LIPING DING, KEITH JONES AND SVEIN ARNE SIKKO

12. AN EXPERT TEACHER'S USE OF TEACHING WITH VARIATION TO SUPPORT A JUNIOR MATHEMATICS TEACHER'S PROFESSIONAL LEARNING

INTRODUCTION

Our study concerns an important issue raised by recent studies of teacher professional development (TPD); that of the process of teacher change. As long ago as 2002, Clarke and Hollingsworth pointed out that the key shift in TPD is "from programs that change teachers, to teachers as active learners shaping their professional growth through reflective participation in professional development programs and in practice" (2002, p. 948). More recently, Goldsmith, Doerr and Lewis (2014) have highlighted that in many existing TPD studies, teachers' learning has typically been treated as an indicator of the effectiveness of the TPD programme rather than the primary object of inquiry. Their research synthesis shows that, to date, few studies have focused on the processes or mechanisms of teachers' learning. Similarly, the latest report from The New Teacher Project (2015) suggests that despite considerable investment in TPD, the evidence base for what actually helps teachers improve remains very thin. Consequently, there is still much to learn about how teachers develop knowledge, beliefs, or instructional practices.

More particularly, the success of pupils from Shanghai, China, in the recent PISA (Programme for International Student Assessment) study has made it important to understand how teacher learning takes place in Shanghai. Our Lesson Design Study (LDS), which focuses on primary school mathematics teacher professional learning through school-based teaching research group activities on lesson design and action, is being conducted in Shanghai (see Ding et al., 2013, 2014, 2015). In this chapter, our research question focuses on how a Chinese expert teacher in Shanghai used the idea of teaching with variation (Gu, Huang, & Marton, 2004) to support a junior teacher (with three years of teaching experience) to develop certain ways of reflecting on her teaching.

LITERATURE BACKGROUND

Given our research question concerning the expert teacher's use of teaching with variation to support a junior teacher to improve her teaching, in this section we chiefly focus on two themes within the existing literature that are relevant to

R. Huang & Y. Li (Eds.), Teaching and Learning Mathematics through Variation, 241–266.

our study: one is teaching with variation; the other is teachers' learning through the social interaction processes within the professional community that leads to sustained learning, together with an understanding of the role of the mentor (or 'knowledgeable other').

Teaching with Variation

Teaching with variation (变式教学 *Bian Shi Jiao Xue* in Chinese) has long been widely practiced by mathematics teachers in China (e.g., Ding et al., 2015; Gu, Huang, & Marton, 2004; Huang, Mok, & Leung, 2006; Li, Peng, & Song, 2011; Sun, 2011). In Gu's early work (the 'Qingpu experiment study' led by Gu from 1977 to 1994 in collaboration with a number of teachers and researchers and focused on improving the effectiveness of teaching and learning of mathematics in the Qingpu district of Shanghai), Gu (1994) noted that the most effective mathematics teachers were those who were able intentionally to arrange what might best be called 'multiple layers of teaching and learning'. Accordingly, Gu, Huang and Marton (2004, p. 319) consider that mathematics teaching largely consists of two types of activities: teaching declarative knowledge (i.e., concepts) and teaching procedural knowledge (i.e., processes). They identify and illustrate two forms of teaching with variation adopted in the two types of mathematics teaching activity, namely *conceptual variation* and *procedural variation*. Within conceptual variation, there are two means of variation: (1) concept variation (i.e., varying connotation of a concept); (2) non-concept variation (i.e., giving counterexamples). Thus, conceptual variation emphasizes understanding concepts from multiple perspectives. In tandem, procedural variation highlights the formation of a hierarchical system of the learner's experiencing process in unfolding mathematics activities, which include steps and strategies for transferring/exploring. In the process of problem solving, for instance, there are three procedural variation approaches: (1) varying a problem; (2) multiple methods of solving a problem; (3) multiple applications of a method (for more details see Gu, Huang, & Marton, 2004, p. 324).

Gu (2014) further explains that it is the procedural variation that plays a key role as *Pudian* (铺垫); that is, in setting up a proper potential distance between previous and new knowledge in a student's learning. Akin to the notion of 'scaffolding', *Pudian* means to build up one or several layers so as to enable learners to complete tasks that they cannot complete independently. In this chapter we aim, in particular, to develop a deeper understanding of how the expert teacher's concrete ideas of teaching with variation were used in helping the junior teacher to develop a deep understanding of the teacher's role of setting up *Pudian* to engage all students in classroom learning.

Teachers' Individual Learning in the Professional Community

In the recent TPD studies there is a growing recognition of the dual nature (both individual and collaborative) of teachers' professional learning (e.g., Murray, Ma & Mazur, 2009; Obara, 2010; Neuberger, 2012; Goos, 2014). Moreover, existing studies have noted that features of the individual teacher's learning, and of the collaborative community within which they work, can be culturally-dependent. In the years since Berliner (2001) noted that lesson study (or coached performance) was limited to some Asian countries, these forms of deliberate practice are now much more widespread (e.g., Hart et al., 2011). As we are interested in practice in China, here we chiefly refer to relevant studies of the concept of teacher professional development, the deliberate practice of particular kinds of school-based TPD models, and the notion and role of an expert teacher in China (e.g., Huang & Bao, 2006; Han, 2013; Li, Chen, & Kulm, 2009; Wong, 2012; Zhang, Xu, & Sun, 2014).

Zhang et al. (2014) point out that, in Shanghai, teacher professional development is defined as a process of continuous learning throughout a teacher's career. Commonly, in Chinese schools, each subject teacher belongs to two groups; a subject-based *teaching research group* and a subgroup of this, the *lesson preparation group* – the latter comprising all the teachers in the school who teach mathematics at the same grade level (Li et al., 2009). The school-based teaching research group (TRG) is the main professional community for teachers, as well as being the basic unit at the different levels (i.e., province, county and school levels) of the teaching research network within the country (Li et al., 2009; Yang, 2009).

Peng (2007) shows how 'lesson explaining', originally a 'bottom up' invention by teachers for their lesson study in the school-based TRG, has become an effective form of TPD particularly for developing teachers' mathematics subject knowledge and the professional community's shared pedagogical content knowledge. Peng illustrates how the fundamental feature of 'lesson explaining' – knowing both 'what' and 'why' in mathematics lesson design – leads individual teachers to reflect and develop their own subject matter knowledge (in Peng's case study on the topic of probability). Moreover, Peng reveals how the individual teacher gains a deeper understanding of mathematics, and develops their pedagogical content knowledge, from studying the textbook and through conversations with a mathematics expert teacher in the 'lesson explaining' community. Other teachers who participated in this form of professional activity also commented that they learnt and reflected on their own mathematics knowledge and pedagogical content knowledge from hearing other teachers articulating their thinking and reflection during the 'lesson explaining' activity.

Through a study of three lessons on the Pythagoras theorem, Yang (2009) analyses how a teacher changed the teaching behaviour during collaborative TRG teamwork: the first lesson emphasized applying the theorem, the second justifying the proposition, and the third producing propositions. Yang quotes from an interview with the teacher that illustrates the teacher's learning in the TRG:

> After the study of teaching, especially the discussion, I think the way of teaching is clearer than that in the textbooks. I have known it well. Where a question should be given to students and where an emphasis is arranged, and

> the teaching details guided by master teacher in discussion, are more useful compared to my own lesson design. (Yang, 2009, p. 295; original translation)

Han (2013) notes that there are several shared forms of teacher mentoring in China, including observing and commenting on the mentees' lessons, inviting them to observe the teaching of the same lessons, and reviewing and revising lesson plan drafts through informal and formal discussions. Through the process of mentoring and deliberate practice of particular kinds, Han (2013) reports on how one teacher's skill in designing a good display on the classroom board was enhanced, while another teacher improved her skill in creating a clear sequence for the lesson that had a suitable structure to promote student learning, and approached instruction with variation.

Li, Huang and Yang (2011) highlight that 'expert teachers' in China are not just experienced teachers; they are part of the teaching culture in China and also play an important role in nurturing that culture. Moreover, Yang (2014) differentiates the multiple roles that an expert teacher plays in China: expert in teaching (i.e., organizing good teaching processes), in researching (i.e., conducting teaching research and publishing papers in professional and academic journals), in teacher education (i.e., mentoring non-expert teachers and facilitating non-expert teachers' professional development), in scholarship (i.e., having a profound knowledge base in mathematics and other areas), in examining (i.e., being able to pose examination problems), and in being an exemplary model for students and colleagues.

Huang, Gong and Han (in press) highlight the critical role played by 'knowledgeable others' (i.e., university professors, subject specialists, etc.) during the lesson study process. It is exactly the mechanisms of how these 'knowledgeable others' work with practicing teachers and develop the teachers' professional knowledge and skills through mentoring during lesson study that is the focus of this chapter.

THEORETICAL FRAMEWORK

There has been criticism that models of professional development oversimplify both teaching and teacher professional growth (e.g., Opfer & Pedder, 2011). We support Clarke and Hollingsworth's (2002) view that teachers' professional growth is more likely to proceed through a series of incremental changes than by a linear path from a single professional development experience via a change in practice to improvement of student outcomes.

We use Clarke and Hollingsworth's (2002) *Interconnected Model* as a tool for categorising the teacher change data we have accumulated in our study. Clarke and Hollingsworth's (2002) model conceptualises individual teacher change within four distinct domains: the personal domain (teacher knowledge, beliefs and attitudes), the domain of practice (professional experimentation), the domain of consequence (salient outcomes), and the external domain (sources of information, stimulus or support) (p. 950). The Interconnected Model particularly identifies the mediating processes of 'reflection' and 'enactment' as the mechanisms by which change in one domain leads to change in another. The term "change sequences" (p. 958) is employed when change in one domain leads to change in another, while the term "growth networks" (p. 958) is used to highlight the occurrence of change that is more lasting change, and thus signify professional growth.

While the Interconnected Model recognizes the multiple growth pathways among the domains, it does not suggest the specific ways of reflecting and enacting. Here we further refer to Gu and Wang's (2003) 'Action Education' Model (briefly named as the *Keli Model* in Huang & Bao, 2006), which enables us to examine the 'change sequences' and 'growth networks' (Clarke & Hollingsworth, 2002) of particular kinds through lesson study activities. The Keli Model emphasizes an exemplary lesson as a means of teacher's action (or enactment), and a whole process that includes three stages of teaching action and two main teacher's reflections between the three teaching stages. Huang and Bao (2006) illustrate the whole process of the Keli model as three stages.

For the first stage, called 'existing action' (or existing enactment), the individual teacher designs the lesson independently and delivers the lesson publicly to a class of students observed by all the Keli group members. After the lesson, the Keli members provide immediate feedback on the teacher's lesson in the first Keli meeting, with the aim to help the individual teacher to reflect and identify the gap between the existing experiences and the innovative design suggested by the curriculum and textbook.

During the second stage, called 'new design', the teacher revises the lesson design according to the Keli members' feedback and re-delivers (or re-enacts) the lesson in another class. The Keli members observe the teacher's second lesson enactment. After the second lesson, the Keli members' discussion with the teacher aims to help the teacher to develop a reflection on the gap between the new design and effective classroom practice (as suggested by the curriculum and textbook) and to improve the lesson design and enactment further. Through the third stage, called 'new action' (or new enactment), the teacher is helped to develop a deep understanding of how students learn in a new style and attain a high quality of learning that is consistent with the goals of the curriculum and textbook.

The Keli Model is also concerned with building up a collaboration that enables teachers and researchers to study theoretical ideas, design innovative learning situations, and reflect on the enactments of teaching and learning within the Keli community (Huang & Bao, 2006). As we have illustrated, we see a teacher's 'action' in the Keli Model as close to the term 'enaction' in the Interconnected Model of Clarke and Hollingsworth (2002, p. 951), in that the teacher's action represents the enactment of something that the teacher has experienced and learned in the Keli community.

In our lesson design study (LDS), we combine both the Interconnected Model (Clarke & Hollingsworth, 2002) and the Keli Model (Gu & Wang, 2003;

Huang & Bao, 2006) for examining teachers' potential change sequences and growth networks in our lesson design study activities. That is, during the lesson study process, we examine the mediating processes (teachers' enactments and reflections) that link the four domains: teacher's lesson design (personal domain), teacher's classroom action (domain of practice), the interactions in the TRG (external domain), and students' classroom learning (domain of consequence).

To illustrate the whole process of our LDS model we use junior teacher Jiyi's (all names are pseudonyms in this chapter) three main teaching cycles that we studied from September to December in 2013 (see L1, L2 & L3 in Figure 1). The first cycle (L1) includes Jiyi's initial stage of lesson design, lesson enactment and reflection. The second cycle (L2) represents the second stage of the re-designed (i.e., re-enacted) lesson of L1. The third cycle (L3) represents the re-redesigned (i.e., re-re-enacted) lesson of L1. Each stage (each cycle in Figure 1) includes a set of the school-based TRG activities, such as Jiyi's classroom teaching, lesson explaining (Peng, 2007) and our study members' observation and the mathematics TRG meetings. In our LDS model we use the term 'cycle' to address the nature of teaching as both comprehension and reasoning, and as transformation and reflection (Shulman, 1987). In Figure 1, T means teacher, LD1 means lesson design 1, action1 is teaching in lesson 1, reflection1 is teacher's reflection after lesson 1, TRG1 is school-based TRG meeting after lesson 1, and so on.

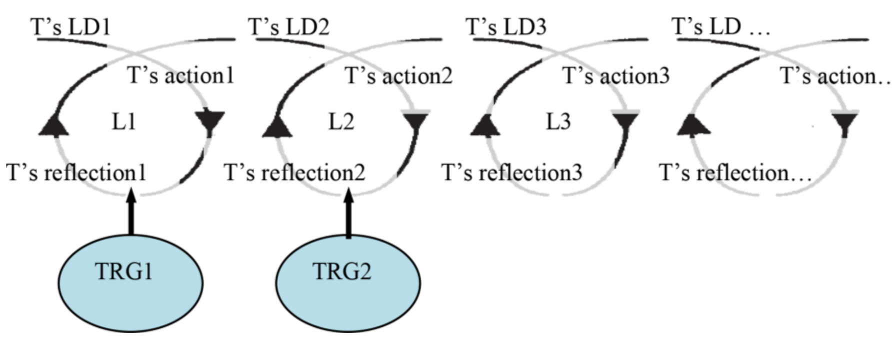

Figure 1. The three main cycles of the LDS model (including L1,L2 L3)

METHODOLOGY AND DATA

Our ongoing LDS study is being conducted through a school-based TRG in a local school located in the western suburb of Shanghai. The school is an international school (Grades 1-9, students age from 6 to 15 years old) funded by the China Welfare Institute with the key mission of launching innovative and laboratory educational classroom studies aimed at improving the quality of compulsory education for children in the country. The school consists of elementary (Grades 1-5) and lower secondary sections (Grades 6-9). Each section has two divisions; one is the domestic division, mainly for Chinese-speaking students; the other is the multi-culture division for both home and overseas students with English as the first language. Our study is conducted within the mathematics TRG in the elementary section. In total, in the elementary section there are 295 students in the domestic division, and 364 students in the multi-culture division. Each class has around 25 students. Students are mixed (both gender and academic attainment) in each class. There are seven mathematics teachers in the elementary section.

Teacher Jiyi had about three years elementary mathematics teaching experience at the time of our study. She was teaching mathematics to Grade 1 and 2 classes. All of her classes at the time of this study were in the domestic division. The class size ranged from 23 to 25 students. In general, students' learning attainment was above average for the school according to the school annual assessment.

Mei is an expert teacher invited by the school to support teachers in our study. The term 'expert teacher' in our study recognizes that Mei is not only an effective teacher of mathematics, but also that she plays the multiple roles that are described by Yang (2014, pp. 271–272). She has over 30 years teaching experience in elementary mathematics teaching in her school district. She has taken the leadership of the in-service elementary mathematics teachers TPD program at her school district level since 2009.

In reporting our findings, we present an analysis of the mediating processes (Jiyi's actions/enactments and reflections) that link Clarke and Hollingsworth's (2002) four domains through the LDS model (see teacher's lesson design, action, reflection and TRG meetings in Figure 1). To develop a deep understanding of individual teacher's learning and professional growth, our particular focus is to examine the interpretive acts and change phenomena that the teacher considered salient (Clarke & Hollingsworth, 2002). Thus, our analysis is primarily based on the following data sources: Jiyi's teaching diary; her reflection notes; her lesson plans (her own design and her redesigned versions); video transcripts of interactions in the TRG meetings; the mathematics textbook; and transcripts of the videoed lessons.

Our data analysis chiefly focused on the following two questions: How the expert teacher's main ideas of teaching with variation were used to create the conditions required to (i) stimulate change sequences, and (ii) foster the junior teacher to reflect on her teaching and changes (as learning) from certain perspectives towards the transformation into growth networks. In terms of the analysis of teaching with variation, we mostly focused on the type of *procedural variation*. We selected this focus because of our aim to understand more sufficiently why Mei emphasized the idea of 'not to lose the chain in learning mathematics' (Ding et al., 2015) in her guidance on lesson design that led Jiyi to make changes in her re-designed lesson and follow-up actions with her class.

Yang and Ricks (2012) argue that 'crucial teaching events' analysis (which is concerned with patterns of the interaction between the teachers and the students, and with the professional judgement of the teachers) is typical in TRG activities. We thus refer to two kinds of analysis of the 'critical incidents' in our analysis of the interpretive acts and change phenomena that both Mei and Jiyi considered salient in the TRG meetings, specifically: (1) analysis of the 'three points' of the lesson plans, namely the lesson's 'key point' (重点 *Zhong Dian*) (content focus), 'difficult point' (难点 *Nan Dian*) (learning focus), and 'critical point' (关键点 *Guan Jian Dian*) (teaching focus) as these 'three points' are used by Chinese teachers when thinking about lesson preparation, lesson enactment, observation and reflection (in a typical lesson plan, the difficult point and the critical point can overlap); and (2) the identification, understanding, and resolution of 'crucial events' of the lesson implementation.

FINDINGS

In this section we present the key findings from the initial data analysis of our study. In the first part of this section, we focus on the first research question, namely how Mei's main ideas of teaching with variation were used to stimulate Jiyi's change sequences. In the second, we turn to the second research question, namely how the junior teacher was fostered to reflect on her teaching and changes (as learning) from certain perspectives towards the transformation into growth networks.

Mei's Use of Teaching with Variation to Guide Jiyi to Redesign L2

1. Using problem variation without consideration of instructional coherence and knowledge connections in Jiyi's lesson 1. In Jiyi's initial lesson plan and action (L1 in Figure 1), she tried the idea of teaching with variation by varying problems (see Tasks 1-4 in Figure 2). The four tasks were relevant to two learning goals of the lesson: (1) to make sense and understand division with remainder in hands-on operations (e.g., drawing, sharing candies); (2) to explore the relationship that a remainder is smaller than a divisor. Noticeably, in Jiyi's lesson plan these two goals were treated both as the key point and the difficult point of the lesson (here the difficult point overlapped with the critical point).

After Jiyi's action in the first lesson, the teaching research group meeting (TRG1) took place (illustrated in Figure 1). Based on the classroom observation of lesson 1, Mei considered that Jiyi's teaching in lesson 1 was likely to lead the students into rote learning. Mei explained that Jiyi did not really understand the role of problem variation in developing lesson coherence through multiple layers of teaching (Mei's own word 'teaching stage') for students' understanding and learning of mathematics. Using Mei's own words in the interaction with Jiyi in the TRG1 meeting, the problem variation through the four tasks did not help to develop students' understanding of the concept of 'division with remainder'—a 'crucial teaching event' (Yang & Ricks, 2012) of the lesson:

Mei: Generally speaking, your lesson (L1) can be seen through several stages (i.e., the four tasks in Figure 2). But you did not really understand what should be done in each of the stages. Thus, the lesson lacks coherence.

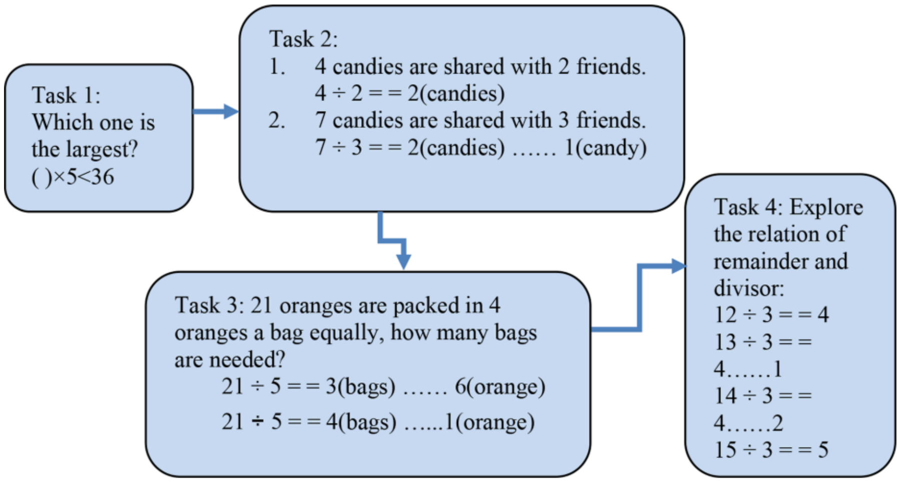

Figure 2. The main lesson structure of lesson 1 (L1)
[note that the six dots ….. indicates the remainder]

> Students did not really understand what the 'six dots' (symbol of 'remainder', see Figure 2) meant on the blackboard. They just imitated what you did. This is a real example of rote learning. [All translations of Jiyi and Mei in this chapter were made by the author team.]

2. Mei's emphasis on 'the chain in learning mathematics' through teaching with variation. To help Jiyi develop a deep understanding of teaching with variation through intentional and systematic practice, Mei highlighted the 'crucial teaching event' (Yang & Ricks, 2012) – developing a deep understanding of the concept of 'division with remainder' through the instructional coherence and mathematics knowledge connections in the lesson; in Mei's own words, "not to lose the chain in learning mathematics" (Ding et al., 2015). Mei pointed out that the chain could be developed according to the teaching framework of three layers of knowledge, which is commonly shared by teachers in China. The specific teaching terms of the three layers of knowledge are (see Figure 3): (1) previously learned knowledge (旧知 *Jiu Zhi*); (2) key points of new learning goal of the lesson (新知识点 *Xin Zhi Shi Dian*; 教学目标 *JiaoXue MuBiao*); (3) future learning according to textbook and curriculum (后续学习 *Houxu Xuexi*; 教材 *JiaoCai*; 教学大纲 *JiaoXue DaGang*).

This teaching framework (illustrated in Figure 3) provided guidance for Jiyi to develop understanding of the connections of mathematics knowledge of two kinds; namely both declarative knowledge (in this case, concepts such as 'division with remainder', 'sharing', 'division', etc.) and procedural knowledge (in this case, the process of division operation) (Gu et al., 2004). Moreover, it enabled Jiyi intentionally to practice and reflect on teaching with variation at two specific levels: the first level was the question of 'how' to teach, namely to simultaneously set up the multiple layers of mathematics knowledge in the re-designed lesson; the second level was the question of 'why' to teach in such a way, namely the theoretical elements of teaching with variation such as 'procedural variation', and *Pudian*, 'the proper potential distance' between students' previous knowledge and the intended learning goals of the lesson and future learning goals (Gu et al., 2004; Gu, 2014; Ding, Jones, Mei, & Sikko, 2015).

Noticeably, Mei strongly helped Jiyi to develop a deep understanding of the connections between two aspects of students' previous knowledge: (1) to analyse students' actual learning (in Mei's own words, 'what have the students learned?'), which is the anchoring part of knowledge (i.e., previous knowledge underpinning learning of the new knowledge and the exploration of new problems, Gu et al., 2004); (2) to analyse the content order prior to the lesson topic in the textbook (see Figure 3). That is, the object of Jiyi's learning is not to critique the problem/task design in the ways that might be done with a poorly-produced textbook, or with questions the teacher has chosen themselves, but to build the coherence of a lesson around given topics in the textbook. This means that the purpose of the 'crucial teaching event' (Yang & Ricks, 2012) highlighted by Mei here is for Jiyi to understand, and exploit the potential for using, the problems/tasks given in the well-designed textbook with her students.

3. Teaching with procedural variation for establishing the chain of learning goals in Mei's guidance of lesson redesign. We use Figure 4 to show the chain of three key learning goals suggested by Mei (for Mei's own design of the same lesson topic, see Ding et al., 2015) – a concrete example for improving the lesson design of the 'crucial teaching event' (Yang & Ricks, 2012) that Mei discussed with Jiyi in TRG1. Noticeably, Mei pointed out that while the first two learning goals are the key points, the second and the third learning goals are the difficult points of the lesson (here the difficult points and critical points overlapped) (see Figure 4). Mei deliberately structured the lesson into three stages, in which each stage had its own learning goal but each progressively developed students' deep understanding of the connections between the concept and the operation of 'division with remainder' through mathematics activities. We consider this process of Mei's intentional, systematic, structured and effortful practice as teaching with procedural variation (Gu et al., 2004).

Figure 4 can be read together with Figure 2 so as to see the changes of learning goals through the multiple teaching stages that Jiyi later adopted in lesson 2 (as illustrated in Figure 1). For the purpose of developing a deep understanding of Jiyi's change sequences as learning in the later sections, in this section, we chiefly focus on Mei's ideas of "not to lose the chain in learning mathematics" (Ding et al., 2015), namely teaching with procedural variation (Gu et al., 2004) through the three tasks for the first learning goal. In a later section of our analysis of Jiyi's learning, we further trace the intentional practice with procedural variation from the first three

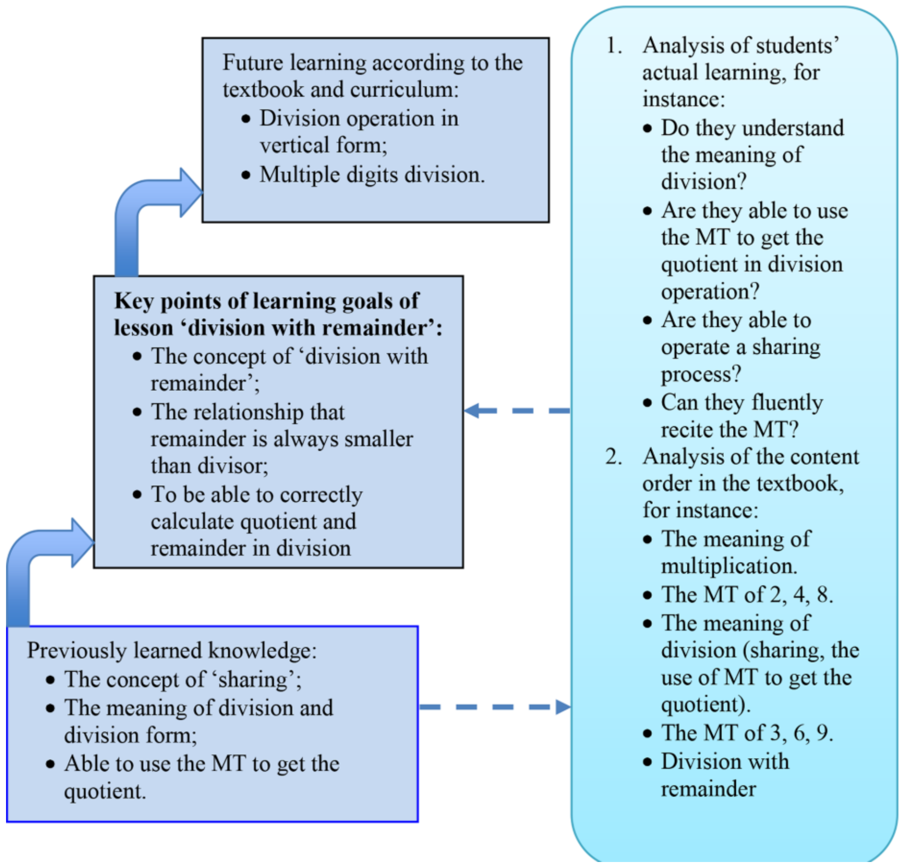

Figure 3. The teaching framework of three layers of knowledge in Mei's guidance to Jiyi of redesigning L2 [MT = multiplication table]

tasks to the fourth task in the second learning goal, together with an explanation of the use of procedural variation to establish the chain for the third learning goal (see Figure 4).

Here we use two quotes from the interactions between Mei and Jiyi in TRG1 to show how Mei explained to Jiyi about the 'crucial teaching event' (Yang & Ricks, 2012), that of developing students' deep understanding of the connections between the concept and the operation of 'division with remainder' by deliberately setting up the multiple layers of teaching with procedural variation. In the first quote (about teaching Task 1 in Figure 4), Mei emphasized that the core teaching stage was to identify the 'anchoring' part of knowledge (Gu et al., 2004); in this instance, students' existing knowledge of the connections between the concepts of division

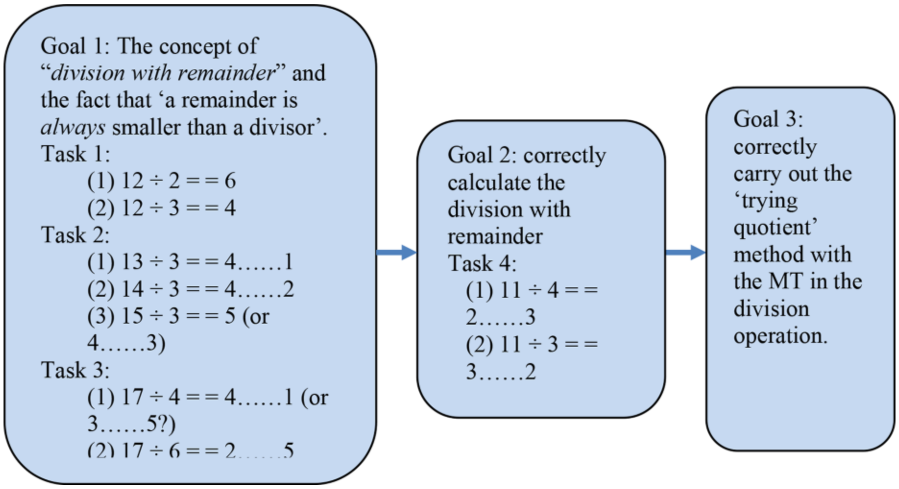

Figure 4. The key learning goals and tasks suggested by Mei

as sharing and as the reverse operation of multiplication. Using Gu et al.'s (2004) theoretical notion, the teaching focus was to set up Pudian (Gu et al., 2004; Gu, 2014) – students' previously learned knowledge of the meaning of numbers 12, 3, 4 in the division form and the use of the method of the multiplication table (MT) to try the division operation in Task 1 as the 'anchoring' part of knowledge for learning the new concept and operation of 'division with remainder' (see Figure 4).

Mei: The first learning goal is preliminarily to know what division with remainder is. The learning process can be divided into two stages. The first stage is of the concept of 'remainder', the other is of the fact that the remainder is smaller than the divisor. In Task 1, the problem is to share 12 peaches. Each monkey is to have 3 peaches. How many monkeys can there be? The purpose of this task is to lead students to review their previously learned knowledge. The teacher should ask students what the numbers 12, 3, 4 mean after they form the division for solving the problem. The second stage is to review how to use the method of the multiplication table to get the quotient [to see the relation of dividend, divisor and quotient in the multiplication operation]. Students should not recite each statement of the factor 3 [i.e., one three is three, two threes are six and so on; here Mei is suggesting that students should have learnt to see the relation of dividend, divisor and quotient in the multiplication table].

Noticeably, in the core of this lesson, Mei did not merely suggest a focus on the repeated subtraction or equal-sharing models (Gu & Wong, 2003) for making sense of the concepts of division and quotient, for this was the students' existing knowledge.

Rather, Mei's intention was focussed on using students' existing knowledge of a specific kind – the concept of division as sharing and as the reverse operation of multiplication – as the anchoring part of knowledge (*Pudian*) for developing a deep understanding of the connections of the new concept and the new operation of 'division with remainder' – this being the 'crucial teaching event' (Yang & Ricks, 2012) of the lesson. For Mei, the proper potential distance between previous and new knowledge in this lesson was for students to see the same kind of relationship between factors and products in the multiplication table and the dividend, divisor and quotient when varying in Task 2 (see Figure 4) and later in Task 4 at the second and the third stages (see the Goals 2 and 3 in Figure 4) of learning of the operation of division with remainder. The implicit *Pudian* (Gu, 2014) becomes evident when Mei addressed the use of students' such previous knowledge in her explanation of Task 4 for the second learning goal (Figure 4) (see the italics we highlight in the quote below).

> Mei: [referring to Task 4 in Figure 4] The second stage is to build up the connection of students' operation of sharing activity to mental calculation activity. Here, 11 ÷ 4, while the class keeps drawing pictures to understand the quotient, some students would be able to use the multiplication table to try the quotient. Then, you [Jiyi] should ask the students how they did so. That is, how they think about the statements of 4 in the multiplication table. Two four is 8, but there is not 8. What to do then? To find a number that is smaller than 11, but closest to 11. In fact, the thinking method is the same as in the task of 'which one is the largest' [see Task1 in Figure 2], but *we should use students' previous knowledge*. Next, 11 ÷ 3 = 3……2. What does each number mean? How is the quotient obtained? Students should be trained to think so in the calculation procedure.

Identifying the Complexity of Jiyi's Learning Through the Change Sequences from Lesson 1 to Lesson 2

In this section, we show the complexity of Jiyi's learning through an analysis of three types of change sequences from lesson 1 to lesson 2 (as illustrated in Figure 1). Our data analysis of change sequences is based on Jiyi's teaching diary, the interactions of Jiyi with Mei in the teaching research group meeting after the second lesson (TRG2), and Jiyi's reflection notes throughout our study. The three types of change sequences we identified are: (1) changes within the teacher's personal domain; (2) changes from the personal domain to the practice domain; (3) a mixed picture of change sequences across the personal domain and the domain of practice.

1. Changes within the personal domain: Understanding the teaching terms of three layers of knowledge for teaching with variation. We found that Mei's guidance "not to lose the chain in learning mathematics" (Ding et al., 2015) in TRG1 first led

Jiyi to reflect on her analysis of the textbook and action in lesson 1, and consequently to make changes in the learning goals and the lesson structure in the lesson 2 plan. Here, we consider the teacher's lesson plan as an explicit realization of the teacher's personal domain (e.g., evidences of the teacher's implicit knowledge and beliefs of teaching and learning mathematics). We illustrate the change sequence within Jiyi's knowledge domain in Figure 5, where E = external domain; K = teacher's personal (knowledge) domain; P = practice domain; S = salient outcome; P-L1 = practice in L1; K-L2 = teacher's L2 lesson plan.

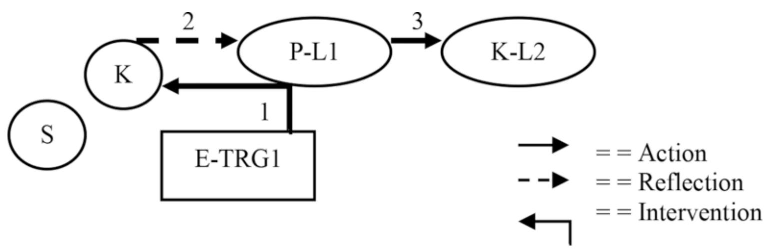

Figure 5. A change sequence made by Jiyi through L1 cycle (see Figure 1)

The first two steps in this change sequence (marked 1 and 2 in Figure 5) is of Jiyi's learning through her reflection on Mei's guidance of using the teaching framework with specific teaching terms of three layers of knowledge for analysing the textbook in the first teaching research group meeting (see TRG1 in Figure 1): (1) previously learned knowledge (旧知 Jiu Zhi); (2) key points of new learning goal of the lesson (新知识点 Xin Zhi Shi Dian; 教学目标 JiaoXue MuBiao); (3) future learning according to textbook and curriculum (后续学习 Houxu Xuexi; 教材 JiaoCai; 教学大纲 JiaoXue DaGang) (see Figure 3). In doing so, Jiyi focused on the 'crucial teaching event' (Yang & Ricks, 2012) of the lesson – developing students' deep understanding of the connections between the new concept and operation of 'division with remainder'. In her teaching diary, Jiyi wrote as follows:

> After an analysis of the textbook, *students have learned the following knowledge* before the lesson 'division with remainder': (1) multiplication of digits from 1 to 9; (2) the concept of 'sharing'; (3) division calculation. *The key points of knowledge of this lesson* (L2) are: (1) the concept of 'remainder'; (2) the meaning of each number in the form of division with remainder; (3) the relationship that remainder should be smaller than divisor; and (4) the calculation process of division with remainder. *Based on the previous knowledge*, students are *to learn the new knowledge*. *To build up the chain of these knowledge points*, I made considerably large changes in the lesson plan. [Italics used to highlight key phrases]

The phrases such as "students' learned knowledge", "the key points of knowledge of the lesson", "based on previous knowledge, to learn new knowledge", and "to build up the chain of these knowledge points" (highlighted in italic), illustrate that Mei's guidance on the use of the teaching framework of three layers of knowledge led Jiyi to develop a specific form of reflection on Mei's idea of teaching with variation, namely "not to lose the chain in learning mathematics".

Moreover, we noted that in her lesson 2 plan Jiyi adopted the three learning goals and the main teaching stages and tasks similar to Mei's guidance in TRG1 (see Figure 4). The third step in this change sequence (marked 3 in Figure 5) shows that changes in Jiyi's understanding of the teaching framework of three layers of knowledge (see Figure 3) for building up the coherence of knowledge chain in students' learning led her to change the learning goals of the lesson plan.

2. Changes from the personal domain to the practice domain: Learning precise teaching language and questioning strategy in teaching with procedural variation. To understand Jiyi's learning as an outcome from the first teaching cycle (illustrated as L1 in Figure 1), Mei suggested a 'lesson explanation' (Peng, 2007) activity (TRG2 in Figure 1) before Jiyi went on to teach the second lesson (L2). While there are considerable positive changes that took place in Jiyi's learning, here we focus on Jiyi's learning to use more precise teaching language and to focus on the learning goal underlying Mei's idea of teaching with procedural variation, namely "not to lose the chain in learning mathematics" (Ding et al., 2015).

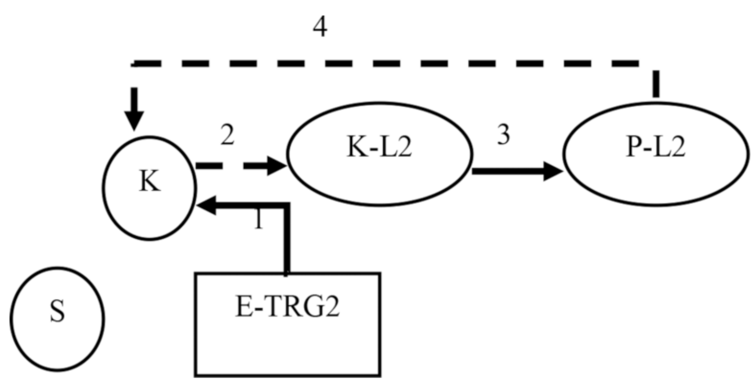

Figure 6. A change sequence made by Jiyi during L2 circle (see Figure 1)

The first two steps in this change sequence (marked as 1 and 2 in Figure 6) are an indication that Jiyi seemed to develop a specific way of reflection, or understanding, of teaching with procedural variation under Mei's support in the TRG2 meeting. That is, Jiyi learned to be more intentional and effortful in using precise teaching language when she explained how she was to teach Task 1 (see Task 1(1) 12 ÷ 2 = 6 in Figure 4). Here, we provide the key interactions between Jiyi and Mei of teaching Task 1 as follows (with italics used to highlight key parts):

Jiyi: Winter is coming; all animals are preparing food for the winter. Let's visit the rabbit family and take a look of what they are *doing*.

Mei: This is a nice start of the lesson. You may consider changing your word here. That is, not to ask students what the rabbit family is doing, but ask them how the rabbits are to *share* the carrots.

Jiyi: Rabbit mother brings 12 carrots to her two baby rabbits to share. I will ask students how many carrots each baby rabbit would have. Then I will ask who can give a mathematical formula and calculate it (12 ÷ 2 = 6). After students give answers, I will invite them to explain the meaning of the numbers 12, 2, and 6. For instance, 12 *represents* 12 carrots, and 2 *represents* 2 baby rabbits.

Mei: It's better not to say '12 *represents*', but to say '12 *means*.' [Note: In Chinese, the word 'represent' (代表 *Dai Biao*) does not request an explicit explanation or reasoning – for instance, a picture can 'represent' a meaning, however vague the meaning may be – while the word 'mean' (表示 *Biao Shi*) clearly requests an explicit explanation or reasoning.]

Mei's emphasis on the use of the precise teaching language such as share and means developed Jiyi's understanding of the important role of teacher's precise teaching language skills to enable students to focus on the key point of learning in the task, in this instance, the 'crucial teaching event' (Yang & Ricks, 2012) of the lesson—developing students' deep understanding of the connections between the new concept and operation of 'division with remainder'.

Next, Jiyi explained how she would teach when varying Task 1(2) (see Task 1(2), 12 ÷ 3 = 4 in Figure 4). Mei's explanation intends to lead Jiyi not only to address the calculation procedure in teaching, but also deliberately to use the questioning strategy to encourage students to explain their calculation method, which was one of the key learning goals of the lesson. That is, not only to enable students to know 'how' to do so, but also to understand 'why' to do so in the division operation. A shift of the calculation method from students' previous knowledge to the new knowledge of the lesson is pinpointed by the teacher's questions of 'how' and 'why' through the teaching with procedural variation (from Task 1(1) to Task 1(2) in Figure 4) – a concrete teaching strategy for dealing with 'the proper potential distance' (Pudian) (Gu et al., 2004; Gu, 2014). Jiyi's clear statement of the term 'Pudian' in the following extract is evidence that she became aware of students' previous knowledge and learning experience as the anchoring part of knowledge in this lesson (with italics used to highlight key parts):

Jiyi: (Task 1: 12 ÷ 3 = 4) Here, I will ask them *how* they get the quotient 4.

Mei: If you ask students "how they get the quotient 4", *how* would students respond in your class?

Jiyi: I will use '*Pudian*' by asking them a question about which statement of the multiplication table they will use [such as, for instance, one three is three, two threes are six, and so on in the multiplication table]. I expect them to respond by 'three fours are twelve'.

Mei: Did you emphasize this method in your previous lessons? If students do not know *how* the quotient 4 comes (that is, why 12 ÷ 3 would get 4), it would be very difficult for you to teach today's topic. The new knowledge in today's lesson ought to be connected to students' previous knowledge.

Mei: 12 ÷ 3 = 4. *How* the quotient 4 comes? Students should understand the [reasoning] method to get 4 here. That is, they need to understand the relationship between the divisor and quotient in the multiplication table (MT). If the divisor were 3, then they would think of the statement with 3 in the MT. *Why* to think of the statement with 3 in the MT? This is because it is students' previous knowledge. And, *why* would students think about the statement 'three four are twelve'? This is because of the relationship between divisor and dividend. Here the dividend is 12, so the statement 'three four are twelve' is considered.

We further identify that Jiyi adopted Mei's guidance of using precise language and the term 'Pudian' as discussed above and intentionally practiced in lesson 2 (see the third step in Figure 6). Jiyi's reflection on her effortful practice with the precise teaching language and proper questioning strategy in lesson 2 was also evident in her reflection note after lesson 2 as follows:

> In lesson 2, I used more precise language, which was more vivid and more suitable for lower grade students [Grade 2 in her class]. Teaching should focus on students' thinking development, so the teacher should play the guiding role in students' learning. In teaching the calculation procedure of division with remainder, I encouraged students to explain their calculation process by questions such as "To think about which set of the statement in the MT (by looking at the divisor)?", "which statement is exactly related to the division?", "why it?", "how to get the remainder?", etc.

Significantly, we found that Jiyi particularly showed her willingness towards improving her teaching language skills in her reflection note after TRG2 (see the forth step in Figure 6). This can be considered as the teacher's commitment to the sustained learning which is a kind of teacher's potential 'growth network' (Clarke & Hollingsworth, 2002).

> The teacher must be aware of using precise teaching language. Particularly to an experienced mathematics teacher, every word should be as precise as possible. Though I know that I am unable to be so precise in every word I say in my teaching, I am now improving my language towards this goal.

3. A mixed picture of change sequences across the personal domain and the domain of practice: The art of teaching with variation. Our data analysis shows a mixed picture of Jiyi's learning in her reflection and action on tackling the relationship between her leading role as teacher and the students' active learning through teaching with variation from lesson 1 (L1) to lesson 2 (L2), as captured in Figure 7.

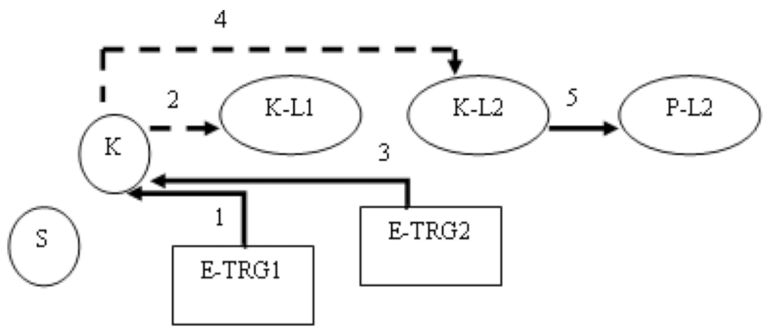

Figure 7. A mixed picture of change sequence made by Jiyi through L1 & L2 circles (see Figure 1)

Here, on the one hand, for the first two steps of the change sequence (labelled 1 and 2 in Figure 7) Mei's guidance of teaching with procedural variation in TRG1 led Jiyi to reflect carefully on the relationship of the three key elements in her initial lesson design (L1): textbook, teaching and learning. Jiyi wrote what she learned of 'the proper potential distance' (Pudian) (Gu et al., 2004; Gu, 2014) in her teaching diary as follows:

> Previously I planned lessons according to my understanding of the textbook content. I rarely thought about that I should deliberately connect what students already learned to what I was to teach in my lesson plan. Now, I think that it is very necessary to do so. Mei's guidance helped me to understand more that teaching should be based on students' existing learning, in order to help them to learn independently. That is, to teach students how to fish rather than giving them fish [an ancient Chinese saying]. So I should give students opportunities to explain what they see, do and think in the learning process.

Nevertheless, on the other hand, our data analysis of the lesson explanation meeting (TRG2 in Figure 1) shows that Jiyi had uncertainties in handling students' learning responses for independent learning, particularly when students' learning responses were not prepared in her lesson plan. Noticeably, it was Mei who helped Jiyi specifically to update her knowledge of *Pudian* by addressing the relationship of teaching and learning of two specific kinds (labelled 3 and 4 in Figure 7): one is to analyse students' potential learning problems and alternative ways of reasoning, which is related to Simon's (1995) notion of 'Hypothetical Learning Trajectory' (HLT); in Mei's own words 'knowing what students are likely to understand and respond' to the teacher's intentional teaching; the second is to improve teacher's teaching language and questioning skills (i.e., the teaching notion of '*follow their response by questioning*' (追问*Zhui Wen*), see the quote of Mei's exchange below) to enable students to focus on the learning goal of the mathematical reasoning development.

It was difficult for Jiyi to develop students' understanding through their independent learning of the division relationship that a remainder is always smaller than the divisor. When Jiyi added one more carrot into the picture for Task 2(1) in Figure 4, it was natural for students to see that the additional carrot in the picture could not be grouped and should remain as a single carrot. This is evident in the following exchange (with italics and bold text used to highlight key parts).

Mei: First, showing the problem [Task 2(1)] before providing a picture. Secondly, encouraging students to *guess* the result after they formed the formula (13 ÷ 3 = ?). After guessing, you can encourage them to *prove* their guess by grouping the carrots in the picture. You should *follow their response by questioning*; for instance, *why* the single carrot that is left cannot be grouped? This question is to encourage them to *prove* their claim. Thus, some students in the class would *explain* to others in the class that it is because each group has three carrots. The one remaining carrot is not enough to be counted as a group.

Jiyi: What shall I do then if some students do not give a clear explanation?

Mei: That is not a problem. You can invite other students to continue until they give a clear explanation. You have to adjust your questions to a *deeper level of teaching*. It is better to invite students to *guess* the result, rather than to *tell* them the result. If you asked them to *tell* the whole class their result, they may worry about giving a wrong answer. But if you encourage a *guess*, they would not worry about a wrong answer, as it is a guess anyway. *Teaching is an art*. Teacher's language plays a very important role in engaging students into deep learning interactions.

In analysing the interactions above, teaching with variation entails not only the precise teaching language for developing students' mathematical reasoning and understanding, but also an art of teaching language for engaging students into active and independent learning processes. On the one hand, as shown above, Mei's precise teaching language, like the words '*guess*', '*prove*', '*why*', and '*explain*' illustrate the important role of teacher's precise language in the development of students' mathematical reasoning and understanding. On the other hand, the teacher's questioning such as '*follow their response by questioning*', plays a significant role to enable students to play an active role in their own individual learning and the whole-class-shared mathematical reasoning. Mei's use of the two different kinds of teaching language shows a sophisticated level of teaching with variation.

The mixed picture of Jiyi's learning is further identified in her learning and reflection on the importance to '*follow their response by questioning*'. On the one hand, Jiyi's '*Oh, Yes!*' as learning is evident in the following interaction with Mei regarding *Task 2(3)* (with italics used to highlight key parts of Mei's response):

Jiyi: If all students can group 15 carrots into 5 groups and no one says 4 groups and 3 carrots left, what I should do?

Mei: Then you can *ask them why not*. Students would tell you that because the remaining 3 carrots can still be grouped. Then you should *follow up their response by questioning* "*why* in the last couple of examples, the remaining carrots were not grouped, but now the remaining ones can be grouped".

Jiyi: *Oh, yes*! This question is very important!

On the other hand, nevertheless, Jiyi confessed her difficulty in such questioning if students' learning responses were out of her lesson plan; that is, the questioning of 'follow their response by questioning' is used in the dynamic teaching process and requests a teacher's impromptu action in the authentic class. Jiyi wrote in her reflection note after TRG2 as follows:

> The lesson plan is only the teacher's hypothesis of students' learning. But I am not sure of what to do if some learning situation out of my lesson plan happens in the class.

The fifth step in Figure 7 thus represents a mixed picture of Jiyi's changes and difficulties as we have illustrated above. She understands some specific ways of teaching with variation (e.g., learning of 'the proper potential distance', Pudian, Gu et al., 2004; Gu, 2014), yet she has difficulty making changes in action (e.g., teacher's language of two levels). As conveyed in Jiyi's teaching diary after L2, though she developed a considerable understanding of the teacher's leading role in the development of students' independent learning, it was difficult for her to do so in action.

> During the process of redesigning the lesson, I found that the amount of content of this lesson is considerably large. After the lesson explanation meeting, I understand that I should guide students to explore by themselves the relationship that the remainder is smaller than the divisor. But I still find it difficult to do so to enable students to make correct conclusion from their own exploration.

DISCUSSION

We have identified three elements for our discussion of the expert teacher's use of the idea of teaching with variation to support a junior teacher's professional learning.

The first part examines the expert teacher's teaching notions that helped the junior teacher to learn the theoretical terms of teaching with variation. The second part of the discussion clarifies the special role that teaching language plays in setting up Pudian in the dynamic process of teaching with procedural variation. The final part highlights the complexity of teacher's professional learning.

The Expert Teacher's Use of Common Teaching Notions to Support the Junior Teacher's Learning of Teaching with Variation

As our data analysis in the foregoing section showed, the expert teacher (Mei) used teaching notions that teachers commonly share and understand in China to create the learning conditions for the junior teacher (Jiyi) to reflect and practice on the specific ways of teaching with variation. In Table 1 we highlight the key theoretical terms of teaching with variation (Gu et al., 2004) that the expert teacher Mei guided the junior teacher Jiyi to learn and to understand in lesson design and action. In Table 2 we summarize the uniqueness of the expert teacher's teaching notions that helped the junior teacher to develop an understanding of the relevant theoretical terms of teaching with variation (as shown in Table 1).

Table 1. The theoretical terms of teaching with variation

Theoretical terms (Gu et al., 2004)
Variation
Teaching with variation
Procedural variation
The proper potential distance and the anchoring part of knowledge
Pudian (akin to scaffolding)

The Complexity of Teaching with Procedural Variation: Building Up the Chain of Knowledge and Setting Up 'Pudian' in the Process of Dynamic Teaching

The expert teacher's teaching notions, summarised in Table 2, created learning opportunities for the junior teacher to understand the complexity of the theoretical notions of teaching with variation, in particular teaching with procedural variation (see Table 1). As pointed out by Gu et al. (2004), procedural variation plays a key role as Pudian in setting up a proper potential distance between previous and new knowledge in a student's learning. Akin to the notion of 'scaffolding', Pudian means to build up one or several layers so as to enable learners to complete tasks that they cannot complete independently. Our analysis in the foregoing sections shows that the complexity of teaching with Pudian requires a teacher intentionally and

Table 2. The expert teacher's teaching notions of teaching with variation

The expert teacher's teaching notions
Coherence
Not to lose the chain in learning
Multiple teaching layers/stages
Students' existing knowledge, the order of textbook content, students' potential learning difficulties/problems and alternative ways of reasoning
The teaching framework of previous knowledge, key points of learning goal, future learning

consciously to practice the following two parts as a whole: (1) building up the chain of knowledge embedded in mathematics textbooks and curriculum; (2) developing the dynamic teaching process with an emphasis on the relationship of the teacher's language and students' understanding and active learning.

To build up the chain of knowledge embedded in the mathematics textbook and curriculum, the notion of 'an anchoring part of knowledge" (Gu et al., 2004) was specifically emphasized through the teacher's analysis of student's existing knowledge and the order of learning content embedded in the textbooks and curriculum. Moreover, it was necessary for the teacher to develop a more sufficient understanding of students' potential learning difficulties/problems and alternative ways of reasoning, which makes resonance with an understanding of the notion of 'proper potential distance' (Gu et al., 2004).

To develop the dynamic teaching process with Pudian, expert Mei highlighted the significance of the teaching framework and teaching language/notions that teachers commonly understand in China as the key elements of effective classroom teaching and learning mathematics. The teaching framework is useful to guide the junior teacher to conduct the analysis of the textbook, to focus on the learning goals of the lesson, and to develop an understanding of students' existing knowledge and potential learning (see Figure 3). We wish to point out that a teacher's teaching language plays a special role in Pudian, apart from setting up the multiple layers of teaching and the well-designed tasks. Our data analysis identifies two levels of teacher language: (1) the preciseness of teacher language, which plays an important role in students' understanding and reasoning in mathematics; (2) the art of teacher language (i.e., 追问 Zhui Wei questioning strategy – 'follow their response by questioning'), which leads students not only to develop active individual learning but also to develop a kind of shared-learning with one another in the whole class. These findings lead us to suggest that the term Pudian in teaching with procedural variation is more specific than the theoretical term 'scaffolding', because it tells new teachers more about how to achieve scaffolding in the authentic classroom.

The Complexity of Teacher's Professional Learning through Intentional, Systematic and Effortful Practice

In the foregoing data analysis section, we showed three types of teacher change sequences: (1) changes within the teacher's personal domain; (2) changes from the personal domain to the practice domain; (3) a mixed picture of change sequences across the personal domain and the domain of practice. We wish to point out that the expert teacher Mei played an important role in guiding the junior teacher to develop reflections in specific ways (i.e., labelled 1&2 in Figure 5; 1&2 in Figure 6; and 1&2, 3&4 in Figure 7). Consequently, the junior teacher Jiyi made change sequences according to her reflections (i.e., labelled 3 in Figure 5; 3 in Figure 6; and 5 in Figure 7). We consider these change sequences as intentional, systematic and effortful practice through professional learning.

Our data analysis also showed a complex picture of Jiyi's professional learning. On the one hand, we identified Jiyi's change sequences as learning and professional growth; on the other hand, we recognized Jiyi's difficulties in the professional learning process. Gu (2014) identifies three stages of teacher's professional learning through various kinds of TPD program: (1) listening [to 'knowledgeable others'] but not understanding; (2) listening and understanding, but not knowing immediately how to act; (3) listening, understanding, and acting. Gu notes that the transition from understanding to action takes a considerable amount of time. Our findings of three types of change sequences support Gu's observation. Our data analysis also leads us to suggest that while teachers' professional growth is more likely to proceed through a series of incremental changes (Clarke & Hollingsworth, 2002), such growth is not straightforward and continuous; rather it is discrete and discontinuous.

CONCLUSION

In this chapter, we sought to address the question of how an expert teacher used the idea of teaching with variation to support a junior teacher to develop certain ways of reflecting on her teaching, and as a result to make 'change sequences' (Clarke & Hollingsworth, 2002) as learning from her teaching enactment and reflection. We identified the expert teacher's significant guidance in the following two sophisticated ways: (1) the use of teaching notions that teachers commonly share and understand in China to understand the theoretical terms of teaching with variation; (2) the use of teaching frameworks and language that teachers commonly understand and practice in the country to understand an emphasis on the fundamental 'chains' in learning mathematics and the dynamic process of Pudian. Our study reveals how the detail of didactics and mathematics pedagogy can be zoomed in on when there is an understood structure within which to do this; in this case the teaching framework (see Figure 3), the lesson structure (see Figure 4) and common understandings of teaching notions and language about variation (see Table 2). Our study makes explicit the possible high-quality expert input in teacher education. It contrasts with other studies where the 'expert' teacher is a mentor or coach who focuses primarily on classroom behaviour and management.

Moreover, we wish to point out that what is also important is that there is a commonly understood structure – the school-based teaching research group (TRG) – in which teachers learn from 'knowledgeable others' for their professional development and network in China (e.g., Huang & Bao, 2006; Huang et al., in press; Peng, 2007; Yang, 2009; Li et al., 2011; Han, 2013).

In focusing on the junior teacher's professional learning through our lesson design study, we found that the teacher modified her lesson plans more than ten times from lesson 1 to lesson 2, according to data from Jiyi's teaching diary and reflection notes. Apart from the redesign of the lesson structure and the improvement of teaching language, there were other considerable changes that were related to our lesson design study, such as the design of number in the tasks (i.e., all numbers in the tasks in Figure 4 were deliberately designed), the amount of tasks, classroom interactions, and so on.

While understanding the 'black box' of teacher's professional learning is in its early stages, the contribution of our study is of the expert teacher's teaching notions (see Table 2) that expands knowledge of using the Chinese practitioner's ideas of teaching with variation to guide mathematics teacher preparation and teacher professional development.

REFERENCES

Berliner, D. C. (2001). Learning about and learning from expert teachers. *International Journal of Educational Research, 35*, 463–482.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, *18*(8), 947–967.

Ding, L., Jones, K., Mei, L., & Sikko, S. A. (2015). "Not to lose the chain in learning mathematics": Expert teaching with variation in Shanghai. In *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education, Vol 2* (pp. 209–216). Hobart, Australia: PME.

Ding, L., Jones, K., & Pepin, B. (2013, July). Task design through a school-based professional development programme. In *Proceedings of the ICMI study 22: Task design in mathematics education*. Oxford: University of Oxford.

Ding, L., Jones, K., Pepin, B., & Sikko, S. A. (2014). An expert teacher's local instruction theory underlying a lesson design study through school-based professional development. In *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education*, *Vol*. 2 (pp. 401–408). Vancouver, Canada: PME.

Goldsmith, L. T., Doerr, H. M., & Lewis, C. C. (2014). Mathematics teachers' learning: A conceptual framework and synthesis of research. *Journal of Mathematics Teacher Education*, *17*(1), 5–36.

Goos, M. (2014). Researcher-teacher relationships and models for teaching development in mathematics education. *ZDM: International Journal in Mathematics Education, 46*, 189–200.

Gu, L. (2014). *A statement of pedagogy reform: Regional experiment and research record*. Shanghai: Shanghai Education Press. [顾泠沅著. 口述教改一地区实验或研究纪事. 上海教育出版社]

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: An effective way of mathematics teaching in China. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–348). Singapore: World Scientific.

Gu, L., & Wang, J. (2003). Teachers' development through education action: The use of 'Keli' as a means in the research of teacher education model. *Curriculum, Textbook & Pedagogy*, *I*, 9–15; *II*, 14–19. [教师在教育行动中成长—以课例为载体的教师教育模式研究. *课程－教材－教法*, 2003, *第一期*, 9–15；*第二期*, 14–19.]

Han, X. (2013). Improving classroom instruction with apprenticeship practices and public lesson development as contexts. In Y. Li & R. Huang (Eds.), *How Chinese teach mathematics and improve teaching* (pp. 171–185). London: Routledge.

Hart, L. C., Alston, A., & Murata, A. (Eds.). (2011). *Lesson study research and practice in mathematics education: Learning together*. Berlin: Springer.

Huang, R., & Bao, J. (2006). Towards a model for teacher's professional development in China: Introducing Keli. *Journal of Mathematics Teacher Education*, *9*, 279–298.

Huang, R., Mok, I., & Leung, F. (2006). Repetition or variation: Practising in the mathematics classroom in China. In D. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries* (pp. 263–274). Rotterdam: Sense Publishers.

Huang, R., Su, H., & Xu, S. (2014). Developing teachers' and teaching researchers' professional competence in mathematics through Chinese Lesson Study. *ZDM: The International Journal on Mathematics Education*, *46*, 239–251.

Huang, R., Gong, Z., & Han, X. (2016). Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study. *ZDM: The International Journal on Mathematics Education 48*, 425–439.

Kaiser, G., & Li, Y. (2011). Reflections and future prospects. In Y. Li & G. Kaiser (Eds.), *Expertise in Mathematics instruction* (pp. 343–353). London: Springer.

Li, Y., Chen, X., & Kulm, G. (2009). Mathematics teachers' practices and thinking in lesson plan development: A case of teaching fraction division. *ZDM Mathematics education, 41*, 717–731.

Li, Y., Huang, R., & Yang, Y. (2011). Characterizing expert teaching in school mathematics in China: A prototype of expertise in teaching mathematics. In Y. Li & G. Kaiser (Eds.), *Expertise in mathematics instruction* (pp. 167–195). London: Springer.

Li, J., Peng, A., & Song, N. (2011) Teaching algebraic equations with variation in Chinese classroom. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 529–556). New York, NY: Springer.

Murray, S., Ma, X., & Mazur, J. (2009). Effects of peer coaching on teachers' collaborative interactions and students' mathematics achievement. *Journal of Educational Research*, *102*, 203–212.

Neuberger, J. (2012). Benefits of a teacher and coach collaboration: A case study. *Journal of Mathematical Behavior*, *31*(2), 290–311.

Obara, S. (2010). Mathematics coaching: A new kind of professional development. *Teacher Development*, *14*, 241–251.

Opfer, V. D., & Pedder, D. (2011). Conceptualizing teacher professional learning. *Review of Educational Research*, *81*(3), 376–407.

Peng, A. (2007). Knowledge growth of mathematics teachers during professional activity based on the task of lesson explaining. *Journal of Mathematics Teacher Education*, *10*, 289–299.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–22.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*, 114–145.

Sun, X. (2011). 'Variation problems' and their roles in the topic of fraction division in Chinese mathematics textbook examples. *Education Studies in Mathematics*, *76*, 65–85.

The New Teacher Project (TNTP). (2015). *The mirage: Confronting the hard truth about our quest for teacher development*. Washington, DC: TNTP.

Wong, J. L. N. (2012). How has recent curriculum reform in China influenced school-based teacher learning? An ethnographic study of two subject departments in Shanghai, China, Asia-Pacific*. Journal of Teacher Education*, *40*(4), 347–361.

Yang, X. (2014). *Conception and characteristics of expert mathematics teachers in China*. Berlin: Springer.

Yang, Y. (2009). How a Chinese teacher improved classroom teaching in teaching research group: A case study on Pythagoras theorem teaching in Shanghai. *ZDM: The International Journal on Mathematics Education*, *41*, 279–296.
Yang, Y., & Ricks, T. E. (2012). How crucial incidents analysis support Chinese lesson study. *International Journal for Lesson and Learning Studies*, *1*(1), 41–48.
Zhang, M., Xu, J., & Sun, C. (2014). Effective teachers for successful schools and high performing students: The case of Shanghai. In S. K. Lee, W. O. Lee, & E. L. Low (Eds.), *Educational policy innovations* (pp. 143–161). London: Springer.

Liping Ding 丁莉萍
Faculty of Teacher and Interpreter Education
NTNU, The Norwegian University of Science and Technology, Norway

Keith Jones
Southampton School of Education
University of Southampton, UK

Svein Arne Sikko
Faculty of Teacher and Interpreter Education
NTNU, The Norwegian University of Science and Technology, Norway