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2. THEORY AND DEVELOPMENT OF TEACHING THROUGH VARIATION IN MATHEMATICS IN CHINA

INTRODUCTION

Chinese students' strong performance in mathematics in various international comparative studies has been noticed for decades (Fan & Zhu, 2004). In particular, Shanghai students' outstanding performances in PISA (OECD, 2010, 2014) have stunned educators and policy makers around the world. Researchers have investigated Chinese students' excellent performance in mathematics from different perspectives (Biggs & Watkins, 2001; Fan, Wang, Cai, & Li, 2004), including societal, socio-cultural perspectives (Stevenson & Stigler, 1992; Sriraman et al., 2015; Wong, 2008), student behaviors (Fan et al., 2004), teacher knowledge, and teacher professional development perspectives (An, Kulum, & Wu, 2004; Fan, Wong, Cai, & Li, 2015; Huang, 2014; Ma, 1999), and classroom instruction perspectives (Huang & Leung, 2004; Leung, 1995, 2005; Li & Huang, 2013).

A close examination of mathematics instruction in China may help better understand why Chinese students can succeed in large class-size classrooms. Typically, Chinese mathematics classrooms have been described as large and teacher dominated, with students who are well disciplined, passive learners (Leung, 2005; Stevenson & Lee, 1995). Classroom teaching in China is polished (Paine, 1990), fluent and coherent (Chen & Li, 2010; Wang & Murphy, 2004), with a focus on the development of important content, problem solving, and proof (Huang & Leung, 2004; Huang, Mok, & Leung, 2006; Leung, 2005). Furthermore, from a cultural and historical perspective, Chinese mathematics instruction has been identified with two fundamental characteristics: (1) two-basics-oriented (basic knowledge and basic skills) teaching, and (2) direct explanation and extensive practices with variation (Li, Li, & Zhang, 2015; Shao, Fan, Huang, Ding, & Li, 2013). Particularly, Gu, Huang and Marton (2004) theorized teaching through variation[1] and argued that teaching through variation is an effective way to promote meaningful learning in mathematics for classes of large size. In this chapter the authors further examine the practice of teaching through variation from a cultural perspective and provide state-of-the-art studies on teaching through variation in China. Finally, the authors discuss how teaching through variation can be implemented to promote deep learning of mathematics in classrooms.

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TEACHING THROUGH VARIATION: A CULTURALLY INDIGENOUS PRACTICES

Teaching and learning mathematics through variation is a widespread idea in China as reflected in the old Chinese maxim, "Only by comparing can one distinguish" (有比较才有鉴别). There are different opinions about using variation in mathematics education. Some focus on using problems with variation in textbooks or curriculum (Cai & Nie, 2007; Sun, 2011; Wong, Lam, Sun, & Chan, 2009) while others emphasize using tasks with variation in classrooms for promoting student learning (Gu et al., 2004; Huang & Leung, 2004, 2005). Teaching through variation in this chapter is aligned with the following definition:

> To illustrate essential features of a concept by demonstrating various visual materials and instances, or to highlight essential characteristics of a concept by varying non-essential features. The goal of using variation is to help students understand the essential features of a concept by differentiating them from non-essential features and further develop a scientific concept. (Gu, 1999, p. 186)

In her study, Sun (2011) argued that the concept of conducting a lesson or practice with variation problems is an "indigenous" feature in China. First, the major traditional philosophical systems such as Confucianism (儒家) imply the variation notion. For example, Confucius said, "I do not open up the truth to one who is not eager to get knowledge, nor help out any one who is not anxious to explain himself. When I have presented one corner of a subject to any one, and he cannot from it learn the other three, I do not repeat my lesson." (The Analects, 7: 8) (举一隅不以三隅反，则不复也) This principle emphasizes the importance of self-motivated inquiry for understanding invariant patterns within different situations. Second, many ancient Chinese mathematics treatises such as *Nine Chapter of Arithmetic Arts*《九章算术》have been organized in a similar structure: concrete examples (stereotype problem) – invariant methods – application (variation problems). In this way, the invariant principles (general methods) were developed through the exploration of the variation of concrete examples and further consolidated by application in a variety of novel problems.

When discussing learning and teaching mathematics, ancient mathematicians also emphasized heuristic strategies through making use of variation. For example, in *Shuan Fa Tong Bian Ben Mo* 《算法通变本末》, Yanghui (杨辉，no details) pointed out that "good learners can grasp the whole category from typical examples; they don't need to teach them all in detail" (Song, 2006). It means that teachers should adopt analyzing typical cases or instances, illustration with diagrams, and drawing inferences about other cases from one instance to help learners to broaden their knowledge from concrete instances. Another example, in *Zhoubi Suanjin*《周髀算经》, a classic mathematics treatise, the following conversation between the teacher (Chenzi) and a student (Rongfang) revealed the teaching philosophy:

Rongfang: *I do not master the Dao (way). Can you teach me?*

Chenzi: *[…] Now in the methods of the Way [that I teach], illuminating knowledge of categories [is shown] when words are simple but their application is wide-ranging. When you ask about one category and are thus able to comprehend a myriad matters, I call that understanding the Dao. Now, what you are studying is the methods of reckoning (the principles of learning mathematics), and this is what you are using your understanding for. […]. So similar methods are studied comparatively, and similar problems are comparatively considered. This is what sorts the stupid scholar from the clever one, and the worthy from the worthless. So, being able to categorize in order to unite categories-this is the substance of how the worthy will devote themselves to refining practice and understanding (Cullen, 1996, pp. 175–178, cited from Sun (2011)).*

The above discussions about learning mathematics focus on using concrete examples to make sense of a category (a concept), grasping ways (generalization) across categories, and developing a hierarchical system of categories. All of these ideas reflect the key notion of using variation problems in learning mathematics.

In addition to the aforementioned traditional cultural values, ancient mathematics treatises and the strategies of mathematics learning, a civil service examination system associated with "educational attainment, career goals, social status, and political ambitions" (Li, Li, & Zhang, 2015, p. 72) has been established since Qin Dynasty (605–1905) in China. In modern China, mathematics examinations exist at all grade levels. In particular, the entrance examination for high schools and colleges are high-stakes and competitive. The high-stakes examination system has contributed to the origin of forming two-basics–oriented mathematics teaching, supported with teaching through variation (Li et al., 2015). Since mathematics teaching and examination focus on basic knowledge and skills that are defined by curriculum standards and the two "basics" are relatively invariant, the exam items have to be designed differently every time, although they have to adhere to standards and textbooks. So, examination items have to be created based on prototype problems in textbooks with varying forms (i.e., many variations while maintaining the same essence, 万变不离其宗). Thus, practices with variation problems surrounding standards and textbooks have been proved in practice to be an effective way to prepare students to succeed in their examinations (i.e., practice makes perfect, 熟练生巧) (Li, 1999).

In addition to the traces of the roots in the ancient Chinese philosophy and mathematics treaties, teaching through variation has been promoted by the examination-oriented education system. Teaching through variation exists in many places without individuals' purposeful awareness.

EARLIER STUDIES ON TEACHING THROUGH VARIATION: CATEGORIZATION OF VARIATION AND MECHANISM OF USING VARIATION

Teaching and learning through variation problems has been practiced for centuries in China. Yet, the practice has only been examined empirically over the last three decades. Gu and his colleagues have explored how to use and theorize teaching through variation (e.g., *Bianshi* Teaching 变式教学) to increase student achievement in mathematics since the 1980s (Bao, Huang, Yi, & Gu, 2003a,b,c; Gu, 1981, 1994; Gu et al., 2004; Qingpu experiment group, 1991). This section describes the major concepts of teaching through variation. First, the authors introduce two essentially different types of variation in mathematics classroom teaching: conceptual variation and procedural variation, based on effective teaching experiences (Gu, 1981). Then, a key concept of potential distance featuring the procedural variation based on empirical studies is discussed (Gu, 1994).

Conceptual Variation

Conceptual variation refers to the strategies that are used to discern essential features of a concept and to experience connotation of the concept by exploring varying embodiments of the concept (i.e., instances, contexts) (Gu et al., 2004). It aims to help students develop a profound understanding of a concept from multiple perspectives. The sections that follow illustrate the critical features of conceptual variation.

Highlighting essential features through variations and comparisons. Students' learning of geometrical concepts is closely related to the following major factors: experience with visual figures that represent the concept and verbal description of the concept. Previous teaching experience in geometrical concepts in middle schools demonstrates that directly defining a concept by describing essential features of the concept may help students memorize the concept. For example, the concept of altitudes of a triangle includes two critical features: perpendicular to one side and passing through the vertex at the intersection of the other two sides. However, the observation and experiment in Qingpu (Gu, 1994) revealed that if a teacher only told students the definition precisely and asked students to memorize the definition, then students were likely to have superficial and rigid understanding. Yet, if a teacher provided opportunities for students to observe and compare deliberately designed variation concept figures such as standard or non-standard position figures, or counterexamples, and then highlighted the essential features of the concept, students are more likely to synthesize the critical features of a concept based on observation of concrete instances. One example of variation figures used for developing the concept of altitude of a triangle is shown in Figure 1.

As shown in Figure 1, a standard figure is used to introduce the concept of altitudes of a triangle that is aligned with daily life experience. But the concept of altitudes in geometry is not equivalent to the perceived meaning of daily life

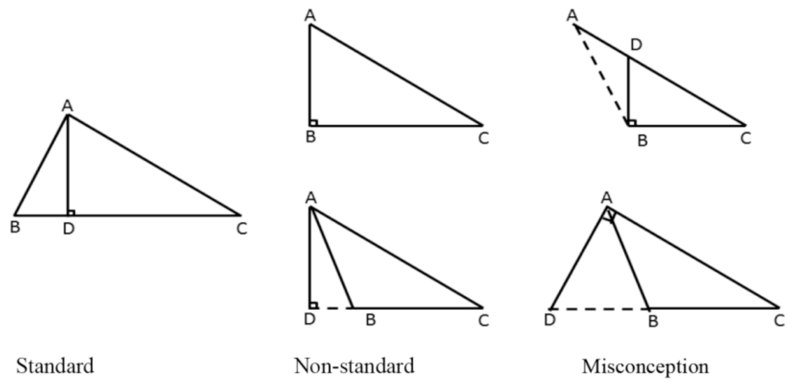


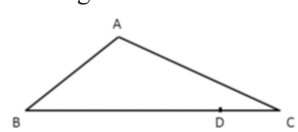
Figure 1. Altitudes of triangle

experience. Thus, identifying altitudes in various triangles (positions and types of triangles) helps students abstract the essential features of the concept. Finally, by contrasting some common misconception figures, the critical features of altitude: "perpendicular to a side and passing through the opposite vertex of the side" are further consolidated.

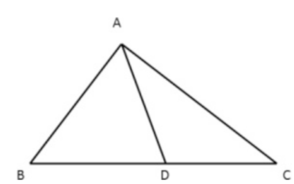
Eliminating the distraction of complex background through transformation and reconfiguration of basic figures. Geometrical figures usually consist of combinations of basic figures through separation, overlapping, and intersection. Sometimes basic figures are embedded in complex situations. The complex background figures often distract, distort, and mask students' perception of embedded basic figures. Thus, essential features of a geometrical concept embedded in complex backgrounds are often hidden and difficult to identify or even subject to being perceived inappropriately. To address this learning difficulty, a traditional strategy was to purposefully isolate geometrical objects explicitly (such as using colors) from complex background figures (including real contexts), which has been proven in practice to be effective. However, the experiment in Qingpu (Gu, 1994) demonstrated that such a strategy might resolve the problem that inappropriate perception of figures constrains appropriate recognition of a geometrical concept. How logical reasoning activities may influence the comprehension of a complex figure is an important issue. These strategies include: analyzing the structure of complex figures or generating a complex figure through transformation (i.e., translation, rotation, reflection, and shrinking and expanding) of basic figures. Through these decompositions and compositions, the focused figures can be separated from complex background figures (See Gu et al., 2004 for details).

Examining the effectiveness of using these variations through quasi-experiments. Since 1980, the Qingpu experiment team has examined the effectiveness of these variation strategies through "identifying effective methods based on implementation", a Chinese version of "design-experiment" (Brown, 1992): repeated within a short period (once a week), which includes an entire cycle of planning – implementation – evaluation – improvement. The effectiveness of these variation strategies has been testified through more than 50 cycles of studies within one year. In particular, quasi-experiment methods (numbers of students in experimental class and control class are similar) were adopted. The experimental studies aimed to examine the effectiveness of using variation strategies. The results of one experiment are discussed below.

In the first experiment, the instructional content is the concept of perpendicular lines. The experimental group is class A (50 students); the concept of perpendicular lines was briefly explained, and then students were provided a set of variation practices. After that, students' errors were identified and discussed based on essential features of the concept. The control group is class B (51 students); the concept (definition of perpendicular lines) was repeatedly explained to students based on textbooks, then simple and repeated problems were provided for students to practice. After the class, a post-lesson evaluation test was conducted. To answer to the question, "What is the distance from a point to a straight line?", the students from the control class mainly recited the definition from the textbook, yet the students from the experimental class explained the definition based on their understanding. The average correct rates on a basic problem of constructing a perpendicular line in both groups were about 70%. However, with the answers to non-routine problems (constructing a perpendicular line in non-standard position triangles, see below), there were significant differences between the two classes as follows:



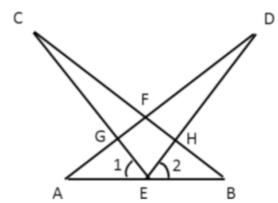
Item 1, in the figure (on the right), asked students to construct a line DE containing D and perpendicular to AD. There was a significant difference ($t = 2.13$, $p < .05$) between experimental class A (mean = 5.80 (out of 10 points)) and control class B (mean = 4.76).



Item 2, in the figure (on the right), asked students to construct the distance segments from B or C to line AD respectively. There was a significant difference ($t = 4.91$, $p < .01$) between experimental class A (mean = 6.04) and control class B (mean = 3.97).

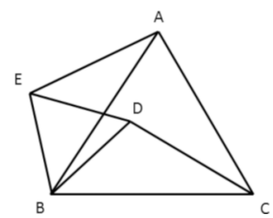
Thus, this study revealed that teaching through deliberate variation problems appears to be more effective than teaching through repeated explanations of a definition.

The second experiment was conducted one month later. The instructional content was the SAS Postulate (Side-Angle-Side): If two sides and the included angle of one triangle are congruent to the corresponding parts of another triangle, then the triangles are congruent. The teaching strategy was swapped: class B was the experimental group and class A was the control group. In experimental class B, the theorem of SAS was briefly explained to students, and then variation problems (with variation figures) were provided for students to practice on. After that, students' errors were discussed and corrected, with particular attention to identifying hidden conditions within a complex figure or context. In the control class A, the teacher explained the theorem (SAS), students restated the theorem, and then students were given several variation problems (without figures, which consist of overlapping or separating basic figures) to practice on. A post-lesson test showed that the answers to two slight variation problems had mixed results; the means in the experimental class were 85% and 67% while those in the control class were 79% and 70%. However, the answers to another two proof problems that included complex variations showed significant difference between the two classes as follows:



Item 3: in the figures (on the right), AE = BE, CE = DE, ∠1 = ∠2. Prove AD = BC. There was a significant difference ($t = 3.18$, $p < .01$) between experimental class B (mean = 8.66) and control class A (mean = 7.12).

Item 4: In the figure (on the right), ΔABC ≅ ΔBDE are equilateral triangles. Prove: $\Delta BCD \cong \Delta BAE$.



There was a significant difference ($t = 2.11$, $p < .05$) between experimental class B (mean = 5.21) and control class A (mean = 3.50).

On item 3, students had to recognize the symmetrical structure of ΔADE and ΔBCE On item 4, the students had to recognize that ΔBAE is rotated left 60˚ from ΔBCD. These results show the effectiveness of using variation figures to help students identify target figures from a complex background figure.

In summary, the experimental studies in Qingpu (Gu, 1994) demonstrated that (1) designing variation problems based on essential features of a concept, and comparing and contrasting concept images and non-concept images could help students clarify connotations and extensions of a concept; and (2) reconfiguring the structure of a complex figure and forming a figure through transformation of basic figures could help students reduce cognitive load and promote their understanding of a concept in depth. Use of these strategies in teaching in a large class could promote more active learning.

Procedural Variation

Mathematical concepts are defined clearly and statically. Yet, obtaining mathematical activity experience and understanding of mathematical thinking methods are a dynamic process. Gu (1981) explored another variation, known as procedural variation. Procedural variation refers to creating variation problems or situations for students to explore in order to find solutions to problems or develop connections among different concepts step by step or from multiple approaches. Based on extensive teaching experience and reflection, Gu (1994) synthesized two critical features of procedural variation as follows (see Gu et al., 2004 for details).

Solving problems through transferring figures. Transferring is one of most important methods of solving problems in mathematics instruction in China. It means to break down a complex problem into simpler problems. The simpler problems provide the foundation for solving the original complex problem. Or reversely, based on a basic problem, through adding constraints, complicated problems can be created. Figure 2 is an example of how transferring methods could help prove a geometrical theorem.

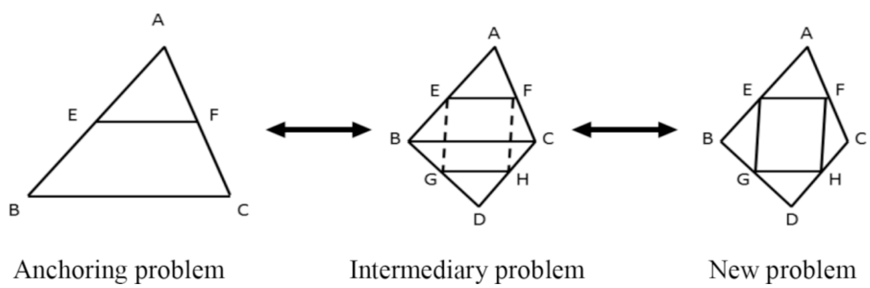


Figure 2. An example of transferring problems

Figure 2 shows how the mid-point quadrilateral that connects four midpoints of a quadrilateral is a parallelogram can be proven based on a simple "anchoring property", which states that the mid-segment of triangle (connecting two midpoints) is parallel and equal to half of the third side.

Building connections among different types of knowledge through categorization and building a hierarchical system of categories. Categorization is an important mathematical thinking method. The key is to ensure that a categorization includes all instances without missing and overlapping. For example, the categorizations of triangles, the categorizations of special quadrilaterals, and the categorization of angles in a circle are typical examples of categorization activities. Another important issue is to build connections among various concepts and various concept figures, and to clarify logical relationships between different concepts. Figure 3 is a typical example of a concept map of angles in circles.

In Figure 3, there are three situations of inscribed angles in circles: the center of the circle is on one chord, between the two chords, or outside the two chords. In addition, there are: relevant angles formed between a tangent and a chord, angles formed inside of a circle by two intersecting chords, and angles formed outside of a circle by two intersecting tangents, two secants, or a tangent and a secant. However, Figure 3, which was presented by a teacher in a unit review lesson, presents the relationships among different angles clearly by adding critical auxiliary lines, both connecting relevant concepts and consolidating these concepts.

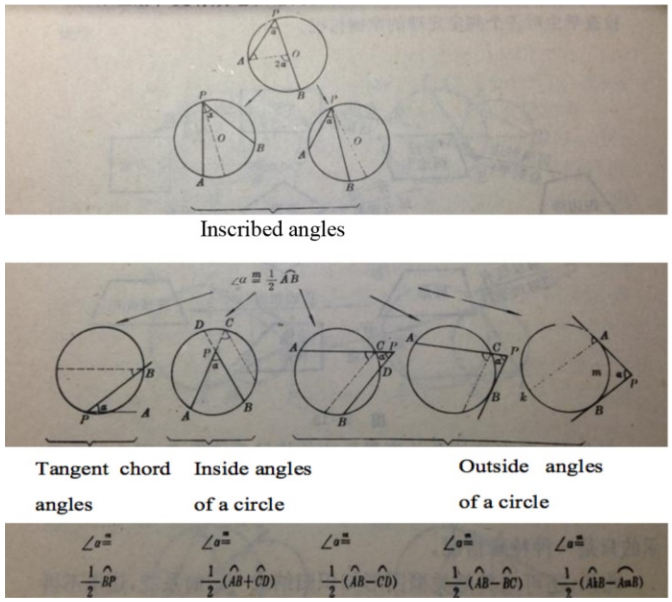


Figure 3. The measurement of angles in circles

Illustration of procedural variation with the analysis of an exemplary lesson. Procedural variation relates to different mathematical thinking, either converging transformation or diverging transformation. Procedural variation is also derived from a prototypical problem, or combination and transformation of representations, or re-recognition or discovering, and so on. Different ways of thinking and multiple representations when creating procedural variations are beyond being dealt with by any conceptual variation. However, making a problem more difficult and complex through extensively varying problems is contrary to the goals of teaching through variation. Varying problems must serve for instruction processes and purposes. In addition to the quantitative results shown previously, we illustrate how to appropriately use variation problems by analyzing an exemplary lesson developed during the Qingpu experiment. The lesson focused on the theorem for determining isosceles triangles. Here, we just describe two segments of the lesson.

Segment 1: Multiple constructions and multiple proofs. In Figure 4: in an isosceles triangle, given the base BC and the angle ∠B formed by a leg and the base, construct the isosceles triangle.

Students provided a variety of constructing methods. Some constructed ∠C = ∠B and extended the sides of the angles so that they intersect at A. Some constructed

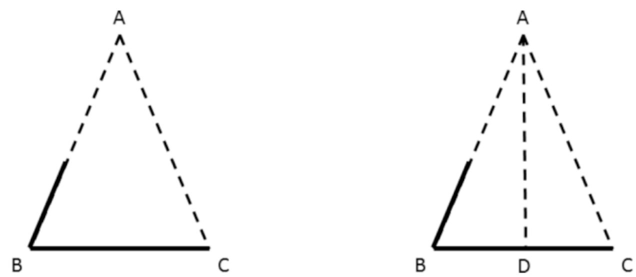


Figure 4. An incomplete isosceles triangle

the perpendicular bisector of base BC and intersect one leg at A. In addition, some students folded BC in half and found the vertex A and constructed the triangle. Based on the constructions of a triangle, the determining theorem of isosceles could be discovered: In ΔABC, if $\angle B = \angle C$, then $AB = AC$. Different proofs of this theorem could be found based on the construction of the figure. For example, the altitude of base BC can be constructed, or the bisector of angle A can be constructed, then prove $\Delta ABD \cong \Delta ACD$; then, AB=AC can be obtained based on the properties of congruent triangles. In addition, students are encouraged to find various proofs: for example, if AB > (*or* <) AC, then $\angle B$ > (*or* <) $\angle C$ based on the property that in a triangle, the longer sides correspond to bigger angles. This is contradictory to the given of $\angle B = \angle C$. So, it is impossible that $AB \neq AC$. This is *Reductio ad absurdum* (indirect reasoning). Moreover, if ΔABC and ΔACB are regarded as two overlapping triangles, then, because $\angle B = \angle C$, $\angle C = \angle B$, and BC=CB; thus, the two triangles are congruent (e.g., ASA) and therefore AB=AC. Based on different ways of constructing the figure, varying proofs were derived which are complementary to a single proof.

Segment 2: Varying the problems hierarchically. Based on previous teaching experience, exploration of multiple solutions to a problem and a set of problems which could be solved by the same method, should be better than seeking a solution to a problem regarding promoting students' flexibility and profoundness of mathematical thinking (Cai & Nie, 2007). However, the Qingpu experiment (Gu, 1994) indicated that exploring hierarchical-progressive variation problems could achieve a much better effect on student learning. The following is an exemplar for illustrating the feature of hierarchical-progressive variation problems. The initial problem is simple: In Figure 5(1), the bisectors of two base angles of an isosceles triangle ΔABC intersect at D, determine whether the ΔDBC is an isosceles triangle.

The answer to the first problem (Figure 5(1)) is obvious. It aims to help students understand how to use judgment theorem and property theorem of isosceles triangles that are the basic knowledge of the content. In Figure 5(2), a segment EF passing through D is parallel to BC (EF||BC). Students were asked to find all isosceles

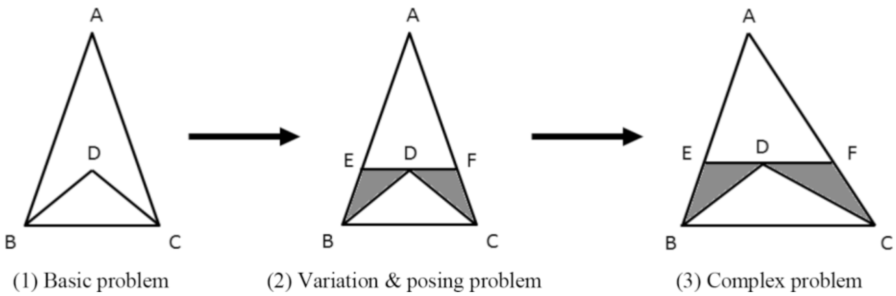


Figure 5. Hierarchical-progressive variation problems

triangles in the figure. ΔDBC and ΔAEF are obviously isosceles triangles, which is easy to prove. Then, students should focus on determining whether ΔEDB and ΔFDC are isosceles triangles. If they are, prove that they must be. In this case, students have to use judgment theorem and identify the common relationships among the bisectors of an angle, parallel lines, and isosceles triangles. Immediately, students were asked to create their own problems based on the relationships and solve them by themselves. Students found the following results: D is the middle point of EF; EF = EB + FC and so on. This procedural variation was used to pose and explore subsequent challenging problems. In Figure 5(3), ΔABC is not an isosceles triangle, but the bisectors of the base angles and parallel lines remain. Students are asked to individually think: among the statements posed in the previous problem, which ones are still tenable and which ones may be not true? This is a relatively complex problem. Repeated experiments showed that about 80% of the students who had experience with hierarchical-progressive variation problems could solve the complex problem, while only about 20% of the students who did not experience this process could solve the problem. Although, all students had similar academic backgrounds at preliminary stage of learning geometry.

In summary, the authors came to the following conclusions: (1) during mathematical activities, careful dealing with hierarchical levels of transferring from a related basic problem to a higher cognitive demand problem, and practicing with relevantly hierarchical-progressive variation problems could advance students' capacity in solving problems step by step; (2) synthesizing common experiences and features during different hierarchical-progressive variation processes, and classifying and connecting these relevant variations could promote students' development of hierarchical and systemic experiences. These strategies have evolved based on a great amount of effective teaching experiences. Actually, dynamic mathematics activities include an important characteristic, namely, the progression of knowledge and skills. This progression could be represented in the forms of hierarchical levels of knowledge or a series of strategies for, or experiences in, doing mathematics activities. Certainly, teaching through hierarchical-progressive variation problems is not the same as rote practice.

Mechanism of Procedural Variation

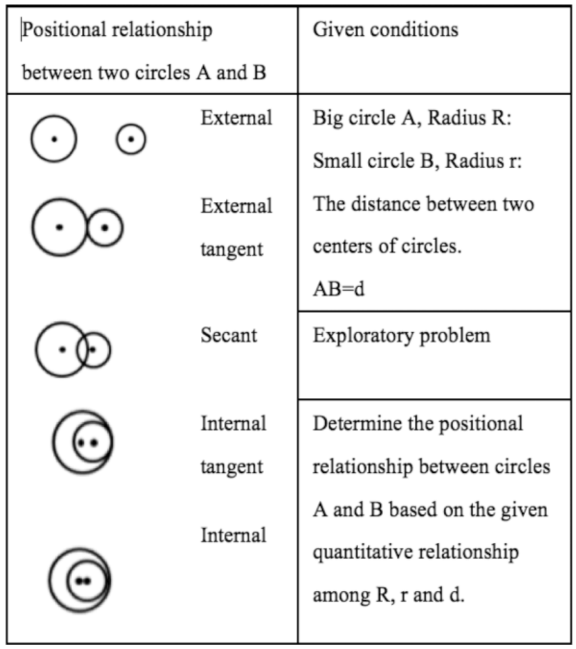
To understand the principles and mechanisms of procedural variation, the Qingpu experiment group (Gu, 1994) conducted a series of studies on student mathematical thinking processes between 1987 and 1988. These studies focused on psychological characteristics of learning through variation and describing progression of knowledge development and essential connections between what students have and what they are supposed to learn. The sections that follow describe the major findings of those studies (Gu, 1994) based on original data analysis (see Gu et al., 2004 for additional examples).

Anchoring knowledge point and new problems. Students' existing knowledge structure is the key factor influencing students' learning of new knowledge. The *anchoring knowledge point* is critical for the success of exploration of a new problem (Ausubel, 1978). *Anchoring knowledge point* refers to the previous knowledge point that underpins learning of the new knowledge.

There were 180 middle school students participating in this experiment. Seventh, eighth, and ninth grade students occupied one-third of the participants equally; male and female averaged half; and the ratio among high, average, and low achieving students is 3:4:3. Using stratified samplings, 60 students participated in the experiment: teaching through variation; another 60 students participated in dissemination of the experiment; yet another 60 students participated in a control group: direct teaching the concept. Activity cards are used as a research tool. One example is shown in Figure 6. There were 6 groups of 5 items, 30 items in total. Groups 1, 3, and 5 included items that can be solved based on visual perceptions (constructing figures based on given data and then making judgments based on visual perceptions) while groups 2, 4, and 6 included items that can be solved based on logical reasoning (Making conjectures based on the given and providing justification).

Regarding the problem in Figure 6, *the anchoring knowledge point* of students of different grade levels were different and therefore, the knowledge distance between the problem and anchoring knowledge point of different grade levels was different. Seventh graders knew about segment diagrams, but had the largest knowledge distance; eighth graders knew about translations of figures (such as two triangles) and had a shorter knowledge distance; and ninth graders knew about the relationship between a line and a circle and had the shortest knowledge distance. The test results showed that the correct rate of students increased as the knowledge distance decreases. This finding reveals that learning new knowledge or solving new problems not only relies on the anchoring knowledge point but also relies on the knowledge distance. This finding also indicates the mechanism of teaching and learning with progression and provides implications for teaching through progressive variation problems.

In addition, students could develop their mathematics cognition as they grow up across grades; how might the cognitive maturity influence students' ability in

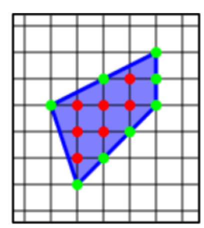


Positional relationship between two circles A and B	Given conditions
External	Big circle A, Radius R: Small circle B, Radius r: The distance between two centers of circles. AB=d
External tangent	
Secant	Exploratory problem
Internal tangent	Determine the positional relationship between circles A and B based on the given quantitative relationship among R, r and d.
Internal	

Figure 6. Example of activity cards (visual perception oriented judgment)

exploring a novel problem? To address this concern, another problem was posed: exploration of Pick's theorem was given to students of three grades (e.g., seventh, eighth, and ninth). The theorem is expressed as follows:

Given a simple polygon constructed on a grid of equal-distanced points (i.e., points with integer coordinates) such that all the polygon's vertices are grid points, Pick's theorem provides a simple formula for calculating the area A of this polygon in terms of the number N of *lattice points in the interior* located in the polygon and the number L of *lattice points on the boundary* placed on the polygon's perimeter:



$$A = N + \frac{L}{2} - 1$$

Although this theorem is totally new to all students, the knowledge needed for exploring this theorem is basic: area of triangle and counting, making the anchoring knowledge point quite similar for students in all grades. Thus, the knowledge distance is quite comparable as well. The incorrect rates (vertical axis) of solving these two problems across grades are displayed in Figure 7.

In Figure 7, the dash-line reveals that, even with the similar anchoring knowledge point for all students in all grades, the correct rate increased as the grade increased; this implies students' mathematical cognition maturity matters. The bold-line

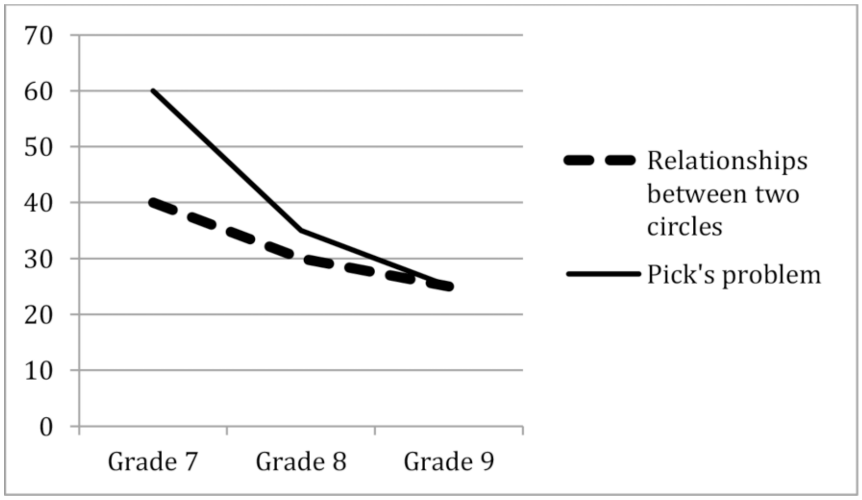


Figure 7. Incorrect rates of solving problems regarding different knowledge

indicates that with different anchoring knowledge point, the correct rate increased tremendously as the grade increased. The gains of correct rate of the two problems across grades are obviously different. This difference may reflect the co-impact of mathematical knowledge distance and cognitive level maturity. The "potential distance" between anchoring knowledge point and a new problem is determined by two factors: mathematical knowledge distance between anchoring knowledge point and the new problem, and cognitive maturity.

Measurability of potential distance. As discussed previously, both the anchoring knowledge point and new problems are related to mathematical content. Thus, the potential distance could be measured through designing appropriate instruments (e.g., mathematical problems) and analyzing test results quantitatively. For instance, in the aforementioned examples (in Figure 7), the potential distance could be indicated by incorrect rate when exploring new knowledge or new problems. The lower the incorrect rate, the lower the potential distance. This is a kind of primary characterization/representation. Of course, further studies could be done through testing different content topics with larger samples and conducting advanced psychometric analysis to build standardized norms. Thus, potential distance is measurable, although more studies are needed in the area.

Differentiation of potential distance. The potential distance between anchoring knowledge point and a new problem could influence the difficulties and achievements of students' exploration of the problem. If the potential distance between new knowledge and anchoring knowledge point is shorter (short distance connection), it is easy for students to understand and master the new knowledge. If the potential distance is longer (long distance connection), the problem can support the development of students' exploratory ability. A teacher could adopt different orientations of instruction: direct, exploratory, or combination according to different potential distances and learning goals.

RECENT STUDIES ON PROCEDURAL VARIATION: CORE CONNECTION AND LEARNING TRAJECTORY

In addition to the definition and features of potential distance described in the previous section, it was noticed that when the potential distance is too long, a majority of students have difficulties in approaching the new knowledge, which we conjectured was due to heavy cognitive load (Gu, 1994). The key questions that need to be addressed include: how can teachers help students build bridges between anchoring knowledge point and new knowledge? How can teachers provide effective scaffolding activities? How can teachers use variation problems to shorten the potential distance, if possible? A second analysis of data taken from the Qingpu experiment (Gu, 1994) reveals partial answers to these questions. The major findings include identifying *core connection* and setting appropriate *Pudian* (i.e., scaffoldings). In addition, based on an attempt to incorporate the western notion of learning trajectory (Simon, 1995) with teaching through variation in Chinese mathematics classroom, it was found that the teaching guided by the combination of learning trajectory and teaching through variation could promote students' understanding of concepts.

Concept of Core Connection

In the Qingpu experiment (Gu, 1994), the teachers in the experiment group had emphasized the integration of numerical and geometrical representations, and invariant features within varying transformations after seventh grade. For instance, in the experimental class, the students were introduced to analyzing the positional relationship between two segments on a line using "segment diagrams" in algebraic lessons as shown in Figure 8.

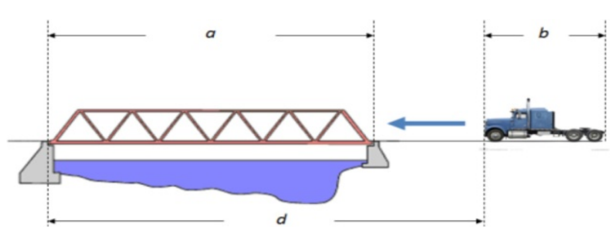


Figure 8. Positional relationships between two segments on a line

In Figure 8, a truck is shown travelling toward a bridge from East to West. The length of the bridge is *a*, the length of the truck is *b*, and the distance between the West end of the bridge and the front of the truck is *d*. To explore the quantitative relationship among *d, a*, and *b,* students need to determine the following relationships between the truck and the bridge: (1) when is the truck not on the bridge? (2) When is a portion of the truck driving on the bridge? (3) When is the truck entirely on the bridge? If students understand these problems clearly, then they can answer the relationship between two circles successfully (see Figure 6). Seventh graders know about segment diagrams and can apply the above process of variation problems to explore relationship between two circles. The longer potential distance of the seventh graders could be *shortened greatly*. Actually, the positional relationship of two circles can be transferred into the positional relationship between two segments (i.e., the distance between two centers of circles, radii). If students understand the positional relationship between two segments, then, they can easily grasp the positional relationship between two circles. It is critical to find the most essential and transferable connections between the anchoring knowledge point and the new problem. We define this type of crucial connection as "*core connection.*" Teaching through variation based on "*core connection*" could result in two unique effects.

Effects of Using Core Connection

The experiment data shows that there are important effects of using *core connection*. First, it could shorten the distance between anchoring knowledge point and a new problem. Second, it could mature cognitive thinking and advance thinking levels.

Shortening potential distance. Based on the experiment of potential distance, a deep analysis of the data shows that using *core connection* could shorten potential distance. Students' explorations of the five relationships between two circles between the experimental group and the control group (around 50 students) were examined and compared. In the experimental group, the teacher emphasized *core connection* by exploring problems with a truck and a bridge (Figure 8). Figure 9 shows students' correct rates in exploring these relationships in the experimental and control group in seventh grade, and control groups in seventh and eighth grades. The results indicated that students' correct rates from the experimental group was much higher than the control groups in seventh grade, and even higher than the control groups in eighth and ninth grades. These results imply that the use of *core connection* could shorten the potential distance significantly, and reduce the students' cognitive load.

Advancing thinking ability. Two types of test items, visual judgment and abstract logical reasoning, are used to examine the correlations between different mathematical thinking levels. The correlations between visual judgment and abstract logical reasoning in seventh, eighth, and ninth grades respectively are 0.390, 0.686, and 0.696. The data appears to imply that seventh grade is a transformative period

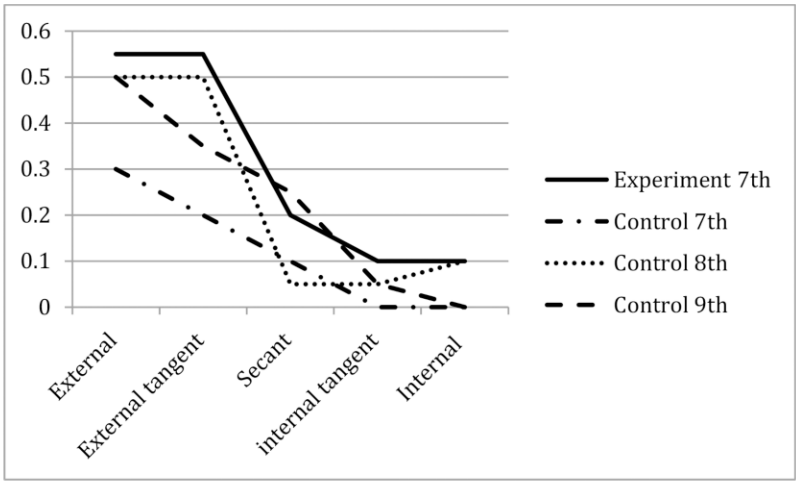


Figure 9. Correct rates of exploring new problems across different groups

from visual to logical reasoning. The scatterplots in Figure 10 further illustrate that students from the experimental group in seventh grade moved toward logical reasoning levels from visual perceptions. This means the transformation from visual judgment to logical reasoning occurred one year earlier (from eighth grade to seventh grade). Thus, variation problem focusing on *core connection* could promote students' transformation from visual judgment to logical reasoning significantly.

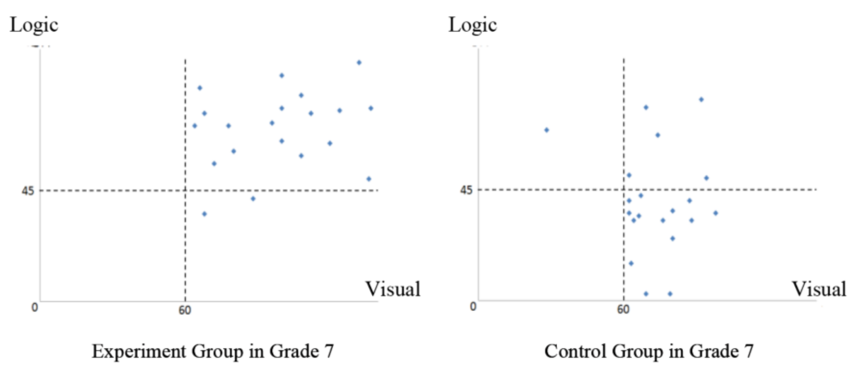


Figure 10. Scatterplots of students' thinking tests in seventh grade

Instructional Pudian

Building on the concept of *core connection* and its importance in procedural variation, this section further discusses another closely related concept of "*Pudian*"

(铺垫). According to Gu et al. (2004), *Pudian* is commonly used in Chinese classroom teaching, which is metaphorically described as "by putting blocks or stones together as a *Pudian*, a person can pick fruit from a tree which cannot be reached without the *Pudian*" (p. 340). Similar to the notion of scaffolding in the West (Wood, 1976), by establishing "*Pudian*", the students can complete the tasks that cannot be done without the "*Pudian.*" In contrast, the *Pudian* emphasizes "the process and hierarchy" of learning (Gu et al., 2004, p. 340). In classroom instruction, *Pudian* could be appropriately applied to instructional design and implementation as follows: Teachers and students move from their existing knowledge and cognitive level toward obtaining new knowledge and solving new problems through effective instructional design (or *Pudian*). The segment diagram in Figure 8 is an appropriate example of how *Pudian* can help students move from existing knowledge toward exploring positional relationships between two circles.

There are multiple strategies to help students move toward higher levels of learning. By utilizing the terminology of scaffolding in the West (*Pudian*, in China), it is crucial to construct appropriate scaffolding when necessary, and remove the scaffolding when unnecessary. In particular, when designing discovering or exploratory learning, appropriateness of constructing and removing scaffoldings is essential. The researchers (Bao, Wang, & Gu, 2005; Huang & Bao, 2006) explored teaching of Pythagoras' theorem by using scaffolding notions (see Figure 11).

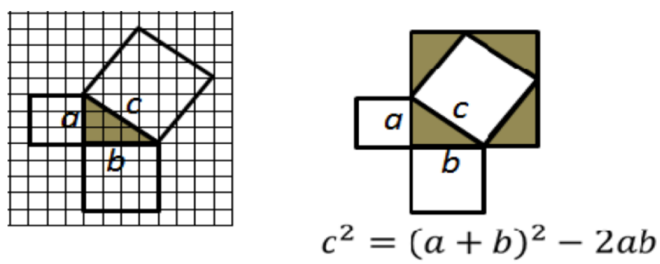


Figure 11. Constructing and removing scaffoldings

In the left figure, when *a, b, c* given various integer values (Pythagoras' number triples), then various data sets of a^2, b^2, $2ab$ and c^2 could be collected; based on this data, many conjectures about the quantitative relationships among a^2, b^2, $2ab$, and c^2 could be made (including Pythagoras theorem, and other conditional equations). After making conjectures, the role of the scaffolding (left grid in Figure 11) is complete, and thus, scaffolding must be removed. The right figure in Figure 11, the sides are labeled as letters *a, b*, and *c*, and calculating the area of the square extending on the hypotenuse is the key. The *core connection* is: Formula of completing square of sum used to calculate the area of a combination figure. From the anchoring knowledge point (area of triangle and square), students can use the scaffolding in the right figure to prove the theorem. This is a creative strategy that has evolved over decades.

This strategy has derived from one of the traditional features of learning: learning and teaching progressively. Teachers usually identify several hierarchical-progressive levels of subject topics, and then employ procedural variation problems (*Pudian*) supporting students to transcend their existing knowledge (anchoring knowledge point) to higher levels of knowledge. In Figure 11, the right figure is a simple and effective scaffolding (i.e., procedural variation) to support students to find proofs. These scaffoldings are interconnected progressively, which is a major strategy in Chinese classrooms. The scaffoldings or *Pudian* are instructional artifacts, which are designed for prompting student learning. Appropriate design and use of scaffoldings requires teachers to be creative designers, supporters, and guiders of student learning. An effective design of scaffoldings in China usually focuses on the progression of mathematical knowledge development and the "*core connection*" of different levels of mathematical knowledge.

Variation, Learning Trajectory, and Student Learning

Traditionally, teaching through variation mainly focuses on subject knowledge structure and teaching strategies from a teachers' perspective. Recently, some researchers explored how teaching through variation could help focus attention on student learning (Huang, Miller, & Tzur, 2015; Huang, Gong, & Han, 2016).

Huang and colleagues (Huang et al., 2015) proposed a hybrid-model for analyzing students' learning opportunities in the classroom. This model includes three hierarchical layers of principles for guiding mathematical instruction in Chinese mathematical classrooms. Teaching through variation (with bridging) is located at a meso-level. A macro-level is *Hypothetical Learning Trajectory* (HLT) and micro-level is known as reflection on activity-effect relationship (Ref*AER). At macro-level, HLT (Simon, 1995; Simon & Tzur, 2004) focuses on three key aspects: (a) *goals* teachers set for student learning in terms of conceptions (activity-effect relationships) they are expected to construct, (b) *sequences of mental activities* (and reflections on them) hypothesized to promote students' transformation of their extant conceptions into the intended ones, and (c) *tasks* designed and implemented to fit with and promote hypothetical reorganization processes from available to intended mathematics. At a meso-level, based on teaching through variation, six components are proposed as being important for effective mathematics instruction. They are (1) tailoring old-to-new; (2) specifying intended mathematics; (3) articulating mental activity sequences; (4) designing variation tasks; (5) engaging students in tasks; and (6) examining students' progress through variation practice. At a micro-level, teachers could monitor students' learning through systemic reflections on activity-effect relationships that include: (1) continually and automatically comparing the effects of the activity with the learner's goal and (2) comparing a variety of situations in which the recorded activity-effect dyads are called upon, which can bring about abstraction of the activity-effect relationship as a reasoned, invariant anticipation. Based on a fine-grained analysis of 10 consecutive lessons taught by a competent teacher in middle school in Shanghai (Clarke et al., 2006) using this framework, the authors concluded that: "our analysis of learning opportunities indicates the power of teaching through variation to deepen and consolidate conceptual understanding and procedural fluency concurrently" (Huang et al., 2015, p. 104).

Moreover, Huang, Gong and Han (2016) explored how teaching through variation and incorporating the notion of learning trajectory could be used as a principle for designing and reflecting upon teaching to promote students' understanding of division of fractions. In their study, a lesson study approach (Huang & Han, 2015) was adopted: a group of teacher educators (practice-based teaching research specialist and University-based mathematics educators) and mathematics teachers worked together to develop lessons on division of fractions based on variation pedagogy and learning trajectory through three cycles of lesson planning, delivering/observing lessons, and post-lesson debriefings. Based on a literature review, a hypothetical learning trajectory on division of fractions was proposed as a foundation for the design of the lessons. Data consisted of lesson plans, videotaped lessons, post-lesson quizzes, post-lesson discussions, and teachers' reflection reports. This study revealed that by building on the learning trajectory and by strategically using variation tasks, the lesson was improved in terms of students' understanding, proficiency, and mathematical reasoning.

Combined, these studies indicate that teaching through variation and incorporating learning trajectory (reflection on activity-effect relationship of student learning) could provide students with opportunities to develop conceptual understanding and procedural fluency concurrently.

INTERPRETATION, IMPLICATIONS AND SUGGESTIONS

In previous sections, we discussed major concepts and principles of teaching through variation that included two types of variations, potential distance, core connection, and *Pudian* (scaffolding). All of these ideas envision a core conception of learning through exploring a series of hierarchical-progressive tasks. This section interprets teaching through variation from other theoretical perspectives and discusses implications for classroom instruction.

Theoretical Interpretations

Gu et al. (2004) explored theoretical interpretations of teaching through variation from multiple theoretical perspectives. First, from the perspective of *meaningful learning* (Ausubel, 1978) that emphasizes establishing the non-arbitrary and substantive relationship between learners' prior knowledge and the new knowledge, they argued that conceptual variation could help students understand the essence of a concept and develop substantial relationships. Meanwhile, procedural variation could help students develop well-structured knowledge and non-arbitrary connections between different types of knowledge. Second, the notion of duality of mathematics learning (Sfard, 1991) proposes that mathematical concepts can be conceived in two fundamentally different ways: structurally (as objects), and operationally (as processes). Gu et al. (2004) claimed that by creating these two types of variation, it would enhance students' understanding of two aspects of a mathematical object: operational process and structural object (these two aspects of a mathematical object are complementary). Third, Gu et al. (2004) also discussed the similarities and differences between scaffolding (Wood et al., 1976) and *Pudian* (i.e., a strategy of procedural variation). Although both scaffolding and *Pudian* emphasize the support for students to achieve higher learning goals within zone of proximal development (Vygotsky, 1978), *Pudian* devotes more attention to core connection and hierarchical progression. Fourth, Gu and colleagues also discussed the relationships among Dienes' theory (Dienes, 1973), Marton's variation pedagogy (Marton & Tsui, 2004; Marton, 2015), and teaching through variation (Gu et al., 2004). Dienes emphasizes "mathematical variability" and "perceptual variability", while Marton stresses the patterns of what varies and what is invariant. Both of them mainly focus on *conceptual variation*. Thus, Gu et al.'s (2004) theory of teaching through variation developed the concept of variation pedagogy by illustrating procedural variation that focuses on developing problem solving ability and building a well-structured knowledge base. In the following section, an additional dimension of teaching through variation, namely, *dimensions of variation*, will be discussed.

Dimensions of variation. Mathematical instruction has often been criticized in the past. For example, in the 1940s, famous mathematicians Courant and Robbins (1941) critiqued mathematics instruction that focused on simple procedural practice, which may develop students' formal operation ability but has nothing to do with profound understanding of mathematics. In fact, precise understanding of mathematical concepts is the foundation of mastering mathematics, and effective problem solving is at the heart of all mathematics. Teaching through variation in China focuses on two fundamental aspects: understanding of concepts from multiple perspectives through conceptual variations and developing problem-solving ability and well-structured knowledge base through purposefully selected procedural variation.

The mechanisms and principles of teaching through variation (hierarchical-progressive learning) include: (1) a measurable, plausibly potential distance between existing knowledge (anchoring knowledge point) and the new knowledge or new problems; adjusting the potential distance based on instructional goals and student learning readiness is critical; (2) both conceptual variation and procedural variation should reflect the core connection between existing knowledge and new knowledge to be learned, and design variation problems should surround the *core connection*. By using appropriate procedural variation problems surrounding the *core connection*, the potential distance could be shortened and learners' thinking ability could be advanced.

Based on the research on classroom instruction reforms and practices over the past three decades, researchers have identified the following three critical aspects of "*core connection*": (1) Situation and application. This aspect is concerned with background and meaning of discovery and development of mathematics. It should be pointed out that background and application should not be treated as simply additional information. Rather they should be carefully considered from the perspectives of mathematical necessity and promoting learners' understanding. For example, the segment diagram in Figure 8 presenting the relationships between a truck and a bridge seem simple, but it reflects the essential quantitative relationship that could be used to present the positional relationship between the truck and the bridge and could be further transferred to present the positional relationship between two circles. (2) Computation and reasoning. These are two basic and fundamental mathematical thinking methods that form a system of mathematical thinking. Mathematical thinking methods reflect the simplicity and convenience of logical connections within variant contexts or situations. For example, in Figure 5, the variation practices regarding isosceles triangles provide an example demonstrating *core connection* in a logical system from a problem-solving perspective. (3) Cognition of learners. Most importantly, student learning should be the focus of all decisions. When designing applications or contexts, it is critical to consider if they could motivate student learning and are conducive to developing students' cognition and thinking. In Figure 11, the scaffolding (left figure) is designed for discovering Pythagoras' theorem by creating several sets of Pythagoras triples; the other scaffolding (right figure) is designed for discovering proofs of Pythagoras' theorem by calculating areas by completing square of sum. These are typical examples on how scaffoldings (*Pudian*) could be designed based on *core connection* between existing knowledge and new knowledge.

In summary, situation and application, computation and reasoning, and cognition level are three relatively independent dimensions, which form a comprehensive space of variation. Of course, when designing a particular lesson, we may focus on one or several dimensions and design greatly meticulous variation in those selected dimensions. Although constructing variation problems should be open, it should focus on essential goals: contexts of knowledge and development of new knowledge; transformation between complex and simple problems; and eliminating rote learning and mastering general and powerful methods.

Implications for Reform of Classroom Instruction

The tradition of teaching through variation has evolved for a long time in China. For further development, attention should be focused on the following two issues.

Variation surrounding core connection. Variation does imply neither "the more, the better", nor "the more difficult, the better." There is an old saying, "ten thousand variation problems remains the same principle (万变不离其宗)". The principle is promoting students' learning of mathematics. Teaching through variation effectively requires addressing students' learning differences. In order to implement differentiated instruction, multiple formative assessments could help teachers understand student learning, and adopt appropriate strategies of teaching through variation. These formative assessments include student-learning worksheets and post-lesson homework sheets, which are developed, based on instruction objectives of units or lessons and used for diagnosing and removing learning obstacles, discussing major problems in class, and designing and making use of post-lesson homework.

In addition, when designing procedural variation, it is crucial to identify and make use of *core connection* about different content. Take one released item on 2012 PISA test, for example (Figure 12).

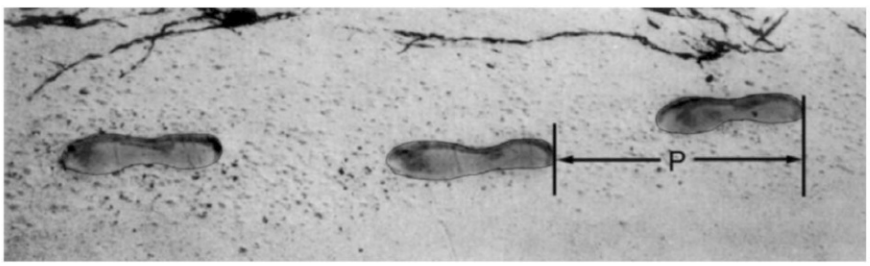


Figure 12. Walking problem on 2012 PISA test

The picture shows the footprints of a man walking. The pace length p is the distance between the rear of two consecutive footprints.

For men, the formula $p/n = 140$, gives an approximate relationship between n and p where, n = number of steps per minute, and p = pace length in meters.

Question 1: If the formula applies to Heiko's walking and Heiko takes 70 steps per minute, what is Heiko's pace length? Show your work.

Question 2: Bernard knows his pace length is 0.80-meters. The formula above applies to Bernard's walking. Calculate Bernard's walking speed in meters per minute and in kilometers per hour. Show your work.

Question 1 is used to test whether participants understand the formula, which acts as scaffolding for solving question 2. Question 2 is used to examine flexibility in using the formula and application of the relationship among distance, time, and velocity in daily situations. Each question demonstrates clear core connection between anchoring knowledge point and a new problem.

Core connection in algebra is abounding. For example, regarding operations with polynomials: the basic concept and skills include factors and like terms. Yet, like terms could be combined or split for different purposes. The purpose of using variation problem practice is not mainly for deriving a specific multiplication formula, or splitting, adding or factorizing formula. Rather it serves for understanding the core thinking methods: applications of operational principles of polynomials through transformation. For instance, first, transformation between multiplication and factorization, namely, $(x – 1)(x – 12) = x^2 – 13x + 12$: from left to right means multiplications (combination of like terms); inversely, it is factorization (including splitting like terms). Second, when discussing quadratic equations through the comparison of two equations: $x^2 + px + q = 0$ and $(x - a)(x - b) = 0$, where a and b are roots of the equation, then, the relationships between roots could be presented as follows: $a + b = -p$, $ab = q$ (Vieta's Theorem): $x^2 + px + q = 0$ can be transformed as $\left(x + \frac{p}{2}\right)^2 = A$ (where $A = \frac{p^2 - 4q}{4}$), thus, the quadratic formula can be derived. Third, when studying the quadratic function $y = x^2 + px + q$, the function can be transformed as: $y = \left(x + \frac{p}{2}\right)^2 - A$, thus, when $x = -\frac{p}{2}$, y = maximum value of the function; Moreover, the monotone and symmetry properties of functions can be analyzed easily. In this way, the operations of multiplication, factorization, and completing the square can lead to the discussion of relationships between roots and coefficients of quadratic equations, monotonous properties, and symmetric features, maximum or minimum value of quadratic functions. This is a typical example in school mathematics of how new concepts can be derived through making use of core connection.

Variation promoting self-exploratory exploration. One possible derivation of using variation in teaching is direct telling. Superficially, using variation knowledge eventually leads to telling rigid and cumbersome formulas. The ultimate goal of improving mathematics instruction is to develop students' self-exploratory learning ability and their ability to learn how to learn by themselves without teaching in the future. Thus, it is necessary to establish a new classroom ecology of harmony in relationships between teachers and students. For example, teaching Pythagoras' theorem for illustrating an ideal classroom ecology. Rigorous proofs of Pythagoras' theorem are difficult for students to understand; "measurement and calculation", or "cutting and pasting" methods are visual and interesting, but the teacher normally provides the results. The following is an example of self-exploratory learning of Pythagoras's theorem (Bao et al., 2005).

As shown in Figure 11, students are asked to make conjectures based on calculating the area of squares in several situations (Figure 13).

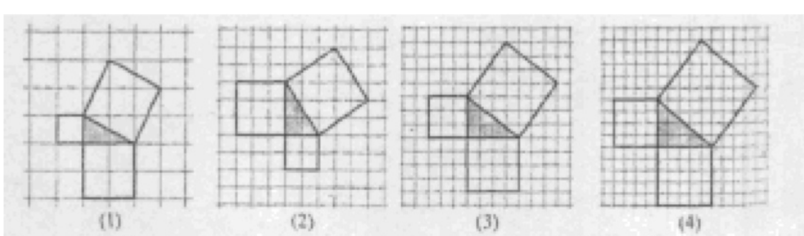


Figure 13. Make conjectures through calculating area of squares

In class, by using calculations on several diagrams (1)–(4) in grids, students created a set of data (see Figure 14).

Area	*Diagram (1)*	*Diagram (2)*	*Diagram (3)*	*Diagram (4)*	
a^2	1	4	9	16	……
b^2	4	9	16	25	……
$2ab$	4	12	24	40	……
c^2	5	13	25	41	……

Figure 14. Data collected based on selected diagrams

Based on these data, students are encouraged to make conjectures (correct and unexpected conjectures). The following is an excerpt focusing on proving and refuting:

1. S1: Based on the data in the table [Figure 14], I find that $c^2 = 2ab + 1$.
2. T: [Surprise! Unbelievable] how can you make this conjecture? Is it possible?
3. S2: I investigated when $a = 2$, $b = 4$, $2ab = 16$, $c^2 = 20$, $c^2 \neq 2ab + 1$.
4. T: Student 2 used a counterexample to refute your conjecture. It disproves $c^2 = 2ab + 1$.
5. S3: Mr. I found that when the difference between a and b is 1, the result is tenable.
6. T: [Thinking in brain: $c^2 = (a - b)^2 + 2ab$, when $b - a = 1$, $c^2 = 2ab + 1$] This suggestion is correct. This is a conditional equation. Good, let us examine the other equation that many of you suggested: $a^2 + b^2 = c^2$.
7. S4: This equation is tenable regarding the given four figures and numbers. But, I think that even if I examine 100 examples and the result is true, I cannot be sure that the equation is true when examining the next situation. So, we have to prove that this is true for all situations.
8. T: Whether $a^2 + b^2 = c^2$ is a theorem? Examining more cases cannot prove it. What do we need to do?
9. Ss: We have to prove.

The previous discourse illustrated that students were actively involving mathematical reasoning activities such as making conjectures ((1)), disproving and refuting ((3)~(5)), and developing proofs ((7)~(9)). The teacher was a facilitator to guide and solicit students' explorations.

Variation and learning trajectory. As discussed throughout this chapter, the core idea of teaching through variation can help students develop profound understanding of mathematical concepts and flexibility in problem solving through forming a well-structured knowledge system using hierarchical-progressive variation problems which surround the core connection between different types of knowledge. Paying attention to student cognitive readiness and development is also one key dimension of core connection. Yet, there are no concrete suggestions about how teachers can pay attention to student thinking and solicit student thinking. To this end, the exploratory studies by Huang et al. (2016) revealed an alternative. That is, to incorporate notions of learning trajectory with teaching through variation. Huang et al. (2016) found that through the combination of two theoretical perspectives, teachers were able to shift their focus on student thinking and solutions during lessons and post-lesson reflections, which eventually resulted in students' development of deep understandings. They further argued that the notion of teaching through variation emphasizes specific strategies in using systematic tasks progressively (content-focused), but it has not paid explicit attention to the route of children's learning. Thus, the incorporation of these two perspectives may provide a useful tool for designing and delivering lessons: Teaching through variation could help teachers strategically design and implement tasks in line with students' learning trajectory.

CONCLUSIONS

This chapter discussed the cultural and historical origin of teaching through variation. The traditional culture value and ancient mathematical learning ideas have afforded mathematical teaching and learning through variation, and the exam-oriented education system has further strengthened this practice. Based on experiences and empirical studies, the core concepts and major mechanisms of teaching through variation have been developed. Two types of variation include conceptual variation and procedural variation. The former focuses on building the essential connections between existing knowledge and new knowledge, developing profound understanding of a concept from multiple perspectives. The latter intends to develop students' problem-solving ability and develop an interconnected knowledge structure. By considering potential distance and *Pudian*, which are associated with core connection between existing knowledge (anchoring knowledge point) and the new knowledge or new problems, teachers are expected to design and implement hierarchical-progressive variation problems to achieve mathematical instructional goals. Appropriate implementation of teaching through variation is likely to develop students' conceptual understanding and procedural fluency concurrently. However, theoretically, more empirical studies on defining and measuring potential distance, and defining and identifying core connection among different types of knowledge are needed. In addition, how to develop teaching through variation by incorporating relevant theoretical perspectives such as learning trajectory (Simon, 1995) and mathematical teaching practices (NCTM, 2014) is a new endeavor worthy of exploring. Practically, implementing teaching through variation effectively requires teachers to possess a profound understanding of content knowledge and rich instructional expertise. It calls for pertinent teacher professional development programs.

NOTE

[1] Teaching through variation is exchangeable with teaching with variation, or *Bianshi* teaching, 变式教学, in this chapter.

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