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# 15. DEVELOPING ALGEBRAIC REASONING THROUGH VARIATION IN THE U.S.

## INTRODUCTION

Historically, algebra in the U.S. has been viewed "as a gatekeeper to a college education and the careers such education affords" (Kilpatrick & Izsák, 2008, p. 11). As such, current curriculum documents emphasize the need to support *all* students in learning algebra (Common Core State Standards Initiative [CCSSI], 2010; National Council of Teachers of Mathematics [NCTM], 1989, 2000). To do so, however, requires a reconceptualization of the preparation students receive for the formal study of algebra (Kilpatrick & Izsák, 2008). In considering this preparation, scholars have indicated that students need opportunities to engage in algebraic reasoning (Blanton & Kaput, 2005; Earnest, 2014; Hunter, 2014; Kaput, 2008; Kilpatrick & Izsák, 2008). Different perspectives exist, though, with regard to the core aspects of algebraic reasoning.

Kaput (2008) characterized algebra in two ways. First, he described algebra as an inherited subject or cultural artifact. Second, Kaput portrayed it as a human activity that requires humans for it to exist. In our work, we focus on the latter and explore Kaput's (2008) view that "the heart of algebraic reasoning is comprised of complex symbolization processes that serve purposeful generalization and reasoning with generalizations" (p. 9).

Within this view of algebra, Kaput (2008) described a core aspect of algebraic reasoning as involving "algebra as systematically symbolizing generalizations of regularities and constraints" (p. 11). Although this core aspect appears in some form across all strands of algebra, we are particularly interested in algebraic reasoning as it supports generalizing a pattern through argumentation for the purpose of building towards functions (Kaput, 1999; Warren & Cooper, 2008). This view of algebraic reasoning has permeated recent international curriculum documents (e.g., Ministry of Education, 2007; Ontario Ministry of Education, 2005) as well as U.S. curriculum documents for over two decades. Table 1 provides an overview of the algebraic presence in U.S. curriculum documents, including *Curriculum and Evaluation Standards* (*CES*, NCTM, 1989), *Principles and Standards for School Mathematics* (*PSSM,* NCTM, 2000), and *Common Core State Standards for Mathematics* (*CCSSM,* CCSSI, 2010).

*R. Huang & Y. Li (Eds.), Teaching and Learning Mathematics through Variation, 321–339.*

The inclusion of algebraic reasoning in U.S. standards is informed, in part, by a literature base that supports a need to develop algebraic reasoning in middle school students (Blanton, 2008; Carraher & Schliemann, 2007; Lins & Kaput, 2004; Soares, Blanton, & Kaput, 2005). Note that we define middle school students as those in grades five through eight, approximately 11 through 14 years old. Additionally, algebraic reasoning is described as the process of building general mathematical relationships and expressing those relationships in increasingly sophisticated ways (Ontario Ministry of Education, 2005; Soares et al., 2005; Warren & Cooper, 2008). Furthermore, Carraher and Schliemann (2007) stated that the role of functions was *the* link between learning algebra from the middle school level through college. Thus, implementing this view of algebraic reasoning in middle grades is substantiated and of "great relevance for mathematics education because it provides a special opportunity to foster a particular kind of generality" (Lins & Kaput, 2004, p. 47) in students' thinking.

*Table 1. Algebraic reasoning in U.S. documents*

| | |
|---|---|
| *Understanding patterns* | |
| *CES* | Analyze tables and graphs to identify relationships (Grades 5–8) |
| *PSSM* | Generalize a variety of patterns with tables, graphs, and words (Grades 6–8) |
| *CCSSM* | Analyze patterns and relationships (Grade 5) |
| *Representing mathematical situations* | |
| *CES* | Represent situations with tables, graphs, and equations (Grades 5–8) |
| *PSSM* | Use symbolic algebra to represent situations and to solve problems (Grades 6–8) |
| *CCSSM* | Represent and analyze quantitative relationships (Grade 6) |
| *Generalizing to functions* | |
| *CES* | Generalize number patterns to represent physical patterns (Grades 5–8) |
| *PSSM* | Identify functions and contrast their properties between quantities and contrast their properties from tables (Grades 6–8) |
| *CCSSM* | Use functions to model relationships (Grade 8) |

Despite the importance of algebraic reasoning demonstrated in both the curriculum documents and the literature, U.S. and international classrooms have fallen short in providing an opportunity for this type of learning (cf. Carraher & Schliemann, 2007; Stacey & Chick, 2004). To address this issue, Blanton (2008) developed curricular materials aimed at supporting teachers as they introduce algebraic reasoning in elementary and middle grades. In these materials, Blanton (2008) described algebraic reasoning as a habit of mind that students acquire through instruction that gives opportunities to "think about, describe, and justify general relationships" (p. 93).

This focus allows for students to engage in algebraic reasoning, a process that is supported by the following teacher practices:

- helping students learn to use a variety of representations, to understand how these representations are connected, and to be systematic and organized when representing their ideas;
- listening to student's thinking and using this to find ways to build more algebraic reasoning into instruction; and
- helping students build generalizations through exploring, conjecturing, and testing mathematical relationships (Blanton, 2008, pp. 119–120).

Through these practices, algebraic reasoning can focus on functional thinking via arithmetic tasks that are transformed into opportunities for generalizing mathematical patterns and relationships (Blanton, 2008; Ontario Ministry of Education, 2005). One way that this can be accomplished is through varying a single task parameter (Blanton, 2008; Blanton & Kaput, 2003, 2005; Ontario Ministry of Education, n.d.; Soares et al., 2006).

Varying a "parameter allows you to build a task that looks for a functional relationship between two quantities" (Blanton, 2008, p. 58) and "can shift the focus from arithmetic thinking to algebraic thinking" (Ontario Ministry of Education, n.d., p. 19). This emphasis on varying a parameter suggests that applying a theory of variation to the design of instruction may be an important means for providing middle school students with an opportunity to engage in algebraic reasoning. Therefore, the purpose of this chapter is to present a case that describes a series of tasks whose development was informed by a theory of variation. Collectively, the tasks align with the vision established in the U.S. curriculum documents and aim to support the development of algebraic reasoning in sixth grade students. In the subsequent sections, a theory of variation will be presented, followed by a description of a four-task sequence, including its implementation in a sixth grade classroom. Finally, a discussion and reflection on the role of variation in the task sequence will be provided.

## THEORY OF VARIATION

According to Marton, Runesson, and Tsui (2004), learning is a process in which students acquire a particular capability or way of seeing and experiencing. In order to see something in a certain way, students must discern critical features of an object. This is known as the theory of variation (Leung, 2012; Marton & Pang, 2006; Marton et al., 2004). The theory of variation can aid teachers in developing students' algebraic reasoning skills by providing students with opportunities to discern critical aspects of what is to be learned, also known as the object of learning (Ling, 2012). While teachers cannot guarantee the lived objects of learning experienced by the students, they can focus students' attention on critical features by providing contrasting experiences that allow students to develop and test conjectures. After all, students can only begin to understand the object of learning once they have seen it in various situations and with varying dimensions (Marton et al., 2004). Therefore, it is imperative that students discern the patterns of what varies and what is invariant in a learning situation (Leung, 2012). It is the main objective of the teacher to reveal these patterns to support students in powerful ways of *seeing* the intended object of learning, which leads to powerful ways of acting (Marton et al., 2004).

There are two features of the object of learning: "the direct and the indirect objects of learning" (Marton & Pang, 2006, p. 194). The direct object of learning is defined in terms of content, such as evaluating algebraic expressions. In contrast, the indirect object of learning refers to "the kind of capability that the students are supposed to develop such as being able to give examples, being able to discern critical aspects of novel situations" (Marton et al., 2004, p. 4). In the paragraphs that follow, we apply this theory of variation to the design and implementation of a four-task lesson sequence that aimed to support the development of algebraic reasoning. We include descriptions of both the direct and indirect objects of learning as evidenced in the design and enactment of the task sequence.

## TASK SEQUENCE

### *Design*

Defined as what the teacher aims for the students to learn, the intended direct objects of learning during this task sequence were for students to be able to generalize a linear pattern given a series of geometric figures, give the generalization as an expression involving one variable (i.e., $an + b$ where $a$ and $b$ are integers), and justify the generalization based on the geometric pattern. This objective supports standard 6.EE.9 from the *CCSSM* (CCSSI, 2010), which states:

> Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. (p. 44)

The intended indirect objects of learning, or capabilities to be developed, during the lessons included seeing the grouping structures within the geometric figures ($a$), relating these groups to the corresponding figure number ($n$), and recognizing the constant as what appears each time in the figure but is not in a group ($b$), where $a$, $n$, and $b$ represent integers in the generalization $an + b$.

The students in these lessons needed to see linear patterns in different circumstances, with certain aspects varying in dimension. Research posits, "The most powerful strategy is to let the learners discern one at a time, before they encounter simultaneous variation of the features" (Lo & Marton, 2012, p. 11). This idea was considered when developing the sequence of tasks. Table 2 provides an overview of the lesson sequence, including the geometric patterns featured in each lesson. In each task, a series of figures is presented and the student is expected to develop a means for determining the number of segments needed to produce the figure, based on the figure's position in the pattern.

*Table 2. Overview of lesson sequence*

| *Task* | *Fig. 1* | *Fig. 2* | *Fig. 4* | *Generalization* |
|---|---|---|---|---|
| 1 |  |  |  | *3n + 1* |
| 2 |  |  |  | *4n + 1* |
| 3 |  |  |  | *4n + 4* |
| 4 | Students are given a generalization and are expected to create a geometric pattern. | | | __ *n + 4* |

*Task 1.* The purpose of this task was to introduce the process of generalizing the pattern. The intent was for Task 1 to provide a common experience on which to build for the students. This included introducing common vocabulary, such as *generalization*, and a particular way of looking for a relationship between the figure number and its corresponding figure. In this lesson, the intent was for students to experience variation with the number of segments in the figure (referred to as fence panels in the problem context and represented by toothpicks) given the number of squares (referred to as corrals in the problem context). Although the corresponding algebraic expression for *n* corrals is *3n + 1,* the goal for this lesson did not necessarily include representing the pattern algebraically, only verbally. The variation in Task 1 was limited to only variation found within the pattern, as students examined Figures 1, 2, and 4 separately. Therefore, there was no *contrast* or anything with which to compare it, perhaps making it difficult to discern what aspects caused the general expression to be *3n + 1.*

*Task 2.* In order for the learners to discern the critical features of the object of learning, Task 1 focused on introducing the idea of finding a generalized pattern. In contrast, Task 2 introduced a different pattern that allowed students to experience the variation of one dimension of the object of learning – the number found in each group. The new pattern held the constant invariant, while the group value changed, leading to the corresponding expression *4n + 1*. In this way, students had the opportunity to see the object of learning under different circumstances and test the validity of any conjectures that they had, seeking to understand the new figures by trying to discern what was critical and what was not (Ling, 2012). Moreover, this task provided students with the opportunity to be aware of the two situations at the same time in order to compare and contrast them, what is known as "diachronic simultaneity" (Marton et al., 2004, p. 17). Based on what the students had experienced before and what they were experiencing in this task, there was the potential for them to develop "separation" (Marton et al., 2004, p. 16) with the group feature and be able to discern it from other features. It was also important that this first variation be situated within a similar situation so that everything else was invariant, making it clear what was affecting change.

*Task 3.* Task 3 was similar to Task 2 in two ways. First, the grouping structure (i.e., the house shapes in the pattern), and thus the coefficient (*a*), remained the same. Second, variation of one dimension of the object of learning was present. However, in this scenario, it was the constant value that was *separated* so that students could experience how the invariant structure within the pattern affects the general expression. The teacher intended to keep the grouping structure the same so that this effect would be clearer. According to Marton et al. (2004), students need to experience the following related to the object of learning: contrast instances, make generalizations from varying appearances, separate each individual aspect, and fuse them together simultaneously (Leung, 2012). In this task, students are separating the last aspect of the object of learning. As a result, they should be able to discern between the two aspects of the object of learning and have a basic understanding of how varying dimensions of those aspects alter the general expression.

*Task 4.* The purpose of this final task was to further develop students "professional seeing" (Marton et al., 2004, p. 11) of generalizing patterns by providing them with the opportunity to experience the object of learning from a novel perspective. In this task, students are asked to create a geometric pattern that satisfies __ $n + 4$. In order to build a corresponding geometrical pattern, students must experience the grouping structure and constant simultaneously and understand how each aspect affects their pattern. Afterward, students are able to compare and contrast solutions, recognize different grouping structures, and see multiple representations of the same algebraic formula.

*Summary.* This sequence of tasks should allow students to become aware of the critical features of the object of learning through carefully selected experiences directed by the theory of variation. Through sequences of contrast, generalization, and separation (Marton et al., 2004), students should be able to enhance their "seeing" (Marton et al., 2004, p. 11) of the intended objects of learning. However, what matters most is what the learner actually encounters and what is possible to learn in the context of the lesson, what is known as the enacted objects of learning

(Marton et al., 2004). In the section to follow, the enacted objects of learning are described through the patterns of variation and invariance that were actually co-constructed by the teacher and the students.

*Implementation*

In this section, we present a summary of the four-task lesson sequence (see Table 2), or enacted objects of learning, that was implemented in a sixth-grade class in a suburban school district located in the southeastern United States. The class had 20 students and met for 55 minutes each day. The first author was the instructor for the lessons. In her role as a university professor, she spends a considerable amount of time teaching demonstration lessons in local schools and has been recognized for her expertise and experience in implementing reform-oriented lessons. The four-task lesson sequence was videoed for the purpose of developing a multimedia case to support teachers' understanding of reform-oriented instruction.

*Lesson 1.* To begin the lesson, the teacher described a problem scenario designed to support the students in understanding the task at hand.

> I have some land that I just bought and I am going to build corrals on the land. We will use toothpicks to represent the corrals. *(Displays a square-shaped corral made with four toothpicks).* That will be one corral. How many panels does it take to build one corral? *(Students respond with four.)* I can build more than one corral but they will be built lengthwise. Now, I am cheap and I do not like to spend money. When I build the second corral, I do not double up on fence panels. *(Displays two corrals made of toothpicks.)* How many fence panels have I used? *(Students respond with seven.)* So here is our problem. I want to build as many corrals as possible on my land but I do not know how long the land is or how many fence panels I will need. This *(pointing to Figure 2)* is two corrals, and it takes seven fence panels. Predict how many fence panels we need for four corrals. Do you have your number? Build your corrals and see if your prediction is correct. *(Students build four corrals with toothpicks.)* How many panels did you need? *(Students respond with 13.)* So here's our task: If I tell you the number of corrals I can build on my land, I need you to tell me how many fence panels I will need.

After supporting students in thinking about the problem scenario, the teacher asked questions aimed to support students' recognition of the structure of the corral pattern.

> T: When you built the corral and then counted the fence panels, how did you count? Think about how you could describe how you counted the fence panels. Jot down how you counted and we will share our strategies in just a moment. *(Students take approximately one minute to write their strategies.)* Let's start with Ben.

S1: I counted the first pen with four and then I added three three times.
T: Do you all understand what Ben said? I am going to ask Candy to repeat Ben's idea.
S2: He counted the first pen with four and then he counted threes.
T: Larry, how did you count?
S3: I counted the left toothpick, then the top toothpick, then the bottom toothpick, then the right toothpick, like all in one box. *(The student illustrates how he counted the remaining toothpicks: top, bottom, right, top, bottom, right, top, bottom, right.)*
T: Did someone count differently?
S4: I counted the ones in the middle, then the ones on the top, then the ones on the bottom.
S5: I counted the top and bottom and then the middle.

Following this exchange, each student was given a number of corrals (i.e., 6, 7, 9, 10, 12) for which they were to figure out the corresponding number of fence panels. After students in their small groups checked each other's work, the teacher asked the students to look across the different problems and identify two or three things that they noticed. The following exchange occurred.

T: What is something that you or your partner noticed?
S1: The number of panels is the number of corrals times three and then you add one.
T: I think I heard a lot of different groups saying something like this. I want you to talk about this – why would this be true? If you didn't see this, check it with your problem. Check it – why would this be true?
S2: We thought because of the four and the rest was three. We didn't have to add any more because the first one was whole.
S3: If we are counting the first four and we take off one and add all the rest together that would bring us to, say nine times three is twenty-seven, and then you add back on the one you took off.
S4: Wouldn't that "three times the number of corrals plus one" – would that be a formula for the problem?
T: I'm going to write that over here. Remember that I do not know how big the land is. What are some other observations?
S5: We noticed that you should make sure that you counted all the panels.
S6: That the number of corrals had an impact on the number of panels.
S7: If you use a simple pattern and you lay it out the long way, it is easier to complete. It is simple to complete.
T: So you are thinking about how you can see the patterns in there. Remember that I do not know how big the land is. They are going to call me up and say, "Hey, we think you could have 200 corrals on there," and I need to be able to immediately say how many panels I need. Which of

our observations is going to help me with that? Talk to your partner about that.

S8: The first one because it would be 200 times three which would be 600 plus one making it 601.

T: Thumbs up if you agree that the first observation is going to be most useful for solving our problem. Ok. Thumbs up if you agree that for 200 corrals we will need 601 fence panels. Wow! I am going to have to challenge you now. Remember this word *formula*. How could you use symbols and a variable to represent this first observation? Talk with your partner.

Students eagerly talked with their partner about how to represent the observation (i.e., the number of panels is the number of corrals times three and then you add one) with symbols. The following expressions were offered: $C \times 3 + 1$; $3n + 1$; $(C*3) + 1$; $3c + 1$. Next, the teacher linked the students' use of the word *formula* to the words *expression* and *generalization*. After some discussion regarding why the generalization was useful for the problem, the teacher asked how many corrals could be built if there were 61 fence panels. The class ended with a discussion of the solution to this problem.

*Lesson 2.* For their homework, students revisited the corral task and responded to the following prompt: *When Sarah looked at the corrals, she said that she saw groups of 3. What do you think she meant by that?* To start the second lesson, the teacher asked the students to take out their homework sheet and compare their responses to this prompt. Then, the following exchange occurred.

T: I would like to have three people share with us what they have written. Alice?

S1: I thought that she started with the first corral and she took out the first toothpick so it would have groups of three toothpicks.

S2: After you have the first set of four toothpicks, you have sets of three toothpicks.

S3: She was thinking about three corrals.

T: My question is: We see how Sarah is thinking about these groups of three. Right? Alice said that toothpick is gone and we have these groups of three. And then Larry said we have this group of four and then we have these threes. And then Alden is talking about these corrals of threes. And so my question that I want you to think about inside your head for just a minute is: How did Sarah's groups of three help us to think about the pattern? *(The students discuss their thoughts in small groups.)* Let's start with Callie.

S4: Take out one toothpick and then there will be threes and then you add the one back.

T: How is this helping you?

S4: Then you can figure out how many toothpicks?
S5: Every time you are going to times it by three and then – take that one panel off then times it and then add the one back.
S6: I thought maybe you could take one out and add three each time.
T: So you take the one out and add three each time. And repeatedly adding the threes is multiplying. *(Teacher points at the multiplication symbol in the generalizations recorded from the previous day.)*
S7: She said that she saw groups, meaning there was more than one group of three. So when you did the formula and taking one out, you would just multiply the number of fence panels times the number of corrals that you have the number of fence panels that you need for all of them.
T: So from this what we are beginning to see is this idea of groups - when we are trying to figure out our generalization it is helpful to think about groups.

Following this exchange, the teacher introduced the task for the day by telling a story, similar to the one from the previous day and using a new shape for the figures, which the students called a house. After asking students to share what they noticed about the pattern, the teacher asked students to think about how this new pattern was different from the pattern explored on the previous day.

S1: Instead of the three in the pattern, we are going to have a four.
T: So you are thinking about multiplying by four. Someone else?
S2: Instead of adding three we are adding four.
T: Good. Another idea?
S3: Houses use four toothpicks.
T: Do you all understand what he is saying? Where are the groups in this pattern? Remember in the homework, Sarah said something about the groups. Where do we see groups in this pattern? Write your ideas down on the paper.
S4: I see groups of four.
T: Will you come up and show us where you see groups of four? *(The student demonstrates at the front of the class the groups of four that she sees.)* Do you all see the same groups?
S5: Each one would have a group of four toothpicks, except for that first one.
T: How can we use our strategies to figure out how many toothpicks are needed for a certain number of houses?

Following this exchange, each student was given a number of houses (i.e., 8, 9, 10, 11, 12, 15) for which they were to figure out the corresponding number of toothpicks. After students in their small groups checked each other's work, the teacher used their number pairs (i.e., number of houses and number of toothpicks) to create a function table. In the function table, she recorded an *n* in the input column and asked the students to think about the corresponding generalization to record in the output table. After time for small group discussion, the students offered the following generalizations: n x 4 + 1; 4n +1; 4h +1; 4 * n + 1. The lesson concluded by finding the output for an input of 50 and finding the input for an output of 81.

*Lesson 3.* On this day, the opening of school was delayed by two hours due to inclement weather. As a result, the original lesson was modified to fit within a 30-minute timeframe. To begin the lesson, the teacher distributed a paper that contained representations of the new pattern. Students noted that a garage had been added to the houses. The teacher asked them to create a function table for the pattern. After several minutes of working, the teacher asked one student to display her work for the class to examine.

S1: I took the one house – it was five toothpicks. And then I added another three for the garage. Then for the second one, I did the two houses, which was nine toothpicks and added three for the garage. Then I saw a pattern – add four each time so that's 8, 12, 16, 20, 24.

T: Tammy, can I stop you a second? Will you all take a look at Tammy's outputs and see if you agree with those? *(Students compare their charts with Tammy's work.)* Ok, so keep going, Tammy.

S1: I did the same thing and then the formula would be *n* times four plus one plus three or to simplify that it would be *n* times four plus four.

T: Tammy, can you tell us again how it is that you figured out the formula or the generalization?

S1: I took the formula that we did yesterday, *n* times four plus one, and I noticed that the garage was just another three sides so all I did was just add three to the formula.

T: And I noticed that some of the other groups did the same thing. They had the plus one and then the plus three, which simplified to 4*n* plus four. So the generalization that she is offering to us is *n* times four plus one plus three or *n* times four plus four. So I want you to do two things in your groups. First, take this generalization and check it. Take an input value, substitute it into the generalization, and see if it produces the correct output. And then second, I want you to think about *why* are we multiplying by four and then why today are we *adding* four when we were adding one yesterday? Talk to your partners. *(Students work in their groups for several minutes, writing down their ideas.)* Let's share out whole group what we are thinking.

S2: I was thinking you would remove this square – the garage- and then you count the pieces of the houses and you get four and you multiply by the number of houses you have and then you add the four back on.

S3: You have four sides on each of the pentagon houses and then you add on the square and that puts the side back on the house.

T: Remember how Sarah saw groups in our problem the other day. Talk to your neighbor about the groups that you see.
S4: She saw groups of four. *(Student outlines the house, missing one side.)*
T: It happens that there is a group of four here in the garage too. This four is different. This number sitting out there by itself is called the constant. So we are looking at what comes in groups and we are looking at the constant – what is sitting here.

With only a few minutes remaining in class, the teacher asked the different pairs of students to develop a pattern for a pre-selected generalization. All generalizations were of the form ___ $n + 4$, where the coefficient of $n$ differed for each group. Students were not able to make much progress, however, as class ended.

*Lesson 4.* Following some discussion of a homework problem, the teacher began class by asking students to look back across the three patterns developed during the previous three lessons. She reminded them of the groups and the constants that had been discussed previously. Then, students began working to develop their own geometric pattern that could be represented by the generalization that was assigned to them. Two groups were asked to present their work to the class. The dialogue from one discussion, which focused on the pattern shown in Figure 1, is featured here.


*Figure 1. Pattern presented by students*

T: Lets give our attention to this group and think about their work.
S1: We got four $n$ plus seven. We thought about a house with two garages. In figure number one, you remove the two garages and count the four panels and then put the garages back on; that is the plus seven.
T: Can you show us where your groups of four are?
S1: The groups of four are right there (outlines part of the house).
T: And where is the constant seven?
S1: It would be here in the garage.

Following the two presentations, students were asked to reflect on the ideas learned over the past four lessons. Students' ideas included: the meaning of the word generalization; it can be hard to find figures given the generalization; the constant; the input/output table.

## THE PRESENCE OF VARIATION IN THE LESSON SEQUENCE

The intended objects of learning for this sequence of lessons were for students to be able to generalize a linear pattern given a series of geometric figures, give the generalization as an expression involving one variable (i.e., $an + b$ where $a$ and $b$ are integers), and justify the generalization based on the geometric pattern. Employing the theory of variation allowed for the intended object of learning to be made accessible to the learners in the classroom. In this section, we present a discussion of the intended objects of learning, the enacted objects of learning, and the lived objects of learning.

### *Intended Objects of Learning.*

The planned sequence of lessons, as represented in Table 3, demonstrates the intentional use of variation to bring attention to the features of linear functions. In the first lesson of the sequence, a toothpick pattern of corrals was introduced in order to provide a starting point for the discussion of linear functions. Then, within the first lesson, only the number of corrals was varied, bringing awareness to the relationship between an input and an output in a linear function. This use of variation established a common experience on which to build understanding of the process of generalization.

*Table 3. Dimensions of variation by task*

| *Task* | *Dimension* | *Variant* | *Invariant* | *Object of learning* |
|---|---|---|---|---|
| 1 | Corrals | Number of corrals (1 to n) | Group size (3) and constant (1) | How the number of corrals relates to the number of fence panels needed |
| 2 | Groups | Group size (3 to 4) | Constant (1) | How the number within each group alters the general expression |
| 3 | Constant | Constant (1 to 4) | Group size (4) | How the additional fence panels alters the general expression |
| 4 | Direction | Given expression instead of picture, group size (4 to __ ) | Constant (4) | Create a geometric figure given a general expression |
| Day 2 Homework | Type of Pattern | Counting shapes instead of sides | Generalization (3n+1) | Transferability |

The subsequent lessons then proceeded to vary one feature of linear functions at a time so as to bring attention to the characteristics of the parts of a linear function.

The second lesson focused on a new toothpick pattern in which the number in each group varied from the corral pattern of the first day. Then the third lesson presented a third toothpick pattern in which the constant was varied. In order to adhere to variation theory, the explored pattern for each day was of the same style (i.e., envisioned as built out of toothpicks), therefore allowing this aspect of the discussion to remain invariant. In addition, only the position in each sequence (or input) was varied within the main activity in each lesson. By keeping these portions invariant, the lessons drew attention to the varied feature, allowing students to separate these features.

Homework was assigned on days two and three in which the visually presented pattern was of a different form than the in-class toothpick models. This variation was intended to provide students an opportunity to extend their thinking about linear patterns into different visual images while maintaining the same generalization that had been explored in class. For example, the homework pattern on the second day was an equilateral triangle with squares built on each side of the triangle (see Figure 2). In counting the number of shapes (i.e., squares and triangles) used to create each "Y," the generalization was $3n + 1$, where $n$ represents the position of the figure in the pattern. This problem required students to count shapes rather than segments but utilized the same generalization that students explored in class on the first day.

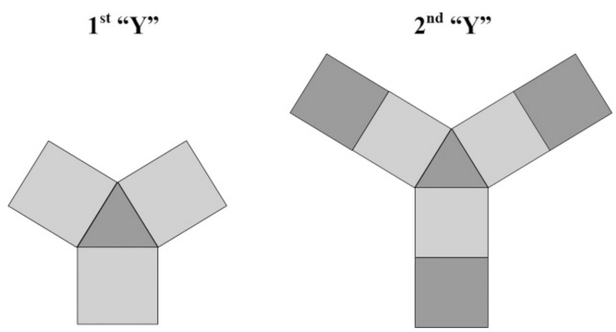



*Figure 2. Homework task pattern for Day 2*

The intention for including a different style of pattern in the homework was to vary the type of pattern with which students interacted while keeping the generalization of the pattern invariant, bringing awareness to the transferability of the concepts of generalizations of linear functions.

*Enacted objects of learning.* Throughout the four lessons in the sequence, the instructor focused student attention on the object of learning with clear questions. In the first lesson, the instructor asked, "When you built the corral and then counted the fence panels, how did you count?" This question encouraged students to consider the different ways in which the panels could be counted and set in motion the possibility for a wide variety of generalizations. In the next phase of this lesson, however, the first student offered that he noticed, "The number of panels is the number of corrals times three and then you add one." This student statement seemed to constrain the ways in which other students later considered the generalization of the relationship. Rather than offering a rich variety of generalizations for the pattern, the generalizations were limited to similar expressions (i.e., C × 3 + 1, 3n + 1, (C*3) + 1, 3c + 1). Although the generalizations were limited, the experience allowed students to focus on the pieces within a linear function and begin to operationalize the ideas of groups and constants as related to them.

At the start of the second lesson, the instructor focused student discussions on the homework by asking, "How did Sarah's groups of three help us to think about the pattern?" This focusing question constrained student thinking to consideration of the groups rather than consideration of the entire linear function. We see the impact of this constraint in the responses of the students during the class discussion as students connected the groups of four in the day two lesson to the groups of three in the day one lesson. Student responses in the class discussion incorporated the language as they said, "I see groups of four," and, "Each one would have a group of four toothpicks except for that first one."

Having established the idea of the role of groups in linear functions during the first two lessons, the planned lessons varied the constant on the third day and held the number of groups invariant. After students generated the function for a new pattern, the instructor asked a focusing question: "Why today are we *adding* four when we were adding one yesterday?" Because the duration of the lesson on this day was shortened (due to weather delays), students did not have enough time to grapple with the idea of the constant and returned to discussion of the groups in their conversation. However, in their presentations on the last day of the lesson sequence, students clearly identified the role of the groups and the constant.

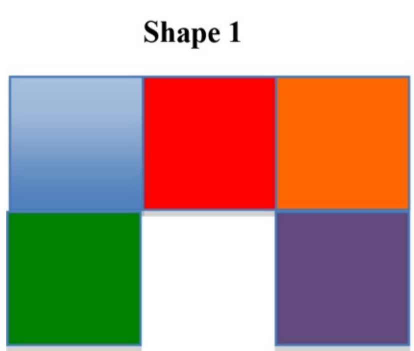
From observations of the enacted lessons, it appears that students were beginning to make sense of the concepts of the role of groups and the role of the constant in linear functions. The choice to use only toothpick structures during the lessons seemed to allow students access to learning about the concepts separately. In a continuation of these lessons, variation concerning the physical structure of the patterns may provide opportunities for students to generalize more broadly.

*Lived objects of learning.* On each day of the lessons, students were assigned homework. We can glean some insight into the lived objects of learning by examining the student work, looking for patterns in learning. On the first homework assignment, students were asked, "When Sarah looked at the corrals, she said that she saw groups of three. What do you think she meant by that?" Student responses to this question varied. Within one group of students who were seated together in class, the responses included: "She saw three even groups of toothpicks;" "That there is a group of four and groups of three connected to it;" "That after you have one set of four toothpicks you have sets of three toothpicks;" and "She saw three panels with three groups." It is clear that there were still a variety of levels of understanding of the concept present in the class.

Homework assigned on the second day required students to draw figures related to the pattern presented in Figure 2 and then generalize the pattern. In most cases, students were able to draw the fourth and tenth figure in the patterns. However, various correct and incorrect generalizations of the pattern were suggested. Provided generalizations included: $3n + 1$ (a correct generalization); $4n$; $n \times n + 1$; $n + 7$; and $4n + 1$. Of the 14 students who submitted the assignment, six of them provided a correct generalization. Of the eight who had incorrect generalizations, three provided responses that did not represent generalizations (i.e., 35 or 4).

Homework assigned on the third day included the following problem:

> Joseph made a pattern using squares. The first figure of Joseph's pattern is pictured below along with his function table. Draw the next two figures in the pattern so that the pattern matches the function table. Then, generalize the pattern.




| Input (Shape Number) | Output (Number of Squares) |
|---|---|
| 1 | 5 |
| 2 | 7 |
| 3 | 9 |
| 4 | ______ |
| . | . |
| . | . |
| . | . |
| 10 | ______ |
| . | . |
| . | . |
| . | . |
| $n$ | ______ |

*Figure 3. Homework task for Day 3*

On this assignment, half of the students provided a correct generalization. In other words, more students were attending to the nature of the role of groups and constants in generalizations of linear patterns. In addition, students were asked to create their own patterns and provide a generalization. Although many students still chose to work with toothpick models, there was more variation in the arrangement of the toothpicks and some students even chose to create a model other than a toothpick model.

Across the assignments and in-class work, gains in understanding were found. Through variation on the object of learning, the students were afforded the opportunities to consider features of linear functions. The lived objects of learning indicate that most students were beginning to make sense of the concepts of linear functions.

## CONCLUSION

Informed by the theory of variation and U.S. perspectives on developing algebraic reasoning in middle grades learners (Blanton, 2008; Blanton & Kaput, 2003, 2005; Kaput, 1999), the sequence of tasks presented in this chapter transformed student noticing into powerful ways of seeing. These tasks provided rich opportunities for students to learn by strategically varying features of the geometric figures being represented. By analyzing what varied and what was invariant, evidence was found of the development of the indirect objects of learning, as students were able to recognize patterns and discern the critical features of the object of learning, (i.e., the aspects and structure of generalized linear relationships). This process utilized variation as a means of building on concepts of pattern and generality, which are typically developed as a path to algebraic reasoning in Western English-speaking countries (APPA Group, 2004).

Moreover, the teacher incorporated questions during the lessons that elicited various strategies for counting the fence panels in order to support students' "professional seeing" (Marton et al., 2004, p. 11). Focusing on different ways of counting provided students the tools by which they could count the fence panels without actually counting them one-by-one. The use of questioning in this way is an example of one of the pedagogical tools suggested as a means for extending knowledge of "numerical concepts to algebraic reasoning" (Hunter, 2014, p. 280). The incorporation of variation in the planning of the lesson tasks allowed for specific areas in which the instructor could press students to make public their thinking about the direct objects of learning (i.e., generalizations of linear patterns), which engaged students at a high level of cognitive function (Hunter, 2014; Kazemi, 1998).

Constructing these generalizations led students to be able to begin to transfer their understanding in order to build linear functions to represent the various geometrical figures. As a result, the series of tasks presented in this chapter collectively align with the vision and aim to support the development of algebraic reasoning in sixth grade students. From a theoretical perspective, careful analysis of the intended, enacted, and lived objects of learning found in this task sequence provides a clear picture of teaching through variation in the U.S. Further, this chapter provides an example that can potentially move U.S. algebra instruction away from a state in which schools do "not adequately prepare students to successfully navigate the significant transition from the concrete, arithmetic reasoning of elementary school to the increasingly complex, abstract algebraic reasoning required for middle school and beyond" (Blanton et al., 2015, p. 76).

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