RONGJIN HUANG AND FREDERICK K. S. LEUNG

8. TEACHING GEOMETRICAL CONCEPTS THROUGH VARIATION

A Case Study of a Shanghai Lesson

INTRODUCTION

Chinese students' superior performance in mathematics in various international comparative studies (Fan & Zhu, 2004; OECD, 2010, 2014) has led to an increasing interest in exploring the characteristics of mathematics instruction in China (Fan, Wong, Cai, & Li, 2015; Li & Huang, 2013). Mathematics classroom instruction in China has been described as being conducted in large classes and teacher dominated, with students being portrayed as passive learners (Leung, 2005; Stevenson & Lee, 1995). On the other hand, Chinese classrooms have also been found to be polished (Paine, 1990), fluent and coherent (Chen & Li, 2010), with a focus on the development of important content, problem solving, and proving (Huang & Leung, 2004; Huang, Mok, & Leung, 2006; Leung, 2005). Gu, Huang and Marton (2004) and Gu, Huang and Gu (2017) developed a theory of teaching with variation and argued that it is an effective way to promote meaningful learning in mathematics in large class-size classrooms. Several examples in geometrical concepts and proofs have been used to illustrate the major features of teaching with variation (Gu, 1992; Gu et al., 2004), but there is a lack of investigation into how the principles of teaching with variation could be applied in teaching geometry that promote students' understanding of geometrical concepts. To this end, we aim to deepen understanding of mathematics teaching in China through examining how particular geometry concepts are taught from the perspective of variation.

LITERATURE REVIEW AND THEORETICAL CONSIDERATION

In this section, we first review the literature on the learning of geometrical concepts from a cognitive perspective. Then, variation pedagogy in general and learning geometry from a variation perspective in particular are discussed. Finally, a framework for this study is described.

Teaching Geometry: A Cognitive Perspective

According to Vinner (1991), a mathematical concept consists of two interconnected components: concept definition and concept image. It is important to introduce a

R. Huang & Y. Li (Eds.), Teaching and Learning Mathematics through Variation, 151–168.

concept by exploring carefully organized sets of examples and non-examples. Through comparing examples and non-examples, the discriminating properties of the concept can be identified. Based on this model, Hershkowitz (1990) proposed a sequence of activities for teaching geometrical concepts that include selecting the critical attributes of the concept that students should discover and the non-critical attributes that students often identify erroneously as an example or a non-example; providing an example and a non-example differing in each critical attribute and examples differing in each non-critical attribute. It was noticed that the prototypical images (such as the upright position of a right triangle, the (interior) altitude in a triangle) could be either a starting point of understanding the concept or a limitation on concept formation (Vinner & Hershkowitz, 1983; Vinner, 1981). Students and pre-service teachers tended to make their judgment based on prototypical examples resulting in incomplete concept images such as failing to draw an altitude when the base needs to be extended (Hershkowitz, 1990). Exploring various non-prototypical images could be used to develop analytical strategies that are based on definition and logical analysis. To process or operate figures in geometry, Duval (1996, 1999) highlighted the ways of reconfiguration, namely, dividing a given whole figure into parts of various shapes and then combing their parts in another whole figure or making new subfigures. For example, a parallelogram is changed into a rectangle, or can appear by combining triangles. Different operations with a figure give different insights into solving a problem.

In sum, from a cognitive perspective, it is essential to explore both prototypical and not-prototypical concept images, and compare concept examples and non-concept examples. In addition, developing the ability of reconfiguration within a given figure is critical for solving geometry problems.

Teaching Geometry: Perspectives from Variation Pedagogy

According to Marton and Tsui (2004), learning is a process in which learners develop a certain capability or a certain way of seeing or experiencing. In order to see something in a certain way the learner must discern certain features of that object. Experiencing variation is an essential experience for discernment, thus significant for learning. Marton and Pang (2006) further argued that it is important to pay attention to what varies and what is invariant in a learning situation. Objects of learning include a general and a specific aspect. The general aspect has to do with the nature of the capability such as remembering, interpreting and grasping. The specific aspect has to do with the subject on which these acts of learning are carried out, such as formulas and simultaneous equations. Teachers are often conscious of this object of learning and they may elaborate it in different degrees of detail. What teachers are striving for is the intended object of learning, which is an object of the teacher's awareness. However, what is more important is how the teacher structures the lessons so that it is possible for the object of learning to come to the fore of the students' awareness, which is called the enacted object of learning (Marton & Pang, 2006).

Interestingly, a theory of mathematics teaching/learning, called teaching with variation, has been developed based on a series longitudinal mathematics teaching experiments in China (Gu, 1994; Gu et al., 2004). According to this theory, meaningful learning enables learners to establish a substantial and non-arbitrary connection between the new knowledge and their previous knowledge (Ausubel, 1968). Classroom activities are developed to help students establish this kind of connection by experiencing certain dimensions of variation. Two types of variation are identified as important patterns of variation for meaningful learning: "conceptual variation" and "procedural variation" (Gu et al., 2004). Conceptual variation aims to provide students with multiple experiences from different perspectives. On the other hand, procedural variation is concerned with the process of forming a concept logically or historically, arriving at solutions to problems (scaffolding, transformation), and forming knowledge structures (relationship among different concepts) (Gu et al., 2004). With regard to teaching geometry, Gu (1994) identified specific patterns of variation. For example, to explore critical features of a geometrical concept, concept figures and non-concept figures have to be compared; and both prototypical and non-prototypical examples should be explored. These are *conceptual variations* serving for developing a deep understanding of concepts from multiple perspectives. To solve geometrical problems, *procedural variations* such as reconfiguring within a given complex figure; or transforming prototypical figures to a complex figure are needed (Gu et al., 2004, 2017).

A Framework for the Current Study

The description of variations in geometry by Gu et al. (2004) is supported by cognitive theories of geometry learning. In addition, Marton and Pang (2006)'s notions of objects of learning provide a lens for examining possible learning opportunities. Thus, both Gu et al.'s (2004) classification of variation and Marton's notions of enacted objects of learning are adopted to examine classroom teaching of geometrical concepts.

A CASE STUDY

Data Source

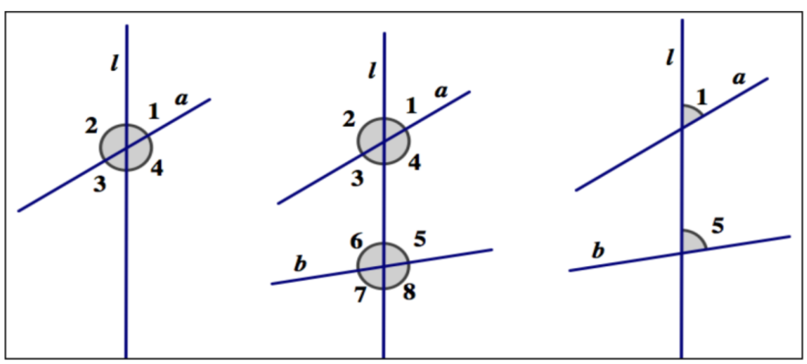
A videotaped seventh grade lesson used as evidence for the Excellent Young Teacher Award in Shanghai in 1999 constitutes the data source for this study. The lesson was taught by a young teacher (less than 5 years of teaching experience) to 56 students in a junior high school located in the countryside of Shanghai. This lesson is a typical and excellent lesson recommended by local teaching research specialists. The lesson was transcribed (in Chinese) verbatim. To ensure the validity of lesson analysis the video recording was referred to when needed. The lesson was analyzed based on Gu et al.'s (2004) classifications.

Description of the Lesson

The topic of the lesson is “Corresponding angles, alternate angles, and consecutive interior angles on the same side of the transversal”. By and large, the lesson included the following stages: review, exploration of the new concept, examples and practices, and summary and assignment.

Reviewing and inducing. At the beginning of the lesson, the teacher drew two straight lines crossing each other (Figure 1(I)) on the blackboard, and asked students to use their previous knowledge (such as concepts of vertical angles and supplementary angles) to answer some review questions. After obtaining correct answers to those questions from the students, the teacher added one straight line to the previous figure (see Figure 1(II)) and asked students how many angles there are in the figure, and how many of them are vertical angles and supplementary angles. After that, the students were guided to explore the characteristics of a pair of angles from different vertices by being asked, “what relations are there between ∠1 and ∠5?”, which actually is the new topic to be explored for this lesson.

Exploring new concepts. In order to examine the relationship between ∠1 and ∠5, a particular figure was isolated as shown in Figure 1(III). Through group discussion, the students found many features about these two angles, such as “∠1 and ∠5 are both on the right side of line 1, and above line a, and b”. Based on the students' explanations, the teacher summarized and stated the definition of “corresponding angles”. Then the students were asked to identify all the “corresponding angles” in Figure 1(II).



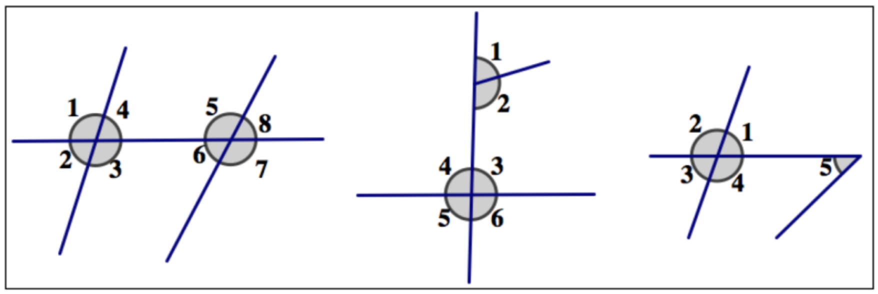
1(I) 1(II) 1(III)

Figure 1. Angle relationship in transversal figures

Similarly, another two concepts, “alternate angles and consecutive interior angles” were explored respectively.

Examples and exercise. After introducing the three angle relationships, students were asked to identify them in different configurations. The problems are as follows:

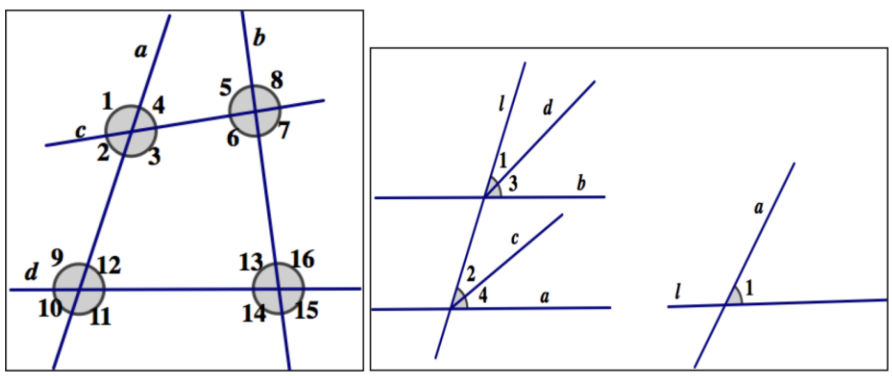
Task 1: Find the "corresponding angles, alternate angles, and consecutive interior angles on the same side of transversal" in Figure 2:



2(I) *2(II)* *2(II I)*

Figure 2. Angle relationship within various transversals

Task 2: Find the "corresponding angles, alternate angles, and consecutive interior angles on the same side of transversal" in Figure 3(I).



3(I) 3(II) 3(III)

Figure 3. Angle relationships in more complex situations

Task 3: In Figure 3(II), (1) Are ∠1 and ∠2 a pair of corresponding angles? (2) Are ∠3 and ∠4 a pair of corresponding angles?

Task 4: Given ∠1 is formed by line *l* and line *a* as shown in Figure 3(III). (1) Add one line b so that ∠2 formed by line l and line b, and ∠1 are a pair of corresponding angles. (2) Is it possible to construct such a line *b* so that ∠2 (formed by line *l* and line *b*) is equal to ∠1?

Summary and assignment. The teacher emphasized that these three types of relationship are related to two angles at different vertices. These angles are located in a "prototypical figure" which consists of two straight lines intersected by a third line. The key to judge these relationships within a complicated figure is to isolate a proper "prototypical figure" which includes these angles in question. Moreover, the teacher demonstrated how to remember these relationships by making use of different gestures as shown Figure 4 below.

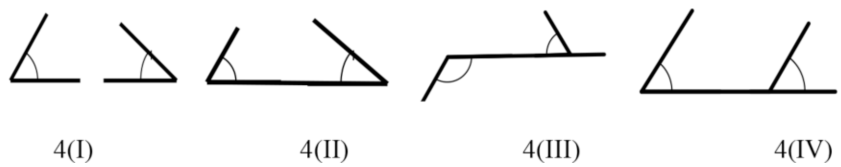


Figure 4. Presenting angle relationship using finger gestures

Finally, some exercises from the textbook were assigned to students.

Enacted Objects of Learning

From the perspective of variation, and in order to examine what learning is made possible, we need to identify what dimensions of variation are constructed. Below we look at the lesson in greater detail from this particular theoretical perspective to identify the enacted object of learning and possible learning opportunities.

Procedural variation 1: Reviewing previous knowledge and bringing the new topic to the fore of students' awareness. At the first stage, *a variation*: varying from two intersecting straight lines to two straight lines intersected by a third one, was created by the teacher's demonstration and questioning. Through questioning students know how many angles there are in the new figures, and what relationships there are among those angles. A cognitive conflict with the previous knowledge reviewed about how to determine the relationship of angles at different vertices was then raised, which is the new topic to be explored in this lesson.

1. T: … now, I've drawn one straight line b to the two intersecting straight line l (see Figure 1(II)), then how many angles are there in the figure?
2. S: Four angles [in unison]
3. T: Good. Increasing by four angles, then, how many angles are there in the figure: two lines intersected by a third line?
4. S: Eight angles [in unison]
5. T: Let's label the added angles as ∠5, ∠6, ∠7, ∠8. We call this figure as "straight line a and b intersected by a straight line l" [the teacher writes the part and highlights it with underline]. Then, there are eight angles. How

many vertical angles are there among them? How many supplementary angles are there among them? [The teacher repeated these questions]. Good, Pan Hong [nominating him]

6. Pan: There are four pairs of vertical angles, and eight pairs of supplementary angles.
7. T: Good! There are four pairs of vertical angles, and eight pairs of supplementary angles. Very good! Good, just now, I reviewed that all pairs of vertical angles and supplementary angles which are formed at the same point. Today, we are going to study the angle relationships among the angles formed at different vertices. For example, ∠1 and ∠5.
8. T: [Demonstrating by transparency as Figure 1(II)] How many angles are there in the figure: straight lines a and b intersected by straight line l?
9. S: Eight angles [in unison]
10. T: Good! Then, we study the positional relationship between two angles, which are at different vertices, such as ∠1 and ∠5. In order to make clear the positional relationship between ∠1 and ∠5, we isolate them from the figure, as showed in Figure 1(III) (demonstrating by transparency). Good! What are the positional features of ∠1 and ∠5.

In the above excerpt, the teacher guided students to construct a "prototypical figure" (e.g., transversal) and review previous knowledge (1~6), then the teacher drew students' attention to the angle relationship located at different points by contrasting with previous concepts: the angles at the same point (8). In order to examine the new relationship clearly, the teacher isolated the focused angles from the complex Figure 1(II), as shown in Figure 1(III). By isolating the focused sub-figure, the teacher tried to help students to clearly identify the characteristics of these angle relationships, and utilize a typical "isolation method", namely, isolating a focused subfigure from a complex figure in problem solving in geometry (Gu et al., 2004).

By opening with this variation (i.e., adding one new line while two intersecting lines remain the same), the relevant previous knowledge was reviewed and the new topic was introduced in a sequential and cognitively connected manner. Thus, this variation is a procedural variation.

At the introducing new concepts stage, two variations were created which are crucial for students to develop an understanding of the new concepts**.**

Conceptual Variation 1: descriptions of new concepts. During the process of forming the new concepts, expressions of the new concepts have been shifted among the following forms: rough description, intuitive description, definition, and schema. After a group discussion, the students were invited to present their observations, and the new concepts were built based on students' descriptions under the teacher's guidance as shown in the following excerpt.

1. T: …good! What are the characteristics of the pairs of ∠1 and ∠5 in terms of their positions in the figure?" (Pointing to Figure 1(III) shown on the transparency). Please discuss this question in groups of 4-students [at once, the 4-student groups were organized: the students at the row in the front turn back so that the 4 students sit around a desk. Then students discuss actively and the teacher circulates around the classroom assisting students occasionally].
2. T: Good! Just now, students have an active discussion. I would like to ask one student to answer: What is the characteristics of the pair of ∠1 and ∠5 in terms of their position? [Pause] Fang Xiuting (who raised his hand), please.
3. Fang: ∠1 and ∠5 are on the right side, and…
4. T: ∠1 and ∠5 are on the right side. Please, explain [it] in more detail. For example, what is the relationship of ∠1 and ∠5 with regard to the straight line l in terms of their positions? Moreover, what is the relationship of them with regard to the straight lines a and b?
5. Fang: With regard to straight line l, ∠1 and ∠5 are on the right side of it. Regarding straight lines a and b, all the two angles are above the two straight lines respectively.
6. T: Good! Very good! Thus, we call the two parts of the plane divided by the line l as two sides of the straight line l [left side and right side], and call the two parts of the plane divided by lines of a and b as two sides of lines a or b [above and below]. Moreover, we define this pair of angles, which possess the previous characteristics as corresponding angles [In Chinese, the angles with the same position]. [Teacher writes down: corresponding angles: ∠1 and ∠5]. What kind of angles are ∠1 and ∠5 [called]?
7. S: Corresponding angles! [In unison]
8. T: Are there other corresponding angles in the figure [Figure 1(II)]? Cheng Dechong, please.
9. Cheng: ∠4 and ∠5 [hesitation for a moment]. No! No! It should be ∠4 and ∠8.
10. T: ∠4 and ∠8 [write down on blackboard], are there any more?
11. Cheng: ∠2 and ∠6.
12. T: ∠2 and ∠6[write down on blackboard], any more?
13. Cheng: No.
14. T: Very good!

In the above discussion, the representation of "corresponding angles" was transferred from the immature description by students (1~3) to a more precise description through the teacher's probing (4~5), then to a formal definition given by the teacher (6), and finally to a schema, which can be applied in simple situations (7~14). This variation of representation of the concept is a conceptual one.

Conceptual variation 2: Different orientations of "basic/standard/prototypical figure". Through questions and answers between the teacher and students, the concepts of three types of angle relationship were constituted in a "prototypical figure": two straight lines intersected by a third line (Figure 1(III)). After that, the teacher provided students with Task 1. By doing so, a new dimension of variation was opened for students to experience how to identify these angle relationships in different figures with various orientations. The teacher purposely varied the figures in terms of their positions and the number of angles in the figures.

1. T…. Next, I vary the picture (see Figure 2 (I)). Can you identify the corresponding angles, alternate angles, and consecutive interior angles on the same side of the transversal? [Present the problem by transparency]. How many angles are there in the picture?
2. S: 8 angles [in unison]
3. T: Then, what is the relationship between ∠4 and ∠5 in these 8 angles? Shi Chihong, please.
4. Shi: [it is a pair of] corresponding angles.
5. T: Now, I vary the picture again [show the picture by transparency like Figure 2 (II)]. Well, how many angles are there in this picture? [point to student S3]
6. S3: 6 angles.
7. T: 6 angles. Then, how many pairs of alternate angles, and consecutive interior angles on the same side of the transversal in the picture? Please [point the student S3].
8. S3: There are two pairs of corresponding angles.
9. T: Which two pairs are they?
10. S3: ∠3 and ∠1. ∠2 and ∠6

By providing students with these variations, the students were exposed to the concepts from different orientations of the figure. It may make student aware that these concepts are invariant although the orientations of the figure vary.

To consolidate the new concepts and develop a method of solving problems, the following procedural variations were constructed.

Procedural variation 2: Different contexts of "prototypical figures". After the students got a rich experience of these concepts in terms of their orientations of the prototypical figure, the teacher then deliberately provided a group of tasks in which "prototypical figures" were embedded in the complex contexts in Task 2 and Task 3. Through identifying the angle relationships in different contexts of the "prototypical figures", an invariant strategy of problem solving, i.e. identifying and isolating a proper "prototypical figure" (i.e., prototypical image) from a complex configuration. In general, isolating a proper sub-figure from a complex figure is a useful strategy of solving a geometric problem (Gu et al., 2004).

After highlighting the common features in the previous questions: identifying the "prototypical figure", the teacher presented a more complex picture [see Figure 3], and asked students to count the number of angles in the figure, and identify the three types of angle relationships in the figure.

1. T: Good. [There are] 16 angles in this figure. As we learned today, there are three types of angle relationships: corresponding angles, alternate angles, and consecutive interior angles on the same side of the transversal in a "prototypical figure" which consists of two straight lines intersected by a third line. In this picture, there are 4 lines. How can we identify these relationships among the angles? Can you identify these three types of angle relationships according to the given condition in this figure? Please, write down your answer on the worksheet [students think individually]. Please, have a close look at which two straight lines are intersected by which line. Which two straight lines are they in question (1)? [i.e., straight lines *a* and *b* intersected by straight line *d*, find all the corresponding angles, alternate angles, and the consecutive interior angles]
2. S: Straight lines *a* and *b* are intersected by line *d*. [in unison]
3. T: [The teacher demonstrates a transparency as shown in Figure 5(I). Students do seatwork individually, whiles the teacher circulated around the class, with occasional assisting of students] Are you ready?
4. S: Yes! [In unison]
5. T: Who would like to answer the question? Yang Ninao, please. How many pairs of corresponding angles are there?
6. Yang: [there are] 4 pairs of corresponding angles
7. T: What are they?
8. Yang: $\angle 9$ and $\angle 13$
9. T: $\angle 9$ and $\angle 13$ [pointing to the relevant angles]
10. Yang: $\angle 12$ and $\angle 13$.
11. T: $\angle 12$ and $\angle 13$[pointing to the relevant angles]

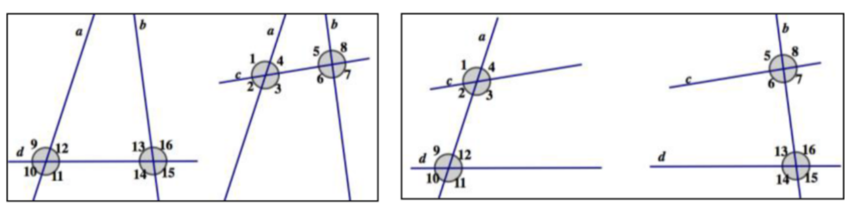
…

After students selected the prototypical figure (1~2), they were asked to identify all three types of angle relationships one by one. The teacher confirmed students' answers by pointing out the relevant angles on the transparency (6~11).

Procedural variation 3: Different directions for applying the new concepts. As soon as the students answered the first question, the teacher posed a new challenging question: "conversely, if $\angle 1$ and $\angle 5$ are a pair of corresponding angles, which prototypical figure contains them?". After allowing individual students to think for a period of time, one student was called on to answer the question. The student gave a correct answer by saying that the prototypical figure is "straight lines *a* and *b* intersected by straight line c"(see Figure 5(II)). The teacher's effort to push students to identify the prototypical figure is evidenced by the following excerpt:

1. T: Think carefully! Which two straight lines intersected by a third line form ∠1 and ∠5, which are a pair of corresponding angles? Are you ready?
2. S: Yes![in unison]
3. T: You, please [point to one student]
4. S2: The straight lines a, b intersected by the straight line c.
5. T: ∠1 and ∠5 are formed by straight lines a, b intersected by line c. Is it right?
6. S: Right! [in unison]
7. T: So, which line in this figure has not been used?
8. S: [Straight line] d.
9. T: In other words, how do we deal with the straight-line d?
10. S: Cover it up!
11. T: [Remove the straight line d from the figure, and form a new figure, see Figure 5(II)]. Is it right?
12. S: Right! [in unison]
13. T: Moreover, if ∠3 and ∠12 are a pair of consecutive interior angles on the same side of the transversal, which two straight lines intersected by a third line form this pair of angles?

Similarly, by searching for a pair of consecutive interior angles on the same side of the transversal of ∠3 and ∠12, students identified a prototypical figure, "straight lines c, d intersected by straight line a" (see Figure 5(III)). Moreover, through identifying a pair of alternate angles ∠13 and ∠7, a prototypical figure, "straight lines c, d intersected by straight line b" was isolated (see Figure 5(IV)).

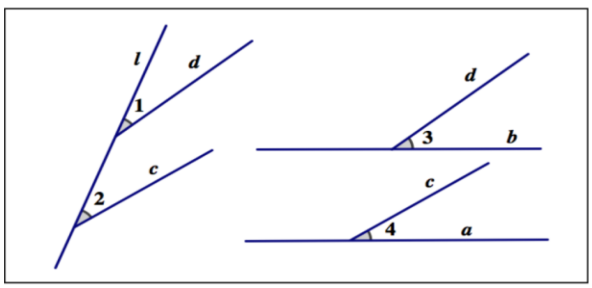


5(I) 5(II) 5(III) 5 (IV)

Figure 5. Identifying angle relationship through decomposing complex figures

After the students identified all the "basic" figures as shown in figure 5, and recognized the relevant angle relationships, the teacher summarized the key points for solving those problems, that is how to isolate a "prototypical figure", for instance, two straight lines *a, b* intersected by a third straight line d by deliberately "hiding" one line *c* from the original figure (see Figure 5(I)). Through identifying the three angle relationships within a given a prototypical figure or isolating a relevant 'prototypical figure' so that the given angle relationship is tenable, the students not only consolidated the relevant concepts, but more importantly, learned the *isolation method* of problem solving, i.e. isolating a basic sub-figure from a complex configuration.

Conceptual variation 3: Contrast and counter-example. After doing extensive exercises, the students might think that they had fully mastered the learned concepts. At this moment, the teacher posed *Task 3* (see Figure 3(II)) to assess whether students had truly mastered the concepts and methods of problem solving. Through isolating a prototypical figure shown in Figure 6(I), students concluded, "∠1 and ∠2 are corresponding angles". However, since students could only identify a figure as shown in figure 6(II), they denied that "∠3 and ∠4 are a pair of corresponding angles". Thus a new dimension of variation of experiencing corresponding angles was opened: example or counter-example of the visual judgment.



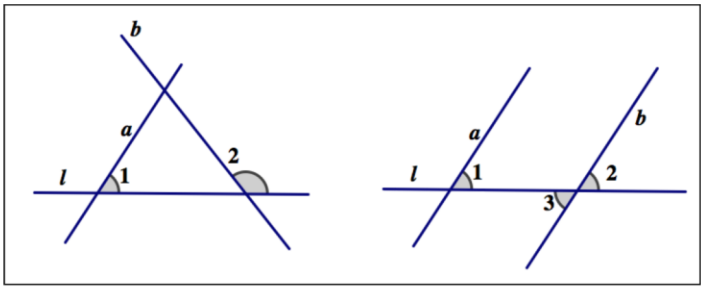
6(I) 6(II)

Figure 6. Contrast with counterexamples

Procedural variation 4: Creating a potential opportunity for learning a new topic. After solving the above problems through observation and demonstration, the teacher presented a manipulative Task 4. First, through playing with colored sticks, the first question was solved (see Figure 7(I), where *a* and *b* intersect). Then, based on drawing and reasoning, the second question was also figured out (see Figure 7(II), where *a* and *b* are parallel). During the process of problem solving, the students' thinking levels were shifted along the following forms: concrete operation (by playing with the colored sticks) (enactive); drawing (iconic); logical reasoning (abstract)

The following excerpt shows how students were guided to reason logically:

1. T: I repeat the question for you: what is the quantitative relationship between the two alternate angles [in Figure 7(b)]? Who will…?
2. S1: it is equal.
3. T: why?
4. S1: because these two angles are equal to 65 degree.



7(I) 7(II)

Figure 7. Exploring a new topic to be discussed in next class

5. T: Because they are equal to 65 degree! Good! Who would like to explain in more detail? [Pointing to student 6].
6. S2: Because ∠3 and ∠2 are a pair of vertical angles.
7. T: ∠3 and ∠2 are a pair of vertical angles.
8. S2: Vertical angles are equal.
9. T: Vertical angles are equal.
10. S2: And ∠1 is equal to ∠2 also;
11. T: Because ∠1 is equal to ∠2 also;
12. S2: So, ∠1 is equal to ∠3 in degrees.
13. T: ∠1 is equal to ∠3 in degree also. Good! Great! Furthermore, if we name the fourth angle as ∠4, what is the relationship between ∠1 and ∠4? Pleas, deal with the question after class.

Although the students found a solution by drawing, it is difficult to explain the solution. The previous dialogue demonstrates the teacher's intention to elicit a reasonable explanation. After the first student stated what he did (1~4), the student was probed for more details (5), and then another student gave a logical explanation by using previous knowledge (7~13). This exercise had two functions, on one hand, the "previous proposition: vertical angles are equal" was reviewed, on the other hand, "a further proposition: if the corresponding angles are equal, then the two lines are parallel" was operationally experienced. That means a potential space of learning was opened implicitly.

Conceptual variation 4: Consolidating and memorization of the concepts. As soon as the key points for identifying the three angle relationships in a variety of different situations were summarized, the teacher skillfully opened a new variation by making use of gestures to help students to memorize the three concepts. If the thumb and forefinger of the left hand form an angle, while the thumb and forefinger of the right hand form another angle, then all three angle relationships can be visually demonstrated by different gestures (see Figure 4) as follows:

1. T: In order to memorize the characteristics of the three angle relationships, I would like to introduce a gesture method. For example, this represents an angle [The thumb and forefinger in left hand form an angle], whilst that also represents another angle [The thumb and forefinger form an angle] (see Figure 4(I)). When the two thumbs are opposite each other, what is the relationship between the two angles? (see Figure 4(II))
2. S: Consecutive interior angles on the same side of the transversal![in unison]
3. T: Good! It is a pair of consecutive interior angles on the same side of the transversal. Then, how to represent alternate angles?
4. [Students try excitedly, some students have got the answer]
5. T: In fact, I just turn the forefinger over (see Figure 4(III)). How to represent corresponding angles? [Students actively take part in trying]
6. T: It is ok, if one angle is against the other one [see Figure 4(IV))

Thus, the students had experienced the three angle relationships in different representations: verbal, drawing, reasoning, and gesturing. These rich representations will benefit students' understanding, memorization and application of these concepts.

Summary

The lesson began with a review by questioning, and then moved forward inducing the new topic by varying an introductory task (procedural variation 1). Through several rounds of teacher-student interactions, the three concepts were built on students' answers (conceptual variation 1). These concepts were immediately applied in a simple situation. After that, the lesson moved to the stage of practice. By addressing a series of well-designed tasks presented by the teacher, the students had an extensive experience of identifying the three angle relationships in various complex situations and learned the isolation method of problem solving in geometry (conceptual variations 2, 3; procedural variations 2, 3). It is worthy mentioning that by solving the last problem, a new topic for further lessons was implicitly introduced (procedural variation 4). During the last stage, a climax of teaching was established by actively imitating the three angle relationships by means of hand gestures (Conceptual variation 4). These dimensions of variation were constructed purposefully to serve different learning goals (See Table 1).

Through exploring various dimensions of variation constructed by classroom interaction (mainly between the teacher and students), the students had been guided to develop and consolidate the concepts conceptually, and apply the concepts in different geometrical contexts, and implicitly explore the potential topics to learn. The lesson had a warm atmosphere with frequent teacher-student interactions and progressed in a coherent manner. The deliberate use of these variations seems to have ensured that the progress of the lesson was both smooth and coherent.

Table 1. Dimensions of variation, their functions, and enactment of objects of learning

Phases of the lesson	*Dimensions of variation*	*Pedagogical effects of the variation*	*Enacted objects of learning*
Reviewing and inducing	Procedural variation 1	Activating previous knowledge; Introducing the new topic	Developing the concepts
Exploring new concepts	Conceptual variations 1, 2	Forming, clarifying and consolidating the new concept	Defining, and consolidating the concepts
Examples and exercise	Conceptual variation 3	Deepening understanding of the new concept by contrasting non-examples;	Deepening the concepts
	Procedural variations 2, 3	Consolidating the new concept; Learning isolation method of problem solving	Consolidating and applying the concepts
Summary and assignment	Procedural variation 4	Creating a potential topic for further learning	Reinforcing the concept; exploring further learning topics
	Conceptual variation 4	Visualizing and memorizing the new concepts	

CONCLUSION AND DISCUSSION

According to the theoretical perspective, it is crucial to create certain dimensions of variation that enact the objects of learning. These objects of learning can be classified into two types. One is the content in question (such as concepts, propositions, formulae), another is the process (such as formation of concepts, or process or strategy of problem solving). In this particular lesson, the objects of learning include development of the concepts of three types of angle relationship involving a transversal (corresponding, alternative, and consecutive interior angles) and problem solving ability by applying these concepts. Two categories of variation, conceptual variation and procedural variation, have been strategically constituted to enact these objects of learning. It was demonstrated that the conceptual variations served the purpose of building and understanding the concept, while the procedural variations are used for activating previous knowledge, introducing the new topic, consolidating new knowledge, developing strategies for solving problems, and creating a topic for further learning.

From a perspective of pedagogy, this lesson was unfolded smoothly and consistently, and was guided by the teacher, which demonstrates the major features of mathematics classroom teaching in China (e.g., Huang & Leung, 2004; Leung, 2005). Yet, if looking at students' engagement and contribution to the generation of knowledge, namely, enacted objects of learning, we cannot say that students are passive learners. The analysis of this lesson indicates that the teacher can still encourage students to actively generate knowledge through strategically creating dimensions of variation. This observation echoes Huang's (2002, p. 237) description of the Chinese mathematics classroom:

> There are teacher, students and mathematics. The teacher presents mathematics and helps students engage in the process of exploring the mathematics by providing proper scaffoldings and asking a series of heuristic questions. The students are eager to listen and engage themselves in the process of learning.

From a perspective of learning geometry, the dimensions of conceptual variations which focus on contrasting concept images and non-concept images, juxtaposing prototypical figures and non-prototypical figures could help students develop a deep understanding of the concept (e.g., Vinner & Hershkowitz, 1983). Moreover, Duval's (1996, 1999) studies support that developing reconfiguration ability when processing geometrical figures is crucial for problem solving in geometry. The dimensions of procedural variation constructed in this Shanghai lesson demonstrate the teacher's competence in setting and implementing deliberate tasks for students' development of this figurative processing ability when applying the learned concepts. Thus, from a perspective of cognitive science, the two types of variation could help students to develop geometrical concepts and problem solving ability in geometry. This reinforces Huang, Miller and Tzur (2015, p. 104)'s assertion of "the power of teaching through variation to deepen and consolidate conceptual understanding and procedural fluency concurrently" based on a fine-grained analysis of 10 consecutive lessons.

In asserting the positive effects of appropriate application of the principles of teaching with variation, a caution of designing and implementing dimensions of variation has been mentioned (Gu et al., 2004, 2017). It is crucial to construct appropriate spaces of learning by exploring relevant dimensions of variation focusing on critical features of the objects of learning with regard to the contexts, reasoning and student learning trajectory (Gu et al., 2017).

REFERENCES

Ausubel, D. P. (1968). *Educational psychology: A cognitive view*. New York, NY: Holt, Rinehart & Winston.

Chen, X., & Li, Y. (2010). Instructional coherence in Chinese mathematics classroom – A case study of lessons on fraction division. *International Journal of Science and Mathematics Education, 8*, 711–735.

Duval, R. (1999, October 23–26). *Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning*. Proceedings of the Annual Meeting of the North American Chapter of the International group for the Psychology of Mathematics education, Morelos, Mexico.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103–131.

Fan, L., & Zhu, Y. (2004). How have Chinese students performed in mathematics? A perspective from large-scale international comparisons. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 3–26). Singapore: World Scientific.

Fan, L., Wong, N. Y., Cai, J., & Li, S. (2015). *How Chinese teach mathematics: Perspectives of insiders*. Singapore: World Scientific.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*(2), 139–162.

Gu, F., Huang, R., & Gu, L. (2017). Theory and development of teaching through variation in mathematics in China. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation* (this volume). Rotterdam, The Netherlands: Sense Publishers.

Gu, L. (1994). *Theory of teaching experiment: The methodology and teaching principles of Qingpu* [In Chinese]. Beijing: Educational Science Press.

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: An effective way of mathematics teaching in China. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–347). Singapore: World Scientific.

Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the International group for the psychology of mathematics education* (pp. 70–95). Cambridge: Cambridge University Press.

Hershkowitz, R., Bruckheimer, M., & Vinner, S. (1987). Activities with teachers based on cognitive research. In M. M. Lindquist (Ed.), *Learning and teaching geometry, K-12: 1987 Year book* (pp. 222–235). Reston, VA: National Council of Teachers of Mathematics.

Huang, R. (2002). *Mathematics teaching in Hong Kong and Shanghai: A classroom analysis from the perspective of variation* (Unpublished doctoral dissertation). The University of Hong Kong, Hong Kong.

Huang, R., & Leung, F. K. S. (2004). Cracking the paradox of the Chinese learners: Looking into the mathematics classrooms in Hong Kong and Shanghai. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 348–381). Singapore: World Scientific.

Huang, R., & Leung, F. K. S. (2005). Deconstructing teacher-centeredness and student-centeredness dichotomy: A case study of a Shanghai mathematics lesson. *The Mathematics Educators, 15*(2), 35–41.

Huang, R., Miller, D., & Tzur, R. (2015). Mathematics teaching in Chinese classroom: A hybrid-model analysis of opportunities for students' learning. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese teach mathematics: Perspectives from insiders* (pp. 73–110). Singapore: World Scientific.

Huang, R., Mok, I., & Leung, F. K. S. (2006). Repetition or variation: "Practice" in the mathematics classrooms in China. In D. J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The insider's perspective* (pp. 263–274). Rotterdam, The Netherlands: Sense Publishers.

Leung, F. K. S. (2005). Some characteristics of East Asian mathematics classrooms based on data from the TIMSS 1999 video study. *Educational Studies in Mathematics, 60*, 199–215.

Li, Y., & Huang, R. (2013). *How Chinese teach mathematics and improve teaching.* New York, NY: Routledge.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *The Journal of the Learning Sciences, 15,* 193–220.

Marton, F., & Tsui, A. B. M. (2004). *Classroom discourse and the space of learning.* Mahwah, NJ: Erlbaum.

OECD. (2010). *PISA 2009 results: What students know and can do: Student performance in reading, mathematics and science* (Vol. I). Paris: OECD Publishing.

OECD. (2014). *PISA 2012 results in focus: What 15-year-olds know and what they can do with what they know*. Paris: OECD Publishing.

Paine, L. W. (1990). The teacher as virtuoso: A Chinese model for teaching. *Teachers College Record, 92*(1), 49–81.

Qingpu Experiment Group. (1991). *Learn to teaching* [In Chinese]. Beijing: Peoples' Education Press.

Stevenson, H. W., & Lee, S. (1995). The East Asian version of whole class teaching. *Educational Policy, 9*, 152–168.

Vinner, S. (1981). The nature of geometrical objects as conceived by teachers and prospective teachers. In Equipe De Recherche, Pédagogique (Eds.), *Proceedings of the 5 th PME international Conferences*, *1*, 375–380.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Dordrecht, Netherlands: Kluwer.

Vinner, S., & Hershkowitz, R. (1983). On concept formation in geometry. *ZDM-The International Journal on Mathematics Education*, *83*(1), 20–25.

Rongjin Huang
Middle Tennessee State University, USA

Frederick K. S. Leung
The University of Hong Kong, China