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14. PRE-SERVICE AND IN-SERVICE MATHEMATICS TEACHERS' KNOWLEDGE AND PROFESSIONAL DEVELOPMENT

INTRODUCTION

As Adler, Ball, Krainer, Lin and Novotna (2005) remarked in their landmark ICME survey report, research into mathematics teacher education was rather sparse until the mid-1990s. From its roots in mathematics and psychology – witness the name of the sponsor organisation of this handbook – the output of researchers in mathematics (or 'mathematical') education had previously been more directed, and often in an anecdotal way, towards learners, curricula, purposes and innovative instruction (Kilpatrick, 1992). The shift of attention towards – or at least, to include – teachers roughly coincided with the advent in the late 1980s of the 'social turn' (Lerman, 2000) and growing attention to professional communities, and to the lives and roles of the actors within those communities. Ponte and Chapman (2006: see p. 462) confirm this chronology in the significant case of PME activity. By the mid-1990s, Tom Cooney, one of the leading researchers in the field, was able to comment: "Although it has been 30 years coming, it appears that the field of mathematics education is poised to seriously consider teacher education as a legitimate field of inquiry" (1994, p. 626). A clear and visible sign of this emergent interest in teachers, and mathematics teacher education in particular, as objects of research, is the 1999 book *Mathematics Teacher Education*, edited by Barbara Jaworski, Terry Wood, and Sandy Dawson. The book was an outcome of activity in a PME Working Group on in-service teacher education between 1990 and 1994. Cooney's claim was further vindicated when he himself was appointed founding editor of a respected journal devoted to the topic: the *Journal of Mathematics Teacher Education*, first published in 1998. One decade later, the coming-of-age of this field of research was evidenced in the four-volume *International Handbook of Mathematics Teacher Education*, with Series Editor Terry Wood.

Considering the scope of this present chapter, it is notable that teacher *knowledge* as such is given only passing attention in the aforementioned PME-rooted book (Jaworski et al., 1999), although some authors make reference to Shulman's identification of the need for a kind of mathematical knowledge beyond confident (or even profound) knowledge of mathematics per se, and the index lists seven references to Shulman's 'pedagogical content knowledge'. The (relatively late)

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 483–520.

emergence of mathematics teacher knowledge as a research field will be discussed further later in this chapter, but it is indicative that 'teacher knowledge' was not listed as a PME research domain[1] in its own right until PME28 in 2004.

Our First Steps: Scoping the Task

Our commission was to write a critical overview of PME research into pre-service and in-service mathematics teachers' knowledge and teaching development, in the years since the publication of the first *Handbook* (Gutiérrez & Boero, 2006). This earlier work included two chapters in a section entitled 'Professional Aspects of Teaching Mathematics': between them, the authors of these chapters surveyed PME research into mathematics teachers' beliefs, knowledge, learning and classroom practices. Our brief is less comprehensive than theirs, and (accordingly) we have just one chapter in which to tell our story.

A team of researcher-colleagues of the first author undertook a content-appraisal of PME Proceedings between 2006 (Prague) and 2014[2] (Vancouver) inclusive, searching on keywords such as *teacher knowledge, teacher belief, teacher education, educator education, professional development, professional growth*. The relevance of the various PME outputs was then confirmed, or otherwise, by a rapid inspection of each paper. The same team then entered key features of each paper, such as student education-phase, teacher career-phase, methodology, sample size (where relevant) and relevant keywords. This search identified 975 candidate outputs to be considered in our survey, and the need to reduce this number significantly was apparent, but it is worth noting here that about two-thirds of these studies concerned in-service teachers. The first blunt instrument to be applied in the reduction process was to restrict reading to four types of papers, namely: Research Reports, Plenary Presentations, Plenary Panels and Research Fora. This was not because we believed that the 'best' research was reported in these outputs, but because the single page made available to authors of other presentations and group activities, such as Short Oral, Poster Presentation, Discussion Groups and Working Sessions, restricts the detail that it is possible to report in a written account (as opposed to the 'live' presentation). Around 530 papers remained, and our next decision was a tighter focus on our brief – teacher knowledge and teaching development – as indicated in the title of our chapter. With this in mind, we eliminated those papers whose focus was on teacher beliefs or teacher practices, unless they also engaged significantly with teacher knowledge and/or teachers' professional development. This was a difficult but pragmatic choice, and does not deny the complex interaction between all these elements in the effort to understand teachers and teaching. Even then, we were left with 130 papers on teachers' professional development and 220 on teacher knowledge, with the intention of reducing both to about 50 papers, in line with the number of papers cited in the Ponte and Chapman (2006).

In the case of the 220 papers on teacher knowledge, this final reduction (to 53) was achieved with the assistance of several colleagues of the second author with relevant expertise, mostly in the UK and Norway, who were able to evaluate each of the outputs in detail against criteria such as the methodological thoroughness and theoretical coherence of the paper, and its relation to established work in the field. In the case of the 130 papers on teacher professional development, the team of the first author prioritised those on the learning of in-service and mathematics teacher educators (MTEs), reflecting a growing interest in MTE learning and professional development. One Plenary Address, two Plenary Panels, and four Research Fora relevant to this focus were retained, together with seven Research Reports related to MTEs' learning. Then, according to the main research questions of the remaining papers, we divided these papers into theoretical reports, teachers' learning outcomes, and learning processes through several rounds of group discussions. Finally, 42 representative papers on teacher professional development are cited in this chapter. The 95 papers which then underpin our survey are included in the reference list for this chapter.

MATHEMATICS TEACHER KNOWLEDGE

As we noted in the introduction to this chapter, the investigation of mathematics teacher *knowledge* is a relative latecomer to the field of mathematics teacher [education] research. The contents pages of early issues of the *Journal of Mathematics Teacher Education* bear out this observation, as does the Editor's retrospective on the first volume in particular (Cooney, 1998) which makes no specific reference to mathematics teacher knowledge. By contrast, and as a rough estimate, something like a third or more of all articles published in *JMTE* in recent years have addressed mathematics teacher knowledge, focusing on aspects such as knowledge to teach particular topics or content domains, the use of particular resources or technologies to develop teacher knowledge, the impact of a particular teacher education program, and efforts to theorise the nature of mathematics teacher knowledge itself; and this trend has been paralleled in the Proceedings of PME Conferences.

Before proceeding to survey the PME outputs identified for close attention, we note that papers with an exclusive focus on either elementary or secondary schooling each accounted for about one third of the 220 papers on mathematics teacher knowledge, with the remaining third mainly unspecified or mixed. The kindergarten/pre-school phase and tertiary education were under-represented by comparison, with three and four papers respectively. While tertiary teaching was the focus of several papers identified in the initial keyword search, few took tertiary teacher knowledge as their principal theme.

These 220 papers with a focus on mathematics teacher knowledge exhibited a geographical bias with a Euro-North American axis. Specifically, the first authors of almost a half were institutionally-located in Europe[3] or the Middle East, and about a quarter in the USA or Canada, with about 10% in each of Australasia and the Far East, 5% in South America, and only one paper originating in Africa. This distribution would account for the dominant voice in the current discourse around mathematics teacher knowledge, within but also beyond PME, in which the influence of Lee Shulman and knowledge-categories (Shulman, 1987) is very powerful. This remark is not at all a criticism of that particular 'take' on mathematics teacher knowledge, but there is the possibility that the deluge of Shulman-influenced papers drowns the particular wisdom and insight to be gained from alternative cultural perspectives on the topic (see e.g., Lee, Huang, & Shin, 2008).

Organisation of the Survey on Mathematics Teacher Knowledge

Scrutiny of the papers targeted for detailed attention revealed a great many characteristics, orientations and topics. In order to organise the survey in a coherent and manageable fashion, the following account of PME research on mathematics teacher knowledge is organised into four main sections, namely: theories of mathematics teacher knowledge; elaboration of mainstream theory; growth of mathematics teacher knowledge; and aspects of mathematics teacher knowledge (in particular the choice and use of representations and examples, teacher noticing and attention to 'big ideas', and teaching with technology). The distribution of space is indicative of the prevalence of these issues in the 53 'representative' papers.

Theories of Mathematics Teacher Knowledge

The seminal work of Lee Shulman and his colleagues in the 1980s underpins most of the frameworks currently in use for conceptualising mathematics teacher knowledge. Shulman's tripartite conception of teachers' knowledge of the *content* that they teach includes not only knowledge of *subject matter*, but also *pedagogical* content knowledge, as well as knowledge of *curriculum*. Subject matter knowledge (SMK) refers to the "amount and organization of the knowledge *per se* in the mind of the teacher" (Shulman, 1986, p. 9); and pedagogical content knowledge (PCK) consists of "ways of representing the subject which makes it comprehensible to others…[it] also includes an understanding of what makes the learning of specific topics easy or difficult …" (Shulman, 1986, p. 9). In addition to his taxonomy of *kinds* of teacher knowledge, Shulman (1986) also draws out three *forms* of such knowledge, *viz.* 'propositional', 'case', and 'strategic'.

A Research Forum at PME33 brought together teams of proponents of three prevalent post-Shulman theories of teacher knowledge, each being articulated first around 2003, together with two commentators (Ball, Charalambous, Thames, & Lewis, 2009b; Ball et al., 2009a; Rowland & Turner, 2009; Davis & Renert, 2009a; Even, 2009; Neubrand, 2009). The first of these theories, *Mathematical Knowledge for Teaching,* (Ball, Thames, & Phelps, 2008; Ball et al., 2009c) refines and re-configures the three kinds of content knowledge – subject-matter, pedagogical and curricular – identified by Shulman (1986). This (MKT) framework, developed by the group at Michigan University, has already been adopted (or adapted) by numerous researchers as a theoretical framework for their own enquiries, and it would be reasonable to describe MKT as the dominant theoretical framework in current research in the field. In the MKT deconstruction of Shulman, SMK is separated into 'common content knowledge' (CCK), 'specialized content knowledge' (SCK) and 'horizon content knowledge' (HCK). CCK is essentially '*learners'* mathematics', applicable in a range of everyday and professional contexts demanding the ability to calculate and to solve mathematics problems. SCK, on the other hand, is knowledge of mathematics content that mathematics *teachers* need in their work, but others do not. On the other hand, they suggest that knowing about typical errors in advance, thereby enabling them to be anticipated, is a type of *pedagogical* content knowledge which they call 'knowledge of content and students' (KCS). In MKT, *horizon* content knowledge includes knowing what mathematical experiences precede those in a given grade-level, and what will follow in the next, and subsequent, grades.

The second theory, the *Knowledge Quartet* (KQ), similarly underpinned by Shulman's work, arose from observation, codification and classification of teachers' actions in the classroom, specifically those that could be construed as being informed by their mathematics subject matter knowledge or pedagogical content knowledge. The KQ identifies three *categories of situations* in which teachers' mathematics-related ('foundation') knowledge is revealed in the classroom: these categories, or dimensions, of the KQ are named 'transformation', 'connection' and 'contingency' (Rowland, Huckstep, & Thwaites, 2005). The first two of these dimensions are evidenced in the ways that the teacher represents and exemplifies the mathematics in focus, and how they sequence material to smooth the path of learning; the third dimension, contingency, attends to how the teacher's knowledge is mobilised as they 'think on their feet' in response to unanticipated events in the course of instruction.

A third approach to understanding mathematics teacher knowledge, *mathematics for teaching* (Davis, 2010), takes a more critical stance towards the legacy of Shulman's theoretical framework, in that the latter suggests (though not necessarily) a cognitive, individual perspective on an entity (teacher knowledge) which is only meaningful in social contexts. (This critique resonates with e.g., Hodgen, 2011; Proulx, 2010). *Mathematics for teaching* is rooted in complexity theory and approaches its enquiries through 'concept studies', a group setting for the collaborative sharing, exploration and enhancement of teachers' knowledge, explained and exemplified in an account (Davis & Renert, 2009b; Davis, 2010) of the collective unravelling by such a group of the concept 'multiplication', in monthly meetings over a two-year period. Concept study enquiry into mathematics-for-teaching begins from the stance that the professional knowledge in focus is mostly tacit, and most profitably viewed in terms of participation, and as an 'active disposition' than an 'in the head' asset.

The task assigned to the three research teams at the PME33 Research Forum was to present a reading of two 10-minute video segments through the lens of their particular theory of teacher knowledge. The three analyses (Ball et al., 2009b; Rowland & Turner, 2009; Davis & Renert, 2009a) are not incompatible, and at times they coincide (e.g., in attention to selecting and sequencing examples) but their emphases are, as would be expected, very different. In her commentary, Even (2009,

p. 148) asks: “Are the different perspectives compatible? Do they complement each other?” In a recent article, Rowland, Turner and Thwaites (2014) have answered the second question in the affirmative with respect to MKT and the KQ, arguing that “In the Knowledge Quartet, the distinction between different *kinds* of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching. In this sense, the two theories are complementary, so that each has useful perspectives to offer to the other” (p. 320). Turner (2010) supports this claim in her PME34 paper, a KQ-based analysis of longitudinal records of one teacher’s approach to teaching addition, by explicit reference to MKT concepts such as SCK and KCS.

In their PME34 paper, Proulx and Bednarz (2010) adopt the situated view of MTK, having already illustrated in Proulx and Bednarz (2009) that such knowledge is embedded in their classroom practice. They present findings from a program for inservice teachers with some features in common with Davis’ concept study. The authors report that their approach leads to new mathematical comprehensions, and that some beliefs concerning ‘mathematical norms’ are being brought to the surface, and challenged.

Although the organization of teacher knowledge into categories of one kind or another might be convenient to try to capture and articulate distinctions between knowledge-types, the boundaries between different categories is usually fuzzy, promoting from the outset disputes about what exactly characterizes different knowledge-types, and indeed whether the supposed boundaries exist at all. In their extensive review, Depaepe, Verschaffel and Kelchtermans (2013) summarise this debate and the attempts to resolve it. In her PME38 plenary address on the professional knowledge of (prospective) mathematics teachers, Gabriele Kaiser (Kaiser, Blömeke, Busse, Döhrmann, & König, 2014) raised once again the ‘paradigmatic differences’ in conceptualisations of mathematics teacher knowledge (and PCK in particular) as ‘in the head’ in one view, and ‘situated’ in another, as captured by Depaepe et al. (2013, p. 22):

> Advocates of a cognitive perspective on PCK believe it can be measured independently from the classroom context in which it is used, most often through a test. […] Adherents of a situated perspective on PCK, on the contrary, typically assume that investigating PCK only makes sense within the context in which it is enacted. Therefore, they often rely on classroom observations ….

Reporting findings from the 16-country *Teacher Education and Development Study* (TEDS-M), and its follow-up TEDS-FU, Kaiser (2014) proposed that these two studies suggest a way that the cognitive and situated conceptions of PCK can be integrated. The TEDS-M theoretical framework of teachers’ ‘professional competencies’ begins from Shulman-type categories, but also includes an affective dimension, and extends (like the Knowledge Quartet) to include beliefs related to mathematics and mathematics teaching and learning, as well as metacognitive factors. Using instruments developed (or adapted) for the purpose, these aspects of prospective teachers' professional competencies were surveyed. Various kinds of comparison of competencies, and some ranking, between countries or groups of countries are reported, and some of these findings are related to cultural characteristics of the countries in question (Hofstede, 1986). For example, it was found that prospective teachers from more collectivistic-oriented countries hold more static views about mathematics (as a theory and a set of rules), whereas prospective teachers from individualistic countries are more associated with a dynamic view (as a process). The follow-up TEDS-FU study investigated how mathematics teachers' professional knowledge develops as they begin their teaching careers, and how this professional knowledge can be investigated in a more performance-oriented way. Practice-oriented, situated indicators of teacher expertise such as 'perception accuracy' (related to 'noticing' – see later), knowledge-based reasoning, and rapid identification of errors in the classroom, were annexed to the existing TEDS-M theoretical framework. These were assessed using web-based instruments requiring participants' responses to items related to short teaching sequences viewed online. The researchers found, inter alia, that the ability to notice classroom situations adequately, and to reason appropriately, are strongly related to both aspects of disciplinary knowledge (both mathematical and pedagogical). On the other hand, the ability to recognise student errors depends more strongly on content knowledge than on pedagogical knowledge.

While the studies reported by Kaiser do indeed integrate both *in vitro* and practice-based approaches to evaluating mathematics teachers' professional knowledge, they both reflect a view that such knowledge can be evaluated – 'measured', in fact – on the basis of teachers' individual responses, out of the classroom, to suitable test/questionnaire items. The 'paradigmatic differences' in conceptualisations of mathematics teacher knowledge, and how teachers are best supported and enabled to grow professionally, remain intact. We return to the issue of knowledge growth later.

Elaboration of Mainstream Theory

Papers presented at PME include a number of proposals for the elaboration, or modification, of extant theories of mathematics teacher knowledge, as outlined in the previous section. While such studies usually add to acronym-overload in the field, some draw attention to gaps or conflicts in the mainstream teacher knowledge discourse. Both Chapman (2012) and Foster, Wake and Swan (2014) take up a critique that Shulman's framework and its derivatives focus on knowledge of mathematical concepts at the expense of problem solving proficiency. Chapman proposes a four-part framework of 'mathematical problem-solving knowledge for teaching' (MPSKT), namely knowledge: of problems; of problem solving; of instructional approaches; and of students as problem solvers. In a study with 11 practising secondary school teachers, it was found that the participants held different (up to six) different conceptions in relation to each of the four PS dimensions.

Foster et al. (2014) propose a more conservative adaptation of Ball et al.'s (2008) MKT framework, in which each occurrence of 'content' is replaced by 'concepts and processes' (thus e.g., 'knowledge of concepts and processes and teaching'). They then report a case study of two problem-solving lessons taught in the context of a lesson study-based professional development program. Their analysis of lesson observations and post-lesson discussions leads them to offer observed aspects of the three PCK-components of the MKT model from a process perspective. In a somewhat similar adaptation, or application, of the MKT framework to pedagogical knowledge of technology, Getenet, Beswick, and Callingham (2015) propose a mathematics – specific version of the TPACK framework (Mishra & Koehler, 2006: see later in this chapter) named Specialised Technological and Mathematics Pedagogical Knowledge (STAMP), which somehow blends the two frameworks to take advantage of the affordances of both.

Cooper (2014) proposes a radical re-versioning of MKT by locating the Michigan theory within Sfard's (2008) commognitive epistemological framework, which views thinking as a form of communication. In this commognitive embedding of MKT, each of the MKT components (CCK, etc.) becomes (or is viewed as) a discourse, and the theory as a whole is renamed *Mathematical Discourse for Teaching* (acronym: MDT). A significant theoretical distinction in Cooper's data analysis is that between discourses (and meta-discourses) concerning mathematics and those concerning pedagogy, each of which has its own keywords, mediators, routines and narratives (with reference to Sfard's characteristic features). He proceeds to an analysis of a PD session on the notion of parity, arguing that two kinds of 'knowing' (about parity) can be discerned in the data, corresponding to the two discourses.

Features of the MKT theory that have attracted considerable attention from researchers are the Common/Specialized content knowledge distinction (CCK/SCK) and, to a lesser expend, horizon content knowledge (HCK). One approach to the CCK/SCK distinction question is theoretical argument. Another, less common approach is to design test items purporting to activate/assess one or either of these constructs, but not the other as far as possible. Of course, the construction of such items will initially draw upon theoretical conceptualisations of the two constructs in the first instance, and eventually *define* them when used as instruments to measure those constructs. Drageset (2009) presented findings from a Norwegian study investigating "the existence of SCK and CCK as two separate constructs" (p. 475) as regards Norwegian primary and lower-secondary teachers. Twenty-seven test items (derived from the Michigan *Learning Mathematics for Teaching* item bank) were administered to 356 teachers; 10 of the items were deemed to test SCK, the others CCK. A rather brief statement of correlation analysis of the test responses concludes that the two constructs are "connected, but still sufficiently different empirically to indicate that there are two different constructs" (p. 479). We note that Michigan-based Schilling (2007) had found that "sometimes SCK shows up as a separate factor in factor analyses and sometimes it does not" (p. 106). The debate continues.

In a paper pre-dating MKT (Ball et al., 2008) and Depaepe et al.'s (2013) PCK survey, Chick, Baker, Pham and Cheng (2006) proposed a literature-based three-part framework for PCK based on the interaction between pedagogical and 'pure' content knowledge (CCK, perhaps). The components are labelled 'Clearly PCK', 'Content Knowledge in a Pedagogical Context' and 'Pedagogical Knowledge in a Content Context'. Several components of each dimension are identified and listed. The adequacy of the framework is tested empirically by reference to questionnaire and interview data concerning decimals from 14 upper-primary teachers. The authors conclude that the framework was adequate, with some redundant components in the case of their sample.

Another group of papers apply and elucidate aspects of the Knowledge Quartet theory of mathematics knowledge in teaching. Petrou (2008) uses the framework in an investigation of Cypriot pre-service teachers' knowledge in relation to their classroom practice. Her case study analysis raises for attention issues in connection with one pre-service teacher's lesson on fractions, in particular concerning representations of fractions and fraction-related division structures. Whereas other PME researchers cite the KQ in their theoretical framework, the most detailed elaborations of KQ-theory are by Turner (2008, 2009, 2011) and Rowland (2010, 2011). The *Contingency* dimension of the KQ – associated with teachers' responses to unplanned and unanticipated events in their mathematics classrooms – receives particular attention in these papers. Rowland (2010) highlights the potential for teacher learning presented by contingent events, especially in post-lesson reflection-on-action (Schön, 1983), and within teacher education programs. He exemplifies this potential with an incident in which a trainee teacher is surprised by a Grade 2 student's division of a rectangle into quarters. Turner (2009) takes up the same developmental theme regarding contingency, with reference to a longitudinal study in which beginning teachers learned to analyse their own teaching using the KQ as a tool. Drawing on an international resource of KQ-analyses of mathematics teaching at elementary and secondary levels, Rowland (2011) presents a taxonomy of 'triggers' of contingent events, the main components being students' responses to questions and tasks, teachers' in-the-moment insights, and the use of pedagogical tools, including technology. Finally, Turner (2008) draws out social, community-of-practice factors in the development of mathematics teaching in early-career teachers, and the interaction of such factors with the kind of critical reflection supported by the KQ.

Growth of Mathematics Teacher Knowledge

Several PME papers address the growth of mathematics teacher knowledge and how it comes about, approaching the issue from a number of directions. Three such papers evaluate the effect of pre-service education, of teaching experience, and of a particular PD program. Blömeke and Kaiser (2008) reported findings from an international study (MT21, a precursor of TEDS-M) of the efficacy of pre-service mathematics teacher education. Participants were 849 German student teachers in three cohorts representing the beginning, middle and end of teacher education (over 5–7 years), who took a situation-based assessment of their knowledge of mathematics, of mathematics pedagogy, and of general pedagogy. Findings confirmed significant knowledge growth between the first and third cohorts, although much less so in general pedagogy than in the two mathematics-specific domains. Blömeke and Kaiser raise the caveat that quasi-longitudinal designs make assumptions about cohort comparability.

In their paper, Doerr and Lerman (2009) address the growth of knowledge for teaching mathematics as a consequence of experience of teaching. Lesson observations and interviews with one teacher participant (Cassie) in a four-year longitudinal study support the claim that the role of commonplace pedagogical routines ('local strategies' such as a particular rubric to support students' mathematical writing) shifted from procedural tools to conceptual *principles* for instruction. Doerr and Lerman point to the vital role of interactions between the project teachers, and between them and the researchers. The roles of reflection and teacher-community participation once again emerge as crucial.

In the third paper, Seago, Carroll, Hanson and Schneider (2014) examine the impact of a topic-specific two-year PD program (*Learning and Teaching Linear Functions* – LTLF) on teachers' understanding and teaching of linear functions. An experimental design involved 63 teachers and 1645 students in California. Multiple instruments, including questionnaires, observations and tests, were used to assess relevant teacher knowledge, teacher practice and student knowledge before and after the intervention. The 'impact analyses' found modest short-term (only) improvements in the intervention teachers' knowledge for teaching mathematics, but student-related aspects of their teaching were enhanced, relative to the control group.

Verhoef and Tall (2011) report research in the Netherlands on lesson study as an approach to mathematics teacher learning. Three upper-secondary teachers took part in two lesson study cycles on 'derivative' over one school year. Questionnaires administered at the beginning and end of the year probed beliefs about educational goals, teaching methods, and associations with the derivative concept. An exit interview elicited views about students' understanding. It was found that the potential benefits of Lesson Study were undermined by other 'controlling' influences such as curricula, ingrained habits, textbooks and student examination preparation. The study seems to confirm the need for caution in transplanting LS to western cultural contexts.

Gilbert and Gilbert (2009) take up the theme of teachers' "systemic, intentional analysis of their own practice" (p. 76) within a Professional Learning Community (PLC) as an effective means of transforming practice. They report findings from a project in which high school teachers worked together on GAMUT[4] tasks designed to highlight the mathematics that teachers use in their teaching. The paper shows how the tasks are sequenced and 'layered' so that each part has potential to deepen and extend participants' mathematical thinking in relation to earlier parts. The same authors develop the notion of school-based PLCs in their PME37 paper (Gilbert & Gilbert, 2013), in which they describe the development within a PLC of Educative Curriculum Materials (ECMs) envisaged as guides for teachers to support teacher learning and lesson planning. Taken together, the two papers illustrate the value of collaborative work on carefully-designed tasks, and of PLC networks, as a means of developing mathematics teacher knowledge in organic and sustainable ways.

In their introduction to the PME35 Research Forum on the use of tasks in mathematics teacher education, Sullivan and Zaslavsky (2011) offer a useful taxonomy of such tasks,[5] making the broad distinction between those that resemble tasks that could be used with school students and those that are peculiar to teacher education (such as analysis of videos of teaching). In a contribution to that Research Forum, Chazan, Herbst, Sela and Hollenbeck (2011) articulate a rather novel approach to the representation of classroom practice in which animations are used to present classroom scenarios for consideration and critique by (in this instance) pre-service teachers. The animation in question concerned a student's unexpected (and correct) approach to solving a particular linear equation. It typifies both a contingent situation (c.f. the Knowledge Quartet) and a provocation of specialised content knowledge (c.f. Mathematical Knowledge for Teaching) with rich learning potential for the PSTs.

Noh and Kang (2007) also explored the contribution of ECMs to the development of mathematics teacher knowledge, but in their case the ECMs were published 'reform-oriented' curriculum materials developed for use with school students, specifically the NSF-funded curriculum *Contemporary Mathematics in Context* (CMIC). Twelve high school teachers participated in individual, task-based interviews with the researchers, using CMIC problems on rate of change. It was found that although many of the participants held a procedural view of derivative, most demonstrated ability to move between different representations of rate of change – a strong feature of CMIC. Although the specific findings reported are appropriately tentative, a social and distributed view of MTK (and its development) necessarily assigns significance to the role of ECMs in professional settings.

The teaching of proof has exercised PME researchers over the years (see Stylianides, Bieda and Morselli, Chapter 9, this volume) but rather less attention has been given to the related teacher knowledge. Drawing on observation and interview data from a three-year case study, Cirillo (2011) presents a beginning teacher (Matt) with a strong mathematics background. Initially, Matt doubted that it was possible to teach proof, but by the third year he likened himself to a 'sherpa' who had climbed the 'mountain' (proof) many times in the past, and who now accompanies his students on the same journey. Cirillo emphasises that Matt's secure subject matter knowledge was not sufficient to enable him to teach his students how to prove, calling for studies of teachers with proven success at doing so.

Whereas reflecting on teachers and teaching practice is now a commonplace means of achieving growth in professional knowledge, the value of studying and understanding learners can be another, lately neglected, means to the same end. In his PME Plenary address, Doug Clarke (2013) pointed to a resurgence of interest in Piaget's clinical interview (e.g., Ginsburg, 1997) in Australia and New Zealand, "as a professional tool *for teachers of mathematics*" (Clarke, 2013, p. 19; emphasis in the original). Clarke reports that regular use of a research-based, one-to-one interview by teachers with their students has contributed to growth of their subject matter knowledge (SCK and HCK in particular) and pedagogical content knowledge, in particular knowledge of students' mathematical understanding, thinking and reasoning.

The *growth* of MTK clearly falls within the remit of the second major focus of this chapter – teachers' professional development – and some of the approaches addressed in this section will be revisited later in this chapter, with the focus directed more towards the development of teaching practice.

Aspects of Mathematics Teacher Knowledge

Apart from the elaboration of theory and attention to the growth of professional knowledge, PME outputs on MTK in the decade under consideration have clustered around particular themes, three of which we review below: namely, teachers' choice and use of representations and examples; teacher noticing and attention to 'big ideas'; and teaching with technology.

Choice and use of representations and examples. In his exposition of the concept of PCK, Shulman (1986, p. 9) referred to "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations", and so the selection and use of representations and examples for pedagogical purposes has been central to the notion of PCK from the outset. These two aspects of PCK are distinguishing components of the Transformation dimension of the Knowledge Quartet (Rowland et al., 2005), and very visible in the exposition of specialised content knowledge in the Mathematical Knowledge for Teaching framework (Ball et al., 2008; p. 400). Although particular PME papers tend to focus on just one of these aspects of mathematics teachers' knowledge and practice, the two are intimately connected. For example, Turner and Rowland (2007) describe how a teacher's last-minute switch from symbolic (numerals) to spatial (100-square) representation of two-digit numbers in a lesson on subtraction caused her prepared examples to 'misfire' in her 'how-to' explanation.

Multiple representations and cross-national comparisons feature in many of the papers on *representations*. Drawing on video classroom data from the Learners Perspective Study, Huang and Cai (2007) report analysis of the representations used by teachers in high-performing schools in Shanghai, China and California, USA, in 10 lessons (each) on linear functions. Huang and Cai (2007) cite NCTM (2001) in stating that teachers' selection of pedagogical representations reflects their knowledge and beliefs about mathematics teaching and learning. It was found that the US teacher drew on multiple representations in most lessons, frequently more than three (e.g., graphic, symbolic, tabular). By contrast, the Chinese teacher normally used only one or two types of representation, with verbal and numerical representations predominating. The US teacher used numerical representations least. The authors suggest that these differences may explain a separate finding regarding Chinese/US students' preference for abstract/concrete representations in problem solving. The US teacher's practice is consistent with the notion inherent in the paper of Koh and Kang (2007) discussed earlier, that the promotion of multiple representations by ECMs such as CMIC is beneficial for both students and teachers.

This cultural difference might explain the claim of Dreher, Kuntzer and Lerman (2012) that "fostering students' competencies in dealing with multiple representations should be a central goal" (p. 212). In another 'inter-cultural' study, British and German pre-service teachers (PSTs) answered a questionnaire in which they rated various items about multiple representations in the teaching of fractions. They detected a difference, deemed to be cultural, in that the British PSTs favoured use of multiple representations, irrespective of their mathematical appropriateness, in the interest of providing for students' individual learning differences. The German PSTs, by contrast, had concerns that multiple representations could confuse students.

Investigating teachers' knowledge to discriminate between different representations to achieve particular learning goals, papers by Barmby and Milinković (2011) and Milinković (2012) explore British and Serbian PSTs' choice between several alternative representations (such as sets, number line, area and arrays) to represent different entities and relations, for students of different ages. Their responses were indicative of the participants' SMK and PCK, but also the stress placed on particular representations in the two countries. Deher and Kuntze (2015), and also Way, Bobis and Anderson (2015), conclude that knowledge about representations, and how to use them in assessing and developing conceptual understanding (of fractions, in these papers) should be an explicit focus in mathematics teacher education.

Concerning the *choice and use of examples* in mathematics teaching, two rather different, extended contributions stand out as 'state of the art' reviews at the beginning and end of the PME decade under consideration. The first, Bills, Mason, Watson and Zaslavsky (2006), is the paper associated with a Research Forum on 'exemplification' at PME30, co-authored by several leading researchers in the field. The scope of the paper includes different meanings of 'example'; a historical survey of the pedagogical use of examples; theoretical perspectives, including the notion of 'personal example space' (Watson & Mason, 2005); teachers' selection and use of examples, with reference in particular to the work of Orit Zaslavsky and her collaborators; the learner's perspective, the role of examples in concept formation and problem solving, including non-examples, counter-examples and generic examples; research perspectives, including instructional design and theory building; and pointers for further research. The notion of generic example (otherwise called a prototype or paradigm) as a provocation to concept formation and reasoning is recurrent throughout the paper, and was subsequently the focus of a Working Session on Generic Proving (Leron & Zaslavsky, 2009) at PME33. In recent years, the notion of 'variation' in pedagogical exemplification has entered more fully into the discourse of instruction, following psychologist Ference Marton's perception that we learn from discerning variation, and what varies in our experience influences what we learn. The provision of examples must therefore take into account the 'dimensions of variation' (Marton & Booth, 1997; Watson & Mason, 2005) inherent in the objects of attention. Western thinking on this notion is also being linked to the practice of *bianshi* ('variation') within Chinese pedagogical practice (Gu, Marton, & Huang, 2004).

At the PME30 meeting, Zaslavsky, Harel and Manaster (2006) also contributed a paper on a secondary teacher's treatment of examples in a lesson on the theorem of Pythagoras as indicative of teacher knowledge, citing work by Zaslavsky and Peled some ten years earlier on teachers generating examples. At PME33, Sinitsky, Ilany and Guberman (2009) reported on pre-service teachers' ability to generalise and explain from fractions-examples.

Drawing on her sustained research into the topic, Orit Zaslavsky gave a PME34 plenary address on mathematical thinking with and through examples (Zaslavsky, 2014). The paper is organised around consideration of three inter-related settings – spontaneous example-use, evoked example-production, and provisioning of examples – with reference to the body of Zaslavsky's work investigating them. Students' (and especially teachers') spontaneously-generated examples can be problematic (Rowland, Thwaites, & Huckstep, 2003) but they can also be productive – Zaslavsky cites the student who wrote 5 + 6 + 7 + 8 + 9 to exemplify a rule for summation of 5 consecutive integers, and then represented the sum as (7 – 2) + (7 – 1) + 7 + (7 + 1) + (7 + 2), thereby providing insight into the rule. The second setting illustrates an expanding comprehension of the concept 'periodic function' at a PD workshop resulting from the provocation formula 'Give an example of…, and another one…, and now another one, different from the previous ones…'. The design of teacher-provided examples relates to aspects of teachers' content and pedagogical knowledge, and needs to take into account what the learner is likely to (and subsequently does, or does not) "see" in the example(s). The key didactic consideration here is 'transparency' and genericity.

Teacher noticing and attention to 'big ideas'. The notion of Horizon Content Knowledge (HCK) made explicit in the MKT framework (and c.f. Shulman's (1986) 'vertical curriculum knowledge') includes a synoptic perspective on mathematics enabling the teacher to look beyond the subject-matter immediately in focus to see the 'big picture'. Kuntze et al. (2011) report two studies from an EU-funded project related, respectively, to assessing and developing German PSTs' (elementary and secondary: N=117) knowledge of Big Ideas. The paper lists key characteristics of (the researchers' perception of) Big Ideas, such as potential to support conceptual understanding and meta-knowledge of the nature of mathematics. Their projects focused on three such Big Ideas (e.g., 'argumentation and proof'). Although the first study identified weak access to content linked to these Big Ideas, the second found that related professional knowledge and awareness of Big Ideas can be built up in professional development courses. Nicol, Bragg and Nejad (2013) report a Canadian study in which six elementary PSTs were asked to adapt a task on reasoning with fractions in order to make it more accessible, or more challenging. Their analysis of the PST's proposals indicates that none takes into account the big mathematical ideas in the original problem, specifically the relationship of a fraction to the 'whole'. These authors frame their finding in the context of teachers' noticing and attention (Mason, 2002) when considering/preparing tasks for the classroom. Papers by Pang (2011) and Vondrová and Žalská (2013) take up this 'noticing' theme, with regard to Czech PSTs' analysis of videotaped mathematics lessons. In a previous study, Vondrová and Žalská (2012) had found that PSTs pay little attention to 'mathematics-specific phenomena' (MSPs) when observing a full mathematics lesson. In this one, six short video clips were shown, so that a greater 'density' of MSPs were present in the material viewed, but the PSTs' ability to notice them was not significantly improved. Rather, their attention was mainly guided by generic motivational concerns. The authors ask the telling question: would practising teachers be more likely to notice the MSPs? Pang's (2011) paper reports very similar findings with PSTs in Korea, although it does note some improvement in sensitivity to mathematics-specific aspects of what they observe later in a case-based teacher preparation course in which such classroom events were regularly analysed and discussed.

Teaching with technology. Despite the very significant presence of digital technology in mathematics education research and practice, little progress has been made to date in integrating pedagogical knowledge of technology into frameworks for mathematics teacher knowledge, or in conceptualising mathematical knowledge of technology-for-teaching. In her PME36 paper, Bretscher (2012) turns to the TPACK framework (Mishra & Koehler, 2006) as a candidate to achieve this integration, and presents an analysis of the use of a PowerPoint presentation, an interactive whiteboard and a spreadsheet by one of three case study teachers in a lesson on nth terms of sequences. She concludes that TPACK is a useful tool for the purpose of including consideration of technology factors in the analysis of mathematics teacher knowledge, but that "the central TPACK construct may be better understood, not as a new category of knowledge …. but rather as a transformation and deepening of existing mathematical knowledge for teaching using technology" (p. 89). Ruthven (2014, p. 380) has subsequently suggested that TPACK "provides a rather coarse-grained tool for conceptualising and analysing teacher knowledge; one that generally needs to be supplemented by other systems of ideas to accomplish analysis to the depth required for effective professional development and improvement".

Two studies by Kuntze and Dreher (2013) used questionnaires to investigate the PCK of 39 PSTs in relation to computer use in mathematics teaching, and how it can develop in pre-service education; and also the views of 65 practising teachers about such computer use, and the extent to which they used it themselves for various purposes. The questionnaires were in part informed by a distributed view of teacher knowledge, to include relevant 'tools', and a framework of Martin (2012) of pedagogical functions of educational technologies (viz. 'connection', 'translation', 'off-loading', and 'monitoring'), which has some potential to enhance the existing technology-free theoretical frameworks for MKT. Findings from the study indicate that the PSTs were moderately optimistic about computer use at the beginning of their course, but that their lack of technology-related PCK rendered them unable, on the whole, to be specific about actual applications. At the end of the course, 25 PSTs who had chosen a computer-related unit showed significant gains in technology-related PCK and positive attitudes, whereas the remainder did not show such gains. As for the practising teachers, it was found that, on the whole, they lacked optimism, experience and PCK in relation to computer use. Clearly comparison between the pre-service and in-service cohorts is problematic, but the first study offers some hope that PCK for technology use is learnable.

Mathematics teachers' professional learning in relation to technology use is taken up in a wide-ranging Research Forum paper by Clark-Wilson et al. (2014) which introduces (with several examples) a number of theoretical frameworks, at different levels of generality, underpinned by the theory of *Meta-Didactical Transposition*, a model for the analysis of teacher education which was itself the focus of a PME37 Research Forum (Aldon et al., 2013). Although teacher knowledge is not foregrounded in the paper, examples of teachers' learning/practice trajectories "provide insight into how the particular features and functionalities of the different digital mathematical tools impact upon teachers' motivation and confidence to integrate them into classroom teaching involving mathematical digital technologies" (p. 102). These 'cases' also illustrate the use of different theories including TPACK (see above) and also Pedagogical Technology Knowledge (PTK) (Thomas & Hong, 2005). In contrast to TPACK, PTK relates specifically to mathematics teacher knowledge, and incorporates the understanding of the principles and techniques that enable teachers to design and manage instruction likely to promote *mathematical* learning with technology.

TEACHER PROFESSIONAL DEVELOPMENT

We turn now to mathematics teacher professional development. Llinares and Krainer (2006) concluded that programs aiming to promote teachers' learning addressed their awareness of mathematical process and content, and of children's mathematical thinking. Llinares and Krainer also identified the factors which promote or hinder teachers' learning as: structure of teachers' learning; mathematical tasks used in teachers' learning; support network; engagement in extended conversation about teaching and learning mathematics; time spent; and action research on teachers' beliefs and practice.

In considering PME research in the decade since Llinares and Krainer's review, we focus on the professional development of in-service mathematics teachers, hereinafter teacher professional development (TPD), in relation to the "knowledge construction or the incremental refinement of practice or both" (Clarke, 2009, p. 85). This part of our review emphasises the *refinement of teacher practice, especially practice influenced by teachers' newly constructed knowledge*. The review of PME studies specific to TPD includes 130 papers with this geographical distribution: about one third of the first authors were institutionally-located in North America, about a quarter each in Europe (6% in UK) and Asia, about 12% in Australasia, 5% in South America, and 2% in Africa. It is worth noting that roughly half of the 130 papers come from English-speaking countries such as America, Canada, UK, and Australia.

Organisation of the Review on Teacher Professional Development (TPD)

Three theories of teacher knowledge were elaborated in the previous section, but theories of TPD are still in process of development. The following sections address, in turn: theoretical perspectives on TPD; *description* of TPD; *interpretation* of TPD; and *prediction* of TPD. We also review PME research on *mathematics teacher educators'* education – an emergent TPD-related theme.

Theoretical Perspectives of Teacher Professional Development (TPD)

In the plenary panel at the PME33, Clarke (2009) summarised the mainstream theoretical perspectives in mathematics teacher education and addressed issues related to the bridge between research and practice via mediation of different theoretical perspectives. Perspectives on mathematics teacher education can generally be described as either *researching TPD from the cognitive or the socio-cultural perspective*; *viewing theory as a static entity or an evolving process*; or *the opposing or complementary nature of theories.* These three perspectives structure the following review of PME studies specific to TPD during 2006–2014.

Researching TPD from the cognitive or the socio-cultural perspective. Ponte (2009) claimed that cognitive theories have been the dominant view in teacher education. For example, Tzur (2007), from a cognitive perspective, asserted that TPD is "progress from intuitive to formal ways of thinking about teaching" (p. 143). He further pointed out that the learning progression does not only refer to behavioural changes but also to a paradigm shift in teachers' thinking from know-what to know-how. Muñoz-Catalán, Climent and Carillo (2009) attempted to make an analogy between student learning and teacher learning about teaching. They adopted the hierarchical stages of interiorisation, condensation and reification from Sfard (1991) to elaborate TPD. Muñoz-Catalán et al. showed that teachers are more inclined to take students' learning difficulties into account and adapt teaching plans that can meet students' need during the condensation stage.

While recognising this dominant cognitive perspective, Ponte (2009) pointed out the emergence of theories that emphasise social processes and how they influence TPD (e.g., social interactions between participants, communities of practice, and activity structures involving participants). Llinares and Krainer (2006) also claimed that investigations of TPD increasingly consider social and organisational aspects. Ponte and Chapman (2006) further elaborated sociocultural theory, based on the work of Vygotsky, which has become prominent in the PME community and has evolved as one of the more productive lines of work regarding teachers' practices. The review of PME studies specific to TPD between 2006 and 2014 also reveals the increasing interest in socio-cultural perspectives. For example, Ohtani (2009) adopted activity theory, one of the socio-cultural theories commonly used to interpret TPD, to argue that Japanese Lesson Study could be a successful approach. Likewise, Jaworski and Goodchild (2006) used activity theory as the framework with which to analyse issues and tensions with respect to the essence of TPD occurring within an inquiry learning community.

Viewing conception of theory as static entity vs. evolving process. In the panel, Clarke (2009) applied the definition proposed by Niss (2007, p. 1308) to elaborate the static conception of theory as "an organised network of concepts (including ideas, notions, distinctions, terms, etc.) and claims about some extensive domain, or a class of domains, consisting of objects, processes, situations and phenomena"; The TEDS-M study (Kaiser et al., 2014) was judged to be conducted under the static perspective which mainly evaluated the content and pedagogical content knowledge, and the learning opportunities, of practising teachers. By contrast, due to the demands of new situations and research purposes, Clarke (2009) proposed that "theories need to be fluid and evolving" (pp. 87–88), e.g., in the context of online distance courses for teachers (Borba & Zulatto, 2006).

The opposing or complementary nature of theories. Different theoretical perspectives on teacher professional development, like those of teacher knowledge, need not be in opposition. Thus Clarke (2009) viewed "alternative theories as potentially complementary rather than necessarily opposed" (p. 91). One example of complementary alternative theories could be seen in the Interconnected Model of Teacher Professional Growth (Clarke & Hollingsworth, 2002), using four domains of teachers' professional growth identified by Guskey (1986). Clarke and Hollingsworth (2002, p. 947) identified "the specific mechanisms by which change in one domain is associated with change in another. The interconnected, non-linear structure of the model enabled the identification of particular 'change sequences' and 'growth networks', giving recognition to the idiosyncratic and individual nature of teacher professional growth".

In this review we find that theories are, for the most part, introduced by the researchers, and that theories of mathematics learning have their analogies in theoretical perspectives on mathematics teachers' learning. If there were be a comprehensive theory of TPD, relevant to multiple contexts and situations, what would be the necessary functions of such a theory? Our review suggests that they would be: to *describe* what teachers have learned; to *interpret* how teachers learn to refine their practice; and to *predict* TPD (that is *to design, evaluate, and research* it). We now consider each of these three functions in turn.

Description of Teacher Professional Development (TPD)

For the most part, PME research which described the refinement of teacher practice also interpreted the processes of refinement. Here we will emphasise the following four foci: *expert teachers*; *beginning teachers*; *inquiry-based teaching*; and *raising teachers' awareness of students' thinking*.

Expert teachers. The meaning of 'expert' is interpreted in different ways because different aspects of teachers' expertise are valued within different cultures and societies (e.g., Berliner, 2001). A PME36 Research Forum on Conceptualizing and Developing Expertise in Mathematics Instruction focused on teacher expertise and its development (Li & Kaiser, 2012), in which Ponte (2012) and Lin (2012) portrayed their perspectives about expert teachers implementation of the new curriculum in Portugal and their long-term participation in teacher professional development programs in Taiwan, respectively. They both considered expert teachers to be being those who employed student-centred teaching, including selecting tasks and conducting classroom discussions before and during teaching. About the selection of tasks, Ponte (2012) claimed the expert teacher "is able to select and perhaps adjust suitable tasks, …, involving students actively in mathematical work, stimulating them to develop their own strategies, concepts, and representations" (p. 126), effectively an elaboration of Lin's (2012) "designing and using tasks that support rich mathematics thinking" (p. 133). About conducting classroom discussions, Ponte (2012) indicated that expert teachers should "conduct classroom discussions that create opportunities for negotiation of meaning, development of mathematical reasoning, and institutionalization of new knowledge" (p.127); consistent with Lin's (2012) "purposely selecting and sequencing students' solutions for whole class discussion; critically questioning and using students' errors or misconceptions for discussion; responding to students' questions adequately" (p. 133).

Lin (2012) further proposed that the expert teacher would create and allocate creative assignments after lessons. Likewise, also in the Forum, Leikin (2012) claimed that "expertise in mathematics instruction is characterised by fluency, flexibility, originality and elaboration" (p. 143): she referred to creative teaching;

giving concrete empirical examples to elaborate these concepts, and to support her claim. It is notable that, in different contexts and from diverse viewpoints, Ponte, Lin and Leikin all proposed similar characteristics of expert teachers. Nevertheless, whether each society has a coherent definition of the expert teacher is still under investigation. The emphasis has shifted to whether there is only one view of what an expert teacher is within one society. Pang (2012) argued that even in the same society, the description of an expert teacher might differ depending on one's role; e.g., Korean mathematics educators usually considered expertise in mathematics instruction from the perspective of "mathematics-specific analysis ability", whereas educators in general in Korea considered expert teachers from the perspective of a "specific case-based pedagogy" (p. 136). This seems to be an important issue when developing expertise in mathematics instruction within one society in the future.

Beginning teachers. At a PME31 Research Forum entitled Researching Change in Early Career Teachers, Hannula and Sullivan (2007) focused on ways in which teacher educators might facilitate effective change in beginning teachers. It was proposed (p. 151) that beginning teachers might be in need of change if they:

1. Have fixed views of the nature of mathematics and limitations in relevant mathematics discipline knowledge;
2. Have anxieties about mathematical knowledge and teaching that can be potentially constraining and even disabling;
3. Are unfamiliar with desired pedagogies and curriculum, not having experienced these as school students themselves; and
4. See learning to teach as a short-term, once-only event as distinct from a career-long process.

Point 1 is similar to Leikin's (2006) intuitive thinking about mathematics teaching. The unfamiliarity with pedagogy and curriculum in Point 3 is the opposite of the "fluency" proposed by Leikin (2012), and the lack of experience as a student seems to be contrary to "awareness of children's mathematical thinking" claimed by Llinares and Krainer (2006). Furthermore, Points 1 and 3 are related to the concept of teacher efficacy. Chang and Wu (2007) studied 64 beginning elementary teachers' sense of efficacy related to mathematics teaching, finding that those who had majored in mathematics or science showed greater efficacy. Hannula, Liljedahl, Kaasila and Rösken (2007, p. 154) summarised the therapies aiming to reduce the mathematics anxiety of pre-service teachers into four types: narrative rehabilitation; bibliotherapy; reflective writing; and drawing schematic pictures. Whether these strategies could also be adopted for beginning (and more experienced) teachers is an interesting issue for future research. Additionally, it should be noted that point 4 deals with societal-based issues, which vary between countries; the correlation between beginning teachers' willingness to refine their teaching and their stance in relation to point 4 is also worth future investigation.

In addition, to enhance teacher efficacy and reduce mathematics anxiety, the Research Forum concluded: "One fruitful approach is to engage innovative mathematics teachers as experts or facilitators (teacher-researchers) for new projects" (p. 175); examples could be seen in Wang and Chin (2007) who investigated the ways mentors intervene in the mathematics teaching of practice teachers, and the principles and underlying values for their interventions.

Inquiry-based teaching. Inquiry-based teaching is widely promoted in mathematics education around the world, e.g., in European countries, implementation of inquiry-based learning in day-to-day teaching has been reported by Maass, Artigue, Doorman, Krainer and Ruthven (2013). Teachers' competence with inquiry-based teaching is often identified as a key indicator of expertise in mathematics instruction. For instance, the view of what makes an expert teacher in Portugal was portrayed by Ponte (2012) as a teacher who is able to select, and perhaps adjust, suitable tasks, especially exploratory tasks, that involve students actively in mathematics work, stimulating them to develop their own strategies, concepts and representations. These are inquiry-based learning tasks. Chapman (2010) maintains that "Inquiry, as a basis of teaching, is being associated with notions of learner-focused, question driven, investigation/research, communication, reflection, and collaboration" (pp. 361–362). Chapman reported the experience of a group of elementary teachers in "researching" how to adopt inquiry-based teaching in their classrooms. They developed an inquiry-teaching model, guided by their mentor to plan lessons for different grades. As a result, Chapman claimed, the teachers gained a deeper and more meaningful understanding of: inquiry-oriented teaching; questioning techniques that guide and enrich student thinking; posing thought provoking questions to motivate students to discuss and understand mathematics at a deeper level; and instructional strategies that allow students to assume ownership of their knowledge and knowledge construction.

Chin at al. (2006) reported a collaborative action research study on implementing inquiry-based instruction in an eighth grade mathematics class. An experienced teacher and a trainee teacher together carried out the action research, supported by an educator. After two-semesters, the trainee teacher gained a deeper understanding of the complex role of a mathematics teacher and had more confidence to conduct inquiry-based teaching on his own. The experienced teacher had also developed from being a novice at inquiry-based instruction to a confident teacher with the intention of communicating the teaching strategy to his peers.

Raising teachers' awareness of students' thinking. Studies focused on the intervention of using students' thinking as the basis of professional development are still ongoing, and some examples are cited here.

Regarding the role of students' mistakes in teachers' learning process, Heinze and Reiss (2007) investigated the effects of teacher training on teachers' ability to handle mistakes and assist students' learning of reasoning and proof in geometry. They conducted a quasi-experimental study and showed that students in the experimental group performed significantly better in the post-test.

Proulx and Bednarz (2009) invited teachers to explore the following fraction division task:

$$\frac{26}{20} \div \frac{2}{5} = \frac{26 \div 2}{20 \div 5} = \frac{13}{4}$$

Is this procedure adequate/correct? Does it always work? How?

A variety of resources, mathematical, didactical and pedagogical, were used by teachers when making sense of this mathematical situation. Some approached the "same" situation from different perspectives, some came at it from different perspectives at different times, and some employed ways that implicitly had a double nature (e.g., mathematical and didactical). All those points of entry appear to play a role.

Goldsmith, Doerr and Lewis (2009) reviewed over 100 studies on teachers' learning to challenge the issue: How do practising mathematics teachers continue to improve their teaching over time? They illuminated the "black box" of teacher learning by exploring teachers' changing attention to and use of student thinking.

Interpretation of Teacher Professional Development (TPD)

In order to interpret how in-service mathematics teachers learn to refine their practices, three contexts in which they learn were identified in PME research: *learning via teaching*, *via researching*, and *via participating in a learning community*. We consider these in turn.

Teachers' learning via teaching. A Research Forum at PME 31 (Leikin & Zazkis, 2007) considered how teachers might learn through teaching. The main sources of teachers' learning through teaching is their interaction with students, use of learning materials (such as textbooks and teachers' guides), communication with colleagues and attending workshops. By giving opportunities for students to initiate interactions and by managing lessons according to students' ideas, teachers also make opportunities for their own learning (p. 124). However, this way of learning is not always made explicit to teachers. Simon (2007) proposed that teachers' current understanding imposes limits on what teachers can learn from their teaching. Tzur (2007) conceptualised such learning in terms of a change in anticipation. That is, whenever teachers direct their activities towards certain goals, such as correcting student mistakes, predicting student responses, providing students with experiences that differ from one's own school experiences, resolving disagreements and/or one's cognitive conflicts, satisfying school's requirement to use software, improving one's own mathematics, etc., they essentially learn through noticing unanticipated ways in which others (e.g., one's students or peers) react to plans the teacher executes (p. 144). Such reactions may become prompts for the teachers' reflection on pedagogical/mathematical activity-effect relationships. That is, the teachers continually consider the extent to which their goal-directed teaching fosters certain effects, effects in the sense of inferred student/peer understanding. Whenever teachers noticed and revisited student/peer unanticipated actions, this prompted further reflection, hence they were learning. These three constructs: anticipation, reflection, and noticing, can powerfully explain the complex mechanisms, contexts, and stages in teacher change via teaching.

Teachers' learning via researching. Research is one of the best methods for teachers to learn how to refine their teaching practice. The focal issues of research in the process of teachers' learning can be collectively summarised from the discussion of the Research Forum at PME34 (Santos-Wagner & Chapman, 2010): (1) the goal is to develop teachers' reflective, analytical and critical thinking, (2) the helpful tools for collecting data from teachers are reflection, noticing and biographical writing, (3) the stimulus to autonomous teacher disposition in relation to mathematics pedagogy, and (4) making use of teachers' classroom practice and learning experience to help them to gain knowledge. In Llinares and Krainer's (2006) review, they suggested that "in the future, we need more of these research-oriented stories, putting an emphasis on explaining phenomena by using empirical evidence as well as theoretical consideration. Action research by teachers (…) and corresponding action research by teacher educators (…), and we need identified efforts in the future (p. 451)." Therefore, teachers' learning from research has been emphasised for its theoretical and practical importance in TPD.

The power of research for TPD is what "practice and theory can offer through learning processes of engaging teachers in *research projects*" (Santos-Wagner & Chapman, 2010, p. 354). In the context of research, teachers can be learners or researchers, depending on the goals they set. Traditionally, there are two methodological approaches to teachers' learning through the process of research in PD; one is *participating in a research project* and the other one is *conducting action research* (including *design tasks* for classroom practice) connected with the teacher's practice. These approaches are equally important in providing opportunities for learning with both theory and practice, although the actions taken in each research project might differ. Through engagement *in* a research project, teachers can quickly receive theoretical support from fellow researchers and follow the arrangement of the research design to learn. The majority of research in TPD can be categorised in this domain. However, with the other approach, conducting action research, teachers have to invest considerable effort in the process of linking theory and practice in order to improve their classroom practice (e.g., Serrazina, 2010) or to refine their teaching (e.g., Chapman, 2010). Generally, action research can function in a mentoring structure where both participants, i.e., mentors and early career teachers, learn together (e.g., Chin, Lin, Ko, Chien, & Tuan, 2006), or in a cooperative community where all participating teachers learn from each other in a systematic arrangement. For example, the experienced teachers learn to design tasks for enhancing their expertise in hierarchical stages to improve their design, take the experiences of their peer colleagues, and then develop further in participating in a design-based TPD program and conducting their own research (see Lin, Chen, Hsu, Yang, & Wu, 2013). This can also be found in lesson study (LS) group learning in which group member-teachers apply LS cycles to continuously refine their lesson (e.g., Robinson & Leikin, 2009). These studies of action research have one characteristic in common, in that they are all trying to stimulate innovation in teachers' professional expertise.

Teachers' learning via participating in learning community. Generally speaking, theories used to interpret TPD in learning communities are oriented to social and cultural perspectives. Review of PME papers with respect to TPD via participation in a community can be categorised into four main research topics: inquiry community, Lesson Study, design-based community, and online learning community.

The notion of *inquiry community* brings together characteristics of "being together" and "exploring" for triggering professional development. Fundamentally, the inquiry community involves an activity system where teachers are able to ask questions and seek answers to discover more about the teaching and learning of mathematics (Jaworski & Goodchild, 2006). To this end, Jaworski and Goodchild (2006) suggested that activity theory based on the work of Vygotsky can well articulate TPD in an inquiry community. They argued that activity theory offers a unit of analysis and the possibility of exploring the mediating elements and dialectical relationship between different tiers of participants and interactions with their environments.

Lesson Study entails a professional community where in-service teachers study lessons in depth on a school basis (Fernandez & Yoshida, 2009). Pang (2015) studied five in-service teachers and argued that lesson study motivates teachers to analyse the strengths and weakness of teaching approaches implemented in one class and to come up with alternatives. In principle, teachers should volunteer to participate in a lesson study community. However, in reality, Krainer (2011) argued that the participation can be regarded as quasi-required because socio-cultural commitment or pressure from principals plays a role in influencing the participation in such professional communities. Thus, Krainer (2011) concluded that culturally-situated theories such as cultural-historical activity theory, anthropological theory of didactics, and community of practice theory become promising theories that can be used to elaborate teacher learning in such professional environments. The use of those theories brings additional lenses in exploring and interpreting new aspects in the Lesson Study community.

Design-based community highlights design as an intervention approach by which teachers are involved in creating instructional tasks for student learning of mathematics. Design-based community does not only highlight learning through participating in practice as in Lesson Study, but also the facilitation of TPD by bridging theory and practice so that teachers have a picture of how theoretical ideas can be incorporated into their teaching. Thus, Lin et al. (2012; 2013) adopted a three-layer structure comprising grand theory, intermediate framework and a design tool (Gravemeijer, 1994; Ruthven, Laborde, Leach, & Tiberghien, 2009) for the design of professional programs, where the intermediate framework and design tool serve to coordinate and contextualise the theoretical insights from grand theory. Being task designers, teachers have opportunities to explore curriculum materials and student learning in detail, as well as to incorporate professional development materials into their designs, all of which become important sources for improving their teaching. The theory adopted for the investigation of TPD by Lin et al. (2012, 2013) is also aligned with situated learning theory, by which teachers development through interaction with others can be identified.

Online environments for teacher professional development have been seen as important for their potential benefits in responding to teachers' needs during the last decade, as compared with face-to-face professional development. It is generally thought that these online TPD programs can provide learning opportunities for teachers at their convenience, and when they are needed (Dede, Ketelhut, Whitehouse, Breit, & McCloskey, 2009). Flores, Escudero and Aguilar (2014) use the term 'online mathematics teacher education [OMTE]' in their literature review of this emerging research area. They found that the main issues investigated include *interactions* among teachers in online settings, and teachers' professional development (*growth*). The *theoretical approaches* employed in this area are partly extrapolations of tools designed originally for face-to-face settings, such as the concept of *community of practice* and *mathematical knowledge for teaching*. Some theoretical approaches are specifically designed for online settings, such as the concept of *humans-with-media* (Borba & Zulatto, 2006) and the instructional model of *online asynchronous collaboration*.

The data collected for supporting teachers' development in online environments include their *written productions*, *teaching materials*, and *mathematical productions* through graphing software, platform resources such as online forums, chat rooms and questionnaires, and digital recording artefacts. Lastly, the transformations of researchers in online environments can be summarised with reference to three aspects. One is that the way they *access data* is less intrusive than the methods used in a face-to-face setting, e.g., observations. Secondly, the efficiency of *data collection and processing* is higher than in a face-to-face setting. Thirdly, online environments create the need for researchers *to create theoretical tools* adapted from face-to-face settings.

Prediction of Teacher Professional Development and Processes—Design, Evaluation, and Research

Our survey found only a few papers which pinpoint the apparatus that can predict TPD. From a socially situated learning perspective, Hsu, Lin, Chen and Yang (2012) proposed a coordination mechanism defined as the ability to innovate for teaching by transforming and coordinating sources of information observed and experienced in different learning environments. Based on this definition, Hsu et al. identified two kinds of coordination, namely coordination as making connections between others and personal ideas in a superficial way; and coordination as integrating sources of information into the creation of novelty. Similarly, Boesen et al. (2014) identified two kinds of interpretation of information in respect of their teaching. One is assimilation, which refers to the ways that teachers interpret information that is in line with their preference. The second is adaptation, which highlights the coordination of information into the learning process.

Lin et al. (2013) further grounded their study on the analysis of teachers' intention to design tasks and evaluate them in alignment with the goals prescribed by professional development programs. Based on a case study, the analysis reveals three stages of teacher growth: self-expression; combining other ideas into personal design; and investigating the essences of mathematics learning. These can be used to evaluate and forecast teacher learning in design-based professional settings.

Although studies on TPD have a predictive orientation, the field in teacher education still lacks fundamental and comprehensive theories that can articulate and predict TPD outcomes appropriately across different professional settings. As suggested by Ponte (2009), this requires new theories about teacher education that can be used to design, evaluate and research processes of teacher education and development. We emphasise that design here does not only refer to planning and arrangement of professional programs by teacher educators, but also to an intervention approach of designing instructional tasks through which teachers have opportunities to improve their practice for better student learning of mathematics.

Various PME papers attempt to conceptualise TPD in terms of elements that can better explain, interpret and predict TPD. Sztajn, Campbell, and Yoon (2009) suggested that TPD should be designed, evaluated, and researched on the basis of four elements: goal, contexts, theories and structure. Goal involves the shared version of mathematics teaching and learning, understanding of mathematics knowledge for teaching, and equity and sense of self as a mathematics teacher. Contexts for TPD include curricular, participant background, teacher engagement in decision-making processes related to the intervention, participation attitudes (e.g., compulsory or voluntary), and the role of accountability in the community. With respect to theory, both teacher growth and instruction are involved. When structuring an intervention, there needs to be consideration of content and format to ensure how opportunities for learning are best organised and presented. Sztajn, et al. argued that conceptualisation contributes to a more careful examination of the fundamental aspects of TPD.

The prediction function also permits design, sustenance and evaluation of professional development programs on a large scale. Marrongelle, Sztajn and Smith (2013) made eight recommendations for the arrangement of large-scale, system-level implementation of professional development programs. They are

(1) to emphasise substance so that teachers have opportunities to engage in practising new content; (2) to enable teachers to create and adapt professional materials; (3) to design professional development programs utilising effective ways to organise learning experiences for mathematics teachers; (4) to build programs which provide a continuous and coherent set of experiences over an extended period of time; (5) to prepare and employ knowledgeable professional development facilitators; (6) to tailor to key role groups (e.g., department chairs, instructional leaders, school administrators and superintendents), ensuring that all understand the new content and practices; (7) to educate all stakeholders such as parents, politicians, school boards and so on; (8) to assess professional development programs continuously. These recommendations would ensure the successful implementation of high-quality professional development programs.

Research on Mathematics Teacher Educators' Education (MTEE)

Llinares and Krainer (2006) identified characteristics underlying research on mathematics teacher educators:

> Mathematics teacher educators' growth is viewed as a learning-through-teaching process supported by reflective practice – growth through practice – and the use of theoretical references generated in the reflection on professional development of mathematics teachers to think and offer explanation on mathematics teacher educators' growth. (p. 447)

They make reference to Zaslavsky and Leikin's (1999, 2004) three-layer action/reflection model, working contexts which allow different levels of autonomy in the development of mathematics teachers and teacher educators (Krainer, 1999), and Tzur's (1999, 2001) four-focus model for MTE development. Consideration of PME studies during the last decade points to what and how mathematics teacher educators learn. Ten research reports, one plenary address and one plenary panel paper were related to mathematics teacher educators (though three of these papers discussed MTEEs' views or dispositions and are not included in this review). Two discussion groups and one working session on mathematics teacher educators' knowledge were held in 2012, 2013, and 2014, respectively, reflecting increasing interest in research on mathematics teacher educators' knowledge.

Six of the nine papers in focus were classified as aiming to reveal or characterise mathematics educators' learning outcomes (what-oriented-questions), the others as aiming to explore or comment on mathematics educators' learning processes (how-oriented-questions). Concerning mathematics educators' learning outcomes, educative power and disposition of mathematics educators are identified as another two categories in addition to knowledge. There were two papers related to mathematics educators' knowledge. One concerned mentors, whose content knowledge, pedagogical knowledge and knowledge of students' cognition were tested as part of their learning outcome (Lin, 2007). The other paper reviewed the main issues investigated in online mathematics teacher education. Two categories were identified as a focus on analysing interactions among teachers, and a focus on teachers' professional development in online settings (Flores, Escudero, & Aguilar, 2014). These two issues can be treated as what mathematics teacher educators should know, and thus be classified as research on mathematics educators' knowledge. However, mathematics educators' knowledge did not extend to mathematical knowledge for educating in PME papers. Thus, Beswick and Chapman (2013) initiated a discussion of mathematics teacher educators' knowledge in 2013, followed by a working session in 2014 (Beswick, Goos, & Chapman, 2014).

Two papers investigated what can be learned from mathematics teacher educators' design, implementation, reflection and revision of their instruction, while one paper investigated mentors' approaches to intervening in the mathematics teaching of trainee teachers. Mathematics teacher educators' or mentors' approaches include metacognitive awareness and discussion (Kalogeria & Kynigos, 2009), documentational work motivated by fieldwork activities (Psycharis & Kalogeria, 2013), and interventions in (trainee) teachers' teaching (Wang & Chin, 2007).

In the remaining four papers concerned with mathematics educators' learning process, three categories of learning process were identified: understanding mathematics education research and practice; cooperatively solving pedagogical and educative problems; and participating in mathematics education research and practice.

Concerning the first category, Rhodes (2009) examined MTE's 'disequilibrium' while observing, analysing, and discussing a mathematics content class for preservice teachers. He found that participants who experienced disequilibrium were analytical in their thoughts and struggled to reconcile their own teaching experiences with their observations. Thus, experiencing disequilibrium is a promising approach to educating MTEs.

Two papers address cooperatively solving pedagogical and educative problems. Reflecting on mathematics education research and its interrelation with mathematics teachers, Krainer (2011) concluded that researchers cannot transmit knowledge directly to practitioners, and proposed viewing researchers as stakeholders in practice and teachers as stakeholders in research as a way to increase the further development of both parties through collaboration. From this point of view, it appears that teachers and teacher educators can mutually support each other to solve pedagogical (how to teach) and educative (how to learn to teach) mathematics problems.

The other paper (Erbilgin & Fernandez, 2011) focused on how one university supervisor (mathematics teacher educator) supported mathematics teachers (mentors) to solve an educative problem, that is, how to mentor student teachers. They found that a program based on educative supervision developed the supervisory knowledge of the mentor and changed the mentor's style of supervisory practice. This study demonstrated how an educative problem can be solved through researchers as stakeholders in practice and teachers as stakeholders in research.

The fourth paper (Liljedahl, Williams, Borba, Krzywacki, & Gebremichael, 2013) discussed the education of young mathematics education researchers, proposing that mentorship is required for them to develop a professional identify as scholars in their field. The issues related to mentoring young researchers beyond supervision were discussed. In particular, Liljedahl proposed that "there is room for, and need of, more explicit and active mentorship of our young researchers within our organization" (pp. 1–90). This implies that the learning of young researchers is viewed as participation in an academic community. That is, we shift our focus away from the individual acting on the world and onto the individual acting in the world (Lave & Wenger, 1991), so young researchers may move from peripheral to full participation in the (PME) community.

FINAL REFLECTIONS

We conclude with some thoughts arising from our survey of PME research on mathematics teacher knowledge and professional development in the decade since the previous overview.

Concerning mathematics teacher knowledge. Interest in mathematics teacher knowledge both within and beyond PME shows no sign of abating at the present time. In our survey we considered PME research concerning: theories of mathematics teacher knowledge; elaboration of mainstream theory; growth of mathematics teacher knowledge; and three particular aspects of mathematics teacher knowledge. Through Research Fora, Plenary Presentations and particular Research reports, mainstream theories of MTK have been thoroughly promulgated, elaborated and exemplified. There is scope for more effort to look for common ground, or complementarity, in the available theories, and PME is an ideal potential forum for doing so in an interactive and collegial context. Fundamental 'paradigmatic differences' between individual/cognitive and situated/social perspectives on MTK remain unresolved, and are perhaps unresolvable (in the sense of reflecting different world views). Most (but not all) theories of MTK naturally follow the lead of Shulman in identifying categories – of kinds of knowledge, or of situations in which it is manifested. The recent trend towards attempting to delineate the boundaries between such categories is interesting, even if potentially futile, but the interdependence of different aspects of knowledge also merits further study.

The theoretical understanding of MTK is intimately linked to designing efforts to promote its growth, and the papers reviewed present several fruitful approaches, of which structured reflection in a (teacher) learning community seems to be especially powerful. Indeed these are characteristics of the lesson study approach to the improvement of teaching and teacher knowledge; we can expect, and welcome, further investigation of the transfer of lesson study to diverse cultural, curricular and praxis contexts. Likewise, a distributed notion of MTK would recognise the crucial contribution of Educative Curriculum Materials to a common-wealth of professional knowledge, and more work can be expected towards theorising and investigating the role of ECMs as a component of MTK, and a stimulus for its development.

Concerning teachers' professional development. This review raises two substantial issues concerning the design, evaluation and investigation of TPD.

The first issue concerns the development of more fundamental and comprehensive theories to better describe, interpret and predict TPD in professional settings. Those studies orienting to culturally and socially situated learning perspectives attempt to articulate TPD in terms of becoming a member of a certain community in which they gradually learn the ability to communicate and act according to its particular norms (Cobb, 1992; Cobb, Yackel, & Wood, 1992; Yackel & Cobb, 1996). However, such studies might be limited in terms of elaborating TPD across different professional settings with different teacher backgrounds and populations. By contrast, studies on TPD orienting to the cognitive and psychological perspective do not consider how teachers appropriate sources of information, and how others such as teacher-colleagues or students play a role in influencing TPD. Developing fundamental and comprehensive theories that embrace both social and cognitive perspectives for better elaborating TPD becomes the emergent issue in teacher education research, along with the identification of fundamental and comprehensive theories to underpin the arrangement and implementation and evaluation of large-scale professional development programs across different mathematics content, teacher attributes and cultural characteristics.

The second issue is about how teachers can learn effectively. Teachers' learning via teaching, researching, and participating in learning communities have been reviewed in this chapter. Teachers' current understanding imposes limits on what teachers can learn from their teaching (Simon, 2007). A design-based community which integrates research and participation in a learning community can better facilitate teachers' learning. The studies of Lin et al. (2012, 2013) point to several requirements for developing such a design-based community: First, to develop a way to link research and practice perspectives in the program. As discussed earlier in this review, Lin et al. suggest a three-layer structure including grand theory, intermediate framework and a design tool (Ruthven, Laborde, Leach, & Tiberghien, 2009) for the design of professional programs. Secondly, to engage teachers in designing instructional tasks and to detect their pedagogical challenges, formulate instructional strategies to overcome these challenges, and then to test whether the strategies are useful or not in interaction with classroom students. In order to facilitate teachers to design tasks, Lin et al. (2102) propose three starting points: student misconceptions, standard 'results' in school mathematics, and engaging with student conjectures, each of which allows teachers to create tasks more easily. Thirdly, to develop strategies for enabling teachers to incorporate theoretical ideas into their design of instructional tasks. The adaptation of the above three considerations could be taken into account for future research on designing TPD in various contexts.

Concerning mathematics teacher educators' education. In general, papers relating to teacher educators' learning paid more attention to learning outcomes than to their learning processes. From our review, 'mathematical knowledge for educating' – the knowledge of mathematics teacher learning or principles of designing educative tasks – has not been well structured. Ideas about investigating mathematics teacher educators' competencies originated in research on mathematics teachers. Nonetheless, mathematics teacher educators' goals, resources and orientations are different from those of mathematics teachers (Schoenfeld, 2011), in addition to their action, reflection, autonomy and networking. Moreover, mathematics educators' power in communicating with teachers and reasoning for solving educative problems and connecting research and practice are less investigated (Yang Hsu, Lin, Chen, & Cheng, 2015). As for mathematics educators' disposition, affective factors are seldom considered.

The meanings and goals of research and practice are different for mathematics teachers and teacher educators, but there can be synergy between them (Krainer, 2011), and there is potential in developing mutually-supportive communities involving both groups.

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NOTES

[1] A published list of research domains, or categories, enables authors to indicate the substantive focus of their research report submissions, and reviewers to indicate their substantive expertise. These domains are reviewed from time to time by the PME International Committee.

[2] A similar search of the 2015 PME proceedings (identifying 27 additional papers for scrutiny) was undertaken after submission of the first draft of this survey, and is reflected in its content.

[3] Of the 530 papers remaining after the one-page contributions had been eliminated (as described), none were from France, and so a distinctive 'didactique' perspective is necessarily absent from this survey.

[4] Guides for Accessing Mathematical Understanding for Teaching.
[5] See also Zaslavsky and Sullivan (2011).

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