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4.8. LEARNING TRAJECTORY

Linear Equations

INTRODUCTION

Iteration has emerged as one of more important methodological processes within the environment of evidence-based Common Core standards. Its importance increases together with the goal to formulate effective student learning trajectories, that is, those theoretical pathways of learning mathematical concepts that come closest to actual student learning. The following definitions of "iteration" from Merriam-Webster and Oxford dictionaries refer explicitly to the successive approximations to a desired solution of the problem. The Merriam-Webster Dictionary defines "iteration" as "a procedure in which repetition of a sequence of operations yields results successively closer to a desired result." The Oxford English Dictionary provides a similar definition emphasizing the term's mathematical undertones: "A repetition of a mathematical or computational procedure applied to the result of a previous application, typically as a means of obtaining successively closer approximations to the solution of a problem."

Educational research needs iteration in order to formulate, refine and tune learning trajectories from a collection of fragmented and diverse research results concerning the concepts in question. For example, Confrey's formulation of the "equi-partitioning learning trajectory" relies on 600 different research pieces (Confrey, 2010). To transform such a large amount of research results into a smooth working teaching sequence facilitating student understanding and mastery of a given concept requires the successive approximation approach to revamp, change and improve the components of the teaching sequence while at the same time creating smooth connections between them.

The iteration methodology used by teachers in the construction of effective teaching sequences is very natural because of the cyclical nature of the teacher's workload assignments (Wittmann, 1999). Teachers can, and often do, teach the same course from one semester to another, or from one academic year to another, creating an environment in which any teaching sequence of a given concept can be iteratively refined over several application cycles. The integration of this natural cycle of work with the teaching-research cycle (TR cycle) discussed in the Chapter 1 creates an extremely powerful methodological tool tailor-made to address the complex question of *learning trajectories*.

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 373–393.

Often, when the current authors' work based on the TR cycle is presented to an audience of educational researchers, the most common question is "What is the difference between your cycle and the design research cycles?"

The difference is subtly profound. The standard design research cycle as well as the APOS theoretical framework cycle (Asiala et al., 1996), that served as the formative basis of the TR cycle created by the current authors, initiate from theoretical models, infer theoretical results, and, then, apply these models to the classroom setting. The TR cycle, on the other hand, starts most often from practice in a particular classroom setting, and its aim is the improvement of learning and related teaching in the very same classroom, and beyond. The theory here is a by-product of iterated practice, and it's not the main objective. Although seemingly insignificant, this change of the starting position for iterated investigation results in significant changes in the research methodologies. Table 1 presents a sample side by side comparison between the methods, aims and results of standard academic research versus the classroom-driven TR model.

LEARNING TRAJECTORIES

The concept of a *Learning Trajectory* has acquired recently new importance as the organizing principle of the new Common Core Standards in Mathematics (CPRE, 2011). There are several definitions of a "learning trajectory" within the research profession (Baker et al., 2012) indicating that the concept didn't yet "condense" (Sfard, 1992) sufficiently in its development. Therefore, one has a certain amount of freedom in focusing one's own investigation on different aspects of the construct. For the purpose of this work the authors adopt Clements' definition:

The learning trajectory (LT) of a particular mathematical concept consists of three components:

- A specific mathematical goal,
- A developmental path along which students' thinking and comprehension develops and,
- A set of instructional activities that help students move along that path (Clements & Sarama, 2009).

The idea of LTs has a wide range of applications. It can be an excellent assessment tool precisely informing the teacher about the successful pathways of mathematical thinking of his or her students as well as about their weaknesses. At the same time, it can serve as a tool, a map or a guide constructed, preferably, by the teacher and for the teacher, providing information about possible trajectories for learning improvement strategies, asked for explicitly by the designers of the approach (Figure 1, Center, Daro et al., 2011). Active implementation of the LT framework in the development of curriculum facilitates intense discussions about the effectiveness of the relationship between abstract research and practicing teachers toward the support of the Common Core effort. "Whose responsibility

Table 1. Comparison of standard academic research and the TR model

Standard research (Design-Based Research) model	*TR model (TR-NYC model)*
Theory-driven:	*Practice-driven:*
"Design-based research can contribute to theoretical understanding of learning in complex settings" (Sandoval, p. 00). Each of the articles by Sandoval, Tabak, and Joseph reveal how the design of complex interventions is an explicitly theory-driven activity.	Teaching-research is grounded in the craft knowledge of teachers that provides the initial source and motivation for classroom research; it leads to the design-based practice and, the primary aim is the improvement of learning in the classroom and beyond.
"In addition, the design of innovations enables us to create learning conditions that learning theory suggests are productive, but that are not commonly practiced or are not well understood" (Author, 0000)	The design of innovation enables the teacher-researcher to establish a creative learning environment based on teacher's craft knowledge that improves learning in the classroom and transforms students' habits (such as misconceptions) into student originality (Koestler, 1964). Learning theories are used as needed to support teachers' craft knowledge. (Prabhu & Czarnocha, 2006)
Cobb and Steffe (1983) assert that the interest of a researcher during the teaching experiment in the classroom is "in hypothesizing what the child <u>might</u> learn and finding [as a teacher] ways and means of fostering that learning".	"…the interest of a teacher-researcher is to formulate ways and means to foster what a child <u>needs to</u> learn in order to reach a particular moment of discovery or to master a particular concept of the curriculum (Czarnocha, 1999)". Since, however, "such moments occur only within students' autonomous cognitive structures, the [constructivist] teacher has to investigate these structures during a particular instructional sequence [in order to be of help to the students]. In this capacity, he or she acts as a researcher" (Prabhu & Czarnocha, 2007)
Articulating, refining and validating is an "iterative process of research synthesis and empirical investigations involving" many types of evidence: Step 1. Meta-research of the concept to create the prototype; Step 2. Iterative refinement of the prototype. (Confrey, 2010)	Use of iteration in the TR-NYC model: Step 1. Process of iteration starting with the first iteration designed on the basis of teaching practice. Step 2. Incorporation of research results as needed in between consecutive iterations. It is the concept of iteration of the design from semester to semester together with the related refinement that can allow for the immediate implementation of the naturally relevant research results illuminating the current classroom situation and providing further insight into the design of appropriate sets of assignments.

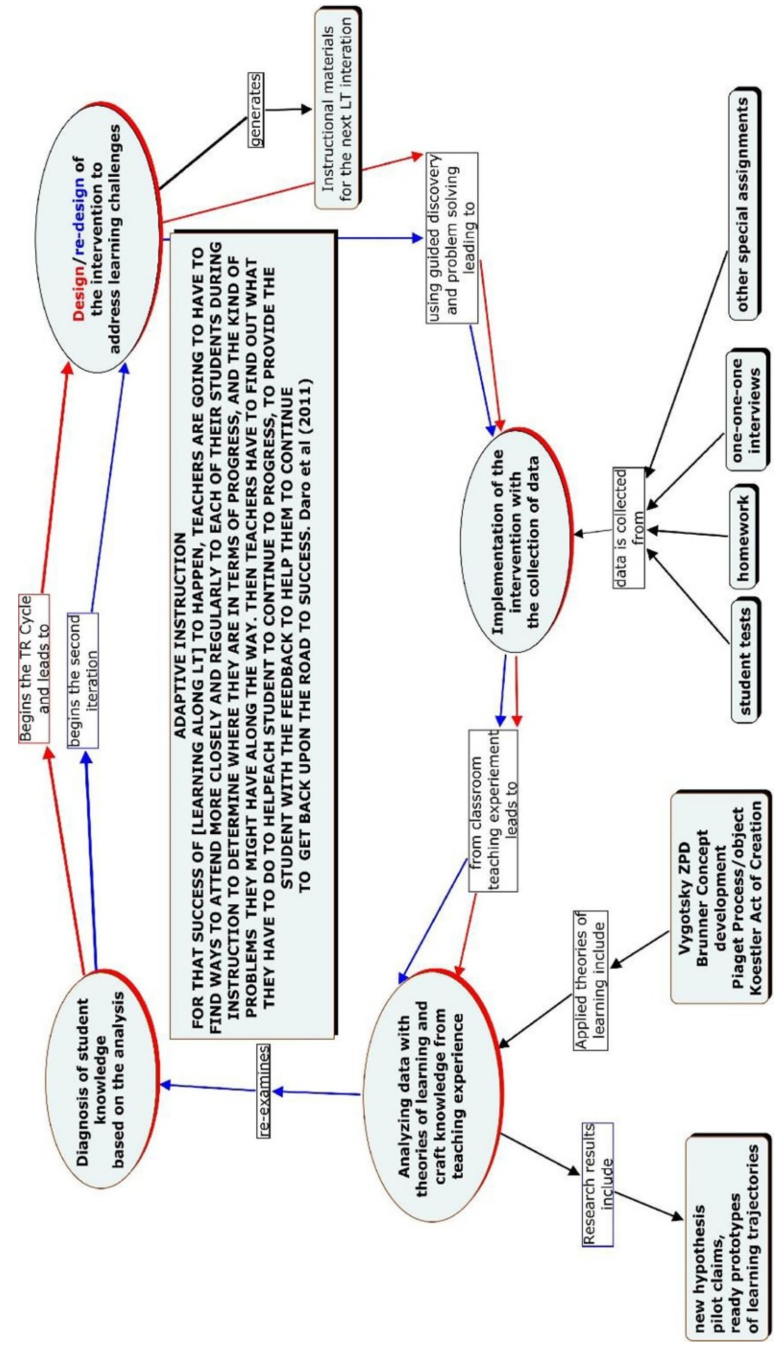

Figure 1. Teaching-research cycle

is it to construct learning trajectories?" asks Steffe (2004, p. 130). Battista (2004, p. 188) states, "to implement instruction that genuinely and effectively supports student construction of mathematical meaning and competence teachers must not only understand cognition-based research on students' learning, they must also be able to use that knowledge to determine and monitor the development of their own students' reasoning." Empson (2011) adds a layer of complexity to the current research on learning and invites one to think seriously about how to support teachers to incorporate knowledge of children's learning into their purposeful decision-making about instruction. Clements and Sarama (2004, p. 85) note, "that learning trajectories could and should be re-conceptualized or created by small groups or individual teachers, so that they are based on more intimate knowledge of the particular students involved…"

Thus, in agreement with Kieran, "it is [only] the teacher who can affect to the greatest extent the achievement of one of the main purposes of the research enterprise, that is, the improvement of students' learning of mathematics" (Kieran et al., 2013). Therefore, the search is on for the most effective routes of joining educational research with classroom teaching (Kieran et al., 2013). Kieran also addresses the variety of differences shared by researchers and teachers that make collaboration challenging (Kieran et al., 2013). It makes sense, therefore, to focus on what is common between researchers and teachers involved in classroom teaching-research. Our assertion is that the concept of iteration as a component of the research methodology is common to both.

THE METHOD OF ITERATION

This presentation is focused primarily on the methodological aspects of the proposed route of research/teaching integration showing an essential methodological trade off necessary (though not sufficient) for teachers' buy-in in the LT approach. The discussion describes the method of iteration for learning trajectories during the process of their research-based construction (Confrey & Maloney, 2010). The TR cycle of the TR-NYC model (Czarnocha & Prabhu, 2006) is the theoretical framework within which iteration is effectuated in classroom teaching-research. Two consecutive examples of the process are presented for the Learning Trajectory for Linear Equations (LTLE) under construction in the context of the Integrated Arithmetic/Algebra Course Teaching-Experiment being conducted at present at an urban community college.

The desired goal is the sequence of instructional problems and strategies that produces the most optimal effective understanding and mastery of the relevant mathematics (linear equations, in this case) in the classroom. Each new iteration of the teaching sequence is produced at the analysis of the data node of the TR Cycle through its major or minor refinement. The refinement may consist in the change of component strategies, their sequencing or the changes in learning environment. The changes are suggested by the analysis of learning in the previous cycle, the craft knowledge of the teacher-researcher as well as through the relevant research results.

The Iteration Trade-Off

Since, generally, every teacher has an option of teaching the same course every semester to a new cohort of students, the TR cycle allows for the continuous process of classroom investigations of the same research question during consecutive semesters or academic years. The TR-NYC model asserts that two such consecutive cycles constitute a single unit of activity explicitly aimed at the improvement of learning (Czarnocha & Maj, 2008). Two cycles are needed to enable the refinement of the particular LT from one iteration to the next. A methodology for construction and validation of a learning trajectory had been thoroughly described by Confrey and Maloney (2010) in the case of the Equi-partitioning Learning Trajectory. According to Confrey and Maloney, articulating, refining and validating is an "iterative process of research synthesis and empirical investigations involving" many types of evidence. Their research sequence starts with the significant research effort in the design of the first prototype. The iterative process is the second step of the research.

Within the TR-NYC model, the iteration becomes the primary methodological tool, while the initial learning trajectory is designed more on the basis of the teaching craft knowledge of the mathematics teacher than on the basis of the relevant research results. The fine tuning of the learning trajectory to the needs of the student cohort through the incorporation of the research knowledge into the design process takes place during the consecutive iteration phases while fulfilling the requirements of adaptive instruction (Daro et al., 2011). It is the concept of iteration of the design from semester to semester together with the related refinement that can produce relevant research results illuminating the classroom situation or providing help in the design of an appropriate set of assignments.

Thus the initial theoretical period of gathering available research required for standard research is not necessary for the classroom teacher-researcher designing learning trajectories because it can be transformed into its "just-in-time" utilization at each refinement node of the TR cycle. The "just-in-time" manifestation occurs along the iteration cycle. This change of emphasis in the role of research as the starting point of investigation to its "just-in-time" consultation is one of the necessary conditions for the incorporation of research into classroom practice.

ADAPTIVE INSTRUCTION

The process of iterative refinement of the teaching sequence associated with a given learning trajectory introduces, in a natural manner, a new type of instruction that adapts itself to students' state of knowledge. It's a promising concept in that it has an application to every student in the class and, thus, it ideally accounts for learning for all students. The process of adaptive instruction outlined by Daro (2011) corresponds to nodes of the TR cycle. For example, "the determination where students are in their progress and the kind of problems they might have along the way" (Daro et al., 2011) corresponds to the Diagnosis node of the TR cycle; "finding out what to do to help students to continue to progress" (Daro et al., 2011) corresponds to Design/Redesign node of the intervention to address learning challenges; providing "students with the feedback to help them to get back upon the road to success" (Daro et al., 2011) corresponds to the Data Analysis nodes followed by the Diagnosis node again, and next the Redesign node. Thus, if there is a need to help students with their immediate problems, the TR cycle may be traversed a couple of times within one class. The paradigmatic example in Chapter 4.1 is a good illustration of several TR cycles taking place within a short classroom dialogue lasting only several minutes. This unity of research investigation and adaptive teaching is possible through the development of thinking technology within the practice of the teacher-researcher touched upon in Chapter 4.1.

CONSTRUCTION OF A LEARNING TRAJECTORY

The construction of a learning trajectory for linear equations through three iterations, demonstrated below, provides an illustrative example of the method.

The Learning Trajectory for Linear Equations (LTLE) has been designed on the basis of algebra classroom teaching craft mathematical knowledge of the teachers and triangulated with the Learning Trajectories Display of the Common Core State Mathematics Standards developed by Confrey et al. (July 2010). The design of LTLE is the adaptive response to the observed challenges of students with the following problem:

Solve for y in terms of x:

$$3x - 2y = 6 \tag{1}$$

Students' recorded solution:

$$\begin{array}{r} 3x - 2y = 6 \\ -3x \qquad\quad \\ \text{- - - - - - - - - - - - - -} \\ -2y = 6 \\ y = -3 \end{array} \tag{2}$$

The First Iteration LTLE, pictured in Figure 2, was designed to respond specifically to student difficulties described above. It outlines the necessary prerequisite and sequential knowledge to understand the central concept "solve for x in terms of y" as well as new concepts dependent on that understanding. The concept map is designed in the environment of the Institute for Human and

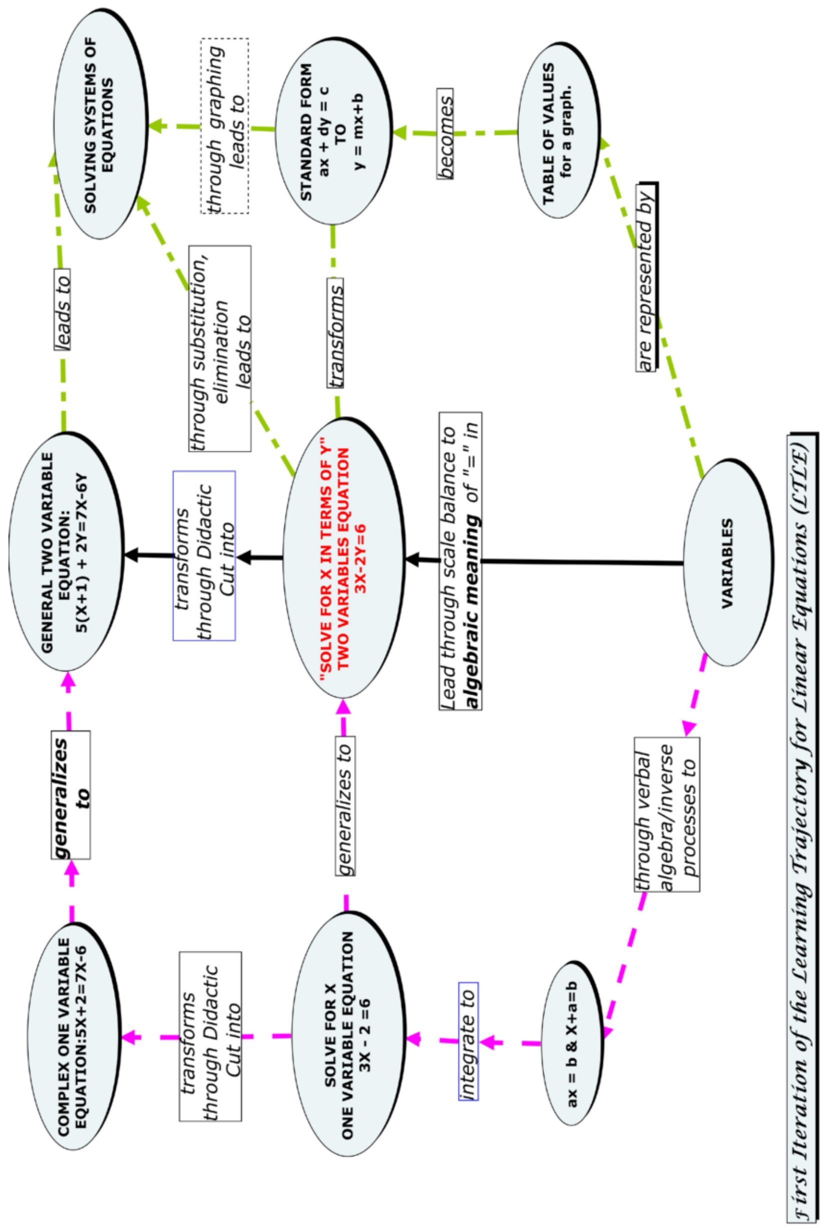

Figure 2. First iteration concept map demonstrating the three component trajectories associated with the learning trajectory for linear equations

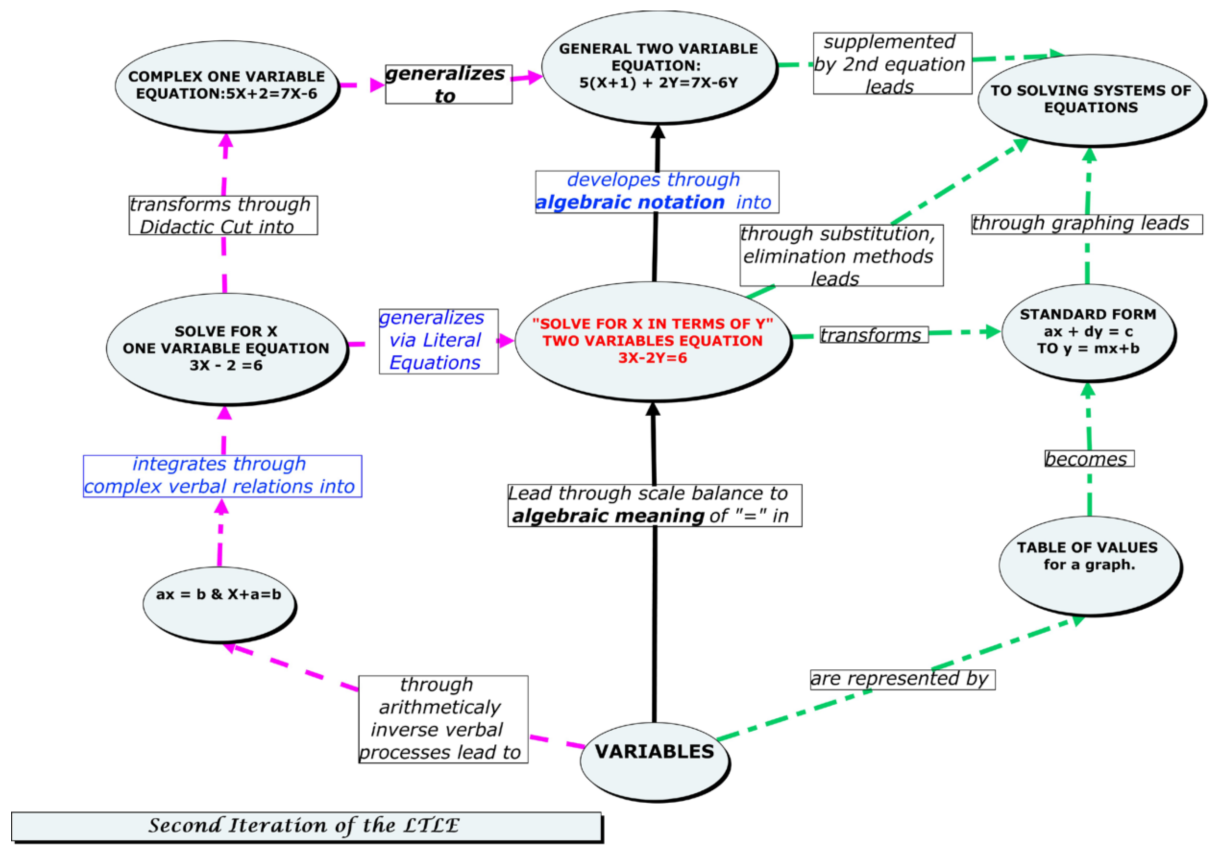

Figure 3. The revised concept map: Second iteration of the learning trajectory for linear equations

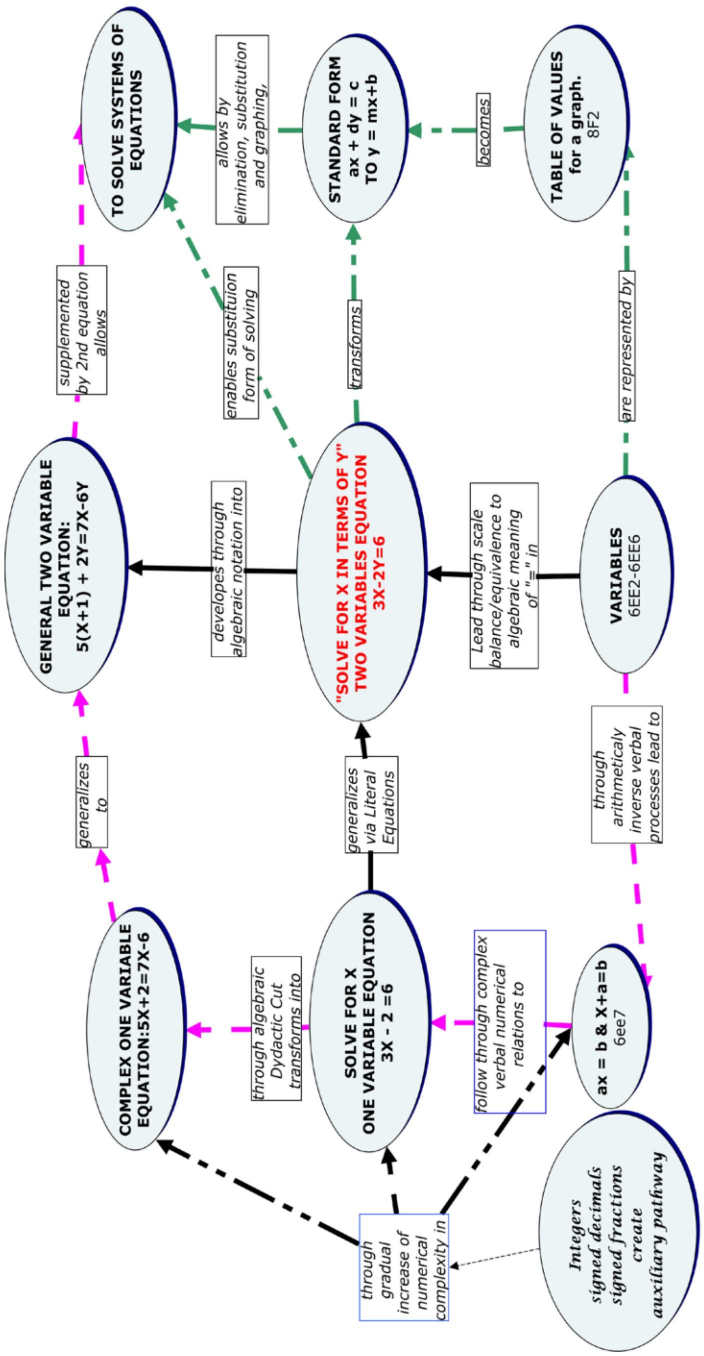

Figure 4. The revised concept map for the third iteration

Machine Cognition CmapTools at http://cmap.ihmc.us/. The oval shaped components represent the concepts, or mathematical objects, that are joined by propositions describing relationships between them. The concepts "solve for x" and "solve for x in terms of y" represent encapsulated or reified procedures.

Teaching-Research Diagnosis

The reasons for the erroneous solution include (a) absence of awareness of the functional relationship between the variables x and y, evidenced by transforming the problem to a simpler equation with one unknown leading to (b) misapplication of the variable as a specific unknown, (c) the absence of understanding the algebraic meaning of the equality symbol "=" evidenced by adding "$-3x$" to one side of the equation only, and, finally, as it was demonstrated by the teacher-researcher Vrunda Prabhu, (d) careless reading. The LTLE consists, therefore, of three separate but connected learning trajectories of (i) the variable as an unknown (broken arrow - - - - -► (pink) in Figure 2 above), (ii) the variable as a general number (black in Figure 2 above) and (iii) the variable in a functional relationship (broken arrow –-·–-► (green) in Figure 2 above) (Ursini & Trigueros, 2011).

The three component trajectories of the LTLE just discussed are shown in different colours on the first iteration concept map above (see Figure 2). The pink - - - - -► one leads along the process of generalization, from a formally similar equation in one variable to a corresponding equation in two variables. This trajectory is useful if the class has mastered solving simple one variable equations. Otherwise, the second trajectory, shown in –-·–-► (green), is available via the graphing component of the schema, that connects the challenge of the problem with its foundations within the concept of a variable, meaning of equality and the functional relationship between x and y. The cognitive fragility of the left upper rectangle in the concept map is well-known in the literature. Filloy and Trojano, for example, observe that the increase of algebraic content along the pink vertical arrow intersecting this rectangle is a serious problem for students because the solution of the more complex target equation departs from that of simpler equations such as $4x + 2 = 6$ (Filloy & Trojano, 1989; Ursini & Trigueros, 2009). The simpler linear equations enjoy more accessible arithmetic interpretations. Filloy and Trojano (1989) coined the term "Didactic Cut" to refer to the associated cognitive step. The two horizontal pathways indicate abstraction from and the generalization of a one-variable equation to a two-variable equation – an arduous process according to many investigations focused on problems that students have with generalization as they begin to study algebra in middle school. Most studies conclude that generalization is a difficult obstacle for the majority of these students (Bell & Malone, 1993; Arzarello et al., 1994; Bednarz & Janvier, 1994; Radford & Grenier, 1996; Bolea et al., 1998a, 1998b). The alternative graphing trajectory, shown in dark grey (green), develops the concept of "solving for y in terms of x" through transformation of a standard form of an equation into a known functional relationship $y = mx + b$.

The third component trajectory, shown in black, joins the concept of the variable as an unknown to the discovered difficulty along the theme of algebraic equality "=" through a series of "scale balance" type of problems. The assumed equilibrium of the scale in such problems is the metaphor for algebraic equality "=". The possibility of distinguishing three different learning progressions within the concept map demonstrates the versatility of such an integrated concept map/learning trajectory for classroom teachers and its usefulness in addressing diverse learners. According to (Ursini & Trigueros, 2009), the best, flexible development of the schema of the variable is to engage, in coordination, the three subschema: (1) variable as a specific unknown, (2) variable as a general number, (3) variable in a functional relationship. This implies the use of all component trajectories, because all three sub-schema are involved in the problem.

Instructional Sequences for the First Iteration

Here, we provide two small instructional sequences, which were used in the design of the first iteration.

We begin with the Teaching Sequence of Mathematical Activities that are meant to propel a student along the pink trajectory of generalization. The trajectory uses a "writing mathematics approach" to increase the meta-cognition and reflection upon the methods of solution. The aim of this sequence is to lead the student in the direction of development of generalization from a simple equation in one variable to the corresponding equation in two variables. The idea is to focus student's attention on the similarity of the solution procedure for one variable to the solution procedure for the task of "solving for y".

Problem 1

Solve for x. As you solve write every step you make in the solution. Look at the three descriptions, collect similar actions in the three examples and write them as one set of steps that apply to all three problems.

(1a) $2x + 7 = 15$

(1b) $-4x + 8 = -28$

(1c) $5x - 3 = 12$

My general set of steps is ____

____

____

Problem 2

Look at the following three examples that are similar but different from the previous set, and <u>solve for x in terms of y</u> by applying your general set of steps from Problem 1 to these three equations. Write your steps carefully and keep careful track of their order.

(2a) $2x + y = 15$
(2b) $–4x + y = –28$
(2c) $5x – y = 12$

Problem 3

Now, solve for y in terms of x (note the change of the instruction) by applying your general set of steps to these three equations. Write your steps carefully and keep careful track of their order.
(3a) $2x + y = 15$
(3b) $–4x + y = –28$
(3c) $5x – y = 12$

Write the general description of steps for the instruction "Solve for y in terms of x"____

Problem 4

Solve for y in terms of x:

(4a) $4x + 2y = 12$
(4b) $6x – 3y = 15$
(4c) $–2x + 3y = 15$
(4d) $–2x + 3y = 15$

What is the critical computational difference between the last two and the first two problems?

____

Instructor's Notes: *The role of Problem 1 is to introduce the solution procedure for a simple and familiar case that consists of subtraction of a number from both sides followed by the division of the result. The role of the Problem 2 is to expose students to the variation in the procedure when an integer from the Problem 1 set is changed into the second variable, y. Problem 3 changes the task from "solving for x" to "solving for y"; students are expected to transfer the procedure from Problem 1 and Problem 2 accounting for the change. In the second iteration, problems (3b) and (3c) were changed from* $–4x + y = –28$ *to* $–4x + 2y = –28$, *and from* $5x – y = 12$ *to* $5x – 2y = 12$, *respectively. The aim of that change was to incorporate the division by the numerical coefficient of the variable y. Two examples of the type are needed to indicate the difference between answers using only integers and those using fractions. Fractions are one of the main obstacles students experience en route* *to algebraic thinking.*

Using the Scale Balance Manipulative: Reinforcing the Meaning of the Algebraic "=" and Extending the Method across the Didactic Cut (Filloy & Trojano, 1989)

The details of the teaching sequence meant to develop the idea of algebraic equivalence are presented here.

A) Solve the equation by removing weights from the scale in such a way so that the scale remains balanced (at an equilibrium). Describe the steps you are taking to keep the scale balanced.
B) Solve the equation algebraically by the Equivalence Principle.
C) What other equivalent equations can you make out of this one?

____

____

D) Solve for x:
$0.75x + 0.5 = 2$
E) Solve for x:
$$\frac{1}{3}x + \frac{2}{3} = \frac{5}{3}$$

The Didactic Cut
A) Solve the equation by switching the weights from one side to another in such a way so that the scale remains balanced (at an equilibrium). Describe the steps you are taking to keep the scale balanced.
B) Solve the equation algebraically by the Equivalence Principle.

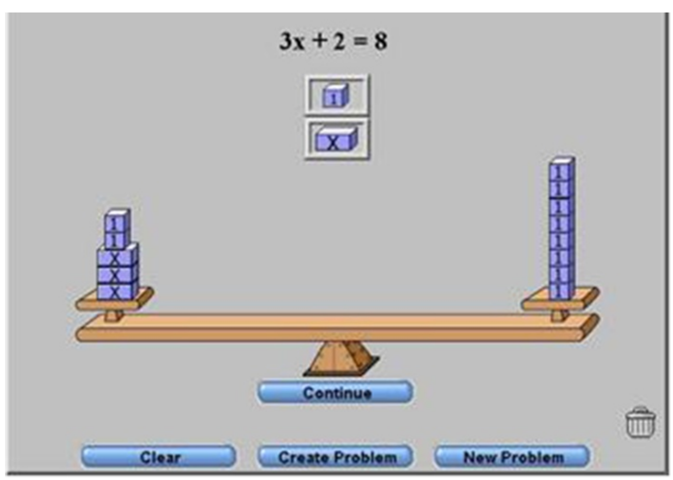

Figure 5. The scale balance manipulative I

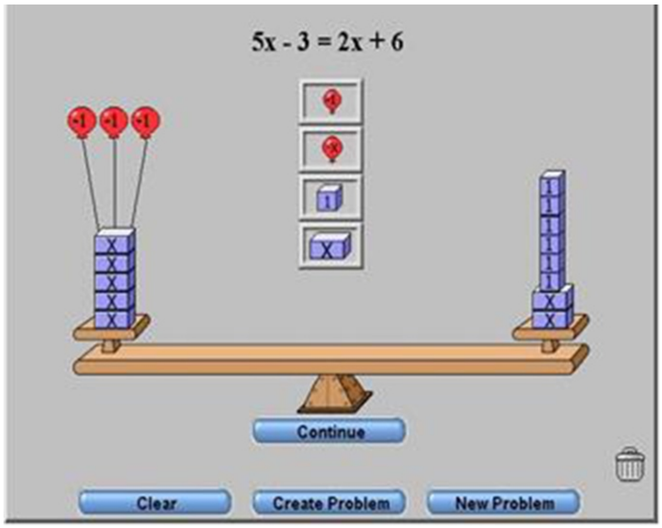

Figure 6. The Scale Balance Manipulative II

C) What other equivalent equations can you make out of this one?

D) Solve for x:
$5.2x - 3.6 = 2.2x + 6.4$

E) Solve for x:

$$\frac{3}{4}x + \frac{3}{8} = \frac{5}{4}x - \frac{5}{8}$$

Instructor's Notes: *Each of the Scale Balance problems starts from the concrete problem that can be solved by changing the weights while keeping the balance at equilibrium followed by the request to solve the same problem algebraically. Description of the steps is intended as the transition to algebraic operations followed by the reinforcement of the Equivalence Principle. Finally, the practice of technique is extended to decimal and fractional numerical coefficients, a well-known Achilles heel of remedial students of mathematics.*

The Second Iteration

The teaching experiment leading to the second iteration had been conducted during the fall 2012 semester at Hostos CC. Analysis of the results of the implementation of the first iteration along with observed student difficulties suggested the following needs:

1. A development of an auxiliary trajectory of algebraic notation;
2. An increase in the complexity of numerical coefficients from integers to signed decimals and signed fractions;
3. A much stronger emphasis on the discovery of numerical relations, and
4. Introduction of literal equations as the scaffold for the procedure "solve for *x* in terms of *y*".

The refinements (1), (3) and (4) are indicated in blue in the Second Iteration concept map (see Figure 3). The need to emphasize numerical relations as the background for algebraic problem-solving suggested a new point of view for the entire curriculum of the Arithmetic/Algebra course. Until this moment the curriculum was based solely on the generalization/particularization relationships between arithmetic and algebra. The new point of view has been provided by the discussion of the curriculum of V. Davydov (Jean Schmittau & Anne Morris, 2004), that takes mathematical relation as the foundation of the approach. The curriculum of the course then became a composition of two principles: generalization (algebraic expressions, polynomials, rational functions) and algebraic relation underlying theory of equations and functional relationships.

Example of Exercises, Which Focus Attention on the Numerical Relationships

The design follows the idea that a process and its inverse reinforce the reflective abstraction, and, hence, the development of the concept; in this case, the concept of the numerical relationships.

Problem1. Translate the verbal statement into an algebraic one:

(1a.) Twice a number is equal to 16 □ ____________________________________

(1b.) 0.5 of a number is equal to 10 □ ____________________________________

(1c.) Twice the number increased by 5 is equal to 11 □ ______________________

(1d.) The negative of twice the number decreased by 8 is equal to negative 4 □ ____________________________________

Problem 2. Express the relations between indicated pairs of numbers verbally:

- Two numbers are related additively if they are related by addition "+"
- Two numbers are related multiplicatively if they are related by multiplication "×"
- Two numbers are related additively and multiplicatively if both addition "+" and multiplication "×" are involved.

(2a) What is the additive relation between the numbers 4 and 15?
(2b) What is the multiplicative relation between the numbers 4 and 15?

(2c) What is the additive relation between the numbers –4 and 15?
(2d) What is the multiplicative relation between the numbers –4 and 15?
(2e) What is the additive relation between the numbers –4 and 15?

Instructor's Notes: *Note that the two problems above are "quasi" inverse processes of each other*: (i) *verbal statement* - - -▸ *algebraic relation, and* (ii) *numerical relation* - - -▸ *verbal relationship. In addition, the second iteration contained a component addressing "literal equations" as a scaffold for the "solve for y" task.*

The Third Iteration

The central improvement for the third iteration was to significantly increase the impact of the "algebraic relations" approach. This resulted in grounding the whole lower half of the trajectory in algebraic problem-solving (see Figure 4). This, in turn, leads up to the algebraic solution methods of systems of simple equations with two unknowns. Inclusion of Davydov's ideas is an example of "just-in-time" employment of new learning theory and related research results. After this basis has been established, the instruction along the upper half of the trajectory readily follows. The "scale balance" manipulative had been taken away for two reasons:

- It didn't make much of an impact on student understanding of the equivalence principle;
- The public software is not sufficiently developed to imitate the algebraic procedure of solving such equations.

Instead, a small algebraic teaching sequence had been designed employing, once again, the process and its inverse method. It is presented below.

Problem 1. Decide which of the pairs of equations below are equivalent and explain the reasons for your decisions?

(1a) E1: $x - 5 = 3$ E2: $x - 5 = 3$

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____

(1b) E1: $x - 5 = 3$ E2: $x + 2 = 11$

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(1c) E1: $x - 5 = 3$ E2: $x = 8$

____

____

(1d) E1: $3x = 9$ E2: $6x = 12$

____

____

(1e) E1: $3x = 9$ E2: $9x = 27$

____

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Problem 2. Each of the two columns below contains a triplet of equations. Is the first equation in each column equivalent to that column's last equation? Explain the reasons for your answers

(A)	(B)
$2x – 6 = 12$	$2x – 6 = 12$
$2x = 18$	$4x –2 = 24$
$x = 9$	$4x = 36$

Conclusion: In order to solve the equation of the type $ax + b = c$ we need to _____

____

____

____

Instructor's Notes: *The problems above require use of the equivalence principle to decide whether the pairs of equations are equivalent. This way the role of the principle is clarified and then it can be applied in the context of a standard set of problems where the principle is used to obtain solutions.*

CONCLUSION

This chapter presents a work in progress. Our aim here has been to demonstrate the process of constructing a formal learning trajectory and to show that a teacher in the classroom can accomplish it. The assessment was primarily done through class observation, results and difficulties of students in their homework assignments and tests. As soon as we arrive at the learning trajectory we are intuitively satisfied with, we will establish more precise assessment measurements and extend their application to other sections of the course led by different instructors. The presence of the teaching-research community in the school described in the Unit 5 is central in the process of tuning and applying the trajectory beyond the initial classroom.

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