *algebraic* equations and basic shapes shown below. Note. These are not the most general forms of the curve, excepting the parabola.

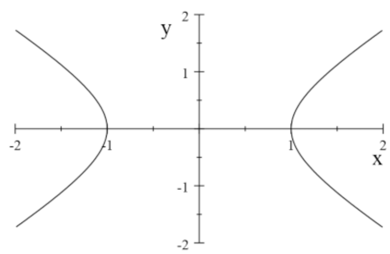
Type	**Form**	**Basic shape**
Circle	$x^2 + y^2 = r^2$	
Ellipse	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$	

G. D. Allen & A. Ross (Eds.), *Pedagogy and Content in Middle and High School Mathematics*, 101–106.

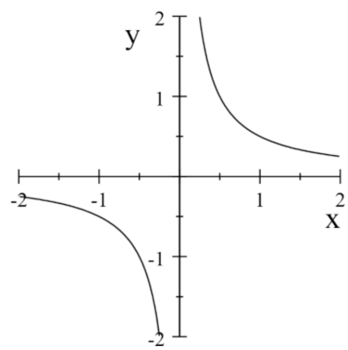
Parabola $y = ax^2 + bx + c$



Hyperbola $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$



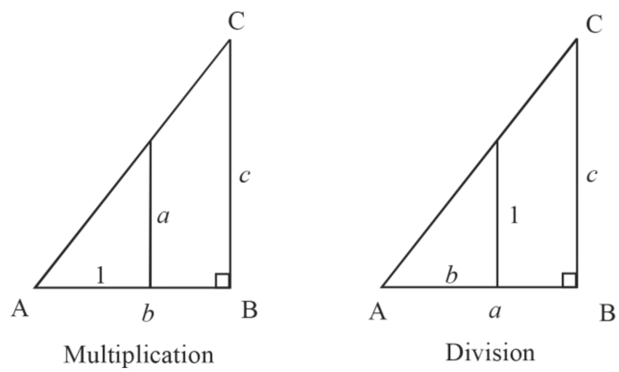
Hyperbola $y = \frac{c}{x}$



BASIC ARITHMETIC

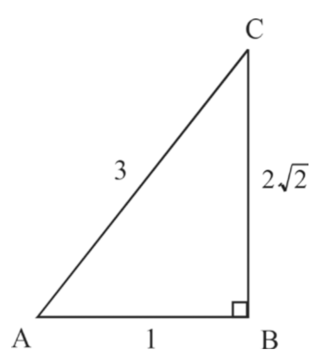
In the solutions of cubics and quartics to follow, we will need to do basic arithmetic, geometrically. To be precise about our constructions, we try whenever possible to use the strict construction rule: use a straight edge and compass only. However, the construction of the conics, except the circle, requires a ruler. Nonetheless, we can do all the basic arithmetic using strict construction. Addition and subtraction are relatively easy; just use the straight edge to draw the line and the compass to add or subtract amounts from it. Doing multiplication and division requires similarity and right triangles. For example, to compute the product, *ab*, construct a unit on the base of the triangle, and at right angles, construct the length, *a*. Now extend the base to length *b*, and again at right angles, construct the length, *BC*. This gives the "Multiplication" triangle shown below. By similarity, we must have

$$\frac{c}{b} = \frac{a}{1}$$
$$c = ab$$

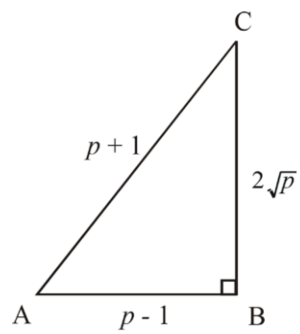


FINDING ROOTS

Square Roots. Let's find $\sqrt{2}$. A simple right triangle will help. We take the hypotenuse to be 3 and the base to be 1. By the Pythagorean theorem, the other leg is $\sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$. Now divide this length by two.



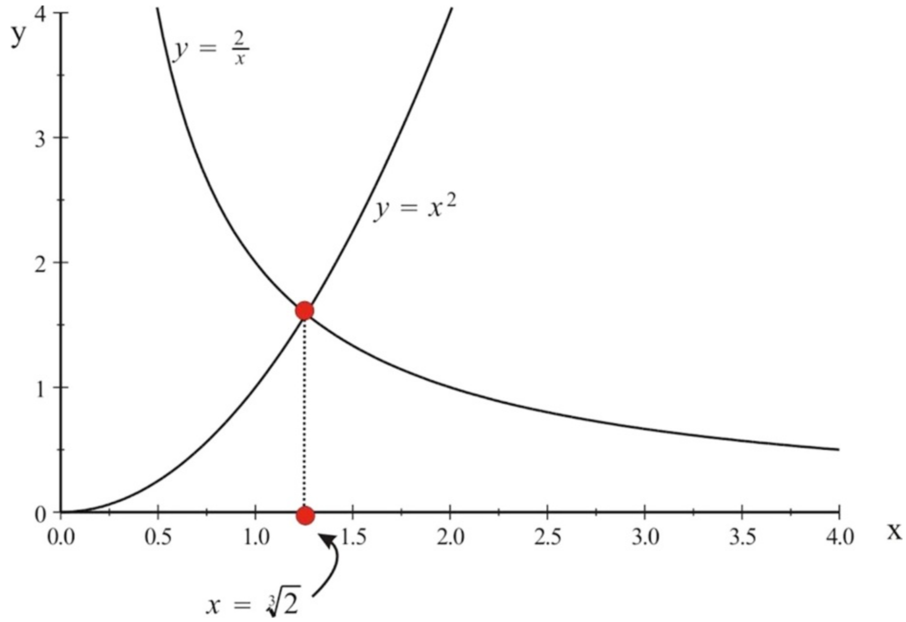
Finding any square root, say $\sqrt{p}$, is achieved similarly. Use the right triangle below, mark the hypotenuse to be $p+1$ and the base to be $p-1$. (Technically, first draw the base of length $p-1$, and then draw a perpendicular from the base at one end and a circular arc of radius $p+1$ from the other. This arc intersects the perpendicular with twice the desired height.) By the Pythagorean theorem, the other leg is $\sqrt{(p+1)^2-(p-1)^2} = \sqrt{4p} = 2\sqrt{p}$. Now divide this length by two.



Root Finding

It is important to note that the number *p* in the construction must be constructable. This means only a compass and straight-edge are allowed to construct it.

Cube Roots. The square root constructions are rather straightforward. Let's jump up a bit now to finding cube roots, first say $\sqrt[3]{2}$. Remember, if $\sqrt[3]{2}$ were actually constructable using a compass and straight-edge, we could solve the ancient Delian problem, "Double the cube." This means to find a new cube with double the volume of a given cube. With a cube of side 1 (hence volume = 1), we would need to find a cube of side $\sqrt[3]{2}$ (Volume = $\left(\sqrt[3]{2}\right)^3 = 2$). This cannot be achieved, as it took 1800 years to determine. And yes, mathematicians continually worked on the problem over these many years. You can learn about the ancient problems at http://www.math.tamu.edu/dallen/masters/alg_numtheory/delian_problem.pdf. This will be our first application of conics. We will intersect the parabola and hyperbola given by



$$y = x^2 \quad \text{and } y = \frac{2}{x}$$

Setting them equal gives

$$x^2 = \frac{2}{x}$$
$$x^3 = 2$$
$$x = \sqrt[3]{2}$$

Finding $\sqrt[3]{2}$

To find any cube root, say $\sqrt[3]{p}$, merely intersect $y = x^2$ with $y = \frac{p}{x}$. Thus $x^2 = \frac{p}{x}$. The solution is $x = \sqrt[3]{p}$. The picture is almost identical to the one above.

Fourth roots. The process of finding fourth roots can be achieved by intersecting conics. However, since we can take square roots, we can merely take the square root of a square root to get the fourth root. That is

$$\sqrt[4]{p} = p^{\frac{1}{4}} = \left(p^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

Nonetheless, we can intersect the quadratic, $y = x^2$, with the reciprocal quadratic, $y = \frac{p}{x^2}$, with the result

$$x^2 = \frac{p}{x^2}$$
$$x^4 = p$$
$$x = \sqrt[4]{p}$$

Note the reciprocal quadratic, $y = \frac{p}{x^2}$, is constructable from the quadratic, $y = \frac{1}{p}x^2$, by simple division – a constructable process.

Other roots. Using combinations of square and cube roots extends to the limit of what we can construct within the context of this discussion. For example, it is easy to find $\sqrt[6]{p}$ by first finding the square root of p and then the cube root of $\sqrt{p}$. Recall,

$$\sqrt[6]{p} = p^{\frac{1}{6}} = \left(p^{\frac{1}{2}}\right)^{\frac{1}{3}}$$

Similarly, we could find $p^{\frac{4}{3}}$ by first finding the cube root of p and then squaring twice as

$$p^{\frac{4}{3}} = \left(\sqrt[3]{p}\right)^4 = \left(\left(\sqrt[3]{p}\right)^2\right)^2$$

Naturally, no one ever said this would be fun or even easy, but possible, it is. It is important to note there was a day when good algorithms for root finding were unavailable, let alone calculators.

We could get other roots such as the fifth root, $\sqrt[5]{2}$, by intersectiing the quartic, $y = x^4$, with the hyperbola, $y = \frac{2}{x}$. However, this takes us too far beyond the realm of the conics. Can you think of another way to compute fourth roots using other curves?

SOLVING QUADRATICS

First, we know how to solve linear equations ($mx + b = 0$) and quadratic equations ($ax^2 + bx + c = 0$) by merely looking for their x-intercepts from their graphs. This is a specialized intersection process similar to those used for root-finding. However, we know how to solve both types geometrically because we know geometric arithmetic (for the linear equation) and the quadratic formula ($x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$) (for the quadratic equation). The ancient Greeks had a complete geometric solution of the quadratic, not involving any formula. It can be found in Euclid's *Elements*. So, with quadratics solved in algebraic and geometric ways, we can move on to solving cubics.

In Part II, we'll show how to solve cubic and quartic polynomials by intersecting various conics. However, the full article, both parts, is available at
http://distance-ed.math.tamu.edu/newsletter/newsletters_new/super_conic.pdf

G. DONALD ALLEN

28. GEOMETRY MEETS ALGEBRA – SUPER-CONIC CONSTRUCTIONS, PART II

INTRODUCTION

In this second part of the conics article, we highlight some of the ways we can do algebra geometrically. In Part I, we found square roots and cube roots and solved quadratics. In this part, we solve cubics and quartics. This effort is achieved by showing that the intersection of various conics can yield the solutions to a great many problems. We also show algebraic and geometric connections. While the full article is available at http://distance-ed.math.tamu.edu/newsletter/newsletters_new/super_conic.pdf, you can also download or read the first part of the full article, or both parts, at http://distance-ed.math.tamu.edu/newsletter/newsletters_new.htm.

Conics. The basic conics we will use are the circle, ellipse, parabola, and hyperbola. Conics have wonderful geometric descriptions through the slicing of a double-napped cone. They also have descriptions in terms of relative distances from one or two *foci* and one or two *directrices*. Their fundamental *algebraic* equations and basic shapes are shown in Part I.

SOLVING CUBICS

Our general cubic problem is to solve the cubic equation, $ax^3 + bx^2 + cx = d = 0$. The general algebraic solution evolved around 1530 with primary discover, Nicolo Tartaglia (1500–1557), which was published by Gerolamo Cardano (1501–1576) in his *Ars Magna*. Here, we focus, once again, on what can be achieved by intersecting simpler curves, i.e. conics. Of course, the direct, though approximate, way to solve such an equation is simply by graphing it and finding the *x*-intercepts, a la the quadratic above. Our goals here are to keep the constructions within the realm of the conics (and reciprocals), as there are wonderful geometric descriptions of them. Consider the system of a parabola and a hyperbola.

$$y = ax^2 + bx + c \text{ and } y = \frac{d}{x}$$

Solving we have

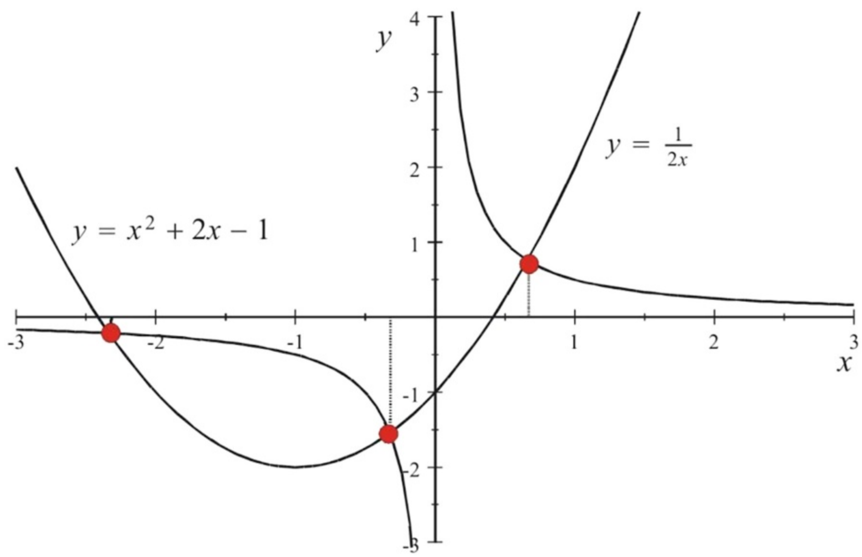
$$ax^3 + bx^2 + cx = d$$

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 107–111.

The example below shows the quadratic, $y = x^2 + 2x - 1$, and the hyperbola, $y = \frac{1}{2x}$, and their intersections. The resulting cubic is

$$y = (2x)(x^2 + 2x - 1) - 1 = 2x^3 + 4x^2 - 2x - 1$$

The approximate solutions are $x = -2.3364, x = -0.32403,$ and $x = 0.66044$.



Intersection of $y = x^2 + 2x - 1$ and $y = 1/(2x)$

It is easy to obtain these approximations with your graphing calculator. Remarkably, it is also possible to solve cubics by using trigonometry.

SOLVING QUARTICS

There are algebraic methods for solving quartics. The first was developed by Ludovicio Ferrari (1522 –1565) in about 1540. But there are also geometric methods, one of which we will now demonstrate. Both solutions require the solution of and "auxiliary" cubic. Let's consider finding the intersection of the two curves

$$y = \frac{f}{x^2} \quad \text{and} \quad y = kx^2 + lx + m$$

Equating them gives

$$\frac{f}{x^2} = kx^2 + lx + m$$

and this yields $0 = -\frac{f}{x^2} + kx^2 + lx + m = \frac{1}{x^2}\left(kx^4 + lx^3 + mx^2 - f\right)$. So, by intersecting the original two curves we get the solution to

$$kx^4 + lx^3 + mx^2 - f = 0 ,$$

provided of course that the solution is not $x = 0$. But this cannot be true, since $x = 0$ is not in the domain of the first function. Notice this quartic does not have a linear term (in x). So, we can't solve just any quartic just yet, only quartics that are missing the linear term.

Our goal is to solve any quartic equation, $x^4 + px^3 + qx^2 + rx = s$. By division, there is no loss by division, of assuming the coefficient of x^4 is one. Let's do a transformation. Using $x = t + \beta$, we can transform this quartic equation to the following

$$x^4 + px^3 + qx^2 + rx = s$$
$$p(t+\beta)^3 + q(t+\beta)^2 + r(t+\beta) + (t+\beta)^4 = s$$

Now expand and collect in powers of t to get

$$\begin{aligned}(t+\beta)^4 + p(t+\beta)^3 + q(t+\beta)^2 + r(t+\beta) &= t^4 + (4\beta + p)t^3 + \left(6\beta^2 + 3p\beta + q\right)t^2 \\ &\quad + \left(4\beta^3 + 3p\beta^2 + 2q\beta + r\right)t \\ &\quad + \beta^4 + p\beta^3 + q\beta^2 + r\beta \\ &= s\end{aligned}$$

The key step is to set the linear coefficient to zero. That is, solve the cubic $4\beta^3 + 3p\beta^2 + 2q\beta + r = 0$ for β. We know this can be done from the previous section. What's next? From the given β, we compute the coefficients of the remaining powers of t. These are the coefficients used in the original equations. For every solution of this cubic we find, another solution of the quartic is determined.

$$\begin{aligned}(\alpha t+\beta)^4 + p(\alpha t+\beta)^3 + q(\alpha t+\beta)^2 + r(\alpha t+\beta) &= \alpha^4 t^4 + \left(4\alpha^3\beta + p\alpha^3\right)t^3 + \left(6\alpha^2\beta^2 + 3p\alpha^2\beta + q\alpha^2\right)t^2 \\ &\quad + \left(4\alpha\beta^3 + 3p\alpha\beta^2 + 2q\alpha\beta + r\alpha\right)t \\ &\quad + \beta^4 + p\beta^3 + q\beta^2 + r\beta \\ &= s\end{aligned}$$

Here is an example of the intersection of the graphs of two original curves, $y = 2/x^2$ and $y = x^2 - 3x + 1$.

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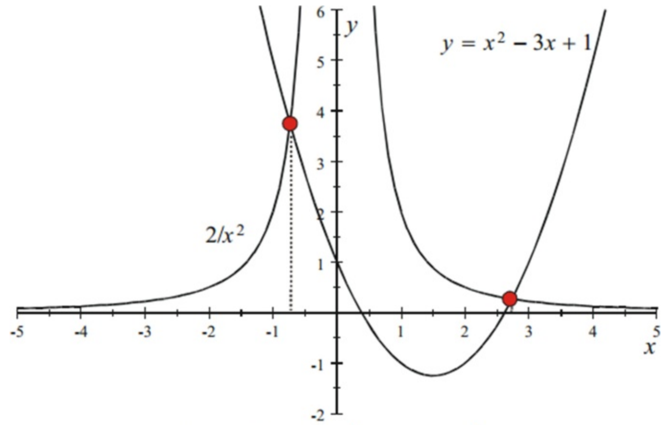


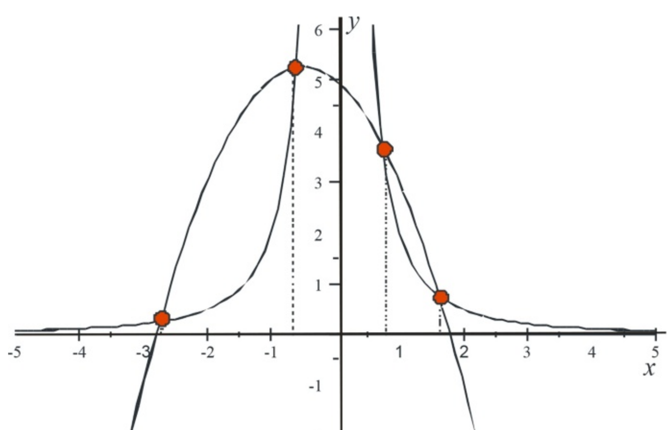
Intersections of $y = 2/x^2$ and $y = x^2 - 3x + 1$
Numerical solutions of the resulting quartic

$$y = \left(x^2\right)\left(x^2 - 3x + 1\right) - 2 = x^4 - 3x^3 + x^2 - 2$$

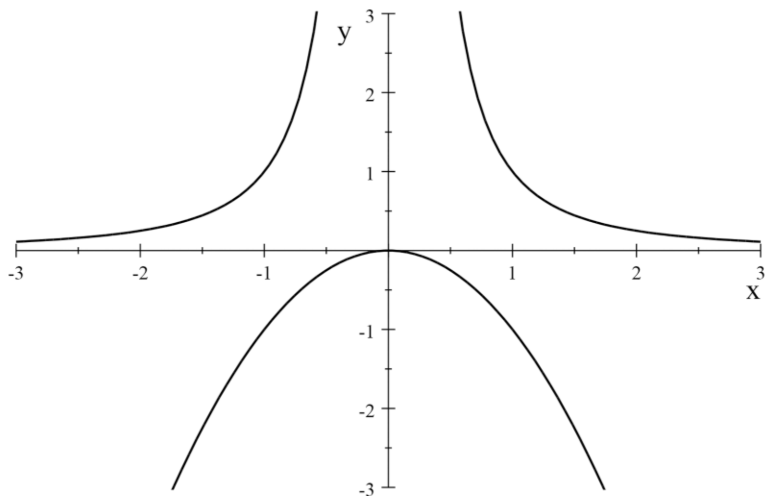
are $x = 2.7321$, $x = -0.73205$, $x = 0.5 - 0.86603i$, and $x = 0.5 + 0.86603i$.

There are just two real intersections; the other two are (conjugate) complex numbers. You can see these approximations on your graphing calculator easily. Can you find the complex solutions on your calculator?

On the other hand, the graphs of $y = -(x+2)^2 + 3(x+2) + 3$ and $y = \frac{2}{x^2}$, shown just below, reveal four real intersections. The relevant quartic is $y = \left(x^2\right)\left(-(x+2)^2 + 3(x+2) + 3\right) - 2 = -x^4 - x^3 + 5x^2 - 2.$ Numerical approximations of the roots are $x = -0.61803$, $x = -2.7321$, $x = 0.73205$, and $x = 1.618$.



Naturally, we don't actually recommend finding these solutions geometrically, except from the graphs. Finally, it is possible to view parabolas and reciprocal quadratics that have no intersections. The interpretation of this is that the resultant quartic has no real solutions. Also, the number of intersections of the two conics that realize a quartic must be 0, 2, or 4. (Why?)



No intersections of the graphs of $y = -x^2$ *and* $y = 2/x^2$

Problem 1. Show how to solve a cubic by intersecting a parabola with a circle.

Problem 2. Show how to solve any quadratic by intersecting a linear function with a hyperbola.

Problem 3. Give an argument using any quadratic and hyperbola that every cubic, $ax^3 + bx^2 + cx = d$, has at least one real solution.

SECTION 4

STATISTICS AND PROBABILITY

G. DONALD ALLEN

29. LET'S MAKE WAR—JUST FOR FUN

Almost every kid plays war of some kind. Aside from the usual cowboy and other war games, rainy afternoons were often the time to get out a deck of cards and play the card game War. As many times as I played this game as a kid, I can hardly ever remember finishing a game. What I can remember mostly is making a "tactical" agreement to quit and do something else.

The other day, I thought to play war again—but on a computer. For a game of pure chance, there is a remarkable amount of mathematics involved. If your mathematical toolkit includes course in probability, then you can appreciate just how complex this simple kids game is. Just determining the expected outcome after playing the first round after dealing is complex enough, let alone subsequent rounds --- when the players have unequal numbers of cards. The game is amazingly complex even when one restricts the size of the deck in almost any way. For example, suppose we have a deck of all suits numbered just two and three. This game, with just eight cards, is very complicated, and way too complicated for a theoretical analysis. What we want to do here is simulate the play on a computer. The results will not be mathematical theorems but rather a collection of averages percentages based on assorted contingencies.

For example, do you always win if you begin with four aces? Do you always win when you begin with four aces and four kings? Answer to both: no. Well, then how often do you win? How many "wars" per game can you expect to play? And very important: how long does take to play a game? Answer: not forever. These are some of the questions we'll consider. The results will be averages based on simulations. For our simulations, we will play over a million games of war, more than the average kid, even an avid war buff.

Certain results can be computed mathematically. We use them to validate our simulations. For example, the average number of aces of the deal should be two. The probability of being dealt three aces is

$$\frac{\binom{4}{3}\binom{48}{23}}{\binom{52}{26}} \approx 0.2497$$

This is determined by counting the number of possible hands of 26 cards with exactly three Aces. First, the number of ways of getting three Aces is (4/3). Second, the number

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 115–123.

of ways of getting the remaining 23 cards is $\binom{48}{23}$. (Note, the use of 48, as opposed to 49; this is to account for the fact that we exclude the fourth Ace from this hand.) The product of these divided by the number of possible 26 card hands, $\binom{52}{26}$, is the desired probability. In fact, the following table gives all the probabilities:

Table 1. Probabilities given the number of aces

Number of Aces	Probability	Approx. Value
0	$\frac{\binom{4}{0}\binom{48}{26}}{\binom{52}{26}}$	0.0552
1	$\frac{\binom{4}{1}\binom{48}{25}}{\binom{52}{26}}$	0.2497
2	$\frac{\binom{4}{2}\binom{48}{24}}{\binom{52}{26}}$	0.3902
3	$\frac{\binom{4}{3}\binom{48}{23}}{\binom{52}{26}}$	0.2497
4	$\frac{\binom{4}{4}\binom{48}{22}}{\binom{52}{26}}$	0.0552

In fact, our simulations give empirical results in very close agreement with these theoretical probabilities.

The Shuffle. Shuffling our computer deck well is very important. First, let's take our cue on how to do it from the mechanical process of shuffling a real deck of cards. This is usually done by riffling the deck. That is, the deck is split into two (almost equal) parts which are then riffled together. With a computer, we can riffle a computer deck perfectly: split the 52 card computer deck into two 26 card halves and interleave the halves to one deck by placing the bottom card of the first half first, the bottom card of the second half second, the second-to-bottom card of the first half third, the second-to-bottom card of the second half fourth, and so on. Now repeat this perfect riffle until the deck is well mixed. In fact what we are doing is permuting the cards. If we start with any deck with cards ordered 1…52, and apply a permutation (of order 52) to it, we achieve a shuffle. The old game of "52 pickup" that everyone learns once amounts to a random permutation of the cards.

For example, consider the small deck of 8 cards. Split it into two halves and riffle them as indicated above. The resulting shuffle can be realized by the following permutation matrix. In fact, this permutation matrix has the property that $P^3=I$, the identity matrix (and also a permutation matrix). This means that after 3 shuffles, the deck is back to where it started. Suppose for example, that our eight card deck is labeled as a vector x:=[1,2,3,4,5,6,7,8]. Then, the first, second, and third shuffles yield:

$$x = [1, 2, 3, 4, 5, 6, 7, 8]$$
$$Px = [1, 5, 2, 6, 3, 7, 4, 8]$$
$$P^2x = [1, 3, 5, 7, 2, 4, 6, 8]$$
$$P^3x = [1, 2, 3, 4, 5, 6, 7, 8]$$

Similarly, for the 52 card deck, it happens that in just eight such perfect shuffles, we arrive back at the original deck configuration. Mathematically, this means $P^8=I$, the identity matrix. One might conjecture that stopping the shuffle midway, say at four riffles, a good mix is achieved. This is not so; the reader may check that such a mix would give all the aces to one player. Other regularities abound. In general, for a deck of any (even) size, a similar riffle can be defined. All are equivalent to permutation matrices, and all satisfy the equation P{k}=I, for some k depending on the deck size. Curiously, for the riffle at hand, the 38 card deck has the greatest k (-36). However, for all deck sizes up to 52, the deck of size 8 yields the smallest number of riffles to return to its original state.

It turns out that the key to great random shuffles is imperfect riffles. Much has been written on how many shuffles to well-mix a deck, particularly for riffling the deck. Under a couple of assumptions, loosely described as:

- The deck is not evenly divided every time,
- The bottom cards are equally likely to be the bottom card after the riffle.

It turns out that about seven riffles suffice for randomly mixing a deck.

Since we are playing our game on the computer, we can use its greater number crunching versatility to mix our deck. Programming perfect riffles is easy --- but not very good. As you have seen above, it is no more complicated than multiplying a vector of numbers by a permutation matrix. Programming imperfect riffles is a little harder. Indeed, programming imperfect riffles has just about the same complexity as simply selecting the sequence of integers, 1,2,…,52 in random order which constitutes a shuffle. And this is what we will do here.

The task of selecting a sequence of numbers in random order requires a random number generator. More precisely, and in the language of mathematics, what we need is a pseudorandom number generator, because any sequence of numbers generated by a computer cannot be truly random, but merely simulate truly random numbers. By this we mean a number generator that spews out numbers that are, say, uniformly distributed over their range according to some preset statistical test or tests. These are the kind of sequences we want. Numbers with other distributions, such the normal or Poisson, are also possible to generate, as well.

One accepted way to produce such numbers is via the linear congruential method which uses modulo arithmetic. Begin with the fixed numbers *a*, *c*, and *M*. Take some number y_0, called the seed, and define the sequence

$$y_{n+1} = (ay_n + c) \bmod \quad M, n = 0,1,2,...,N$$

To obtain pseudorandom numbers in the interval [0,1), take

$$x_n = \frac{y_n}{M}$$

This method, with the fixed numbers *a*, *c*, and *M*, prudently selected, and there is a whole theory for this, generates a sequence of numbers that satisfies rigorous statistical tests for randomness --- though such methods have defects. The particular values we used were

$$M = 2^{31}-1,\ a = 16{,}807,\ c = 0.$$

In fact, the whole generator that produces the x_{n}, in Fortran code, is given by

```
function random(seed)
        integer seed
        double precision dl
        d = dmod(16807.0d0*dble(seed),2147483647.0d0)
        seed = idint(d)
        d = d *4.6566128752458d-10
        random = sngl (d)
        return
        end
```

What we did was compute 52 pseudorandom numbers using this generator and place them in an array. To get the randomly ordered integers 1,…,52, we used the relative positions of these numbers. For instance the "position" of second smallest number would be assigned the value 2 in some deck array. Finally, the deck would be reset into four suits by removing multiples of 13 from each of the numbers. Precisely, for each i=1,…,52, the $new\_deck_i = deck_i$ mod13 + 1, the plus 1 to give values between 1 and 13, not 0 and 12. A "13" is the Ace, a "12" is the King, and so on.

The game. With the deck well shuffled, we deal the cards and play the game. Programming the game is relatively straight forward. It is just of loop of compare statements with a subroutine for a "war." Let's look at the results. First, we played many, very many, games. One feature of the program is that the user can enter the number of games to be played in a set. For example, you could request it to play 100 games, or 1000, or even one million games. Call this a set. The output is a collection of statistics about the play. We selected sets of 100, 500, 1000, 2500, 5000, 10,000, 50,000, 100,000, 1,000,000 and 15,000,000 games. Depending on how many games are in a set, there were a number of interesting features. Below we list features more or less common to most of the sets.

- Each player won approximately, in fact, very nearly, half the games. Indeed, for the longest set of games, player A won 7,498,859 games and Player B won 7,501,141 games.
- The average length of a game was 296 plays, with a war counting as one play. However, the standard deviation was 218. This indicates quite a wide spread in the number of plays per game. In games played with 2 down cards for a war, the average length was about 225 plays with a standard deviation of 171 plays.
- The average number of "wars" per game was about twenty—and about fifteen for games played with two down cards.
- Approximately 25% of all games ended in default. That is, during a "war", the opponent did not have enough cards to complete the play - scored as a loss. Thus, about 75% of all games ended with one player's last card being captured. These percentages change to 40% and 60%, respectively, when a war is played with two down cards.
- The longest game of a set naturally increased as the number of games in the set increased. For example, in a set of 100 games, the longest game was 1428 plays. In the set we played of fifteen million games, the longest game was 4511 plays. To predict the expected duration of longest game in a set of say N games is a rather complicated mathematical problem, and requires considerable information about the probabilistic nature of the game itself. By the way, the shortest game played in the set of two million was just 27 plays, a short game indeed. (Question. Using our definition of a play, what is the shortest possible game? Answer: one play. How?)
- This game, in probability expert circles, is often considered a deterministic game. The randomness is all in the shuffle. This means, once the deck is shuffled, the outcome is completely determined. This does not imply one can make a function from the shuffle to the outcome. After all, there are $52! \doteq 8.08\times10^{67}$ shuffles, a huge number greater than the estimated number of atoms in our galaxy! Accounting for the hierarchy, an ace is an ace independent of suit etc.; we should note there are in this game $52!/(4!)^{13}$= 92024 242230271 040357108 320801872 044844750 000000000 9= 2024×10^{49} actual different shuffles, still a mighty big number.

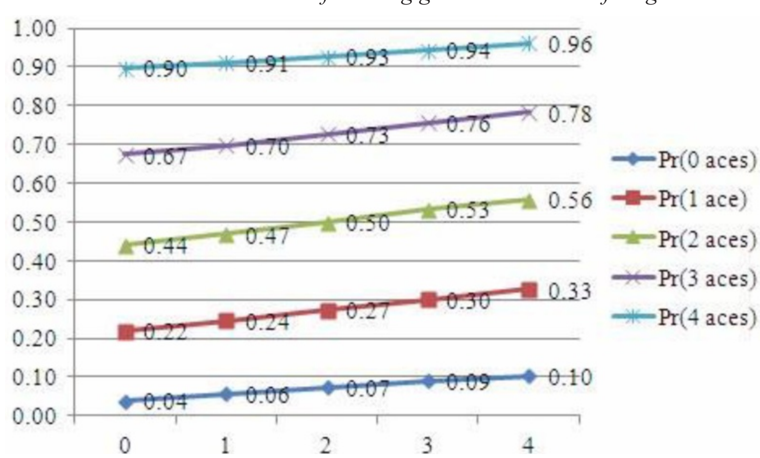
Now, onto the main subject which centers on the question: If you start with all the Aces and Kings, will you always win? Not too surprisingly, the answer is, no, but the probability of winning if you begin with all Aces and Kings is about 0.962. This comes from the table below which lists the results from playing a set of two million games. The overall probability of winning when you are dealt all four Aces is 0.924. The complete table of percentage wins from which we conclude probabilities is given below.

Table 2. Wins

Percentages wins with Aces and Kings							
		Aces					
		0	1	2	3	4	Overall
Kings	0	3.8%	21.7%	43.9%	67.4%	89.8%	48%
	1	5.7%	24.4%	46.9%	69.9%	91.0%	49%
	2	7.4%	27.3%	50.0%	72.7%	92.6%	50%
	3	9.1%	30.0%	53.2%	75.7%	94.4%	51%
	4	10.4%	32.6%	55.8%	78.3%	96.2%	52%
	Overall	7.6%	27.5%	50.0%	72.5%	92.4%	50%

Another fact to observe is that there appears to be a strict ranking of percentage wins according to the order of Aces and Kings which puts a_1 Aces and k_1 Kings ahead of a_2 Aces and k_2 Kings if $a_1 \geq a_2$ or if $a_1 = a_2$ and $k_1 \geq k_2$. For example, this makes the player dealt four Aces no Kings a more likely winner than the player that begins with three Aces and four Kings. It may be inferred that an Ace has four times the value of a King. As is evident from the graph, the probability increases almost linearly within quadrades of kings. You can see the probability of winning given four aces and no kings on the top graph to be 0.90.

Table 3. Probabilities of winning given the number of kings



Does the computed distribution of Aces and Kings follow theoretical expectations? This we actually can compute. As indicated above, the probability of being dealt *a* Aces is

$$\frac{\binom{4}{a}\binom{48}{26-a}}{\binom{52}{26}}$$

Similarly, it can be demonstrated that the probability of obtaining *a* Aces and *k* kings is

$$\frac{\binom{4}{a}\binom{4}{k}\binom{44}{26-a-k}}{\binom{52}{26}}$$

Data from our 15,000,000 game set illustrates that the initial distribution of Aces and Kings matches nearly exactly these values. In the two charts below, we give the theoretical probabilities and the actual values for our experiment.

Table 4. Theoretical probabilities

Theoretical probabilities given Aces and Kings							
		Aces					
		0	**1**	**2**	**3**	**4**	**Overall**
Kings	**0**	0.0021	0.0114	0.0213	0.0162	0.0042	0.0552
	1	0.0114	0.0568	0.0974	0.0679	0.0162	0.2497
	2	0.0213	0.0974	0.1527	0.0974	0.0213	0.3902
	3	0.0162	0.0679	0.0974	0.0568	0.0114	0.2497
	4	0.0042	0.0162	0.0213	0.0114	0.0021	0.0552
	Overall	0.0552	0.2497	0.3902	0.2497	0.0552	

Table 5. Computed probabilities

Computed probabilities given Aces and Kings							
		Aces					
		0	**1**	**2**	**3**	**4**	**Overall**
Kings	**0**	0.0021	0.0113	0.0213	0.0162	0.0043	0.0552
	1	0.0114	0.0569	0.0973	0.0681	0.0162	0.2499
	2	0.0213	0.0974	0.1528	0.0973	0.0213	0.3900
	3	0.0162	0.0679	0.0973	0.0568	0.0114	0.2497
	4	0.0042	0.0162	0.0213	0.0113	0.0021	0.0552
	Overall	0.0553	0.2498	0.3901	0.2497	0.0552	

How did the distribution of games by number of plays look? While in our 15,000,000 game simulation, the longest game was 4511 plays, we show the distribution only up to 2500+ plays. It does reveal a "Poisson-like" distribution, though it actually isn't. You see here the great preponderance of games was right at the average. The horizontal scale is the game length, and the vertical scale is the number of games at that length. Note over all in this note, we have used graphical, tabular, and charts to tell this interesting story. See Figure 1. The vertical axis is the number of games with the given length (horizontal axis).

Timings. How fast did the computer play a game? This again depended on the number of games to a set and the computer. On a typical PC, a Dell, with the program compiled by the Absoft Fortran 90 Compiler, the run time was just over 3917 seconds, or about 65.3 minutes. Translated to smaller numbers, the computer played about 3829 games per second! Much of that time, was spent shuffling the pseudodeck.

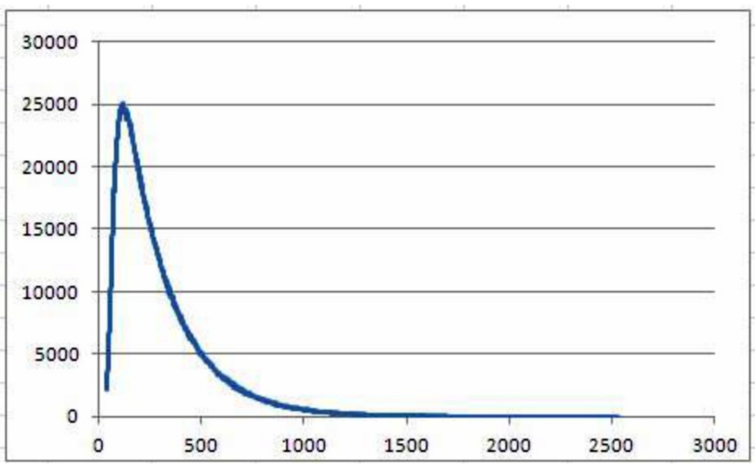


Figure 1. Distribution of number of games by game length

Rules for the games of war.

1. Shuffle the deck and deal an equal number of cards to each player. Players are not permitted to select specific cards for play.
2. Each player plays a card from the top of the deck; the winner of the play is the player with the higher face value; he takes both cards and places them at the bottom of his deck.
3. In the event both players play a card of the same face value, a "war" is declared. In this case, both players play a new card face down and a second card face up. The winner of the "war" is the player with the higher face value for the second card. The winner takes all six cards (and places them at the bottom of the deck).

If the second cards have the same face value, a new "war" is declared. The above process is repeated, and repeated again until one player has a higher second card. The winner collects all cards.

4. When one player's cards are depleted, the other player is the winner. If one player runs out of cards during a war, the other player declared the winner. If both players simultaneously run out of cards, the game is considered a draw. This is the only condition for a draw. In our simulations, not a single draw occurred.
5. One game variation is to play wars with two down cards. Another variation is to play three down cards and the player selects one of them as the up card.

REFERENCES

Keller, J. B. (1995). How many shuffles to mix a deck. *SIAM Review, 37*, 88–89.

Bayer, D., & Diaconis, P. (1992). Trailing the dovetail shuffle to its lair. *The Annals of Applied Probability, 2*, 294–313.

Niederreiter, H. (1992). *Random number generation and Quasi-Monte Carlo methods*. Philadelphia, PA: Society for Industrial and Applied Mathematics.

AMANDA ROSS

30. A BRIEF LOOK AT PROBABILITY

Middle and high school students often have great difficulty understanding the concept of probability, the difference between experimental and theoretical probability, and how to apply probability to solving real-world problems. Many students, as well as teachers, simply memorize some algorithms for different situations and apply them where applicable, without fully understanding why the probabilities are calculated in such a manner. This article serves to explain the concepts of simple and compound probability, as well as to provide different examples of variations of the same sort of probability problem. In doing so, the teacher will be given additional resources for understanding, as well as ways of teaching, the concept of probability. This article is not meant to discuss or present all aspects of probability concepts studied in middle and high school mathematics; it is merely a brief examination of the topic, with a specific focus on compound probability.

DEFINITION OF PROBABILITY

Probability is defined as the likelihood that an event will occur. Probability ranges from 0 to 1, with a probability closer to 1 representing a stronger probability. A probability of 0 represents an impossible event, while a probability of 1 represents a certain event.

The study of probability includes theoretical and experimental probability. Theoretical probability equals the ratio of possible number of outcomes of an event to the total sample space. Experimental probability is based upon actual trials. For example, tossing a coin 100 times and recording the total number of heads would be an example of experimental probability. Note. As the number of trials increases, the experimental probability approaches the theoretical probability.

SIMPLE PROBABILITY

Simple probability is the probability of a single event. It is defined as the ratio of the number of outcomes in the event to the total number of possible outcomes. Examples of simple probability include the roll of a die, a coin toss, drawing a card from a deck of cards, spinning a spinner, and drawing a marble out of a bag.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 125–130.

COMPOUND PROBABILITY

Compound probability is the probability of two or more events, called compound events. Compound events may be mutually exclusive or non-mutually exclusive, or independent or dependent. Compound events include the union and intersection of two or more events.

MUTUALLY EXCLUSIVE AND NON-MUTUALLY EXCLUSIVE EVENTS

Mutually exclusive events are events that cannot concurrently happen. In other words, the events cannot happen at the same time. Examples of mutually exclusive events are rolling a 2 and a 4, getting a head and a tail, and rolling a 1 and an even number.

Non-mutually exclusive events are events that overlap. In other words, the events have at least one element in common. Examples of non-mutually exclusive events are rolling a 2 and an even number, drawing a jack and a spade, and rolling a prime number and an odd number.

When speaking of the probability of mutually exclusive and non-mutually exclusive events, the union of events A and B is examined. This probability is denoted as P(A or B).

The probability of mutually exclusive, or disjoint, events, A and B, may be represented as $P(\text{A or B}) = P(A) + P(B)$.

The probability of non-mutually exclusive events, A and B, may be represented as $P(\text{A or B}) = P(A) + P(B) - P(\text{A and B})$.

INDEPENDENT AND DEPENDENT EVENTS

Events are said to be independent, when the outcome of one event does not depend on the outcome of another event. In other words, the outcome of a second event is not affected by the outcome of a first event. Events are said to be dependent events, when the outcome of the second event does depend on the outcome of the first event.

Examples of independent events include rolling a die and tossing a coin, tossing a coin and drawing a card from a deck of cards, spinning a spinner and drawing a marble from a bag, and drawing a card from a deck of cards, replacing the card, and drawing another card. Examples of dependent events include drawing a card from a deck of cards, not replacing the card, and drawing another card, or drawing a marble from a bag, not replacing the marble, and drawing another marble.

When speaking of the probability of independent or dependent events, the intersection of events A and B is examined. This probability is denoted as P(A and B).

The probability of independent events, A and B, may be represented as $P(\text{A and B}) = P(A) \cdot P(B)$.

The probability of dependent events, A and B, may be represented as $P(\text{A and B}) = P(A) \cdot P(B|A)$.

SAMPLE PROBLEMS

Example 1

A Single Coin Toss

a. What is the probability of getting heads or tails?

The events are disjoint. Thus, the probability may be represented as $P(\text{heads or tails}) = \frac{1}{2} + \frac{1}{2}$ or $P(\text{heads or tails}) = 1$.

Multiple Coin Tosses (Flipping the Same Coin More Than Once)

b. What is the probability of getting heads on the first flip and tails on the second flip?

Multiple coin tosses are independent events because the outcomes of the subsequent tosses do not depend on the outcome of the previous tosses. Thus, the probability may be represented as $P(\text{heads and tails}) = \frac{1}{2} \cdot \frac{1}{2}$, or $P(\text{heads and tails}) = \frac{1}{4}$.

c. What is the probability of getting three tails?

The probability may be represented as $P(\text{tails and tails and tails}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$, or $P(\text{tails and tails and tails}) = \frac{1}{8}$.

Example 2

The Roll of a Single Die

a. What is the probability of rolling a 1 or an even number?

The events, rolling a 1 and rolling an even number, are disjoint. Thus, the probability may be represented as $P(1 \text{ or even}) = \frac{1}{6} + \frac{3}{6}$, or $P(1 \text{ or even}) = \frac{2}{3}$.

b. What is the probability of rolling a 2 or an even number?

The events, rolling a 2 and an even number, are not disjoint. Thus, the probability may be represented as $P(2 \text{ or even}) = \frac{1}{6} + \frac{3}{6} - \frac{1}{6}$, or $P(2 \text{ or even}) = \frac{1}{2}$.

The Roll of a Die Two Times

c. What is the probability of rolling a 1 on the first roll and an even number on the second roll?

The rolls of a die are independent events because the outcome of the second roll does not depend on the outcome of the first roll. Thus, the probability may be represented as $P(1 \text{ and even}) = \frac{1}{6} \cdot \frac{3}{6}$, or $P(1 \text{ and even}) = \frac{1}{12}$.

d. What is the probability of rolling a 2 on the first roll and an even number on the second roll?

The events are again independent. Thus, the probability may be represented as $P(2 \text{ and even}) = \frac{1}{6} \cdot \frac{3}{6}$, or $P(2 \text{ and even}) = \frac{1}{12}$.

Example 3

Drawing Two Cards from a Deck of Cards

The table below presents the solutions for four sample problems, given that the first card is, or is not, replaced.

Problem	*Probability when First Card is Replaced*	*Probability when First Card is Not Replaced*
What is the probability of drawing an ace and then a jack?	$P(\text{ace and jack}) = \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$	$P(\text{ace and jack}) = \frac{4}{52} \cdot \frac{4}{51} = \frac{4}{663}$
What is the probability of drawing an ace and then another ace?	$P(\text{ace and ace}) = \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$	$P(\text{ace and ace}) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$
What is the probability of drawing a face card and then a 2?	$P(\text{face and 2}) = \frac{12}{52} \cdot \frac{4}{52} = \frac{3}{169}$	$P(\text{face and 2}) = \frac{12}{52} \cdot \frac{4}{51} = \frac{4}{221}$
What is the probability of drawing an ace and then a spade?	$P(\text{ace and spade}) = \frac{4}{52} \cdot \frac{13}{52} = \frac{1}{52}$	$P(\text{ace and spade}) = \left(\frac{12}{52} \cdot \frac{4}{51}\right) + \left(\frac{1}{52} \cdot \frac{3}{51}\right) = \frac{1}{52}$

When looking at the probability computations for a replaced card, it can be generalized that the sample space for the probability of the second card draw is not reduced; it remains 52. However, when the first card is not replaced, the sample space for the probability of the second draw is decreased by 1. Furthermore, when the card is the same, the number of possible outcomes is also decreased by 1.

The last problem shows more computational difficulty, for non-replacement. In fact, this sort of problem is often omitted from mathematics texts and other mathematics resources. It appears to be a type of problem that is not explicitly taught to students. Thus, a bit more time will be spent on examining this problem. The probability of drawing an ace and then a spade shows an overlap, since there is an ace of spades. The two events are not mutually exclusive. Thus, the probability should be computed in two steps. First, compute the probability, considering that an ace of spades is not drawn. This probability is equal to the product of the probability of drawing a spade that is not an ace and the probability of drawing an ace, with the sample space reduced. Next, compute the probability, considering that an ace of spades is drawn. This probability is equal to the product of the probability of drawing an ace of spades and the probability of drawing an ace, with the number of possible outcomes and sample space reduced.

Example 4
Drawing Marbles from a Bag
A bag contains 4 green marbles, 2 yellow marbles, 3 blue marbles, and 5 orange marbles.
With Replacement

a. What is the probability of drawing a green marble and a blue marble?
The events are independent. Thus, the probability may be represented as $P(\text{green and blue}) = \frac{4}{14} \cdot \frac{3}{14}$ or $P(\text{green and blue}) = \frac{3}{49}$.

b. What is the probability of drawing two green marbles?
The events are again independent. Thus, the probability may be represented as $P(\text{green and green}) = \frac{4}{14} \cdot \frac{4}{14}$ or $P(\text{green and green}) = \frac{4}{49}$.
Without Replacement

c. What is the probability of drawing a green marble and a blue marble?
The events are now dependent, since the first marble is not replaced. The sample space for the second event will decrease by 1. Thus, the probability may be represented as $P(\text{green and blue}) = \frac{4}{14} \cdot \frac{3}{13}$ or $P(\text{green and blue}) = \frac{6}{91}$.

d. What is the probability of drawing two green marbles?
The events are again dependent. Also, in this case, the two marbles are the same. Thus, the sample space AND number of possible outcomes will each decrease by 1. The probability may be represented as $P(\text{green and green}) = \frac{4}{14} \cdot \frac{3}{13}$ or $P(\text{green and green}) = \frac{6}{91}$.

Example 5
Finding P(A or B) or P(A and B) Using Addition Rule
The probability of the union, or intersection, of two events may be found by using the addition rule for non-mutually exclusive events, namely $P(\text{A or B}) = P(A) + P(B) - P(\text{A and B})$.

b. The probability that Josh will attend a Division 1 University is 0.40. The probability that he will attend a Division 2 University is 0.45. The probability that he will attend a Division 1 and Division 2 University is 0.10. What is the probability that he will attend a Division 1 or Division 2 University?

By evaluating the addition rule for the given probabilities, the probability may be represented as:

$$\begin{aligned} P(1 \text{ or } 2) &= P(1) + P(2) - P(1 \text{ and } 2) \\ &= 0.40 + 0.45 - 0.10 \\ &= 0.75 \end{aligned}$$

c. Theoretically speaking, the probability that Eric will choose a latte is 0.25. The probability that he will choose an espresso is 0.10. The probability that he will choose a latte or an espresso is 0.30. What is the probability that he will choose a latte and an espresso?

By evaluating the addition rule for the given probabilities, the probability may be represented as:

$$\begin{aligned} &P(\text{latte or espresso}) = P(\text{latte}) + P(\text{espresso}) - P(\text{latte and espresso}) \\ &0.30 = 0.25 + 0.10 - P(\text{latte and espresso}) \\ &0.30 = 0.35 - P(\text{latte and espresso}) \\ &0.05 = P(\text{latte and espresso}) \end{aligned}$$

The probability of choosing a latte and espresso is 0.05 or 5%.

CONCLUSION

In order to promote a conceptual understanding of probability, students must be exposed to many different variations of the same type of probabilistic problem. Students should be encouraged to ask questions and seek answers to non-standard probability problems. A think-pair-share activity would be helpful here, as students ponder questions for which they do not know the answer, discuss with a partner, and then discuss with the whole class. This list of challenging problems may then serve as an ongoing class research project that could be posted to the class website. If students only know how to approach the most simplistic probability problems, they will be greatly hindered in college and in real-world situations that require probabilistic thinking. This brief article attempts to jumpstart this sort of thinking on the part of the student, as well as the teacher.

AMANDA ROSS

31. AREA UNDER THE NORMAL CURVE

The normal curve serves as a mystery for many students. Students are often shown examples of normally distributed data, in the form of a bell-shape. This curve is called "normal" or "symmetrical." This normal distribution may show a mean other than 0. Then, students are shown a standard normal curve, with a mean of 0 and a standard deviation of 1. They are asked to use this curve to find areas under "the curve." Students are asked to compute some "z-scores" and use them to find these "areas" that may be applied to the sample at hand. For any high school student (or college student, for that matter), these ideas may be quite daunting, if not presented in a clear manner.

This brief article will focus on providing a conceptual overview of the normal curve, as well as exemplars for calculating different areas, given real-world scenarios.

A NORMAL DISTRIBUTION

First of all, a normal distribution is simply a distribution of data values that is symmetrical. The mean, median, and mode of normally distributed data will all fall at the same place. The distribution will not be skewed. For a non-standard normal distribution, the plotted data will represent actual raw scores that have not yet been standardized. (Standardizing scores will be explained next.)

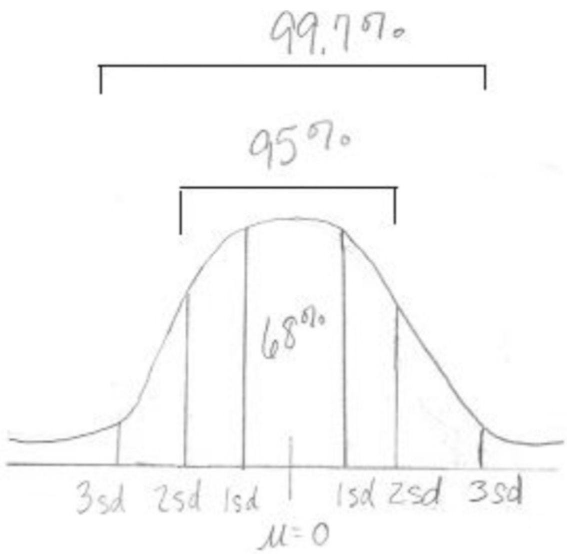
STANDARD NORMAL CURVE

The standard normal curve is a type of normal distribution, or normal curve. It is also symmetrical, but represents standardized scores. The raw scores are standardized as z-scores, which represent the number of standard deviations a score falls above or below the mean. By converting to z-scores (or standardized scores), the variation of scores is taken into account, when comparing scores to the mean.

The standard normal curve has a mean of 0 and a standard deviation of 1. The total area under the curve is 1, with an area of 0.5 above and below the mean. The standard normal curve is bell-shaped.

A diagram of the standard normal curve, or standard normal distribution, is shown below. Note. The diagram is not drawn to scale.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 131–140.



Notice that 68% of the area falls within ±1 standard deviations, 95% of the area falls within ±2 standard deviations, and 99.7% of the area falls within ±3 standard deviations. These areas can be confirmed using the normal z-table, i.e., for a z-score of 1, the mean to z area is 0.3413. Thus, the area between z = −1 and z = 1, is 2 × 0.3413 or 0.6826, or approximately 68%. This is an important concept to explain to students, so they know how these percentages are derived.

The z-scores are compared to find areas under the normal curve. A standard normal distribution table (or z-table) gives the mean to z area for z-scores from 0 to 4, as well as the larger and smaller portion areas for each z-score. This information may be used to find the areas between two z-scores, the area above a z-score, the area below a z-score, and the area between the mean and a z-score. The most helpful first step in finding such areas involves drawing a diagram.

Thus, in order to summarize the aforementioned information, prior to delving into some examples, we have the following:

- The standard normal curve is a type of normal distribution, with a mean of 0 and standard deviation of 1.
- The areas between ±1 standard deviations, ±2 standard deviations, and ±3 standard deviations are 68%, 95%, and 99.7%, respectively.
- The standard normal curve allows you to compare scores that have been standardized, which takes into account the variation of scores about the mean. (By converting two raw scores to z-scores, the areas between the two scores may be appropriately interpreted.)
- A standard normal distribution table (or z-table) is used to compute areas under the normal curve.

CALCULATING A Z-SCORE

A z-score is represented as $z = \frac{X - \mu}{\sigma}$, where X represents the score, μ represents the population mean, and σ represents the population standard deviation. Literally speaking, the z-score represents the number of standard deviations a value falls above or below the mean.

USING THE Z-TABLE

A z-score may be located in the z-table. For example, suppose you have calculated a z-score of 1.25. (This score represents a raw score that is 1.25 standard deviations above the mean.) You are asked to find the mean to z area for this z-score. An excerpt of the z-table is shown below:

z	*Mean to z*	*Larger Portion*	*Smaller Portion*
1.20	0.3849	0.8849	0.1151
1.21	0.3869	0.8869	0.1131
1.22	0.3888	0.8888	0.1112
1.23	0.3907	0.8907	0.1093
1.24	0.3925	0.8925	0.1075
1.25	0.3944	0.8944	0.1056
1.26	0.3962	0.8962	0.1038
1.27	0.3980	0.8980	0.1020
1.28	0.3997	0.8997	0.1003

The table shows the mean to z area to be 0.3944, or 39.44%. This area can also be determined by finding the smaller portion area and subtracting it from 0.5, or one-half of the area under the normal curve (0.5 – 0.1056 = 0.3944). It is important that students understand the relationship between the whole area under the curve, one-half of the area under the curve, and the placement of the z-score. This conceptual understanding will allow them to find given areas, using more than one approach.

EXAMPLES

Now, let's look at some examples that students may encounter.

Example 1

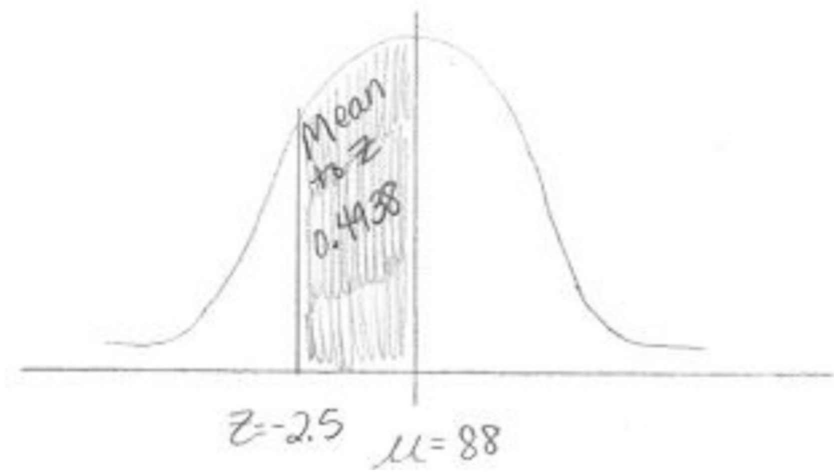
The class average on a final statistics exam is 88, with a standard deviation of 4 points. Amy scores 78 on the exam.

(For each problem below, a diagram will be drawn, in order to illustrate the scenario. Note. The drawings are not drawn to scale.)

a. What percentage of students scored between 78 and the class average?
   Step 1: Calculate the z-score.

$$z = \frac{78 - 88}{4} = -2.5$$

   Step 2: Interpret the z-score.
   Amy's score is 2.5 standard deviations below the mean.
   Step 3: Draw a diagram, labeling the correct area.

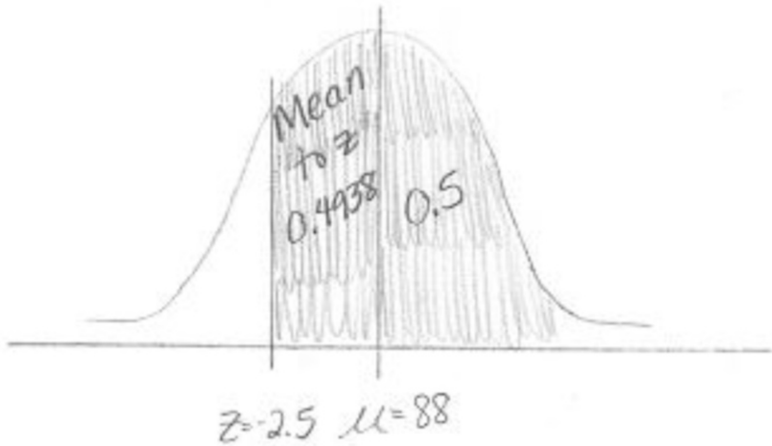


The area is found using the z-table.
An excerpt of the z-table is provided below:

z	*Mean to z*	*Larger Portion*	*Smaller Portion*
2.45	0.4929	0.9929	0.0071
2.46	0.4931	0.9931	0.0069
2.47	0.4932	0.9932	0.0068
2.48	0.4934	0.9934	0.0066
2.49	0.4936	0.9936	0.0064
2.50	0.4938	0.9938	0.0062
2.51	0.4940	0.9940	0.0060
2.52	0.4941	0.9941	0.0059
2.53	0.4943	0.9943	0.0057

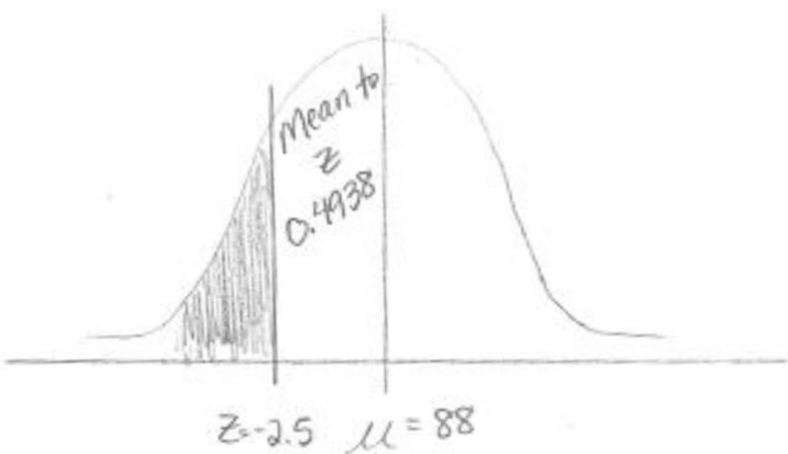
The mean to z area for a z-score with an absolute value of 2.5 is 0.4938. Thus, 49.38% of the students scored between 78 and the class average.

b. What percentage of students scored higher than Amy?



The area above this z-score equals the sum of 0.5 and 0.4938, or 0.9938. Thus, 99.38% of the students scored higher than Amy. The larger portion area, given in the z-table, represents this same area.

c. What percentage of students scored lower than Amy?



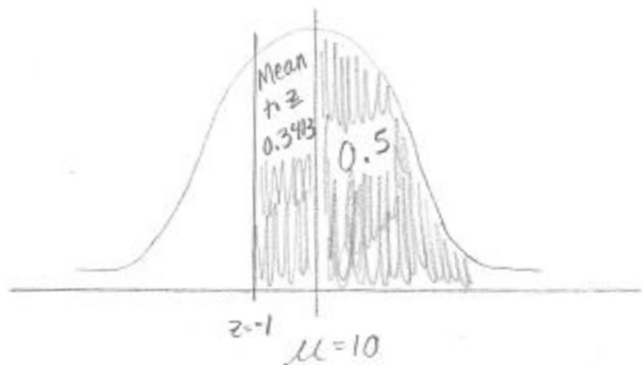
The area below this z-score equals the difference between 0.5 and 0.4938, or 0.0062. Thus, 0.62% of the students scored lower than Amy. The smaller portion area, given in the z-table, represents this same area.

Example 2

The average amount of lotion per bottle is 10 ounces, with a standard deviation of 0.2 ounces. One randomly selected bottle contains 9.8 ounces.

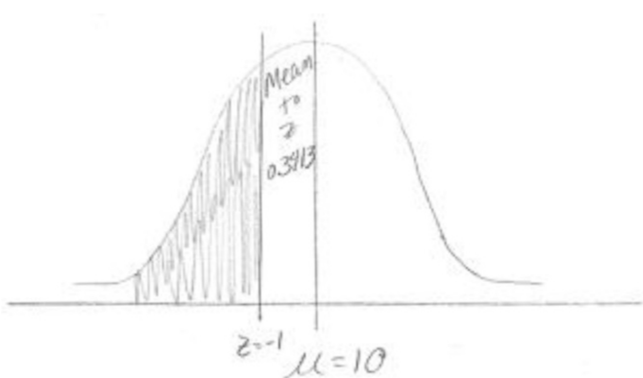
a. What percentage of bottles contained more ounces than this bottle?

The z-score is represented as $z = \frac{9.8 - 10}{0.2} = -1$.



We know that the mean to z area for a z-score with an absolute value of 1 is 0.3413. Thus, the total area equals the sum of 0.5 and 0.3413, or 0.8413. (This is also the larger portion area for a z-score with an absolute value of 1.) So, 84.13% of the bottles contained more ounces than this randomly selected bottle.

b. What percentage of bottles contained fewer ounces than this bottle?



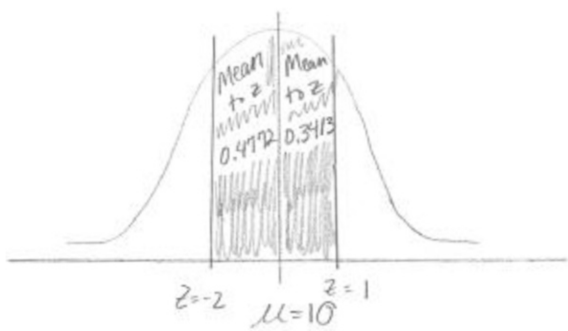
This area is equal to the difference of 0.5 and the mean to z area of 0.3413, or 0.1587. (This is the smaller portion area for the z-score.) So, 15.87% of the bottles contained fewer ounces than this randomly selected bottle.

c. What percentage of bottles contain between 9.6 and 10.2 ounces?

In this case, the piece of information about the randomly selected bottle is not needed. We are concerned with two new scores and must calculate two new z-scores. The z-scores are shown below:

$$z = \frac{9.6 - 10}{0.2} = -2$$

$$z = \frac{10.2 - 10}{0.2} = 1$$

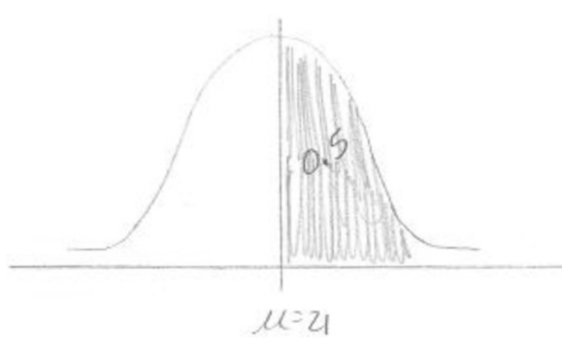


Since we want the area between these z-scores, we simply find the mean to z areas for each z-score and add them. The mean to z area for a z-score with an absolute value of 2 is 0.4772. The mean to z area for a z-score of 1 is 0.3413. Thus, the area between the scores is 0.8185, or 81.85%. So, 81.85% of the bottles contain between 9.6 and 10.2 ounces.

Example 3

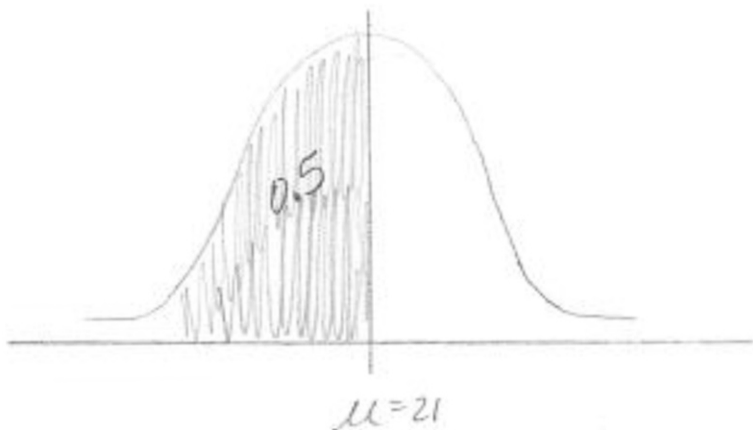
Suppose the average ACT score is 21, with a standard deviation of 3 points.

a. What percentage of students score higher than 21?



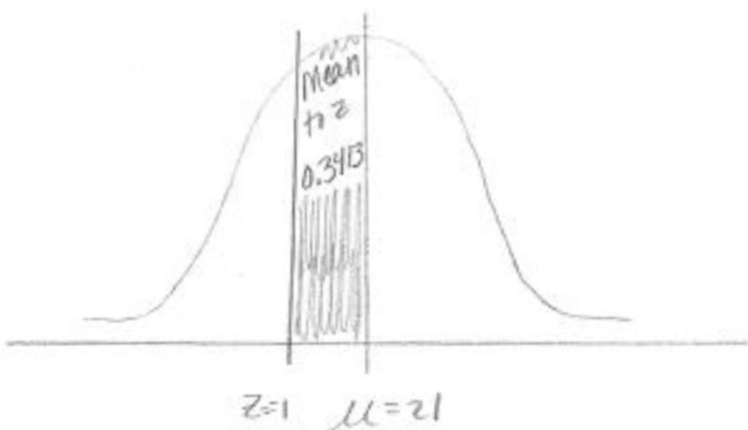
This area above the mean is 0.5. Thus, 50% of the students score higher than 21. This is a good problem because students will have to think about what the z-score represents.

b. What percentage of students score lower than 21?



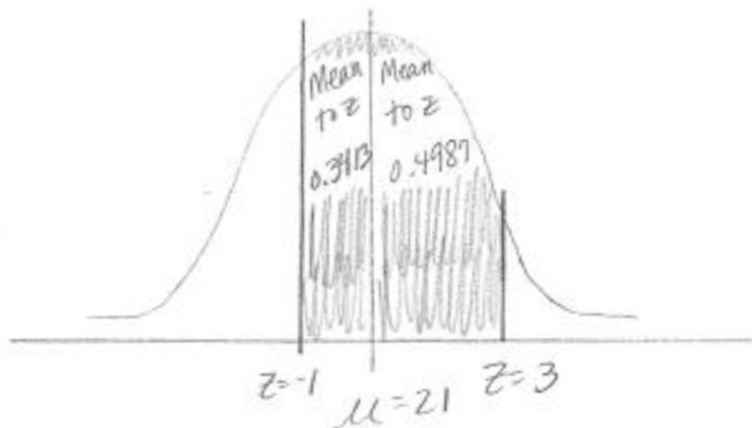
The same type of thinking is needed here. The area below the mean is also 0.5. Thus, 50% of the students score lower than 21.

c. What percentage of students score between 18 and 21?
The z-score for the raw score of 18 is −1.



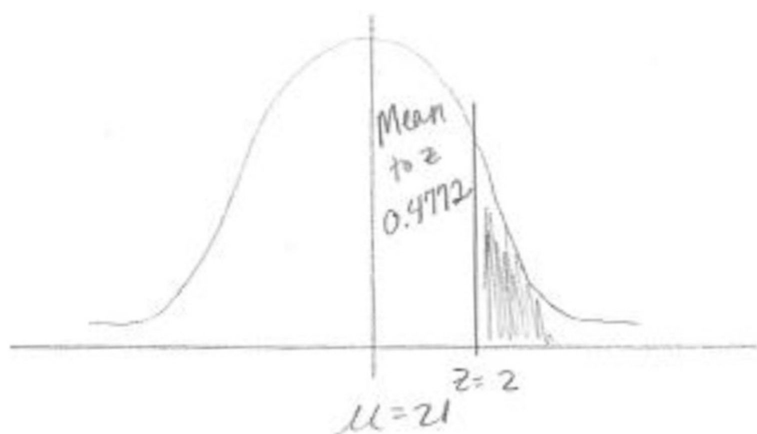
The mean to z area is 0.3413 or 34.13%. Specifically stated, the percentage of students who score between 18 and 21 (the average) is 34.13%.

d. What percentage of students score between 18 and 30?
   One new z-score must be calculated. We know that the z-score for the raw score of 18 is −1. The z-score for the raw score of 30 is 3.



The mean to z area for a z-score with an absolute value of 1 is 0.3413. The mean to z area for a z-score of 3 is 0.4987. Thus, the total area between these scores is 0.84, or 84%. So, 84% of the students score between 18 and 30.

e. What percentage of students score higher than 27?
   The z-score for a raw score of 27 is 2.



The mean to z area for a z-score of 2 is 0.4772. Thus, the area above this z-score is equal to the difference of 0.5 and 0.4772, or 0.0228. (This is the smaller portion area for the z-score.) So, 2.28% of the students score higher than 27.

This brief article describes the concept of a normal distribution and a standard normal distribution, the differences between raw scores and standard scores, and the ways to use standard scores to find areas under the normal curve, given different real-world scenarios. The key to a conceptual understanding of these concepts involves drawing diagrams and verbally explaining to which area the problem is referring. In addition, in reference to part e above, the student should be able to explain that the z-score of 2 relates to the raw score of 27. In other words, a score of 27 is 2 standard deviations above the mean, or 21. In closing, students should be given opportunities to discuss, ask questions, and compare ideas related to the normal curve.

Note. This article focused on manual computation of areas under the normal curve. An upcoming article will discuss how the area under the normal curve may be calculated, using a graphing calculator. Students should be adept at finding the area under the normal curve, using both approaches.

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32. FINDING THE AREA UNDER THE NORMAL CURVE USING THE GRAPHING CALCULATOR

This brief article serves as a follow-up to the article, "Area Under the Normal Curve." In this article, the area under the curve for each sample problem will be calculated using a graphing calculator. The previous article focused on the manual approach for calculating such area.

USING THE GRAPHING CALCULATOR

There are two methods for finding the area under a normal curve, using a graphing calculator.

You may use the normalcdf function or the ShadeNorm function. The latter function will show the shading of the area under the normal curve. Both approaches involve the input of a lower bound, upper bound, mean, and standard deviation.

When using the normalcdf function, follow the steps below:

1. Select 2nd Vars and normalcdf(.
2. Enter the lower bound, upper bound, mean, and standard deviation.
3. Press Enter.

When using the ShadeNorm function, follow the steps below:

1. Select 2nd Vars and ShadeNorm(.
2. Enter the lower bound, upper bound, mean, and standard deviation.
3. Press Enter.

When using the ShadeNorm function, a graph of the shaded area under the normal curve will appear with the area, lower bound, and upper bound given. The window will need to be set appropriately.

The examples below show how to use the normalcdf function to find the areas.

Example 1

The class average on a final statistics exam is 88, with a standard deviation of 4 points. Amy scores 78 on the exam.

a. What percentage of students scored between 78 and the class average?
   Enter normalcdf(78, 88, 88, 4).

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 141–143.

The area is 0.4938. Thus, 49.38% of students scored between 78 and the class average.

b. What percentage of students scored higher than Amy?
Enter normalcdf(78, 100, 88, 4).
The area is 0.9924. Thus, approximately 99% of the students scored higher than Amy. Note. Using the normal distribution table, the area is shown to be 0.9938. When using a much larger upper bound, such as 1000, the graphing calculator also gives this same area.

c. What percentage of students scored lower than Amy?
Enter normalcdf(0, 78, 88, 4).
The area is 0.0062. Thus, 0.62% of students scored lower than Amy.

Example 2

The average amount of lotion per bottle is 10 ounces, with a standard deviation of 0.2 ounces. One randomly selected bottle contains 9.8 ounces.

a. What percentage of bottles contained more ounces than this bottle?
Enter normalcdf(9.8, 20, 10, 0.2).
Note. An arbitrary upper bound was chosen. The area is 0.8413. Thus, 84.13% of the bottles contained more ounces than this bottle.

b. What percentage of bottles contained fewer ounces than this bottle?
Enter normalcdf(0, 9.8, 10, 0.2).
The area is 0.1587. Thus, 15.87% of the bottles contained fewer ounces than this bottle.

c. What percentage of bottles contain between 9.6 and 10.2 ounces?
Enter normalcdf(9.6, 10.2, 10, 0.2).
The area is 0.8186. Thus, 81.86% of the bottles contained between 9.6 and 10.2 ounces. (This is slightly different from the area calculated using the z-table, simply due to rounding.)

Example 3

Suppose the average ACT score is 21, with a standard deviation of 3 points.

a. What percentage of students score higher than 21?
Since the question asks for the percentage of students who score higher than the mean, it is known that the answer is 50%. However, the normalcdf function may be used to show this area.
Enter normalcdf(21, 40, 21, 3).
The area is 0.5. So, 50% of the students scored higher than 21.

b. What percentage of students score lower than 21?
This answer is also known to be 50%. However, the lower and upper bounds, mean, and standard deviation may be entered as shown below.
Enter normalcdf(0, 21, 21, 3).
The area is again 0.5. So, 50% of the students score lower than 21.

c. What percentage of students score between 18 and 21?
Enter normalcdf(18, 21, 21, 3).
The area is 0.3413. Thus, 34.13% score between 18 and 21.

d. What percentage of students score between 18 and 30?
Enter normalcdf(18, 30, 21, 3).
The area is 0.8399. Thus, approximately 84% score between 18 and 30.

e. What percentage of students score higher than 27?
Enter normalcdf(27, 40, 21, 3).
Note. An arbitrary upper bound was chosen. The area is 0.0228. Thus, 2.28% score higher than 27.

Each of these answers may be compared to the answers, given in the previous article. Students should be shown both approaches and be adept at solving using either a manual or technological method. It would be helpful for students to first solve, using the z-table and then check their answers, using the graphing calculator. Students may also compare the shaded areas, drawn by hand, with the shaded areas, given by the graphing calculator. This cross-reference may be used to solidify their conceptual understanding of the topic.

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33. HYPOTHESIS TESTING

Hypothesis testing is the process of testing the truth value of a null hypothesis. When undergoing hypothesis testing, it may be assumed that the mean is not equal to the null hypothesis, less than the null hypothesis, or greater than the null hypothesis. The researcher will determine what is to be tested. Generally, the test involves mu (μ), not equal to the null hypothesis.

Statistics is broken up into the realms of descriptive statistics and inferential statistics. Hypothesis testing is found under the umbrella of inferential statistics. Such testing involves testing the mean between two groups, comparing a sample mean to a population mean, calculating a confidence interval, comparing the difference between proportions, comparing a sample proportion to a population proportion, and comparing frequencies of categories to some expected values. Specific tests include, but are not limited to, t-test, one-way ANOVA (analysis of variance), two-sample t-test, one-proportion z-test, z-interval, and chi-square test. The aforementioned tests are on the lower level of analysis complexity and will be the ones briefly examined in this article.

Each section will explain the meaning of the test, provide an example, show how to perform each analysis, and how to interpret the results. The z-table, t-table, and chi-square table may be used to undergo such testing. However, this article will focus on graphing calculator or Excel usage.

T-TEST

A t-test compares the mean of a sample to a population mean. With such a test, the sample mean and sample standard deviation are used and compared to a population mean.

Example

A professor records the overall average on his statistics final exam as 87. A random sample of 30 students shows a final exam average of 89, with a standard deviation of 3 points. Decide if a statistically significant difference exists between the population average and the random sample average.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 145–150.

On the graphing calculator, enter the following:

$\mu_0 = 87$
$\bar{x} = 89$
$s_x = 3$
$n = 30$
$\mu \neq \mu_0$

Click Calculate.
The results are:

$t = 3.65$
$p = 0.001$

The p-value is less than the alpha level of 0.05, thus the null hypothesis may be rejected and a statistically significant difference declared. Note. The researcher sets the alpha level (or level of confidence), but 0.05 is a standard level often used. (By lowering the alpha level, the researcher can reduce the risk of Type I error, or rejecting the null hypothesis, when it should *not* be rejected.)

Summary: There is a statistically significant difference between the overall recorded final exam average and the final exam average for the sample.

ONE-WAY ANOVA

A one-way ANOVA compares the interval outcomes of two or more nominal groups, for one variable. Thus, two or more sample means are compared. This example shows two means being compared. However, an ANOVA allows for the means of more than two groups to be compared.

Example

The exam scores of students in two different groups are compared. Group A received supplemental instruction via a blended learning format. Group B did not receive such supplemental instruction.

Group A scores:
97, 60, 88, 97, 98, 78, 92, 66, 74, 72, 61, 68, 91, 61, 99,
79, 65, 98, 75, 92, 79, 94, 97, 88, 72, 63, 97, 61, 62, 95

Group B scores:
72, 87, 94, 75, 67, 61, 70, 61, 98, 68, 99, 76, 67, 95, 60,
97, 81, 92, 73, 61, 75, 80, 64, 79, 69, 83, 73, 84, 71, 73

Determine if a statistically significant difference exists between the groups.
In Excel, do the following:

1. Enter Group A scores in Column 1.
2. Enter Group B scores in Column 2.
3. Select all data in both columns for the Input range.
4. Choose Group by Columns.
5. Choose desired alpha level; default is 0.05.
6. Click OK.

The p-value is 0.269. Since the p-value is greater than the level of confidence, or 0.05, the null hypothesis will not be rejected, thus showing there is not a statistically significant difference between the groups.

Summary: There is not a statistically significant difference in the scores of students who received the supplemental instruction and those who did not receive the supplemental instruction.

TWO-SAMPLE T-TEST

A two-sample t-test is similar to a one-way ANOVA, with the exception that the means of only two groups may be compared.

Example

A random sample of 30 cereal boxes has a mean of 15.8 ounces, with a standard deviation of 0.2 ounces. A random sample of 35 cereal boxes has a mean of 15.9 ounces, with a standard deviation of 0.1 ounces. Determine if there is a statistically significant difference between the two samples.

On the graphing calculator, choose Stats and enter the following:

$\bar{x}_1 = 15.8$

$s_{x1} = 0.2$

$n_1 = 30$

$\bar{x}_2 = 15.9$

$s_{x2} = 0.1$

$n = 35$

$\mu_1 \neq \mu_2$

Choose Pooled Variance. Click Calculate.

The results are:

$t = -2.60$

$p = 0.012$

The p-value is less than the alpha level of 0.05, thus the null hypothesis may be rejected and a statistically significant difference declared. (The absolute value of the t-value is also greater than the critical t-value.)

Summary: There is a statistically significant difference between the average amounts of cereal contained in the boxes, in the two samples.

ONE-PROPORTION Z-TEST

A one-proportion z-test compares a sample proportion to a population proportion.

Example

On average, 75 percent of the students pass the geometry entrance exam, on the first try. This year, 60 out of 85 students pass the exam, on the first try. Decide if there is a statistically significant difference between the sample proportion and population proportion.

On the graphing calculator, enter the following:

$p_0 = 0.75$

$x = 60$

$n = 85$

$prop \neq p_0$

Click Calculate.

The results are:

$z = -0.939$
$p = 0.348$

The p-value is greater than the level of confidence of 0.05. Thus, the null hypothesis should not be rejected.

Summary: There is not a statistically significant difference between the average proportion and the proportion of students passing on the first try, for the current year.

Z-INTERVAL

A z-interval is a confidence interval that is computed, using a sample mean and population standard deviation. The true mean (or population mean) is likely contained within the interval. A z-interval may be used to test a claim. (The margin of error may be manually calculated and used to manually create the confidence interval.)

Example

A bottle manufacturing company claims to include 11.8 ounces of beverage in each container, with a standard deviation of 0.6 ounces. A random sample of 100 bottles shows a mean of 11.7 ounces. Using a 95% confidence level, construct a confidence interval and interpret the results.

On the graphing calculator, enter the following:

$\sigma = 0.6$

$\bar{x} = 11.7$

$n = 100$

$C - level = 0.95$

Click Calculate.

The confidence interval is (11.582, 11.818). Notice the claim of 11.8 ounces is included in the confidence interval.

Summary: Since the population mean lies within the confidence interval, it may be determined that the company's claim is true.

CHI-SQUARE TEST

A chi-square test compares frequencies of observed outcomes to expected outcomes. The test is used to determine if a statistically significant difference exists between observed and expected values.

Example

The average numbers of Freshmen, Sophomores, Juniors, and Seniors participating in Theatre are 45, 62, 50, and 68. This past year, the numbers were 40, 68, 52, and 74, respectively. Determine if there is a significant difference between the average numbers of grade-level students participating and the numbers for the current year.

In Excel, do the following:

1. Enter the observed values in Column A.
2. Enter the expected values in Column B.
3. Click an empty cell and type "=CHITEST(A1:A4,B1:B4)".
4. Interpret the resulting p-value.

The p-value is approximately 0.63, which is greater than 0.05. Thus, there is not a significant difference between the observed and expected frequencies.

A manual calculation of the chi-square statistic is shown below:

$$\chi^2 = \Sigma \frac{(O-E)^2}{E} = \frac{(40-45)^2}{45} + \frac{(68-62)^2}{62} + \frac{(52-50)^2}{50} + \frac{(74-68)^2}{68}$$

$$= \frac{25}{45} + \frac{36}{62} + \frac{4}{50} + \frac{36}{68} \approx 1.75$$

For degrees of freedom of 3, a chi-square statistic of 1.75 has a probability greater than the alpha level of 0.05. Thus again, we have shown that there is not a significant difference between the frequencies.

Summary: The participation frequencies for this past year are not statistically significantly different from the average frequencies recorded.

Each of these tests is an example of hypothesis testing, an important element of inferential statistics. It is the process for rejecting or failing to reject a null hypothesis. Once students understand the foundations of these tests and applications thereof, they will be ready to use more complex analyses, such as regressions, HLM, SEM, and MANOVA.

AMANDA ROSS

34. A BRIEF LOOK AT EXPECTED VALUE

The concept of expected value is a mystery to many students (and many teachers, as well). Most students understand that it is an average, of some sort. Specifically stated, for one trial, the expected value is the *average of possible outcomes*.

Stated even more explicitly, the expected value is the *weighted average of a random variable*. In other words, the expected value equals the sum of the products of each value assigned to a random variable and its corresponding probability. The corresponding probability may be theoretical, or have an assigned empirical probability. Expected value may be represented as a weighted average, using the formula, $E(X) = \sum x_i \cdot P(x_i)$, where x_i represents all possible values for a random variable and $P(x_i)$ represents the corresponding probability, be it theoretical or assigned. This formula may be expanded as $E(X) = (x_1 \cdot P(x_1)) + (x_2 \cdot P(x_2)) + (x_3 \cdot P(x_3)) + \ldots + (x_i \cdot P(x_i))$.

For x trials, the expected value is the *average of the averages* of the trials. A sampling distribution of means may be plotted. This distribution represents the means of n trials. The mean of these means is the expected value. As the number of trials approaches infinity, the mean of the sampling distribution, or expected value, approximates the population mean. The calculation of each probability, for a probability situation with more than one trial, requires the binomial probability formula, or $P(X) = {}_nC_x \cdot p^x \cdot q^{n-x}$, where n represents the number of trials, x represents the number chosen at a time, p represents the probability of success, and q (or $1 - p$) represents the probability of failure. As stated above, the expected value then equals the sum of the products of each probability, $P(X)$, and the number, x.

The general formula for finding the expected value of an event, for n trials, is $E(X) = n \cdot p$, where n equals the number of trials and p equals the probability of the event, in question. In this case, the expected value represents the number of times you would expect to get a certain outcome, after conducting more than one trial. (For example, the probability of rolling any number on a die is $\frac{1}{6}$, so you might expect $150 \cdot \frac{1}{6}$ or 25 2's, when rolling a die 150 times.)

It is important that students make the connection between the Law of Large Numbers, theoretical probability, sampling distributions, and expected value. The *Law of Large Numbers* states that as the number of trials increases to infinity, the experimental probability will approximate the theoretical probability. This law also

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 151–154.

states that the mean of a sampling distribution approximates the population mean, as the number of samples increases to infinity.

Now that you have an understanding of expected value, let's look at some sample problems. (The solution is presented below each problem.)

FOR ONE TRIAL

1. What is the expected value of a coin toss?
   H = 1
   T = 0

$$E(X) = \frac{1+0}{2} = \frac{1}{2}$$

2. What is the expected value of a die roll?

$$E(X) = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$$

3. What is the expected value of spinning a spinner, with eight equally-spaced sections?

$$E(X) = \frac{1+2+3+4+5+6+7+8}{8} = \frac{36}{8} = 4.5$$

4. Tyler will win \$50 if he pulls a vowel from a hat. He will lose \$10 if he pulls a consonant from a hat. What is the expected value of his winnings?

$$E(X) = \left(50 \cdot \frac{5}{26}\right) - \left(10 \cdot \frac{21}{26}\right) = \frac{250}{26} - \frac{210}{26} = \frac{40}{26} \approx 1.54$$

The expected value of his winnings is \$1.54.

5. Adrian has won a raffle entry. He may now claim some prize by choosing one of four boxes. Each box contains some prize dollar amount. If he chooses Box 1, he will win \$5. If he chooses Box 2, he will win \$10. If he chooses Box 3, he will win \$20. If he chooses Box 4, he will win \$50. What is the expected value of his prize?

$$E(X) = \left(5 \cdot \frac{1}{4}\right) + \left(10 \cdot \frac{1}{4}\right) + \left(20 \cdot \frac{1}{4}\right) + \left(50 \cdot \frac{1}{4}\right) = \frac{85}{4} = 21.25$$

The expected value of his prize is \$21.25.

6. Hannah is asked to spin a spinner, with eight equally-spaced sections. She will win \$20, if she rolls a 2 or 4. She will lose \$2, if she rolls an 8. She will win \$4, if she rolls any other number. What is the expected value of her winnings?

$$E(X)=\left(20\cdot\frac{2}{8}\right)-\left(2\cdot\frac{1}{8}\right)+\left(4\cdot\frac{5}{8}\right)=7.25$$

The expected value of her winnings is \$7.25.

7. The table below represents the probabilities and nightly costs, associated with four different hotels. What is the expected value of a night's stay?

	Hotel 1	*Hotel 2*	*Hotel 3*	*Hotel 4*
P(X)	0.25	0.20	0.15	0.40
Nightly cost	\$85	\$140	\$165	\$120

$$E(X)=(85\cdot 0.25)+(140\cdot 0.20)+(165\cdot 0.15)+(120\cdot 0.40)=122$$

The expected value of one night's stay is \$122.

FOR X TRIALS

1. A coin is tossed 200 times. How many times can you expect it to come up heads?

$$E(X)=200\cdot\frac{1}{2}=100$$

2. A die is rolled 1,200 times. How many times can you expect to roll an even number?

$$E(X)=1,200\cdot\frac{1}{2}=600$$

3. A bag contains 4 red marbles, 3 blue marbles, and 9 green marbles. Marcus pulls 2 marbles from the bag. What is the expected value for drawing a red marble?

$$P(0)=\binom{2}{0}\left(\frac{4}{16}\right)^{0}\left(\frac{12}{16}\right)^{2}=0.5625$$

$$P(1)=\binom{2}{1}\left(\frac{4}{16}\right)^{1}\left(\frac{12}{16}\right)^{1}=0.375$$

$$P(2)=\binom{2}{2}\left(\frac{4}{16}\right)^{2}\left(\frac{12}{16}\right)^{0}=0.0625$$

$$E(X)=(0\cdot 0.5625)+(1\cdot 0.375)+(2\cdot 0.0625)=0.5$$

4. Carlos must draw 3 cards, from a standard deck of cards. What is the expected value for drawing an ace card?

$$P(0)=\binom{3}{0}\left(\frac{4}{52}\right)^{0}\left(\frac{48}{52}\right)^{3}=0.7865$$

$$P(1)=\binom{3}{1}\left(\frac{4}{52}\right)^{1}\left(\frac{48}{52}\right)^{2}=0.1966$$

$$P(2)=\binom{3}{2}\left(\frac{4}{52}\right)^{2}\left(\frac{48}{52}\right)^{1}=0.0164$$

$$P(3)=\binom{3}{3}\left(\frac{4}{52}\right)^{3}\left(\frac{48}{52}\right)^{0}=0.000455$$

$$E(X)=(0\cdot 0.0.7865)+(1\cdot 0.1966)+(2\cdot 0.0164)+(3\cdot 0.000455)\approx 0.2308$$

This article has briefly touched upon the concept of expected value, as taught at the high school level. Review the ideas and formulas to make sure you have a conceptual understanding of expected value and how to approach different probability problems. Brainstorm additional real-world problems that may be given to your students. Encourage your students to ask questions and make connections between the different expected value formulas. Also, make sure that they have an intuitive sense of expected value and the Law of Large Numbers.

PART II

ASSESSMENT

SECTION 5

CLASSROOM FORMATIVE AND SUMMATIVE ASSESSMENTS

G. DONALD ALLEN

35. THE EVIL TWINS – TESTING AND STRESS

We make here a few comments on the nature and necessity of big-time testing and the concomitant stress upon students. Our point is that both are needed, both are necessary.

An easy case can be made that the current testing method, particularly high-stakes testing, is flawed. Problems include: (a) not accounting for the current mental state of the student, (b) including test questions that are inaccurately stated or unclear, (c) measuring only a small fraction of the material studied, and (d) placing undue stress upon the students. All true, true, true.

The alternatives to each of these objections create their own problems. In the following, let us exclude the issues of inaccuracy and clarity. Each of them has the same issue which denies their suggested solution. Each solution is prohibitively expensive. Each involves a lot of higher level intervention in the examination of student work. Each involves a larger examination to more fairly cover expected outcomes.

Specifically, every test creates stress. Think back to the time when you passed your driver's licensing road test. Whew! This is what you said when it was over and you passed. This is the case with tests of all types, and taken throughout all of life. If anything, critics should celebrate the simple fact that life is a series of tests, for good or bad. Even the military officer wishing to advance in the ranks must past performance tests. With no prior experience, the stress is even higher than it might be. Indeed, you may be challenged to find any profession where some sort of advanced (and stressful) test is NOT involved.

Does testing distort the learning process? This question was recently posted in the message box on our online journal website (http://disted6.math.tamu.edu/newsletter/index.htm.) Of course it does. So what? Without the testing, the teachers would distort the learning process by simply emphasizing the curricula they deem more necessary. Without these tests, students in class A who learned from a distorted curricula would be disadvantaged when they get to course B - taught by another teacher. Without the testing, the new Common Core Curriculum would be a farce. With no summary testing, there could be no assurance the curricula is being taught as prescribed.

As to the "current mental state" of the student on test day, this is an unfortunate artifact of any and all testing systems. The system in the USA allows students, year by year, to recompense their testing scores, hopefully minimizing errant results based on aberrant mental states.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 159–160.

I'm a teacher, you're a teacher. If I don't give exams, I cannot know how my students are doing, if they are studying, or if they understand what they have studied. I cannot expend hundreds of hours assessing students' abilities with one-on-one interviews. I must give these delimited, unfair, and stressful tests. Do you?

> *If all teachers were doing their jobs as defined and desired, and if all student were doing as well, there would be no need for the design and determination of performance measures.*

G. DONALD ALLEN

36. UNFAIRNESS IN TESTING – MATHEMATICS AND SIMULATIONS

We constantly hear that testing is unfair. It is. For multiple choice tests, the list of whys is endless, but there are other unfair practices that widen the gap between good and poor[1] students. In some way, the practice of testing of almost every student type is unfair by its very nature. This applies from taking your driver's test, to passing the medical boards, to testing for a G12 position in government. Simply recall when you took your driver's road test. Some of us were terrified and our terror may have resulted in mistakes; others by their nature were more relaxed, clearly focused and therefore performed better. It also and most particularly applies to the K-20 schools – our present subject.

To frame our intent at examining more interesting and novel unfairness issues, we review some of the more standard issues of unfairness and limitation of multiple-choice tests. There are many including misinterpretation of the questions, the evaluation of knowledge beyond the range of options provided, difficulty in phrasing for identical interpretation by all students, the encouragement of guessing, anxiety, and errant test-taking strategies.

Now, let's look at just a couple of the testing unfairness issues not generally considered in the vast literature on testing. First of all, the good student can better discriminate difficult questions from easier ones. After all, for the poor students, more or all the questions seem difficult. This makes the test taking strategy of good students better vis-a-vis allocation of time for a problem. This is rather important, for the misallocation of time spent on a given problem can profoundly affect performance on other problems.

The good student is better at guessing. This implies dually that incorrect choices for good students are more revealing about both the problem difficulty and traps than for poor students. When none of the answers given offer any clue whatever, the good student is "tipped" that this problem may require some thinking, while the poor student experiences the no-clue situation frequently. Good students usually develop a "sense" of incorrectness about some answers. For example, if the question asked the students to find the perimeter of a semicircle of radius 10, (answer is about 51.4) and if one of the answers is 60, the good student realizes the answer cannot be this large and discounts it immediately. Such a (quick) decision by the poor student is more problematic. The implication here implies that teaser distractors are biased against poor students. Tests with teasers, i.e. traps, therefore discriminate. Tests looking for misconceptions are therefore deeply biased against struggling students.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 161–164.

Such tests should absolutely not be used for normative testing. The basic question here is whether the examiner wants to determine if a student can (eventually) solve a problem or can beguile him/her into making a mistake.

How should this affect scoring correction? The good student is better at eliminating incorrect distractors. This could imply a greater scoring correction should be used for the better students. Such a practice would be highly controversial, far more than already controversial and conventional scoring correction methods.

Is there a possibility of a fair test? The fairest form of test I've ever seen is an oral exam where the examiners try hard to balance the pressure of taking an exam in the first place with a sympathetic outlook to genuinely understanding what the student knows, to having clear criteria on what is passing. The format is to begin with a simple question most everyone should know and then progressively complicate it until there is no answer or a wrong answer. Then move to another topic. It takes time and the resources of two or three examiners, but it is mostly fair in assessing the examinee's knowledge of the subject. Yet, aside from technical subjects there is an intrinsic unfairness here as well, though the direction of the bias is not entirely clear. However, the bias is seemingly toward students with quick recall, the ability to speak clearly, and the ability to carefully discern what the examiners are looking for. Recall, clever Hans.[2] As well, such procedures cannot be rapidly instituted as in a regulation. It takes possibly years of experience for the average student to become facile in this format. Shy or easily intimidated students may never master it. Moreover, the costs of mass implementation of such testing would be astronomical.

For multiple choice math tests, here are some steps to follow. (1) No distractor should betray itself as a candidate for elimination. This could greatly increase the test creation time for non-objective answers. Poor students and good students solving it incorrectly or having a misconception about how to solve will find no teaser distractor to select. (2) It would be good to indicate the level of difficulty (easy, medium, hard) of each problem. This slightly helps the poor student over the good students. Yet, it also gives advantage to good students, giving ready knowledge for an optimal testing strategy. But no research has been done on this.

Facts about poor students. They do not study their exams to learn about mistakes. Often they stuff the returned exam into a folder never to be looked at again. They strongly believe good students are luckier than they. On the other hand when poor students actually solve a problem correctly, they do not credit their ability. They credit luck. This informs their study practice.

MATH EXAMPLES

1. You have a rope that stretches about a circular building with radius 100 feet. But you need to extend the length to include a sidewalk around the building which is six feet wide. How much extra rope do you need?

Set A	Set B	Set C
a. About 100 feet	a. About 5 feet	a. About 3.14 feet
b. About 200 feet	b. About 6 feet	b. About 6.05 feet
c. About 6 feet*	c. About 7 feet	c. About 12.10 feet
d. About 12 feet	d. About 8 feet	d. About 15.3 feet
e. About 24 feet.	e. About 9 feet	e. About 18.0 feet

**Correct.*

Analysis. This question is easy if only the student will use the formula for the circumference of the circle. However, in Set A, the obvious traps are distractors a and b. The hurried student may select one of them. In Set B, there are no traps but all answers, and the correct one can be easily eliminated. In set C, there appears to be no give-away.

2. Find all values of x for which $x > |-3|$ and $x <|5|$.

Analysis. There are three possibilities. You want to see if the student knows what absolute value means and then apply it, you want students to get the correct answer, or you want to set up poor students to blunder. Poor students can often do the problem correctly, but if they are way laid by a trap, this may not be evident.

Essay questions are often singled out as an honest and fair method to test students' knowledge. For simple college entry exams mostly about the student himself, these are relatively easy to address and understand. For course specific questions the answer may not follow what the instructor requested.

Example. The executive checks the power of the American legislature. The constitution grants the president much power and its importance must be documented. Explain how the president can check the Legislature.

Analysis. There are several aspects of this question. The ability of the student to read and precisely understand what the question is asking. A. The ability to assemble information into a format to provide an answer to the question. B. The ability to construct the answer in complete paragraphs. C. The ability to understand what the instructor may be looking for. So, what is this question testing, and to what degree? The answer may be all of them, but this works in a bias to poor and particularly anxious students. It clearly biases against students with reading and/or writing under pressure issues, and it biases against students that haven't understood the nuances of the instructor's position on the subject. Most significantly, it biases against the unprepared student. This last point is the reason for the question. But the obstacles in getting a coherent answer from a student, based on knowledge of the subject, are clearly present.

NOTES

1 Definition. Poor students are those that perform poorly on normal tests. It has nothing to do with economic circumstance. While much of educational leadership denies there are such students, they

exist, and have forever. Some students are better at math than others, or at any other subject for that matter.

[2] Clever Hans was a horse in the early 20th century that was claimed to have the ability solve various math problems. In fact, Hans was watching the human observers and was responding directly to involuntary cues in the body language of the human trainer, who knew the solution to each problem.

REFERENCES

Aguinis, H., Culpepper, S. A., & Pierce, C. A. (2010). Revival of test bias research in pre-employment testing. *Journal of Applied Psychology, 95,* 648–680.

Arnold, B. R., & Matus. Y. E. (2000). Test translation and cultural equivalence methodologies for use with diverse populations. In I. Cuellar & F. A. Paniagua (Eds.), *Hand-book of multicultural menial health: Assessment and treatment of diverse populations* (pp. 121–136). San Diego, CA: Academic Press.

Arvey, R. D., & Faley, R. H. (1988). *Fairness in selecting employees* (2nd ed.). New York, NY: Addison-Wesley.

Berk. R. A. (Ed.). (1982). *Handbook of methods for detecting test bias.* Baltimore, MA: Johns Hopkins University Press.

Camilli. G., & Shepard. L. A. (1994). *Methods for identifying biased test items.* Thousand Oaks, CA: Sage.

Shultz K. S., Whitney D. J., & Zickar M. J. (2014). *Measurement theory in action: Case studies and exercises* (2nd ed.). New York, NY: Routledge.

AMANDA ROSS

37. ONLINE ASSESSMENTS IN MATH

Online assessments are quite prevalent in the mathematics classroom. Online assessments can take many forms, including, but not limited to teacher-created online assessments via course management systems and quiz/test creation websites, classroom response systems, online math practice, games/applets, online practice tests, online portfolios, online surveys, and so on. Students can take online assessments on a laptop, tablet, or smart phone. These assessments may be formative or summative, in nature.

What are the benefits of online assessments? How do they help teachers? How do they help students? Online assessments may be designed to offer instant feedback to the teacher and/or student. The feedback may show whether a choice is correct or incorrect, or it may show both the correct answer and the chosen answer. Some online assessments go a step further and show incorrect and correct feedback, meaning that they show the student why a correct answer is correct and provide guiding instruction to help students select the correct response, based upon each plausible distractor. This type of feedback is especially helpful with formative assessments, such as checks and quizzes. Students are offered scaffolded learning through such feedback/prompts, while teachers are provided with detailed assessment information that may be used to differentiate instruction and change planned classroom instruction, accordingly.

In this article, we will briefly examine some important categories, related to the discussion of online assessments. These categories are outlined below. We will look at course management systems, quiz/test creation websites, classroom response systems, other types of online assessments, types of items available, and advantages and disadvantages of online assessments.

Course Management Systems

There are many course management systems/platforms out there, including several that are free to use. Moodle is one of the free course management systems. With Moodle, a teacher can create a course and design assessments, just as college professors do. With a course management system, a student goes online to the class Moodle page, logs in, and then takes any assessments, reads/contributes to discussion threads, views and uploads files, and much more. The assessment options in Moodle are quite vast, including opportunities for self-assessment and assessment of peers.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 165–168.

Quiz/Test Creation Websites

Online assessments may also be accessed via quiz/test creation websites, instead of directly through a course management system or platform. QuizStar is one such quiz creation tool that is easy and free to use. A teacher can create a class, create quizzes, assign quizzes to a class, and then view the assessment results under the report manager tab. Assessment results may be viewed on an individual basis, per question, or across the class. The teacher can set the maximum number of attempts, time allotted for the quiz, and type of feedback given.

Classroom Response Systems

Classroom response systems are polling systems that provide instant, real-time results to the teacher and students. In the past, the student had to purchase a clicker that was used as the device for answering questions. Now, the student has the option of using a different device, such as a smart phone, for responding to polling questions. Online polling systems, such as I-clicker and Poll Everywhere, require an internet connection and allow the student to respond to questions, using a device of his or her choosing. Such systems provide the teacher with real-time data, regarding the comfort level of the students, with different mathematics topics. The teacher may then modify the intended instruction, based upon the results. Teachers can also look at individual results and thus differentiate instruction, as needed.

Online Math Practice, Games, and Applets

Interactive mathematics games and applets provide instant feedback to the student, regarding whether or not there is a clear understanding of the topic at hand. These tools are not as pertinent to the teacher, in terms of grading. However, the teacher can allow students a specific amount of time during the day to explore games and applets, as both a learning opportunity and extension. The teacher can then use the games/applets as a means of formative assessment to quickly determine if, and to what degree, the students seem to grasp the material. IXL Math is a free website that provides online math practice, for students in grades Prekindergarten through high school. The NCTM Illuminations website is also free and offers mathematics applets and explorations. The National Library of Virtual Manipulatives (NLVM) is another such resource. PBS offers mathematics games. These resources are forms of assessment. They serve to provide a means of self-assessment for the student, rather than as formal assessment to be used by the teacher. Self-assessment is an integral part of the learning process.

Online Practice Tests

Full-length online practice tests are invaluable resources, necessary for mathematics success. For example, when students register to take the MCAT exam or GRE, they most likely seek some sort of practice tests, in order to self-assess how well they currently understand the material. In middle and high school mathematics, there are free full-length practice tests available online. In fact, practice tests, aligned to the Common Core Standards can be found online. Again, readily available, online practice tests are intended for self-assessment by the student, rather than as a means of formative or summative assessment by the teacher. When teachers wish to formatively and/or summatively assess their students, they can create their own quizzes/tests, using a quiz creator tool or course management system.

Online Portfolios

Online portfolio assessments, or e-portfolios, are often used to get a more in-depth picture of the level of understanding realized by a student. E-portfolios also provide a longitudinal look at how a student's level of understanding progresses/evolves, over a period of time. Such assessments provide excellent qualitative data, pertaining to how students problem solve and how well they can describe and prove mathematical relationships. The teacher may also wish to have students create power point presentations to cover class material. The student can then use this power point to teach the class. The power point would serve as a great piece/artifact for the student's e-portfolio. Moodle has a portfolio module. E-portfolio hosting sites for institutions (schools and colleges) are also available; some are free, while others charge. Some grants have been used to create free e-portfolio hosting.

Online Surveys

Online surveys may also be used as a self-assessment tool. The teacher can create a survey and include questions that relate to students' comfort level and understanding of mathematics content. Free online tools, such as SurveyMonkey, may be used by the teacher and student. The survey would serve as a self-assessment checklist for the student and as a type of formative assessment for the teacher.

Types of Online Assessment Items

The platforms and resources mentioned above allow for different kinds of assessment items. These items include multiple choice (MC), multiple correct response (MCR), short response (SR), extended response (ER), and technology-enhanced items, such as drag-and-drop, classification, matching, graph plotting, etc. With quiz/test creator tools, the item types may be limited to MC and MCA. Other online assessment programs use their own html editor to create technology-enhanced items. Students certainly benefit from assessment via a variety of online assessment types.

Advantages of Online Assessments

There are many advantages of online assessments, including but not limited to, positively changed instruction and learning, instant and detailed feedback, for both the student and teacher, availability of online assessment records and progress, and compatibility with the flipped classroom model. Classroom response systems promote engagement in the classroom, by allowing students to respond to questions at any time, during the lesson. Students must pay attention to the content because questions may be asked at impromptu times throughout the given instruction. Teachers can instantaneously see how well students comprehend the material, thus providing real-time data that can be used for instruction modifications.

Disadvantages of Online Assessments

There are some disadvantages of using online assessments, such as time limit constraints and the inevitability of technology problems. Students with visual impairments may also be at a disadvantage, when taking such assessments, unless proper audio is provided. Some online assessments may not allow for more open-ended responses, such as essay responses.

Summary

Online assessments are a mainstay, in the mathematics classroom. There are many types, from which to choose, and the teacher has a lot of resources of which are available, free of charge. Providing students with opportunities to interact with such assessments and become tech-savvy, in middle and high school, will undeniably give them an edge, when starting a technical program, or college.

G. DONALD ALLEN AND SANDRA NITE

38. WHEN IT COMES TO MATH, WHAT EXACTLY ARE WE TESTING FOR?

So many platitudes; so many directives; so many hopes; so much preparation; so much criticism; so much condemnation; so much money. All this applies to modern high-stakes testing, and particularly because the US performs poorly in international competitions, and not that well here at home. While the main point of this note is the title, we take a bit of time to fill in the scope of high-stakes testing, the single biggest driver in schools today. There are several stakeholders: governments, districts, schools, principals, teachers, and students. Low or high performance can make highly significant outcomes for any and all.

There is a hierarchy of testing – scope, domain, and specifics. The scope of assessment in mathematics is general, easy to understand, and simple to state. Although there are many, many types of assessments, the two foremost are the *formative* and *summative* varieties. In fact, in all subjects, these are the super-categories of the assessment.

SCOPE FOR TESTING

Basically,

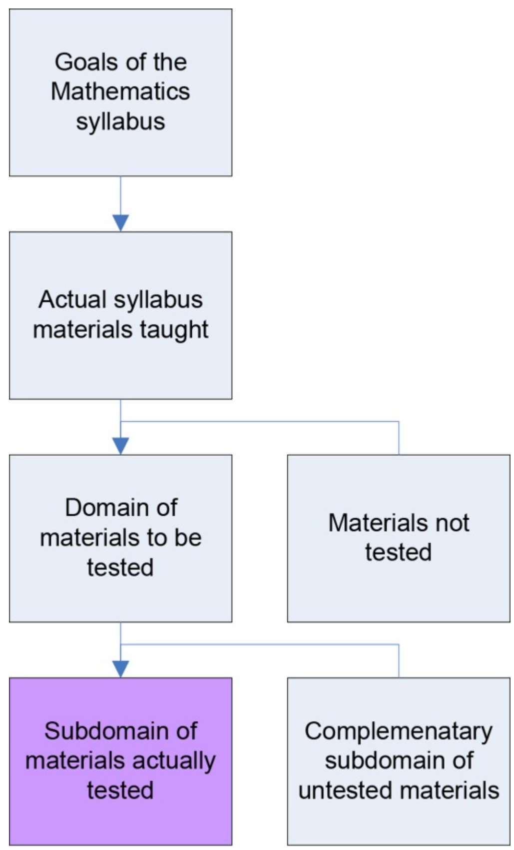
- Formative assessment is utilized to immediately determine whether students have learned what the instructor intended. It is also intended to be a learning instrument for the student. Theoretically, this type of assessment is intended to enlighten teachers on which materials need further instruction. Theoretically, this type of assessment should not be used to evaluate and grade students. Formative assessment is to inform, not evaluate.
- Summative assessment is cumulative in nature and is utilized to determine whether students have met intended goals or have achieved student learning outcomes by the end of a course.

DOMAIN OF TESTING

This sounds fine, until we come to precisely what is tested. Following the work of Campbell, Baker, and others, we must consider the *incomplete measurement of desired outcomes*. In short, this means by the necessities of time and expense, and other logistics, the full domain of the taught syllabus cannot be tested. This implies

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 169–175.

all performance measures are incomplete. And this means that distortions may arise in the conclusions of such assessments over what would be an ideal measure. Educators and students then respond to the exigencies of the situation by focusing on that which will be or assumed to be measured at the expense of the true goals of the system. Of course, this is well understood by practitioners in the field of measurement, and has been so for decades. Nevertheless, it is useful to consider a simple diagram of how this plays out.



It is only the left bottom box that gets the attention, the contents of which are guessed and double guessed by curriculum experts and teachers. The materials in this box, far smaller than the actual full syllabus, is the focus of test preparation directly preceding to the test and indirectly throughout the school year. Usually, it consists of those topics easier to understand and certainly easier to formulate as a general test question. In short, the math course full evaluation is placed on a small subset of available topics. However, there is some variation from year-to-year between what is actually tested, though not a lot. A careful look at these test shows that a three year window will usually suffice to get the full scope of problem types a student might see in any given year. This is quite a small amount of materials.

The consequences of all this can be expressed in a few questions:

1. Is the quality of these test materials up to the standards of what should be measured and even restricted to the final amount?
2. Can the use of this small amount of materials be validated as representative of all materials?
3. Can it be established or validated that given the restricted amount of materials that will be tested, this does not impact the instruction of the full syllabus?

We give only brief answers. For question one, we note in most cases, the high-stakes tests are constructed by private corporations that successfully bid for the contract. State education agencies rely on their expertise – and often do not have the expertise to find or remove any but the most egregious errors. Even with multi-million dollar contracts in hand, the contractors have committed massive reported errors on the tests. Most notably, the New York State exams have had numerous problems over the years. For math, the NY Daily News[1] recently reported

> *On the eighth grade test, one question had no correct answer, and schools are instructed to alert students.*
>
> *And on the fourth grade exam, one question has two correct answers. But in this case, schools are directed to tell students about the problem only if they ask questions about the item.*

Often States do not release exams for some years, perhaps knowing they will be examined very carefully.

For the second question, the quality and validation are evaluated in terms of the initial representativeness of the tested sample, using data obtained before any operational administration of the test. This means educators focus on a somewhat smaller and unrepresentative sample than is actually tested. In practice, this means a much greater risk of narrowed and otherwise degraded instruction, inappropriate test preparation, and inflated scores.[2]

For the third question, partly because schools are now evaluated on the basis of how their students perform, it is now widely established that many schools insist on teachers expending considerable effort to teach a select syllabus in the four to six weeks leading up to the test. This obviously impacts the teaching and the curriculum overall. The pejorative, "Teaching to the Test," is now dismissed outright. One group, those advocating the new Core Curriculum State Standards (CCSS) argue the new curriculum will obviate this need. Nonsense. High-states testing is just that, and educators will try to enhance the performance of their students in any way they think appropriate.

Not to appear as an apologist for "teaching-to-the-test" but this is exactly the prime study method used by all students (in this case "study-to-the-test") for their high-stakes exams. Just a few examples–the GRE, LCAT, MCAT, Bar Exams, Medical Exams, qualifying exams of all types, and public service exams. Moreover, students prepare to take major national and international math exams, such as the Putnam Exam in the USA and the Math Olympiad, worldwide, by studying past exams, in specialized courses. The emphasis is a little different with much more independent work required by the students. Students learn to do by doing. Now, teaching-to-the-test this way has profound formative results, with spectacular summative performances.

Under the low-stakes radar of testing, most of these did not matter. The goal was only to gain a possibly incomplete measure of how students were doing. Under high-stakes testing, testing matters to all stakeholders. Under high-stakes testing, which means high-stakes – are involved, there is a certain possibility of corruption. We cite Campbell's law,[3] "The more any quantitative social indicator is used for social decision making, the more subject it will be to corruption pressures and the more apt it will be to distort and corrupt the social processes it is intended to monitor." Corruption is rampant in the US at all levels for high-states exams, likely the least by the students themselves.

TESTING SPECIFICS

In particular, what are we testing for? More than may be apparent.

There are three categories of questions typically asked on math exams, knowledge, abilities, and skills. Here is a partial list.

1. General mathematical knowledge
2. Breadth of knowledge
3. Depth of knowledge
4. Ability to solve problems
5. Ability to apply mathematics
6. Ability to make an inference
7. Ability to parse language into mathematics
8. Ability to parse mathematics into language
9. Ability to communicate mathematics
10. Ability to approximate
11. Ability to generalize
12. Ability to abstract
13. Ability to solve novel problems, to innovate
14. Misconceptions
15. Skills at computation
16. Skills at manipulation

According to the National Research Council, Mathematical proficiency[4] consists of five points, arranged from the most basic to the most complex.

Conceptual understanding
Procedural fluency
Strategic competence
Adaptive reasoning
Productive disposition

There is some overlap between these categories, but they also stand as independent testing aspects. In terms of our list of testing goals, we give the following chart.

Math proficiency	Testing Categories
Conceptual understanding	1,2,7,8,9
Procedural fluency	10,15.16
Strategic competence	3,4,6
Adaptive reasoning	5,6
Productive disposition	11,12,13

For various high-stakes tests, we illustrate the types of assessment questions that are assessed on these different exams.

1. General mathematical knowledge, ACT/SAT/GRE
2. Breadth of knowledge, none
3. Depth of knowledge, Course exams
4. Problem solving ability, Math Olympiad, etc
5. Misconceptions, EOC exams
6. Ability to make an inference
7. Ability to parse language into mathematics, EOC exams, most exams
8. Ability to parse mathematics into language
9. Ability to communicate mathematics, show your work type problems
10. Ability to approximate, EOC exams, most exams
11. Ability to generalize, Course exams
12. Ability to abstract advanced course exams
13. Skills at computation, most exams
14. Skills at manipulation, most exams
15. Applications, Novelty, Innovation

In a small survey of middle and high school teachers, we see how they view their testing practices.

Which of the following do you test for in your mathematics classroom?		
Answer Options	**Response Percent**	**Response Count**
General mathematical knowledge	100.0%	14
Breadth of knowledge	42.9%	6
Depth of knowledge	71.4%	10
Problem solving ability	92.9%	13
Misconceptions	64.3%	9
Ability to make an inference	35.7%	5
Ability to parse language into mathematics	14.3%	2
Ability to approximate	50.0%	7
Ability to generalize	35.7%	5
Ability to abstract	28.6%	4
Skills at computation	100.0%	14
Skills at manipulation	71.4%	10
Applications, novelty, and/or innovation	21.4%	3
Ability to communicate mathematics	50.0%	7
Ability to convert verbal statements/expressions into	85.7%	12
Ability to convert mathematical statements/expressions	78.6%	11

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Misconceptions	64.3%	9
Ability to make an inference	35.7%	5
Ability to parse language into mathematics	14.3%	2
Ability to approximate	50.0%	7
Ability to generalize	35.7%	5
Ability to abstract	28.6%	4
Skills at computation	100.0%	14
Skills at manipulation	71.4%	10
Applications, novelty, and/or innovation	21.4%	3
Ability to communicate mathematics	50.0%	7
Ability to convert verbal statements/expressions into mathematical statements/expressions	85.7%	12
Ability to convert mathematical statements/expressions into verbal statements/expressions	78.6%	11

As to their perceptions of what kind of questions they ask by percentage on a typical test, we see interesting results.

About what percentage of your test questions address each of the following?											
Answer Options	0-10%	11-20%	21-30%	31-40%	41-50%	51-60%	61-70%	71-80%	81-90%	91-100$	Response Count
General mathematical knowledge	2	4	1	3	2	0	1	1	0	0	14
Breadth of knowledge	5	1	1	1	0	1	1	0	0	0	10
Depth of knowledge	3	3	1	4	0	1	1	0	0	0	13
Problem solving ability	2	2	0	5	0	1	2	1	1	0	14
Misconceptions	4	3	2	0	0	0	2	0	0	0	11
Ability to make an inference	5	1	3	0	0	0	0	1	0	0	10
Ability to parse language into mathematics	4	0	1	2	0	0	0	0	0	0	7
Ability to approximate	5	2	1	0	0	0	1	0	1	0	10
Ability to generalize	8	0	0	0	0	0	2	0	0	0	10
Ability to abstract	6	0	0	0	0	0	1	0	1	0	8
Skills at computation	2	3	3	1	0	1	2	0	1	0	13
Skills at manipulation	4	2	3	1	0	2	0	0	1	0	13
Applications, novelty, and/or innovation	5	1	0	0	1	1	0	0	0	0	8
Ability to communicate mathematics	3	1	2	0	1	0	0	0	1	0	8
Ability to convert verbal statements/expressions into	5	2	2	1	1	0	1	0	1	0	13
Ability to convert mathematical statements/expressions	6	3	1	0	1	0	1	1	0	0	13

These results are a little surprising in that so many of the respondents do indicate they ask questions that assess whether students can think abstractly and generalize.

Conclusion. It is clear that testing in mathematics can contain many types of questions. However, if a typical end-of-course exam consists of sixty questions (typical), only about 3-4 of each type of question can be given. This is notwithstanding the scope of the domains, wherein there must be basic questions of types 1,2,7,8, and 9. This leaves little room to explore anything near a depth of understanding.

NOTES

1. *New York Daily News*, More mistakes on New York State tests. Fourth and eighth-grade math exams have errors http://www.nydailynews.com/new-york/education/mistakes-new-york-state-tests-article-1.1066649#ixzz21MkE63Wt April 24, 2012.
2. Koretz, Daniel, Some Implications of Current Policy for Educational Measurement, Exploratory Seminar: Measurement Challenges Withinmthe Race to the Top Agenda, December 2009. Online at http://www.k12center.org/rsc/pdf/KoretzPresenterSession3.pdf
3. Campbell, D. T. (1979). Assessing the impact of planned social change. *Evaluation and Program Planning, 2*, 67–90.
4. *Adding it Up*, National Research Council.

SECTION 6

LOW-STAKES AND HIGH-STAKES TESTING

AMANDA ROSS

39. ASSESSMENT VIA HIGH-STAKES TESTING

DEFINING ASSESSMENT

When you think of *assessment*, what comes to mind? Perhaps you think of a combination of quizzes, tests, problem-solving activities, explorations, creative design projects, or cumulative mathematics missions. When thinking of testing a student's knowledge of transformations of functions, you may visualize a graphing calculator activity that requires a strong foundational knowledge of functions and the ability of the student to shift those functions with relative ease. Students might be responsible for replicating a figure drawn on graph paper by determining the functions used to complete the outline of the figure.

Any of the assessments mentioned above could be meaningful indicators of student understanding and possible catalysts in curriculum reform. Do each of these assessments truly reveal student understanding? Does a combination of the assessments build mathematical literacy and comprehension? You may contend that "assessments" constitute an array of tools, including selected response, constructed response, performance assessments, and verbal communication (e.g. interviews, debates, etc.).

HIGH-STAKES TESTING

Did you think of *high-stakes testing* when the word "assessment" was mentioned? Were the other examples of assessment precluded at the mere mention of the word? What and who do these types of tests impact? Well, for starters, they impact students' future success, including grade retention, promotion, tracking, and graduation. They also impact teachers' promotion and salary. These tests even impact a school's visibility, in terms of success and academic achievement. The community is thus affected, as families and teachers often move to more successful districts to live and teach. Note. By definition, high-stakes test are tests that "…carry serious consequences for students or educators" (Marchant, 2002, p. 2).

ADEQUATE YEARLY PROGRESS STATUS

In order to expand the ideas mentioned above, we should explore the technical jargon used to statistically analyze the success level of a school, district, or state. Each learning entity is given an Adequate Yearly Progress (AYP) rating, which includes

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 179–183.

factors, such as attendance rates, graduation rates, and Academic Performance Index (e.g., score calculated from state standardized test results). States have specific procedures put into place to handle Title I schools that do not attain the desired AYP level in the same subject for two consecutive years. Serious consequences result when the AYP is not raised to the specified level, including staff replacement or seizure by the state. With such stringent requirements and outcomes, related to high-stakes testing, a thorough analysis of concerns must be completed. However, let's first visit the origins of high-stakes testing.

ORIGINS OF HIGH-STAKES TESTING

The Committee of Ten was created by the National Education Association (NEA) in 1892 to strengthen the current school curriculum. One of the tasks of that committee was to form another committee, the College Entrance Examination Board. Out of this establishment, came standardized testing in colleges and universities (Moran, n.d.).

So, when did high-stakes testing first make its appearance in elementary and secondary schools?

High-stakes testing found a place in schools shortly after college entrance exams were created. They were first used to sort students and place in appropriate groups. In the 1970s, the focus shifted to usage for promotion and graduation. Due to objections, the movement died down, but in the early 1980s, interest in high-stakes testing was rekindled. After *A Nation at Risk* was released in 1983, policymakers scrambled to find a way to handle the accountability issue. High-stakes testing was again deemed as the answer (Moran, n.d).

High-stakes testing served two important functions:

a. cross-comparisons of schools
b. monitoring of poor school districts that were essentially not accountable to any one or any agency

The creation and implementation of high-stakes tests provided a new wealth of data for measuring and monitoring student performance. The high-stakes testing movement also advanced the testing profession in the areas of format, accuracy, dependability, and utility (Phelps, 2005).

CRITICISMS OF HIGH-STAKES TESTING

Mathematics education professionals have argued (and will likely continue to argue) over the need for this political push, initiated by No Child Left Behind. Administrators, teachers, college faculty, educational researchers, and consultants stand on both sides of the aisle, when speaking about this issue. Empirical evidence is provided in support of, and against, high-stakes testing. We will briefly examine the research base written in opposition to high-stakes tests. It should be noted that this research analysis entails a broad overview of the research completed in this area and is not intended to be construed as all-encompassing.

There are several criticisms directed towards high-stakes tests. High-stakes tests have been linked to destructive consequences (Au, 2007; Goldberg, 2004; Goldberg, 2005; Heilig & Darling-Hammond, 2008; Marchant, 2002; O'Malley Borg, Plumlee, & Stranahan, 2007; Perna & Thomas, 2009; Popham, 2001; Smith & Fey, 2000; Thomas, 2005). Disdain and apprehension for these tests can be grouped into four categories:

1. misplaced curriculum focus
2. increase in dropout rates/barrier to future success
3. problems with test construction/scoring
4. abandonment of morals

Let's briefly visit each of these categories

What is meant by misplaced curriculum focus? Researchers and practitioners alike have found that time spent on test preparation replaces time needed for higher-level thinking problems and activities (Au, 2007; Goldberg, 2004; Goldberg, 2005; Marchant, 2002). *It should be noted that the misplaced focus is not a fault of the tests or testing agencies, but of those who use the results*. Au (2007) stated, "The primary effect of high-stakes testing is that curricular content is narrowed to tested subjects, subject area knowledge is fragmented into test-related pieces, and teachers increase the use of teacher-centered pedagogies" (p. 258). In addition, high-stakes tests provide no feedback to students, and thus offer no potential for improved learning (Marchant, 2002). Tests alone do not improve achievement; an amalgamation of sources is needed (National Research Council, 2001).

High-stakes tests are also arguably linked to higher dropout rates and barriers to future success, namely (a) in school and (b) entry into college. O'Malley Borg, Plumlee, and Stranahan (2007) suggested that failure on the high school exit exam will prompt minority students to drop out of school, due to a new setback added on top of already existing obstacles. Grade retention, mandated from a failing test score, provides an enormous educational setback for years to come and possibly results in dropout (Marchant, 2002). Perna and Thomas (2009) addressed effects on college enrollment by stating,

> High school exit exams shape college enrollment by limiting high school graduation; diverting attention away from ensuring that students are academically prepared for college toward ensuring that students obtain the minimum academic requirements for graduating from high school; reducing time for college counseling; and reducing students' real and perceived academic qualifications for college. (p. 472)

Problems with test construction, including limitations in form and complexity, are also cited (Garland, 2006; Goldberg, 2004; Goldberg, 2005; Keng, McClarty, & Davis, 2008; Marchant, 2002; Popham, 2001; Smith & Fey, 2000). Problems with validity, including construct validity and consequential validity, reveal the arduous and daunting task of creating valid, and thus interpretable, tests (Smith & Fey, 2000). The administration medium is a component of test construction, and Keng, McClarty, and Davis (2008) examined the benefits of the paper and online versions. All of these points lead to problems with past administrations of the New York Regents Exams of 2003 and 2006. Arenson (2003a, 2003b) noted the problems associated with the design of the 2003 exam. In another article that year, she discussed the questionable handling of failing scores on the exam. Garland (2006) questioned the rigorousness of the 2006 exam. Problems and inconsistencies, related to alignment of the test to state standards and difficulty of problems presented, are carefully considered by the testing agencies. Nonsensical authentic problems have also been noted in past administrations of the New York Regents Exam, and it appears that steps are being taken to prevent such items in the future.

Unbelievably so, problems with scoring errors have also increased within the past decade. Scoring errors have occurred in both high school graduation exams and college entrance exams. Fratt (2005) reported on grading errors in a Minnesota high school that resulted in incorrect failure of 8,000 students in 2000. Scanning issues, preventing proper answer detection, as well as failure to recheck answer sheets, have resulted in incorrect SAT scores, both higher and lower (Arenson & Henriques, 2006; Honawar, 2006).

Finally, ethics and morals are challenged due to the advent of high-stakes tests. Morals are often abandoned, due to the high-pressure atmosphere, related to jobs, bonuses, reputation, and real estate values (Au, 2007). For example, pressure to hide failing scores, assist students, etc. is an ethical struggle in the school system. Heilig and Darling-Hammond (2008) offered another startling revelation, regarding accountability. The authors bring to light exclusion of certain populations from school and tests on test day.

In the next article, we will visit specific problems with construction of high-stakes tests and examine guiding principles and suggestions for improvement.

REFERENCES

Arenson, K. W. (2003a). Math failures are raising concerns about curriculum. *The New York Times*. Retrieved December 7, 2009, from http://nytimes.com/2003/07/02/nyregion/02REGE.html.

Arenson, K. W. (2003b). Scores on math regents exam to be raised for thousands. *The New York Times*. Retrieved December 7, 2009, from http://www.nytimes.com/2003/08/30/nyregion/scores-on-math-regents-exam-to-be-raised-for-thousands.html.

Arenson, K. W., & Henriques, D. B. (2006). SAT errors raise new qualms about testing. *New York Times*. Retrieved December 7, 2009, from http://www.nytimes.com/2006/03/10/education/10sat.html?_r=1

Au, W. (2007). High-stakes testing and curricular control: A qualitative metasynthesis. *Educational Researcher, 36*(5), 258-267.

Fratt, L. (2005). Graduation exams under the microscope. *District Administration, 41*(4), 17.

Garland, S. (2006). Math scores fall: Klein blames a harder test. *New York Sun*. Retrieved December 7, 2009, from http://www.nysun.com/new-york/math-scores-fall-klein-blames-a-harder-test/41435/

Goldberg, M. F. (2004). The high-stakes test mess. *Education Digest, 69*(8), 8–15.

Goldberg, M. F. (2005). Losing students to high-stakes testing. *Education Digest, 70*(7), 10–19.
Heilig, J. V., & Darling-Hammond, L. (2008). Accountability Texas-style: The progress and learning of urban minority students in a high-stakes testing context. *Educational Evaluation and Policy Analysis, 30*(2), 75–110.
Honawar, V. (2006). SAT glitches prompt broader testing worries. *Education Week*, *25*(28), 8–9.
Keng, L., McClarty, K. L., & Davis, L. L. (2008). Item-level comparative analysis of online and paper administrations of the Texas assessment of knowledge and skills. *Applied Measurement in Education, 21*(3), 207–226.
Marchant, G. J. (2002). What is at stake with high stakes testing? A discussion of issues and research. *Ohio Journal of Science, 104*(2), 2–7.
Moran, R. F. (n.d.). *Sorting and reforming: High-stakes testing in the public schools*. Ohio, OH: University of Akron. Retrieved December 29, 2009, from http://www.uakron.edu/law/lawreview/v34/docs/moran341.pdf
National Research Council. (2001). *Understanding dropouts: Statistics, strategies, and high-stakes testing*. Washington, DC: National Academy Press.
O'Malley Borg, M., Plumlee, J. P., & Stranahan, H. A. (2007). Plenty of children left behind: High-stakes testing and graduation rates in Durval County, Florida. *Educational Policy, 21,* 695–716.
Perna, L. W., & Thomas, S. L. (2009). Barriers to college opportunity: The unintended consequences of state-mandated testing. *Educational Policy, 23*(3), 451–479.
Phelps, R. P. (2005). *Defending standardized testing*. Mahwah, NJ: Lawrence Erlbaum Associates.
Popham, W. J. (2001). *The truth about testing: An educator's call to action*. Alexandria, VA: Association for Supervision and Curriculum Development.
Smith, M. L., & Fey, P. (2000). Validity and accountability in high-stakes testing. *Journal of Teacher Education, 51*(5), 334–344.
Thomas, R. M. (2005). *High-stakes testing: Coping with collateral damage.* Mahwah, NJ: Lawrence Erlbaum Associates.

AMANDA ROSS

40. HIGH-STAKES TESTING

Test Construction

In the previous article on high-stakes testing, we looked at the origins and criticisms of standardized tests. In this article, problems with test construction and guidelines for improvement will be briefly visited.

Extremely precise measures of item creation are undertaken by testing agencies. However, there is inevitably room for error. Goldberg (2004) stated, "Even if it were possible to create the ideal test, it would still be vulnerable to everything from wrong answer keys to programming errors" (p. 13). Let's first explore some noted problems with test construction.

A BRIEF GLANCE AT PROBLEMS WITH TEST CONSTRUCTION

The illustrious undertaking of designing a high-stakes test realizes the following difficulties:

1. Mechanics of the test (e.g., wording, length of questions, questions with more than one answer) (Arenson, 2003)
2. Nonsensical situations/data (Arenson, 2003; Raimi, 2004)
3. Non-alignment to standards (Tucker, n.d.)
4. Content too narrow or too broad (Arenson, 2003; Tucker, n.d.)
5. Cultural insensitivity (Moran, n.d.)

It should be noted that the ever-increasing size of the curriculum poses additional hardships on testing agencies who must closely match the test to the content learned in the classroom.

GUIDELINES FOR IMPROVEMENT

In examining the aforementioned problems, it is conceivable that a set of guiding principles/suggestions for improved test construction is needed. We will look at this from the points of view of a researcher and a group of practitioners. These two examples are simply a subset of a large amount of literature, designated to the improvement of high-stakes tests. Plake (2002) identified six criteria for evaluating high-stakes tests, including *alignment to test specifications, opportunity to learn, freedom from bias and sensitive situations, developmental appropriateness, and appropriateness of mastery*

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 185–186.

level cut points. The author also discussed score consistency and reliability. Yeh (2006) conducted several teacher interviews and compiled a list of questions teachers viewed as beneficial to evaluating high-stakes exams. The questions included:

> Do tests provide diagnostic information?
> Do they avoid contrived writing prompts?
> Do they focus on substance, rather than essay format?
> Are they sufficiently challenging?
> Do they focus on analysis and evaluation, rather than simple summarization?
> Do they require students to use multiple sources of information to answer questions?
> Do they avoid questions that require memorization?
> Are they fully aligned to the state curriculum standards?
> Do they avoid excessive quantities of instructional objectives?
> Do they help teachers to prepare all students for college? (p. 109).

In summary, the purpose of this short article was to illuminate the categories of problems apparent in the testing arena, as well as sample criteria used to formulate the "most ideal" high-stake tests. The author believes practitioners will become more aware of test design and allowable implications/learning opportunities from results after pondering the information presented.

REFERENCES

Arenson, K. W. (2003). Math failures are raising concerns about curriculum. *The New York Times*. Retrieved December 7, 2009, from http://nytimes.com/2003/07/02/nyregion/02REGE.html

Goldberg, M. F. (2004). The high-stakes test mess. *Education Digest, 69*(8), 8–15.

Goldberg, M. F. (2005). Losing students to high-stakes testing. *Education Digest, 70*(7), 10–19.

Moran, R. F. (n.d.). *Sorting and reforming: High-stakes testing in the public schools*. Ohio, OH: University of Akron. Retrieved December 29, 2009, from http://www.uakron.edu/law/lawreview/v34/docs/moran341.pdf

Plake, B. S. (2002). Evaluating the technical quality of educational tests used for high-stakes decisions. *Measuring and Evaluation in Counseling and Development, 35*(3), 144–152.

Raimi, R. A. (2004). *High stakes examinations and educational foolishness*. New York, NY: University of Rochester. Retrieved December 29, 2009, from http://www.math.rochester.edu/people/faculty/rarm/highstake.html

Tucker, A. (n.d.). *The New York Regents math test problems*. New York, NY: Stony Brook University. Retrieved December 29, 2009, from http://www.ams.sunysb.edu/~tucker/MathAPaper.pdf

Yeh, S. S. (2006). Can rapid assessment moderate the consequences of high-stakes testing? *Education and Urban Society, 39*(1), 91–112.

AMANDA ROSS

41. LOW-STAKES TESTS AND LABELS

If you live anywhere in the world of education, then low-stakes standardized tests and high-stakes standardized tests are all around you. This article focuses on low-stakes standardized tests, but it is important to briefly review a bit of background information. Standardized tests are tests that use a uniform scoring procedure, i.e., standard scoring procedure. Standardized tests may be norm-referenced or criterion-referenced. A norm-referenced test is one that compares a student's scores to those of other students. A criterion-referenced test is a test that measures a student's understanding and skill set, related to certain criteria. A high-stakes test defined simply is a test, whereby the stakes are high. Alternatively, a low-stakes test is defined as a test, whereby the stakes are low (subjectively labeled). A common high-stakes outcome is grade promotion/grade retention, while a common low-stakes outcome is a performance label. It is within this arena, where this article takes aim. (This article serves as an opinion piece, with an intended consequence to impact change, related to low-stakes tests and labels.)

In PK-12 mathematics, we have Common Core standards, including Standards for Mathematical Practice, other state standards, NCTM standards, and College and Career Readiness Standards. Students work all year long, learning new mathematics content, new strategies, and real-world applications. They work to meet all of these standards. They fervently try to show their teacher and parents all that they have learned. They have a love and enthusiasm for learning. They try with all their heart. Then, at the end of the year, they are given one or more "low-stakes" tests that will be used to measure performance. The test(s) will be used to determine each student's mathematical performance level and LABEL. The student's performance on other assessments throughout the year will not be taken into account, when giving this label. This label can permanently squash the student's love and enthusiasm for learning!

Imagine that these tests are administered to kindergartners during a noisy, inopportune time. The student's success on two timed tests, each one minute, in length, is used to show a performance label for a blank on a report card, called "Math Skills." Suppose a student gets a "below average" rating for math skills for the year, while all individual math components on the report card, show complete mastery. How on any platform does that make sense? How should the kindergartner feel, seeing a "below average" label, on his or her report card? At that age, such a label can be detrimental to self-esteem and level of confidence. The student will not

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 187–188.

understand that all other math components were mastered, and that this is the result of a couple of low-stakes tests, administered once, at the end of the year. To reiterate, the student is given this label, due to the administration of two 1-minute timed tests, given during a busy, noisy time, in the school day. It is the author's position that this procedure is not how education and assessment should be aligned.

The last sentence serves as a good segueway into the author's views on proper alignment of education and assessment. First and foremost, results from any low-stakes test should be used in conjunction with results from other mathematics assessments, administered throughout the school year. One test, or a couple of tests, administered at one time, on one day, should not be used to identify a performance label for a student's mathematics skills. The blank could be modified to state the name of the test, so that it is clear that one test or set of tests was used to determine such a label. Also, the author opposes the idea of using labels of "below average," "average," and "above average." This system of labeling needs to be changed. There isn't any reason for such subjective labeling; the precise scores may be shown, instead.

So, what are our alternatives? It is easy. What do we have at our disposal? We have performance assessments/tasks, portfolios, e-portfolios, projects, student interviews, and teacher observations, as well as other formative and summative assessments, given throughout the year. These, combined with any standardized scoring results, should culminate into a final scoring and "level of performance," not "label of performance." The author's view is not that standardized testing is bad. However, is should be one indicator of success, not the sole indicator of success. If we continue to assess our students in this unfair manner, they will be turned off to education and learning, and we might as well refrain from professional development because it will be moot. Assessment is supposed to positively impact instruction, not be used to unfairly label our students.

PART III

TECHNOLOGY, GAMES, AND TIPS

SECTION 7

TECHNOLOGICAL APPLICATIONS

G. DONALD ALLEN

42. WHAT TECHNOLOGY WORKS FOR TEACHING MATHEMATICS AND WHY – A PERSPECTIVE, PART I

One important question that is really not asked much is "What technology works for teaching?" It is generally assumed that calculators, graphing and others, are "it." But are they really? In this brief report, we consider just a couple of aspects of technology such as

What's important?
Old Technologies
Research says?
General technology principles
Not-so-old technology
New and emerging technologies
Household technologies

What's important for students? The three "T's," of course: training, testing, and tutorials. What's important for teachers? There is teaching mathematics, guided assessment, identifying misconceptions, and tracking progress. And what's important for the schools? We offer successful students, quality education, teacher quality, and elevated standardized test scores. The fundamental question relates to how technology helps make this happen. So, just how is technology used in the schools?

Technology for students is a multi-faceted affair. We have

Student learning (including interactive applets, video tutorials, and calculators)
Student remediation
Student communication
Student to student interaction including collaboration (including communication)
Diagnosing student ability and learning issues
Assessment of students
Distance learning—synchronous and asynchronous

Technology for teachers is likewise, multi-faceted, and we observe the following types of applications

Teaching students
Diagnosing student ability and learning issues
Assessment of students
Assessment of teachers

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 193–195.

Teachers communicating with parents and teachers
Management of student/teacher progress
Concept discovery
Guided assessment

Covering all these points would involve a book-length article. So, to confine ourselves to just a couple of pages, we consider just one question: "What should technology do?" More particularly, what should technology do in the mathematics classroom? Here are the basic points:

Be an integral part of the problem solving process.
Enable understanding – not just give an answer.
Not do the thinking. Thinking must be key to the process.
Have a gentle learning curve with more power coming with more study, but powerful even at the entry level.
Be interactive, engaging, empowering, and even cool.

This sounds good, but is not that easy to achieve – especially the last point. Is the technology "engaging, empowering, and even cool?" Do students see a need for it after graduation? Can it be used to solve problems? Do their parents use it? Do their "heroes" use it? Is it important?

Let's look as some old technology, such as pen and paper algorithms, look-up tables, counting boards – abacus, slide rule, adding machines, calculator, and chalk – slate blackboard. How many are still used today – even after thousands of years? What made them successful is that they were fun, engaging, and empowering. They enabled us to solve problems better/faster, consider more complex problems, and led to enhanced algebraic thinking (e.g. programming). Moreover, what they did was remarkable. They sped and facilitated problem solving, allowed the mechanical parts of problems (e.g. arithmetic) to be faster, allowed more realistic problems to be solved, empowered the user, and were used ubiquitously. Once again, the last point is most important. If a technology is regarded only for classroom use, students will relegate it just to the classroom and not as a part of their lifelong learning.

Just as how important the technology is and what it can do, we have the flip side of what it does *not* do. Namely, good technology does not substitute for thinking, interfere with the mathematics understanding, detract from mathematics learning, or have steep learning curves (mostly). *All* of these are key points. Technology must not get in the way of actual learning and thinking! So what does the research say?

Technology has been proven effective when…

Implemented carefully
Leadership is on board
Professional development is given
Appropriate curricular design is achieved

But many educators have

Miscalculated the difficulty of implementation
Over-promised the deliverables
Over-promised the ability to extract a learning return on technology

Read the Cisco-Metri report. It is enlightening. See http://www.cisco.com/web/strategy/docs/education/TechnologyinSchoolsReport.pdf

We will continue this discussion in the next article.

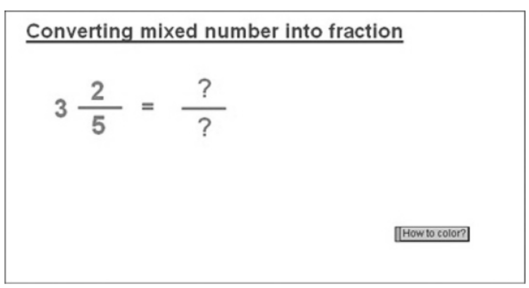
ANTONIJA HORVATEK

43. USING TECHNOLOGY IN TEACHING MATH

I am a math teacher in Croatia. I work in a primary school, teaching grades $5^{th} – 8^{th}$, working with kids of 10 to 15 years of age. Unfortunately, in my school, there is not enough equipment for each student to have his or her own computer. Thus, I compensate for the lack of computers, by using a projector. I started using it about 7 years ago and found it to be very useful. I have used several programs, including GeoGebra, Geometer's Sketchpad, and Power Point, when making materials for my classes. In this article, I will describe ways to use some of these materials.

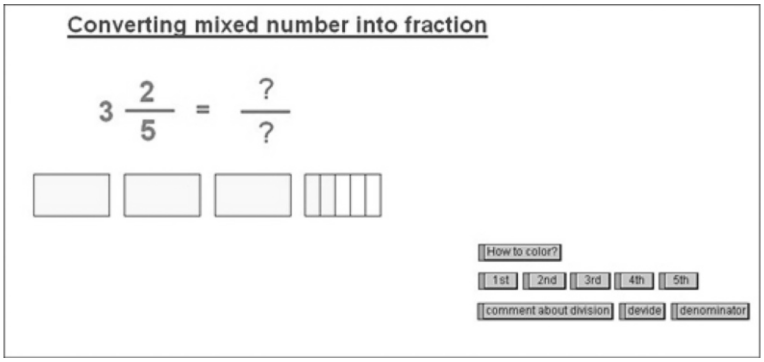
INVESTIGATION - CONVERTING A MIXED NUMBER INTO A FRACTION

Our kids work with fractions in the 5th grade. Computers prove to be very useful in this section. For example, we use the projector to illustrate the procedure for converting a mixed number into a fraction. For this purpose, we use The Geometer's Sketchpad file: http://www.antonija-horvatek.from.hr/materials-English/Fractions/Converting-mixed- number-into-fraction.gsp .

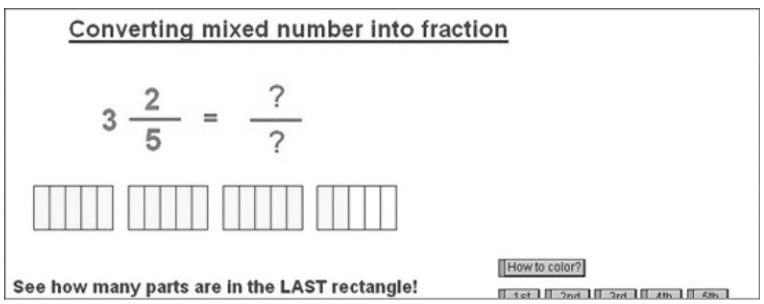


Here, we have the mixed number, $3\frac{2}{5}$. We are looking for the improper fraction that is equal to $3\frac{2}{5}$. To identify the fraction, we look at a picture of the mixed number. Pupils are asked to color $3\frac{2}{5}$ of a rectangle. Pupils realize they must color 3 whole rectangles, draw another rectangle, divide it into 5 equal parts and color 2 of the parts. With each part of the process, the accompanying portion of the illustration is completed.

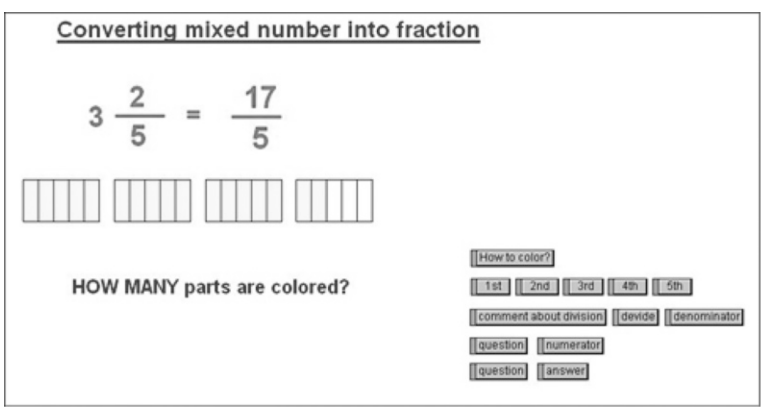
G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 197–204.



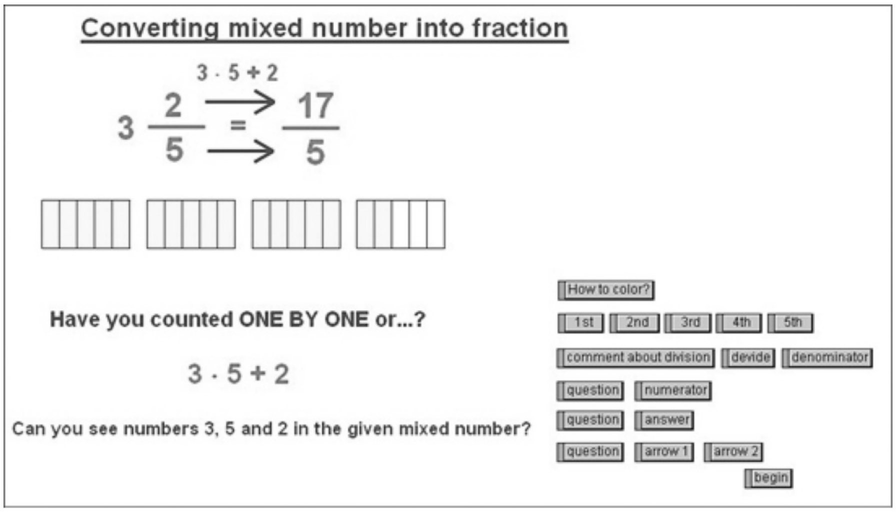
Now, we have an illustration that represents the mixed number. Pupils must determine how to present the colored part with a **fraction**. In order to do so, they need to count the number of total parts in the partially shaded rectangle, as well as the total number of shaded parts in the four rectangles (see the picture below).



Pupils may now see that the denominator of the fraction is 5, since each rectangle may be divided into 5 equal parts. The numerator is 17, since that is the total number of shaded parts, shown in all four rectangles.



Next, we discuss procedures students may use, when calculating the number that goes in the numerator.



At the conclusion of the activity, students have found the rule for converting a mixed number into a fraction, $a\frac{b}{c} = \frac{a \cdot c + b}{c}$.

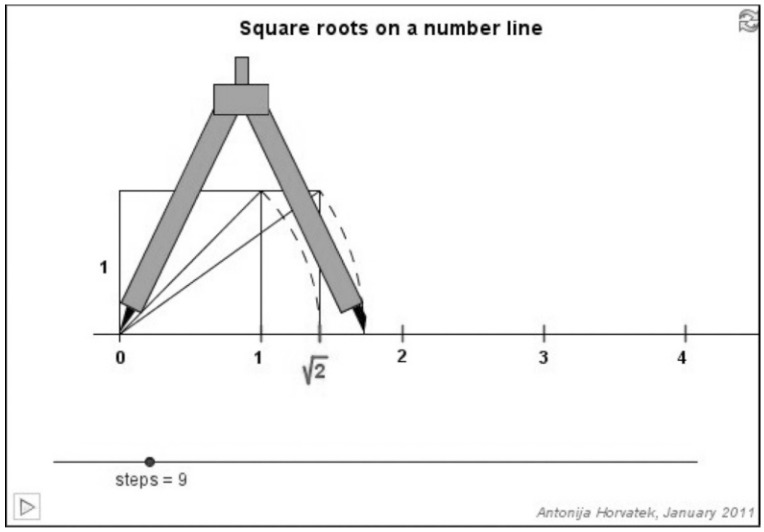
PRESENTING SOME STEPS – RADIAL EXPRESSIONS ON A NUMBER LINE

In Croatia, square roots are presented in 8th grade. In our curriculum, it is necessary to teach the graphing of $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5},$ etc., on the number line, but it is not necessary to graph numbers like these:

- $-\sqrt{2}, -\sqrt{3}, -\sqrt{4}, -\sqrt{5},$ etc.
- $\sqrt{n}, 2\sqrt{n}, 3\sqrt{n}, 4\sqrt{n},$ etc.
- $a\sqrt{n} + b$

So, I use GeoGebra applets that I created to show all of these graphed expressions. The applets present the constructions, in a very short time. You can find these applets on this page: http://www.antonija-horvatek.from.hr/applets/real-numbers/Square-roots-on-number-line.htm .

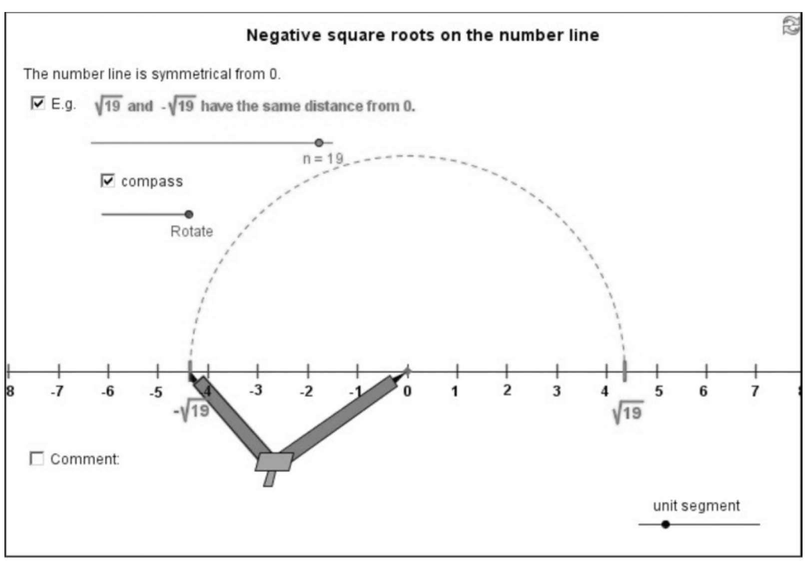
The first applet shows how to graph $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5},$ etc. on the number line.

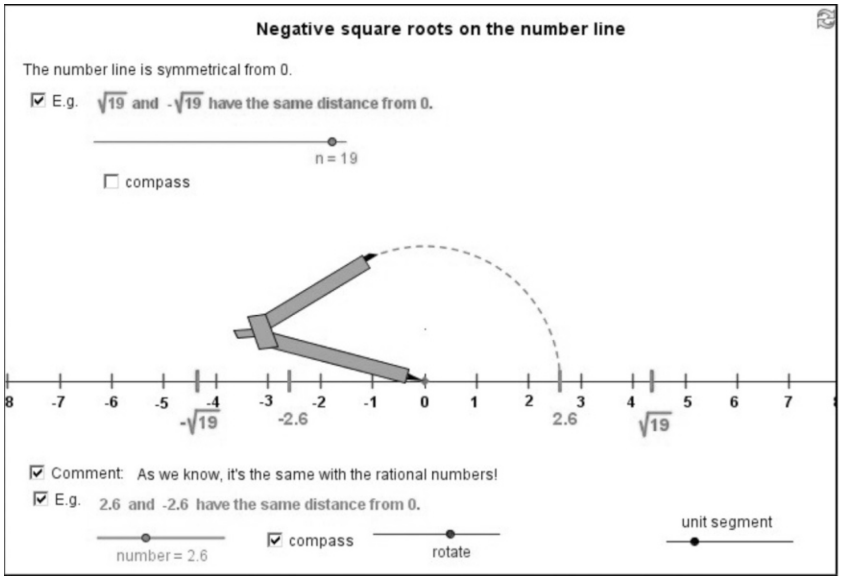


I also like to show the construction on the blackboard. *It is clearer for some kids when the teacher explains the process, while drawing the graph on the blackboard.* At the same time, kids draw the graphs, in their notebooks.

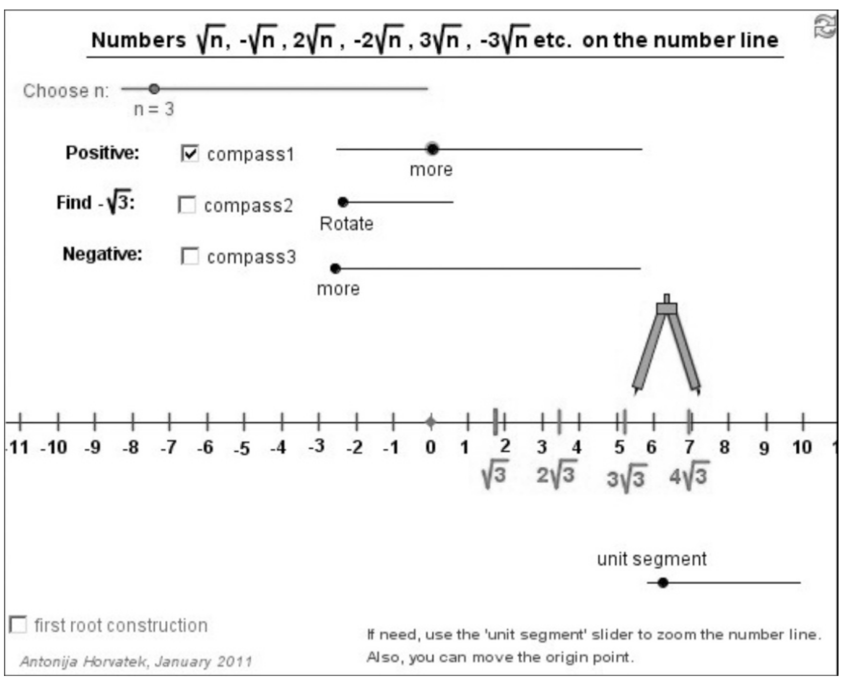
The graphing of other radical expressions are presented via projector:

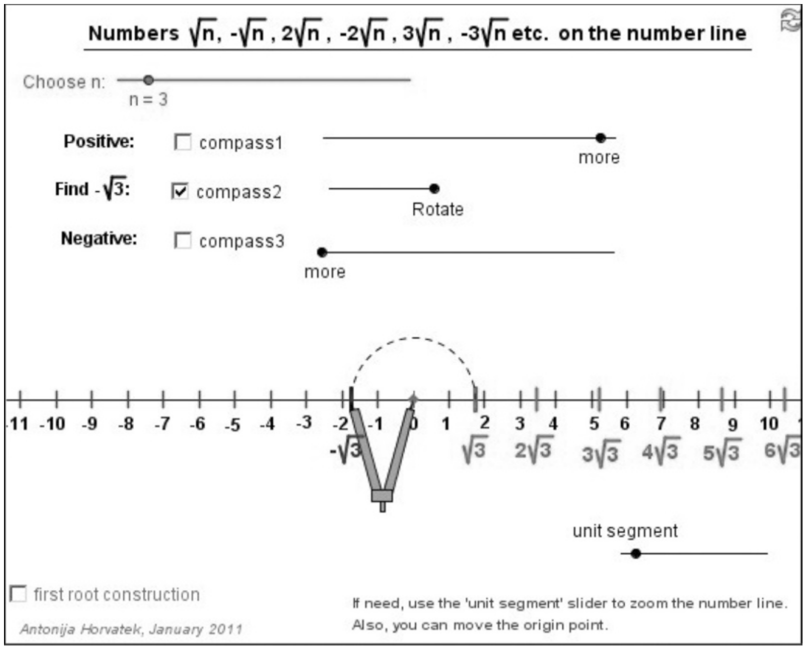
NEGATIVE SQUARE ROOTS

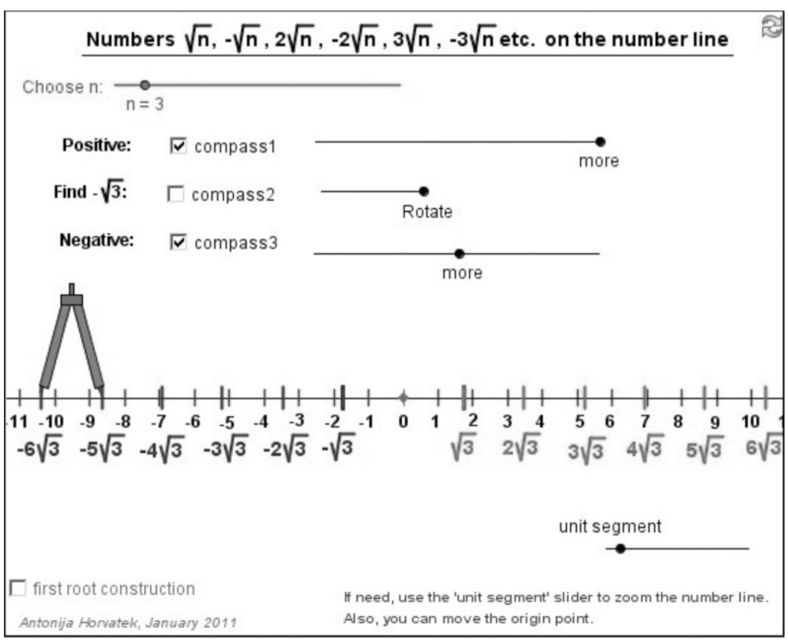


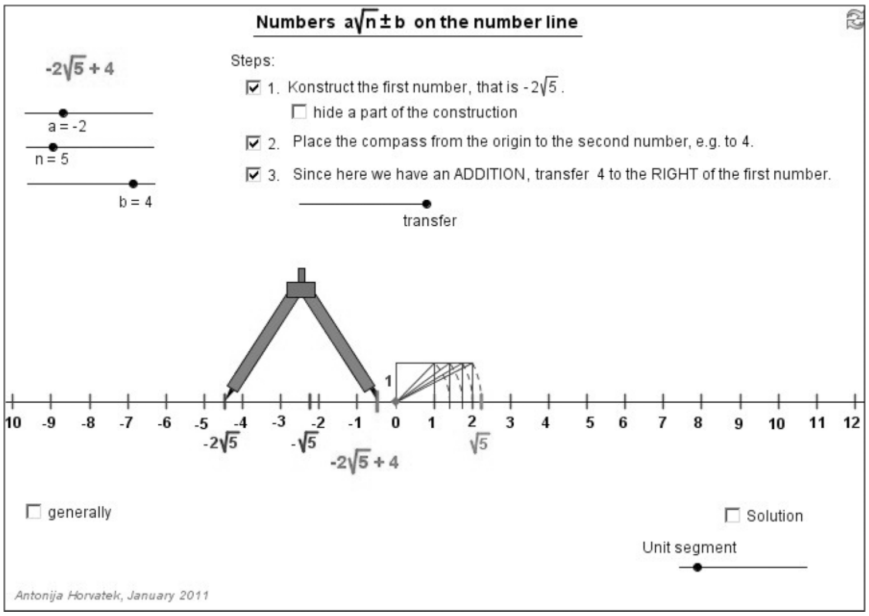


Other Radical Expressions




Numbers √n, -√n , 2√n , -2√n , 3√n , -3√n etc. on the number line
Choose n:
n = 3
Positive:
compass1
more
Find -√3:
compass2
Rotate
Negative:
compass3
more
-11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
-√3
√3 2√3 3√3 4√3 5√3 6√3
unit segment
first root construction
If need, use the 'unit segment' slider to zoom the number line.
Also, you can move the origin point.
Antonija Horvatek, January 2011


Numbers √n, -√n , 2√n , -2√n , 3√n , -3√n etc. on the number line
Choose n:
n = 3
Positive:
compass1
more
Find -√3:
compass2
Rotate
Negative:
compass3
more
-11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
-6√3 -5√3 -4√3 -3√3 -2√3 -√3
√3 2√3 3√3 4√3 5√3 6√3
unit segment
first root construction
If need, use the 'unit segment' slider to zoom the number line.
Also, you can move the origin point.
Antonija Horvatek, January 2011



In each of these cases, I ask pupils appropriate questions and try to guide them to the correct conclusions. Also, I give the web address with these applets and ask them to visit it, to study it again, and to solve some (optional) exercises at home.

REVISION - FRACTIONS

The link below shows how we use a Power Point, when discussing revisions: http://www.antonija-horvatek.from.hr/materials-English/Fractions/Fractions-revision.pps

This presentation can be used at home, as well.

OTHER WAYS TO USE TECHNOLOGY IN MATH TEACHING

All of the materials presented in this article, plus many others, may be found at: http://www.antonija-horvatek.from.hr/materials-English.htm.

Many of the materials are in Croatian and are not translated into English. Given in Croatian, here is an example that covers the solving of systems of two linear equations in two variables:

http://www.antonija-horvatek.from.hr/7_razred/08_Sustavi_jednadzbi/Sustavi_jednadzbi.zip.

I often use Power Point presentations as a framework, with links to GeoGebra files. Here is an example of integration of these two programs:

http://www.antonija-horvatek.from.hr/7_razred/09_Jednadzba_pravca/Uvod-Jednadzba_pravca.zip. Note. The example is in Croatian.

http://www.antonija-horvatek.from.hr/

G. DONALD ALLEN

44. WHAT TECHNOLOGY WORKS FOR TEACHING MATHEMATICS AND WHY – A PERSPECTIVE, PART II

> Oh, the little more, and how much it is! And the little less, and what worlds away.
>
> (Robert Browning)

In his recent book, *Oversold and Underused*, Larry Cuban argues that technology has not transformed teaching or learning and has not affected the productivity gains that educators and others had hoped for. In educational technology, despite the vast sums expended, progress has been slow. Many teachers/administrators

- Do not have reliable and current technology
- Lack access to research
- Have poorly articulated vision
- Have had technology fidelity issues
- Have difficulty identifying true impact technologies from the loser technologies

In consequence

- The real potential (of technology) has been untapped.
- The educational infrastructure is becoming wary.
- Educators are questioning the value of using technology.

Yet, in many cases, teachers are compelled to use unproven technology, having minimal training. The unstated paradigm is that it is cutting edge, it is expensive, and thus it should work. The unstated and perhaps unrecognized fact is the profession of teaching has been with us for millennia – and is a highly polished craft.

It becomes possible to distill, but not guarantee, successful teaching and use of technology for learning to three points:

- Fidelity with which technology is implemented for improved learning
- Evaluation and monitoring of impact and use of data
- Planned flexibility to adjust when results don't align with targeted improvements

It is easy to list these, but it is difficult to realize them systematically and institutionally. The last may be the most difficult, as once the training is completed, there are those in the system fully invested in the technology and methods – and may be rather unwilling to admit it isn't working, let alone make needed changes.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 205–207.

GENERAL PRINCIPLES ON TEACHING WITH TECHNOLOGY

Here, we are focusing on using technology to teach. The other aspect is using technology to learn. The latter is a far more difficult subject with complex issues. In using technology to teach, it is important to consider

- General teaching/technology strategies
- Planning your technology usage strategy
- What are the quantitative aspects?
- Should supplementary services be available?
- What about faculty involvement?
- Who else should be involved?

We can manage only a sketch of these points. A full explanation of each point is really beyond the scope of this essay. The aspect from faculty involvement is very important, given acceptance that nothing good comes of just placing the technology in the classroom without a serious plan or diligent training.

On the usage strategy, we need to consider the best way to use the technology. For example:

- Modes
  - Lab
  - Classroom demonstrations
  - Posts to the Internet
- Student Perception
  - Added-on work?
  - Steep learning curve?
  - Is there any fun/profit/importance to this?
- Time to learn

As to the quantitative issues we note just a few points

- How much time is allocated?
- Well functioning, modern equipment
- Individual technology units for students or pairs of students
- Faculty prep time (negative faculty do more damage than good)

The lynchpin of making technology work is the teacher. A negative teacher attitude can be paralyzing to any success venture in the classroom, technological or pedagogical.

Are there supplementary services such as mentors/tutors, video tutorials, and even peer mentoring? Overall, adequate help should be available. Don't underestimate the power of video tutorials, particularly in problem solving. They are easy to make; in most workshops we've conducted, teachers can be skilled at making their own videos in just a half-day.

What about teacher involvement? This is most important. Here are just a few points, posed as questions, to consider.

- Should there be a select group of faculty to do technology? Such faculty are strong advocates.
- Is there time available for teachers to perfect their use of technology?
- Are faculty encouraged or conscripted to join in?
- Are there serious professional development and hands-on workshops? Teachers need to actually do it before they use it!

The teacher that "wings it" with technology in the classroom takes a huge risk of many possible dimensions. For expertise with technology, teachers need time:

- Time to explore the technology – to determine how best to use it
- Time to learn the subtleties
- Time to become facile
- Time to learn to use the technology "on the fly"
- Time to study suggested activity materials

AMANDA ROSS

45. THE GRAPHING CALCULATOR

A Brief Look at What It Can Do

Graphing calculators – how wonderful they are! My older TI-83 Plus can do lots. Unfortunately, I don't have the latest and greatest TI-89 yet. You can visit https://education.ti.com/en/us/products#product=graphing-calculators to find information on all TI graphing calculators. The intent of this article is to provide a brief overview of the capabilities of a graphing calculator. The focus will primarily relate to algebraic and statistical functions of the graphing calculator. All instructions/procedures will relate to the TI-83 Plus. However, the TI-89 procedures will be similar and shouldn't be hard to find.

Some of these functions are shown in the table below.

Algebraic functions	*Statistical functions*
• Enter functions into the y = screen, make table of values, and graph • Find minimum, maximum, and intercepts • Evaluate exponential functions • Evaluate logarithmic functions • Evaluate trig functions • Use inverse trig functions to calculate angle measures • Perform matrix operations • Write a program to graph functions and transformations of functions • Find derivative • Solve systems of linear and non-linear equations	• Calculate combinations and permutations • Run statistical tests, i.e., z-test, t-test, two-sample z-test, two-sample t-test, one-proportion z-test, two-proportion z-test, z-interval, t-interval, two-sample z-interval, two-sample t-interval, one-proportion z-interval, two-proportion z-interval, chi-square test, two-sample F-test, linear regression, ANOVA • Determine fit of different function models (linear, quadratic, cubic, quartic, exponential, power, logistic) • Calculate the area under the normal curve • Create scatterplots and find lines of best fit • Generate random data • Calculate summary statistics (measures of center and spread)

Let's look at a few of the capabilities, for each domain. We will look at some examples and procedures.

- Enter functions into the y = screen, make table of values, and graph
- Find minimum, maximum, and intercepts
- Find a derivative

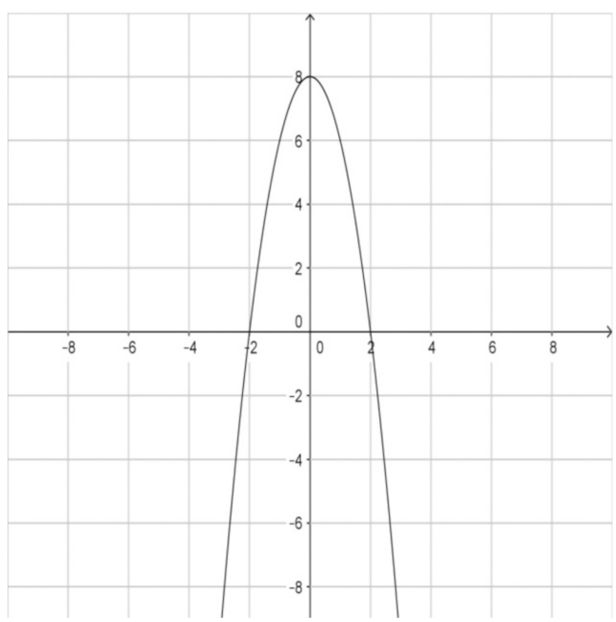
G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 209–213.

ENTER A FUNCTION INTO THE Y = SCREEN
Press y =
Enter function below
$y = -2x^2 + 8$

MAKE A TABLE OF VALUES
Press 2nd Graph
Scroll through to find values of interest.

x	y
−2	0
−1	6
0	8
1	6
2	0

GRAPH THE FUNCTION
Press Graph
(Set window by pressing Window and adjusting x- and y-values min and max and scale.)



**Note. This is a png file from GeoGebra.*

MINIMUM OR MAXIMUM

Press 2nd Trace
Select minimum or maximum.
Press Enter.
Choose left bound of the parabola. Enter.
Choose right bound of the parabola. Enter.

Maximum: (0, 8)

INTERCEPTS

Look at graph OR
To find x-intercepts,
Press 2nd Trace.
Select zero to find the x-intercepts.
Choose left bound and right bound of each x-intercept. Enter.
To find y-intercept,
Press 2nd Trace.
Select value.
Type 0 for x-value.
Press Enter.

x-intercepts: (−2, 0) and (2, 0)
y-intercept: (0, 8)

DERIVATIVE

$y = -4x$
Algebraic manipulation shows the derivative to be.

The numerical derivative, for any x-value, may be found, using the graphing calculator.

Enter function into y = screen.
Press 2nd Trace.
Select 6: dy/dx.
Choose a value for x on the graph, or type a value for x.
Press Enter.

The calculator will show the derivative, evaluated for the given x-value. For $x = 2$, the derivative is −8. You can also use the nDeriv function of the graphing calculator to find the numerical derivative. What does the derivative mean? It is the rate of change, at a point, on a function. It is a limit, more or less.

CALCULATE COMBINATIONS AND PERMUTATIONS

Example: 6 out of 10 kids will be randomly chosen for placement in a picture. Find the number of different ways for the picture to come out.

$_{10}C_6$
Enter 10 into the calculator.
Press Math.
Scroll over to PRB.
Choose 3: nCr.
Type 6.
Press Enter.

There are 210 ways for the picture to come out.

T-TEST
A professor at University Y claims that the average score on his final exam is 82. A random sample of 20 student final exam scores shows an average of 80, with a standard deviation of 2 points. Is there a statistically significant difference between the professor's claim and the average of the sample?
Press Stat.
Select Tests.
Choose 2: T-Test.
Select Stats option.
Enter the following:
μ_0*:82*
$\bar{x}$*:80*
s_x*:2*
n:20
Choose mu is not equal to μ_0.
Press Calculate.
The p-value is less than 0.01, which is less than 0.05, the level of significance often chosen by researchers. Thus, there is a statistically significant difference between the professor's claim and the average of the sample. The calculator output also shows the t-value to be approximately −4.47. The absolute value of this t-value, or 4.47, is greater than the critical t-value in the t-distribution table, or 2.093. (Recall that degrees of freedom or df equals n minus 1.)

CALCULATE THE AREA UNDER THE NORMAL CURVE

The ACT average of a sample of students was 23, with a standard deviation of 2 points. Michelle scored a 21. What percentage of students scored lower than Michelle?

Select 2nd Vars.
Choose 2: normalcdf(.
Following the prompt, enter 0, 21, 23, 2, which represents the lower bound, upper bound, mean, and standard deviation.
Press Enter.
The calculator returns the approximate output of .1587. So, approximately 15.87% of the students scored lower than Michelle.

This can be checked by manually calculating a z-value and finding the area to the left of the z-score, in a z-distribution table.

$$z = \frac{X - \mu}{\sigma}$$

$$z = \frac{21 - 23}{2} = \frac{-2}{2} = -1$$

Finding a z equal to the absolute value of of -1 in a z-table shows the same area to the left of the z-score, or 0.1587.

Alternatively, the ShadeNorm function of a graphing calculator may also have been used to find the area under the normal curve.

SECTION 8

MATHEMATICAL GAMES OR "RECREATIONAL GENRE PROBLEMS"

G. DONALD ALLEN

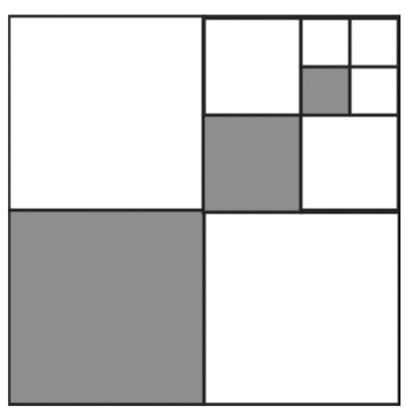
46. PROBLEMS IN MATHEMATICAL RECREATION

In about the twelfth century, with the advent of wide scale commerce between Europe, Asia, and the Middle East, there came a need to understand and make currency transactions and understand other aspects of banking and commerce. At this time, a number of schools were created to train the children of this new commerce class in “reading, writing, and arithmetic.” It is arithmetic – or math – we focus on here. In particular, at that time, a number of books on mathematical recreation were written. They provided a sort of mathematical entertainment for students, teaching them problem solving outside the context of the standard *quadrivium* (arithmetic, geometry, music, astronomy), the curriculum of the day.

This brief article contains a few problems of the “recreational” genre that don’t fit really anywhere in modern curricula/standards, but clearly stimulate students. They’re suitable for all grades 6–12. In most cases, original and sometimes clever ideas, not learned methods, will work in their solution. You will note that while algebra can help solve some of these, most can be-resolved by other means. The solutions are posted on our website. When the high stakes exams are over and the school year is ticking down, you may have a little breathing room to let your students explore some interesting problems.

The most important aspect of these problems is to “see” how to solve them without laborious algebraic computations.

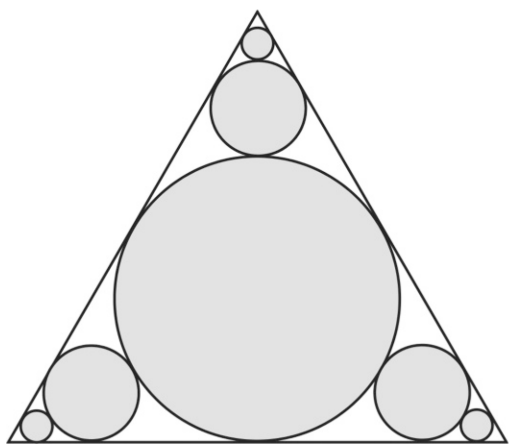
1. Consider the shaded square in the diagram below and assume that shading goes on indefinitely.



G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 217–219.

Assume the square is one inch square. How much of the total area of the square is shaded? Thoroughly justify your thinking/explanation. This issue in this problem is not finding the answer ($A_{shaded} = \frac{1}{3}$), but explaining that the answer is correct.

2. Given any number x less than 1. Argue that the repeated decimal $0.9999... = 0.\overline{9}$ is greater than x.
3. Your friend has a new cat which he claims is $\frac{1}{6}$ Yellow Himalayan, which comes from selective breeding of the Yellow Cheshire and the Blue Himalayan cats. Is this possible? How many generations must be required to achieve a 1/6 mixed breed? How does this differ from the claim that the cat is, say $\frac{15}{16}$ Yellow Himalayan?
4. An equilateral triangle has a circle inscribed whose radius is known to be $r = 2$. In the picture below, you are to determine the area of the shaded region, assuming an infinite sequence of circles converges to each vertex. The questions below pertain to the totality of the (infinity) of circles.



a. Predict whether the area is finite. Of course, you can compute the area by infinite series and determine if it is finite.
b. Show whether the series converges.
c. Determine the area using ideas of similarity.
d. In what way does the area depend on the initial radius, if at all?
e. What is the sum of the radii? Determine the area and sum of the circumferences.

POSITIVE CLASSROOM PRACTICES

- High quality power point presentations with appealing and appropriate graphics or well prepared and clearly delivered lessons

- Attention grabbing introduction to lesson/activity
- Review of previously taught skills needed for successful participation
- Showing a relationship of lesson/activity to real life situations
- Demonstrating a caring and personal interest in students and their lives outside the classroom
- Demonstrating fairness in calling on students for responses
- Planning hands-on lessons to actively involve students
- Allowing students a glimpse into one's own personal life
- Recognizing that certain days/weeks aren't prime times for instruction/learning and gearing the lesson accordingly (e.g., days right before holidays and Friday afternoons)
- Showing enthusiasm for the subject and the lesson
- Monitoring activities to ensure success for everyone
- Providing clear and concise instructions
- Having all materials ready before each class period
- Closing the lesson rather than ending abruptly when bell rings
- Establishing a comfortable rapport and working relationship

DIANNE GOLDSBY

47. A MILE OF PENNIES

There are 84,480 pennies in a mile. Yes, that is correct – 84,480 pennies! An interesting fact to know: 16 pennies laid side by side constitute 12 inches in length, or one foot, and since 5,280 feet equals one mile, that gives us 84,480 pennies in a mile.

5,280 feet of pennies = 84,480 pennies = math teaching/learning opportunities

Today, approximately 150 billion pennies of the 288 billion in circulation are estimated to be in use. The rest are in jars and sacks, made into jewelry, rest in loafers, or thrown into fountains. Notably so, pennies can be collected for charity or school projects and used to teach math concepts. For example, pennies can be utilized to practice and review measurement, area, volume, circumference, decimals, and fractions. They are an inexpensive and readily available manipulative.

Some interesting *penny* (i.e., one-cent coin) *facts* which can serve as springboards to activities are:

- The penny weighs 2.5 grams.
- It is 19 mm in diameter and 1.55 mm thick.
- The composition is 97.5% zinc and 2.5% copper.
- The mint has produced over 288.7 billion pennies.
- Approximately 1,040 pennies are produced every second (30 million a day).
- From January–April of 2009, about 864.4 million one-cent coins were produced.
- A typical mint bag of pennies contains about $4,000 worth.
- The first one-cent coin was made of pure copper and was larger.

Note. There has been much discussion about whether or not to continue making the penny. A bill was proposed in 1989 to phase out the penny by requiring all cash transactions to be rounded to the nearest five cents. The group, Americans for Common Cents, was founded to actively oppose and defeat the legislation. The bill was defeated. In 2006 when zinc prices were at record highs, the bill was introduced again but defeated. Just think of all the math you could practice with that bill!

Some *specific activities* include:

- Find the area of a penny.
- Find the volume of a stack of pennies.
- Estimate the number of pennies it would take to circle the earth.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 221–222.

- If it costs about 0.81¢ to make a penny, what would it cost to make a mile of pennies? If the inflation rate is 5% per year, how much would the mile cost next year? In three years?
- Estimate the number of pennies in a quart jar.
- How many times would the pennies minted circle the earth if put side-by-side?
- How much would the mint shipping box of pennies weigh? What could the dimensions of a box be, in order to hold these?
- What would the original penny cost today to mint?

The first Lincoln penny was introduced in 1909 and replaced the Indian-head cent. In 2009, the new Lincoln penny was introduced. In 2005, President Bush signed a bill to mandate a commemorative Lincoln penny for Lincoln's 200th birthday. The new coin has 4 new designs highlighting parts of Lincoln's life. A special version of the new one cent coin contains the metals used in the original 1909 cent – 95% copper and 5% tin and zinc. (What is that coin worth?)

If you're in need of a mathematics school project, consider having a "Mile of Pennies" campaign to collect pennies for charity or the school. The Mu Alpha Theta chapter (at a school where I previously taught) collected pennies for math manipulatives for various classes. Each year, students asked when the collection would start again. Classes competed for pizza parties or chocolate chip cookie cakes (donated by the local pizza place and chapter members). Math activities were incorporated into the collection and were really enjoyed by all!

Resources for this topic include the US Treasury website (http://www.ustreas.gov/education/) and the US Mint website, which has lesson plans for grades 6–8 math and science integration (http://www.usmint.gov/kids/teachers/coinCurricula/01centCoin.cfm).

G. DONALD ALLEN

48. EASY PROBLEMS

I am reading Daniel Kahneman's remarkable book, *Thinking, Fast and Slow* (Farrar, Straus, & Giroux, 2011). The broad theme of this book is that people are intuitive thinkers, but in many cases the intuition is both wrong and biased. He takes the brain and divides it into two systems, called System 1 – the intuitive and fast part of your brain, and System 2 – the more deliberate and analytical part. He gives numerous examples of how the one system overrides the other, when System 1 seems to rule and when System 2 does. I recommend this book to you, with enthusiasm.

What I want to highlight is an example from the book, a sample math problem that is totally simple and most all would get it right, if not for the impetuous System 1. Here's the problem. Let your intuition figure it out. Be quick!

A bat and ball cost $1.10.
The bat costs one dollar more than the ball.
How much does the ball cost?

A number came into your mind; it was 10 as in 10 cents. The answer seems correct, and not needing further analysis by that drudge upstairs, System 2, you report it in. But you are wrong. You intuition jumped in there quickly and you agreed with it. In fact, the answer is 5, as in 5 cents. Check it out. (The solution is shown below.)

$x + y = 1.10$

$x = y + 1$

Use substitution.

$y + 1 + y = 1.10$

$2y = 1.10 - 1$

$2y = 0.10$

$y = 0.05$

The upshot about this really simple problem is that if you give such problems to your students, their System 1 will ferret out a reasonable response quickly, and then applying the *principle of least effort*, neglect to call on System 2, for confirmation. Make it too simple, and you may find too many wrong answers on the quiz.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 223–224.

This question is not new. Many thousands have taken it. The shocking result is that more than 50% of Harvard, MIT, and Princeton students gave the intuitive but incorrect answer, 10. At less selective institutions, the percentage incorrect jumped to the 80% range.

As Kahneman notes, “Many people are overconfident, prone to place too much faith in their intuitions. They apparently find cognitive effort at least mildly unpleasant and avoid it as much as possible.”

AN EQUATION FOR LIFE

An important equation. Let A = (your) attitude, P = (your) persistence, and I = (your) intelligence. In terms of your quantified versions of these

$$A_q + P_q \geq I_q$$

Your attitude quotient plus your persistence quotient exceeds your intelligence quotient. This simple inequality demonstrates your attitude and persistence applied to your life can have more effect than your native intelligence. Indeed, look around you. Note how some people, perhaps less intelligent, in your mind, have achieved what you desire. Now look at their attitude and persistence. That’s how they did it. Indeed, I believe in many cases

$$P_q \geq A_q$$

The lesson here is not to give up because you think you don’t have the smarts. Keep at it with a strong and positive attitude, with a relentless spirit for achievement.

AMANDA ROSS

49. MATHEMATICAL GAMES AND LEARNING

Learning games are available for every subject, at every grade level. These games may be online or tactile. Some are good, and some are not as good. Some learning games are free, while others cost money. This article will examine mathematical games and their impact on learning.

Let's consider some guiding questions, when thinking about the role of mathematical games, both inside and outside, the mathematics classroom. Such questions include:

- How will mathematical games impact student learning?
- What are the characteristics of good mathematical games?
- Do they increase level of engagement, and if so, how?
- Are online games better than tactile board games or those created by the teacher and/or student?
- When is the optimal time for playing mathematical games? Before a lesson, during a lesson, after a lesson, or after school? How many minutes per day should be allotted for play with mathematical games?
- How should mathematical games be incorporated into the mathematics classroom? What should be the role of the teacher, during this time?
- What follow-up activities should be used to accompany mathematical games?
- What technological devices are available for use?

This article will touch upon each of these questions. Note that the ideas are that of the author. So, this article serves as a current position statement on the usage of mathematical games and their impact on learning. Also, note that in order to accurately assess each of these questions, a valid research study should be conducted. Qualitative and/or quantitative data should be gathered and analyzed, resulting in conclusions that may be applied to the population, from which the sample was drawn.

IMPACT ON LEARNING

Mathematical games, both online and tactile, will likely have a positive impact on student learning. The success of such utilization heavily depends upon the degree of usage and how the games are used, in relation to classroom instruction and activities. If used properly and as a supplement to, or component of, classroom instruction or learning modules, mathematical games will reinforce understanding and allow

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 225–228.

for self-assessment of understanding of content. It also goes without stating that teachers should select good, strong mathematical games.

CHARACTERISTICS OF GOOD MATHEMATICAL GAMES

The denotation of good mathematical games is subjective, of course. Some teachers may view one set of games as the most effective, while other teachers may view another set as better. However, it is most likely that teachers will agree upon which games are good and which are not. A brief list of characteristics of good mathematical games is shown below:

- Valid and reliable, in design
- Provides instant and accurate feedback to students
- Provides sound effects for correct and incorrect answers
- Provides opportunities for additional attempts
- Includes a variety of item types or applets
- Offers questions at differing levels of difficulty
- Includes content that is aligned to mathematical standards, for given grade levels
- Designed according to specific learning objectives/outcomes
- Includes animation and real-world content, pertinent to students' interests

IMPACT ON LEVEL OF ENGAGEMENT

Mathematical games can increase engagement, if they are designed to be more than the rote answering of questions. If the content is too easy and does not increase in difficulty, the student will inevitably get bored and be less engaged than if doing some other activity, such as a performance task; this goes for both online and tactile games. However, if designed properly, such games will offer students an added layer of mathematics content, with which to engage, which will increase level of engagement. When interfacing with online games, students will also be given the opportunity to work with different technological devices, which will likely be quite appealing.

ONLINE GAMES VS TACTILE GAMES

Online and tactile games may both positively impact students' mathematical learning. Online games are more convenient and may be accessed anywhere and anytime. Tactile games are more appropriate for classroom usage. Tactile games do have many benefits that online games do not. For example, tactile games allow student to touch and handle and move objects and shapes. There is a lot of benefit to working with an actual balance scale, when learning about solving equations in one variable. The students get to manually move the objects and see how the scale adjusts. Students can certainly create their own games via manual construction or programming, as with a calculator. All in all, as long as the games are engaging and well-designed with specific learning objectives in mind, they will have a positive impact on students.

THE MATTER OF TIME

Mathematical games can actually be played at different times, throughout the learning process, and deliver positive, albeit different, results. Whether a game is played before a lesson, during a lesson, after a lesson, or after school, it will likely solidify current understanding and result in increased and deeper understanding of the mathematical content. The timing of when the student interacts with the game will help the student, in different ways, however. It is actually good for the student to interact with mathematical games, at different times, during the learning process, in order to be most successful with fully understanding the mathematics content. For example, playing a game before a lesson will focus the lesson and provide the student with a self-assessment opportunity. Once the lesson commences, the student may then ask the teacher any questions he or she has, regarding content not understood during the game. Playing a game during a lesson is a type of formative assessment that serves as a check for understanding, akin to a classroom response system, or clicker system. Playing a game after a lesson, or after school, will provide additional practice on mathematics topics, opportunities to challenge and extend understanding, and engage with higher-level problem-solving, related to the learning objectives.

The optimal number of minutes of play per day is a subjective matter. The main component to consider is that the gaming time should supplement, not replace, the instructional time. Time allotted for lesson and module instructional content should be primary, with time spent on online games and other games being secondary. Having games interspersed throughout an online mathematics module is an optimal situation. These interspersed games serve as great checkpoints or formative assessments.

INTEGRATION INTO THE CLASSROOM

This idea was touched upon, in the section above. Games should be used to solidify understanding, extend curriculum, challenge students, provide self-assessment information, and engage students. Games may be used to focus a lesson, formatively assess student understanding, and summatively assess student understanding. The teacher should serve as facilitator, during this time. The teacher should be available to answer questions, regarding content and design.

FOLLOW-UP ACTIVITIES

There are many activities that may be used to follow the play of a mathematical game. For example, students may be given a survey to determine how their understanding of the content has changed or improved. On the survey, students may also discuss any aha moments or areas of confusion, discovered via interaction with the game. The teacher might also give students the opportunity to create a different game that assesses the same learning objectives, as those encountered during the game(s) played. For example, students might write a program for a game for a graphing calculator. The game might assess an understanding of functions and outputs. Students could think about the game played, evaluate its strengths and weaknesses, think of modifications, and then employ those changes, in a self-created game. Game creation, either tactile or programmed, would certainly deepen students' understanding of a mathematical content area.

TECHNOLOGICAL DEVICES

Mathematical games can be played on many different technological devices, including laptops, smart phones, iPads, other tablets, and graphing calculators. Games can even be played on Smart Boards. The use of a Smart Board allows all students to see the game and be involved in the thinking process/discussion. The opportunities are many, and learning is at the touch of a student's fingertips.

SECTION 9

TEACHING TIPS AND RESOURCES

G. DONALD ALLEN

50. TWO GREAT SOFTWARE TOOLS (WINPLOT AND MATHTRAX)

For those of you teaching in the schools or colleges, here is some great software for doing math graphics. Both are easy to use and free.

Winplot. WinPlot is a very easy to use for general-purpose plotting utility, which can draw (and animate) curves and surfaces presented in a variety of formats. You can paste your work directly into PowerPoint or Word. The user interface is very intuitive; you'll be up and running applications in minutes. http://math.exeter.edu/rparris/winplot.html

MathTrax: MathTrax is a graphing tool for middle school and high school students to graph equations, physics simulations or plot data files. The graphs have descriptions and sound so you can hear and read about the graph. Visually impaired users can access visual math data and graph or experiment with equations and datasets. http://prime.jsc.nasa.gov/MathTrax/

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 231.

SANDRA NITE

51. USING POLYA'S PROBLEM SOLVING PROCESS IN THE MATHEMATICS CLASSROOM TO PREPARE FOR TAKS

Do you and your students need a boost to get ready for TAKS and optimize student performance? Here is an idea that can help. In working with high school students, we often find that they have a considerable amount of knowledge about algebra and geometry, and they also know a lot of the strategies for working through problems. What they do not know is how to pull it all together to use a problem solving process successfully. We found that they often do one of two things wrong: (1) immediately start carrying out a plan before they have fully devised it and bubbling the first answer choice that has a number they find in combining numbers in the problem, and (2) when a problem seems complex either because it has a lot of words or because it does not seem straightforward and similar to what they have done before, they skip it.

We have found that working with students using Polya's four-step problem solving process can boost students' confidence and performance on the TAKS. First, we discuss the steps in the process and put them on the board, a poster, or individual handouts like the one below.

Problem Solving

1. Understand the Problem.
Write it in your own words; identify the unknowns; write information from the problem

2. Devise a Plan.
Decide on one or more strategies to use

Some Strategies to use When Devising a Plan:
Look for a Pattern
Make a Table
Guess and Check
Work Backward
Write an Equation
Use Indirect Reasoning
Break it Down—Examine a Related Problem, Simpler Case, or Subgoal

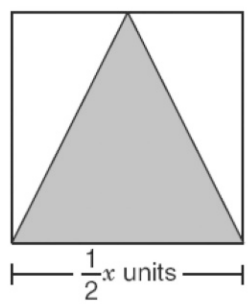
3. Carry out the Plan.
Follow through on the strategies, keeping a good record of your work.

4. Look Back/Check.
Check the results in the original problem; interpret the solution in terms of the original problem to see if it makes sense and answers the question asked

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 233–235.

Then, we facilitate practice on released TAKS items by guiding questions and helping them learn to ask themselves questions to dissect the problem and solve it. In working with them, we have to keep focusing back on “Understand the problem” because they want to jump to the next step before they have thoroughly explored the problem. Consider the problem below from Grade 10 TAKS 2006, which only 43% of students in Texas answered correctly:

43 A triangle is inscribed in a square, as shown below.



What is the area of the shaded triangle inscribed in the square?

A $\frac{1}{4}x^2$ units^2

B $\frac{1}{2}x^2$ units^2

C $\frac{1}{8}x^2$ units^2

D $\frac{1}{16}x^2$ units^2

Some guiding questions in understanding the problem (with desired response in brackets) could be:

1. What does the question ask us to do? [find the area of a triangle]
2. How do we find the area of a triangle? [one-half base time height]
3. Do we know the base? [Yes – ½ x]
4. Do we know the value of *x*? [No]
5. Do we need to know the value of *x* to answer the question as it is here? [No]
6. Why not? [All the answer choices still have *x* in them.]

7. So what will we use in the formula for the base? [½ x]
8. Where is the height? (have someone draw it or describe how to draw it)
9. What is the figure the triangle is inscribed? [square]
10. What do you know about a square? [all sides congruent]
11. Do we know the height of the triangle? [Yes]
12. How do you know? [It is the same as the length of the side of a square.]
13. What is the height of the triangle? [½ x]
14. Now do we understand the problem well enough to devise a plan?

At this point, the plan is devised and carried out. Then we check and look back to see if it makes sense. Here we have focused on the first step in the process because we have found it to be the critical part for students to master and the step they usually skip. The idea is to continue to ask questions to draw out responses from students, never telling them how to do it. In this way, they learn what to think about and how to ask themselves questions to break down the problem. We check to make sure all students understand as we go through, having them explain to each other as necessary. Whenever possible, we ask for alternative methods for solving the problem. We try to exhaust every possibility so students will have a lot of tools at their disposal when solving problems on TAKS. We have found this approach to be very successful in increasing student confidence and ability to perform well on TAKS.

G. DONALD ALLEN

52. PROBLEM-SOLVING STRATEGIES – A QUICK CHECKLIST

Many challenges confront both the new and experienced teacher. They include the old standbys such as classroom management, discipline problems, curriculum inadequacies, administration issues, parent involvement, and more. Newer ones include particular teaching with technology. For the mathematics teacher, the pedagogical issues from constructivism to skill and drill seem to keep coming back – swinging like a pendulum. Should the student learn to construct their own knowledge or to become an expert practitioner of established knowledge and algorithms? If a student has a misconception about say fractions or algebra, how does the teacher overcome it? All of these issues and problems are background noise when the teacher must show students the techniques and art of *problem solving.* Problem solving is the centerpiece, the common denominator, the root, and the underlying foundation of the entire mathematics curriculum.

In this short article, we will consider it in a general way. We will consider strategies for understanding, simplifying, and solving problems. Important for every student is the creation of a mental environment where problems can be solved. This varies by student.

We'll proceed by giving the strategies as a checklist, to the point and easy to remember. To the beginning teacher: Teaching problem—solving can be among the most difficult of our many missions as math teachers.

STRATEGIES TO UNDERSTAND THE PROBLEM

- Clarify the problem – Do you understand what you are to solve in the context of the problem setting?
- Identify key elements of the problem – What are the important facts, their interdependencies, and there connections?
- Draw a picture or diagram of the problem – Is it possible to draw a picture or diagram of the problem situation? It is possible to draw a diagram of how the data/parameters of the problems are connected?
- Consider specific examples – If there are variables in the problem, does the problem make more sense or become clearer by examining specific examples?
- Consider extreme cases – Have you considered all possible cases? What are the extreme cases in the problem beyond which the problem becomes meaningless?

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 237–239.

STRATEGIES TO SIMPLIFY THE PROBLEM

- Simplify the problem – Have you ordered carefully the important aspects of the problem and made the problem more transparently simple? One way to do this is reframe or rewrite the problem in your own words.
- Solve one part of the problem at a time – In multi-part problems, can you solve just one part first? Can you see in the problem the order in which the smaller parts needs to be solved?
- Rewrite the problem in other terms – Can you recast the problem using alternate or other terms? This helps clarify just what is there. Do you understand the problem so you can explain it to anyone?

STRATEGIES TO SOLVE A PROBLEM

- Reason by analogy in using what you have learned about similar problems – How much problem solving experience do you have? Can you use reasoning from another problem on this one? Experience is absolutely important. No one can be a good problem solver without experience.
- Use deductive and/or inductive reasoning – What type of reasoning works for this problem? Inductive reasoning is not rigorous but is intuitively helpful. Reasoning by analogy is very helpful in problem understanding.
- Question your assumptions – Are your assumptions correct? Valid? Necessary?
- Guess, check, and revise – Have you guessed what the answer should be? Did you check your answer for reasonableness? Do you need to revise any of your assumptions, simplifications, or variables? Did you do a check on the "units" in the problem? For example, if the answer must be measured in meters, does your answer give meters?
- Work backward – Did you work backwards? This is difficult to teach.

Finally, what workplace environment is most conducive to solving problems? How can your students operate at their peak and be successful?

STRATEGIES TO YOUR PEAK PERFORMANCE

- Consider alternatives – Your first strategy may not work. Some problems are written so the "obvious" strategy doesn't work. Look for alternatives.
- Avoid distractions – Distractions can occupy just that part of the mind that is working on solving the problem. Do you know what your distractions are?
- Change your work setting, but be comfortable
- Think clearly and confidently – It is always easy to say this, and with practice you can do both.
- Take a break – Didn't solve the problem right away? Don't worry; your mind is still working on it. Taking a break can help you get out of a "problem-solving rut."

- Be Persistent – another key to success – *Don't underestimate persistence. Don't give up. For the teacher: don't give hints too quickly for they can defeat the learning process.*

A basic question for the teacher is which of these can be taught, and which are those a student develops independently? This varies by teacher. Always note the two key strategies: (a) experience in problem solving, and (b) persistence at problem solving.

DONNA LACKEY

53. WHAT IS MENTAL MATH?

When we discuss the value and importance of students being able to do "mental mathematics," what is it that we actually mean? I am not sure that there is a single "official definition." However, here are some thoughts about my perception of the subject. Mental math is about doing paperless mathematics, but not all paperless mathematics is really "mental math." Mental math is about being able to visualize the problem, using a "traditional" approach in your head, as well as other methods. Mental math is about using creative techniques based on understanding of concepts and acquired knowledge. It is about "thinking outside the box" for a non-traditional approach to a traditional problem – an approach that can easily be done mentally. Mental math can include, but is certainly not equivalent to estimation.

WHAT DO STUDENTS NEED IN ORDER TO PERFORM MENTAL MATHEMATICS?

In order for students to be able to do "mental math," they must first have a full understanding of the mathematical concepts related to the problem they are solving. This understanding could include such concepts as place value, fraction and decimal representations, and construction of geometric figures. Second, students need exposure to activities that will help them develop mental images that they can reference. Examples of this might include work with dot patterns (such as on dominoes) and other drawings and diagrams, as well as various manipulatives, such as fraction bars and fraction circles. They would benefit from work with solid geometric figures, experience with various measuring devices (cups, quarts, gallons, 12" ruler, yardstick), and experience with clock faces. Third, in order to do mental math, students need to know the "math facts." These would include addition, subtraction, multiplication, and division facts, as well as knowledge of many square number and square root facts. A student that does not know most of these facts is going to have a struggle with mental math (just as they will struggle with paper math, unless they have a handy calculator).

WHAT ARE SOME EXAMPLES OF "MENTAL MATH" PROBLEMS?

Example 1: $2 - \frac{3}{4} = 1\frac{1}{4}$

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 241–243.

Solution: A problem such as the one above can be answered mentally, based on an image and understanding of the concepts of fractions, i.e., being able to picture the two whole circles and realizing the amount left when ¾ of one is removed. It is not really mental math if you simply have a talent for picturing the entire problem worked out in your head in a more routine, traditional fashion such as: $2 - \frac{3}{4} = \frac{8}{4} - \frac{3}{4} = \frac{5}{4} = 1\frac{1}{4}$.

Example 2: "It is 9:20. John reads for 1 hour and 35 minutes. What time does he finish reading?"

Solution: The answer acquired through mental math could include visualization of the clock face, with an hour later being 10:20. The process might continue with the image of the minute hand on the four, proceeding with visualization of the passing of 30 minutes as the minute hand moves around to the ten, and then passage of five more minutes, putting the minute hand on the eleven. The new time is pictured to be 10:55. It is not really "mental math" for someone to picture and do a traditional columnar time addition problem in his head, mentally working out:

	9 hours	+	20 minutes
+	1 hour		35 minutes
=	10 hours	+	55 minutes

The ability to do this addition problem mentally, without pencil and paper may be praiseworthy, but still not really what we hope "mental math" really represents.

Example 3: 57 + 49 = 106

Solution: To add 57 and 49, one might be able to do the columnar adding "mentally" without pencil and paper. However, again, that is not what we really hope for. What we would really like to see is a creative technique such as the following reorganization:

$$57 + 49 = 57 + (50 - 1) = (57 + 50) - 1 = 107 - 1 = 106$$

Being able to do various kinds of mental math is very important for students to be truly proficient and successful in mathematics. In addition to problem-solving skills, we must model and teach mental math. We must start early and help students develop their mental math skills through every grade level. At the beginning of this article, I gave some of my perceptions about what mental math is and what it is not. However, to get obsessive over or quibble over the exact definition of "mental math" will not serve any good purpose. If you can identify something as a worthy mental (versus paper) skill for students to have, then teach it!

WANT TO TRY A FEW MENTAL MATH PROBLEMS?

Attempt these without pencil and paper, and use something other than a traditional approach. Look for creative approaches.

Add this group of numbers:

\$0.52 \$3.90 \$5.40 \$4.60 \$0.67 \$0.48 \$9.33 \$3.10

Add: 34 + 28 + 49 + 52 + 21

Add: 4246 + 550 + 3244

Subtract: 329 – 134

Add: $4\frac{2}{3} + 5\frac{1}{4} + 2\frac{1}{3} + 7\frac{3}{8}$

Subtract: $32\frac{1}{2} - 7\frac{5}{8}$

Multiply: 17 x 25 =

Multiply: 16 x 75 =

Multiply: 24 x 26 =

Divide: 5628 ÷ 4 =

TINGTING MA

54. A DIGITAL SIMULATION FOR PROSPECTIVE TEACHERS TO TEACH ALGEBRA CONCEPTS FOR EQUITY

INTRODUCTION

In this article, we describe the exploratory work of developing a simulated middle school algebra teaching environment in *Second Life (SL)* in the first phase of a 5-year NSF- funded project. The project aims to design, develop and test technology-enriched teacher preparation strategies to address equity in algebra teaching and learning for all students. This study aims to explore the design and implementation of a digital simulation for prospective teachers to teach algebra for equity, from the perspectives of Graduate Research Assistants (GRAs), as middle-grade student avatars.

SL, a multi-user interactive virtual environment, has been documented as having great potential for providing interactive, effective and enjoyable teaching and learning opportunities (Bransford & Gawel, 2006). *SL* offers exciting opportunities for simulated teaching and learning in teacher education programs. The affordances provided by *SL* for education include communication, experience (Hew & Cheung, 2008), scaffolding, and professional development (Cunningham & Harrison, 2010).

THE SETTINGS: WHEN, WHERE AND WHO

The project is embedded within a semester-long "Problem Solving in Mathematics" course at a Southwestern university, involving seventeen prospective teachers at the junior or senior level, majoring in middle school math and science education. At the beginning of the semester, *SL* orientation was provided for the prospective teachers to learn and practice basic tools and skills in *SL* and construct their own teacher avatar. Towards the end of the semester, the prospective teachers were asked to prepare and submit a 15-minute lesson on proportion, slope, ratio or rate, to be taught in *SL.* These topics were identified as important algebraic ideas in the middle grade mathematics curriculum (AAAS, 2000; Kulm & Capraro, 2008). The lessons were then assigned to GRAs, within a team, to review and identify possible algebraic misconceptions and to prepare to act as a diverse group of middle grade student avatars.

To support their work, the GRAs attended weekly project meetings, where they discussed the objectives and hypotheses of the study, identified research-based algebraic misconceptions, attended *SL* training seminars specifically designed for them, and constructed their own avatars to represent a group of diverse middle

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 245–248.

grade students, with specific social-cultural characteristics and algebraic knowledge (Brown et al., 2011; see Figure 1). After reviewing the lessons submitted by the prospective teachers, GRAs attended a pre-simulation planning meeting to discuss how to interact with prospective teachers in *SL* as student avatars.

Figure 1. A research team meeting held in a SL conference room

The prospective teachers presented their lessons over two 90-minute class periods (see Figure 2). Each GRA acted as a student avatar in at least nine of the lessons. After the lesson presentations, the GRAs were asked to reflect on their experience of role playing middle grade students in *SL* and to complete a survey, composed of open-ended questions. A follow-up focus-group interview was held and attended by the GRAs to further explore their reflections in a more in-depth way.

Figure 2. An episode of teaching simulation in SL. A teacher avatar (a prospective teacher) was teaching her lesson on proportions for a group of diverse middle grade student avatars (GRAs) in a SL classroom

PREPARATION AND DESIGN FOR TEACHING ENVIRONMENT SIMULATION IN SECOND LIFE

A major component of preparing for and designing the simulation in *SL* were the pre-simulation planning meetings. During the meetings, each GRA led discussions about their assigned lessons; a mathematics education professor contributed to the discussion.

For example, a prospective teacher provided a scenario of a dartboard game in an amusement park in her lesson "solving for x in a linear equation":

> Mark went to the amusement park and was given $10.00 to spend on snacks and games. The dartboard game is his favorite. It costs $1.00 for 3 darts and each additional dart is 25$. If he wants to spend no more than $5.00 on the darts game, how many darts will he get?

The prospective teacher used the equation, $y = 1 + 0.25(x–3)$, to solve the problem. While reviewing the lesson in the meeting, GRAs raised the following question: "What if x is less than 3 for $y = 1 + 0.25(x–3)$?" It is clear that the prospective teacher did not have a clear understanding of functions. The research team decided to challenge the prospective teacher by asking a specific question during the *SL* simulation: "If Mark buys nothing, why does he still have to spend 25 cents?" It was expected that the prospective teacher could explain that *x* must be greater than or equal to 3 or in the context, Mark must buy at least the first 3 darts. In addition to the question on algebraic knowledge, the research team decided to design and assign a challenge on equity. Another GRA stated, "I have never played dartboard. I don't know what a dart game is." the statements were made to see whether... the prospective teacher could take the equity issue into account.

The meeting addressed some challenges in the simulation, helped GRAs get well prepared, and to some extent, led to effective interactions with prospective teachers in the simulation. The participants provided reflections on challenges, procedures and suggestions for improvement. Suggestions to better address the challenges included identifying algebraic misconceptions from literature and real middle grade mathematics classes and assigning specific algebraic misconceptions to each GRA prior to the simulation. In addition, participants commented that the time allocation for simulated *SL* teaching should be extended.

In order to address the challenge of allowing prospective teachers to know their student avatars better, one participant suggested, "give the pre-service teachers a bio of every [student] avatar including misconception types, cultural background… displaying in text or in video…so they can know who we are and know the misconceptions that we have to challenge them in their *SL* teaching." Another participant shared a similar view. What the participants suggested is to not only construct student avatars with characteristics in misconceptions and culture, but also provide this information for prospective teachers before they teach the student avatars in *SL*.

This study explores the design and implementation of a simulated teaching environment in *SL* for prospective teachers to teach algebra for equity, from the perspectives of GRAs. As an initial effort, this study endeavors to help address one of the most critical questions about how to best prepare teachers (e.g., using approximations of practice) (Grossman et al., 2005; Grossman & McDonald, 2008). Future studies are required on the digital simulation of a teaching environment.

REFERENCES

American Association for the Advancement of Science [AAAS]. (2000). *Middle grades mathematics textbooks: A benchmarks-based evaluation*. Washington, DC: Author.

Bransford, J., & Gawel, D. (2006). Foreword. In D. Livingston & J. Kemp (Eds.), *Proceedings of the first second life education workshop* (p. iii). Paisley: The University of Paisley.

Brown, I. A., Davis, T. J., Lewis, C. W., & Kulm, G. (2011). Preservice teachers' knowledge for teaching algebra for equity in the middle grades: A preliminary report. *Journal of Negro Education, 80*, 266–283.

Cunningham, C. A., & Harrison, K. (2010). The affordances of second life for education. In G. Vincenti & J. Braman (Eds.), *Multi-user virtual environments for the classroom: Practical approaches to teaching in virtual worlds* (pp. 94–119). Hershey, PA: IGI Global.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2005, April). *Teaching practice: A cross-professional perspective*. Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada.

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal, 45*(1), 184–205.

Hew, K. F., & Cheung, W. S. (2008). Use of three-dimensional (3-D) immersive virtual worlds in K-12 and higher education settings: A review of the research. *British Journal of Educational Technology, 41*(1), 33–55.

Kulm, G., & Capraro, R. M. (2008). Textbook use and student learning of number and algebra ideas in middle grades. In G. Kulm (Ed.), *Teacher knowledge and practice in middle grades mathematics* (pp. 255–272). Rotterdam, The Netherlands: Sense Publishers.

G. DONALD ALLEN

55. TEACHING IS A BALANCING ACT

In teaching or learning a new topic, or designing a curriculum, decisions must be made. How much time is available for the learning tasks? What is the scope of what is to be learned, i.e., the content and curriculum? Just as important is to teach or learn, according to a balance between *understanding, procedure, and skills*. These can be divided by time and effort devoted to each. As shown in Figure 1, all three are shown as equally balanced.

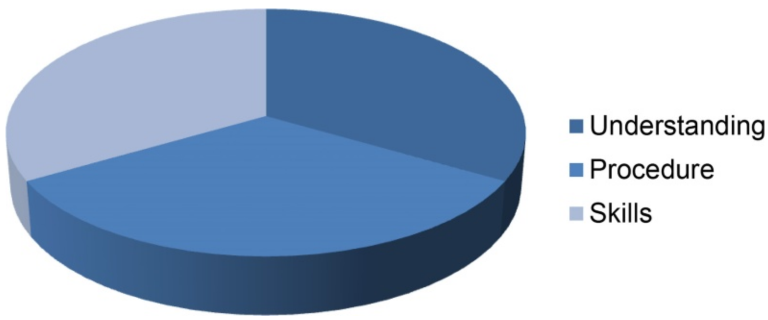


Figure 1.

To some extent, each of these is decided by the teacher. Each of these is important, in a competitive way. Avoiding one or two of them, and essentially nothing of value can be learned. As a math professor, I often hear the lament, "I understand the material; I just can't solve the problems." Imagine if your doctor says something similar, "I understand you have an intestinal problem, I just don't know how to treat it." Or the airline pilot who says, "I have my pilot's license, but I've just never flown this airplane." In each of these, the learning and likely teaching was not balanced, as it should have been.

We hear of teachers who can solve fraction problems (procedure and skill), but can't explain what they've done and why. We hear of students who can multiply 12 × 13 (skill), but don't understand what it means. These and countless more examples stimulated and energized the emergence of inquiry-based learning, sometimes referred to as discovery learning or constructivism. This pedagogy is child-centered, with the teacher becoming a facilitator. Special pedagogies, e.g. 5E method, have been developed for this.[1] Inquiry-based learning means the student

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 249–252.

more or less discovers his or her own knowledge as he or she learns – with guidance from the teacher. It is a powerful technique, but not for the faint of heart. It requires the teacher to have a consummate knowledge of the subject, to be dedicated to student learning, to have patience to let the student explore and learn, and to have the class time to allow for such discovery. It requires a manageable class size. The key point here is the need to engage the student in his or her own learning.

Inquiry is not good for everything. For example, in math, it is important to teach procedures and skills. Attempting this by inquiry can bog down the student into a confusing morass of half-baked, even incorrect, procedures, and little or no skills. In the USA, there was a concerted attempt to use inquiry for almost everything, even to the point of allowing students to develop their own algorithms for skill-tasks such as multiplication. The result was a host of stories about students who just couldn't do anything. Fortunately, the US educational system has recognized the importance of procedures and skills. Indeed, we now often see research papers, where conceptual understanding and procedural skills seem to complement one another.[2] We conclude in teaching and learning…

> There must be a balance between the big three: understanding, procedures, and skills. Learning is not something only for now; it is for the long term.

It is curious and important that the mind works toward understanding and to solve problems in a variety of ways. For the long term, the conceptual ideas, procedures, and skills must tap into all aspects of thinking and learning.

Focus on Math. It is now widely accepted that the big three: understanding, procedures, and skills should share prominent roles in learning new things, and therefore the teacher should give balance to all three, not simply teaching skills alone, or any of the others.

Now imagine a medical school based completely on the inquiry method. Medical students learn by inquiry and discovery about the human body, organs, and their functioning, discovering their own medical procedures and developing their own skills. Of course, no one wants this doctor. Imagine a medical school where the only teaching is the treatment of dozens of diseases, with nothing of why or how they occur, or how treatments interact, or consequences of one treatment over another. The doctor will do something, but will not know why. You probably, you don't want this doctor either.

Problem Solving: Problem solving is one of the key reasons to learn mathematics. All individuals, no matter their profession, are concerned with problem solving throughout their life. The principles outlined here apply. First, we look in the abstract.

Understanding: To see or understand what the problems is, what is it means, why it should be solved, and how the problem components interrelate.

Procedure: What procedures should be used to solve the problem, set up the key equations, and prepare the problem for solution?

Skills: To carry out procedures. Often, there are several procedures one can follow. Skills help with selecting the best procedure.

Let's consider a few examples. Problem solving is the cornerstone of just about every profession, from science to accounting, to practice of the law. This applies to politics, medicine, law, engineering, shop-keeping, and even the trades, such as brick-laying, plumbing, and the rest. For some, the common term "practice" is used. Even the word "practice," as applied in the legal and medical professions means the application of "procedures and skills," with the understanding of components apparently a given. Now, let's focus on problem solving and what the triad of understanding, procedure, and skills means in particular.

A. Mathematical optimization problems
Understanding: To see or understand what the problems is, what is to be optimized, and how the problem components interrelate.

Procedure: What procedures should be used to solve the problem, to set up the key equations, and to make ready the problem for solution?

Skills: To carry out procedures. In this component, the skills to handle in an efficient manner all that needs to be done, whether to use calculus, graphing, tables, etc.

B. Adding/Multiplying Fractions
Understanding: To understand what a fraction means, through the various models (area, set, linear), to understand equivalent fractions, to understand that fractions are numbers and any of the regular mathematical operations apply. By the way, there are multiple aspects of fractions students must learn. These include: Basic Fractions, Equivalent Fractions, Adding Fractions, Subtracting Fractions, Multiplying Fractions, Dividing Fraction, Comparing Fractions, Converting Fractions, Reducing Fractions, and Relationships. No wonder they are difficult for students to learn.

Procedure: For addition, it is important to teach the procedure of determining equivalent fractions and then adding the numerators. For multiplication, the procedure is easy, but understanding the foundation or reasons why it works, comprises the big issue here. Remember, without the understanding, these procedures will ultimately not be retained long term.

Skills: To carry out procedures. In this component, the skills to handle in an efficient manner all that needs to be done. Remember this: While emphasizing skill and drill, students may learn to perform well on these operations. However, related to the long term, many students may forget and then be lost on what to do, or misremember the procedure, and then get the wrong answer. Without the understanding, the procedures and skills will not stick.

C. Measurement[3]
Understanding: Measurement is a part of a child's development from the earliest years. It connects math to other sciences, i.e., is integrative. It is the determination of the size or magnitude of something. Important components are the idea of the "unit," the act or process of assigning numbers to phenomena according to a rule. This involves the understanding of grouping some attribute, grouping by a prescribed

attribute, ordering objects by size, counting, and measuring with nonstandard units (e.g. paperclips). Concatenated with this are partitioning, unit iteration, transitivity, conservation, accumulation, and numerical connection.

Procedure: Depending on what is to be measured, the procedures can vary. For length, our example, the student is instructed on measuring the line, with decimals and/or fractions. For area, volume, weight, time, and energy, there are similar procedures. Without any understanding, procedures are meaningless.

Skills for length: The student must learn to make length measurements with accuracy (this is key) every time and to know when only a certain accuracy can be achieved.

Summary. For each learning environment, the proper balance of these big three: *understanding, procedure,* and *skills*,[4] is critical for good teaching and long term learning. This is the foundational goal of good teaching. If you have a very large class (about 50 or more) or a small class (less than 30), the rules are different. Remember, all take valuable time. The key is to determine the best balance. The goal is long-term retention.

NOTES

1 Allen, G. D. (2010). *Designing a mathematics lesson using the 5E model*. Retrieved from at http://disted6.math.tamu.edu/kenya-tz2010/teachingpractice/desiging_a_5e_lesson.htm

2 Riddle-Johnson, B., Sigler, R. S., & Alibali, M. W. (2010). Developing conceptual understanding and procedural skills in mathematics: An iterative process. *Journal of Educational Psychology*, *93*, 346-362. [Note in the paper, the authors discuss only conceptual understanding, and procedural skills. In this note, we separate procedural skills into both components: procedures and (computational) skills. One can have the one without the other.]

3 Teachnology, online resources at http://www.teach-nology.com/themes/math/measure/

4 For drill and skill worksheets, downloadable and free, see http://academicsolutions.com/

JENNIFER GRAHAM, MICHELE WARD AND LARRY JOHNSON

56. TAMU'S PEER PROGRAM PROMOTES STEM THE WORLD OVER

The Partnership for Environmental Education and Rural Health program (PEER; http://peer.tamu.edu) at the Texas A&M University College of Veterinary Medicine has been making waves recently. Established in 1999, the PEER program provides a helping hand to teachers through their unique and creative lesson plans tailored to individual teacher requests. Originally intended for teachers in rural parts of Texas, the PEER program has been steadily growing since their start up and has received requests from teachers in just about every state. Recently, they received their first international request, stretching their reach from all over the continental U.S. to worldwide.

PEER Program and Website Features:

- A multistep review process for every lesson plan and materials created
- Grants teachers in remote areas of the state, nation, and the world access to request activities, websites, PowerPoint presentations, and even customized lesson plans
- Provides teachers with specific information and materials they need through direct requests
- Allows the PEER program to direct our efforts to fulfilling teachers' needs with material that will be used in classrooms
- Directs teachers to other teachers' ideas and requested materials

The educational materials PEER generates are free and available for download from their website 24/7, which also hosts a distance learning community and online mentoring for teachers. On average, 200 of their available custom lesson plans are downloaded weekly. PEER also has video presentations of their activities, a videoconferences series of scientists' presentations, and interviews of undergraduate and graduate students conducting research projects that are available for viewing. One of PEER's goals is to spark an interest for research in young students to cultivate awareness in the learning, research, and career opportunities available through continued education.

PEER's unique lesson plans and teaching materials come from the creative minds of the undergraduate, graduate, and veterinary student fellows PEER employs; all hail from a diverse background of majors to cover any possible requested subject. In the past, they've had: biomedical and civil engineering, physics, communications, English, mathematics, and accounting majors, in addition to biomedical science,

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 253–254.

agricultural education, biology, and wildlife and fisheries majors. The PEER program organizes videoconferences with veterinarians, students, and a myriad of professionals here at the university to give students everywhere an opportunity to interact with professionals and get a behind the scenes look at the veterinary school and research laboratories. The majority of teacher requested resources are for lessons for Science, Technology, Engineering, and Mathematics (STEM) education and more recently for Agricultural Science. Of the 600 requested lesson plans online, over 80 cover mathematics lessons.

PEER fellows create a unique lesson plan tailored to the teacher's specific requests for either a presentation on a subject intended for a certain grade level, fun hands-on activities to engage the students while teaching the TEKS (or national education standards), or just links to videos or websites that would make a good computer activity for the students. The PEER fellows strive to create comprehensive lessons that require minimum preparation for the teachers, while attempting to maximize the learning experience for the students. The PEER fellows create supplemental worksheets, lesson-specific assessment tools, activities to help introduce and close out the lessons, and class projects.

PEER prides itself on making the material fun. One of the more popular downloadable activities is a game created by an undergraduate biomedical engineer fellow called Medopoly. Medopoly demonstrates clinical trial process and teaches the steps of scientific research in drug and appliances development toward treatments for diseases. Mathematics lessons include those entitled, "The Mathematics of Music," "Shoebox Apartments," "Rational Code Cracking," and "Seasonal Mathematics."

PEER has a vast network of educators nationwide to help ensure the lesson plans are accurate and grade appropriate. Once a requested resource is completed by the undergraduate or graduate fellow, the created lesson plan and associated materials go through a stringent review process with careful scrutiny from graduate fellows, TAMU faculty members in that subject's department, and finally certified teachers for fact checking and readability for that grade level.

The PEER program is headed by Dr. Larry Johnson, a professor at the veterinary school who teaches scientific ethics, science communication, and histology. Dr. Johnson was awarded grants from the National Science Foundation (NSF), National Institute of Environmental Health Sciences, and the Office of Research Infrastructure Programs at the National Institutes of Health (NIH) for this innovative program to integrate science and math learning in rural middle schools, but the PEER program has grown to include community outreach programs that provide experiential learning opportunities for youth and the sharing of veterinary medicine expertise.

PEER plans on branching out with their teaching materials and are looking at utilizing other forms of media for their educational materials; currently an application (app) is in the works for computers, smart phones and tablets.

Their long-term goal is to ensure that the next generation of researchers in academia, industry, and government are aware of, and sympathetic to, the challenge and opportunities of K-12 education and how they can help.

SANDRA NITE

57. ENERGY MATH LESSONS FOR MATH CLASSES

Mathematical applications abound in the world around us. In fact, other STEM (science, technology, engineering and mathematics) fields depend upon mathematics as a foundation for most of their applications. Energy math is a fact of our lives; without it these days we could not even find a new reservoir. Technology tools include seismology, hydrophonics, computer modeling, geological formations, and more. Mathematical tools include tomography, 3D mapping, fracturing models, data analysis and more. Indeed, the subject is no longer intuitive. Those days are long gone. It is absolutely scientific, and moreover very, very mathematical. The mathematical facets of energy are immense.

See: http://sciencenetlinks.com/lessons/searching-for-oil-the-role-of-science-and-technology/. Halliburton, a diversified energy company based with the gas and oil company, has an educational component that funds many ventures in bringing the energy business to the student. In 2011, Dr. Allen and I received grant funds to oversee development of classroom lessons and PBL's related to mathematics in the oil and gas industry.

Under our direction, high school teachers created activities suitable for middle school through high school precalculus classes. Some lessons are designed to last for one class period, about 50 minutes, and some are Project Based Learning (PBL) activities that will last for several days. They are based on the 5-E Instructional Model (see http://www.bscs.org/bscs-5e-instructional-model), an inquiry model adapted to mathematics from the science field.

Gas and oil and other energy topics addressed in these activities include fractional distillation of crude oil, geothermal energy, oil well production, designing storage tanks (PBL), hydrostatic pressure, purchase price and energy efficiency in appliance shopping (PBL), barrel unit conversions, hydrostatic pressure, oil and gas well analysis, production costs and breakeven point, country population versus oil production, calculating the volume of a well, transportation cost decisions (PBL), using the Pythagorean Theorem to calculate distance, and oil consumption and production. This list seems weighty, but they are merely applications of what every teacher covers in grades 8–12. They bring the science to the mathematics classroom.

The mathematical topics that are part of these activities are proportions, unit conversions, Pythagorean Theorem, evaluating expressions, bar graphs, circle graphs, scatter plots, using functions to model data, transformations of functions, linear functions, quadratic functions, solving linear equations, systems of linear

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 255–256.

equations, literal equations, matrix operations, solving systems of equations with matrices, volume of surface area of a cylinder, and volumes of concentric cylinders.

Dr. Allen designed the website, *Energy Math*, to house the materials developed by the teachers. Each of the lessons or activities includes the TEKS addressed as well as the national standards (from NCTM). Lessons are free to all teachers and can be accessed at http://disted6.math.tamu.edu/halliburton/index.html.

If you have questions, you can contact us at energymath@math.tamu.edu.

G. DONALD ALLEN

58. MULTIPLE REPRESENTATIONS, I

DEFINING MULTIPLE REPRESENTATIONS – THE BASICS

In this multiple part discourse, we consider the fundamentals of multiple representations, one of the cornerstones of modern mathematical educational theory. They form various ways of explaining concepts, important for the teacher, and multiple ways of learning concepts, important for the student. They constitute an analogue to perception by the senses. For example, when you both hear and see an event, it has stronger comprehension properties. Similarly, when you can see data expressed and also graphed, it has greater cognitive significance. The general outline of this paper is

i. Defining Multiple Representations – the Basics
ii. Visual Cognition in Multiple Representation
iii. Multiple Representations and the TEKS

WHAT ARE MULTIPLE REPRESENTATIONS?

Multiple representations of information and functions are critically important for students to learn because each representation adds a new dimension of understanding to the situation at hand. While this lesson discusses the algebraic connections between multiple representations, there is one graphic that particularly demonstrates the importance of knowing multiple representations of information to develop better communication. When graphing data in a spreadsheet, one is confronted with a buffet of possible representation types.

In this simple graphic (Figure 1), we are confronted with the dozens and dozens of ways data can be represented. It is important for us to know which to use to communicate the message we intend. For example, if our data represents a quadratic function, it may be best not to use a bar chart. See Figure 2.

Both charts illustrate the same data set, but the one on the right is easily identified as the parabolic shape. Moreover, the spreadsheet functionality automatically creates a function for us by "interpolating" the data using a type of spline algorithm. We could actually get the formula for the function by computing the regression line and displaying the equation with the spreadsheet or with a graphing calculator.

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 257–262.

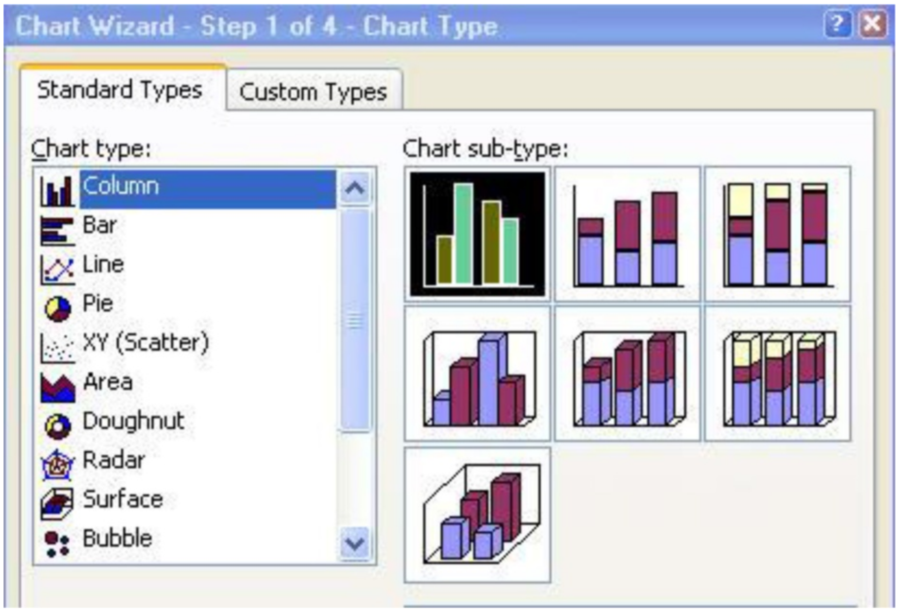


Figure 1. Selecting a chart type

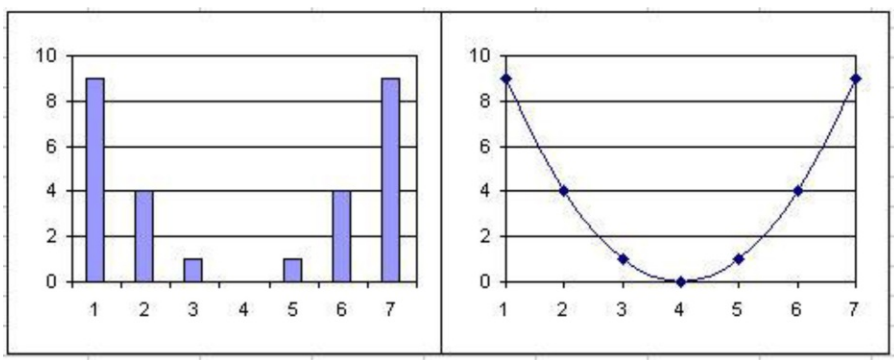


Figure 2. Quadratic funtion representations

If you understand this simple example, you show a considerable knowledge of multiple representations of information, particularly functional information.

Normally, there are numerous representations of functional information. They include

1. Concrete
2. Picture
3. Verbal

4. Numeric (tabular)
5. Graphic
6. Algebraic
7. Abstract

Often it is possible to group the picture and graphic together; we shall do so here. The other two important notions we will consider below are the numeric and algebraic. The concrete representation can often be folded into one of the others. So, the multiple representations considered in this lesson are the *big three*: algebraic, graphic, and numeric.

Multiple representations are but one facet of a larger question, that of multiple venues of knowledge. Among them, there are

- Different types of knowledge
- Different ways of representing knowledge
- Different ways of using knowledge
- Different and structured methods to assimilate and interpret knowledge

So, learning multiple representations in mathematics parallels or is contained by a much broader aspect of human cognition.

MULTIPLE REPRESENTATIONS AND LEARNING TRANSFER

Multiple representations of functions are a recurring theme throughout mathematics at all levels. Its position, vis-à-vis understanding functions, is unique and worthy of examination. Yerushalmy and Gafni (1992) stated, "Function in its multiple representations is the fundamental object of algebra which ought to be presented through the learning and teaching of any topic in algebra." As important as this is, fewer studies have been conducted involving transfer between representations (Knuth, 2000).

For students to develop a likely understanding of functions, they must be given opportunities to solve problems that require them to transfer between algebraic, numeric, and graphic representations. This is called the "transfer of learning." Research has confirmed student difficulties with certain types of transfer problems and has suggested instructional factors as a possible cause. In one study, algebra teachers were surveyed to determine the amount of class time they use to teach the different types of transfer problems. They were also asked how many times these problems appear on their teacher-made assessments. Results establish that teachers dedicate less class time to graphic-to-numeric transfer problems than to any other types, and that these problems appear less frequently on assessments. These are exactly the types of transfer problems that pose the most difficulty for students. It was conjectured that teachers' familiarity with these problems, combined with assumed student mastery, contribute to this mismatch (Cunningham, 2005). In a study of calculus, students enrolled in three different courses, one traditional, one using graphing calculators, and one using the computer algebra system called Mathematica, Porzio (1999) made a well-documented case that students need to be able to solve each type of transfer problem.

Note: In educational research experiments, there are often two groups of subjects to employ, the experimental (or treatment) group that experiences the intervention and the control (or comparison) group that does not receive the intervention. Every effort must be made to keep the two groups as similar as possible.

Pre-algebra students learned about functions in a unit that emphasized (a) representing problems in multiple formats, (b) anchoring learning in a meaningful thematic context, and (c) problem-solving processes in cooperative groups. In post-test results, treatment students were more successful in representing and solving a function word problem and were better at problem representation tasks, such as translating word problems into tables and graphs than were comparison students (Brenner et al., 1997).

Teachers, regardless of curriculum and textbook, make the ultimate instructional and assessment decisions on how to integrate transfer problems into their courses.

THREE DIMENSIONS OF FUNCTIONAL UNDERSTANDING

The three dimensions of functional understanding are perspective, object, and representation. On the *perspective dimension*, students need to be able to view a function from the process perspective, which means that for each x value there is a corresponding y value.

From the *object perspective,* a function is considered an entity that can be rotated or translated as a whole.

The *representation dimension* consists of the three most prominent representations for functions: algebraic, tabular (numeric) and graphical (graphic). Competency with functions also requires flexibility to move between these representations to solve problems.

Movement along the perspective dimension or movement along the representation dimension, the primary focus of this study, should be central to mathematical and curricular investigation.

TWO LEARNING TRANSFER TRIADS

Two triads of learning transfer illustrated in Figures 3 and 4 should enlighten our understanding of the multi-channeled processes involved with multiple representations. The first is the application of concepts of cognitive psychology to the reconstruction of instruction. The purpose is to transpose concepts of cognitive psychology to a curricular and instructional framework. New ideas are emerging, including cognitive learning concepts that provide a foundation on which to build a learning environment that achieves integration and facilitates the acquisition of higher-order skills (Herschbach, 1998).

Previous investigations in this area (Stein et al., 1983) indicated that people do not spontaneously transfer clues to a representation problem, even though the necessary information is available. It has been substantiated with success that when an individual is presented with a problem, he or she encodes the information to form a mental representation of the problem. These associations are established with conceptual knowledge (Yackel, 1984).

In brief, there are three important processes at work in the transfer of learning: understanding, representation, and experiences. These can be arranged as the vertices of the first learning transfer triad (Sutton, 2003).

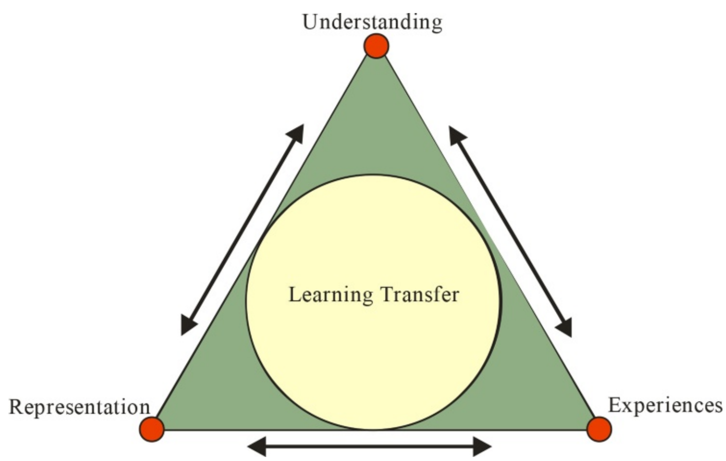


Figure 3. Learning transfer triad (Sutton, 2003)

The second triad pertains directly to learning transfer between multiple representations of functions: algebraic (functional), graphic, and numeric. Between each pair, there are two separate and distinct directional channels, for which problems and learning can be quite different. In Lessons 2 and 3, we will discuss a variety of problems considering each of these channels. While three specific representations will be considered here and another three have previously been mentioned, there is another of importance in advanced mathematics – the abstract representation.

Interpreting graphs is much more difficult than understanding the rule-oriented process of solving inequalities given in algebraic form. In the section below, on visual cognition, we will discuss the host of issues involved with any graphic type of function representation. The knowledge, and background, of the student must be considerable to fully use visual cognition in conjunction with algebraic and numeric representations.

What is important is that teachers, especially algebra teachers, need to be aware of the complexities of student learning about the multiple representations of functions. Real certainty that learning has taken place requires a more careful look. Student

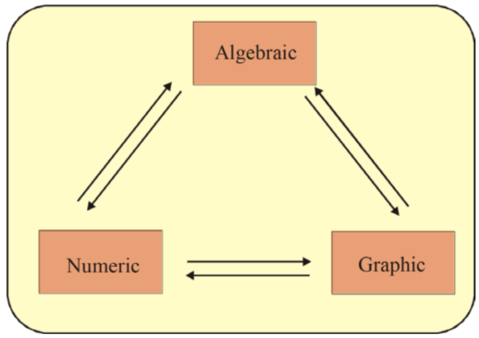


Figure 4. Six types of transfer

writing, though seldom used, could provide a clearer picture of student learning. In addition, teachers need to provide students with sufficient instructional and assessment opportunities to solve transfer problems. Moreover, they should be given with increased dedication to those topics that are most problematic, yet necessary for student understanding and success in mathematics.

REFERENCES

Allen, G. D. (2001). Online calculus, the course and survey results. *Computers in the Schools, 17,* 17–30.

Barthes, R. (1981). *Camera Lucida: Reflections on photography*. (R. Howard, Trans.). New York, NY: Hill and Wang.

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Duran, R., Reed, B. S., & Webb, D. (1997). Learning by understanding: the role of multiple representations in learning algebra. *American Educational Research Journal*, *34*(4), 663–689.

Cunningham, R. F. (2005). Algebra teachers' utilization of problems requiring transfer between algebraic, numeric, and graphic representations. *School Science and Mathematics, 105*, 73–81.

Herschbach, D. R. (1998). Reconstructing technical instruction. *Journal of Industrial Teacher Education*, *36*(1), 36–61.

Knuth, E. J. (2000). Student understanding of the cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500–507.

Lamb, A. (2001). Digital glyphs: Imaging ideas for a visual world, activate. *The Journal of Technology-Rich Learning, 1*(2).

Stein, B. S., Benningfield, S., & Hedgecough, C. A. (1983). *Constraints on the automaticity of transfer on problem solving*. Paper presented at the meeting of the Southeastern Psychological Association, Atlanta, GA.

Sutton, M. J. (2003). Problem representation, understanding, and learning transfer implications for technology education. *Journal of Industrial Teacher Education, 40*(4), 36–61.

Wilson, M. R., & Krapfl, C. A. (1994). The impact of graphing calculators on students' understanding of function. *Journal of Computers in Mathematics and Science Teaching, 13*, 252–264.

Yerushalmy, M., & Gafni, R. (1992). Syntactic manipulations and semantic interpretations in algebra: The effect of graphic representation. *Learning and Instrction, 2*, 303–319.

G. DONALD ALLEN

59. MULTIPLE REPRESENTATIONS, II

LEARNING TRANSFER FOR GENERAL, LINEAR, AND QUADRATIC FUNCTIONS

Our objectives in this note are to demonstrate several types of multiple representations as they pertain to general, linear, and quadratic functions. The principle types of such representations include the Algebra ↔ Graphic, the Algebraic ↔ Numeric, and the Numeric ↔ Graphic. There is much to learn even for linear and quadratics. However, learning essentials for these two basic functions forms a foundation for all functions. We conclude with a selection of discussion questions and activities.

1. INTRODUCTION

In a previous article we reviewed the essential learning transfer issues for general multiple representations. We learned that a part of the learning transfer problem of multiple representations is the lack of emphasis some teachers place on one or more of the elements of the triad or tripod of functional forms: algebraic, graphic, or numeric. Another part of the problem is that the connections are not as self-evident as might be thought. In this article we consider recapitulate those notions and include particular issues of each connection within this tripod.

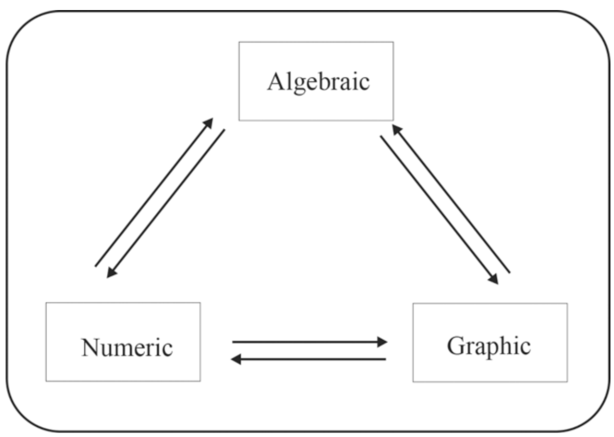


Figure 1. Multiple representations tripod

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 263–276.

Important in this article are the two basic forms for a straight line.

- **Slope-intercept form**. The slope *m* and the y-intercept b are given. The formula is

$$y = mx + b$$

- **Two-point form**. Two points (x_1, y_1) and (x_2, y_2) are given, with $x_1 \neq x_2$. The line passing through these points is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} x - x_1$$

The slope is $m = \frac{y_2 - y_1}{x_2 - x_1}$ Simplifying, we have

$$y = \left(\frac{y_2 - y_1}{x - x_1}\right) x - \left(x_1 \left(\frac{y_2 - y_1}{x - x_1}\right) + y_1\right)$$

from which the y-intercept is evident. If $x_1 = x_2$, then we have the vertical line $y = x_1$ This is not a function, though sometimes students believe it to be so.

2. ALGEBRAIC TO GRAPHIC REPRESENTATIONS

a. Algebraic to Graphic. A typical question might be to graph the linear equation

$$-x + 2y = 12$$

using the slope and *y*-intercept. To solve, we must find the slope, and this means rewriting the equation in the slope-intercept form

$$y = mx + b$$

This point must be emphasized *over and over* again. *To get the slope and intercept it is essential to convert the equation to the slope-intercept form*. Indeed, students not realizing this often simply pick one of the coefficients of *x* or *y* as the slope and the "free" number as the intercept. This mistake persists into college. Not only is this a mistake but it is the misconception that using the data in the problem within a known formula will result in a correct answer. For the problem at hand we solve

$$\begin{aligned} -x + 2y &= 12 \\ 2y &= 12 + x \\ y &= \frac{1}{2}x + 6 \end{aligned}$$

The slope is therefore $\frac{1}{2}$ and the y-intercept is 6. Another point to emphasize is that in the slope-intercept form of a line the intercept is in fact the y-intercept.

b. Algebraic to Graphic. Functions interpreted as mappings occur frequently on examinations. Here is a typical, though simple, question. Which mapping best represents the function $f(x) = 2x^2 + x - 3$ when the replacement set for x is {0, 2, 3, 5}? Answers from which the student must determine the answer appear as shown in Figure 2. Of course the answer is d.

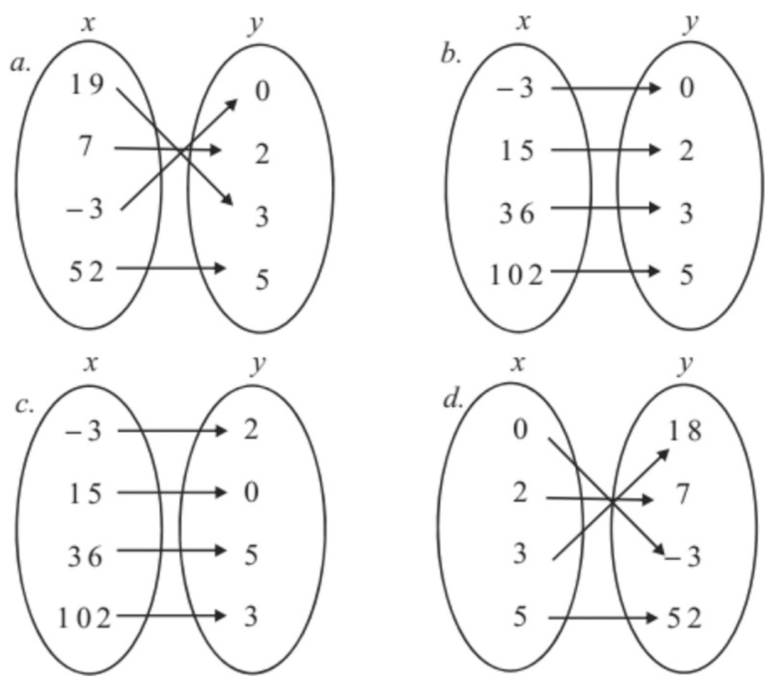


Figure 2. Showing replacement sets

This alternate type of graphic is very important for students to master. Note however, only a few of the domain values are given. This can be misleading as it is not the full representation, in the sense that there are many functions that interpolate exactly these points. Nonetheless, this is a part of algebraic to graphical multiple representations.

c. Graphic to algebraic. A typical question is to determine the equation of a line from the shown graph. For example, find the equation of the line from Figure 3 below. (Assume each tick mark represents one unit on the graph)

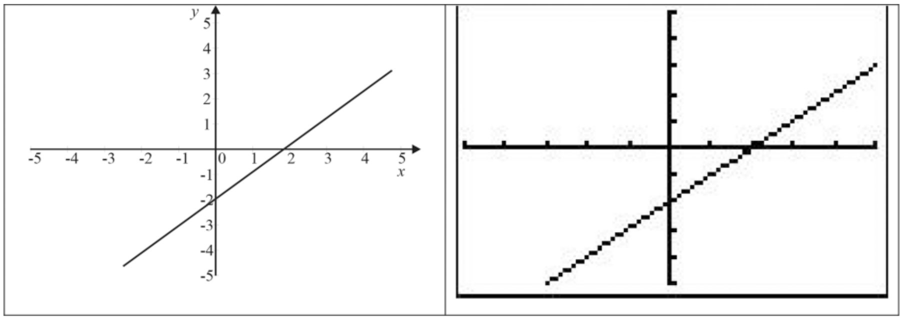


Figure 3. Actual vs. calculator version of the same graph

Shown here are two versions of the same graph, the left one an accurate rendering using high resolution imagery, and the right one from a TI-84 graphing calculator. Here, two side-by-side representations of essentially the same thing present to the student learning challenges of their own. Note in both graphics the horizontal and vertical scales differ—another learning challenge.

The student must identify two points on the graph, and then apply the two-point form for the equation of the line. Alternatively, the student must identify two points on the graph to compute the slope and also the y-intercept. For this problem, we see that two points are (0,-2) and (2,0). Hence the slope is

$$m = \frac{0-(-2)}{2-0} = \frac{0+2}{2+0} = \frac{2}{2} = 1$$

Notice all the steps taken above. It would have been very simple to write

$$m = \frac{0-(-2)}{2-0} = 1$$

Some students do not see this quickly. Taking the extra steps gives them a little additional time for needed cognition. As it happens one of the points is the y-intercept. So we can apply the point-slope form to write the equation of the line as $y = x - 2$. Following a little further, we can use the two-point form to write

$$y - (-2) = m(x - 0)$$

And with $m - 1$ already computed we obtain

$$\begin{aligned} y - (-2) &= x \\ y + 2 &= x \\ y &= x - 2 \end{aligned}$$

This is a typical but rather simple example. It is possible to make the problem more challenging by moving the line. For example, in Figure 4 below, it is not a simple matter to determine two points. Indeed, they must be estimated. This brings to bear the difference between the exact and approximate.

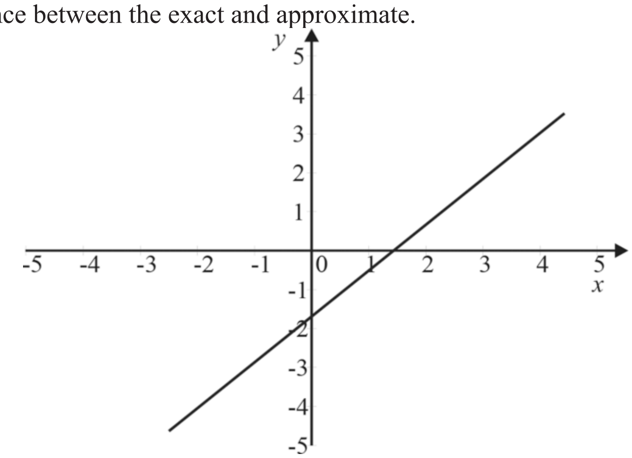


Figure 4. Graphical representation

For this problem, it is advisable to add a grid as shown Figure 5.

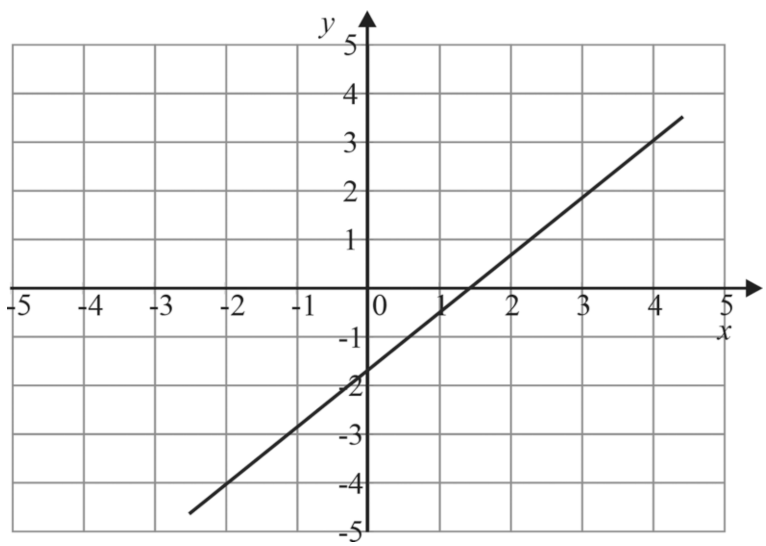


Figure 5. Same as Fig. 4 with grid

Here the student can see the line passes through the points (-2,-4) and (4,3). For this problem the slope will be (7/6). It is nearly impossible to accurately read the intercept from the graph. So, using the two point form for the line is essential in this problem. The alternative is that the student will obtain an approximate algebraic solution.

3. ALGEBRAIC TO NUMERIC REPRESENTATIONS

a. Algebraic → Numeric. Given the linear equation $-x + 2y = 12$, construct a table displaying ordered pairs that are solutions to the equation. This problem is particularly simple for linear functions. One may merely select a set of x-values and compute the corresponding y-values. Below we show two such tables. For the first we have selected the integers {0,1,2,3,4,5}. For the second we chose even integers {0,2,4,6,8,10}. Moreover, we show the result for the first set in two different forms, one with decimals and the other with fractions.

Table 1. Using even numbers input

x	y
0	6.0
1	6.5
2	7.0
3	7.5
4	8.0
5	8.5

x	y
0	6.0
2	7.0
4	8.0
6	9.0
8	10.0
10	11.0

Note the advantage of selecting the even numbers. The y-values are all integers. Now, since we know the solution is $y = (1/2)x + 6$, we can obtain the answers in fraction form.

Table 2. Using odd numbers input

x	y
0	6
1	$6\frac{1}{2}$
2	7
3	$7\frac{1}{2}$
4	8
5	$8\frac{1}{2}$

While this is simple to do, it is important for students to maintain which variable is the one chosen to be the independent variable.

b. Numeric → Algebraic. Given a set of two ordered pairs, determine the equation of the line that passes through them. This problem recounts the two-point form for a straight line as given above. Similarly, we might give a slope and a single point and ask for the line. It is very important for students to be conversant in both forms. For example, suppose the points are (-1,-2) and (3,2). The two-point form of the straight line equation is

$$\begin{aligned} y-(-2) &= \frac{2-(-2)}{3-(-1)}\left(x-(-1)\right) \\ &= \frac{2+2}{3+1}(x+1) \\ &= x+1 \\ y &= x-1 \end{aligned}$$

That is straigtht forward. Struggling students need to have at their disposal (or preferably committed to memory) the form of the equation and to be strongly encouraged to followed it rigorously. Many of the problems students have in more advanced math courses seem to stem from their careless and imprecise use of procedures and formulas. Always review the proper algebraic form and use standard mathematical notation.

Another important point about this problem is that there are many functions that fit the data. For example, the quadratic $y = x - 1 + (x + 1)(x - 3) = x^3 - x - 4$ also fits the given data set. Figure 6 below shows both curves.

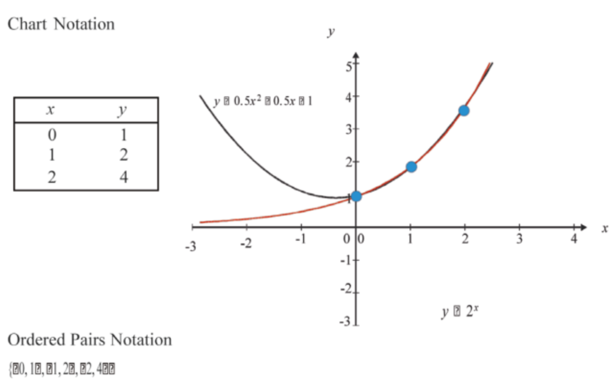


Figure 6. Quadratic and linear functions

Why is this important? Because in the numeric to algebraic representations, it is important that only a model of the data is being represented.

c. Numeric → Algebraic. Given a table of ordered pairs determine the function that agrees with them. This is a somewhat greater challenge, and when the function is not linear, it is greater still. First, consider the linear case. In the table below we have taken successive differences of the *y*-values. That these differences are constant indicates there is an underlying linear function.

Table 3. Differences

x	y	Difference
0	3.0	
		-2.0
1	1.0	
		-2.0
2	-1.0	
		-2.0
3	-3.0	
		-2.0
4	-5.0	
		-2.0
5	-7.0	

What is important for this difference rule to work is the x-values be equally spaced. If not, divided differences must be used. When the differences are constant, the equation of the line can be determined from any two points. However, the slope can be determined by dividing the constant difference by the difference in the x-values. For the problem at hand the slope is -2, and from the table, the y-intercept is 3. Thus the line has equation $y = -2x + 3$. The problem for quadratics is one level of complexity greater. We will need to use two differences in this case. Inspect the table below.

Table 4. First and second differences

x	y	First Difference	Second Difference
0	-3.0		
		-1.0	
1	-4.0		2.0
		1.0	
2	-3.0		2.0
		3.0	
3	0.0		2.0
		5.0	
4	5.0		2.0
		7.0	
5	12.0		

When the second differences are constant, it indicates the function is a quadratic. There are several ways to find the quadratic.

4. NUMERIC TO GRAPHIC REPRESENTATIONS

a. Numeric to Graphic. A typical question is this: Given a set of ordered pairs, graph the linear equation that passes through them. On numeric to graphic problems, there are two ways to proceed. The first way is to plot the points on the coordinate plane and then connect the dots either smoothly or with line segments. When given a set of ordered pairs, there are only two points to plot and the only line that makes sense is a straight line. What must be observed is that the student is creating a graphic model of data. This means in the case of a set of ordered pairs that unless it is known there is an underlying straight line, the line drawn is only a model representing the data.

Chart Notation

x	y
-2	3
3	-1

Ordered Pairs Notation

$\{(-2,3),\ (3,-1)\}$

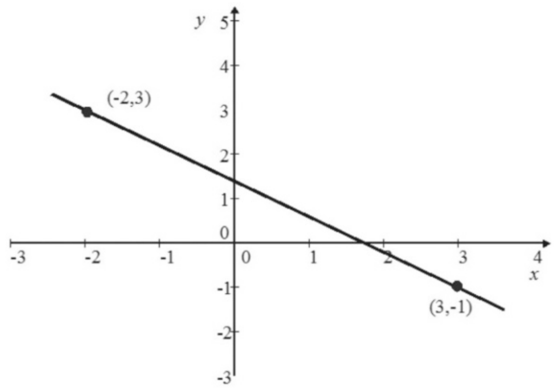


Figure 7. Multiple representations

When the data set includes more than two points, this is even more evident. In the example below we have a quadratic. Or is it an exponential?

Chart Notation

x	y
0	1
1	2
2	4

Ordered Pairs Notation

$\{(0,1),(1,2),(2,4)\}$

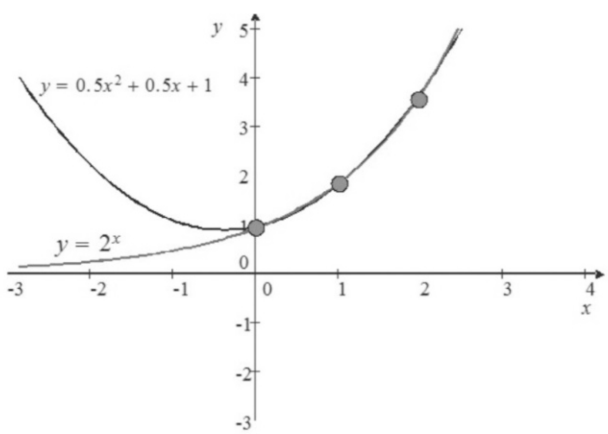


Figure 8. Two functions with same data

Both the quadratic $y = 0.5x^2 + 0.5x + 1$ and the exponential $y = 2^x$ are valid algebraic interpretations of the data. If only because of their experience with "connect the dots" childrens' games, many students understand intuitively there may be multiple algebraic representations of data. It is important to explain this in the classroom, though perhaps not dwell on it.

b. Graphic to Numeric. Given the graph shown in Figure 9, determine an ordered pair that satisfies the linear equation $-1.5x + 2$.

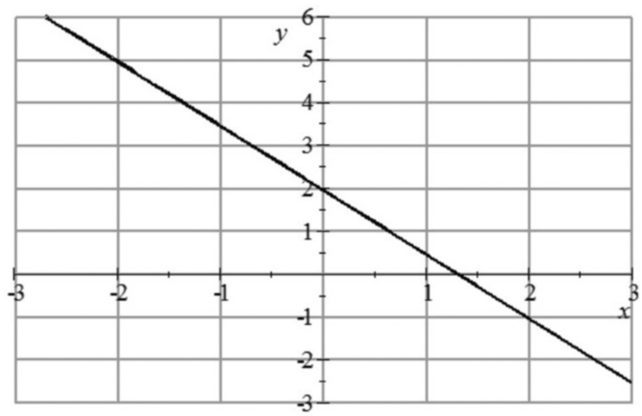


Figure 9. Simple linear graph connecting two points

It is easy to see that the points in Table 5 lie on the graph. Notice how the values are written with some as decimals and others as whole integers. You may need to work with your local standards as to whether all decimals or even fractions are the accepted manner to express the values.

Table 5. Linear data

x	y
−3	6.5
−2	5
−1	3.5
0	2
1	0.5
2	−1
3	−2.5

The same graph is shown below without the grid lines. This makes the measurement problem somewhat more challenging.

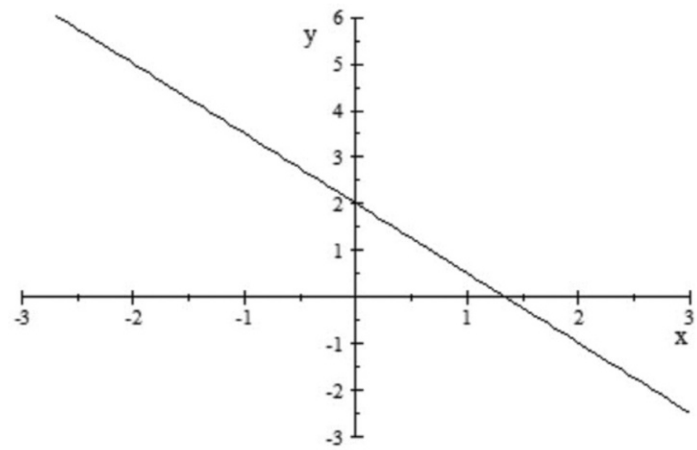


Figure 10. Linear equation showing the same data – but more

c. Graphic to Numeric. Given the graph shown in Figure 11, determine an ordered pair that satisfies the quadratic equation $y = x^2 - x + 2$. (Notice the horizontal grid lines are at two unit intervals while the vertical gridlines are at one unit intervals.)

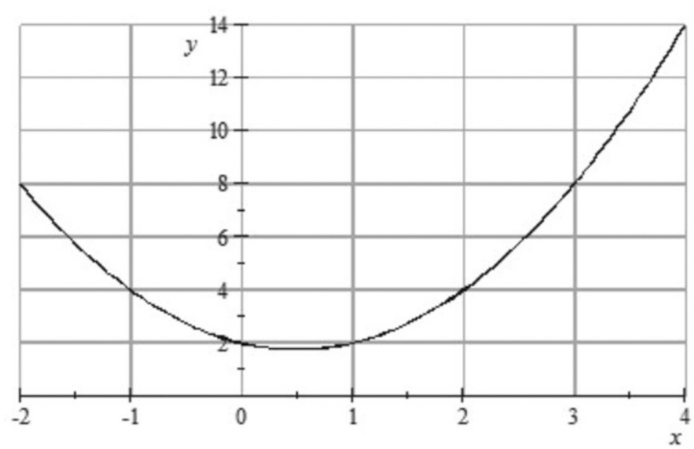


Figure 11. Quadratic

It is easy to see the points in Table 6 lie on the graph.

Table 6. Quadratic data

x	y
−2	8
−1	4
0	2
1	2

Find other data points on the graph. Note that the same graph in Figure 11 below (without the grid) presents measurement problems for some students. Expect students to make approximations.

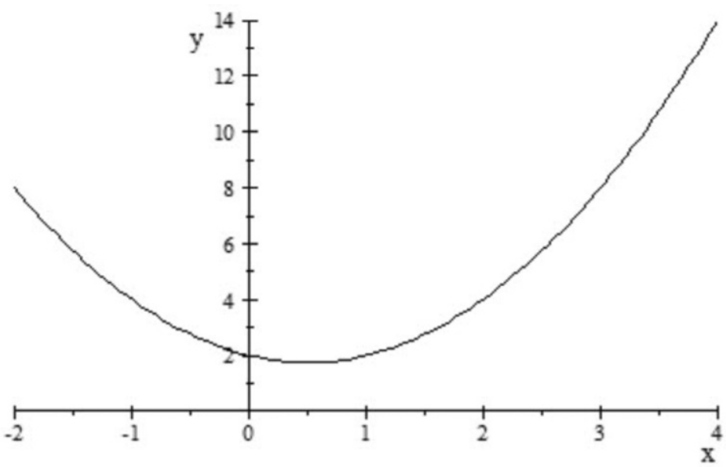


Figure 12.

d. Graphic to Numeric. In this type of problem the student is given a graphic and asked to determine some type of numerical information about it. For example, a typical question may ask the student to determine the x-intercepts or vertex of the parabola shown in the graph below.

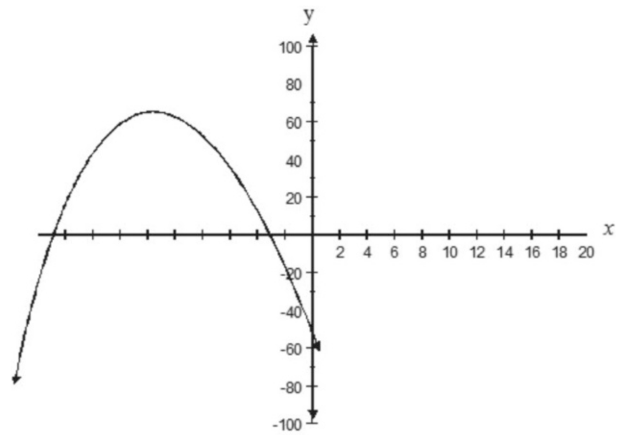


Figure 13. Graph to data example

The answer selection is one of the following

a. (–1.5, 0), (–9.5, 0)
b. (–3, 0), (–19, 0)
c. (–11, 64), (0, –57)
d. (0, –3), (0, –19)

The correct answer here is *b*, though it appears it could be *a*. The reason *a* is not correct is that the labels are by twos, not units. In the real world no one tells you these little details.

DISCUSSION QUESTIONS AND ACTIVITIES

1. Given an assessment of which of these six learning channels may be the most difficult to teach.
2. Given an assessment of which of these six learning channels may be the most difficult to learn.
3. Give a review of all the mathematical tools with which students should be familiar to fully understand all six learning channels.
4. In the Algebra ↔ Graphic section, how would we approach instruction for graphing a quadratic? What additional tools must students have?
5. Construct a number of mapping-type questions involving exponential, linear, and quadratic expressions.
6. Use the function machine graphic as a type of generalized graphic representation to make a lesson on functions with multiple representations.

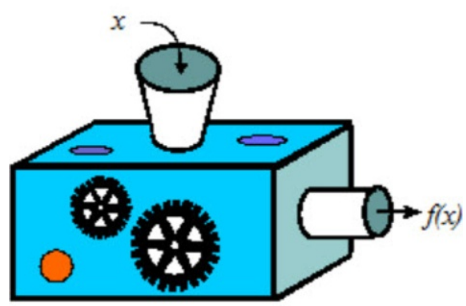


Figure 14. Diagram of a function as machine

7. In the Numeric → Algebraic subsection, what should we do if we have a problem such as $x + 2y^2 - 3y = 4$. We are given values of x. This is a little advanced for the grade nine, but for Algebra II and pre-calculus, the problem is reasonable.
8. Develop a complete module of learning transfer covering the topics of this lesson: Numeric ↔ Graphic, Algebra ↔ Graphic, and Algebraic ↔ Numeric. Estimate the number of teaching hours for each component.
9. Give examples of Numeric ↔ Graphic learning transfer situations where there are multiple solutions such as the quadratic/exponential problem above.
10. A good example for an Graphic to Numeric problem is to ask students to identify the vertex of a parabola from its graph. Construct three problems.
11. In a numeric to algebraic example above we noted that both $y = x - 1$ and $y = x - 1 + (x + 1)(x - 3) = x^2 - x - 4$ fit the data (-1,-2) and (3,2). Discuss as you would with an Algebra II class how it is possible to generalize this and produce many functions that fit the same data. (Note: the idea of "fitting" data is also mathematically called "interpolating" data.) Clearly, the answer to this question is to add to the basic linear function (y=x-1) any function that has a value of zero at each of the given data x-values. How can this be made clear to students? A typical approach is to show students several examples and expect them to get the idea. How can you add to this typical teaching prescription?
12. What essential differences are involved in teaching the numeric to algebraic learning transfer between Algebra I, Algebra II, and Pre-calculus? Rephrase this question for each of the other five other learning transfer problems.

G. DONALD ALLEN

60. MULTIPLE REPRESENTATIONS, III

In this article, we probe carefully some of the many problems students are asked to solve in their class and particularly on their exit and end-of-course exams. It is a fact that most mathematics curricula are similar. It is a fact that most students see entirely similar problems. It is a fact that what is taught in any curricula is defined by clarifying examples and from the high-stakes exams given. Therefore, it is of great value to give a close analysis of typical problems. In this paper, we consider easy and medium level questions connected with multiple representations, together with an associated analysis.

INTRODUCTION

Much can be learned from what the State believes should be taught within the context of their curriculum by examining the questions appearing on their high-stakes exams.

Every high-stakes test is developed over several years. It is comprised of several activities. Normally, a committee of educators identifies the student expectations for each grade and subject area assessed that should be tested on a statewide assessment. Then a staff committee incorporates these selected student expectations, along with draft objectives for each subject area, into the grade exit level surveys. Based on many inputs from teachers and collegiate educators, and sometimes mathematicians, draft versions of the tests are created. The thinking includes the vertical alignment of objectives across grades 2 through 10. At all levels, state educational leaders rely

Math Objectives	
Objective 1	Functional Relationships
Objective 2	Properties and Attributes of Functions
Objective 3	Linear Functions
Objective 4	Linear Equations and Inequalities
Objective 5	Quadratic and Other Nonlinear Functions
Objective 6	Geometric Relationships and Spatial Reasoning
Objective 7	2-D and 3-D Representations
Objective 8	Measurement
Objective 9	Percents, Proportions, Probability, and Statistics
Objective 10	Mathematical Processes and Tools

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 277–287.

on educator input to develop items that are appropriate, as well as valid measures of the objectives and the student expectations the items are designed to assess. Exit and end-of-course exams are normally divided by objectives, though the curriculum often does not contain these objectives. The objectives – which can be vertically stated – are categories into which specific curriculum can be grouped. For Grade 11, the usual objectives are:

The precise definitions with clarifications of the Objectives can be found. Within the categories of the Objectives, there is one further grouping of the objectives themselves of Algebra I, Geometry and Measurement, Percents and Proportional relationships, Probability and Statistics, and Mathematical Processes. These are exemplified in the table below.

Objectives	**Content**
Objectives 1-5	Algebra I
Objectives 6-8	Geometry and Measurement
Objective 9	Percents, proportional relationships, probability, and statistics
Objective 10	Understanding of mathematical processes

The categories of student thinking, multiple representations, and problem solving all fall within the scope and context of the objectives and course subcategories. In this article, we will focus on questions involving multiple representations.

It is important to note that because the high school standards, e.g. in Texas the TEKS (Texas Essential Knowledge and Skills), are based on courses and because there is no state-mandated course sequence, some of the high school TAKS (Texas Assessment of Knowledge and Skills) mathematics objectives contain student expectations from earlier grades. Moreover, although not all the TEKS are tested on the TAKS examination, they are nonetheless critical for student understanding and must be included in classroom instruction.

Scoring Correction, Guessing, and all that. The TAKS examinations, like many high school and collegiate examinations are multiple choice. Some students have developed a very keen ability to guess by eliminating answers. This is an important component to success on the TAKS. On the other hand, one could say there are far too many traps among the answers on the TAKS exams. A trap is an answer that is what a student may obtain by working a problem in a well-known but incorrect manner. For example, when asking for the area of a rectangle, one answer may be the sum (not product) of the length and width. As to the most efficient and effective balance between traps and non-traps, the answer is unknown. Indeed, this very important question has not even been studied. The traps are included to inform teachers of what they need to focus on. While this is an admirable goal, many teachers never see a detailed analysis of how their students perform.

In reading this article, the teacher is not encouraged to teach to the TAKS (aka standards); the methods and techniques needed to be mathematically successful require a more comprehensive and holistic approach to the subject. Attempting to teach to the standards by reviewing multiple problems can be counterproductive, though again no definitive study has been made.

EASY LEVEL QUESTIONS WITH ANALYSIS

In this section, we consider examples of some typical questions that are relatively straightforward. Most involve some aspect of multiple representations.

Question 1. What are the approximate roots of the function graphed in Figure 1?

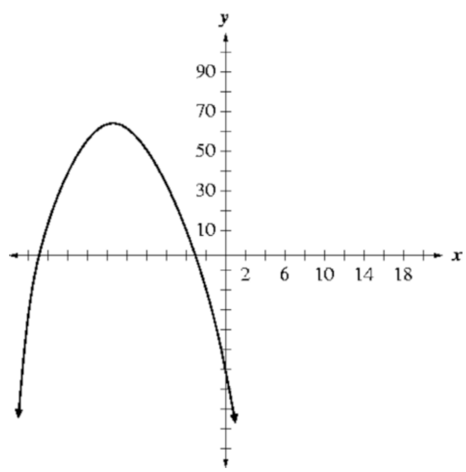


Figure 1. Determining roots from a graph

a. (-1.5, 0), (-9.5, 0)
b. (-3, 0), (-19, 0)
c. (-11, 64), (0, -57)
d. (0, -3), (0, -19)

Analysis: (Graphic to Numeric)

a. Incorrect, because the *x*-intercepts of the parabola were found, but the choice is not correct because the *x*-scale is counting by 2's, not by 1's. This would be particularly hard to plan for if attempts are made to teach to the TAKS. There are just too many variations of the same problem.
b. Correct. Since the *x*-intercepts of the parabola were found using an *x*-scale counting by 2's. Note this problem could be made much easier if the axis was labeled by one's instead of two's.

c. Incorrect, because (–11, 64) is the vertex (maximum value), and (0, –57) is the y-intercept. The student must know what is being sought, and may select this answer if he/she recognizes these are essential points for the graph but fails to recognize precisely what the question seeks.
d. Incorrect, because (0, –3) and (0, –19) have the x- and y-values switched around—it should be that y is equal to 0. The student needs to know accurately the notation for points.

Question 2. Which mapping best represents the function $f(x) = 2x^2 + x - 3$ when the replacement set for x is {0, 2, 3, 5}?

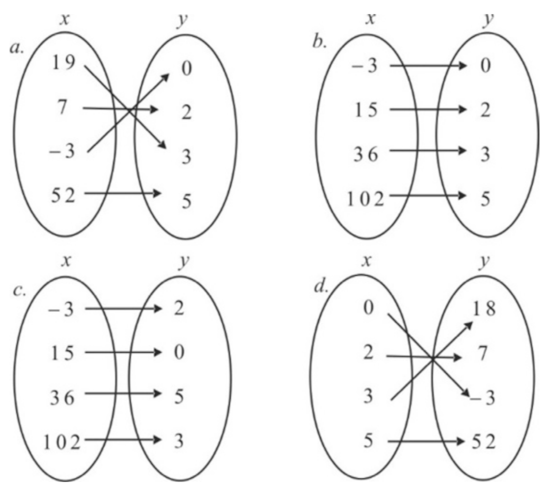


Figure 2.

Analysis: (Algebraic to Numeric and/or Algebraic to Graphic)

a. Incorrect since the x- and y-values are reversed. Students commonly make this mistake when they are in a hurry. Teaching them to be careful is a component of good teaching.
b. Incorrect since the x- and y-values are reversed, and the order of operations was not performed correctly to obtain the y-values; with $2x^2$, you must first deal with the squaring, then multiply by the constant, not the other way around. The issues with the $2x^2$ are most common.
c. Incorrect because although the x- and y-values are in the correct places, the order of operations was not performed correctly to obtain the y-values; you must first deal with the squaring, then multiply by the constant, not the other way around.
d. Correct since the x- and y-values are in the correct places, and the order of operations was performed correctly (the squaring was performed first followed by the product with the constant).

Question 3. Eric studies the parabola below.

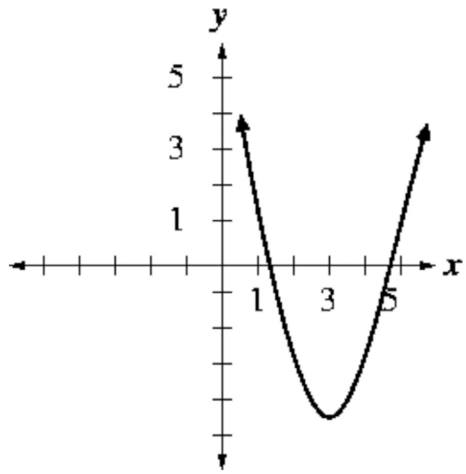


Figure 3.

What is an accurate conclusion Eric could make about this parabola?

a. The vertex is at (–4, 3).
b. The minimum value is at (5, 1).
c. The maximum value is at (3, –4).
d. The axis of symmetry is the line $x = 3$.

Analysis: (Araphic to Numeric)

a. Incorrect since the *x*- and *y*-values are reversed. Incorrect because (5, 1) is simply a point on the parabola, not the minimum value. Clearly, this is an answer that students can eliminate. Such elimination enhances to the opportunity get the correct answer without actually knowing how to solve the problem.
b. Incorrect because (3, -4) is the minimum, not the maximum value.
c. Correct since you could "fold" the parabola about the vertical line at $x = 3$, and both halves of the parabola would match up.

MEDIUM LEVEL QUESTIONS WITH ANALYSIS

In this section, we consider examples of some TAKS-like questions that are not quite as straightforward as the previous ones. All involve some aspect of multiple representations.

Question 4. In the graph of the function $f(x) = 2x^2 + x - 3$, which describes the shift in the vertex of the parabola if, in the function, 4 is changed to –2?

a. The vertex is translated down 2 units.
b. The vertex is shifted 2 units to the left.
c. The vertex is shifted 6 units to the right.
d. The vertex is translated down 6 units.

Analysis: (Algebraic to Graphic)

Here the student must visualize the graphics of a parabola or sketch the graph of the parabola in question. Another issue here is the use of the term "shift." Another term is "translate." On some versions of exams the term "translate" is used exclusively. The term "shift" is also used in instruction. Both terms should be used as both are used in common practice.

a. Incorrect because the shift down to –2 was a total of 6 units, not 2
b. Incorrect because a change in the constant results in a shift up or down
c. Incorrect because a change in the constant results in a shift up or down
d. Correct because a change in the constant results in a shift up or down; specifically, to go from 4 to –2 is a change of 6 units

A similar question also applies to any function *g(x)* by considering $g(x-10)+4$. In this case, the student must be aware from an algebraic or graphic perspective, the graph of $g(x-10)+4$ has the same form as $g(x)$ but shifted (translated) 10 units to the right and four units up. With the replacement of the 4 by -2, the translation of the vertex or minimum is downward by six units.

Translation problems are extremely important when studying algebra from the parent function viewpoint. The reason is that the parent form of the function is relatively easy to understand graphically, numerically and algebraically.

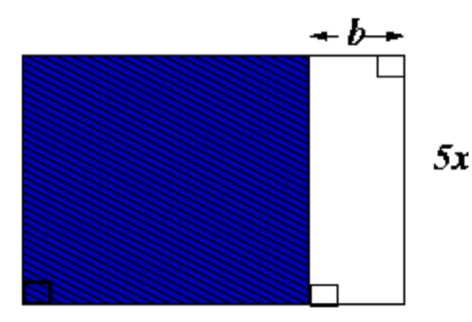


Figure 4. Graphics and quadratics

Question 5. In the figure below, assume b = 5. Find an expression for the shaded area.

a. $25x$ b. $5x(x-c)$ c. $5(x-c)$ d. $2(x-c+5)+10x$

Analysis: (Graphic to Algebraic)

If anything, this example illustrates the wide diversity of problems involving the six types of informational transfer. The value of b is irrelevant to this problem. Many students feel that when information of any type is given, it must be used in the resolution of the problem.

a. Incorrect because $25x$ is the area of the unshaded rectangle
b. Correct since the area is the length times the width, and the correct length and width were multiplied together
c. Incorrect because 5 is the width for the unshaded rectangle
d. Incorrect because the *perimeter* of the entire rectangle (shaded plus unshaded parts) was found

This problem can be made more challenging, for example, by asking what values of *x, b* and *c* are required to make the shaded area have some specified value. In this case, the student must realize that *b* is still irrelevant and any value of *c* can be specified to solve for the value of *x*. Or any value of *x* can be chosen and the resulting product $x(x-c)$ for (c).

Question 6. Find the value of *w* to make the area of the figure below equal to 13.

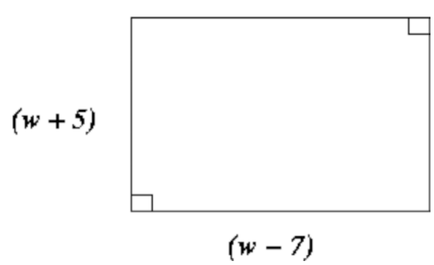


Figure 5. Area as a quadratic

a. 6 b. 7.5 c. 8 d. 7

Analysis: (Araphic to Algebraic to Solution)

In this example, we begin with a graphic from which the student is to construct a quantity, the area, and then determine an unknown so that the area satisfies some condition. Hence, this requires two fundamental steps. The area is $(w+5)(w-7)=13$. Expanding and simplifying, we have $(w-8)(w+6)=0$.

a. Incorrect because both factors are non-zero. This means the product cannot be zero. Additionally, with the answer 6, one of the side lengths is negative.
b. Incorrect because the length and width were added and set equal to 13, rather than multiplied and set equal to 13.
c. Correct because the length and width were multiplied correctly and set equal to 13; the equation was set equal to 0, re-factored and *w* solved for correctly.
d. Incorrect because the length and width were not multiplied correctly; the middle term was left out; also, the answer is approximated, not exact.

This problem can also be analyzed by noting that the side length, $x-7$, is zero or negative for all answers except c. The problem can be made a little simpler by relating the graphic to the perimeter. This will give a linear equation to be solved. It could be a good idea to precede the quadratic (area) problem with the linear (perimeter) one. For perimeter problems, more complex figures can be considered. Consider this Question.

Find the value of *x* so that the perimeter of the region shown is 10.

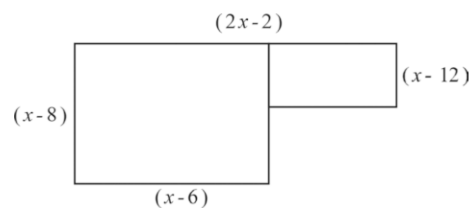


Figure 6.

The task here is to find the side lengths not given and then to add all lengths and solve the resulting perimeter for the value of *x* giving the perimeter to be 10. This rather simple problem is made difficult only by the number of separate steps required to solve it.

Question 7. The national sales record for a recent hit DVD movie is shown on the graph below. Which statement best describes the sales of this DVD?

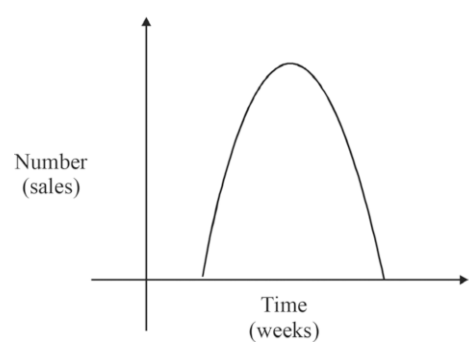


Figure 7. Complex of rectangles and quadtratics

a. Sales rapidly increased, reached a peak, and then gradually decreased.
b. Sales gradually increased, reached a peak, and then leveled off.
c. Sales rapidly increased, reached a peak, and then rapidly decreased.
d. Sales remained constant throughout the time period.

Analysis (Graphic to Numeric)

More precisely than graphic to numeric, this problem asks for qualitative information about the sales graph. Clearly answer (c) is correct. However, with just a cursory reading, the student may select answer a. Here, we have another situation where hastiness on the part of the student, or unclear thinking about the words, can lead a student to a wrong answer. We see in this example the importance of consistent accurate verbal descriptions of graphical aspects. A variation on the previous problem has more subtle answers.

Question 8. The sorghum production in Texas, in millions of metric tons, is illustrated in the graph below. What conclusion can be made from this graph?

a. More fertilizer should be used in 1997 to increase production.
b. The 1996 decline in production indicates a future trend for the next few years.
c. Production now exceeds demand.
d. There is a trend indicating an increase in the production of sorghum.

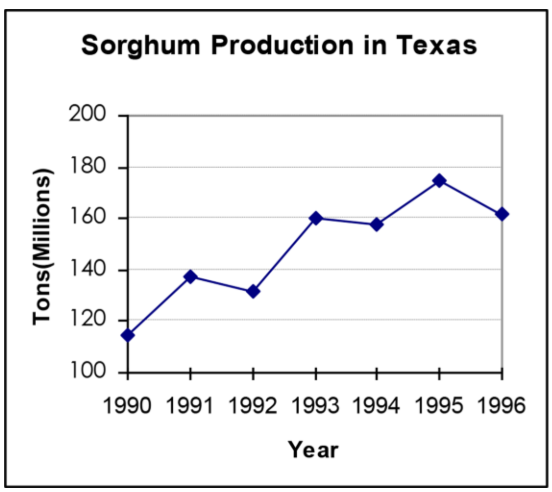


Figure 8.

Analysis (Graphic to Numeric)

This is a problem almost directly taken from the deprecated TAAS exam. More precisely than graphic to numeric, this problem asks for qualitative information about the agricultural production graph. Arguably, the answer could be argued to be a. However, answer d is more accurate to the situation at hand. Note here a qualitative judgment about the trend of information is given. Note, now if we had given numeric data rather than graphic data, the question is more difficult to answer.

Sorghum Production

Year	Tons
1990	115
1991	137
1992	132
1993	160
1994	158
1995	175
1996	162

The graphic more clearly represents the information in the form where a trend can be discerned.

CONCLUSIONS

It is very important that teachers understand student difficulties is parsing a question and successfully solving it. It is also important for students to have a comprehensive understanding of the various forms of multiple representations. It is important for teachers to understand that simply teaching to the test obviates the serious issues that such instruction merely gives students a key to parsing questions that are rather similar to those studied, without any deep understanding. The obvious consequence is that such instruction shorts students' ultimate needs. On the other hand, it is well-known that the study of past exam problems does provide considerable benefit to students if this study is more general in its nature, requiring them to independently see solution methods. Some teachers give their students hints too soon. This practice defeats the goals. Learning is time-consuming, more for some than others. Learning by students and patience by teachers are sometimes the two sides of the same learning equation. Learning multiple representations is a significant part of student understanding. Indeed, multiple representations form a significant part of the analysis of *big data*,[1] the topic of the next in this series of articles.

Discussion Questions and Activities

1. Make three variations of Question 5 involving functions other quadratics, e.g. linear functions and cubic functions.
2. Make two graphics, one of which is translated horizontally and vertically from the other. Formulate a graphic to numeric question about either the *x*- or *y*-intercepts.

3. In general, classify the types of problems of the graphic to numeric variety about two graphics, where one is translated horizontally or vertically from the other. Develop a succinct lesson plan that encompasses all of these together. This lesson plan should be independent of the particular function under study. That is, it should apply to all functions.
4. List the qualitative properties of functions. Pose questions that ask for transfer of knowledge about graphic functions and qualitative properties.
5. What are the issues behind generalizing from tabular data to a function and functional representation?
6. For each of the actual problems above, specify which of the standards/objectives for Algebra I fits the problem.

NOTE

[1] *Big data* is often defined for data where its size is part of the problem.

G. DONALD ALLEN

61. MULTIPLE REPRESENTATIONS, IV

Estimating π. There are numerous methods for estimating π. For one thing, there are various formulae, as we've seen in the previous paper on some historical valuations. These have been known since ancient times. Our list was incomplete; there are others, as well. But... Let's fast forward to the more modern era. In recent times, i.e. only a couple of centuries ago, and using *calculus,* there newly appeared one of the first series type formulas for approximating involving the arc tangent function. We know $\tan^{-1} x = \int \frac{1}{1+x^2} dx,$ and we know the fraction $\frac{1}{x^2+1}$ can be expanded as the *geometric series*

$$\frac{1}{x^2+1} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - x^{14} + \ldots$$

This series is called *alternating* because the signs of the terms alternate between positive and negative. We also know

$$\tan^{-1} 1 = \frac{\pi}{4}$$

Thus integrating the infinite series we obtain

$$\int \frac{1}{x^2+4} dx = \int \left(1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - x^{14} + \ldots\right) dx$$
$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \frac{1}{13}x^{13} - \frac{1}{15}x^{15} + \ldots$$

Evaluating at x = 1, then multiplying by 4, we have

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$$
$$= 4\left(\sum_{j=1}^{\infty} \frac{(-1)\, j+1}{2j+1}\right)$$

The partial sums, namely

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 289–294.

$$S_n = 4\left(\sum_{n=1}^{n} \frac{(-1)^{j+1}}{2j+1}\right)$$

are what is needed for our computation. While we know lim $S_n = \pi$, this does not help much with the actual numerical approximation. For the record, to 50 places

$$\pi = 3.14159265358979323846264338327950288419716939937511$$

We will never use this number of digits, but there is it. Indeed, only 32 digits of accuracy are required for every possible cosmological need for the computation of π – no matter what the size of the (finite) universe may be. However, from this formula, we know it is possible to approximate π to any number of places, provided you use a sufficient number of digits. How good is this approximation? To determine this, consider summing up to just 26 terms. Shown in Figure 1 below, we see the approximation is *not good.*

n	Partial Sum		n	Partial Sum
1	4		14	3.070254618
2	2.666666667		15	3.208185652
3	3.466666667		16	3.079153394
4	2.895238095		17	3.200365515
5	3.33968254		18	3.086079801
6	2.976046176		19	3.194187909
7	3.283738484		20	3.091623807
8	3.017071817		21	3.189184782
9	3.252365935		22	3.096161526
10	3.041839619		23	3.185050415
11	3.232315809		24	3.099944032
12	3.058402766		25	3.181576685
13	3.218402766		26	3.103145313

Figure 1. Approximating pi using the arc tangent function

As you can see, the approximations alternate between a larger number and a number smaller than π. As you may also see, none of these approximations are very good. In Figure 2, we look at the graph of these approximations. The horizontal line is vertically placed exactly at π. These are not the terms of the approximating series, but the actual partial sums. It clearly demonstrates the oscillatory behavior just described.

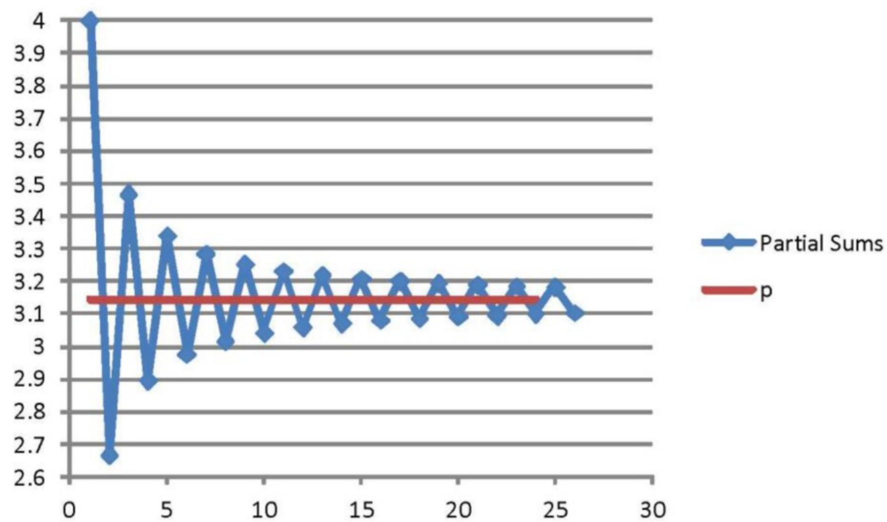


Figure 2. Partial sums shown in graph

What you should notice is the the oscillations above and below are very, very close. Indeed, as the series is alternating, we know that at the nth term, the error of approximation is less than the magnitude of the (n + 1)st term and of the opposite sign. This notwithstanding, an intuitive approach to improving the approximation would simply take the averages of pairs of successive terms. This improves the calculation remarkably. Before showing the calculation, we computed the partial sums to 1000 terms using twenty decimal places of accuracy to get the approximation

$$\pi = 3.14059265383979292\,60$$

with the error of approximation to be .0009999997500003125 or about 10^{-3}. This was a lot of computation to get such a poor approximation. Indeed, this approximation is just marginally better than using 22/7, which you know from high school days. However, if Archimedes had know this series, he could have achieved a far better estimate of π than he did and with far less effort.

Now, let's look at the partial sum and averaging data. First we show the data in chart form, with the error of approximation from π. We will notice something quite remarkable.

n	Partial Sums	Error	Averages	Error
1	4	0.858407346	4	0.8584073
2	2.666666667	-0.474925987	3.3333333	0.1917407
3	3.466666667	0.325074013	3.0666667	-0.074926
4	2.895238095	-0.246354558	3.1809524	0.0393597
5	3.33968254	0.198089886	3.1174603	-0.0241323
6	2.976046176	-0.165546478	3.1578644	0.0162717
7	3.283738484	0.14214583	3.1298923	-0.0117003
8	3.017071817	-0.124520837	3.1504052	0.0088125
9	3.252365935	0.110773281	3.1347189	-0.0068738
10	3.041839619	-0.099753035	3.1471028	0.0055101
11	3.232315809	0.090723156	3.1370777	-0.0045149
12	3.058402766	-0.083189888	3.1453593	0.0037666
13	3.218402766	0.076810112	3.1384028	-0.0031899

Figure 3. Approximating pi using the arc tangent function

n	Partial Sums	Error	Averages	Error
14	3.070254618	-0.071338036	3.144328	0.002736
15	3.208185652	0.066592999	3.139220	-0.0023725
16	3.079153394	-0.062439259	3.143669	0.0020769
17	3.200365515	0.058772862	3.139759	-0.0018332
18	3.086079801	-0.055512852	3.143222	0.00163
19	3.194187909	0.052595256	3.140133	-0.0014588
20	3.091623807	-0.049968847	3.142905	0.0013132
21	3.189184782	0.047592129	3.140404	-0.0011884
22	3.096161526	-0.045431127	3.142673	0.0010805
23	3.185050415	0.043457762	3.14060	-0.0009867
24	3.099944032	-0.041648621	3.142497	0.0009046
25	3.181576685	0.039984032	3.140760	-0.0008323
26	3.103145313	-0.038447341	3.14236	0.0007683

Figure 4. Approximating pi using the arc tangent function

First, notice that by using only 26 terms, we have achieved a better estimate using averages than by using the 1000th partial sum. Second, notice the oscillating behavior of the error of the original partial sums, this time numerically. And as before, notice the error of approximation is not improving by much. By the 26th computation, the error is about -0.038 which is, terrible for all the work you've done. But now look at those successive averages and the associated errors. True to our guess, the averages are much better approximations to π. What is so remarkable about those errors is that they also alternate; so, maybe we can try the averaging trick again.

In fact we can. We will call these the second order averages. We will note that the errors also oscillate leaving us to compute averages again. This continues on and on. In the table below, we show all of these higher order averages and we do it for just the first thirteen terms in the partial sums. The results, averages, averages of averages, averages of averages of averages, and so on are shown in Figure 5.

Typ^e	**Approximation to pi**	**Error of Approximation**
Partial Sum	3.218402766	7.681011E-02
Averages	3.138402766	-3.189888E-03
Averages (2)*	3.141881027	2.883732E-04
Averages (3)**	3.141549764	-4.288973E-05
Averages (4)	3.141602069	9.414942E-06
Averages (5)	3.141589762	-2.892040E-06
Averages (6)	3.141593864	1.210287E-06
Averages (7)	3.141591970	-6.830949E-07
Averages (8)	3.141593175	5.217845E-07
Averages (9)	3.141592104	-5.492194E-07
Averages (10)	3.141593481	8.277856E-07

* denotes averages of the averages
** denotes averages of the averages of the averages

Figure 5. Partial sums up to n = 13

What is so remarkable is that we ultimately approximate π to within 7 decimal places. The approximation keeps getting better until the 10th level of averaging. Using higher precision arithmetic, the approximation is better still. One important note here is that we use *terrible data to achieve very accurate results.* It is quite another matter, and beyond the scope of this note, to demonstrate (i.e. prove) that these complex averages of averages converge to π.

SUMMARY

What we have constructed here with only our simple intuition and the multiple representations is called a *summability* method. More precisely, we have derived an acceleration method. An acceleration method speeds the convergence. Mathematicians love these, and have been studying them for the last three centuries. This one is rather new, as the author could find no previous reference. As a more theoretical development shows the binomial expansion is part of the formulation, we may call it *binomial summability.* As a top-notch approximating method for π, it is not that good. Some methods, those used in practice to achieve 10 trillion digits (about the current known accuracy) of accuracy, are a more sophisticated, some achieving 25 digits of accuracy at each step. However, our method is quite effective for everyday computation. It can be applied to many alternating series, particularly those that converge slowly. However, the bottom line is that without the multiple representations of the data at hand, it may not have been possible to conceive its very possibility. In this situation, it involved merely a moment's reflection, and then verification that averaging can go a long way.

NOTE

[1] Spreadsheets are available upon request: dallen@math.tamu.edu

REFERENCES

Brezinski, C. (1980). *Pade-type approximation and general orthogonal polynomials* (International Series of Numerical Analysis, Vol. 50). Birkhauser, Basel: Springer Basel AG.

Calculus. Consult any calculus book for information on alternating series.

Drummond, J. E. (1972). A formula for accelerating the convergence of a general series. *Bulletin of the Australian Mathematical Society, 6*, 69–74.

Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. (2007). *Numerical recipes 3rd edition: The art of scientific computing.* New York, NY: Cambridge University Press.

Smith, D. A., & Ford, W. F. (1979). Acceleration of linear and logarithmic convergence. *SIAM Journal on Numerical Analysis*, *16*, 223–240.

G. DONALD ALLEN AND DIANNE GOLDSBY

62. THE TWO-MINUTE LESSON

Most instruction in the schools is highly formalized, complete with curricular specifications, implying lesson plans, talking points, and testing measures. Teachers are constrained by the formalism; teachers are almost powerless to teach beyond the specs. In some cases, they must follow a rote curriculum purchased by the school district. However, the teacher does have some latitude; the teacher can sneak in lasting instruction if only with a few spare classroom minutes. In this note, we consider a personal experience and then add a few suggestions based on it.

A personal experience. We frequently see "man on the street" segments on various channels of how ignorant Americans are about geography. We're not discussing where Burkina Faso is on the map but where the heck is Iowa? Many, too many Americans just don't know. The why is simple; it is simply not taught well in the schools.

Here's how I taught my kids the States and the world years ago. I posted a map of the USA on the wall near the breakfast table. This map contained only the states without names. Every morning while the kids were scarfing down their corn flakes, I would point to a state and ask its name. By and by, they knew every state, including the little ones. This process took years. So, repetition over years does work. Indeed, this is the way, for example, that math is taught, though in a more formalized and ponderous way.

We call this "reinforced or repetitive learning." It is the way the coach teaches the team, and the best coaches are almost always excellent teachers.

Expanding the experience. So, let's make suggestions to teachers for grades K-8 and beyond, even to college. We are advocating the two-minute lesson, a tool having roots in antiquity but deprecated by the formal world of curriculum and textbooks.

For geography, every morning, show the map of the USA without names. Point to states and ask the class which it is. Begin with the local state, move to its neighbors and then beyond. Some kid will always know, but which kid it is may vary. In a single school year, a teacher can cover this map at least four times. The process takes two minutes max. Now multiply this effect over eight years. The good students will learn them all, and these mini-lessons will last a lifetime. Even the dullest student will pick up a lot, and know something important, namely the lay of our Land.

On the matter of vocabulary, the teacher can also stage a "word of the day" event. It could be made exciting by giving "big" words such as "exemplary." Give a word and then, give the definition. Repeat two or three times a day. Reinforce throughout the week. The words and definitions can be written on sentence strips (or

G. D. Allen & A. Ross (Eds.), Pedagogy and Content in Middle and High School Mathematics, 295–297.

cash register tape) and posted in a vocabulary section on the wall. You could even have an “Outlaw Word” section – words students should avoid such as “borrow” in subtraction. At this age, students’ brains are sponges, and they learn despite their best efforts not to.

In teaching shapes, such as squares, triangles, trapezoids, ellipses, and the like, a similar procedure is applicable. This couple of minutes per day intervention can cement into the student’s mind the notions. No technical details are needed, though some attention to comparative issues should be included. For three-dimensional shapes, similar activities can be designed using actual shapes, perhaps made of Styrofoam. When the lesson is very short and when there are no other items to come, the student quickly absorbs the content. This is, or can be, incorporated into “Calendar Time” in the primary grades and “Do Nows” or warm-up problems/questions at the beginning of class in the intermediate, middle, and high school grades. These student-centered activities tend to decrease student attention lapses and increase student attention during lecture portions which follow (Bunce, Flens, & Neiles, 2010) and serve to focus attention at the beginning of class.

> Repetition is the mother of learning and is an essential key to the physical development of a child’s brain” (PBS, 2010, p. 1) as evidenced by the success of PBS shows for children and the impact these have had.

“As a skill is practiced or rehearsed over days and weeks, the activity becomes easier and easier while naturally forcing the skill to a subconscious level where it becomes permanently stored for recall and habitual use at any time” (PLB, 2016). Practice if properly conducted can be effective – not the mindless drill and kill but systematic practice and review. Repeated experience is what wires a child’s brain, say neuroscientists (Nash, 1997).

In teaching elements of algebra combined with geometry, simple interventions to clarify the notions of slope, midpoint, and distance between points can not only expose misconceptions but resolve them. Similarly, the notions of local and absolute extrema can be illustrated – and taught – and learned. Again, a quick review problem to start the class can keep these concepts before students and help them retain through multiple reviews. Also, a small number of review problems, say 4 or 5, on previous topics can be included in the homework. A sheet of such problems can be given at appropriate times and students can reference as part of the homework assignment. At the beginning of class, the review of these problems can be done quickly. This repeated review and practice can strengthen and cement these ideas for students.

In biology, the notions of cell, nucleus, cell wall and other cell geography can be taught years before the formal biology class. Pre-and-informal knowledge is a most important learning tool, preceding the confrontation and the rigors of the actual knowledge. When done effectively, these mini-lectures or two minute lessons, can transmit new information in an efficient way, explain or clarify ideas, organize concepts and thinking, challenge beliefs, model problem solving, and foster enthusiasm and a motivation for learning (Kumar, 2003).

The power of the two-minute lesson is without rival. It is efficient and effective. It is a cheap extremal in the teaching and learning process. It can be used to reinforce what has been taught, to suggest new concepts, and to preview what is to come. It works!

REFERENCES

Bunce, D. M., Flens, E. A., & Neiles, K. Y. (2010). How long can students pay attention in class? A study of student attention decline using clickers. *Journal of Chemical Education, 87*(12), 1438–1443.

Kumar, S. (2003). An innovative method to enhance interaction during lecture sessions. *Advances in Psychology Education, 27*(1), 20–25.

Nash, J. M. (1997, February). Special report: Fertile minds from birth, a baby's brain cells proliferate wildly, making connections that may shape a lifetime of experience. The first three years are crucial. *Time*, *3*.

PBS. (2010). *Learning through the early years: The benefits of repetition and variation.* Retrieved from http://kbyutv.org/kidsandfamily/readytolearn/file.axd?file=2011%2f3%2f10+Learning+Through+the+Early+Years.pdf

Professional Learning Board. Retrieved January 21, 2016, from https://k12teacherstaffdevelopment.com/tlb/learning-by-repetition-does-it-work/