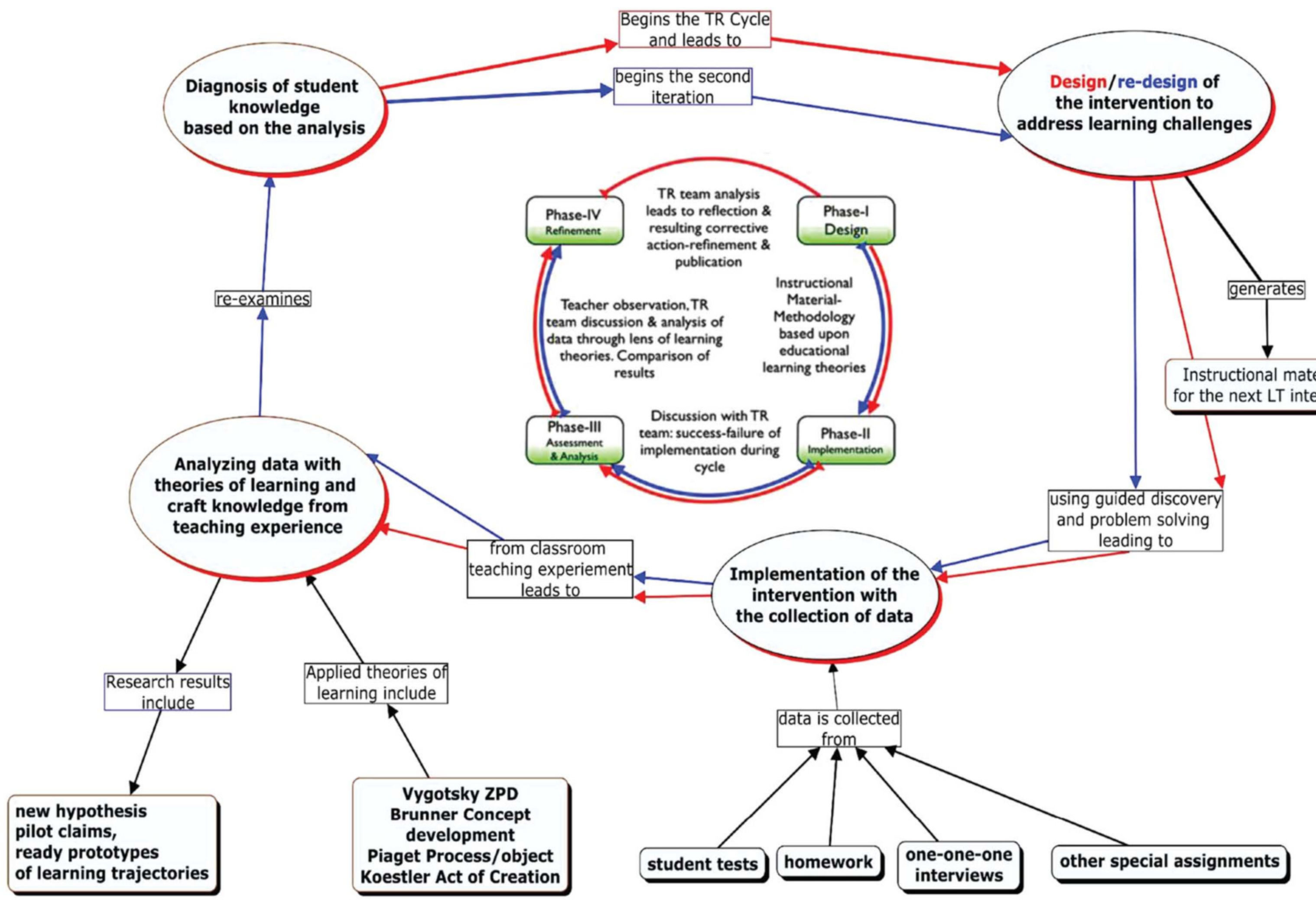
The Creative Enterprise of Mathematics Teaching Research

Elements of Methodology and Practice – From Teachers to Teachers

Bronislaw Czarnocha, William Baker, Olen Dias and Vrunda Prabhu (Eds.)

Teaching – Research Cycle



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Elements of Methodology and Practice – From Teachers to Teachers

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We dedicate this book to Vrunda Prabhu
One of the founders of the
Teaching-Research Team of the Bronx
In gratitude for leaving behind
The Universal Signpost:

Creativity is the way out!

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FOREWORD

This book is about creativity in students working with mathematics in classrooms and teachers developing mathematics teaching for the benefit of their students. It is written for "teachers of Mathematics and researchers in Mathematics Education" by a team of mathematics teacher-researchers, focusing on the education of "underserved" students in the Bronx, USA. The authors write:

> We are addressing ourselves to the teachers of mathematics who want to use research to reflect upon and improve their craft practice, and to researchers who are interested in uncovering riches of classroom teaching for research investigations. And most of all we are interested in those educators who see the urgent need for creative synthesis of research and teaching.

The book is eclectic and wide-ranging, drawing on literature, theory and research in Mathematics Education over several decades. It develops a theoretical model for practice referred to as the TR/NYCity model. The authors present the antecedents of the model, the philosophy of practice on which it is based and examples involving practice with teachers and students. They propose that the model addresses several 'gaps': the Achievement Gap between different groups of students (e.g., Pisa, 2012), the Teaching Gap between methods of teaching (e.g., Stigler & Heibert, 2000) and the gap between Research and Practice whereby research is undertaken by and reported to academic researchers, having little relevance or interest for teachers for whom 'teaching practice' is their central concern (e.g. Hargreaves, 1996).

The model is underpinned by a number of themes: Arthur Koestler's theory of bisociation, the main instantiation of which is the "Aha moment"; Laurence Stenhouse's theory of research as a basis for teaching in which a 'Stenhouse Act' is both a teaching act and a research act; and theories of creativity, traceable to Poincaré and Hadamard. In particular the model proposes a synthesis of Koestler and Stenhouse theories by which a teacher seeks to create Aha moments of mathematical understanding through Stenhouse acts. Central to the model is the use of student and teacher reflection as part of the act in which they engage. Fostered by the teacher, student reflection enables students to think beyond the procedural means of getting the solution to a problem, to the rationale for procedures they use and the possibilities of alternative solutions. An aim for student learning is that engagement at this meta-level promotes aha moments of understanding (the cognitive) from which students derive pleasure and motivation (the affective). The teacher meanwhile is reflecting in a similar way, looking critically at her own approaches to fostering students' creative involvement, and seeking alternative approaches to achieving student understanding.

Creativity in the classroom is centred on problem-solving approaches through which students are introduced to mathematics and engage in dialogue with each other and the teacher. The associated practice of teaching is fundamentally a research process, in which a teacher engages in design of tasks and classroom activity, and the reflective interaction with students that feeds back to the design process. Several chapters include teaching sequences in which this process is exemplified and critiqued. The authors emphasise that the interactive process, fostering aha moments, has a cognitive/affective duality in which students' reasoning is challenged in ways that reduce their antipathy for mathematics.

While the TR/NYCity model, focuses centrally on students' learning through critical reflection in problem solving, its more global significance lies in its teacher-researcher dialectic. The TR-act is a research act designed simultaneously to offer a classroom approach (the teaching act) rooted in dialogic mathematical problem solving and to critique the approach (a research act) in and from practice. Both acts are inquiry-based (involving the asking of critical questions), with a meta-level of inquiry (the TR act). The teaching sequences offered are annotated throughout to point out the stages of this reflective critiquing process.

It is hard to do justice, in a short space, to the complexity of the ideas in this book. I encourage the reader to engage with these ideas, taking a critical position and reflecting at the same time on their own teaching/research practices.

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Hargreaves, D. (1996). *Teaching as a research-based profession: Possibilities and prospects*. Teacher Training Agency Annual Lecture, Teacher Training Agency, London.

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Equally important are the words of gratitude in the dedication to Vrunda Prabhu, who sadly passed away prior to the publication of this book. We are grateful for her having introduced us to Koestler's Theories and the teaching practice exploiting their insights.

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We are grateful to Ruslan Flek and Edme Soho, the copy editors of the volume.

Note of information: Concept maps in the volume are usually in color as are photos of contributors, however the original colors can only be seen in the e-version of the book.

INTRODUCTION

The book Creative Enterprise of Mathematics Teaching-Research submitted for readers' consideration and enjoyment presents the results and methodology of work of the teaching-research community of practice, TR Team of the Bronx. It is directed towards two audiences, teachers of Mathematics and researchers in Mathematics Education. We are addressing ourselves to the teachers of mathematics who want to use research to reflect upon and improve their craft practice, and to researchers who are interested in uncovering riches of classroom teaching for research investigations. And most of all we are interested in those educators who see the urgent need for creative synthesis of research and teaching. The two central themes of the book are the methodology of TR/NYCity model of teaching-research and creativity, more precisely, creativity of the Aha moment formulated by Arthur Koestler (1964) in deeply powerful but little known theory of bisociation exposed in his work The Act of Creation. Both themes are introduced in Unit 1 as the basic thematic threads permeating and organizing this exposition. Unit 1 contains also Chapter 1.3 which describes the student population of the Bronx as one of many educationally "underserved" communities in US and in the world plagued by the increasing Achievement Gap in general, and in mathematics learning in particular (Pisa in Focus, #36, February 2014). TR/NYCity model has been formulated with the focus on improvement of learning mathematics within "underserved" student population necessary to bridge the Achievement Gap. Thus TR/NYCity is a framework of inquiry in Mathematics Education characterized by the substantive quality of Stenhouse (Rudduck & Hopkins, 1985), that is its "acts of finding" are undertaken to benefit directly and in equal measure not only research community, but also others outside of that community, in our case, students in ours and others' classrooms.

Arthur Koestler defines Aha moment that is bisociation as "the spontaneous flash of insight, which…connects the previously unconnected frames of reference and makes us experience reality at several planes at once…" (p. 45). We define the bisociative framework as the framework composed of two or more "unconnected frames of reference", which might be joined by the discovery of a "hidden analogy" through Aha moment of bisociation (p. 179). Since teaching-research is composed of two generally, and unfortunately, separate "frames of reference", teaching practice and education research, ripe in our opinion with "hidden analogies", TR/NYCity Model is a bisociative framework of inquiry with enhanced possibility to facilitate creative Aha moments both for students and teacher-researchers.

The first coordination of teaching-research practice with Koestler theory of bisociation is done by Vrunda Prabhu (Unit 2) to whom this volume is dedicated. Unfortunately she passed away during her work on Koestler creativity in the classroom. Prabhu coordination and recognition of TR/NYCity as a bisociative framework has profoundly transformed our view on the methodology itself and its role in Math Education, especially on the role of student creativity in the improvement of learning mathematics among "underserved" student population and, as a consequence, in bridging the Achievement Gap. One of the central aspects of Prabhu's work has been the exploitation of cognitive/affective duality characteristic for Aha moments (Czarnocha, 2014) in service of eliminating negative habits and attitudes of students to mathematics. She found support for her quest in Koetler's assertion that "The creative act…is an act of liberation – the defeat of habit by originality" (p. 96).

The recent papers of the team collected in references of the Epilogue show the further directions of investigations into implications and role of Koestler creativity in mathematics classrooms.

TR/NYCity model finds it "niche" within the inquiry frameworks formulated by Eisenheart (1991) and re-introduced into Mathematics Education by Lester (2010). These authors postulate existence of three different framework of inquiry: theory-based framework, practice-based framework and a conceptual framework. Whereas the first two frameworks are in general separate from each other, the third one incorporates elements of both, while pointing to the essential role of justification in this framework. Our insistence on the balance in the work of TR/NYCity model between research knowledge of the profession and craft knowledge of the teacher finds its expression in the conceptual framework of Eisenheart and Lester with the bisociation leading to creative Aha moments as its central justification.

Incorporation of bisociation into the definition of TR/NYCity model allows to understand Stenhouse acts which are "at once educational act and a research act" as bisociation-in-action (Rudduck & Hopkins, 1985). Important examples of classroom pedagogies, which have the quality of Stenhouse acts are discussed in Unit 3 Tools of Teaching-Research.

We can formulate now the new definition of TR/NYCity Model as the conceptual bisociative framework of Design Research conducted by the classroom teacher, whose aim is to improve the process of learning in the classroom, and beyond – the characteristic of the "substantive nature" of teaching-research. The details of this methodology are obtained as consequences of the definition. The original TR/NYCity definition, which led us to contemporary understanding, is placed at the opening of Chapter 1.1.

Our exposition in the volume is divided into five units, which introduce the reader into practice of TR/NYCity methodology, as well as into results obtained with its help. Following Unit 1, which sketches the main thematic threads of the volume, we present Unit 2 Creative Learning Environment (CLE). Unit 2 contains, in its majority, a collection of TR reports of Vrunda Prabhu describing the process of her search for CLE in classrooms of mathematics, which culminates in the coordination of her practice with Koestler theory in Chapter 2.4. Coordination of teaching practice with an appropriate theory is an essential bisociative act within TR/NYCity model. We expand the discussion of this process in Chapter 3.2.

Unit 3, Tools of Teaching-Research introduces reader to methods and techniques utilized later in Unit 4. It is composed of two parts, Chapters 3.1–3.3, which focus on preparation, conduct and assessment of a classroom teaching experiment, and Chapters 3.4–3.9 which focus on the chosen set of pedagogical strategies: Teaching Research Interviews, Concept Maps and Discovery method of teaching. All three strategies support the formation of Stenhouse acts and they express the bisociativity of TR/NYCity model. We direct this unit to the attention of teachers of mathematics as a point of entry into mathematics teaching-research. The TR tools introduced here open the pathway through Unit 4 of TR Design, which together with Unit 2 constitute the central nucleus of the presented work.

Unit 4, Teacher as Designer of Instruction: TR Design invites the reader to the exploration of the principles and practice of the Teaching-Research Design as one of the methodologies of Design Research in Mathematics Education.

The types of TR Design are discussed in the extensive introduction to the unit. The introduction continues the discussion of different frameworks of inquiry within Design Research initiated in Chapter 1.1 and applies it to the characterization of the types of TR Design and related with it, classification of Stenhouse acts. The collection of teaching experiments and teaching-research investigations in Unit 4 contains examples for all three types.

Unit 4 starts with the discussion of Koestler's theory in the context of problem solving in Chapter 4.1 and leads toward synthesis of bisociation with Piaget theories of conceptual development. It demonstrates that processes of reflective abstraction such as constructive generalization and interiorization may be built on bisociative foundations. Although each chapter in Unit 4 typically has its own theoretical framework, Chapter 4.1 is designed to unify these different frameworks of concept development with Koestler's theory of creativity.

The chapters of the unit present three modes of teaching-research activity. Chapter 4.2 presents our "daily" TR activity in the context of teaching rates and proportions, Chapter 4.6 reports two teaching experiments focused on the iterative classroom design of learning trajectories, and last two chapters present two teaching experiments of different TR Design types.

Unit 5 Teaching-Research Communities illuminates TR/NYCity model from the point of view of teaching-research community of practice and its development. Whereas different aspects of the TR community of practice were touched upon in different chapters of the book, here we focus on the expansion of the TR community to new academic disciplines and cultural/educational environments The introductory review of learning communities sets the ground for the first two chapters of the unit, which report on the expansion of TR community of practice from its original domain of mathematics into mathematics/English interphase (Chapter 5.1), and expanding further to the learning community of three courses "linked" together through the mathematically-based theme Part of a Whole (Chapter 5.2). The closing chapters of the unit as well as of the whole book address the professional development of teacher – researchers (PDTR) in two very different cultural educational environments: amongst the teachers of community schools in Dalit villages of Tamil Nadu, India and among the mathematics teachers of five European countries participating in the international project supported by the Socrates program of the European Community. Chapter 5.3 signals further development of TR/NYCity Model into TAR, that is Teaching-Action-Research composed of Teaching-Research in the school and Action Research in the surrounding it village community conducted by the teachers of the school. This development took place during the PDTR in Tamil Nadu, India.

The complexity of the mathematical classroom has been recognized since the works of Anne Brown (1992), Collins (1992) and Wittman (1995) introduced the principles of Design Research into Mathematics Education. The variety of themes addressed by TR/NYCity Model conveys the degree of complexity encountered in the mathematics classroom of "underserved" student population. Consequently, the book and its story can be accessed in several ways: through particular articles and their specified themes, through the inquiries into specific themes e.g. interaction between learning mathematics and language, development of proportional reasoning as a gateway to algebra, problem solving in the context of Koestler theory, the affective role of creativity in transforming negative attitudes to the subject e.t.c. It can also be read as the guide or the handbook facilitating individual or team entry into teaching-research.

Many of the chapters contain teaching sequences whose effectiveness has been investigated through several iterations. They can be adapted to particular conditions of the classroom, expanded and experimented with at will. Generally, presented teaching sequences are accompanied by teacher annotations made either on the basis of professional craft knowledge or appropriate theory or both. The aim of annotation is to bring the reader closer to our work, and in particular to the "technology of thinking" (Chapter 1.1) – the process of integrating the teaching craft knowledge with theories of learning, conceptual development or Koestler creativity. Stenhouse acts are the products of that integration.

The work presented here has taken 15+ years of classroom investigations, designs, reflections and re-designs within the community of TR Team of the Bronx. With this volume we take the voice in the discussions on the role of teachers in research as well as on the role of researchers in the classroom; more generally on the teaching practice-research divide. The highlights of the discussion show surprising absence of knowledge of, and respect for the work of the teacher. Our colleague, Erich Wittmann (1999), in his effort to convince research community to take on research upon teaching sequences, which hasn't been noticed till now, observes that

> …the design of teaching has been considered as a mediocre task normally done by teachers and authors of textbooks. To rephrase Herb Simon: why should

> anyone anxious for academic respectability stoop to designing teaching and put himself on one level with teachers? The answer has been clear: he or she usually wouldn't.

One can similarly ask, what self-respecting master teacher of mathematics would be interested in collaboration with the academician who needs to "stoop" to collaborate and think together? The answer has been clear: he or she usually doesn't. The result is a deep divide between research-and teaching practice which undermines any effort to improve learning, and to large degree it is rooted in the issue of the "academic status."

Interestingly enough, 5 years later, there appear calls for the active participation of teachers in the design and construction of Learning Trajectories in the context of Common Core effort in US. Clements and Sarama (2004, p. 85) note "that learning trajectories could and should be re-conceptualized or created by small groups or individual teachers, so that there are based on more intimate knowledge of the particular students involved…". Since teaching sequences are the essential component of learning trajectories and their design depends on the level of knowledge of particular cohorts of students, teachers' designs are necessary for any planned improvement of learning. However, as Susan Empson (2011) notes "We know very little about how teachers do these things, in contrast to what we know about children's learning…"; "these things" being to "understand, plan, and react instructionally, on a moment-to-moment basis, to students' developing reasoning" and coordinate these interactions with learning goals (Battista, 2010) of the learning trajectory.

The present volume responds to these concerns by demonstrating that teachers of mathematics, if freed a bit from the negative constraints of academic respectability, can design learning trajectories as they always did, and investigate their design through the iterations natural to teachers' semester or yearly work using JiTR methods. It shows how we, teachers, think, how we design and how we teach, and investigate at the same time with the help of TR/NYCity methodology, especially how we "understand, plan, and react instructionally, on a moment-to-moment basis, to students' developing reasoning" (Empson, 2010) and coordinate these interactions with learning goals called for by Battista (2010). Thus the answer to Steffe (2004, p. 130) who asks "Whose responsibility is it to construct learning trajectories?" is: it is the responsibility of teacher-researchers working with theTR/NYCity Model as the conceptual bisociative framework of classroom Design Research. The TR/NYCity Model's emphasis on the balanced relationship between research and craft knowledge of teacher, eliminates the issue of "academic status" and substitutes it by the bisociative creativity of Aha! Moments.

UNIT 1

THE MAIN THEMES OF THE BOOK

INTRODUCTION

The aim of this introductory unit is to present the foundations of the work, which are based upon two conceptual frameworks, TR/NYCity Model and Koestler theory (1964) of the Act of Creation The coordination between teaching practice and theory (here Koestler theory) – a fundamental component of teaching-research has taken place during the teaching experiment by Vrunda Prabhu and became one of the thematics threads throughout the book. The next two chapters present each of the frameworks separately, and the last chapter throws light upon the theme that appears in many chapters of the book, namely the cognitive and affective learning challenges, encountered in education of the "underserved" student population.

Chapter 1.1 presents the "skeletal" (Eisenheart, 1991) structure of TR/NYCity model, which underlines the full volume. We see here TR/NYCity model as a synthesis of practical and theoretical frameworks of inquiry reaching its completion as a bisociative conceptual framework. The bisociative nature of TR/NYCity reveals itself, with the help the Koestler's theory of the Act of Creation, as the natural environment to support the creative teaching and learning. We lead the reader through its historic development as well as the constructive connections with work of Lewin, Vygotsky, Stenhouse and Eisenheart, each contributing to the formation and versality of the approach culminating with the Act of Creation of Koestler. It is the natural connection of TR/NYCity methodology with creativity that makes it a strong candidate as a methodology to close the Achievement Gap characteristic for the underserved student population.

We expand upon the TR Cycle and point out that its iterative nature joins the practice of research and the practice of teaching-research. We formulate the concept of Just-in-Time Research (JiTR) in analogy to (JiTT) Just-in-Time Teaching as a feedback loop between classroom challenges of the teacher and research knowledge of the profession. We argue that such a bottom – up relationship of teaching practice with research is necessary for research to be accepted and incorporated in the classrooms. Two themes complete the presentation of the methodology: the discussion of characteristic tools of the methodology (1) Discovery method of teaching and (2) development of thinking technology, which lead to Stenhouse TR acts as one of their expressions. The Discovery theme, while present in most of our TR investigations is discussed in Chapter 3.9, while Stenhouse TR acts we encounter again in the Unit 4.

Chapter 1.2 presents an overview of contemporary research on creativity as a background to Koestler's theory of creativity, the second main theme of the book. A central concept of creative research and Koestler's theory turns out to be the affective-cognitive duality of the Aha moment (Czarnocha, 2014). The affective component of the duality suggests that facilitation of Aha moments might be a powerful pedagogical tool in changing student attitude to mathematics as well as to themselves as mathematicians. Since the experience of Aha moment is known to members of general population, its facilitation in any regular classroom creates the possibility of opening the research on, and practice of mathematical creativity by all students, in addition to gifted ones. The chapter discusses the issue in the section the Transition from Genius to the Classroom focusing subsequently upon bisociation theory of creativity by Koestler (1964). The author, Bill Baker expands the discussion to delve into the cognitive-affective duality where the mechanism of bisociation enters into Piagetian framework. This discussion is continued in Chapter 4.1 in the context of problem solving. The affective aspect of the duality is discussed within Goldin (2009) work in the area. However it is the Unit 2 of Vrunda Prabhu that needs to be read in conjunction with this chapter as the development-in-practice of the very same duality.

Chapter 1.3, the last chapter of Unit 1 sketches the socio-economic-ethnic background of student population we teach in the Bronx together with the main academic difficulties our student experience. The chapter points to the fact that other similar centres of "underserved" student population in US, Europe and Asia demonstrate similar characteristics, so that the dydactic designs and artefacts formulated in the context of Bronx student population can work elsewhere, can be generalized. The issue of generalizations of the methodology is further developed in Chapters 1.1 and 2.3. The author addresses critical pedagogical issue of what works among the population and shows that the focus on creativity through facilitating Aha moments might indicate the correct route to close the Achievement Gap which at present can be detected throughout the world. The role of community of TR practice for that process is developed in the Introduction to the Unit 5.

BRONISLAW CZARNOCHA

1.1. TEACHING-RESEARCH NEW YORK CITY MODEL (TR/NY CITY)

TR/NYCity Model is the methodology for classroom investigations of learning, which synthetizes educational research with teaching practice. It is conducted simultaneously with teaching and it aims at improvement of learning by the teacher of the class in the same classroom, and beyond.

INTRODUCTION

TR/NYCity Model is based on the careful composition of ideas centred around Action Research (Lewin, 1946) with the ideas centred around the concept of the Teaching Experiment of the Vygotskian school in Russia, where it "grew out of the need to study changes occurring in mental structures under the influence of instruction" (Hunting, 1983). From Action Research we take its focus on the improvement of classroom practice by the classroom teacher and its cyclical instruction/analysis methodology, and from Vygotsky's teaching experiment we take the idea of the large-scale experimental design based on a theory of learning and involving many sites – different classrooms (Czarnocha, 1999; Czarnocha & Prabhu, 2006). Vygotsky teaching experiment methodology introduced the possibility of viewing the classroom teacher as a member of a collaborative research team investigating the usefulness of research based classroom integration. The integration of these two distinct frameworks re-defines the profile of a teacher-researcher:

1. as an education professional whose classrooms are scientific laboratories, the overriding priority of which is to understand students' mathematical development in order to utilize it for the betterment of the particular teaching and learning process;
2. who as a teacher can have the full intellectual access to the newest theoretical and practical advances in the educational field, knows how to apply, utilize and assess them in the classroom with the purpose of improving the level of students' understanding and mastery of the subject;
3. who as a researcher has a direct view of, and the contact with the raw material of the process of learning and development in the classroom, acts as a researcher in the context of the daily work and uses that process to design classroom improvement and derive new hypotheses and general theories on that basis.

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 3–21.

The implicit vision underlying the profile is the conceptual and practical balance between researches and teaching, where both components of the educational profession are given equal value and significance; both the research knowledge of the researcher and the craft knowledge of the teacher are resources for the teacher-researcher.

Admittedly, the proposed profile is ambitious, yet it's doable, especially in the context of community colleges whose full time mathematics faculty have PhD level experience in mathematics, physics or engineering research and can relatively easily transfer those skills into classroom-based investigations of learning. On the other hand, given the progressing collapse of public education in US, the majority (80%) of freshman students who enter every semester into our colleges require remediation to be able to get to college level courses. The remediation starts on the level of arithmetic through algebra it constitutes 80% of our "bread and butter" courses. The placement into, and exit from remediation is decided by the university wide – standard exam. Consequently, the mathematics faculty of community colleges are intimately familiar with the issues of school mathematics. The composition of research skills with the craft knowledge of teaching elementary mathematics is at the basis of the formulation of TR/NYCity Model.

HISTORICAL BACKGROUND AND DEVELOPMENT OF TR/NYCITY MODEL

Stenhouse TR Acts

TR/NYCity owns its formal origins to Action Research of Kurt Lewin (1946) and Teaching Experiment methodology of Vygotsky. TR/NYCity model finds its completion in the bisociation of Koestler (1964) leading to the Stenhouse TR acts (Rudduck & Hopkins, 1985).

Lewin proposed the Action Research methodology in the context of the quest for improvement of "group relations", a euphemism for interracial relations in US of 30ties and 40ties. He saw it as "…a comparative research on the conditions and effects of various forms of social action, and research leading to social action." His Action Research cycle consisted of the stages (or steps) of diagnosis with plan for action, implementation of action, its assessment providing at the same time the basis for "modifying overall plan" and leading to the next cycle. It was however Stenhouse who introduced Action Research methodology into education profession as teaching-research in the inaugural lecture at the University of East Anglia in 1979 presentation "Research as basis for teaching" – a theme whose importance has steadily grown till contemporary times. Already in early seventies of the last century he recognized that one of the possible explanations for the failure of research

> …to contribute effectively to the growth of professional understanding and to the improvement of professional practice… was the reluctance of educational researchers to engage teachers as partners in, and critics of, the research results. (Rudduck & Hopkins, 1985)

The extracts from the transcripts of seminars with the part-time MA students reveal his understanding of Action Research in terms closely related to TR/NYCity model arrived at spontaneously through our work. He understood Action Research primarily as "the type of research in which the research act is necessarily a *substantive act;* that is an act of finding out has to be undertaken with an obligation to benefit others than research community" (p. 57), in our case, students in ours, and other classrooms. However, it's the concept of "*an act* [which is] *at once an educational act and a research act*" (p. 57), that completes a stage in our development of thinking technology, that is the process of integration of research and learning theories with the craft knowledge of the profession anchored in practice. The bisociative framework (see below) of TR acts produces new mental conceptions, the product of thinking technology. These conceptions (e.g. schema, ZPD, hidden analogy, bisociation) become part of the discourse within the community of teacher-researchers, tools to design methodology for improvement of classroom craft and for deepening one's research interest.

It is surprising Stenhouse did not utilize Action Research cycles. It could be because the curriculum research he envisioned as conducted by teachers, apart from case studies, was to test hypotheses arrived at by curriculum research outside of the teacher's classrooms (p. 50).

The second root of our methodology is anchored in the methodology of the Teaching Experiment of Vygotsky, which had a professional research team together with teachers investigate the classroom and was conducted "…*to study changes occurring in mental structures under the influence of instruction*" (Hunting, 1983). Interestingly, introduction of professional research into classroom by Vygotsky and his co-workers in the thirties was the fulfilment of the first part of the Stenhouse's vision of the seventies who demanded "*In short, (1) real classrooms have to be our laboratories, and (2) they are in command of teachers, not researchers*" (p. 127). For the second part of Stenhouse vision we propose classrooms, which are in the command of teacher-researchers as the synthesis of both methodological efforts.

The Teaching Experiment methodology reappeared in the work of Steffe and Cobb (1983) as a constructivist teaching experiment, which was appropriated by Czarnocha (1999) for teaching purposes in high school class of mathematics, already as a tool of a teacher. Czarnocha (1999) realized that the constructive teaching experiment can easily become a teacher's powerful didactic instrument when transformed into guided discovery method of teaching.

Design Science

The interest in the work of the professional practitioner of whom teacher is but one particular example has been steadily increasing in the second half of the previous century since the work of Herb Simon (1970), the Design of the Artificial. His work proposes the design as the "*principal mark that distinguishes the professions from sciences*" (pp. 55–58). Kemmis and McTaggart (2000) developed the principles of Action Research, while Schon (1983) investigated the concept of a Reflective Practitioner through the process of reflection-in-action. Both frameworks had found applications in the work of teachers and researchers through joint collaborations, however the research/practice gap hasn't been yet bridged.

The terms Design Experiment, Design Research or the Science of Design are often interchangeable and they refer to the professional design in different domains of human activities. It was introduced into research in Math Education by Ann Brown (1992), Collins (1992), and Whittmann (1995). Anne Brown had realized during her exceptional career that psychological laboratory can't provide the conditions of learning present in the complex environment of a classroom and transformed her activity as a researcher directly into that very classroom as the leading co-designer and investigator of the design in the complex classroom setting. In her own words: "As a design scientist in my field, I attempt to engineer innovative classroom environments and simultaneously conduct empirical studies of these innovations" (A. Brown, 1992). She provided this way one of the first prototypes of design experiments which, theoretically generalized by Cobb et al. (2003), "entail both "engineering" particular forms of learning and systematically studying those forms of learning within the context defined by means of supporting them…". The profession has followed her lead seeing the classroom design experiments as theory based and theory producing. Paul Cobb et al. (2003) assert that Design Experiments are conducted to develop theories, not merely to empirically tune what works. Design research paradigm treats design as a strategy for developing and refining theories (Edelson, 2002). Even Gravemeyer (2009) who defines "the general goal of Design Research to investigate the possibilities for educational improvement by bringing about and studying new forms of learning" hence stating it closer to substantive quality formulated by Stenhouse, yet he warns us that "great care has to be taken to ensure that the design experiment is based on prior research…" eliminating this way the designs anchored in prior practice. Unfortunately, the educational research profession cuts itself off by these restrictions from the source of profound knowledge contained in the tacit and intuitive craft knowledge of the teachers. Clearly, if the goal is improvement of learning, a more general framework is needed which recognizes both education research and teaching practice as two approaches of comparable significance, value and status.

Frameworks of Inquiry and the Unity of Educational and Research Acts

We find such a framework within the three frameworks of inquiry identified by Margaret Eisenhart (1991): theoretical, practical, and conceptual (Lester, 2010). Following Eisenhart, Lester (2010) posits three types of frameworks used in Math Education, first, the theoretical framework based upon theory i.e. the constructivist, radical constructivist and social constructivist theories discussed second, a practical framework, "… which guides research by using 'what works' … this kind of research is not guided by formal theory but by the accumulated practice knowledge of practitioners and administrators, the findings of previous research, and often the viewpoints offered by public opinion" (p. 72). The third is a conceptual framework that can pull from various theories as well as educational practice.

The theoretical framework guides research activities by its reliance on a formal theory; that is, a theory that has been developed "on the theoretical, conceptual, and philosophical foundations" (Lester, 2010) by using an established, coherent explanation of certain sorts of phenomena and relationships—Piaget's theory of intellectual development and Vygotsky's theory. However, as soon as such a theory- based design undergoes a TR cycle, the initial determinative role of theory changes into the JiTR-approach (Just-in Time-Research; see below), which allows for the participation of craft knowledge based on the teaching experience in equally significant manner.

The Practical Framework is employed in what we refer to as 'action research' and as discussed, it has some common components with teaching-research.

> For Scriven, [quoted in Lester (2010)] a practical framework guides research by using "what works" in the experience of doing something by those directly involved in it. This kind of framework is not informed by formal theory but by the accumulated practice knowledge of practitioners and administrators, the findings of previous research, and often the viewpoints offered by public opinion. Research questions are derived from this knowledge base and research results are used to support, extend, or revise the practice. (Lester, 2010)

However, the distinction that we make with Lester's description of a practical framework and a framework for teaching research is that we, as researchers, view the goal of teaching-research to inquire into how theory and models of learning reflect upon what the teacher and student experience in the classroom. The question for the teacher researcher and supportive TR community is what needs to be transformed or changed in the existing theories or models in order to improve the fit between these frameworks and classroom practice?

The third and final framework considered by Lester is that of

> a *conceptual framework* [that] is an argument that the concepts chosen for investigation, and any anticipated relationships among them, will be appropriate and useful given the research problem under investigation. Like theoretical frameworks, conceptual frameworks are based on previous research, but conceptual frameworks are built from an array of current and possibly far-ranging sources. The framework used may be based on different theories and various aspects of practitioner knowledge. (Lester, 2010)

We argue that amongst the three frameworks for research present in philosophy of education research only the conceptual framework allows for the possibility of bisociative synthesis between teaching and research through Stenhouse TR acts.

Of special importance in working with conceptual frameworks is the notion of *justification*. A conceptual framework is an argument including different points of view and culminating in a series of reasons for adopting some points and not others. The adopted ideas or concepts then serve as guides: to collecting data, and/or to ways in which the data from a particular study will be analysed and explained (Eisenhart, 1991).

According to Lester (2010) "…too often educational researchers are concerned with coming up with "good explanations" but are not concerned enough with justifying why are they doing what they are doing…" (p. 73).

Our insistence on the balance between research and teaching practice, the basis for the unified Stenhouse TR acts, finds its justification and fulfilment in the bisociation of Koestler (1964) that is in "*a spontaneous leap of insight which connects previously unconnected matrices of experience*" (p. 45). A bisociative framework is the framework composed of "two unconnected matrices of experience" where one may find a "hidden analogy" – the content of insight (Chapter 1.2). Given the persistent divide and absence of deep connections between research and teaching practice, TR/NYCity constitutes a bisociative framework composed of "unconnected [in general] matrices of experience" of teaching and research, within which one can expect high degree of creativity on the part of the teacher-researcher through leaps of insight leading to the unified Stenhouse acts defined above. The process of coordination of TR/NYCity with Koestler bisociation theory is the guiding theme of Unit 2: Creative Learning Environment. Unit 2 presents the search for classroom creativity by Vrunda Prabhu during which this coordination has taken place revealing "hidden analogy" between Koestler theory and Prabhu's teaching practice.

We can state now a new definition of TR/NYCity methodology:

> *TR/NYCity Model is the conceptual bisociative framework of Design Research conducted by the classroom teacher, whose aim is to improve the process of learning in the classroom, and beyond – the characteristic of its "substantive nature".*

TR bisociative framework facilitates integration or, still better, synthesis of practice and research through instances or sequences of instances of Stenhouse acts which are "at once an educational act and a research acts" (Rudduck and Hopkins, p. 57). In what follows we will call them Stenhouse TR acts. The Stenhouse TR acts are the foundation stones of "thinking technology" discussed below within which their unity is naturally positioned. The facilitation of longer or shorter instances of Stenhouse TR acts can be reached from either teaching practice or from application of research to practice, as well as from both simultaneously. The "skeletal structure" (Eisenhart, 1991) of the TR/NYCity conceptual framework can be obtained as requirements and conclusions from the definition.

We discuss different designs of teaching experiments and TR investigations in Unit 4, The Teacher as a Designer of Instruction: TR Design, while in Chapter 3.2 we discuss "nuts and bolts" of classroom teaching experiment. The Introduction to Unit 4 develops the "skeletal structure" of TR/NYCity as the consequence of the definition.

TEACHING-RESEARCH CYCLE (TR CYCLE)

Just-in-Time Teaching (JiTT) and Just-in-Time Research (JiTR)

Teaching-Research cycle is the fundamental instrument in our work, which allows for the smooth integration of research and teaching practice within our conceptual framework. The difference from other similar cycles of Action Research or of the Design Experiment (Cobb et al., 2003) is simple: it allows the teacher-researcher to enter the classroom investigation from either of both directions, from research and from teaching. There is however, an important methodological trade off: whereas a Design Experiment researcher prepares the design of classroom intervention on the basis of prior research, the teacher-researcher uses Just-in-Time approach, that is research literature consultation takes place during the TR cycle, generally at the Analysis and Refinement nodes, when we either compare the results to assumed theory of learning, or when we search for adequate theoretical framework to understand the learning situation, or in any other unclear classroom situation.

Just-in-Time Teaching (JiTT) as expressed by Novak et al. (1999) is a teaching and learning strategy based on the interaction between web-based study assignments and an active learner classroom. Students respond electronically to carefully constructed web-based assignments which are due shortly before class, and the instructor reads the student submissions "just-in-time" to adjust the classroom lesson to suit the students' needs. Thus, the heart of JiTT is the "feedback loop" formed by the students' outside-of-class preparation that fundamentally affects what happens during the subsequent in-class time together. JiTT has been used well together with Peer Leader methodology (Mazur & Watkins, 2009).

Analogically, Just-in-Time Research (JiTR) is research and teaching strategy based on the "feedback loop" formed between the didactic difficulties in the classroom encountered by a teacher, and educational research results that may throw light into the nature of these difficulties. At this moment, the classroom teacher makes contact with the bisociative framework of TR/NYCity model.

Anchoring TR in TR cycle

It is in the introduction of educational research into the classroom that we differ from Action Research. The JiTR approach differs from standard educational research in that theory is repositioned from being a required foundation to the Just-in-Time solution for didactic difficulties in the mathematics classroom.

William J. Harrington, describing his work of a teacher-as-researcher in Laura R. Van Zoest (2006) states that, "Teachers do informal research in their classrooms all the time. We try a new lesson activity, form of evaluation, seating arrangement, grouping of students, or style of teaching. We assess, reflect, modify, and try again, as we consider the perceived consequences of changes we made." Hence, there is a natural pathway that extends these informal activities into systematic research, offered by the

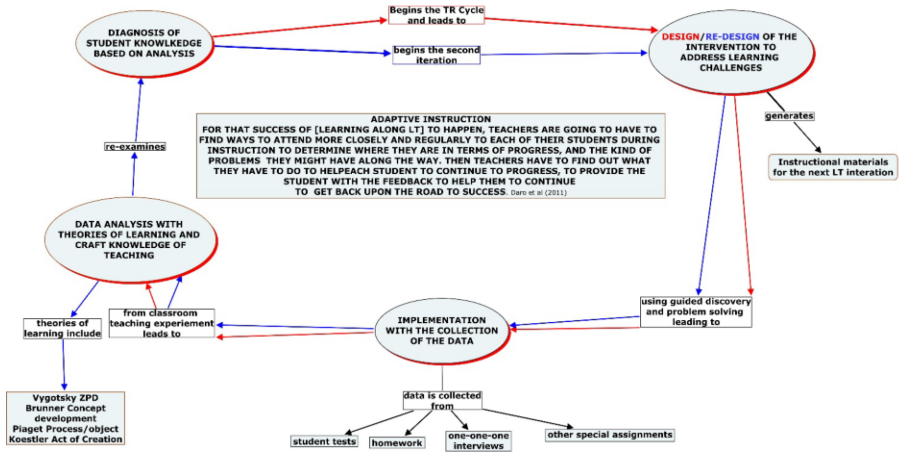

Figure 1. The TR Cycle

TR/NYCity model that successively progresses along *Teaching-Research* (TR) cycles of diagnosis, design of instruction in response to diagnosis, collection of relevant data and its analysis, and, ultimately, with the help of relevant external research results through JiTR approach, the redesign of interventions. The TR cycle, the explicit generalization of Action Research principles in the classroom, is particularly well fit into our work because of our work's naturally cyclic structure via semesters or academic years. Since every teacher has an option of repeating to teach the same course to a new cohort of students, the TR cycle allows for the continuous process of classroom investigations of the same research question during consecutive semesters. The sequential iteration of TR cycles is one of the main methodological research tools of the TR/NYCity Model facilitating the process of integration of teaching and research into a new unit of professional classroom activity, teaching-research.

TR/NYCity requires a minimum of two full TR cycles within a context of a single teaching experiment to fulfil the requirement of improvement of instruction. In its insistence on the improvement of learning through cycle iteration, TR/NYCity incorporates and generalizes the principles of Japanese and Chinese Lesson studies (Huang & Bao, 2006).

Consequently, every teaching experiment of the TR/NYCity Model has a main teaching-research question, composed of two sub-questions:

- What is the state of the students' knowledge under the impact of the new intervention?
- How to improve that state of knowledge?

The duration of the TR cycle can vary depending on intervention. In can last a year, a semester, and a couple of days or even one class. In its rudimentary form we can find it even in teacher-student inquiry dialogs (see example in Chapter 4.1).

The bisociative creativity of the teacher reaches its fulfilment during this period of reflection and redesign spurred by the simultaneous consideration of data analysis results, relevant teaching experience, relevant JiTR results from professional literature and appropriate theories of learning or conceptual development. It is precisely at this moment when the new teaching-research hypotheses are formed, leading to new theories and investigations. The focus of this teaching-research activity is the investigation of student learning followed by the design of teaching, whose effectiveness is often investigated in the subsequent TR cycle.

Instructional Adaptability of the TR/NYCity Model via TR Cycle

The increased degree of flexibility created by this integration of teaching and research within a single "tool box" helps teachers reach new levels of instructional adaptability to student learning needs. In fact, the comparison of the adaptive instruction described by (Daro et al., 2011) with the TR cycle reveals a very high degree of correspondence:

> For that [success of LT framework] to happen, teachers are going to have to find ways to attend more closely and regularly to each of their students during instruction to determine where they are in their progress toward meeting the standards, and the kinds of problems they might be having along the way. Then teachers must use that information to decide what to do to help each student continue to progress, to provide students with feedback, and help them overcome their particular problems to get back on a path toward success. This is what is known as adaptive instruction and it is what practice must look like in a standards-based system.

Every TR cycle consists of the following components:

- (1) The design of the instruction/intervention, in response to the diagnosis of student knowledge,
- (2) Classroom implementation during an adequate instructional period and collection of data; this incorporates problem-solving, guided discovery classroom discourse and design of interventions for diagnosed difficulties,
- (3) Analysis of the data, in reference to existing experimental classroom data, appealing to the general theory of learning through J-i-T approach and the teacher-researcher's professional craft knowledge,
- (4→1) Design of the refined instruction based on the analysis of the data obtained in steps 1 through 3, leading to the hypothesized improvement of learning. The symbol "4→1" is intended to convey that the 4th step in the cycle is equivalent to going back to the 1st step in the cycle.

As a result, every such 1→2→3→4→1 is an instance of adaptive instruction – finding the level of students' understanding through tests, homework assignments and one-on-one interviews, responding to the difficulties by the re-design of the intervention, implementation and assessment. Consequently, the TR cycle is called for, as the theoretical framework of the teacher's work in a mathematics classroom driven by the Common Core Standards. Transformations of the teacher's pedagogy and improvements, based on research and evidence, have to take place exactly within such a framework. Chapters 4.2, 4.9 and 4.10 provide detailed examples of two (or more) full cycles of such an approach.

Generalization in TR/NYCity Model

One of the central questions asked of frameworks related to action research is the question about the generality of our assertions. How general is TR/NYCity? Why and how that what we understand in the Bronx, has any bearing anywhere else? In terms of the original definition at the beginning of the chapter, what is the nature of the word "beyond" in that definition?

TR/NYCity has three ways to generalize its findings:

1. By coordination with a theory whose correctness has been asserted in the profession. If we coordinate our findings with a theory, then they acquire degree of generality afforded to the theory, that is one can draw conclusions from the findings in terms of the coordinated theory of learning. These conclusions might be relevant, with proper modifications to any classroom situation to which that theory applies.
2. By running an artefact used in a TR investigation through many iterations with different cohorts of students. As a result, the artefact acquires large degree of generality, which provides the basis for its application to different new situations (Chapter 2.2).
3. A special window of generalizations opens up when we consider student populations with similar socio-economic status to the one in the Bronx. The similarity of the socio-economic status results in similar cognitive/affective challenges experienced by students to which similar adaptive interventions are needed (Kitchen et al., 2007) The successful generalization of TR/NYCity artefacts has been reached amongst Indian Dalits (downtrodden) of Tamil Nadu (Chapters 2.2 and 5.3) and in Poland amongst rural students of Southern Poland (Czarnocha, 2008).

The discussion of artefacts in the context of Design Research (Unit 4) brings forth an important clarification that its generalization can be obtained by expanding its application to similar student populations.

Thinking Technology

The dictionary definition of technology is "the application of scientific knowledge for practical purposes, especially in industry." Thinking technology in TR/NYCity model is the process of integration of research results and framework with craft knowledge of the teacher. This spontaneous process inherent for TR/NYCity model finds its elegant expression in Koestler bisociation theory and Stenhouse TR acts. It is a very subtle process, in which scientific concepts such as "hidden analogy" of Koestler become the critical tools, metaphors with the help of which we start to identify classroom situations, the term becomes a phrase with the help of which we, members of the TR team start communicate with each other in our own new language. In fact, by making the connection between scientific meaning and classroom situation we create the analogy between two generally separate matrices of thinking – hence the connection itself is a new bisociation, a possibility of new meaning.

One could conjecture that any process of coordination (as distinct from application) of a theory of learning with elements of teaching practice is the bisociative creative process during which new connections and therefore new meanings are made.

The process of coordinating research and teaching practice is facilitated by the duality inherent in the teacher-researcher work (Malara & Zan, 2002). The practice of teaching-research duality creates a new mental attitude promoting a novel design of instructional methodologies while, at the same time, requiring an investigative probe into student thinking, on the basis of which consequential teaching and research decisions are made. This duality is explored deeper in Units 2 and 4. The exploration together with utilization of the duality is conducted by the classroom teacher-researcher. In this process, teachers are not solely engaged in research on learning, they are also engaged in the transformation of teaching on the basis of, and through that research. This means that they do not simply incorporate the results of research into their teaching practice but rather allow methods of research to become the methods of teaching leading to Stenhouse TR acts. Thus the route towards Stenhause TR acts is through the process of integrating research knowledge and craft knowledge in practice of teaching. In this process, teachers do not switch into a role of researcher, instead, they oscillate between the role of a teacher and the role of a researcher and fuse their efforts toward a new unit of professional activity – bisociative teaching-research with its Stenhouse TR acts.

TR/NYCITY AND THE DISCOVERY METHOD OF TEACHING

The discovery method of teaching has been the preferred instructional method by the teacher-research team working with and developing TR/NYCity methodology since its inception. The Discovery method of teaching has a fundamental role in the TR/NYCity model. This method was introduced into TR/NYCity via the Texan Discovery method created and formulated by R. L. Moore, a topologist brought up by the Chicago school of mathematical thought of the thirties. B. Czarnocha and V. Prabhu adopted this method during their NSF grant in calculus 2002–2006. However, our understanding of its role in TR classrooms came with time through many TR investigations and teaching experiments. Using different approaches such a "guided discovery method", "inquiry method" or "inquiry leading to discovery", it has appealed to our imagination and practice as teacher-researchers because with its help we could lay bare student authentic thinking for our investigations.

On the one hand, from the educational aspect Discovery method provides learning environment best suitable for facilitation of bisociation. According to Koestler (1964) subjective, individual bisociation are more often encountered in the condition of "untutored learning". The Discovery method is one of the closest classroom approximations of this condition. This approach to teaching relies on designing situations and using techniques, which allow the student to participate in the discovery of mathematical knowledge. These are authentic moments of discovery with respect to student's own knowledge, which in the further development of methodology are related to subjective Aha! Moments of Arthur Koestler (Chapter 1.2).

On the other hand, from the research point of view, it is the best instrument, which opens the nature of student thinking to us, teacher-researchers for investigation through careful interaction. It allows us to investigate and to extend the scope of students' ZPD, to help in eliminating misconception as well as in facilitating bisociations. Thus the process of TR together with Discovery method of teaching constitutes an extended in time Stenhouse TR act.

Creativity: From Bathos to Pathos – From Habit to Originality

The institution of creativity as the structural component generated within the learning environment provided by teaching-research has significant consequences beyond its cognitive importance.

Vrunda Prabhu has found out (Chapter 2.4) that student success in her classroom depended on three closely connected components of (i) cognition, (ii) motivation and (iii) self-regulated student learning (Prabhu, 2006). More specifically, when creativity is explicitly nurtured and facilitated in a mathematics classroom in the context of such an integrated learning environment, it can transform the habit of distaste toward mathematics into mathematical originality supporting Koestler's assertion that "creativity means breaking up habits and joining the fragments into new synthesis" (p. 619). Moreover, according to Koestler:

> The creative act, by connecting previously unrelated dimensions of experience, enables him [the inquirer] to attain a higher level of mental evolution. It is an act of liberation – the defeat of habit by originality.

Habitual dislike of mathematics is, at present, one of the main student obstacles for success in mathematics learning that could be eliminated with the help of that "act of liberation" providing a pathway from Bathos to Pathos, using Koestler metaphor (p. 96).

Summary of the Argument

To summarize the argument, TR/NYCity is the generalization of Action research and of the Design experiment methodology (Design experiment methodology is seen here as the further development of the Teaching Experiment of Vygostsky school in Russia). In its original vision it was seen as the bridge between the two methodologies, which eliminates the limitations of both – a new integrative conceptual framework. By the same token, TR/NYCity is designed specifically to bridge the gap between research and teaching practice – one of the fundamental obstacles in the effective transformation of mathematics education. The need for such a bridge was indicated by the report of US National Research Council, How People Learn-Bridging Research and Practice (Donovan et al., 1999). We review below essential components of the research/teaching practice gap in our profession as seen by contemporary reports.

GAP BETWEEN RESEARCH AND PRACTICE

English (2010a) notes that the complexity of educational theory and philosophy, has lead to a gap between educators and researcher based upon concerns about the relevancy of such philosophies to educational practice,

> The elevation of theory and philosophy in mathematics education scholarship could be considered somewhat contradictory to the growing concerns for enhancing the relevance and usefulness of research in mathematics education. These concerns reflect an apparent scepticism that theory-driven research can be relevant to and improve the teaching and learning of mathematics in the classroom. Such scepticism is not surprising…claims that theoretical considerations have limited application in the reality of the classroom or other learning contexts have been numerous…it remains one of our many challenges to demonstrate how theoretical and philosophical considerations can enhance the teaching and learning of mathematics in the classroom… (p. 66)

Harel (2010) and Lester (2010) both note that government funding agencies and panels created to direct government research efforts are increasing restricting their attention to quasi experimental-control group efforts with a goal of what works i.e. action research. They advance the hypothesis that more attention to research frameworks would perhaps counter the ideology that all research should be practical-statistical i.e. scientific based methodology based upon a p value indicating success or failure i.e. 'what works.' Harel's (2010) claim that attention to frameworks is lacking in educational research is due in part to his belief that there exists "…a feeling on the part of many researchers that they are not qualified to engage in work involving theoretical and philosophical considerations" (pp. 88–89). The issue that arises for those of us advocating for a more active role of teachers in integrating educational research and craft is that, if researchers feel they are not qualified then how much more likely those teachers feel unqualified. That is, how can practical research methodology such as that used in action research be expected to integrate theory and practice in a meaningful way when its practitioners may feel unqualified to engage in theoretical considerations? This question is particularly relevant to us because we strongly believe in order for reform efforts, indeed, any research based pedagogy to actually improve education there must be a sustained effort in the school and that any such effort must involve the teacher and the researcher working together or a teacher-researcher to determine what works as well as to reflect upon why it does or does not work from both a practical craft level as well as through the lens of theoretical framework.

Another reason reform effort to improve mathematics education through theoretical considerations has floundered is that mathematical education theories are often appear impractical to the craft practitioner to implement i.e. theories that provide little guidance for instructional design but within the research community there are often contradictory positions about such efforts. The result is that reform efforts and counter reactionary movements tend to arise and disappear like last year's fashion statements. Sriramen and English (2010) comment on an early attempt by mathematicians to change traditional mathematics called New Math which in the 50's and 60's tried to change the rigidity of traditional mathematic through a top down approach to pedagogical change. "One must understand that the intentions of mathematicians such as Max Beberman and Edward Begle was to change the mindless rigidity of traditional mathematics. They did so by emphasizing the whys and the deeper structures of mathematics rather than the how's but in hindsight…it seems futile to impose a top-down approach to the implementation of the New Math approach…" (p. 21). Goldin (2003) notes how behaviourism led to a back to basics counter movement within mathematics education: "behaviourism was fuelling the 'back to basics' counterrevolution to the 'new mathematics', which had been largely a mathematician-led movement. School curricular objectives were being rewritten across the USA to decompose them into discrete, testable behaviours" (p. 192). Goldin (2003) also notes that constructivism has more recently displaced this back to basic reactionary movement. "Radical constructivism helped overthrow dismissive behaviourism, rendering not only legitimate but highly desirable the qualitative study of students' individual reasoning processes and discussions of their internal cognitions" (p. 196). Yet he warns that the excessive of radical constructivism will render it impractical and unsuitable "Constructivists excluded the very possibility of 'objective' knowledge about the real world, focusing solely on individuals' 'experiential world'" (p. 193).

The point being that a top-down approach to educational reform by research experts has not succeeded and we venture will never succeed without first teacher buy in, but this is not near enough, in order for the craft practitioner to continue to implement reform methodology and to design instruction based upon theory, when the researcher goes back to academia the teacher must internalize the theory and even more how such theory relates to design of instruction. Yet we consider that even this is not enough to sustain reform efforts especially with underserved populations that demonstrate serious negative affect with mathematics. The approach to educational research in which experiments have a beginning and an end is founded upon an underlying assumption that some truth can be found that will dramatically change educational practice. This assumption needs to be re-evaluated if educational craft practice is to actualize the benefits of research. We consider that a constant collaboration between educational researchers and teachers is needed and provides the best hope of actualizing change in educational practice to close widening gap between research and theory and the scepticism it has caused. Boote (2010) comments on the need for continual teacher development based upon design research in improving educational practice: "Indeed, the professional development of all participants may be more important and sustaining than the educational practices developed or the artefacts and knowledge gained" (p. 164). Examples of such an international professional development of teacher-researchers based on TR/NYCity methodology are discussed in the Unit 5.

THE COMPARISON BETWEEN TEACHING-RESEARCH AND DESIGN-BASED RESEARCH

The discussion in this section is the continuation of the theme found in the section Frameworks of Inquiry and the Unity of Educational and Research Acts, which gets further clarification in the Introduction to Unit 4. Our aim here is to provide a detailed comparison between theoretical and practical frameworks as seen from the point of view of TR/NYCity, which we see as the conceptual framework creating the bridge between the two via TR cycle.

Research, in particular, design-based research	*Teaching-research, in particular TR/NYCity model*
Theory driven: (EDUCATIONAL PSYCHOLOGIST, 39(4), 199–201 Copyright © 2004, Lawrence Erlbaum Associates, Inc. William A. Sandoval, Philip Bell Design-Based Research Methods for Studying Learning in Context: Introduction.) Design-based research can contribute to theoretical understanding of learning in complex settings. Each of the articles by Sandoval, Tabak, and Joseph reveal how the design of complex interventions is an explicitly theory-driven activity.	*Practice driven:* (*Professional Development of Teacher-Researchers*, Rzeszow University, Poland, 2008) (*Teaching Experiment NYCity Method*. 2004) Teaching-research is grounded in the craft knowledge of teachers that provides the initial source and motivation for classroom research; it then leads to the practice-based design. Its aim is the improvement of learning in the classroom as well as beyond.

Use of Theories of Learning in Design-Based Research: (Educational Researcher, Vol. 32, No. 1, pp. 5–8), (Design-Based Research: An Emerging Paradigm for Educational Inquiry by The Design-Based Research Collective, 2003) In addition, the design of innovations enables us to create learning conditions that learning theory suggests are productive, but that are not commonly practiced or are not well understood.	*Use of Theories of Learning in Teaching-Research:* (Dydaktyka Matematyki, 2006, v. 29, Poland, Teaching-Research NYCity Model. B. Czarnocha, V. Prabhu) The design of innovation enables the teacher-researcher to create the Creative Learning Environment based on teacher's craft knowledge, which improves learning in the classroom and transforms habits such as misconceptions, into student originality (Koestler, 1964). Learning theories are used as needed to support teachers' craft knowledge.
Focus of the Teaching Experiment in Design-Based Research: (Journal for Research in Mathematics Education. 14(2) pp. 83–94, 1983, Cobb, P. and Steffe, L. P., The Constructivist Researcher as Teacher and Mod el Builder) Cobb and Steffe assert that the interest of a researcher during the teaching experiment in the classroom is "in hypothesizing what the child *might* learn and finding [as a teacher] ways and means of fostering that learning".	*Focus of the Teaching Experiment in Teaching-Research:* Proceedings of the epiSTEME Conference, Bombay, Homi Bhabha Institute, 2007, B. Czarnocha, V. Prabhu Teaching-Research and Design Experiment – Two Methodologies of Integrating Research and Classroom Practice) …The interest of a teacher-researcher is to formulate ways and means to foster what a student *needs* to learn in order to reach a particular moment of discovery or to master a particular concept of the curriculum (Czarnocha, 1999). Since, however, "such moments occur only within students' autonomous cognitive structures, the [constructivist] teacher has to investigate these structures during a particular instructional sequence [in order to be of help to the students]. In this capacity, he or she acts as a researcher".
Use of Iteration in design-based research: (ICLS, 1, pp. 968–975, 2010, Confrey, J., Maloney, A., The construction, refinement and early validation of the equi-partitioning Learning Trajectory) …articulating, refining and validating is an "iterative process of research synthesis and empirical investigations involving" many types of evidence. Step 1: Meta-research of the concept to create the prototype. Step 2: Iterative refinement of the prototype	*Use of Iteration in TR/NYCity model:* Step 1: Process of iteration, starting with the first iteration designed on the basis of teaching practice. Step 2: Incorporation of research results as needed in between consecutive iterations. It is the concept of iteration of the design from semester to semester together with the related refinement that can bring in now relevant research results illuminating the classroom situation or providing help in the design of appropriate set of assignments.

The TR cycle through its natural iteration of teacher's activity from semester to semester provides the opportunity to move beyond the narrow "chicken or the egg" question of "What is the primary, or the more important realm,―research or practice?" and to creatively integrate design-based practice and design based research (see Unit 4).

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1.2. CREATIVITY RESEARCH AND KOESTLER

An Overview

INTRODUCTION

This section provides an overview of research on creativity in mathematics education: Koestler's understanding of creativity, the role of creativity with mathematics students, especially those who do not view themselves as gifted and a discussion on the how to support a creative learning environment within the mathematics classroom. Leikin (2009a) notes that, creativity and giftedness are underrepresented topics in mathematics education research; one specific obstacle she specifies is a lack of shared understanding of creativity within the mathematics classroom and ways to promote such creativity. In this present book the authors put forth their work on creativity based upon the definition and supporting theory of creativity due to Koestler (1964) within the framework of teaching research experiments supported by learning communities in an effort to promote creativity research within the classroom-learning environment. At the heart of Koestler's theory is the view that creativity involves the synthesis of previously unrelated frames of reference, hidden analogies that become apparent through one's intuition in an often unconscious process during incubation. That process leads to the conscious illumination or realization of this previously hidden analogy. Furthermore, this process has a transformation effect upon an individual's affect towards the area or subject of thought, in this case mathematics. Following Koestler we consider this process of synthesis to describe both the creative thought of professional-eminent mathematicians as well as the development of cognitive structures in both gifted and ordinary students. We are particularly interested in the benefits for student affect during the process in which an individual discovers for themselves new material or previously hidden analogies with students one might characterize as being resistant to mathematics i.e. those who have a negative affect or understanding of their ability in mathematics. Another central component to the work of Koestler is that there are strands of creative thought that cross the three domains of; humor, scientific and mathematical research and also, literary and artistic endeavours. Koestler refers to this notion as a Triptych. The role of the Koestler's Triptych in this volume (Chapter 2) is not so much as to argue that creativity crosses domains rather that there

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 23–41.

is an affective relationship between creativity in these domains that can be beneficial for student with a negative or resistant attitude towards their mathematical ability.

THE NEED FOR RESEARCH IN CREATIVITY IN EDUCATION

Leikin (2009a) notes that there is a dearth of research on creativity and gifted students and the relationship this has to talent loss,

> nurturing mathematical promise is directed at preserving human capital, it allows the community to grow new generations of creative mathematicians scientists, and high-tech engineers who contribute to the further development of sciences, technologies, and various branches of mathematics…talent loss is a major challenge for every society. (p. 387)

Although we agree completely with Leikin's thesis on the need for research with gifted individuals our goal is more in line with Silver's (1997) view that creativity has a psychological aspect i.e. creativity is a disposition that is not restricted to a few gifted students, and as a result creativity should be introduced through-out instruction in mathematics,

> persons creative in a domain appear to possess a creative disposition or orientation toward their activity in that domain. That is creative activity results from an inclination to think and behave creatively…this view suggests that creativity-enriched instruction might be appropriate for a broad range of students, and not merely a few exceptional individuals. (pp. 75–76)

Furthermore, we argue that accepting an educational culture in which a majority or significant minority of the population have no experience with the beauty and satisfaction of learning and discovering mathematics for themselves would result in a society with two tiers, in the lower tier are individuals who have access to ever more sophisticated technology yet less and less understanding of the basic principles upon which such technology is based. As eloquently stated by Koestler:

> Modern man lives isolated in his artificial environment…his lack of comprehension of the forces…the principles which relate his gadgets to the forces of nature, to the universal order…his refusal to take an interest in the principles behind it. By being entirely dependent on science, yet closing his mind to it, he leads a life of an urban barbarian. (Koestler, 1964, p. 264)

Sriraman et al. (2011) points out that research on creativity within mathematics education with students who do not consider themselves gifted is essentially non-existent, "There is almost little or no literature related to the synthetic abilities of 'ordinary' individuals, except for literature that examines polymathy" (p. 120). The issue of polymathy or giftedness across domains will not be directly tackled but is inherent in the work we do following Koestler's notion of the Triptych and his thesis that there are grey areas between the three domains of humor, science and art-literacy. Specifically, we consider the role of humor and literacy endeavours in promoting affective aspects of students in their studies of mathematics. Our thesis is that when presenting mathematics to students from an underserved population or to any student with a negative attitude towards mathematics a creative and supportive learning environment must be established in order to transform the habitual negative affect expressed in the commonly heard phrases, "I hate math" or "I such at math" with a positive affect that allows for learning and even appreciation of the mathematical cognitive content. Furthermore, as Vrunda Prabhu was fond of stating, when an instructor begins to establish such a creative learning environment founded upon education research in creativity and cognition it will simultaneously engender creativity on the part of the students.

ROOTS OF CREATIVITY RESEARCH

> Creativity plays a vital role in the full cycle of advanced mathematical thinking…Yet it is external to the theory of mathematics…Most mathematicians seem to be not interested in the analysis of their own thinking procedures…only a few explicitly attempt to describe ideas related to mathematical creativity. (Ervynck, 1991, p. 42)

The roots of contemporary research in mathematical creativity focus on the 'creative person' (Leikin & Pitta-Pantazi, 2013) and are often traced to the work of H. Poincaré, and J. Hadamard. As noted in Liljedahl (2013), "Hadamard's treatment of the subject of invention at the crossroads of mathematics and psychology is an extensive exploration and extended argument for the existence of unconscious mental processes. To summarize, Hadamard took the ideas that Poincaré had posed and, borrowing a conceptual framework of Wallas turned them into a stage theory. This theory still stands as the most viable and reasonable description of the process of mathematical invention" (p. 254). Wallas was a Gestalt Psychology and the resulting stage theory of creativity within a problem-solving environment includes: initiation (preparation) incubation, illumination and verification. The first stage is characterized by, "an attempt to solve the problem by trolling through a repertoire of past experiences" (Liljedahl, 2011, p. 52).

A problem-solving situation in which no idea or solution is forthcoming may lead to the nest stage of incubation, "one forces oneself consciously to work hard on a new problem or an idea. When no solution is forthcoming, the problem is put aside and one's mind needs to relax to make the necessary connections" (Sriraman et al., 2011). The importance of the initial-preparation stage is because it, "creates the tension of unresolved effort that sets up the conditions necessary for the ensuing emotional release at the moment of illumination" (Lilijedahl, 2013, p. 254). In the view of Poincaré and Hadamard putting aside conscious work when blocked allows the solver to "begin to work on it at an unconscious level" (Liljedahl, 2013, p. 254) during the incubation stage. This creates the possibility of illumination, "the manifestation of a bridging that occurs between the unconscious mind and the conscious mind…a coming to the conscious mind of an idea or solution." Colloquially this is often referred to as the 'Aha' experience (Liljedahl, 2013, p. 255).

Koestler studies the creative experience of many eminent mathematicians and scientists he comments that Poincaré believed in divine influence or unconscious intuition that during incubation selects out of countless combinations of patterns and thoughts only the relevant or beautiful ones i.e. "the aesthetic sensibility of the real creator" (Koestler, 1964, p. 163). The following quote from the musing of Poincaré is used by Koestler to demonstrate what he considers the two essential characteristics of creativity first, an affective component in which, "the self is experienced as being a part of a larger whole, a higher unit-which may be Nature, God, Mankind, Universal Order…an abstract idea…the participatory or self-transcending tendencies" (p. 52). The second component is the synthesis of two frames of references previously considered independent:

> One evening, contrary to my custom I drank black coffee and could not sleep, Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination. By the next morning I had established the existence of a class of Fuchsian functions, those that come from hypergeometric series. I had only to write out the results, which took but a few hours…and I succeeded without difficulty in forming the series I have called theta-Fuchsian. Then, I turned my attention to the study of some arithmetical questions apparently without much success…disgusted with my failure I went to spend a few days at the seaside, and thought of something else. One morning, walking on the bluff, the idea came to me…with just the same characteristics of brevity, suddenness and immediate certainty, that the arithmetic transformation of indeterminate ternary quadratic forms were identical with those of non-Euclidean geometry. (pp. 115–116)

THE TRANSITION FROM GENIUS TO CLASSROOM

Liljedahl (2013) points out that implicit in this view of creativity as an original product is that the discovery can be, "assessed against other products within its field, by the members of that field, to determine if it is original and useful" (p. 255). Liljedahl (2013) comments that as a result of Koestler and other's treatment of creativity is that, "…creative acts are viewed as rare mental feats which are produced by extraordinary individuals who use extraordinary thought processes" (p. 255) i.e. "… creative processes are the domain of genius and are present only as precursors of the creation of remarkably useful and universal novel products" (p. 256). We note that the sentiment that creative mathematics is tied to mystical geniuses has been linked to limiting creativity in the math class. For example, Silver's rather striking comment about the 'genius' view of creativity and how it has limited research into creativity in mathematics education:

> The genius view of creativity suggests both that creativity is not likely to be heavily influenced by instruction and that creative work is more a matter of occasional bursts of insight than the kind of steady progress towards completion which tends to be valued in school. Thus, there have been limited attempts to apply ideas derived from the study of creativity to the education of all students. (Silver, 1997, p. 75)

We note that any instructor of a remedial or first-year mathematics course frequented by non STEM (science, technology, engineering and mathematics) majors can probably relate to the frustration of having students equate their lack of mathematics fluency with a lack of God given talent and their strong conviction, that it is therefore not worth making a concerted effort.

Leikin and Pitta-Pantazi (2013) are even blunter than Silver in their critique of this so-galled genius approach to creativity as they comment on how the perception of creativity has changed with the dawn of research into creativity in the classroom: "Initially creative ideas were considered to be generated mystically…subsequently, the mystical approach was replaced by a pragmatic approach which was mainly engaged in ways of developing creativity" (p. 160).

Although we completely agree with the thesis put forth by Silver, Liljedahl and Leikin and Pitta-Pantazi that assert creativity should be part of every student's experience in mathematics, we note that Liljedahl's comment that Koestler's treatise on creativity lends itself to the genius approach obscures the fact that both Koestler and Liljedahl emphasise the affective component of the creative experience, regardless of whether the product is original to the mathematical community or only novel to the individual solver. "Minor, subjective bisociative processes do occur on all levels, and are the main vehicle of untutored learning" (Koestler, 1964, p. 658). We also note that Koestler's work provides a precise description of a mechanism that underlies the transformation between incubation and illumination; a mechanism that appears to be lacking in existing mathematical educational literature on creativity. This mechanism for the synthesis of planes of reference allows for a theoretical framework to study creativity within the social (classroom) situation as students create meaning of mathematics. That is when the individual discovers a product-result that is new to themselves but known to the instructor, tutor etc.

The transformation of creativity from an analysis of eminent mathematicians to the classroom i.e. the transformation from the creative person and creative process in the field of mathematics to the creative person, creative process and creative environment (in the classroom) has led to a multitude of definitions and approaches to studying creativity.

Definitions of Creativity in Mathematics Education Research

> The significance of creativity in school mathematics is often minimized because it is not formally assessed on standardized tests, which are designed to

> measure mathematical learning. The problem with relating to students' work as 'creative' is rooted in the definition of creativity as a useful, novel, or unique product…Although according to the traditional view of creativity students' work would not be considered as creative, the researchers agree that students' discovery may still be considered creative if we examine the issue of creativity from a personal point of view, namely, whether the students' discoveries were new for them. (Shriki, 2010, p. 162)

As noted by Shriki, "Eminent mathematicians such as Jacques Hadamard and George Pölya argued that the only difference between the work of a mathematician and that of a student is their degree" (p. 161). The integration of creative pedagogy within classroom mathematics for gifted or ordinary students of mathematics raises the issue of how does one define and then using this definition measure creativity. Sriraman et al. (2011) like Shriki propose that, "a differentiation be made between creativity at the professional and school levels" and that creativity at the school level should include:

1. The process that results in unusual (novel) and/or insightful solutions(s) to a given problem
2. The formulation of new questions and/or possibilities that allow an old problem to be regarded from a new angle (p. 120).

We are entirely in agreement with Sriramen's intention to extend the range of creativity to include classroom mathematics. However, the practical and theoretical issue that arises is not whether there should be a definition for creativity in educational mathematics but the plurality of definitions. Mann (2006) notes that educational research included over 100 definitions of creativity he claims that, "the lack of an accepted definition for mathematical creativity has hindered research efforts" (p. 238). R. Leikin has done much work on definitions and assessment of creativity focused on gifted students. For example, Leikin (2009b) notes two formulations of creativity in mathematics educational research that have been used to assess an individual's propensity for creativity. The first is the ability for convergent and divergent thinking due to J. Guilford. "Convergent thinking involves aiming for a single correct solution to a problem, whereas divergent thinking involves the creative generation of multiple answers to a problem or phenomena, and is described more frequently as flexible thinking." Her review also notes the definition suggested E. Torrance i.e. the capacity of an individual for flexibility, fluency, novelty and elaboration.

> Fluency refers to the continuity of ideas, flow of associations, and use of basic and universal knowledge. Flexibility is associated with changing ideas, approaching a problem in various ways, and producing a variety of solutions. Novelty is characterized by a unique way of thinking and unique products of a mental or artistic activity. Elaboration refers to the ability to describe, illuminate, and generalize ideas. (Leikin, 2009b, p. 129)

Silver (1997) did not like the focus on spontaneous illumination if it meant any diminishing of attention to the need for hard work and instruction, “creativity is closely related to deep flexible thinking in content domains; is often associated with long periods of work and reflection rather than rapid, exceptional insights; and is susceptible to instructional and experimental influences” (p. 75). Silver went on to propose that creativity is an essential component of problem solving and problem posing. We note that most of these definitions and the research based upon them are designed to separate gifted-creative students of mathematics from those who are not necessarily gifted. That is to distinguish between those students who are fluent-proficient and flexible problem solvers capable of understanding multiple representations that often diverge from the instructor presented approach. Silver (1997) admonishes educators to consider that a creativity-enriched instruction is appropriate for students not considered gifted and that instruction is an important components of establishing a creative learning environment, which we refer to as the attempt to democratizes creativity (Prabhu & Czarnocha, 2014). Shriki (2011) goes a step further stating that creativity should be part of all students learning experience:

> It is widely agreed that mathematics students of all levels should be exposed to thinking creatively and flexibly about mathematical concepts and ideas. To that end, teachers must be able to design and implement learning environments that support the development of mathematical creativity. (p. 73)

It is important to point out here that both flexibility and fluency may not reinforce student creativity but diminish it. Koestler (1964) points out that that flexibility as a component of a “rigid and flexible variations on a theme” (p. 660) contributes to the formation of a habit, hence it may diminish originality and with it, creativity of an individual. Since “The creative act is…the act of liberation – the defeat of the habit by originality” (p. 96). Thus, acting out of a habit diminishes originality. The possible decrease of originality due to flexibility had been noted by Leikin et al. (2013) and by Kyung Hee Kim (2011). The theoretical issue that that arises, is whether Koestler’s definition can provide for a foundation to focus on the creative aspect of student originality, even with students who do not demonstrate fluency and whose flexibility and divergent thinking patterns are intermittent. In other words, how can creativity be integrated into a student’s attempt to give meaning to material presented in the classroom. A constructivist perspective would argue that all learning is essentially the self-discovery of what is known by others. For example, Sriramen et al. (2011) points out “The ability to create an object in mathematics is an example of mathematical creativity” (p. 121).

One of the tenets of this volume is that creativity is needed to reach all students of mathematics especially, those students who are resistant or do not consider themselves gifted in mathematics. Moreover, for sustained implementation of a creative learning environment teachers need to be involved in all phases of research; in our view the underlying deficiency in mathematics education is that researchers continue to believe that a holy grail can be found in a new pedagogy based upon a learning theory that will be designed, implemented and assessed with input from teachers only at the implementation phase. This model has not worked in changing mathematics pedagogy from the traditional model of an active instructor lecturing to passive students to a creative learning environment because it cannot be sustained without direct input from teachers in design, implementation, assessment and a refinement cycle. Creativity and originality in the classroom must come from both the instructor and the students. Students must take responsibility for their own learning and instructors must take responsibility for researching that learning process to improve it while exploring theoretical aspects of their craft. In this situation the students will pick up the enthusiasm and motivation of their instructor and break their habits of learned failure through the creative originality. In turn their attempts to overcome their limitations at whatever level they are will inspire the instructor. In like manner educational research will benefit from the craft knowledge of instructors relating their work and findings to theories of learning and creativity. Another basic tenet of this work is that a sustained effort to implement a creative learning environment in the classroom requires a learning community of instructors, a support network of other like minded teacher-researchers to share the many joys, disappoints and the assessment of what works and does not work and to assist each other relate their work and findings to theoretical aspects of learning and creativity.

Question that arise and that will be addressed include; what is the nature of a creative learning environment, what is the relationship between creativity and conceptual or critical thinking? How does creativity enter into theories of learning? Certainly all students can be original and this original thought can diverge from what is presented in the classroom, students have intuition that comes into play in their effort to understand mathematics. The question of how to design, implement and assess such an environment using Koestler's work underlies much of the material presented.

CREATIVITY AND KOESTLER

Koestler (1964) considers creativity as situated within a problem solving environment which he defines as, "bridging the gap between the initial situation and the target" (p. 649). The initial effort or preparation to bridge the gap occurs while, "keeping my eyes both on the target and on my own position" the search for a solution is characterized as, "searching for a matrix, a skill which will bridge the gap" (p. 651).

Bisociation

Koestler (1964) describes the main mechanism of creativity in terms of an analogy between two or more previously unrelated frames of reference: "I have coined the term bisociation in order to make a distinction between the routine skills of thinking on a single plane as it were, and the creative act, which … always operates on more than one plane" (Koestler, 1964, p. 36).

The terms matrix and code are defined broadly and used by Koestler with a great amount of flexibility. He writes, “I use the term matrix to denote any ability, habit, or skill, any pattern of ordered behaviour governed by a code or fixed rules” (p. 38). He uses the same definition later, substituting the phrase pattern of activity in place of pattern of ordered behaviour. The encompassing nature of these phrases allows one to include most processes used in the mathematics classroom, the caveat being that there must be some underlying order to the patterned activity. Indeed, as Koestler states, “all coherent thinking is equivalent to playing a game according to a set of rules” (p. 39). It follows that the term matrix can be applied to all coherent, logical or rule-based thought processes employed by an individual learning mathematics:

> The matrix is the pattern before you, representing the ensemble of permissible moves. The code which governs the matrix…is the fixed invariable factor in a skill or habit, the matrix its variable aspect. The two words do not refer to different entities; they refer to different aspects of the same activity. (Koestler, 1964, p. 40)

Thus, for Koestler (1964), bisociation represents a “spontaneous flash of insight… which connects previously unconnected matrices of experience” (p. 45). That is a, “transfer of the train of thought from one matrix to another governed by a different logic or code” (p. 95). Examples of bisociation in Koestler’s work abound, and range from humour to some of the most significant scientific discoveries. He describes one such humorous illustration in a story about a student cutting and replacing the legs of a pompous science teacher’s chair. In this case, the matrices were the professor’s attitude of absolute authority and the law of gravity the science teacher was lecturing about, which the student understood well enough to apply in his prank. Bisociation is also used by Koestler to describe original inventions; for example, when Gutenberg fused together two matrices to invent the printing press, “the bisociation of the wine-press and seal, when added together, became the letter-press” (p. 123).

The focus of the Act of Creation Theory is on the bisociative leap of insight, that is, an Aha! moment, or a moment of understanding,—a phenomenon that can be observed amongst the general population, and hence emphasizes, for example, the creativity of all students in a classroom setting or if one prefers the joy of learning. In this sense Koestler’s theory can be seen as a foundation for understanding educational research on creativity for all students including those who do not necessarily consider themselves gifted in mathematics.

Incubation and Illumination

> Max Plank the father of quantum theory wrote…that the pioneer scientist must have a vivid intuitive imagination for new ideas not generated by deduction, but by artistically creative imagination. (Koestler, 1964, p. 147)

The concept of the Incubation period embodies the belief that, "after attempting to solve a problem which needs wider knowledge and insight…a solution is more achievable if work on the problem is interrupted" (Sriraman et al., 2011, p. 123). It has been noted that there is a lack of research on the gestalt theory leading to a Eureka moment in the classroom:

> The period of incubation eventually leads to an insight on the problem, to Eureka or the 'Aha' moment of illumination…Yet the value of this archaic Gestalt construct is ignored in the classroom. This implies that it is important that teachers encourage the gifted to engage in suitably challenging problems over a protracted time period thereby creating the opportunities for the discovery of an insight and to experience the euphoria of the "Aha" moment. (Juter & Sriraman, 2011, p. 48)

Koestler's mechanism of bisociation could provide a useful tool for clarifying if not the process of incubation then instead the mechanism leading to illumination. Sriraman et al. (2011) note that the mechanism for incubation is poorly understood and advocates that a study of this mechanism would be beneficial for mathematics education. He reviews several theories of incubation and characterizes mechanisms underlying incubation as vague and complex. Liljehdahl (2009) reviews mathematicians whose most original discoveries 'Aha' moments come to them in non-mathematical activities such as, "showering, walking, sleeping, talking, cooking, driving, eating, waking, and riding the subway" (p. 65). Moments of illumination were reported through intuition in states between consciousness awareness and unconscious sleep and through the use of imaginative visual thought with pictures. Eureka moments were reported during mathematical activities involving the review and questioning of previous work especially where they notice gaps in their understanding. Sriramen notes the importance of illumination is that it can lead to insightful or creative thinking. "One theoretical reason for studying incubation is because it is closely associated with insightful thinking… understanding the role of incubation period may also allow us to make use of it more efficiently to foster creativity in problem solving, classroom learning and working environments" (Sriramen et al., 2011, p. 125).

Liljedahl (2009) highlights the affective aspect of illumination among prominent mathematicians notes that:

> The prevalence of anecdotal comments pertaining to this strong emotional response along with the uniform absence of any mathematical detail in these same anecdotes leads to a final conclusion that the AHA! Moment is primarily an affective experience. That is what sets it apart from other mathematical experiences is not the ideas, but the affective response to the appearance of the ideas. (p. 67)

He reinforces and elaborates on the connection between the cognitive and affective components of creativity in particular illumination:

> That is, what sets the phenomenon of illumination apart from other mathematical experiences is the affective component of the experienced, and ONLY the affective component. This is not to say that illumination is not a cognitive experience, for clearly it is. After all, it is the arrival of an idea that, in part, defines the phenomenon…while the affective component of illumination is consequential to the differentiation of it from other mathematical experiences, the cognitive component is not. (p. 264)

In his work with pre-service teachers who were 'resistant' to mathematics he notes that illumination or the "unexpected presentation of a solution filled them with positive emotions, precipitating changes in beliefs and attitudes, and encoding the details of the experiences" (Liljedahl, 2013, p. 264).

One might question the hypothesis that the affective component of illumination is of equal value to the cognitive for example the affective response may not last if the insight is determined not be valid or the material one thought one learned cannot be recalled during an exam. Is there a sharp distinction as painted by Liljedhal or a gradual shading of distinction a matter of degrees of affect as well as cognition that mark learning and creative work in math?

DEMOCRATIZING CREATIVITY

Prabhu and Czarnocha decided to bring the idea of teaching research experiments and their interest in creativity in mathematics education to students who had intermittent if any formal education in mathematics in the rural villages of India. This fruitful effort required that children who had limited exposure to or training in mathematics begin to engage in reasoning leading to formal math concepts. To accomplish this Prabhu focused on 'democratizing' the research and theory of creative research within the Teaching Research framework to provide a support learning environment that fostered creativity and ownership of ideas for students both women and their children who had limited experience with formal mathematics.

Much of the research on creativity in mathematics education has been focused on gifted students who have the talent to become research orientated mathematicians, the effort to bring such research into underserved communities and ordinary students has received less attention. The following assessment of a contemporary research effort on creativity notes:

> Missing is information on what initiatives are in place to develop and facilitate mathematical creativity in underserved and under-identified populations. This type of discussion would be informative to the field of gifted education and counter the criticism that field is not inclusive. (Chamberlin, 2013, p. 856)

Our hypothesis is that, in order to bring such research into underserved communities one needs to address the democratization of creativity i.e. the role of creativity in learning with ordinary even resistant students who do not consider themselves gifted. Yet much of the assessment and the underlying definitions of creativity focus on gifted students. Leikin and Pitta-Pantazi (2013) review educational research on gifted students which focuses on the affective domain of giftedness, that is the effect upon creativity of 'personality variables' such as self-concept and anxiety as well as 'personal-psychological attributes' such as: self-regulated learning, self-evaluation, responsiveness to extrinsic rewards, mathematical inclination, self-promotion, and the ability to learn how to play the game as well as risk taking. They note that the literature is divided upon whether creativity is a subset of, or independent of such giftedness (p. 160).

If one considers that creativity is defined or strongly characterized by cognitive abilities such as: flexibility, fluency and originality and affective-personal properties such as motivation, self promotion, risk taking, the ability to be taught etc. then one is more likely to view creativity as a subset of gifted students or eminent mathematicians who display these qualities.

For Koestler (1964) the defining characteristics of creativity are both the affective self-transcendent 'Aha' moment and the cognitive synthesis of two previous independent matrices. These two criteria, one affective and one cognitive, both require a high degree of conscious attention what he refers to as the transition from habit to originality. In contrast, fluency for Koestler is related to what he refers to as an exercise in understanding, which like the cognitive theorist Anderson (1995) notion of 'proceduralization' all too often has an inverse relationship to conscious attention i.e. it can promote habit: "We may then, somewhat paradoxically, describe awareness as that experience which decreases and fades away with our increasing mastery of a skill" (Koestler, 1964, p. 155). In regards to flexibility, Koestler notes that, "some highly developed, semi-automatized skills have a great amount of flexibility – the results of years of hard training; but their practitioners are devoid of originality" (Koestler, 1964, p. 157). In our effort to bring creativity into the classroom environment with 'ordinary' students we, like Koestler, consider that attributes such as fluency and flexibility, which are used to identify gifted students, are some markers of creativity but do they do not define it. The act of creation, is the spontaneous leap of insight that connects frames, which are disconnect defines creativity and provides a framework for its facilitation. Thus, we have a theoretical foundation to consider creativity as a critical component of any self- learning process that contains both gifted and ordinary or even resistant students of mathematics.

The sentiment that the Eureka experience can be used and beneficial in the mathematics classroom during problem solving and learning with students in mathematics (not only gifted) is expressed by Sriramen et al. (2011):

> understanding the role of incubation period may also allow us to make use of it more efficiently to foster creativity in problem solving, classroom learning, and working environment. Educators try to incorporate incubation periods in classroom activity in temporal pauses during classroom discourse or extended time periods for project related assignments…Incubation should

> not be neglected in the classroom. Students should be encouraged to engage in challenging problems and experience this aspect of problem solving, till a flash of insight results in the 'Eureka' or 'Aha' moment and the solution is born…The benefits of incubation are completely evident. (Sriramen et al., 2011, p. 125)

The concept that creativity is a central component of learning is not new, "Indeed, there is a sense in which all reasoned thinking, all genuine acts of figuring out anything whatsoever, even something previously figured out, is a new making, a new series of creative acts" (Paul & Elder, 2008, p. 8). The question that arises then is what is the relationship between learning and creativity? Our analysis of this issue which will involve a review of the learning environment according to Piaget, Koestler, Vygotsky and the cognitive theorist Anderson with a focus on how creativity and creativity research integrates into what Koestler would refer to as progress in understanding and Piaget would refer to as accommodation i.e. how an individual builds structures or schema. Specifically we are interested the relationship between Koestler's bisociative mechanism of creativity and Piaget's mechanism of learning 'reflective abstraction.'

KOESTLER: LEARNING ENVIRONMENT FOR CREATIVITY

Koestler (1964) notes the lack of student engagement in math and science education due to scripted learning pedagogy based upon modelling problems from textbooks followed by students reiterating the techniques presented:

> The same inhuman-in fact anti-humanistic-trend pervades the climate in which science is taught, the classrooms and the textbooks. To derive please from the art of discovery, as from the other arts, the consumer-in this case the student- must be made to re-live, to some extent, the creative process. In other words, he must be induced, with some proper aid and guidance, to make some of the fundamental discoveries of science by himself, to experience in his own mind some of those flashes of insight which have lightened its path. (pp. 265–266)

Koestler would allow that bisociation or originality (in the sense of Sriraman) is the essence of untutored learning: "Minor, subjective bisociative processes do occur on all levels, and are the main vehicle of untutored learning" (p. 658). This statement demonstrates two important viewpoints of Koestler that: (1) bisociation is an essential mechanism in the learning process, and (2) the subjective learning environment must allow for and approximate the conditions of 'untutored learning'. We believe, as did Koestler, that students cannot engage in mathematical problem-solving until they, in some sense, discover mathematics for themselves. This leads us to explore the mechanism of bisociation as foundational to learning mathematics.

For Koestler, bisociation explains the individual's use of analogies in learning and discovery. The distinction between the Eureka moment of originality and routine thinking is in the degree of novelty or unexpectedness of the analogy used. He writes, "one of the basic mechanisms of the Eureka moment is the discovery of a hidden analogy; but hiddenness is a matter of degrees. How hidden is a hidden analogy" (p. 653).

CREATIVITY AND LEARNING: VON GLASERFELD

Von Glaserfeld's position expressed here is a paraphrase of his work (Glasferld, 1998) and represent in our view a striking parallel to Koestler's theory of creativity. This supports the hypothesis that, "learning itself may be seen as a creative process in which meaning is constructed by the learner" (Bodin et al., 2010, p. 144). In theorizing how an individual learns new knowledge, Von Glaserfeld postulates that the ability to search for patterns, regulations, groupings and rhythms are innate. This process is accomplished through schemes, which we take to be essentially the same nature as Koestler's matrix. A scheme from Von Glaserfeld's view contains three components first, the recognition by an individual of a problem situation second, the association of an activity with this situation and third, an anticipated result from applying this activity. When an individual is presented with a problem that fits a scheme, they recall the appropriate activity and the present situation is assimilated into the existing structure what Koestler refers to as an exercise in understanding. In contrast, progress in understanding or the Piaget's equivalent of accommodation takes place when the existing situation requires modification of the schema or no schema can be found that applies. Creativity enters the learning process in this last situation. As Koestler would describe it, the solver runs through all available matrices or schemes that may apply, not finding any that fit. Then they go through the incubation period in which selective attention is applied to features or characteristics of the situation that were previously overlooked or not focused on. Von Glaserfeld employs the Piaget terms of perturbation and disequilibrium to describe this. At some point intuition, good fortune or hints from a tutor leads the solver to an analogy between some characteristic of this problem to a previously overlooked matrix or schema that is relevant. What Koestler refers to as the hidden analogy, Von Glaserfeld describes this creative moment as an analogy to a previous situation that allows one to construct a hypothetical rule or concept that explains or sheds new light on the current situation. This hypothetical rule or principle if sufficiently verified becomes the code of the new matrix or scheme a process referred to as accommodation. This creative leap based upon analogy requires the ability to recall previous situations reflect upon their relevancy and compare these previous schemes to the present situation. The simultaneous selective attention required to focus on and compare aspects of the present situation and previous relevant-analogous schemes is for Von Glaserfeld a conceptual step, a generalization. This generalization from a specific problem situation is creative in that the analogous rule is hypothetical that is a hypothesis created by the solver to be verified. Koestler's cognitive mechanism underlying bisociation, i.e. the discovery of the hidden analogy, though without the affective component, appears to be similar to Von Glaserfeld description of creativity. The synthesis of two matrices certainly allows for the abstraction of a rule from a hidden matrix to apply to a present situation that result in the development of a new code and this process is certainly creative as well as developmental in Koestler's model. In this light Von Glaserfeld's description of creativity presents a foundation one may use to situate bisociation within the context of learning theory based Piaget, which is the content of Chapter 4.1.

Pedagogy: Affect and Creativity

Throughout the collaboration of Prabhu, Czarnocha, Dias and Baker, the TR team collectively reflected upon the challenge of providing an appropriate learning environment for students in remedial classes of mathematics in the City University of New York (CUNY) system. Although these students have had exposure to mathematics in their secondary education they have failed mathematics placement exams and required a review before entry into college level mathematics (open admissions). In this situation their background in mathematics was frequently not the issue but instead the affective issues of lack of consistent motivation, their anxiety and at times apathy for mathematics. These students tend to view mathematics as a rule based experience and interpreted problem solving through an only-one-method-allowed frame in which the instructor as the authority was to provide to them (Woods et al., 2006). Prabhu conceived of the concept of the creative learning environment as methodology to motivate students and promote positive affect. The goal being to engage these students more fully in the learning process to get them to take ownership of their learning process, and to realize that their acceptance of failure is not helpful.

Shiriki's (2010) claim that students at every level should be exposed to thinking creatively, and the attempt to democratize creativity, even for resistant students results in the need to deal with student affect. Yet research on affect is arguably less available than research on creativity: "Affective issues in mathematics teaching and learning have long been under-represented themes in research" (Cobb et al., 2011a, p. 41). This is perhaps due to the belief among mathematicians and educators that learning mathematics should exclusively focus on cognitive development:

> Learning mathematics has been almost exclusively understood as a rational cognitive process of acquisition more or less along the lines the structure of mathematics makes available…Only when learning has failed, when it was difficult to understand why it failed due to some assumed misconception blocking the way, was it necessary and appropriate to think about the social/emotional aspects of mathematics. (Seeger, 2011, p. 207)

In this work the relationship between affect and creativity is explored in two aspects. The first is the creative learning environment; the social setting that provides support in the classroom for students to feel secure and valued enough to try their best. The creative learning environment is founded upon the social contract or relationship between the instructor and student. The goal of the TR team is a collaborative effort to reflect upon, and analyze pedagogy: classroom discourse, curricula and methodology in order to improve learning in the classroom. Thus, the TR team by supporting one another's effort to establish a creative learning environment in our classrooms brings about improvement of learning. The second aspect of affect and creativity studies is the relationship between affect and cognition during the creative 'Aha' moment of understanding. It has been noted that the creative experience of illumination has a transformative effect on student affect. For example, Sriramen et al. (2011) asserts that the experience of illumination can transform student affect "the 'Aha' experience has a helpful and strongly transformative effect on a student's beliefs and attitudes towards mathematics and their capability to do mathematics" (p. 124). While Leikin's understanding of illumination and other cognitive based experiences in learning by gifted students is eloquently stated, "the realization of intellectual potential by individuals improves their self-perception, crates a positive emotional background and causes satisfaction through the achievement of goals and by overcoming multiple challenges. The realization of potential determines to a great extent the future of the individual" (Leikin, 2009a, p. 386).

The hypotheses put forth by Leljehdal (2009) and (2011) that cognitive learning in general does not have the same affective component, as the experience of illumination is perhaps as controversial as it is tantalizing. The question that arises is whether the implementation of a creative learning environment that supports both the cognitive and affective aspects of illumination i.e. bisociation can transform students learned habits of failure to willingness to learn and ownership of their learning.

Prabhu's work is profoundly shaped by her realization that the central problem of learning encountered with underserved populations in the mathematics classrooms is not so much cognitive as affective. It became clear to her that the main issue our students encounter is how to access their own intelligence and talent. Thus, Prabhu realized many students were limited more by their self-identification as failures in mathematics than by actual cognitive deficiencies. This self identity as a habitual failure was built upon painful memories of mathematics. The effort to democratize creativity is founded upon the belief that creativity within the learning process is important for remedial underserved mathematics students: "A remedial algebra student can exhibit creativity as often and as clearly as an advanced calculus student" (Applebaum & Saul, 2009, p. 276). Indeed creativity is not only important for learning it may well be a requirement for learning, as Goldin (2009) states, "An indifferent person cannot be a creator" (p. 184).

We now look at the question of what is the appropriate environment to promote creativity within cognitive development that promote positive affect:

"Affective pathways are likely to interplay importantly with inventive mathematical behaviour-curiosity or puzzlement, together with a sense of courageous adventure, may evoke invention, while feelings of excitement, wonder or dogged determination may contribute to …inventive acts" (Goldin, 2009, p. 187).

Creative Learning Environment: Social Contract

Sarrfay and Novotná (2013) relate Brousseau's didactic contract to creativity within a mathematics classroom. They note that creativity when viewed as originality (Koestler) or inventiveness (Goldin, 2009), cannot be taught by an instructor: "Obviously a teacher can never teach the ability to invent new solutions (at least not directly); he/she can ask for it, encourage it, but cannot require it. This is one of the fundamental paradoxes of the whole didactical relationship" (p. 281). Sarrfay and Novotná (2013) characterize this didactic contraxt:

> The didactic contract should not be understood as a real contract formulated and signed by the teacher and his/her students, but as a didactical relationship that is established between the teacher and the student show act as if such a contract existed. (p. 283)

Prabhu (Unit 2) interpreted the didactic contract as a handshake and a compromise, in which the student's role was to live up to their potential, through active engagement in the class discourse. This involves changing the traditional lecture format with its implied didactic contract of a teacher-authority figure displaying the correct, absolute and unalterable truth of mathematical knowledge to passive students with a didactic contract in which students take responsibility for their learning and the reaching of their potential.

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1.3. UNDERSERVED STUDENTS AND CREATIVITY

INTRODUCTION

In this section we discuss the educational issues of immigrant students who frequently make up a sub-cultural often characterized by different language, race and a higher rate of poverty than the predominant culture. In addition to language these students frequently struggle with mathematics both cognitively and in the affect domain, alongside identity issues of what college means to the individual often being the first generation in their family to go to college. We also discuss the role of the instructor in promoting a creative learning environment that can transition students from habits of failure to discovery of their own excellence. A transition that is vital when working with students who come from poverty and frequently lack exposure to the dream and expectations of college.

HOSTOS COMMUNITY COLLEGE AND DEVELOPMENTAL MATHEMATICS

Eugenio María de Hostos Community College was established in the South Bronx an inner city college in the City University of New York (CUNY) system to meet the higher educational needs of people who historically have been excluded from higher education. That is to provide access to higher education leading to intellectual growth and socio-economic mobility. The college student profile indicates that the student body is approximately: 66% Female, 59% Hispanic, 23% Black, with 13% not known. Approximately 85% of entering student require remedial mathematics about 59% of students have HS degrees while 15% have G.E.D and 21% have HS degrees from another country. In general they are young adults, as well as returning students of variety of ages. It should be noted that while the students represent a variety of immigrant and native cultures, language and racial background, a common denominator of the Bronx is poverty. New York State Department of Health data for 2012 indicates that the poverty rate of the Bronx was about 31% compared to 16% in the state as a whole; with about 44% of the children below age 18 living in poverty compared to 23% for the rest of the state. The effect of poverty on completion of a High School (HS) degree is noted in (Kewal Ramani et al., 2011) as being even more important than race. Thus while Hispanic and blacks have almost twice the drop- out rates as white children; low income children are five times more likely to drop out of High School as high income children. Although the Bronx is increasing the home of charter schools the reality of children living in poverty and going to public schools mirrors a nationwide

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 43–55.

trend that has been exasperated by the economic collapse of the global markets in 2008 as reported in Washington Post article (Lyndsey Layton, May 28, 2015).

Tough (2015) notes that two trends in education in the U.S. the first is that many including minorities and immigrants make it to college but drop out all too often with debt and the second is that the wealth of the parent is a huge indicator of college graduation rate. Santiago and Stettne (June, 2013) report that Hispanic students indeed many minority students that do graduate from High School attend community college yet their graduation rate in community colleges is low and a real concern. There are many reasons for low graduation rates among minority, immigrant and low income students the high number of single mothers, students working to survive while trying to simultaneously earn money, court cases and homelessness are factors. Leonhardt et al. (2015) report on a study that documents the importance of the income level of the neighborhood you grow up in on determining ones graduation and success rate. An important academic factor for underserved students who often attend community colleges is the need for remediation in language and mathematics. Colleges place student in remedial mathematics when they are not proficient with H.S. algebra typically use placement exams. As pointed out by Lu (2013) while 60% of students entering community college require remedial courses many of these do not graduate: "Only 28% of two-year college students who took at least one developmental course earned a degree or certificate within 8.5 years, compared to 43% of non-remedial students." The article goes on to say that states such as Florida, Texas, Connecticut and Colorado have made such courses optional. Indeed, as the recent article in the *New York Times* article "Is Algebra Necessary?" (Hacker, 2012) demonstrates, U.S. society is struggling with the issue of whether proficiency in algebra is indeed necessary. In addition to algebra placement exams cover pre-algebra or basic arithmetic skills and for those entering at this level of remedial (developmental) need the outlook is even less bright. A study conducted at the City University of New York (CUNY) in 2005 (Akst, 2005) revealed that among those students who start their mathematics developmental sequence with arithmetic only 37% pass the subsequent course in developmental algebra. From this it may be inferred that the central source of their difficulties is what is known as the cognitive gap between arithmetic and algebra (Filoy, Trojano, 1985; 1989). This gap is real factor or barrier in these students' educational process and subsequent social mobility. The U.S is not alone in this issue a report from PISA 2012 (PISA in Focus, #36, February 2014) documents the difficulties that European Union as well as some Asian countries have with mathematics and the negative effects that a lower socio economic level have on the results of the test.

IMMIGRANT-MARGINALIZED STUDENT GLOBAL ISSUE

The issues of teaching mathematics to diverse students within an inner city are not unique to the South Bronx, USA. Civil et al. (2012) state that, "The underachievement of certain immigrant groups has become a globalized phenomenon in the modern world" (p. 267). Civil and Planas (2011) consider how language differences can engender, "language policies that are directed to immigrant students which are politically charged" (p. 38). Yet despite governmental pressures they note that second language minority students across the globe "find ways to overcome political restraints" (p. 43). Alrǿ et al. (2009) note that while:

> Cultural Diversity has always been present in Denmark-as much as any other society. However, the recent increase immigration of people from non-European, non-Western countries has exacerbated the discussion of cultural difference and multiculturalism, unfortunately the tone of such discussions has not necessarily been positive. (p. 14)

Civil (2010) reviews research on immigrant students in Europe and U.S. she quotes educators who characterize the state of minority students education as underserved: "African American and Latino students and poor students, consistently have less access to a wide range of resources for learning mathematics, including qualified teachers, advanced courses, safe and functional schools, textbooks and materials" (p. 1449). César and Favilli (2005) note that Italy, Spain and Portugal were emigrant countries until the late 70's at which time African children began to immigrate to these countries with second language issues. They note the special role of teacher education in creating a, "social disposition that facilities the existence of an inclusive society" (p. 1163). In writing about equality in mathematics education Esmonde (2009) argue that, "mathematics plays a central part in governmental and corporate decision making and these decisions disproportionately affect marginalized people-the very people who are less likely to have access to quality mathematics education: "Mathematical knowledge can therefore be an important component of struggles for social justice at home and abroad" (p. 1008). Tate (2005) furthers this argument stating that, "a disproportionate percentage of African American students are using curricula designed for low ability or non-college bound students" (p. 179).

William et al. (2009) look at immigrant students in a poor neighbourhood of U.K. they note the students who do decide to go to college are often the first and only in their family indeed their block to make that decision. For these students the decision to go to college is typically to either: a) become someone, b) the personal satisfaction of doing what interest them c) to obtain a vocation and hence satisfy the need of making money. For these students, as in many immigrant populations around the world, education is seen as a ticket out of poverty for themselves and often as single mothers for their new family. Difficult with remedial mathematics and entry into college math can be a rite of passage one, which they are often the first perhaps only members of their family to embark on.

MARGINALIZED STUDENTS AND POSITIVE SOCIAL INTERACTIONS

Social-cultural identity plays a major role in student acceptance or rejection of the mathematics classroom environment as well as the teacher acceptance or rejection of student behaviour,

> Students who come to school already speaking the school Discourse and behaving as teachers expect are provided more opportunities to learn than their peers are. …Those who do not embody these privileged ideas are often positioned as being deficient and difficult to teach and so report that their teachers do not seem to care whether they learn mathematics. (Edmonde, 2009, p. 1018)

For this reason a major component of reform pedagogy that is often applied to marginalized communities focuses on small group discussion, and discovery learning in which student create meaning working together. Edmonde (2009) reviews literature about the social interaction specifically in small groups of minority and marginalized students. He notes specific social processes that literature supports help with learning and achievement that include, "asking question, discussing problem-solving strategies, observing someone else's problem-solving strategies, teaching a peer, resolving a disagreement or conflict, and explain one's thinking" (p. 1016). Other critical thinking skills that Edmonde lists as important for developing conceptual understanding include time spend reviewing and discussing, "multiple strategies and solution paths…explaining their thinking, asking specific questions and making connections" (p. 1028). Clearly the quality of the critical thinking and reasoning is the important factor. In contrast, Civil and Planas (2004) note that when a (middle school) teacher employed pedagogical activities that encouraged student discussion and participation in the mathematics classroom more popular students (those involved in athletics) were the centres of attention until the discussion became more academic when those who were recognized as gifted in mathematics became the focal point. They conclude that social exclusion of certain groups or types may occur during such pedagogy.

Creativity and Marginalized Students

In a review of literature on what stimulates creativity among marginalized students (Haley et al., 2006) suggests several characteristics: authenticity of the themes and tasks, novelty the sense that it was different, the role of a mentor and the freedom to explore. These authors study one program that prmpoted higher order thinking skills with disadvantaged students through the use of: computers, drama and Socratic thinking i.e. creative and critical-logical dialogue between teacher and pupil that required more than one word answers on the part of the student. The effects of this program included enhanced ability to: explain ideas; engage in conversation, problem-solving skills and increased confidence and motivation.

It was the firm belief of Prabhu that the pathway to reach underserved students was through their creativity; i.e. when students begin to enjoy thinking and reasoning within mathematics they transition from their habitual acceptance of failure to ownership of their learning and excellence of their potential. This creative process is both cognitive as students began to construct meaning of mathematics for themselves and affective as they struggle with their image of themselves as someone who cannot do math or as a family member of immigrants and/or a racial group that traditionally has not gone to college. Common themes that run through literature on what works with serving the needs of these disadvantaged students include: quality student-teacher relationships, supportive administrations-schools or colleges and a sense of community for students eliminate within the context of high expectation of success. As a learning of community of teacher researchers in the South Bronx this work contains some of the results of our focus on the design of instruction and pedagogy within the mathematics classroom to encourage student creativity and engage them in their learning process.

Prabhu's realization that underserved students in remedial mathematics must be reached through the affective domain simultaneous with the cognitive domain to complete a transition from habits of failure to excellence highlights the importance of the teachers role in promoting a creative learning environment as well as the relationship between affect and cognition.

The Importance of Affect on Student Cognition

As noted by Furinghetti and Morselli (2009): "The most important problem in research on affect in mathematics is the understanding of the interrelationship between affect and cognition" (p. 72). Studies on affect tend to employ at least the following three components beliefs, attitudes and emotions, "emotions are most intense/least stable, beliefs as most stable least intense and attitudes in between," (Rosetta et al., 2006). Goldin (2009) attributes the lack of positive affect to be the main reason student drop out of challenging mathematics, "it was the affective dimension that in my view played the primary role" (p. 182). Goldin hypothesizes an internal representation system for an individual's affect that is central to the relationship to problem solving and mathematics, "human affect serves as an internal representational system, encoding meanings, facilitating communication, and (like the cognitive representational system) contributing to or impeding mathematical power" (pp. 182–183). Goldin goes beyond the hypothesis that creating or illumination has a powerful effect upon an individual's affect in asserting that "states of emotional feelings do not merely accompany cognition. Rather, the emotional feelings themselves have signification; that is they encode information or carry meanings" (2009, p. 186).

Polya's first stage of problem solving 'understanding the problem' involves both reading the problem and relating or processing the information in a meaningful manner, successful solvers' often reformulate the problem through, "…gestures, words, pictures, symbols sketches, examples and so on" (Furinghetti & Morselli, 2009, p. 72). Students who cannot associate the problem information with an appropriate matrix or scheme will not be able to assimilate the information into the scheme. Either because they do not recall a relevant matrix or because the scheme they attempt to employ is inappropriate. At this point student beliefs about their own ability can already be a factor many students chose the simplest strategy available, the one done immediately before by the instructor or focus their attention on superficial problem information that is within their safe zone. They may feel frustrated when their choice of strategy doesn't work out particularly if they perceive that other students can solve the problem but they cannot. "The frustration may evoke useful problem-solving heuristics leading to…such as trying a simpler, related problem. On other occasions…the feeling may rapidly give way to anxiety or despair, and evoke avoidance strategies" (DeBellis & Goldin, 2006, p. 133). Thus, the affective experience of frustration during problem solving i.e. blind alleys because there is not available schema or matrix to assimilate the problem information can lead to searching for a new strategy. As noted by Schlöglmann (2009) humans when confronted with a situation that leaves negative affect tend to forget the experience we tend to suppress unpleasant memories and try to avoid such situation in the future. This statement about human nature and affect could help to explain the short memory span and avoidance behaviour of some students for mathematics problem-solving and mathematics classes in general.

The feeling of frustration during problem solving is what Goldin refers to as local affect "the rapidly-changing states of feeling that occur when participating in an activity…engaging in mathematics" (Goldin, 2009, p. 187). Strong emotions such as frustration or elations during problem solving can lead to global affect or the individual's attitude towards mathematics i.e. math anxiety, the common reframes of "I hate math" or "I hate fractions." Thus the period incubation when the solver cannot find an readily available matrix or scheme to assimilate the problem information can lead to the creative and transformative affective-illumination and cognitive-accommodation experience or to individuals who question their problem solving ability and a society that questions the importance of mathematics to real life. We now look at the question of what is the appropriate environment to promote creativity within cognitive development that promote positive affect.

TEACHER'S ROLE

"Traditional teaching methods involving demonstration and practice using closed problem with predetermined answers insufficiently prepare students in mathematics. Students leave school with adequate computational skills but lack the ability to apply these skills in meaningful ways. Teaching mathematics without providing for creativity denies all students, especially gifted and talented students, the opportunity to appreciate the beauty of mathematics" (Mann, 2006, p. 236).

We began our consideration of the role of the instructor with some reflections on constructivist's pedagogy that underlies a shift from a classroom methodology based upon an active instructor lecturing to passive students to an environment with dynamic interaction and active student engagement As Sawyer (2004) notes, "The basic insight of constructivism is that learning is a creative improvisational process" (p. 14). Sarrazy and Novotná (2013) state the constructivist position on the role of the teacher as bring the guiding consciousness that helps students bring dead math back to life: "To achieve this revival, teachers must create situations in which they can show students the use, the interest, and other aspects of the mathematics" (p. 281). We consider that Koestler would share this view that, the role of education is to bring the thrill of original discoveries to the student by a reconstruction of the discovery process within the class. Bodin et al. (2010) argue that, "the teachers role is the key to creative thinking in the classroom" (p. 145). These authors agree with the constructivist assertion of Von Glaserfeld that creativity is integral to the learning process. That is, an individual is being creative whenever he/she creates new meaning for themselves, "children are being creative… when they produce something new to themselves, as when they construct meaning for symbols, signs and operations, make sense of a mathematical problem, devise a way of solving it." Yet as noted by these authors pre-service teachers find it, "difficult to be specific about encouraging and assessing creativity in mathematics lessons" (Bodin et al., 2010, p. 144).

Leikin et al. (2013) note that "teachers consider themselves as a key factor in developing mathematical creativity without holding themselves accountable for concurrently hindering creativity…they are more likely to blame the educational system…" (pp. 210–211). Thus, the vagueness or lack of a clear definition of creativity within the learning process for ordinary students and pressure to complete a syllabus focused on procedural skills appears to leave those who search for creativity within the classroom at a loss. Shriki (2010) states that, "Although most teachers would agree that it is important to develop students' creativity the literature indicates that creativity is not normally not encourage at schools" (p. 161). More to the point Shriki conclude that, "it should be noted that I could not find specific recommendations or guide lines aimed at providing teachers with an assessment tool for evaluating students' creativity" (p. 162). The lack of a specific commonly accepted definition of creativity and its role in learning and by this we include the affective component results in teachers who teach the way they were taught. Goldin (2009) describe the affective side of the mathematics classroom experience that most of us can relate to:

> School mathematics often presents an affective context for mathematics that is not very conducive to trust or intimacy…sometime the teacher seemed to place the highest value on speed and accuracy of routine computations…with painful negative consequences for the children's self-esteem as they were neither especially fast and accurate nor neat. Opportunities for inventiveness or creativity were relatively infrequent, since mathematics tended to be presented as systems of rules to be learned and procedures to be followed. (p. 182)

Sawyer (2004) describes a creative learning environment based upon socio-constructivist approach as one in which learning within the classroom is a process of co-construction where communication between members of the group provides meaning for new concepts. He describes creativity within the classroom in terms of a transition from the mindset or didactic contract of, "The teaching as performance metaphor" to one an environment characterized as "improvisational performance" (p. 12). The importance of creative improvisation is that without such openness to student's thoughts and ideas i.e. when the instructor scripts the discourse by following a text, "students cannot co-construct their own knowledge" (p. 14).

Norton and D'Ambrosio (2008) deliberate on the socio-constructivist viewpoints of Bruner and Vygotsky in which meaning is internalized through classroom discourse. They note that the teacher must be engaged in the assessment and refinement cycle in order to prepare lessions appropriate to support student creativity and learning i.e. "the teacher must continually establish meaning of the students' language and actions so that the students' actions guide the teacher…developing new hypotheses about students'; cognition while remaining open…in order to design tasks to provoke creative activity in the students" (p. 225).

Prabhu frequently notes the uniqueness in the Teaching Research cycle of the refinement stage where a bisociation occurs between two frames of reference: that one from the past cycle with the information about its instructional effectiveness with the one to envision for the next cycle, which aims at the elimination of the ineffective components in order to refine curriculum material classroom pedagogy and to reflect upon how the results gleamed reflect upon learning theories. We point out that this process necessitates the involvement of the teacher-researcher synthesizing the role of researcher with their craft experience to bring creativity into classroom and to reflect upon how what occurred in the classroom reflects upon learning theories. Thus, one goal is that an inspired instructor will bring motivation into the classroom and thus, students may experience the energy and enthusiasm for mathematics and the process of learning mathematics engendered by this process. In a teaching research experiment designed to support a creative learning environment the goal is for learning theory to assist teacher-researchers construct creative curriculum and pedagogy with assessment and refinement as part of the process. An equally important goal is to reflect upon the relationship of what occurs in the classroom to educational research on creativity and learning. Cobb et al. (2011) note that: "In my view, the most important contribution that theory can make to educational practice is to inform the process of making pedagogical and design decisions and judgements in particular cases" (p. 112). The guiding philosophy of teaching research to implement a creative learning environment is that teachers-researchers must simultaneously exist in both frames of reference (bisociation) otherwise creativity cannot be sustained, the transformation of students occurs one day at a time, brilliant insights and multi-year projects that provide for teachers to drink from the source of educational literature and theory is not sustainable. In the same article Cobb et al. (2011) states:

> The increasing importance that we came to attribute to the teachers' central mediating role is at odds with the way in which the teacher is backgrounded… the teacher's initiatives and her responses to students are treated as ancillary to the focus on students' learning. (p. 114)

A creative learning environment necessitates a creative teacher-researcher not as an ancillary focus but as part of the central focus on the dynamic of the classroom and as an integral part of the research team. For none other than the teacher can bring creativity into the classroom with a sustained effort through assessment and refinement.

The teaching research paradigm for promoting a creative learning environment is founded upon teacher involvement in researching his/her own craft, as does action research (Benke et al., 2008). The distinction being that in teaching research the curricula and pedagogy of the instructor can change with time and circumstance as long as they are committed to a methodology to inspire students engagement in their classroom, reflect in a meaningful way on their actions and results and relate this to educational research on the phenomena way of teaching and student learning.

The effort to encourage student engagement within the classroom requires openness to student ideas, input and suggestions even when they take the instructor off topic, the instructor has to encourage and prod student to engage from day one promoting a social contract in which participation is expected however introductory college classes are all too often characterized by large size and thus students who have negative or low affect towards mathematics tend towards non-participation. In secondary education negative student affect is often observed during the transition from the small class size of primary schools to the large impersonal size of mathematics classes in middle or junior high school. This transition frequently marks the end of student's positive affect towards mathematics (Athanasiou & Philippou, 2009). Even in small classes with students who are motivated it is frequently the case that one or two students dominate the classroom dialogue while weaker student and those with negative self images wait passively for another student to answer or go to the blackboard. Thus, the expectation that all students will participate is an essential component of the social contract that must be established in day one and reinforced throughout the class in order for students to live up to their potential i.e. their excellence.

The relevant question, at the heart of the social-constructivist argument that the instructor's role is to create an environment in which knowledge is co-constructed through classroom dialogue is eloquently posed by Norton and D'Ambrosio (2008) "how a teacher can know whether a student has meaningfully imitated an action or whether she has simply mechanically repeated observed action that she was trained to follow. When does a teacher's assistance generate meaningless habits, and when does it promote development" (p. 221). Student learning especially in social situations can be illusive, active participation in a discussion and supplying steps for the instructor on one day is often followed by 'the next day effect' in which blank stares greet an instructor's attempts to get students to solve the same question.

Norton and D'Ambrosio (2008) suggest that an essential tool in the instructors' repertoire is scaffolding which they characterize as, "selecting an appropriate task, directing the students attention, holding important information in memory and offering encouragement" (Norton & D'Ambrosio, 2008, p. 223). This description of scaffolding provides a foundation for the learning community that was formed by Prabhu, Czarnocha, Baker and Dias to study the effects of inquiry based learning during problem solving with remedial students in the South Bronx (C^3IRG CUNY Grant). Scaffolding and structuring of exercises can been seen in the work of Prabhu in the creative learning environment as well as the work on proportional reasoning with rate by Czarnocha, Dias and Baker. In particular scaffolding is important to support the foundational cognitive groundwork for bisociation as the instructor identifies the schemes-matrices that are to be synthesized and the concepts that are to be bisociated.

One goal of social constructivist learning is to support the creative process of students providing meaning to new concepts during problem solving. This goal focuses attention on how students understand and reformulate problem information. That is the process of "decoding" the original problem and the subsequent process in which the individual must "self represents a model" (Singer & Voica, 2013). Koestler (1964) points out that much of the reasoning that mathematicians accomplish during illumination is through visuals and only much later expressed in and with much effort expressed in language. Prabhu employed the Fraction Grid a self-made visual display of the fractions in order to assist students give meaning to the measurement subconstruct of the fraction. While Professor Dias and Baker employed the two sided number line as a visual in order to give meaning to proportional reasoning for students transitioning into elementary algebra.

A final component of the Teaching Research Methodology to support and sustain a Creative Learning Environment developed by Prabhu and Czarnocha was a team of collaborative teacher-researchers. The collaboration occurred within the class as instructors worked with colleagues within mathematics and counsellors from student development to assist students with affective issues and self-regulated learning. However, it was the collaboration as a team of teacher-researchers that is most remarkable and distinctive. This team of collaborative researchers would be referred to as a community by Lin and Ponte (2008) that is, as a self-grouping through personal interest in educational research. "Communities are regarded as self-selecting, their members negotiating goals and tasks" (Krainer, 2008, p. 5).

The learning community (Unit 5) is not simply instructors lead by an educational researcher. A model we note would leave the instructors passive following the lead of the expert. The top-down model of an active instructor lecturing to passive students has been rejected by social constructivists as not being effective in producing a creative learning environment in the classroom. Following this logic the top down model of a researcher telling teachers how to create and assess pedagogical curriculum we believe needs to be reconsidered. That being said an excellent example of a top-down approach in which the researchers encouraged the teachers to be actively involved is the learning community described by Jaworski (2008). In this community the experts or didactians constantly met with and included the teachers in all phases: curricula and lesson plan development, implementation and assessment and of course refinement of the results.

In our teaching research community the instructors function as both researchers and teachers the creativity and motivation engendered within the community is then brought into the classroom in an effort to infect the students through like minded dialogue with one another. The learning community has much in common with action research (Benke et al., 2008) however the goal is not just to improve student fluency. The teacher researcher operates on two frames of reference simultaneously reflecting upon curricula and pedagogy to influence student affect, engagement and creativity but also reflecting upon the learning theories that underlie learning and creativity within the classroom experience.

The bissociative nature of the TR NYCity model provides a well-equipped conceptual framework for understanding and analyzing both affective and cognitive aspects of student's creativity in the learning process. As such it provides a foundation to motivate student transition from habits of failure to excellence.

The community of the TR team of the Bronx supports the transition from habits of failure to success. The members share the successes, joys and disappoints as well as the insights into how theories of learning can be interpreted and translated into craft practice. The importance of a learning community is not only to inspire the instructor to try new approaches, assess and refine her methodology in an effort to inspire student; it also provides a support network when the frustrations that students and instructors feel at low performance and inevitable poor test results lead to discouragement. The axiom that an instructor cannot give up on low performing students first, because everyone else already has, and second, because if you do they will give up on themselves necessitates a community.

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UNIT 2

CREATIVE LEARNING ENVIRONMENT

INTRODUCTION

Creative Learning Environment (CLE) contains the teaching-research (TR) reports of Vrunda Prabhu, the first among the Teaching-Research Team of the Bronx, who realized that the central problem of learning we encounter in the mathematics classrooms is not a cognitive one but an affective one. It became clear that the main issue our students encounter is how to access their own intelligence and smartness blocked by the negative affect, or old memories or even because the absence of elementary critical thinking skills which would open their minds (and hearts) to themselves. Ultimately Prabhu asserts: "*Creativity in teaching remedial mathematics is teaching gifted students how to access their own giftedness*".

Five chapters of the unit provide an unusual opportunity to see the full scope of the development of CLE, the impact that process has upon the teacher-researcher herself (Chapter 2.5) and the manner in which her reflections are impacting practice of younger members of the team (Chapter 2.6).

Vrunda Prabhu led her teaching-research investigations along two parallel environments: in her home institution of the Bronx Community College (BCC) as well as in community schools of Dalit villages of Tamil Nadu, India. While both routes are intertwined in each chapter, Prabhu addresses the closely related question of generalizability of TR work directly in Chapter 2.2.

Having understood the role of negative affect in blocking student access to their own excellence, as she would put it, she started the design process of Creative Learning Environment through several directions. First, she deconstructed the concept of didactic contract to eliminate the ambiguity present in the term it acquired since the important work of Brusseau (1996). The Brusseau formulation of the contract is somewhat paradoxical as it cannot be recognized unless it is broken: the students can learn only when they accept that they will not to be taught everything; that they accept the necessity of engagement in an activity in which they can learn mathematics. As Sarazy and Nowotna (2013) point out, the "didactical contract should not be understood as a 'real' contract formulated and signed by the teacher and his/her students". However, nothing short of true reality can anchor education in the Bronx or in Dalit villages of Tamil Nadu. Prabhu's understanding of the didactic contract gets directly into the core of the paradox of how to facilitate true and wilful engagement in mathematics in the context of generally negative student attitude towards the subject. That is why Prabhu is seeking here the Didactic Contract as a Handshake and a Compromise, whose content is the direct question to the class "What is your *excellence*, and are you falling short in fulfilling it?" following the goal stated at the first meeting: "Your *excellence* is the goal of this classroom. Are you ready to pursue this aim, jointly, throughout the course of the semester?" The aim of this approach is to get at the very centre of students' resistance to learning, and turn it into their self-fulfilment through the commitment to develop their own potential. Ultimately, it is the Handshake of the student with himself/herself in mutual understanding of the commitment to reach their best. Compromise is needed as the strategy of dealing with one's own resistance in the context of the Handshake. Together they express the conviction that without active, conscious, committed student participation in learning, no learning can take place, especially in college. Handshake and Compromise provide an inspiration and at the same time delegate an irreducible responsibility for maintaining the inspiration in the hands of the student. Therefore, the second aim of the teacher-researcher became the creation of such a learning environment in which student active participation will be assured. Her natural predisposition was to look for didactic inspiration in connecting mathematics with art and drama. She saw drama situations created in the mathematics classroom as the mediation milieu between mathematics and students. Creativity, doing art, participating in drama serves the purpose of engagement by students (see Chapters 2.2 and 2.3).

Creativity, Literacy and Numeracy approach of Chapter 2.2 was originally created by Prabhu for the women centred program Montessori for Mothers in Tamil Nadu, India and discussed closer in Chapter 5.3.1. The design of that program, which she saw as the generalization derived from separate classroom teaching experiments and investigations, prompted Prabhu's deep reflection. She searched for the general answer to the question "where is the general character of the TR work hidden?" She found the Fraction Grid (FG) she designed as the didactic artefact to be the pure expression of generality derived from many different teaching situations and refined through craft knowledge and/or research knowledge of the profession. The integration of mathematics and art within FG shows also fascinating hyperbolic patterns, which allow to construct converging sequence of fractions, and to find the general pattern of those sequences already as a conscious practice of generalization by students. She formulated the new concept of artefact generalization within teaching practice for mathematics education. The general process of artefact generalization was formulated by Nagel (1979) as "an experimental law" or still earlier by Merton (1968) as an "empirical generalization" similar to other branches of science—"an isolated proposition summarizing observed uniformities of relationships between two or more variables" (Merton, 1968, p. 149). In the case of Prabhu's considerations, the process of empirical generalization occurs on two levels: (1) as the results of refinement of the same artefact through many didactic activities, and (2) with the two variables have been the classroom environment in the Bronx and the classroom environment in community schools of Tamil Nadu.

Her work warrants a new conjecture for the existence of two distinct routes of generalization within the teaching-research craft, one through the design of the artefact which embodies the experience of at least two or more didactic activities – the "empirical generalization", and the second one, and through the creation of, or the coordination with a general theory, which can be discerned in specific situations under considerations.

She proceeded further in the chapter to characterize the mediation role of both art and drama between mathematics and the student, pointing out at the same time that the very process of mediation creates the favourable conditions for the fulfilment of Handshake and Compromise didactic contract. The search for the best theoretical framework to place Fraction Grid connected Prabhu with Brunner's learning theories (Bruner, 1966) applied in the context of Zone of Proximal Development (ZPD) of Vygotsky (1987). She framed her classroom approach in terms of the Discovery method of teaching making contact with the realization of Brousseau and Novotna (2008) that the teacher cannot teach (at least directly) this ability of creating new solutions: he/she can demand it, expect it, motivate it, but cannot require it. The Discovery method, natural for TR/NY city model is an excellent method for the facilitation of student efforts in creating new solutions (Chapter 1.1).

This precisely was the Prabhu's motivation for framing her teaching in terms of the discovery method; as a teacher-researcher she preferred the Discovery method of teaching aimed at the facilitation of the development of student schema of thinking. Appendix to that chapter contains the approach to teaching fractions based on the Fraction Grid whose initial idea led to the Creativity-Literacy-Numeracy program. She realized that FG as an artefact together with FG approach to fractions represents generalization derived from the particularities of each classroom situation, which can be effectively applied, therefore, in different learning environments.

Many authors have investigated the role of artifacts and their mediation in learning since the work of Vygotsky (1978) on semiotic mediation. In particular, Rabardel (1995) introduced the distinction between the artefact and the instrument paying significant attention to the development of its two aspects:

- Instrumentalisation, concerning the emergence and the evolution of the different components of the artifact, e.g. the progressive recognition of its potentialities and constraints.
- Instrumentation, concerning the emergence and development of the utilization schemes.

Both of them are dialectically related to each other within TR cycles leading to the Thinking Technology defined in Chapter 1.1.

Chapter 2.3 presents the second methodological tool Prabhu utilized for the formulation of CLE in the classroom that is *problem posing/problem solving* dynamics. This tool helped her to focus student's attention on mathematics for a sufficient period of time, so that the students start enjoying the subject through the use of cognition, positive affect, and desire to develop self-learning skills as a consequence of their problem solving activity. The new component was important to be included in the structure of CLE, as Prabhu's classroom observations led her to the realization that the Handshake and Compromise didactic contract had to be supplemented by more direct attention to the issue of affect. Thus, her trajectory of CLE development intersects here with the work of Goldin and DeBellis (2006) that analysed in detail the relationship between cognition and affect.

Posing problems/solving problems is one of basic tools of teacher-researchers, with the help of which they can probe the inner structure of students' *cognitive* ZPD in order to design our questions within, what Murray and Arroyo (2002) called, *affective* ZPD of the student. Moreover, the tool plays a standard double teaching-research role: for the teacher-researcher, it allows to probe student understanding while investigating the scope of student cognitive ZPD relatively to the mathematics in question. For the students, it creates often necessary scaffolding leading to understanding and/or grasping the concept in question. When joined together with drama nature of interaction in the class, problem posing becomes classroom norm, so that next level exercises, when students design and pose new problems by themselves is easily assimilated and solved. Using revised Bloom's Taxonomy (Krathwohl, 2002) where the Synthesis at the top of the Bloom's pyramid has been substituted by Creativity, she can demonstrate how drama as the didactic tool in mathematics classroom can help to penetrate that pyramid top down, instead of passing through the traditional upward development.

Consequently, Chapter 2.3 presents the convergence and development of the threads introduced earlier, while at the same time it prepares the ground for the coordination of Prabhu's practice with the Koestler's theory in the next chapter. This subtle process of integration/preparation became possible when Prabhu amplified the instructional staff of her classroom to include two new teacher-researchers, the Vice President for Students Development in the Bronx CC, a specialist in student affect, as well as the librarian who was interested in self-regulated learning. Hence, the third component of the CLE developed by Prabhu was simultaneously collaborative teaching with two or more instructors in her class, each fulfilling its special role based on the individual's expertise. That process merged the creativity of the instructors with the creativity of students. Note that each instructor, a librarian, student counsellor or a mathematician was expressing its own take upon the particular mathematics of the classroom, each with the same goal: to engage student in the process of understanding. Thus, the whole classroom worked with several different frames of discourse, that of instructors and that of mathematics and art. It is not surprising then that Prabhu, in her continuous creativity literature search, paid attention to the Koestler condition of the presence of two separate frames of reference for the occurrence of Aha! Moments – the spontaneous leaps of insight, the essence of creativity.

Chapter 2.4 of the unit presents Vrunda Prabhu full integration of the path she traversed through the search of CLE for the Bronx students with the Koestler theory of the Act of Creation as the proposal for the CLE based on cognition, affect and self-regulated learning. The objective of the proposal has been to reverse the culture of failure through development of student ownership and enjoyment of mathematics. Three instructors were teaching the course collaboratively and simultaneously in one classroom: mathematics instructor, the Vice President for Student development who provided affective understanding of the situation while being mathematics friendly, and the librarian, the specialist in self-regulated learning. Together they have integrated cognition, affect and self-regulated learning into the basic core of the syllabus and its philosophy. Among many theories of learning coordinated with different aspects of the teaching experiment, Koestler theory together with Polya's approach to problem solving was responsible for the design of the Creative Problem Set so that "bisociation was facilitated, as the creative leap that occurs when several frames of reference are held in simultaneous scrutiny and insight, apparent from the various simultaneous perceptions conveyed by the students…" Note the effective approach of the Prabhu/counsellor pair in creating slight perceptual shifts to keep the focus of inquiry described in Chapter 2.3. "The counsellor method was to switch the frame of reference, while keeping the underlying mathematical focus constant. For example, if the task were to calculate 1/2 + 1/3, the counsellor would switch the frames of reference from pizza to cookies to something else…" creating this way several different bisociative frameworks. Prabhu offered a series of assignments, which in her experience of the teaching experiment, helped students to develop an intuition and interest in bisociative thinking. She ended her discussion of this experiment which integrated her experience as a teacher-research in search of the CLE with Koestler theory with the following words: *Creativity in teaching of remedial mathematics is teaching gifted students how to access their own giftedness.*

We complete the presentation of Prabhu's search for the Creative Learning environment with some pages from her Teaching-Research journal written contemporaneously with the conduct of the teaching experiment. She investigated in these pages the impact of the creativity principle, its reflective action upon herself as a teacher-researcher. She pointed to this unique spot of the TR cycle, the refinement stage where bisociative framework is obtained by confronting two frames of reference: the design and its effectiveness from the past cycle with the one to envision for the next cycle aiming at the elimination of the ineffective components, possibly with the help of an appropriate learning theory and relevant research results. The creativity principle applied to teaching is aimed at the transformation of the habitual teaching into the teaching of the teacher-researcher whose two components, teaching and research do constitute a powerful bisociative framework of action and thought. Prabhu proceeded to identify areas of her practice impacted by that principle. She saw it in the development of concept maps with the help of which she was transforming the design of instruction as impacted by the continuously changing knowledge of student mathematical thinking, the choice of the relationships within the schema of the relevant mathematics concepts to be emphasized, and through the educational research results that might throw light upon the best pedagogical approach. The concept maps of Prabhu represent the next example of artefact empirical generalization, which we will call Prabhu Generalization. The unit closes with the work of the teacher-researcher apprentice from the younger generation to see how the principles of didactic contract formulated by Prabhu are shared and interpreted by the members of the team (Chapter 2.6).

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VRUNDA PRABHU, BRONISLAW CZARNOCHA
AND HOWARD PFLANZER

2.1. THE DIDACTIC CONTRACT, A HANDSHAKE AND A COMPROMISE

A Teaching-Action-Research Project

SUMMARY

Having understood the role of negative affect in blocking student access to their own, as she would put, excellence, Prabhu et al. started the process of Creative Learning Environment design through several routes. First, she deconstructed concept of didactic contract to eliminate the ambiguity present in the term it acquired since the important work of Brusseau. As Sarazy and Nowotna (2013) point out the

> didactical contract should not be understood as a "real" contract formulated and signed by the teacher and his/her students, but but as a didactical relationship that is established between the teacher and the students who act as if such a contract existed…

However, nothing short of true reality can anchor education in the Bronx or in Dalit villages of Tamil Nadu. That's why Prabhu et al. is seeking here a Didactic Contract as a Handshake and a Compromise, whose content is the direct question to the class "What is your *excellence*, and are you falling short in fulfilling it?" following the goal stated at the first meeting: "Your *excellence* is the goal of this classroom. Are you ready to pursue this aim, jointly, throughout the course of the semester?"

The Choice, a drama participatory script is an important example of the technique and the degree to which mathematics friendly drama scenes can impact student mathematical creativity. It was the tool with the help of which she was able to reinstate the Didactic contract of the Handshake. Handshake is an act of friendliness between the two people; Prabhu's task has been to transfer that handshake to the space between the student and himself/herself. As a result, in the context of the enactment of the drama within the classroom, students relax their tightly held resistance to learning and convey their interests in a few words, thus reading the world of mathematics independently. Drama is the reconciling element (Freire, 1973) mediating between mathematics and the learner.

It facilitates in rethinking the existing tension of prior mathematical experiences and resistance to learning that is a block preventing full access to creative memory.

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 63–73.

Drama alleviates this resistance, freeing moments of access to creative memory. She finally attacks the sources of student resistance and she finds them in the traditional negative student classification in the classroom as well as in the society at large. Drama/mathematics connection gives her a handle on the negativity in the classroom; to address negativity at large, she proposes, imported from Tamil Nadu, TAR (teaching-action-research) as generalization of TR (Chapter 5.3), that is the process through which teachers of the school becoming at the same time community organizers along community's transformation in the direction that takes the interest of children as its main criterion. One of the ways she proposes to impact the society at large is through New York Subway poster exhibition series Mathematics in Motion.

INTRODUCTION

In our classrooms, there might not readily exist a complete and clear understanding of the concept of the *Didactic Contract* (Brousseau, 1997) without the need for further explanation. The concept of *Compromise* might be interpreted differently among different individuals (Czarnocha, 2009). The word *Handshake* avoids these ambiguities. Two individuals start to move from their static position and work toward a handshake, making contact with each other's point of view. Each evaluates the satisfaction of that handshake independently. In the present context of The Didactic Contract, a Handshake and a Compromise, a learner starts from his or her spontaneous (Vygotsky, 1986) understanding expressed in daily classroom actions and works toward a desired goal. The goal is the same for the instructor/teacher-researcher and the student: excellence.

It is the beginning of the semester, and the entire first class meeting, if needed, is spent on many informal discussions including a reading for the semester (PROM/SE, 2006) centred around the theme of a *Handshake*. "What is a Handshake?" is the persistent question posed by the teacher-researcher. Expressed more formally, the question to the students is "What is your *excellence*, and are you falling short of fulfilling it?" The first day of class is the most important. It is the day when the teacher-researcher is firmly stating the objective: "Your *excellence* is the goal of this classroom. Are you ready to pursue this aim, jointly, throughout the course of the semester?" Different forms of support, developed over the course of many teaching-experiments in a variety of classrooms over a period of more than five years are utilized. This includes Self-Assessment Reports, which allow a day-by-day self-evaluation of one's own work ethic and attitude toward learning. The evolving learning or a lack thereof, is the gauge of satisfaction with one's own Handshake with him or herself in the quest of one's own *excellence*.

STATEMENT OF THE PROBLEM

Disenfranchisement from mathematics is at staggering proportions in developmental mathematics classes in the community colleges of the Bronx, and so is the accompanying resistance to learning. Resistance toward mathematics, evident societally, creates an obstacle to learning that, in the classroom, presents a serious hurdle to counteract. Hence, there exists an urgent need for the *handshake* and *compromise* of the *didactic contract* – a critical need to reverse the resistance to learning by creative means of engaging learners to reconnect with the natural enjoyment of mathematics. Drama provides one such medium through which this communication can be initiated.

The Didactic Contract as a Handshake and a Compromise is the aim of the teacher-researchers in the classroom where the goal of the TR/NYCity methodology of teaching-research is the improvement of learning. Without active participation in one's own learning, or, at times, with an active resistance to it, one violates one's Handshake with oneself. As a result, the Handshake in the classroom environment is affected and each participant, teacher-researcher and student, can gauge the extent of its non-satisfactory status, with each party observing one's own level of participation and continuously questioning one's own dedication to their excellence.

SIGNIFICANCE OF THE PROJECT TO THE ACADEMIC DISCIPLINE

The present collaboration has had prior trials in developmental classes of the Learning Communities Project at Bronx Community College, where the professors staged a one-session participatory performance called *The Choice*. Students with tremendous *bonding*, created by a group of just nine taking three classes together as a learning community, had learned to avoid each other's fears of mathematics well. Through that one-session performance, the instructor was able to cultivate the students' courage and guide their first attempts at making contact with those aspect of mathematics they enjoyed. Via the participatory, dramatic intervention of *The Choice*, the team of teacher-researchers made great strides toward a possible a return to the students' goal of *excellence*.

The booklet of end-of-term activities presented by each student to the Learning Communities, called Excellence in Discovery of Number was born through the intervention accomplished by *The Choice*. Having found pride in one's own work, even while working to not allow it its full expression, the students had proudly described their own mathematics through such projects as *Natasha's Dots*, *Stephanie L's Primes*, *Giselle's Triangle*, just to name a few. The break in the resistance to learning, permitting students to see and recognize the vision of their own mathematics through clear readings and writings, that occur in a friendly, public space of the classroom, even in the span of a single semester, works significant miracles in changing students' attitudes toward mathematics. The significance of the collaborative work lies in its contribution to (i) the formation of a mathematics-friendly community and a positive attitude towards mathematics in the classroom, and (ii) the investigation of the process of employing drama to involve people in mathematics. It is not known if it is commonly practiced as a teaching strategy in other disciplines; however, it is definitely the case that drama is not a commonly embedded aspect in the teaching of mathematics.

BACKGROUND

The collaboration of two mathematics professors and a drama professor from three different colleges of the City University of New York (CUNY), one senior college and two community colleges located in the Bronx, was designed and implemented in 2008, to address common learning difficulties for students in a developmental mathematics class, as part of the Learning Communities Project at Bronx Community College, described earlier.

In the situation described above, the students' combined resistance to learning obstructed the classroom didactic contract, or the handshake. This prevented the instructor from finding an inroad into the specific difficulties the students were having. By the enactment of *The Choice*, students' spontaneous discussion about whether they prefer the flat screen TV or a visit to grandma in California, the situation in the classroom lightened from the usual mathematics they were accustomed to seeing and ignoring, while the instructor, careful to intervene when an opportunity presented itself, was able to pick up on students' expressed interests. For example, Stephanie who in regular class was unable to tear herself away from her phone, relaxed her short attention span, and, for the most part of the hour *forgot* her phone, and started talking about the patterns in prime numbers. While a conversation about primes continued between the instructor and Stephanie L., Natasha, who was extremely timid, allowed herself to relax and talk about simpler patterns found in counting numbers. Sylvester, a self-proclaimed *hater* of mathematics, usually extremely reticent in class, decided to stand alone and defend his choice of a visit to grandma versus the flat-screen TV, insisting on firmly expressing his position.

The opportunity for classroom relaxation of long-held fears obstructing the possibility of a handshake in the learning process between instructor and students was created by the enactment of *The Choice*, and the conversations among the three instructors and the group of students.

Since December 2004, the TR/NYCity methodology developed and successfully practiced in the Bronx was incorporated into community development projects (TAR) in Tamil Nadu (TN), India. With lessons learned from *The Choice*, the team imagined many further avenues of collaboration in Tamil Nadu, India and the Bronx.

Utilizing Vygotsky's ideas as the theoretical inspiration for the navigation along the Zone of Proximal Development (ZPD), aiming to achieve the passage from spontaneous to scientific concepts in the span of one semester to fulfil the goal of the improvement of learning, the TR/NYCity approach has consistently utilized the existing educational knowledge base for a state-of-the-art *fix* for the diagnosed need in the classroom. The latest such endeavour was a request to the program officer from one of the oldest philanthropies in India to design an adult literacy program for women in rural TN. Literacy of adult women is considered a *failed enterprise* in some philanthropic circles, and even by some women's organizations. However, the finding of the TAR project is to the contrary. Women are most vocal in expressing a deep desire for learning how to read and write, and, in turn, teach their children:

> We like your methods, now we know we can draw … teach us your methods, we feel humiliated by the thumbprint.

THE OBJECTIVE

The objective of the collaboration has been to create an environment in which learners can overcome their long-held fears of mathematics and, in the light setting of drama and easy conversation, find their own desire to reconnect with mathematics. In such orchestrated conditions, the resistance to learning mathematics is weakened and, perhaps, even dropped. The conscious public absence of resistance to one's own natural inclination for concepts such as *Number* and *Space* is the starting point for sustained action of eliminating one's fear of mathematics and the subsequent resistance to its learning.

Through the acquired common knowledge of the background of the learners, the teaching-research team brainstorms and devises an everyday possible scenario, in which learners could easily imagine themselves, and in which they could casually begin participating. As they participate, the team accordingly adjusts the *script* to allow the learner further room for expression. Natural conversations start occurring, and the learner breaks through his or her resistance; the foundations for the handshake are beginning to establish.

The approach of utilizing drama to attain the intended objective, with the theoretical supports indicated above, while being useful to our own work, is validated by the work of Simon and Hicks (2006), among others. Pflanzer (1992) found that the creative approach works best in drawing out the latent imaginative possibilities of participants through the learning process by accessing their creative memories. Simon and Hicks (2006) make a case for "use the creative arts as a bridge to facilitate inclusion and open doors to those previously disenfranchised in the education system". Role-playing, acting out the obscured troublesome issue that lies in one's memory preventing learning to occur, is the tool that allows the team to create an effective bridge of assistance toward the act of learning and the creation of strategies to strengthen its affect. Quoting Gardner:

> The current educational climate, with its emphasis on targets, standards, predetermined objectives and outcomes, favours a cognitive, rational style of learning, more dependent upon linguistic or logical-mathematical intelligences than, for example, musical, bodily-kinaesthetic, intrapersonal or interpersonal intelligences. (Gardner, 1999)

Allen (1995) points out that "our imagination is the most powerful faculty we possess". Drama, utilizing the imagination, reconstructs our innate capability of connecting with mathematics, and one's latent interest.

THE METHOD AND ITS THEORETICAL BASIS

TR/NYCity model is the methodology of investigating student learning simultaneously with teaching, whose explicit goal is the improvement of learning in the classroom, and beyond (Czarnocha & Prabhu, 2006). TR/NYCity model operates on a sequence of cycles: design of the intervention, its implementation, collection and analysis of the data, redesign of the intervention, next implementation, and so on. It is expected that after a finite number of such cycles one find a successful intervention that leads to satisfactory level of understanding. However, after several such cycles in the developmental classes at the colleges of the Bronx, it became clear that the process is not converging to the satisfactory level of learning because of the students' resistance to learning mathematics itself. The discussions with different cohorts of students in remedial classes revealed that their resistance is grounded in early childhood memories as well as in the social peer pressure realized by the perceived *coolness* of mathematically illiteracy. The teacher-researchers understood that their transformative teaching-research activity has to leave the confines of the classroom and transform the environment around the students. At that moment they transform themselves into teacher-action-researchers (TAR) – that is, teachers who undertake action research in order to transform the community surrounding the students.

In both TR and TAR,[1] and whether in the design of instructional sequences or classroom teaching, the principle of discovery is essential for navigating the ZPD. Mahavier (a student of R.L. Moore) advises:

> Always start in a way that allows each student an entry point, that is, a connection to students' spontaneous concepts. Challenge and intrigue by changing the level of difficulty of problems, and increasing the complexity. (Mahavier, 1999)

The role of language in the development of concepts and in particular, mathematical concepts is extensively studied by Vygotsky (1986). In the present context, language working in the service of mathematics, impacts a change in the social environment of an individual impacting the individual's learning.

The Vygotskian-supported drama scripts as a natural thought process facilitating the reading and writing of one's self-learning aid mathematics. Given the inherent, perhaps, latent mathematical sense in each one, utilizing the scripts, the team enhances the possibilities of creating teaching-learning situations, whereby, the learner in the dramatized environment begins critical enquiry into his or her own learning, and, hence, reading and writing about oneself, or, one's actions. Thus, the process of meta-cognition is introduced into the learner's learning environment. The navigation of the zone of proximal development takes place in this engineered script; this dramatization provides the opportunity for an epiphany regarding one's own learning potential.

The project is named *the Didactic Contract, a Handshake and a Compromise*. The terms have had common usage in the learning and teaching for several semesters as follows.

The Handshake

The Learning Community: What's Up? – A Handshake is the name given to every developmental mathematics class taught at Bronx Community College by one co-pies since the spring of 2007, and at Hostos Community College since the fall of 2007 by another co-PI. The need in the classroom arose from teaching developmental mathematics classes after a successful application of the methodology in Calculus classes (NSF-ROLE #0126141). Students in developmental classes exhibited such resistance to learning that more than half of the semester was spent trying to create an environment conducive to learning. It took that long for students to gain faith in the instructor. Until that point in time, the students' resistance to learning was so high that the struggle to undermine the instructor's attempts to teach was won by the students. The need for a didactic contract an active, conscious effort to take interest in one's own learning, was solidified and the game plan was made clear to the students from the first day of class. Knowing the public fear of mathematics through the reading assigned on the first day (PROM/SE), students were in the new classroom environment where each was an active participant and supporter of the learning of the others. By the middle of the semester, any remarks of the kind, "I'm no good at math" had diminished significantly, almost to nil, without any intervention from the teacher-researcher, but from the immediate remark of a fellow classmate, stating, "Please do not offend your own self" (We're paraphrasing here.) The sense of self-respect for one's own capability to *do mathematics* was born. The *Handshake* was understood as it was intended, that is, as a *handshake* with oneself, one's own *Number Sense,* and the classroom environment.

The Compromise

The *Compromise* is the composition of two rival principles, in which part of each is sacrificed to make the composition possible. In our context, the two rival principles are evident: people are very turned off by mathematics, and, yet, mathematics is a natural part of everyday life. One has to tune in to that reality in spite of the resistance. By acting the conflict out, with the help of drama exercises, students engage in learning, beginning to form the didactic contract with them. By acting out the situation during which students can join in the discussion, an avenue is created for students to honestly begin learning.

The collaboration between mathematics and drama is continued in the development of two external funding possibilities: NSF-ISE (Friends Are Everywhere: Building a Mathematics Friendly Community) and S.D. Tata Trust

(Educational Community of the Future: Mothers as Teachers and Learners, A New Perspective on Rural Education) for the teaching-research and teaching-action-research projects in the Bronx and Tamil Nadu. In the submission to NSF, it is envisioned that the theme *Mathematics in Motion* will artistically embed mathematics into public life through posters displayed on MTA buses and subways in the Bronx as a pilot to be scaled up to all of New York City, if successful. In the exhibition of mathematically oriented posters in the subways, it is imagined that the current team would work with students of the project in dramatizing 3–5 minute scripts to be *performed* on the subway. All scripts have the same objective, of rekindling the natural love of mathematics and number puzzles that is not explicitly expressed.

The posters are colourful and attractive, like Polish theatre posters, and deal with mathematical ideas in daily life in unique ways such as simple puzzles, optical illusions and some basic but challenging mathematical ideas that are surprising for subway riders to think about. Topics may also include the inner workings of computers, using lively images of patterns in a binary system (using 0s and 1s). Another possible idea is for students to learn how to read distances on a street map using a key (let's say a map of Manhattan), and then try to compare distances travelled underground to their destination on a subway map of Manhattan as a class project in one of the courses at the CUNY community colleges located in the Bronx.

A First Script-Sketch Example of a 2-Min Drama

Person 1 (P1): Number, its everywhere, isn't it?
Person 2 (P2): Oh yeah, show me one place it is here.
P1: Number in you, in me, in us, and look up there – Number in *Mathematics in Motion*
P2: 1, 2, 3, 4, la, la, la
P1: Counting numbers, eh?
P2: I have one and you have two, what ratio is me to you?
P1: Let's ask this fellow here. Kind Sir, I have one bread and you have two? What is the ratio of your breads to mine?
P2: Oh, yeah, he's pretty smart. Ask him another. How about this, I have three breads and she has four, what is the ratio of the breads of each of us? 1 to 2 to 3 to 4.
P1: I have no bread, what is the ratio of my bread to all of yours?
P2: He likes Zero.
P1: I like Zero and Zeno.
P2: Who's Zeno?
P1: The guy who said if Achilles runs faster than anyone else, who would win in the race with him and the Tortoise?
P1: Who would?
P2: Think about it, for your homework. Email me your answer at the interactive Coffee Shop Community.

Drama is a catalyst for creative problem solving because students can make decisions viscerally and logically as they negotiate a dramatic scene. In the interactive scene *The Choice* there was a logical progression leading to a choice for using financial resources available to a family for buying an HDTV or visiting their beloved grandma who lives in California. There was also the possibility of doing both in a more limited way with other options taken into consideration using a mathematical approach for making a life decision. This process applied to mathematical thinking in the student's lives allowed them to practically divide the actual resources available and enhance their understanding of fractions and decimals in a real world decision.

The use of drama in the classroom helps students to overcome their fears of mathematics and graphically shows the relevance of mathematical concepts in their lives. Using drama as a means for unlocking the mathematical process can combine intuition, interaction exploring different approaches to a problem enhancing logical symbolic development in the fractional use of resources and time. There is a natural and understandable transitional bridge between drama and mathematical concepts that can promote understanding of mathematical relationships and their development from a new and different perspective.

A defined theatrical situation in the classroom enables the mathematical process. Theatre opens the mind emotionally/conceptually for the student to the creation of and use of mathematical symbols and numbers providing a basis for the teacher-observer's response to the learning process and the redefinition of it through this interaction. It facilitates a more Frerian learning approach involving the teacher and the students in an ongoing practical dialogue relating to the abstract realm of mathematical concepts. Mathematical understanding and the student's possession of the tools to negotiate life situations are essential for survival in this complex contemporary society. The mathematical process is logical and can be empowering when presented in a dramatic context. There can be an "aha" moment when the abstract numerical concept makes sense in the real world.

CONCLUSION

It is possible for the daily life of the classroom to be the basis for the Didactic Contract, A Handshake and Compromise, to be played out in the very same classroom. The Handshake and the Compromise is the "real" didactic contract with the class, using the words of Sarazy and Novotna (2013). Its aim is to offer the goal of excellence to students and to transfer the responsibility for maintaining it into students' hands during the semester of the course. By utilizing drama explicitly in the classroom, or in the envisioned subway dramatization of *Number* and *Space*, the team attempts a rekindling of the natural love and understanding of these mathematical concepts that each of us and our students inherently possess, establishing a didactic contract, as a handshake and a compromise with oneself.

Combining Drama with Mathematics is part of a general integrative theme *Creativity – Language – Numeracy* being implemented in the teaching-research and teaching-action-research community development project in TN, India. Creative memory (O'Hara et al., 1975) is seen as a key element in engaging researchers in discussion of adult learning. In the teaching-research approach, creative memory serves the dual process of being embedded in teaching and influencing learning. In the context of the enactment of the drama within the classroom, students relax their tightly held resistance to learning and convey their interests in a few words, thus *reading* the world of mathematics independently. Drama is the reconciling element (Freire, 1973) remediating between mathematics and the learner.

It facilitates in rethinking the existing tension of prior mathematical experiences and resistance to learning that is a block preventing full access to creative memory. Drama alleviates this resistance, freeing moments of access to creative memory. In the participatory style of the enacted drama, where teacher-researchers are on the lookout to build upon moments of discovery by students, the inroad created by the drama in the classroom, is the beginning of the mathematical joint exploration by student and teacher-researcher. Through drama students find their own voices in mathematics, they start naming the world of their mathematical experiences. In *reading* and *naming* the numerical/mathematical thoughts via the participatory drama, students break through the resistance to their own learning, creating/revealing moments of access to their creative memories.

The teacher-researchers pick up on the *read*/internalized numerical thought and build the necessary scaffolding for students to develop their newly expressed interests.

A collective annihilation of the fear of mathematics does not occur in the duration of a one-session participatory performance. It sows strong seeds of change on which the teacher-researcher has to work for the duration of the semester to foster the climate created by the participatory drama intervention. The outcome is rewarding.

And then we understood
It is not this
And it is not that
It is
        Neither this nor that
It is
        Both this and that
The Didactic Contract, a Handshake and a Compromise

NOTE

[1] There is an essential difference between the two. Teaching-Research is the activity of teachers in the context of the mathematics classroom. Teaching-Action-Research (TAR) is the activity of teachers who decided to extend the focus of their work to include the village community around the school in order to transform it. See Chapters 1.1 and 5.3.

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2.2. FOCUS ON CREATIVITY – LITERACY – NUMERACY

INTRODUCTION

The TR/NYCity model of teaching-research has its origins at Hostos Community College (HCC) in the Mathematics/English-as-a-Second-Language (ESL) linked course teaching experiment, entitled *ESL/Math Collaboration Toward Building the Family Model at Hostos CC*[1] (Chapter 5.1). In its 11th year of evolving development, a blending of themes has been its hallmark, with mathematics remaining a steady focus. In the teaching experiment above, the question was,—would closely knit coordination of syllabi between the algebra course, with its algebraic language, and intermediate-level ESL, help students better learn English? The hope was that learning algebra with its highly syntactical language in coordination with learning the syntax of the natural language would help in mastering the latter.

TR/NYCity, designed as the composition of *Action Research* and *Teaching Experiment*, is useful for the conduct of classroom investigation, and, ultimately, yields utility beyond the classroom; the richness of the particularity of the classroom transcends its limitedness of scope, and the gulf between the particularity of the classroom, and the existing generality for which a solution is desired, is crossed slowly. In each repeated cycle, the particular results acquire sharper focus. Hence, in the *teaching-action-research* community development project, the request, in January 2009, was to design an adult literacy program for women, and it is the tools developed over several teaching experiments in mathematics classrooms that were of help—the elements learnt from the particularity of each classroom situation that can serve the development of the broader general focus of literacy.

How does a methodology targeted to address the particularity of the classroom situation, generalize? What is the progression from particularity to generality? Answers to these questions outline the development of the *Creativity-Literacy-Numeracy* theme, the evolution of the *Learning-to-Learn*[2] materials and the power of the methodology.

The persistent theme in mathematics classrooms has been focused on how to engage students and continue to keep their own interest in learning. Creativity, in the form of observing and creating art, or participating in drama, serves the purpose of drawing in learners, to begin the engagement (engagement in learning requires the

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 75–93.

suitable learning environment, and the desire on the part of the learner to utilize the created learning environment) in the learning of mathematics.

NSF-ROLE #0126141 (*Introducing Indivisibles into Calculus Instruction*) provided the climate for the development of the TR/NYCity model in depth. In a four-year-long Calculus teaching experiment, through repeated iterative attempts to get to the root of learning difficulties surrounding the concept of the definite integral, the methodology matured. The invitation of the TR/NYCity methodology for the professional development of teachers of community-based schools in rural Tamil Nadu, India via a presentation at an international conference (Czarnocha & Prabhu, 2004), reporting on successes in Calculus classrooms in the Bronx, extended the scope from academic classrooms to the community-based school. The community-based school is itself a classroom, and the boundaries are opaque. Children may range widely in age and across several grades, the teacher may require content and pedagogy reinforcement, and resources may be very scarce. The request of the historian of mathematics and community organizer to the teacher-researchers of CUNY, was to deliver and incite "debates in education to the teachers of the community-based school." Probing along directions opening up by interest of community members through repeated field visits and ongoing work, the way proposed itself by the expressed needs of adult women for their own learning, and the teaching of their children. This created the theme of *Literacy*. Then, naturally, since all women are daily agricultural labourers, and use numbers when dealing with money earned, or travelling along differently numbered bus routes, the link to *Numeracy* was readily made. Finally, the *Creativity-Numeracy-Literacy* theme emerged from within the work as the one that could meet the needs. Simultaneously, development of a literacy program was expressed by philanthropy to the teaching-research team, as an unmet need.

Art and Neuroscience are emerging fields of research (Ramachandran & Blakeslee, 1998). Reading and writing, that is, recognition with comprehension of meaning, of printed symbols, and writing with comprehension and proper combination of symbols, at its core, is about decoding and forming a schema of symbols. In this context the symbols assume forms of letters or numbers, or art forms. Art, in its visual appreciation and as a hands-on immersion, develops observation skills and attention. Art, in one's own experimentation with it, develops ease using tools like the pencil, the eraser, or the chalk. It develops the needed muscular dexterity that may not be common among adults who have never written or are not in the habit of writing. In its regular practice it develops discipline, and stillness of mind. In its progressive enjoyment, art is already in the early developmental stages of mathematical thinking. Art, as the mediator between letter and number, or to letter or number, has the following organizational principles along with the necessary attentional principle:

- Ratio
- Symmetry

- Similarity
- Perspective
- Part-Whole Relation

Within the academic setting, the theme linking Art, Mathematics and Language has also been steadily emerging and developing. In Summer 2005, working with students from NYC public schools who had failed the Regents test, and would retake it after the immersion program (NSF-MSP#0412413), it was found that students' difficulties with fractions originated from an absence of clarity about the relative sizes of fractions. Hence, in comparing 1/2 and 1/3 (of a unit), the common tendency was to focus on the numbers 2 and 3 and use the fact that $2 < 3$, to arrive at the same relation between 1/2 and 1/3. Students were unable to visualize the relative sizes of fractions of a given unit. When, on a poster board two equal sized line segments were drawn, and one was divided into halves, and the other into thirds, relative sizes are made explicit. Starting with such preliminary classroom aids, the Fractions Grid was created by hand. The Fractions Grid (FG), shown below, is a series of 20 equally spaced parallel line segments of equal length, with successively increasing number of divisions,—each divided piece indicated by tick marks and labels (symbolic representation of the fractions *a half* and *a third*). Its usefulness was evident in students' work (CUNY Collaborative Incentive Award, C^3IRG 3 – Investigating Effectiveness of Fractions Grid and Fractions Domino in Community Colleges in the Bronx). Since then, it has been made electronic, and is a steady feature in use in the early study of fractions in developmental mathematics classrooms in the Bronx, and in the teaching-action-research project in India. The didactic tool connects art to mathematics quite naturally. While the entire class observes the FG, each one finds something that interests them on the tool, and discussion starts with what is observed in the *art-form* (the tick marks, the labelling, the space between tick marks, the shapes created on the paper), and, in an informal setting, this discussion is *linked*: what is observed is *linked* with the mathematical meaning of the observations. Language, via classroom discussion, forms the mediator, and the creation of word-meaning begins (a unit of analysis outlined by Vygotsky) for linking that which is seen through the art of the FG with the mathematics behind it. Hence, for every operation, such as 1/2 + 1/3, the accompanying drawing runs parallel to the symbolic computations, and students, initially reluctant to draw, begin to see that the absence of clarity in computation is closely linked to an absent mental image of the computation. Thus, the iconic (Bruner, 1966) aspect of the Fractions Grid acts as the mediator in computations. As the instructional interchange continues, the *art-form* of the diagrams is a required supplement for operations on fractions.

Reluctant learners also begin to see the advantage the drawing has to offer, and those engaged in regular drawing begin to stop, saying, "I see it in my head." An example of computations and the parallel drawings for addition of fractions is shown below.

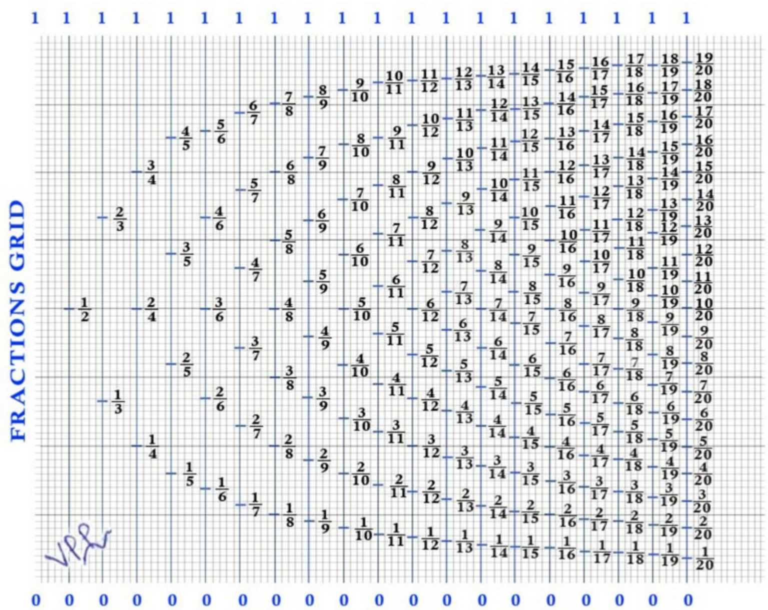

Figure 1. The fractions grid

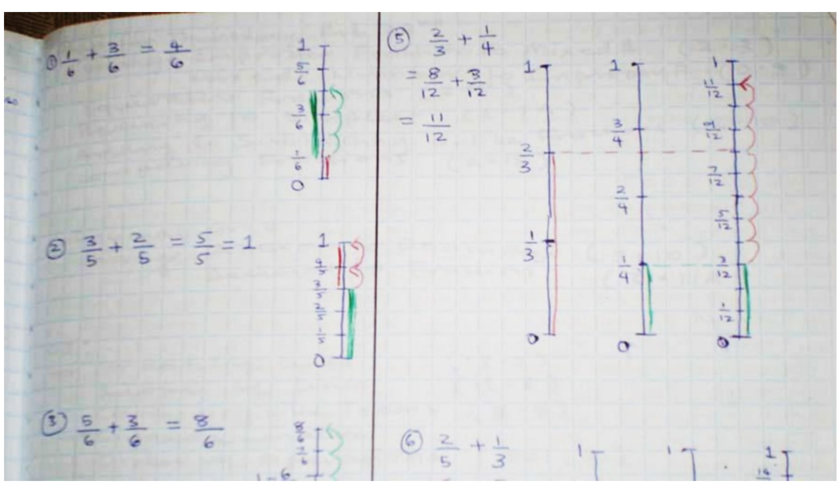

Figure 2. Adding fractions using the fractions grid

Elements of a discovery-based instructional sequence, entitled *Story of Number: Fraction* (C[3]IRG 3: *Investigating Fractions Grid, Fractions Domino in Community*

Colleges of the Bronx) is included at the end of the chapter, where readers can see the integrative theme at work.

METHODOLOGY

The objective of the teaching-research methodology is improvement of learning. To know what needs to be improved, one studies the early stages of learning, and one diagnoses from the start the learning needs of students demonstrated by their thinking. The first figure in Chapter 1.1 supports the phase-cycle representation below.

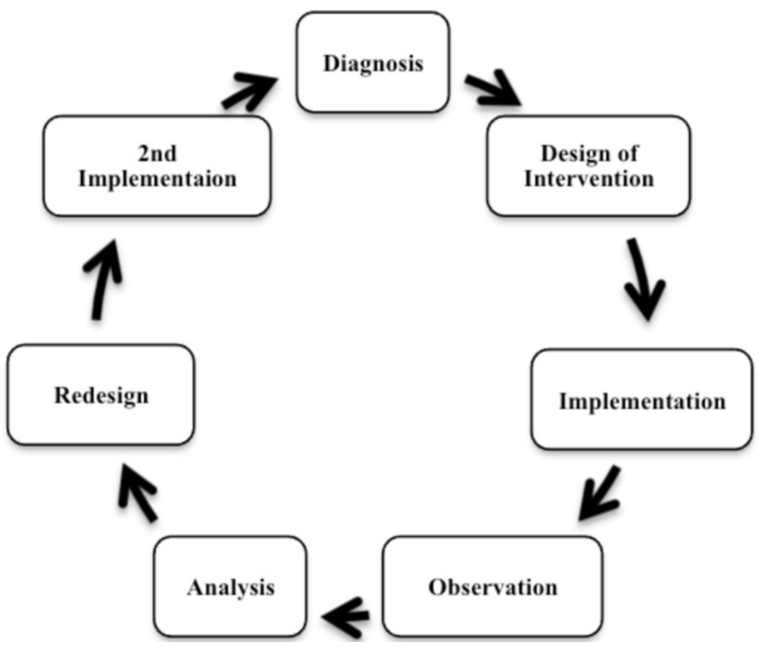

Figure 3. A Representation of the TR Cycle as outlined by the TR/NYCity Model

TR/NYCity model is based on a careful composition of ideas centred around *Action Research* with the ideas balanced on the concept of the teaching experiment of the Vygotskian school of thought in Russia. While *Action Research* relies on the individual improvement of the teaching and learning in the classroom by the teacher, the teaching experiment of Vygotsky views the same classroom as juts one of the sites of a large-scale investigation of learning processes that may be designed using a general theory of learning (Czarnocha and Prabhu, Volume 2, #2, MTRJ, Logical Thinking in Calculus). Improvement of learning being the goal, requires knowledge of learning theories to adapt to the path of matching diagnosed, or expressed, needs of learners with the goal. The impact of the utilization of learning theories finds its way into the instructional materials; the learning materials are designed with the theoretical aspects of the learning theories, and via the teaching-research cycle (where investigation of learning is simultaneously being carried out along with teaching), these materials are tested against the promises offered by the theories of learning. The differences between what is promised and what is observed is witnessed, and, accordingly, refinements are made. Recalibrations of the initial design occur in the process of teaching. Since there are two iterations in any given learning semester, the generated materials in just one semester inform significantly.

What is the role of each of the theoretical facets of the employed methodology?

1. When we, as teachers, have in front of us, the task of educating our students to the maximum of their intellectual potential, we are confronting the problem of the navigation of the *Zone of Proximal Development* (ZPD) (Vygotsky, 1986). The ZPD is defined as the conceptual distance between what the student can accomplish by himself/herself and what the student can accomplish with the assistance of the teacher. In order to help students to reach the scope of their potential, we have to understand, and instructionally address, the nature of students ZPDs. Hence, navigation of the ZPD forms an integral part of the design of instruction as well as its conduct in the actual classroom. The instructional material has to contribute to the creation of such a learning environment. Bruner's phases of *concept formation* (reversed, or in any unstructured way) utilized in the learning materials, positively influence learning, demonstrating the impact on the ZPD.
2. Bruner's phases of concept formation along the *concrete/enactive – iconic – symbolic* strands are taken into account in the design of the instructional sequences (The instructional sequence, or discovery-based problem set, is *Story of Number: Fraction,* appended at the end of this article). Fractions Grid is an iconic instrument, which serves a double role: on one hand, through counting of the units, it promotes concrete/enactive facility of the Bruner's developmental spectrum, and on the other hand it is the springboard to the development of the symbolic mastery on the opposite end of Bruner's developmental spectrum. This is possible, since Bruner's phases do not have rigid boundaries but continuously transform enactive into iconic and into symbolic. With the instructional sequence (*Story of Number: Fraction*), note that one goes from iconic, that is, from the Fractions Grid, to enactive, that is, counting the units – the reverse process of going from picture to enactment. In fact, counting as an elementary operation helps in operations on fractions. When each of the phases appears in the instructional material, the learner can also observe the parallel representations of the same concept, hence, not only does learning occur, but also there is scope for the learner to see greater connections, and inter-relationships, and begin the process of building the schema (Bruner, 1966) of the concept in question.

3. Discovery-based learning (*Story of Number: Fraction* is a discovery-based instructional sequence) jumpstarts discovery. The mode of enquiry of all participants (students and teacher-researchers) is on the learning from the thinking process of the other. When the curiosity is fired enough to want to learn, one questions that which one does not understand, is willing to state openly one's own understandings, and, in the ensuing dialog, one clarifies and rethinks one's point of view, and discovers. The Mahavier-Moore (Mahavier, 1999) discovery-based instruction has been implemented since NSF-ROLE #0126141.
4. Schema of a concept is the network of the relationships between different components of the concept – procedures, rules, other concept-objects and so on. The mastery of the schema of a concept is expressed through the ease in navigating between different components and their relationships. The instructional materials and classroom discourse is geared toward the development of the schema.
5. Cognitive obstacles diagnosed in student learning are addressed via instructional interventions, and also have the possibility of determining whether the cognitive obstacle is an epistemological obstacle. The number line, generally documented in developmental mathematics textbooks as made of points, is the cognitive obstacle, which upon viewing the mathematical development of the concept, is also seen to be an epistemological obstacle.

CREATIVITY-LITERACY-NUMERACY SCHEMA

A schema of learning, loosely speaking, entails ways to tie together all loose ends so that which one learns is whole, well connected, coherent, cohesive and easy to put together, pull apart, treat in isolated pieces or reconnect some pieces together. What schema is inherently attempted to be developed via the *Creativity-Literacy-Numeracy* integrative theme? It is the schema of *meaning*, inexpressible spontaneously. The schema of symbol-decoding and understanding connections between symbolic expressions is the mathematical aspect of the theme. Language mediates concept/schema formation from its art-forms to the symbolic forms. In the expression, by each student, of the art-form either observed or constructed by themselves, the communication clarifies, enables, strengthens nascent concepts, and then, symbols/computations link themselves with the art and the word, thus giving rise to *word meaning* (Vygotsky, 1966). Integration of the three aspects of the theme produces the supportive structure for *word meaning* or unit of analysis.

How does teaching-research prod into multiple beneficial aspects from diverse sources and integrate them to address classroom diagnosed needs? The theoretical and practical strength of the approach rests on the integration of theory and practice. For the *Creativity-Literacy-Numeracy* theme, three components are linked together:

- a medium other than a symbol
- a letter as a symbol
- a number as a symbol

Number is symbolized art, and the symbol, both as letter or number, in the integrated theme, is deciphered and coordinated with art. The visual appeal offers greater receptivity/less resistance, and the nature of art is harnessed to be the scaffold to the abstract nature of letter and number as symbol.

Through repeated teaching of same, or similar, elementary mathematics concepts in remedial classes of mathematics to adult students, drama is seen as another creative aspect that could be utilized in the learning and teaching of mathematics. In combining drama and mathematics via a one-session participatory theatre performance (*The Choice*, Pflanzer, et al.), students, in an informal conversation about the drama being performed, live, reveal their interests, such as (student) Stephanie's fascination for primes, (student) Natasha's discovery of the simplicity of the counting numbers, (student) Nancy's decoding a difficult pattern in the triangular numbers, or (student) Sylvester finally taking interest in fractions. These hints are enough for the teacher-researcher to make the next step, which in this case involved creation of exercises named for each student to continue to keep their interest and to introduce tantalizing mathematics, challenging and intriguing. Students, with their named concept (Stephanie's primes, Natasha's Dots, Nancy's triangular numbers, Sylvester's continued fractions) present their work on a poster board to the end of semester Learning Community Cohort. Students bring forth their learning along both the cognitive and affective track. The cognitive outshines the affective, the desire for consistent, sustained work is absent. The theme *Creativity-Literacy-Numeracy* casts a wide net, reaching out to learners to catch/attract them to learning. Within the means and possibilities of a semester or the trips of the teaching-action-research project, the theme moves forward slowly.

Number and art are the mediators in the design of literacy in Tamil; Art and Language, as communicating mathematical thought, is the mediator in the development of mathematical expression and mastery of computations with fractions among adult students in the Bronx. The theme *Creativity-Literacy-Numeracy*, so termed for the community development project in Tamil Nadu, is suitable in and out of the classroom in the Bronx. Under the *Creativity* category we include Art and Drama. Under the category of *Literacy* we include *Language* as means of communication of mathematical thought, and under the category of *Numeracy* lies reasoning with numbers, or ease of negotiating mathematical thought.

CONCLUSION

Over successive teaching experiments, learning of the teacher-researchers grows significantly, so that each little activity created to address perhaps just one student's problem (perhaps isolated in the class, for example, comparing $\frac{1}{2}$ and $\frac{1}{3}$ NSF-MSP), over a span of several teaching experiments acquires a focus sharp enough for the activity to become a didactic tool. The tool itself when used by individual teacher-researchers has different results in the individual classrooms, and instructional materials, like *Story of Number: Fraction*, are created as the different team members teach to utilize the didactic tool better. The inclusion of GIS-based projects into the instruction further enhances the interconnectedness between Art, Language and Number in addition to continuing to seamlessly weave within the three representations (*concrete/enactive-iconic-symbolic*; Bruner, 1966). The aim is connecting students' spontaneous concepts to their maximal reach, navigating the ZPD. Within the semester in which the activity was initiated, it goes through at least two teaching-research cycles and the activity acquires class-wide implementation, that is, what started as a navigation of ZPD of one student who could not determine the relative sizes of $\frac{1}{2}$ and $\frac{1}{3}$ (of a unit) acquires utility and learning for the entire class, and in the ensuing teaching-research the class group ZPD (Brown, 1992) begins to be navigated.

As the semesters proceed, the materials get further refined and so build up the Learning to Learn library. Thus, particularity to generality, that is, a TR-intervention in one classroom for a couple of students acquiring general utility, is an example of the power of the methodology. From the difficulty in comparison of relative sizes of $\frac{1}{2}$ and $\frac{1}{3}$ (of a unit) by a single student, one has arrived at the integrative theme of *Creativity-Literacy-Numeracy*, a theme applicable for a variety of general audiences:

1. Mothers of young children in TN, India to learn Tamil
2. Organizers in TN to learn English and engage in meaning of didactic contract
3. Creation of the Mathematics-Friendly Community in the Bronx

Diagnosed difficulty creates need for intervention by teacher-researchers. The intervention is studied for effectiveness and is tuned. While initiated for a single student, the whole class naturally uses it. Thus, as group ZPD is being studied, intervention is refined to address all learning difficulties that arise. Multiple diagnosed problems and multiple interventions by teacher-researchers are carried out, and the refinements to the intervention create a didactic tool (for example, Fractions Grid). Over semesters, in preparation for teaching-research materials for succeeding semesters, the tool and generated student data result in more refined tools/materials for the new semester, and these materials are now geared toward the whole class. The utilization of the materials in the classroom is already at a much more different level than in the previous teaching-research cycles, and hence learning is of a different kind and is helped by prior learning. Hence, Fractions Grid through the CUNY Collaborative Incentive Award[3] tried out FG, Fractions Domino and GIS-based projects simultaneously with only Fractions Domino being an intervention that was never tried out before.

Art and Number has an acquired meaning through FG, GIS-based projects, Fractions Domino, and it is seen how helpful Art is in the meaning creation in the context of Number. When literacy is requested by philanthropy in India at HBCSE, 2009, literacy is seen first as language, and then also for the art in the underlying letters, and the commonality is explicitly articulated between *Creativity – Literacy/Language – Numeracy/Mathematics.*

The power of the methodology is the clarity of understanding arising from the careful methodical progression and subsequent action of the phases of the TR-cycle. Implementation of the intervention(s) in the classroom is followed by observation of moments of understanding, and learning from students' learning creates the new materials. In this article, the power of the methodology is illustrated via the brief discussion of how the integrative theme of *Creativity-Literacy-Numeracy* evolved over several cycles of teaching experiments in the mathematics classrooms of the Bronx, and via the community development teaching-action-research project in rural Tamil Nadu, India. The spiral of learning evolves (i) as a focused, investigation targeted to the particularity of the classroom, (ii) over several teaching experiments, knowledge learnt from prior cycles necessarily generalizes to address these learning difficulties across learners. The precise focus, the scientific approach, the investigative quality of a classroom-based teaching experiment takes material and knowledge gathered from cycle to cycle and gears these features themselves for broader application. Thus, the source of the general emerging theme *Creativity-Literacy-Numeracy* can be identified with the particular difficulty of identifying relative sizes of fractions (of the unit in question). Applications of such materials to the general problem of literacy or numeracy, thus, have at their disposal all the didactic tools effective for particularity.

The bidirectional nature of the TR/NYCity model results from the integration of two approaches, *Action Research* and *Teaching Experiment* of Vygotsky, into one coherent methodology. We call the methodology bidirectional because it allows to investigate student learning followed by the design of learning based on that investigation following the route: teaching practice →research →teaching practice.It offers the possibility of developing hypotheses and theories out of teaching practice, and at the same time, it offers the possibility of applying the results of research directly into the mathematics classroom practice through the properly designed classroom teaching experiment (Czarnocha and Prabhu, Vol. 2, #2, MTRJoL, Logical Thinking in Calculus).

In the present article, learning from several teaching experiments forms the basis for the design of materials on the basis of the *Creativity-Literacy-Numeracy* theme. Learning theories enter the context of the teaching experiment in the specific requirements to be fulfilled to address diagnosed learning needs.

NOTES

1 Hostos CC is the only bilingual English/Spanish College of the CUNY system with 80% Latino student population who need scaffolded transition from Spanish Basic Competency level to Academic proficiency. ESL/Math teaching experiment explored the role of mathematics as a mediator between these two registers.

[2] Learning to Learn is the collection of materials generated from the 7+ years of teaching experiments devised to address diagnosed learning needs in the classroom.

[3] The CUNY Collaborative Incentive Award, *Investigating the Effectiveness of Fractions Grid, Fractions Domino at Community Colleges in the Bronx*, awarded to Prabhu, Czarnocha and Watson, was the understanding that for our students' ratio is a distinct concept from part-whole relationship that underlies fraction. That knowledge is in agreement with the mathematical analysis of both concepts.

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APPENDIX A

Development of the Fractions Grid

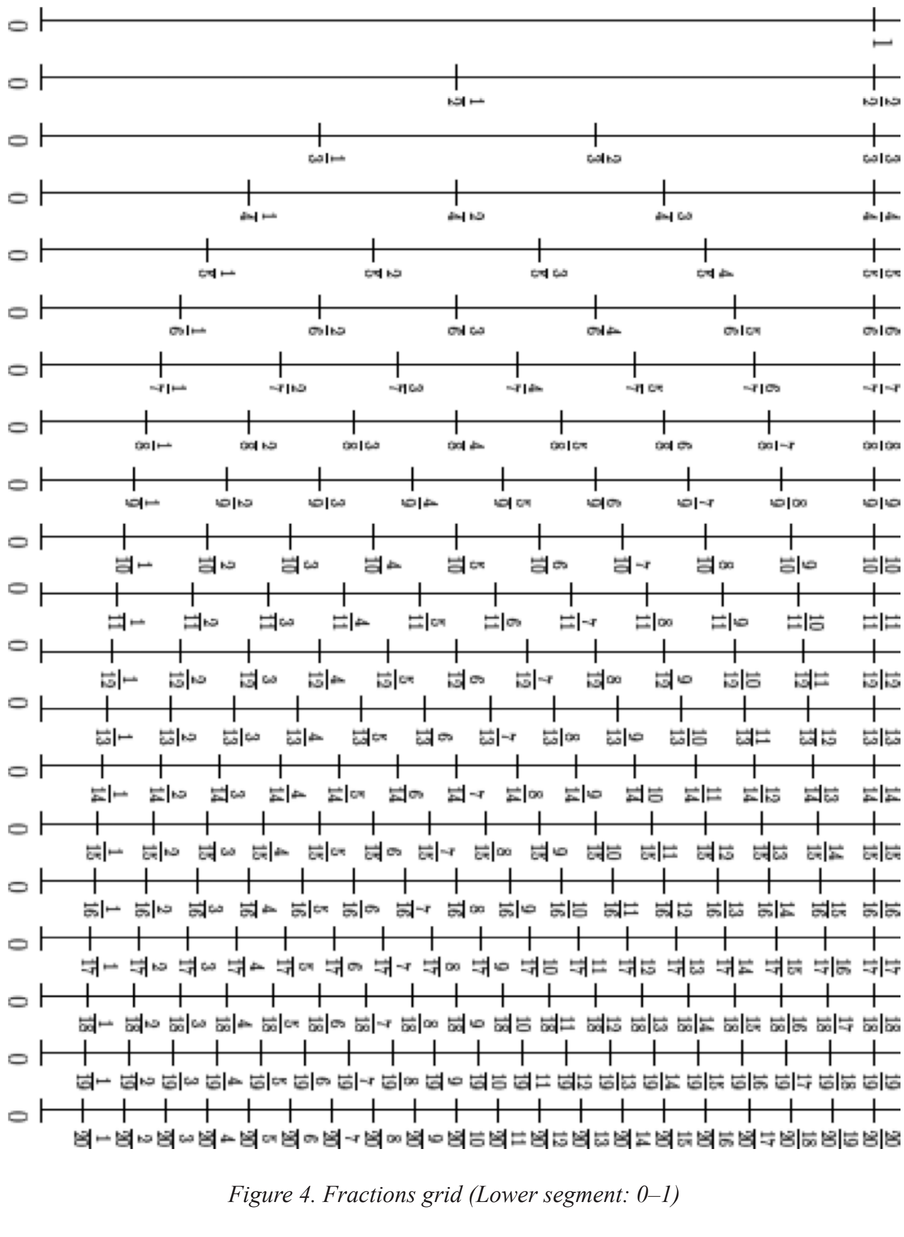


Figure 4. Fractions grid (Lower segment: 0–1)

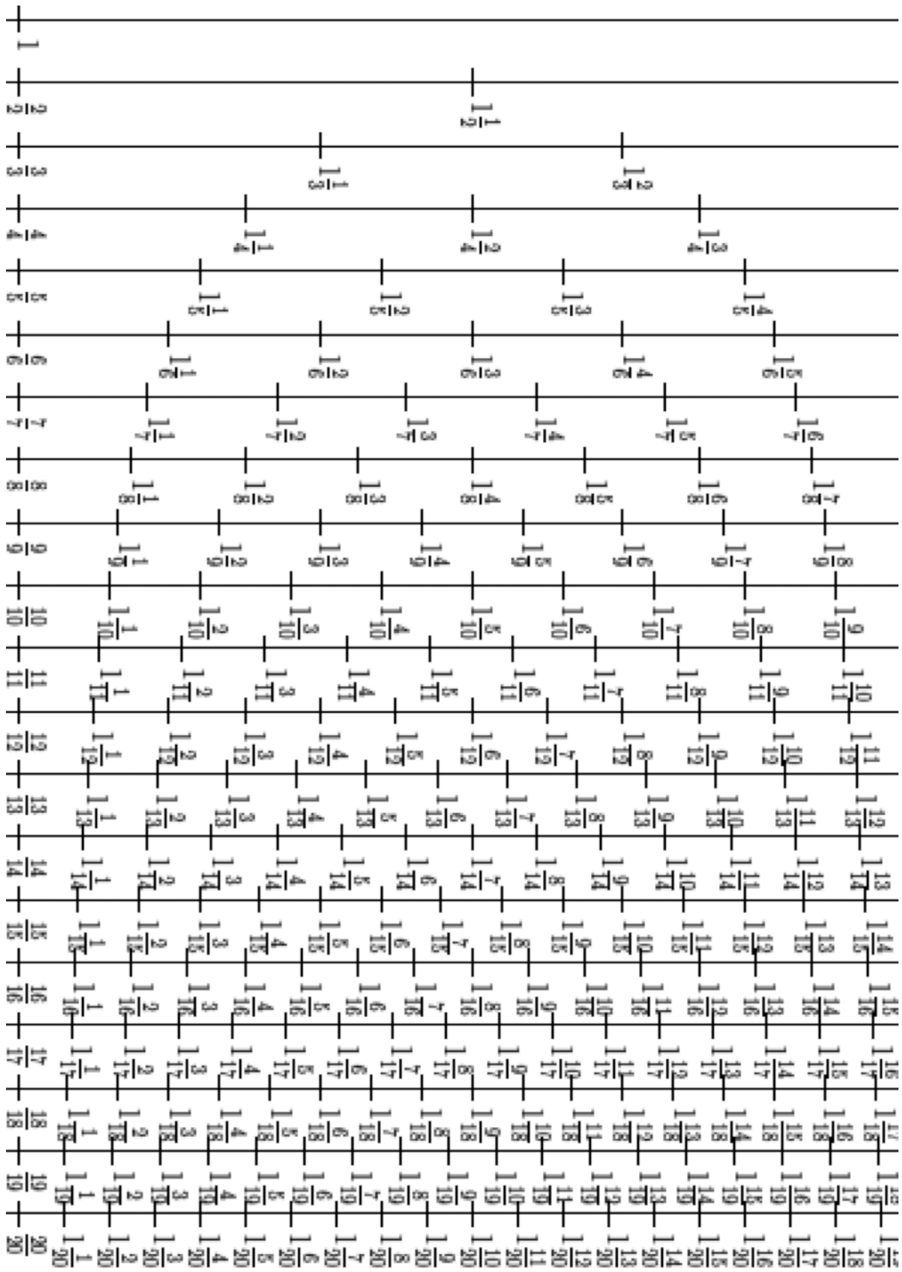


Figure 5. Fractions grid continued (Upper segment: 1–2)

APPENDIX B

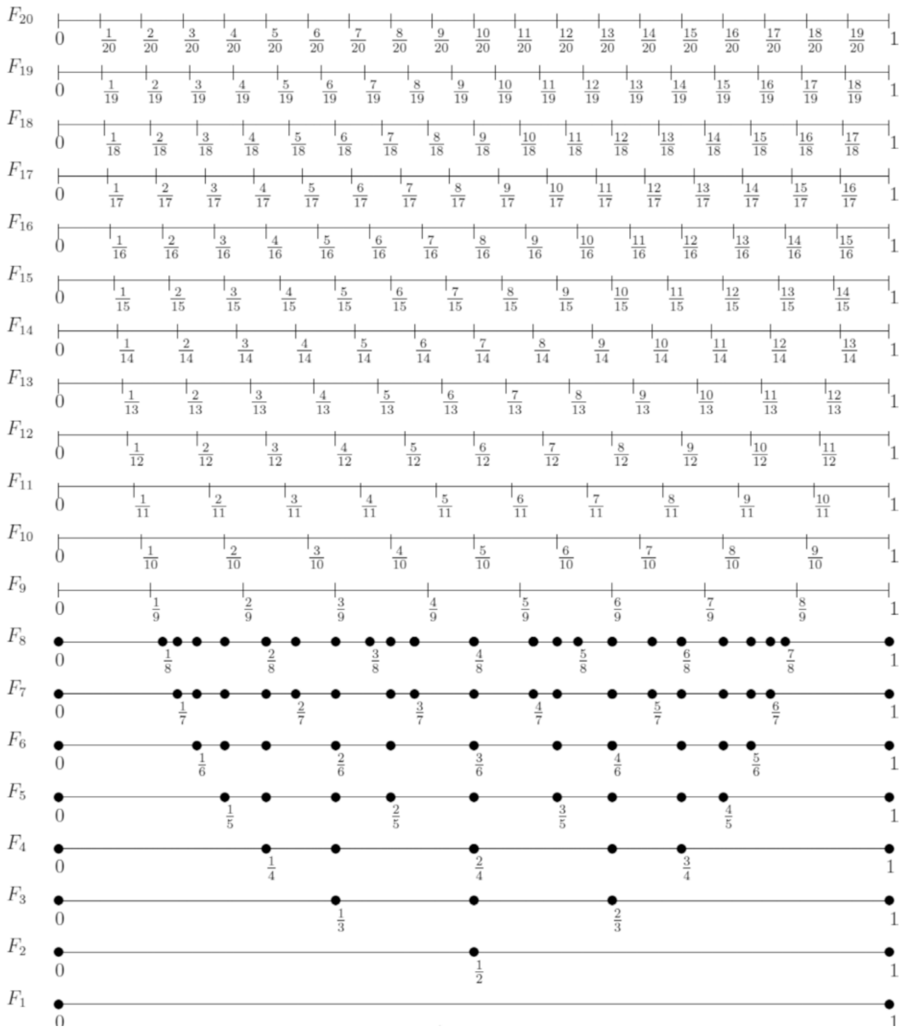


Figure 6. Addition with fractions grid

APPENDIX C

Story of a Number

Story of Number : Fraction

Fraction Definition A fraction (or rational number) is a number that can be written as $\frac{a}{b}$, where a and b are integers and b is not zero. Thus, we have

(i) a pair of integers a and b, or written as $\frac{a}{b}$ and

(ii) a Whole.

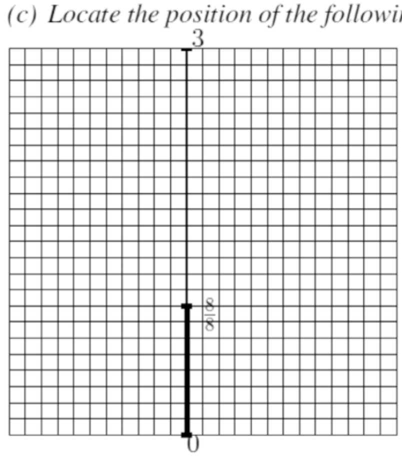
The number b informs about the total number of equal parts the Whole is divided into; the number a informs about the number of those equal parts under consideration. b is the denominator and a is the numerator of the fraction $\frac{a}{b}$.

Problem 45. *Use the definition of fraction above to determine if the numbers are rational numbers:* $\frac{3}{4}, 5, 0, -6$

Problem 46. *(a) Note the dark line segment in the figure below indicates the size of the fraction $\frac{8}{8}$. The division mark and label $\frac{8}{8}$ indicate the position of the fraction $\frac{8}{8}$. What is another number representation of $\frac{8}{8}$?*

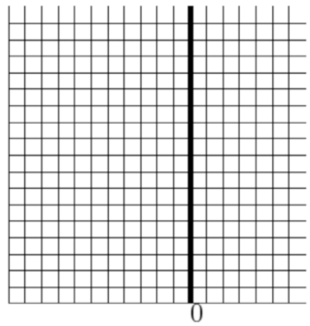
(b) In what ratio does the $\frac{8}{8}$ divide the line segment of length 2 units?

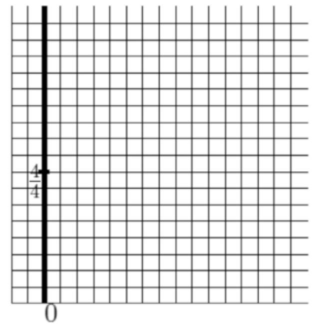
(c) Locate the position of the following fractions $\frac{1}{8}$, $1\frac{3}{8}$, $\frac{11}{8}$, $\frac{13}{8}$ in the figure below.



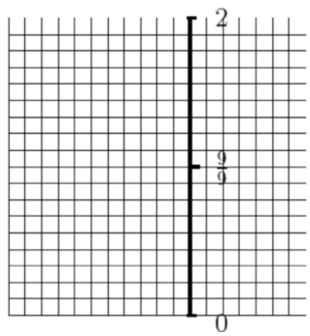
Problem 47. *In each of the grids below, locate the following numbers:*

$1\frac{3}{4}$, $1\frac{1}{4}$, $\frac{9}{4}$





Problem 48. *Estimate the position of the following fractions* $1\frac{1}{9}$, $1\frac{2}{3}$, $\frac{1}{3}$, $\frac{4}{3}$ *on the same number line in the grid below:*



Problem 53. *Find a collection of fractions from the FractionsGrid that have the same size as the ones indicated below: (Hint: Find the equivalent fractions for the ones given below)*

(a) $\frac{1}{2}$

(b) $\frac{1}{4}$

(c) $\frac{1}{3}$

(d) $\frac{3}{4}$

(e) $\frac{2}{3}$

In each case what operation is needed to transform the given fraction into an equivalent one you found? The fractions are called equivalent because they represent the same length or size written in different fractional units.

Problem 54. *On the FractionsGrid, find all fractions equivalent to* $\frac{4}{6}$

Problem 55. *Which of the following pairs of fractions are equivalent, i.e., of the same size?*

(a) $\frac{4}{6}$ *and* $\frac{10}{15}$

(b) $\frac{21}{28}$ *and* $\frac{12}{16}$

(c) $\frac{20}{48}$ *and* $\frac{35}{84}$

(d) $\frac{5}{16}$ *and* $\frac{34}{111}$

Problem 56. *Which of the two fractions is smaller, and why? Hint: Use the FractionsGrid, if needed.*

(a) $\frac{4}{10}, \frac{9}{10}$

(b) $\frac{1}{16}, \frac{1}{4}$

(c) $\frac{5}{12}, \frac{1}{3}$

Problem 57. *Demonstrate by geometry the equality below:*

(a) $\frac{1}{2} = \frac{3}{6}$

(b) $\frac{1}{3} = \frac{3}{9}$

(c) $\frac{2}{5} = \frac{6}{15}$

Problem 58. *Which of the three fractions is largest?*

(a) $\frac{1}{7}, \frac{6}{7}, \frac{4}{7}$

(b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{3}$

(c) $\frac{3}{4}, \frac{7}{8}, \frac{3}{16}$

(d) $\frac{4}{15}, \frac{2}{3}, \frac{5}{9}$

Compare the different approaches possible to determine the orer in each case.

Problem 59. *Arrange the following fractions in ascending order:*

(a) $\frac{11}{12}, \frac{5}{8}, \frac{3}{4}$

(b) $\frac{2}{3}, \frac{4}{9}, \frac{5}{6}$

(c) $\frac{1}{3}, \frac{4}{7}, \frac{1}{2}$

(d) $\frac{7}{11}, \frac{5}{6}, \frac{2}{3}$

Problem 60. *Arrange the following in descending order:*

(a) $\frac{3}{4}, \frac{5}{6}, \frac{5}{9}, \frac{7}{12}$

(b) $\frac{3}{4}, \frac{4}{5}, \frac{7}{10}, \frac{11}{12}$

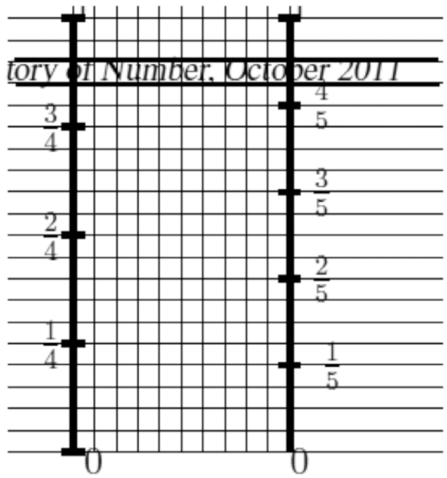
(c) $\frac{2}{3}, \frac{5}{12}, \frac{1}{2}, \frac{5}{8}$

(d) $\frac{7}{9}, \frac{5}{6}, \frac{13}{18}, \frac{2}{3}$

Problem 61. *How would you compare the fractions $\frac{4}{7}, \frac{5}{9}$ according to size?*

Problem 62. *How would you compare the fractions $\frac{1}{a}$ and $\frac{1}{b}$, if we do not know the numerical values of a and b? How can we formulate a general rule for adding fractions when the denominators are distinct numbers?*

Problem 65.



(a) The smallest division in the picture above is the unit. How many units are there in each of the line segments? What is the size of each units?

(b) How many of those units are in $\frac{1}{5}$?

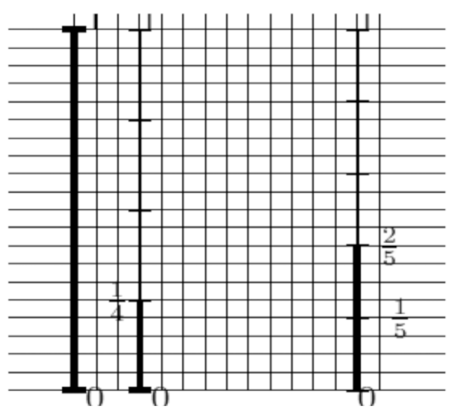
(c) How many in $\frac{1}{4}$?

(d) How much more is $\frac{1}{4}$ than $\frac{1}{5}$?

(e) What is $\frac{1}{4}$ - $\frac{1}{5}$?

(f) What is $\frac{1}{4}$ + $\frac{2}{5}$?

Problem 66. *Compute* $1\frac{1}{4} - \frac{2}{5}$



(a) Which is bigger: $\frac{2}{5}$ *or* $\frac{1}{4}$*?*

(b) Can you subtract $\frac{2}{5}$ *from* $\frac{1}{4}$*?*

(c) Can you subtract $\frac{2}{5}$ *from* $1\frac{1}{4}$*?*

VRUNDA PRABHU, PETER BARBATIS AND HOWARD PFLANZER

2.3. THE POZNAN THEATRE PROBLEM

SUMMARY

The Poznan Theatre problem chapter is the first in the sequence of Prabhu's TR reports, which inform about the discovery of Koestler's bisociation for mathematics education. However, bisociation is mentioned as a "think aside" issue to be fully explored in the next chapter. Here she reports on the problem posing/problem solving dynamics as the didactic tool through which "to establish and sustain the learner's attention on the practice of mathematics". She experimented with this tool in the framework of ZPD characterized not solely by its two cognitive and affective dimensions, but also by the emergence of, as yet hidden, analogy with Koestler's bisociation which acts also along those very same dimensions through the "defeat of habit by originality" connection. Poznan Theatre problem is then an example of the mediation role of drama in relation to mathematics. It represents the path from Creating to Remembering in the revised Bloom taxonomy, showing at the same time "how the creative learning environments invite student to think together, and, thus, contributes to the development of the thinking technology (Chapter 1.1)." What's equally interesting is the process of classroom collaboration of three instructors, each a specialist in different domain of education process, creating together that environment in action. The third author posed a problem, the second and third author together, scaffolded the problem posing/problem solving process. Ultimately she states the answer to the general problem of "how do we bridge the disproportionate achievement gap existing among our Bronx students?" by posing and solving the problem through iterative refinements toward a steady state – which is, ultimately, the daily work and the underlying thinking technology of the enterprise of partnership between students and teacher-researchers."

INTRODUCTION

Mathematics, as the creative expression of the human mind, is, intrinsically built on the processes of questioning, wondering why and how, and, through reflection and contemplation, gaining insight, supported by careful justification, into the answers to the questions posed. Problem-posing and problem-solving are, thus, the core elements of *doing mathematics*. In contemporary discussions about teaching and learning mathematics, this central aspect is hidden from sight; instead, the syllabus, the learning objectives, the learning outcomes, and similar structures are more

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 95–106.

prominent, making mathematics seem like a set of objectives and, at times, explicitly referred to as *a set of skills to be mastered by the student* who is then considered proficient or competent in those skills. The high failure rate in mathematics starting as early as third grade of MSP-Promyse (2006), dislike of mathematics reflected not just among students, but societally, the low number of students seeking advanced degrees in mathematics are reflective of mathematics not being appreciated for what it is – the quest of the human mind toward knowing, and wanting to know why and how.

In the particular context of Community Colleges of the City University of New York (CUNY) located in the Bronx, and, analogously, for the large percentage of high school graduates who need remedial/developmental mathematics courses in college, problem-posing has to be frequently directly connected with the classroom curriculum. The objective is urgent: closing the achievement gap. The problem, as it exists, is that absence of proficiency in mathematics, measured by placement tests and mandated by the university, could actually prevent students from attaining proper general college education. The question is how to change this trend.

The *Teaching-Research/NYCity model* (TR/NYCity), described in detail in Chapter 1.1, is a methodology spearheaded by Bronislaw Czarnocha, initially, for the purpose of addressing the existing situation in the mathematics classrooms of the Bronx, which has evolved to speak to the needs of the much larger mathematics education community. TR/NYCity is the simultaneous investigation of learning and teaching, with the express purpose of improvement of learning in the immediate classroom and beyond. Given its deep and broad nature, finding inroads into the particularly difficult situation in the Bronx, the model is readily applicable to all classrooms with similar difficulties; for example, the relationship between *particularity* and *generality* is a built-in flexible mechanism within the methodology. The goal of improvement of learning entails the perpetually posed question,—what could be done to further facilitate and enhance learning? The TR/NYCity methodology is based on the *Teaching-Research cycle* presented in Chapter 1.1. This cycle encompasses a recurring nature that iterates over (i) the immediate learning difficulties diagnosed within the classroom carrying out a *teaching experiment*, as well as over (ii) time, such as semesters or years. Through both of these types of iterations, learning develops and materials are generated, which are, in turn, embedded back into the learning environment that gradually acquires greater robustness.

Over the period from 2006 to 2012, teaching experiments were conducted in remedial mathematics classes at two CUNY community colleges,—Hostos Community College (HCC) and Bronx Community College (BCC). These remedial mathematics teaching experiments were undertaken following the success of a calculus teaching experiment that took place in these same colleges (NSF-ROLE#0126141). Between 2006 and 2010, the cognitively challenging materials continued to develop, and, in 2009, it was discovered that the markedly absent but necessary *didactic contract* (or *handshake*) on the part of the learners toward their own learning continued to inhibit satisfactory progress. In 2010 the teaching-research team expanded to include the Vice President of Student Development at Bronx Community College (BCC) who is also a counsellor. Affect, in the psychological sense, referring to student attitude, became a steady issue addressed daily, and with persistence and know-how. Success started becoming tangible. Over a two-year period, from 2010 to 2012, it became clear that for a learning environment to be effective, a careful simultaneous integration of attention to cognition, affect and self-regulatory learning practices is essential (Prabhu & Czarnocha, 2014).

In the present chapter, we provide two distinct examples of the interplay of the *problem-posing/problem-solving* dynamic. The first is named the *Poznan Theatre Problem*. A drama professor and an arithmetic course teacher-researcher posed the problem in the fall of 2010. The inclusion and content contribution of a drama professor in a mathematics classroom, taught by Vrunda Prabhu, was the second such intervention. In both cases, the intervention had a specific purpose described in Chapter 2.1. The second example of the *problem-posing/problem-solving* dynamic is explored through an elementary algebra course teaching experiment focusing on solving applications using rules of exponents.

WHY PROBLEM-POSING?

In his 2010 work, Knott states:

> Recent developments in mathematics education research have shown that creating active classrooms, posing and solving cognitively challenging problems, promoting reflection, metacognition and facilitating broad ranging discussions, enhances students' understanding of mathematics at all levels. The associated discourse is enabled not only by the teacher's expertise in the content area, but also by what the teacher says, what kind of questions the teacher asks, and what kind of responses and participation the teacher expects and negotiates with the students. Teacher expectations are reflected in the social and socio-mathematical norms established in the classroom. (Knott, 2010)

The quote above sets the stage for problem-posing as an important constituent of everyday mathematics teaching. Of particular importance is the need to take into account the missing interest in mathematics, impacted by prior experiences and failures. In such a grim environment, there is a span of one short semester, during which the instructor has the opportunity to reverse this trend of negativity, develop a self-directed questioning attitude, and facilitate enjoyment and mastery of the intended mathematics material.

Vygotsky (1978) describes the *Zone of Proximal Development* (ZPD) as "the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or collaboration of more capable peers". In the classroom environments we encounter, where, to be effective, the environment requires a careful integration of attention to cognition, affect and self-regulatory learning practices (Prabhu & Czarnocha, 2014), the ZPD has to be "characterized from both cognitive and affective perspectives. From the cognitive perspective we say that material should not be too difficult or easy. From the affective perspective we say that the learner should avoid the extremes of being bored and being confused or frustrated" (Murray & Arroyo, 2002).

A major cause of the lack of satisfactory performance in remedial mathematics classes is absence of interest and the resulting absence of attention. Hence, a problem for the teacher-researchers is how to establish and sustain the learner's attention on the practice of mathematics. During the 2010–2012 period, through formation of the elements of the creative learning environment, much insight was gained on this front. The counsellor was able, through his craft knowledge, to hold and sustain student attention on the problem or topic being taught/discussed by the mathematics instructor. The counsellor's method was to switch the frame of reference, while keeping the underlying mathematical focus constant. For example, if the task was to calculate 1/2 + 1/3, the counsellor would switch the frames of reference from pizza to cookies to something else, maintaining student attention while focusing on the underlying mathematical concept.

DEFINING PROBLEM – POSING: FROM RESEARCH TO EFFECTIVE PRACTICE

Mathematics incorporates thinking technology, in which posing problems and attempting to solve them to the extent possible relying back onto the scope of knowledge accessible within it, is the foundation and basis of the discipline. Its clarity and transparency makes itself known to persons choosing to explore its language and challenge. Mathematics addresses the questions of why and how it uses minimal building blocks on which its edifice is constructed. Thus, at any level of study of mathematics, problem-posing and problem-solving are inextricable pieces of the discipline.

Within the TR/NYCity model, the effect of the TR cycle is that particularity of the classroom and the means undertaken to improve the nature of the learning in the given classroom become *usable* and *adaptable* tools for classrooms facing similar learning issues. Therefore, generality of usefulness from the particularity of usefulness is naturally achieved via the creation of teaching experiments designed and conducted with precise attention to the particular classrooms under consideration. Teaching experiments are also generally collaborative; in our case (a) two different Bronx community colleges face similar learning difficulties, and (b) a team approach is required to tackle the difficulties.

Problem-Posing Illustration 1

This is the second of the Drama and Mathematics collaborative interventions staged by a drama playwright Howard Pflanzer in a mathematics classroom of Vrunda Prabhu

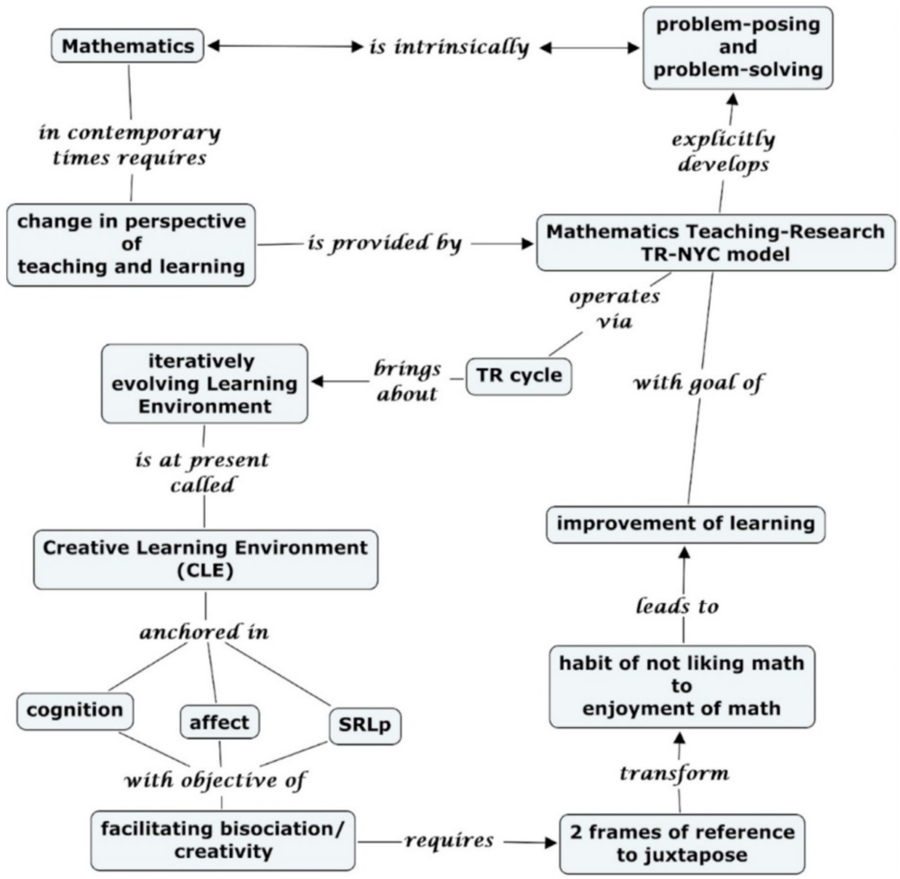


Figure 1. The role of problem-posing and problem-solving in the improvement of mathematics learning

at BCC. The intent is to increase student participation through active problem-posing in conjunction with problem-solving in a safe public space of the classroom. Students are given a set of ten problems based on the curricular topics but that are also of general interest. The problem set is called the *Creative Problem Set* and, in Koestler's terms, this could be called *thinking-aside* tools, that is, they provide opportunity for free thinking and exploration on the part of the students, thus, increasing individual interest and voluntary engagement. Two issues are addressed here: the importance of this creative exercise from a developmental perspective, and its promotion for the establishment of an individually motivated work ethic on the part of the student. From the developmental perspective, Victoria Purcell-Gates lessons on literacy utilize a process of learning that students with weak mathematical thinking skills need. To tackle students' undeveloped work ethic in studying mathematics, self-regulated learning is scaffolded, and a satisfactory didactic contract is intended. Problem-posing in conjunction with problem-solving serves many important goals in the development of the totality of a successful educational endeavour.

The scenario was created by Howard Pflanzer, applied to the Creative Problem Set, with the intention of designing a lesson for the arithmetic class. This is the second instance of such a drama/mathematics collaboration in the classroom. The first one took place in the fall of 2008. The Choice: Grandma or Flat Screen TV, as it was called, was intended to create an avenue for a group of nine Learning Community students to let go of their scholarship resistance and begin to actively participate in learning. This staging was successful; for more details see Chapter 2.1. The Poznan Theatre Problem had a similar objective of increasing student participation in their own learning, as well as the additional goal of preparing the class for an impending visit by the Chancellor of the University.

Scene 1

A NYC artist visits Poznan. He has no phone with a GPS. He has a street map of Poznan. He has the following itinerary:

- Start at the Theatre of the 8th Day in centre – indoors – 7 pm.
- Production – abandoned soccer field – outdoors – 9 pm.
- Production – The Golem in courtyard of abandoned slaughter house – 11 pm.

How fast should he walk to be on time?

This scene together with the map of the region in Poznan, Poland became the source for a series of problem posing/problem solving activities.

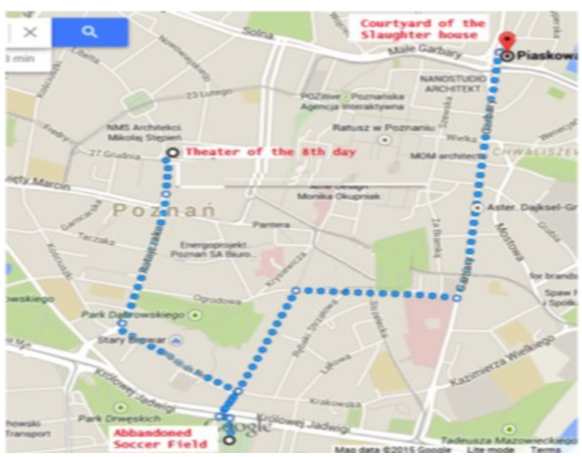


Figure 2. Map of the artist's walk for the problem in the classroom

The artist is Howard Pflanzer, the third author of this chapter. He and the first author, Vrunda Prabhu, are collaborators in the Math and Drama scheme in teaching mathematics in the classroom and the community. Pflanzer has been requested to assist in the mathematics classroom. The class is expecting a visit from the Chancellor of the University and Pflanzer is preparing the class for it. The class is a group of about 25 bright youngsters, turned off from mathematics who have failed the mathematics placement exam for college level math course eligibility, and have been placed in a remedial arithmetic course. The students are considered *at-risk*, that is, their further college education depends on them being able to master the mathematics in question. The nation-wide results for remedial students are not encouraging. The second author, Peter Barbatis, is the Vice President for Student Development at the college. He has prior experience teaching remedial mathematics, and his expertise includes counselling. The first and the third author are collaborating on changing students' attitudes toward learning of mathematics. He and the class instructor collaboratively teach the class, 50% of the instructional time each. The class trusts both instructors. There is an environment of enjoyment of mathematics created by the two collaborating instructors, and the students, to varying degrees, have displayed engagement and amusement.

Scene 2

From the large map of Poznan, brought to class, HP provides the following information:

On this map, 2 inches = 1050 meters

He begins asking the class questions. First, he reminds the class that the map uses different units for measuring length, and asks for the conversion between *km* and *miles* as well as between *cm* and *inches*. No one, including the instructor, knows the conversion. HP offers the next piece of information:

1 kilometre = 0.62 miles

Instructor uses the opportunity to just observe the class, imagining being one of the students. HP continues:

> *The distance from my hotel to the Theatre of the 8th Day is 2.5 inches, on the map. What time do I have to leave the hotel to be at the theatre at 7 PM, if I am walking at the speed of 3 mph? Also, the distance between the theatre and the abandoned soccer field, where the production is supposed to take place outdoors, is 4 inches, on the map. How fast should I walk if I am supposed to be there by 8:30 PM?*

The questions of HP are not answered with all the gusto that PB wishes, so, pretty soon, he jumps up and says, “Conversion! Conversion between inches and meters. What does it mean?” PB begins prompting the students, and the class, whose members trust him, begins answering. The play continues in this way, with HP asking the questions he’s created, and PB, guiding students from that question to the knowledge of facts and curricular material required to get to the answer of the question being asked.

What really happened here? Both were “experts” relative to the students. The experts created a path from the top of Blooms’ taxonomy, ploughing through intermediate phases to the bottom. Except two, previous to this example, most students had not had much of an opportunity to problem-solve in this way.

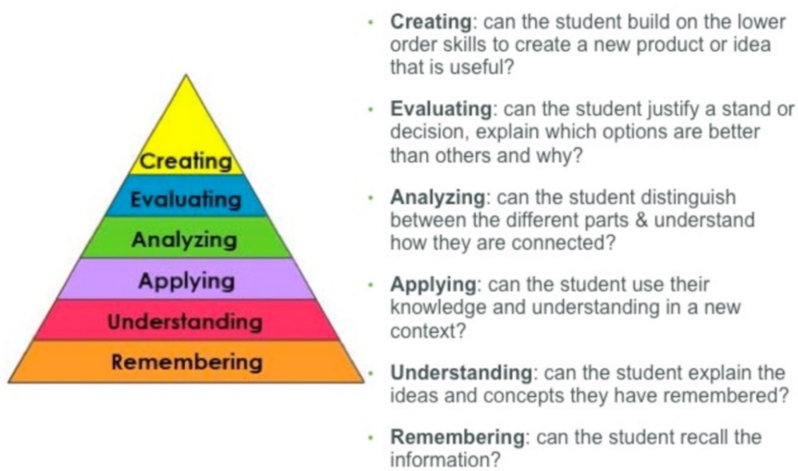


Figure 3. Revised Bloom’s taxonomy (Krathwohl, 2002)
http://laurieodonnell.co.uk/wp-content/uploads/2011/03/Slide13.jpg

Students’ fear of mathematics, especially in the Arithmetic class, is the main inhibitor in their successful completion of the course. They have been placed into the course on account of their scores on the placement test which indicates non-mastery of operations on numbers. Given this background, the learners are to be drawn into problem-solving via the Poznan Theatre Problem, where the problem to be solved is a story *created* by the second author, and, in a way, *directed*, in the classroom, by the third author, while the first author is the official mathematics instructor. The title of the course was Creative Problem Solving.

On the first day of class students were asked to answer the following question aloud: What is mathematics? Spontaneous answers included, “It’s numbers” and “It’s problem solving.” In reply to the second response, the class was asked if problem-solving extends beyond the mathematics classroom, and at a later time, the students were asked to explain their understanding of the term *Creative Problem Solving*. Ultimately, the objective was to get students to voice their interpretations of the meaning of the word *creative*.

Further Comments and Reflections

1. The following have to be taken into consideration when attempting to draw reticent learners out from silence, and, on a broader scale, to find an avenue for learners to surmount their emotional hesitation and learn *how to learn*, affirming their inherent capability to do and learn mathematics:
   - *Affect*: the conscious subjective aspect of feeling or emotion; the experiencing of affective and emotional states;
   - *Psychomotor*: reaction, idea, of or relating to mental states that affect motor capabilities, i.e.;
   - *Self-regulatory learning*, where *regulation* is to be understood as a principle or condition that customarily governs behaviour;
   - *Cognition*: knowledge, noesis, the psychological result of perception, learning and reasoning.
2. The Poznan Theatre Problem represents a path from *Creating* to *Remembering* in Bloom's 2001 revised taxonomy of cognitive skills. It is an example of how the creative learning environments invites students to think together, and, thus, contributes to the development of the thinking technology. The second author posed a problem, and the second and third author, together, scaffolded the problem-solving and problem-posing process.

Students ability to, unconsciously, navigate through the problem, learning to incorporate the phases of Bloom's taxonomy, or breaking up the problem from its entirety via Polya's strategy, is realized by the Pflanzer-Barbatis enactment of the Poznan Theatre Problem.

Problem-Posing Illustration 2

In this example, the setting was an Elementary Algebra class. Students had trouble determining which rule of exponents needs to be applied to a given problem. There was a tendency to use anything arbitrarily without justification. The class problems were followed by a quiz, in which students had much difficulty in determining which rule was applicable for the exercise under consideration. Again, it was a matter of not being able to slow down the thinking sufficiently to observe the structure of the problem and the similarity of the structure with one or more rules. Students were asked to work on the following assignment:

Given the following rules of exponents

$$a^n \times a^m = a^{n+m} \tag{1}$$

$$\frac{a^n}{a^m} = a^{n-m} \quad (2)$$

$$(a^n)^m = a^{n \times m} \quad (3)$$

$$a^0 = 1 \quad (4)$$

$$a^{-n} = \frac{1}{a^n} \quad (5)$$

Make up your own problems using combinations below:

- *Rules 1 and 2*
- *Rules 1 and 3*
- *Rules 1, 2 and 3*
- *Rules 1 and 4*
- *Rules 2 and 4*
- *Rules 1, 2 and 5*
- *Rules 1 and 5*
- *Rules 1, 2, 3, 4, and 5*

Then, solve each of the problems you created.

In the work that students submitted they created problems that had only one term that required the use of, say Rule 1 (for example, $a^n \times a^m = a^{n+m}$), and another term that required the use of Rule 2 (for example, $\frac{a^n}{a^m} = a^{n-m}$), but there were no problems that had one term requiring the use of *both* rules (for example, $\frac{x^8 \times x^9}{x^7}$). This gave the instructor a point from which to develop problem-solving through deeper problem-posing, that is, through dialogic think-aloud face-to-face sessions. Students were asked to observe the structure of the given problem and state the nature of the similarity to all those rules where similarity was observed. This interaction led to examples of posed questions that, in turn, allowed the teacher-researcher to construct more complex exercises.

CONCLUSION

The Poznan Theatre Problem is an example of how didactic contract, elaborated in Chapter 2.1 is attempted. Exponent is an example of active problem-posing leading to its successful integration by the learners. The mastery of the language of mathematics through self-directed attention to reading comprehension is an example of how the repertoire needed for problem-posing and solving needs to be consistently enriched.

Continuous repetitions of the problem-posing/problem-solving dynamic increases learners' catalogue for recognizing their own moments of understanding and the emerging patterns of internalized comprehension. Writing, as a medium utilized for learning to write, and writing to learn, makes the understanding enduring, concrete and reusable by learners.

Problem-posing is a constant in the discovery-oriented enquiry-based learning environment. For example, operations on integers such as addition and subtraction visualized via the number line forms the basis for ongoing questioning and interactive problem-posing between students and the teacher-researcher.

Algebra as the field of making sense of structure simultaneously with making sense of number provides opportunities for problem-posing along the path of *Particularity* ⇔ *Abstraction* ⇔ *Generality* on the Arithmetic-Algebra spectrum. In Algebra classes, it is harder to utilize concept scaffolding, and problem-posing occurs solely on the side of the teaching-research working to include triptychs in the learning environment mix. In the process, the triptych rows evolve into *simpler* usable forms.

The root of the general problem being posed is: how do we bridge the disproportionate achievement gap existing among our Bronx students? The TR/NYCity model approach provides a mechanism – posing and solving this problem iteratively with ongoing refinements toward a steady state – which is, ultimately, the daily work and the underlying thinking technology of the enterprise of partnership between students and teacher-researchers.

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2.4. THE CREATIVE LEARNING ENVIRONMENT

SUMMARY

The chapter is the description of the teaching experiment designed to integrate cognitive and affective components of learning, supported by self-regulating learning practices, whose objective has been the reversal of the culture of failure through the development of student ownership of learning and enjoyment of mathematics. Prabhu presents three examples of students in her class illustrating different affective and learning issues in her classrooms. She presents an unusual team of TR composed of herself as the instructor, the VP for students – as an affective mediator – and the Librarian – the specialist in self-regulating learning practices – who taught together during the course. Each of them was contributing to classroom teaching situations within their own expertise creating together many bisociative Aha! Moments with students. The connection between Koestler's theory and Prabhu's classroom practices has been made by Prabhu during this teaching experiment.

The instructor taught both experimental and control classes. The data in terms of attendance, midterm results as well as the results of motivation scales (LASSI and MSLQ) show the extend of success in terms of improved attitudes to mathematics, goal setting, level of engagement and increase (not yet sufficient) in persistence. The description of methodology for such classroom collaboration is especially interesting as well as the examples of triptychs modelled to certain degree by the general Koestler's triptych (Chapter 1.2). Prabhu's triptychs were the first instructional assignments designed explicitly to facilitate student awareness of the possibility of bisociation. Students were exploring two different "matrices of experience" searching for hidden analogies and differences through the discussion of relationships between two pairs of the concepts in any of the triptychs. Through the integration of classroom practice with the Koetler's theory, an avenue is created for learning beyond the semester of the course. The Koerstler's triptych provides the context within which (1) ending of fear of mathematics as well as (2) the transformation of harmful repeating habits to originality via creative expression have become a real possibility. Finishing her expose in this chapter Prabhu asserts that "Creativity in the teaching of remedial mathematics is teaching gifted students how to access their own giftedness."

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 107–126.

INTRODUCTION

Students in basic remedial mathematics classes are alienated and at risk of being lost from STEM fields. The teaching-research enquiry since 2006, for a conducive learning environment (LE), found that affect and cognition must be addressed in tandem, since the affective and cognitive pathways can mutually inhibit cognitive performance. Ownership of learning through a satisfactory didactic contract requires a learning environment based on the design principles of cognition, affect and self-regulatory learning practices to reverse the currently negatively skewed culture surrounding students' engagement with mathematics, and, consequently, their successful achievement of the desired learning objectives. Preliminary results and the significance of such a creative learning environment in our urban educational situation are discussed.

OBJECTIVE

Students in remedial mathematics at community colleges are at risk. Their success in higher education depends on overcoming obstacles to learning, many of which stem from attitude, related to affect perception, and detrimental to cognition. Nationally, approximately one-third of students entering colleges need remediation (Byrd & McDonald, 2005); as many as 41% of all community college freshmen are enrolled in remedial courses (Hoyt, 1999; McCabe, 2003). Differences between under-prepared college students and college-ready students include lower high school GPAs, lower confidence, lower self-predictions for completing college education; indicators that perpetuate a cycle of minimal accomplishment and low self-esteem (Boylan, 1999; Boylan, Bonham, & Bliss, 1994). Nationally, 47% of students requiring remediation graduate, while only 24% of students needing three or more developmental courses complete their program (Adelman, 1996). Failure to complete developmental classes remains the stumbling block to success (Boylan, 1999; Kraska, Nadelman, Manier, & McCormick, 1990). Ownership of learning is absent.

In this article we sketch out the design of a potential learning environment (LE) within a multi-cycle Teaching Experiment (TE) in remedial mathematics, more specifically, arithmetic, at an urban community college, whose objective has been to reverse the culture of failure through development of student ownership of learning and enjoyment of mathematics.

The teachers-researchers are a multidisciplinary team, comprised of:

i. A mathematician who is the instructor of the Basic Mathematics classes in question,
ii. The Vice President for Student Development who is a mathematics-friendly counsellor, and
iii. Academic Librarian whose expertise includes self-regulated learning.

A successful learning environment integrates cognition, affect and self-regulatory learning practices.

THEORETICAL FRAMEWORK

The cyclic Teaching-Research NYC (TR-NYC) model (Czarnocha, 2002; Czarnocha & Maj, 2008) forms the theoretical framework of the teaching experiment. Revisiting the contents of Chapter 1.1, Figure 1 below sketches out the teaching-research cycle. The TR-cycle begins with diagnosis of learning. The red arrows indicate the first run through the phases of the cycle, while blue arrows indicate subsequent runs. Teaching-research integrates craft-knowledge of the team with research base of the profession. Individual teaching practice of each teacher-researcher is our craft-knowledge, based on experience and a well-developed "know-how" of teaching. The research knowledge we introduce into our teaching depends on the improvement task. Thus, pursuing the development of the guided discovery, we utilized the Zone of Proximal Development (ZPD) methodology of Vygotsky (1986) and the Moore-Discovery approach; to align teaching with the natural path of concept development we consulted the constructive pedagogy of Bruner laid out in The Acts of Meaning (1990), and to address affective obstacles we learned from Brousseau's concept of didactic contract. The problem-solving approach of the current TE-cycle has integrated work of Polya, Bloom and Koestler in the design of the creative problem-solving environment facilitating student ownership of learning through integration of affective and cognitive components of learning (see Figure 1). Since such integration simultaneously addresses the two fundamental aspects of learning, it has a chance to close the achievement gap.

Students in Basic Mathematics classes have been exposed to the topics under consideration before, perhaps several times. According to Bloom:

> To be physically (and legally) imprisoned in a school system for ten to twelve years and to receive negative classifications repeatedly for this period of time must have a major detrimental effect on personality and character development. (Stringer & Glidewell, 1967)

The effects of "negative classifications" are manifested in current mathematics classrooms as resistance to learning through disengagement. The classroom climate in its absence of readiness to learn is in urgent need of creating factors conducive to learning. Accustomed to a culture of failure, learners in developmental mathematics classes are prone to not knowing what they know. Repeated failures undermine learners' trust in their own possibilities, distancing reality of achievement.

MODE OF ENQUIRY

The teaching-research mode of enquiry is discovery-based, that is, enquiry leading to discovery on part of all learners – students as well as instructors.

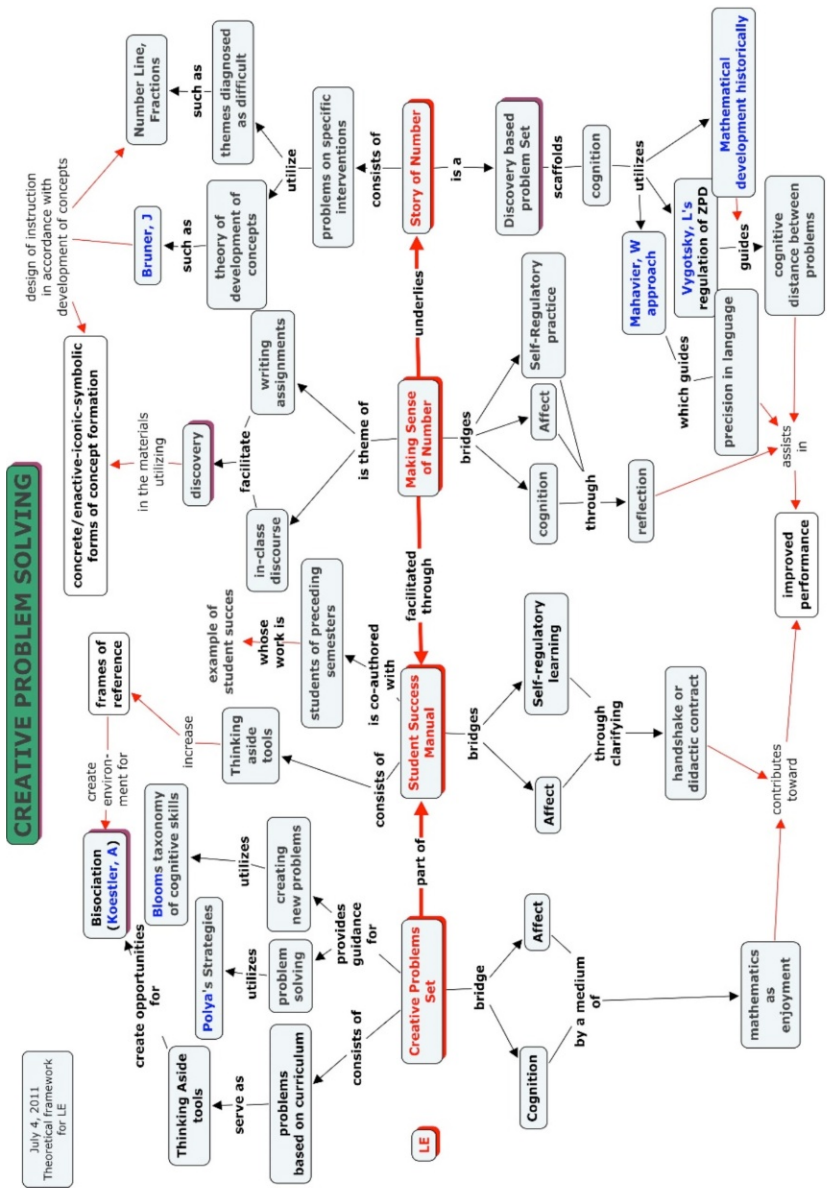

Figure 1. Theoretical supports for the design of the learning environment

What is the present classroom climate? What are the specific needs that warrant the designed learning environment?

Meet Yra, a freshman:

> Yra has almost perfect attendance. Yra seems reasonably attentive (does not talk with others, fidget on phone, daydream, etc.) but never asks questions. When asked to do a problem on the board, Yra does it well; sometimes with a little help. She attends review sessions. There is no reason to be upset about Yra's performance. But at the last review session, Yra turns worrisome. The Vice President asks if there are any questions, to which she responds, "Yes." When asked what the question is, she states, "Fraction". The mathematics instructor continues asking what exactly is troubling Yra about fractions, and she answers with the dreaded words, "Everything. Fraction".

A student such as Yra is not just in need of the best teaching methodology that can help her cognitively, fluidly advance, Yra needs two other skills: (i) knowing how and when to ask questions, and (ii) not waiting to ask for help. The Student Success Manual planned for the next TR-cycle embeds a module on self-assessment.

Meet Ayn, also a freshman:

> Ayn is a good student. She thinks well and has good attendance. When she had to be in the Dominican Republic for a court case she asked for work so she would not lose academic standing. Ayn is afraid of mathematics. Ayn had a turnaround moment. It was when the Librarian did his spontaneous speech on self-regulated learning, and the mathematics instructor followed by saying that she was not picking on her but wanted to let her know she was a very good student and she should not use phrases as "I am scared" because of the impact they could have on someone else in class. Ayn never said words of a similar nature. However, Ayn had not dropped her fear. At a review session at which Ayn was the only student to attend,[1] the productive session had ended with Ayn and the instructor having a conversation about her performance, and the instructor mentioning that she was excellent on all counts except one; to which Ayn had replied with half a smile, "Yes I know, I am afraid".

Ayn is not obstructed by absence of self-regulation in learning. She possesses reasonably good self-regulatory learning practices. Her thinking capability is openly visible to the entire class; hence, there are no cognitive hurdles. Ayn is anxious based on the fact that this is a "math" class. Ayn is acting under affective inhibition. Integration of the writing theme of Making Sense of Number with the concept map of the course has been included in the Student Success Manual as an effective tool to eliminate this type of an affective inhibition.

Meet Lida, another student in the class:

> Lida dedication to assigned work, is well known in class. In class, it is certain her voice will be heard asking questions and answering questions. Lida could not do well on a two-hour test. She was exhausted. Lida was intent on

remembering, and the semester-long repeated requests of "just tell me how to do it" had not altered until the end of the semester. Lida had not been able to allow herself to undertake greater reflection on her computations and choose a strategy prior to engaging in computations.

Problem-posing pedagogy discussed at length in Chapter 2.3 and introduced through the Creative Problems Set in spring 2011, is being introduced as a regular classroom feature for group-work and development of metacognition to overcome cognitive blockage.

The three typical students under consideration demonstrate the need for a classroom climate conducive to accessing their own knowledge hindered by cognitive aspects such as not knowing where to start and how to proceed,[2] affective aspects, such as fear of mathematics, and an absence of sufficient study skills. These three factors, addressed simultaneously, emerged as essential foundations for establishing a favourable atmosphere for effective learning (Barbatis et al., 2012); more specifically:

- Cognition, achieved by construction and implementation of well-scaffolded educational materials and classroom discourse, utilizing the theory of the Zone of Proximal Development (ZPD). ZPD was utilized via meaningful questioning in the classroom and instructional resources designed in accordance with Bruner's theory of development of concepts along the concrete, iconic and symbolic forms, in this case, the Story of Number (Bruner, 1978).
- Affect, that is, classroom discourse and independent learning guided by development of positive attitude toward mathematics through instances and moments of understanding and enjoyment of problems at hand, extended by self-directed means of keeping up with the changing attitude toward mathematics and its learning, in this case, the Creative Problems Set, and
- Self-regulated learning practices that include "learning how to learn", productive and careful note-taking, daily attention to homework and asking questions, paying attention to metacognition and independent work, in this case, the Student Success Manual

The result of the one-time visit in fall 2010, by the Vice President with counselling expertise was promising; a sustained intervention was carried out in spring 2011 in the experimental class of Basic Arithmetic taught by the mathematics instructor. Another section taught by the same instructor without direct intervention of the team was the control class. The experimental class was conducted with the help of inquiry leading to discovery method, procedural-conceptual balance and with attention to problem-solving and problem-posing. The creation of trust between the teacher and students, based on the absence of negative classification and attention paid by all three instructors provided elements of the didactic contract. These, in turn, facilitated an increase in engagement, high-goal-setting and increased attendance. However, students did not develop enough persistence in achieving their learning goals. The next TR cycle will incorporate the conclusions of Barbatis (2006) dealing with persistence.

Referring back to Figure 1, the centre of the map is the LE composed of four instructional components:

- Creative Problems Set, a set of 10 problems as thinking-aside-tools (Koestler, 1964) for schema-building utilizing problem-solving and problem-posing;
- Story of Number, an instructional-sequence, utilizing Bruner's concrete-iconic-symbolic stages guides concept-formation;
- Making Sense of Number, the theme started explicitly in fall 2010 in the context of short essay writing assignments.
- Student Success Manual developed for Fall 2011 facilitated:
  - Sustainability of creative moments occurring in the classroom;
  - Thinking aside tools, analogous to Victoria Purcell-Gates' findings in development of literacy;
  - Building undeveloped skills (such as multiplication tables via the prime number pyramid, concept of the unit through Achilles and Tortoise race, etc.)

Note the newly present integration of the three design principles of cognition, affect and self-regulation, woven across the components of the created learning environment. The Creative Problems Set was found to be interesting, and students enjoyed discussions with the teacher-researchers whether it was about the decimal system or the race. However, the "know-how" to continue these explorations independently is still needed.

THE DATA

The control group exhibited self-defeating attitudes early in the semester, poignantly reflected in sparse attendance. Students with sporadic attendance were questioned, and had failed the class before. They attributed non-attendance to prior failure. Performance of the control group is evident in the significant difference in the midterm grades of the two classes. Both classes were administered the LASSI in the second week of the semester and the Motivated Strategies for Learning Questionnaire (MSLQ) in the last week of classes. The teacher-researchers had already detected absence of self-regulation in learning practices and the MLSQ was selected over the LASSI to assist and assess. The Motivated Strategies for Learning Questionnaire (MSLQ), is extensively used to assess college students' motivational orientations and their use of different learning strategies. MSLQ was chosen because it gives relatively good information about the level of student motivation and self-regulation of cognitive activity, which, accordingly to Pintrich et al. (1993) correlate well with the final grades in the course. Both qualities are in dire need of reinforcement among the majority of students in community colleges of the Bronx. Our aims were to find out to what degree a pedagogy focused on student creativity in the class can impact levels of their motivation and self-regulation, and as a consequence, their test scores as well.

Table 1. Students' background

	Experimental group	*Control group*
Gender	61% female	64% female
Ethnicity	72% Latino; 7% Black; 21% Unidentified	78% Latino; 4% Black; 11% Unidentified; 7% White
Average Age	21 (range from 18 to 30)	23 (range from 19 to 36)

Table 2. Attendance

	Experimental	*Control*
Initial Enrolment	28 students	28 students
# with perfect attendance	7 students	1 student
Withdrawals	2	5
Attendance Rate	81.3%	61.3%
Midterm Grade Average	2.53	1.96

The intent was to measure critical thinking, motivation for conceptual change, beliefs about knowledge, intrinsic and extrinsic motivation, and willingness to seek help for class. The instrument helps faculty members know what study skills/ affective characteristics should be explored with students; further, it provides a better insight on our students. The theories of Bloom and Koestler where it expected that students own their learning by understanding the affective and cognitive components of learning will assist as they overcome their "negative classification" and self-perception.

Table 3. LASSI scores

	Experimental	*Control*
Anxiety	50.5	64.4
Attitude	40.7	37.3
Concentration	46.9	66.9
Info. Processing	53.1	58.4
Motivation	44	45
Self-Testing	40.2	51.9
Selecting Main Idea	44.5	67.4
Study Aids	34.8	58.5
Time Management	47.8	58.3
Test Strategies	47.7	63.4

The MLSQ is an 81 item self-reporting instrument containing 6 motivational sub-scales and 9 learning strategies scales as follows:

Table 4. MLSQ distribution

Motivational scales	*Learning strategies scales*
Intrinsic Goal Orientation (4)	Rehearsal (4)
Extrinsic Goal Orientation (4)	Elaboration (6)
Task Value (6)	Organization (4)
Control of Learning Beliefs (4)	Critical Thinking (5)
Self-Efficacy for Learning and Performance (8)	Meta-cognitive Self-Regulation (12)
Test Anxiety (5)	Time and Study Environment Management (8)
	Effort Regulation (4)
	Peer Learning (3)

In the table below, scores from a preliminary MSLQ version are shown.

Table 5. Scores from a preliminary MLSQ

	Motivational beliefs			*Self-regulated learning strategies*	
	Self efficacy anxiety	*Intrinsic*	*Test value*	*Cognitive regulated strategy use*	*Self-learning*
Experimental	5.0	5.4	4.7	4.1	4.0
Control	5.0	5.6	5.1	4.8	4.2

RESULTS

Successes were clear – improvement of attitudes toward mathematics reflected in attendance, goal-setting[3] was more evident, higher level of engagement and increased but still not sufficient persistence. Preliminary data shows that early and sustained attention to affect in conjunction with cognition is essential in student retention, engagement and Just-In-Time study skills facilitation. Creativity-Literacy-Numeracy, a useful medium for design of instructional interventions, found a powerful integration of craft-knowledge of teacher-researchers with their own wealth of research through the theory of Act of Creation (Koestler, 1964). Koestler's approach of facilitating creativity is important in positively affecting both cognition and creating a mutually supportive affective-cognitive base. The affective-cognitive bridge, and the affective-cognitive-SRL practices bridge continue to play an important part in the emerging model of Students-as-Partners-in-Learning.

Classroom Methodology

The classroom had an open community environment. The class met twice a week. The instructor was present at all instructional sessions. She taught collaboratively with the VP/counsellor once per week. In the collaborative teaching sessions, the VP/counsellor found ways to keep the focus on enquiry while changing the perceptions involved through slight shifts of attention. For example, in the context of addition of fractions, the instructor's emphasis was consistently on the number line to visualize the operation under consideration, while VP/counsellor intervened with several different examples such as one about a candy bar shared by two people each eating the fraction under consideration and questioning how much of the candy bar is left, the pizza that was meant to be shared between friends and planning time for painting the wall of a room. Through these gradual perceptual shifts of attention in which the focus of enquiry is held constant and the class environment is made light through some humour, the mathematics literacy base of learners was exposed to new situations with constancy in conceptual thinking. The environment had been effectively created for the bisociative act of Koestler to occur. Study skills are embedded within the Just-in-Time method, that is, students are provided with a supportive environment whenever they need it, and independent thinking was allowed and encouraged to flow.

The third collaborator, the librarian, provided library resources for the class, and there were sessions when the entire team interacted with the class at the same session; once, exclusively for the purpose of discussing the first author's concerns about study habits. From this session he discovered that self-regulation in study skills would be a helpful inclusion into the classroom environment.

Three instructional approaches emerged, each arising from the natural inclination toward mathematics and problem-solving of each teacher-researcher on the team. There, of course, were differences in individual approaches, one being more procedural, another more conceptual, however, the commonality across instructional approaches, is the commitment and intent for learners to discover the underlying mathematical structures called for in each problem situation. The instructional approaches can all be explained using the theoretical perspective created by Arthur Koestler. Bisociation was facilitated, as the creative leap that occurs when several frames of reference are held in simultaneous scrutiny and insight, apparent from the various simultaneous perceptions conveyed by the students. Koestler's work provides a theoretical foundation for creativity within in mathematics education; this innovative approach that will be explored further in succeeding semesters. Noting the effective approach of the VP/counsellor pair in creating slight perceptual shifts to keep attention on the focus of enquiry, it was also used while learning operations on fractions. The topics naturally progressed from problems involving operations on fractions to those embedded with rules of exponents and fractional exponents. It was an exercise enjoyed by students; their attention was periodically directed to the use of rules of operations on fractions as a skill being learned. Further, the VP/counsellor scaffolded bisociation by bringing students' attention explicitly to the distractors in the problems under consideration, and, jointly with students, examined ways of improving focus.

Interventions of drama and math in the classroom in the form of the Poznan Theatre Problem (Chapter 2.3) are further examples of the open community environment of the classroom. Professor Howard Pflanzer, a playwright and a drama professor, was invited to stage a one-session classroom interactive activity. His selection involved topics in the course curriculum in the context of theatre appearances in Poznan, Poland. He engaged students by asking them to solve the puzzle of reaching several theatres locations without a GPS, and only a street map in hand (see Chapter 4.2 for more details).

Making Sense of Number is the explicit overarching theme whether in classroom discussion or in the instructional materials, including homework. It is essential to maintain a multi-frame-of-discourse mode for the balance between enjoyment, cognitive penetration, and aesthetic appreciation. Creativity had emerged within the classroom as an organic development of the craft knowledge of the teaching-research team; however, it was the support of Arthur Koestler's The Act of Creation (1964) that provided the rich theoretical base where thinking and the development of creativity were rooted.

Design of Triptych – Based Assignments

The Act of Creation (Koestler, 1964) defines bisociation, that is, "the creative leap [of insight], which connects previously unconnected frames of reference and makes us experience reality at several planes at once." Consequently, the creative leap of insight, or bisociation, can take place only if we are considering at least two different frames of reference or discourse.

How do we facilitate this process? Koestler offers a suggestion in the form of a triptych, which consists of "three panels…indicating three domains of creativity which shade into each other without sharp boundaries: Humour, Discovery and Art." Each such triptych stands for a pattern of creative activity, for instance:

Comic Comparison ⇔ Objective Analogy ⇔ Poetic Image

The first is intended to make us laugh, the second to make us understand, and the third to make us marvel. The creative process to be initiated in our developmental and introductory mathematics urgently needs to address the emotional climate of learners, and here is where the first panel of the triptych comes into play – humour. Having found humour and the bearings of the concept in question, the connections within it have to be explored further to "discover" the concept in detail, and, finally, to take the students' individual breakthroughs to a level where their discovery is sublimated to Art.

Here's an example of the triptych assignment used by Vrunda Prabhu in her Introductory Statistics class:

Trailblazer ⇔ Outlier ⇔ Originality[4]
⇔ Sampling ⇔
⇔ Probability ⇔
⇔ Confidence Interval ⇔
⇔ Law of Large Numbers ⇔
Lurker/Lurking Variable ⇔ Correlation ⇔ Causation

The triptych below is an example of student work:

Trailblazer ⇔ OUTLIER ⇔ Original
Random ⇔ SAMPLING ⇔ Gambling
Chance ⇔ PROBABILITY ⇔ Lottery
Lurking Variable ⇔ CORRELATION ⇔ Causation
Testing ⇔ CONFIDENCE INTERVALS ⇔ Results
Sample Mean ⇔ LAW OF LARGE NUMBERS ⇔ Probability

Triptych assignments facilitate student awareness of connections between relevant concepts and, thus, further support understanding. However, what maybe even more important is the accompanying discussions that help break the "cannot do" habit and transform it into original creativity; below is a triptych completed by a student from a developmental algebra class:

Number ⇔ Ratio ⇔ Division
Part-Whole ⇔ Fraction ⇔ Decimal
Particularity ⇔ Abstraction ⇔ Generality
⇔ Variable ⇔
Multiplication ⇔ Exponent ⇔ Power

The use of triptychs in the mathematics classroom brings back the game and puzzle-like aspects inherent in mathematics. What is the connection between the stated concepts? What other concepts could be connected to the given concepts? Given the largely computational nature of the elementary classes, and the students' habit of remembering pieces of formulas from previous exposures to the subject, a forum for making sense and exploring meaning is created to help connect prior knowledge with new synthesized reasoned exploration. The question "how", answered by the computations, is augmented with the "why" through the use of mathematical triptychs. Student responses to the algebra triptych designed and implemented by the teacher-researcher Bronislaw Czarnocha indicate the germs of student creativity:

Table 6. Student responses to an Algebra Triptych

Factory ⇔	⇔ *Factor* ⇔	⇔ *Polynomial*
1. Starting point for the factor.	A variable for the polynomial	
2. Breakdown into factors	One way to solve the…	…polynomial is to factor.
3. The relationship between a …	… factory and a factor is that the factory has sameness	…equation and equality
4. Puts together an equation	Breaks down into smaller numbers	Numbers turn into a different equation
5. A factory…is an equation of some sort that you need to factor to get an answer	Then factor has to do with polynomial, it's an equation like polynomial which you have to factor	To get the answer to whichever polynomial problem.
6. A factory has many numbers of boxes so it can be distributed to the stores	A factory and factor is the same because…	… A polynomial is distributed to the unknown factor.

The creativity of students came into play, especially on the transition from the first column to the second, when they wanted to establish a relationship between a factory, – a place that produces things, with the concept of a factor in mathematics. The cognitive links can be seen through such phrases as "Factory is an equation…", "has many boxes to be distributed…", and "One way to solve the polynomial is to factor." Through this analogy, the student is conveying and solidifying his or her understanding that the polynomial gets distributed, or, in other words, factored into "those boxes."

Each triptych needs to undergo several TR cycles of design refinements, triangulated by short semi-structured interviews, to get to the precise meaning of the used metaphor. Nevertheless, even from the first cycle of student responses, one can already see the potential richness of creative mathematical thinking geared towards finding connections between related concepts of the triptych. However, positioning the triptych within a sequence of classroom work or homework assignments in an efficient way, so that it provides the necessary facilitation of the related concepts discussed in the classroom, does require special creative attention on the part of the instructor.

Significance

The study provided the potential to positively benefit many students, combatting the ubiquitous and long-standing disinterest in mathematics accepted by society at large, as well as the disenchantment and disenfranchisement from education, especially, of young men and women in urban public universities such the City University of New York.

The “win-win” situation for participants was created by the presence of the counsellor in the classroom allowing:

- On the cognitive front, to devise a curricular resolution to the long standing epistemological obstacle of the real number;
- On the affective front, to address the resilient student perceptions of long-standing apathy toward mathematics;
- The creation of a comprehensive learning environment that integrates cognition-affect-SRL-practices (Barbatis, 2008) and extends the model of changing student-perceptions through the teaching-research methodology, Students-as-Partners-in-Learning;
- The recognition of creativity as the primary engine of discovery in learning, whether mathematics or metacognition, that facilitates the ownership of learning, and where challenges are well-scaffolded to enhance student achievement (Csikszentmihalyi, 1997).

Through the integration of classroom practice with Koestler’s bisociation, an avenue is created for learning beyond the semester. Koestler’s triptych with the three planes – Humour, Discovery and Art, supporting the bisociative act of creativity, provides the path from Bathos to Pathos, the needed ending of fear of mathematics and the transformation of repeated habit to originality via creative expression.

Teaching-research is existentially bisociative, balancing perspectives, understanding local complexity in isolated precision, achieving a holistic coherence, choosing most fitting research footholds for learning difficulties. This approach adapts existing research, to produce better student learning in the classroom and to achieve the goal of actual improvement of learning. The design principles of the theoretical framework reflect the integrative unity across academic disciplines for a “win-win” situation that, in benefiting the particularity of the classroom, in its refinement through adaptation, stands to benefit a general problem.

Creativity in the teaching of remedial mathematics is teaching gifted students how to access their own giftedness.

A NOTE ON HOW TO READ THE LEARNING ENVIRONMENT CONCEPT MAP

Naturally, the concept map, Figure 1, is very complex because it reflects the very complex reality of a remedial mathematics classroom. Yet within that complexity, one can distinguish three separate primary LE components surrounding the main content component of the course, the Story of Number – a teaching sequence addressing arithmetic and algebra. Each of the first three LE components is designed to bridge affect, cognition and self-regulation skills; the Creative Problems Set and the methodology of teaching called Making Sense of Number are integrated within the Student Success Manual, which guides students through the development of self-regulatory skills. More precisely, the concept map shows upward directed branches stemming from each of these three primary components explaining the content or the method utilized by each, and downward directed branches showing the relation each component makes with cognition, affect and self-regulatory learning, the three essential factors described in the article. Each branch can be investigated separately paying attention to particular concepts (in boxes) and to connecting phrases (no boxes). One can proceed down or up along the branches, depending on preference. For example, the Creative Problems Set connects cognition with affect by facilitating enjoyment in problem-solving and, thus, in learning mathematics, that, in turn, leads to improved performance. Continuing upwards, one can see that the Creative Problems Set is composed of standard curriculum problems, problem-posing and problem-solving and facilitates creation of new problems by students themselves. Each of the components is supported by research literature such as Blooms taxonomy of cognitive skills, Polya's method of development of strategies and Koestler bisociation. All three primary components contribute to the increase of enjoyment with, reflection upon and ownership of mathematics content. On the other hand, the fourth and main cognitive component, the Story of Number teaching sequence, is constructed with the help of the Discovery Method guided along classroom ZPD, where the degree of cognitive challenge is regulated via precision of language and historical development of the concept in mathematics. Like the first three, it contains an analogous branching structure.

To summarize, the methodology of each LE component is sketched in the upper part of the concept map. The Story of Number is the teaching sequence in arithmetic and algebra built out of smaller particular sequences, with the help of a particular theory of concept development. The Creative Problems Set acts, naturally, through a problem-solving approach that leads to the development of conception accordance with Bloom's recently refined taxonomy. The Student Success Manual consists of student sample work from previous semesters, which are used as "thinking aside tools". Making Sense of Number is an approach that facilitates concept development through discussions and writing assignments.

NOTES

1. It was a very important session because, it was only that session that allowed the instructors to complete all topics on the Math 01 and Math 05 curriculum.
2. This is the question that is explored in the problem-solving grant, more specifically, Polya Steps 2 and 4.
3. One student in the experimental class of Arithmetic determines to master Arithmetic and Algebra and starts a general interest in the class both experimental and control to do so.
4. Two examples of facilitating the discussion along the triptych Trailblazer—Outlier – Originality are in the Appendix to this chapter. For a more detailed explanation detailed explanation of the Triptych methodology in action see Chapter 2.5.

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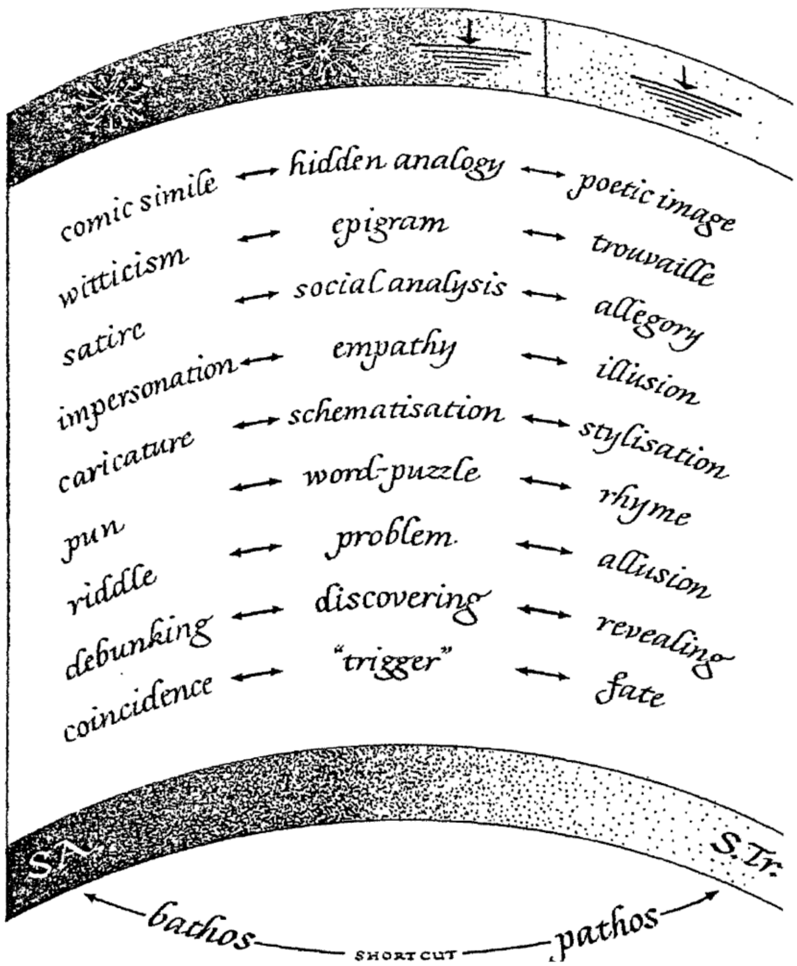
APPENDIX

Koestler Triptych

The three panels of the rounded triptych…indicate three domains of creativity which shade one into each other without sharp boundaries: humor, discovery and art… Each horizontal line across triptych stands for a pattern of creative activity, which is represented on all three panels;



comic comparison – objective analogy-poetic image.

The first is intended to make us laugh; the second to make us understand; the third to make us marvel. The logical pattern of the creative process is the same in all three cases: it consists in the discovery of hidden similarity. But the emotional climate is different in the three panels:: the comic simile has a touch of aggressiveness, the scientist's reasoning by analogy is emotionally detached, i.e. neutral.; the poetic image is sympathetic or admiring, inspired by a positive kind of emotion…The panels on the diagram meet in curves to indicate that there no clear line dividing them (Koestler, 1964, p. 27).

Examples of the Triptych Methodology in Action

The application of the triptych methodology depends on the teacher's interpretation and didactic need. The central role of the methodology is to facilitate students' discovery of hidden analogies between two pairs of concepts along the triptych with the aim to free them from some of the affective obstacles and to release their originality in the classroom. We are presenting two examples of application of the triptych methodology by two different instructors, the first one having more emphasis on the cognitive aspects of the path, while starting with a comic representation of the notion of the Trailblazer. The second example recognizes the presence of the triptych pathway within the general classroom approach and it focuses primarily on affective issues along that path.

One can use the elements of the definitions of three concepts found in a standard dictionary such as Miriam Webster or Oxford dictionaries of English language:

- Trailblazer: a person who makes a new track through wild country, a pioneer, innovator, ground breaker.
- Outlier: a statistical observation that is markedly different in value from the others in the sample; a person whose place of business and residence is far removed from others.
- Originality: the ability to think independently and creatively; the quality of being novel or unusual

Example 1. The Donald Duck as a trailblazer

A comic example of the trailblazer could be the image, a short video from a YouTube or digitally designed for the class of Donald Duck trailblazing new skis on a slope covered with snow, while other ducks related to the character are observing the process and attempting to follow. The comic aspect is reached by counterpoising an idea of "the person who makes a new track through the wild country, a pioneer" with the known qualities of the Donald Duck. The aim of the discussion is to find the hidden analogy between the scene and "a statistical observation that is markedly different in value from others in the sample". For example, students might make the connection between the outlier as a statistical observation with the reality and the uniqueness of the Donald Duck. They can ask whether every trailblazer is an outlier or whether anyone living far away would be a trailblazer. Having reached that understanding along the first bisociative transition, students turn their attention to the second bisociative transition in the context of the relationship between outlier and originality. They may inquire whether every outlier is an original person, possibly making the connection between the quality of a trailblazer as a pioneer and the quality of originality via being an outstanding outlier.

The aim of the assignment was to focus student attention on the bisociative frameworks and the discovery of hidden analogies between them.

Example 2. Classroom Outliers

The second instructor who used the triptych in the classroom changed the position of trailblazer and outlier in her understanding of the classroom dynamics to

Outlier–Trailblazer–Originality

The three components of the triptych are the three stages of learning that students have to pass during the class. As instructors, it is our responsibility to help students to traverse this pathway from left to right. along the triptych. This can be achieved through the class dialogue using humour, drama, contract and compromise. Every instructor has basic contract with the class regarding syllabus, grading policy, lateness, absentees, homework etc. which is considered the class norm.

Classroom Outlier: We know that the main obstacle for our students' success is their own negativity towards learning, fear or anxiety about math, lack of self-confidence, lack of motivation etc. in all different levels. Some of them have no idea how to be a student, hence, for example, unable to see the connection between punctuality, absentees and passing the class. They never complete homework, are not prepared, do not participate and hence *feel like outliers* in the classroom.

The instructor can focus on their plight by simply addressing one of the issues at a time using humour, not discipline. For example, giving compliment if the outlier participates or asking him/her something outside the subject matter that has the relevance for the student. Once the outlier is closer to the class norm then the work starts on the trailblazer quality.

Classroom Trailblazers: I believe that all students have a potential to become a pioneer or an innovator even though some are unaware of it. Both humour and spontaneous drama during the class can be used to recognize or enhance the trailblazer quality. Suppose that after the test-1 the instructor asked the students to compare the questions in the test with previous class notes, homework exercises or review notes. Most students agree that they look similar and one student (outlier) says they are not same. The chances are very high that the class will try to convince the outlier how these are similar, comparing exercises, steps, rules etc. helping the student to find the "hidden analogy" between the two. Once awareness is created about the similarities, there is a clear strategy for improvement. Success of students as trailblazers in the class environment is within their reach. During this drama, the lecture time might be compromised but this social interaction is excellent tool for addressing the affect issues of the class.

Classroom Originality: Students' increased belief into their own trailblazer quality in the context of their class, they start producing their original work. Their success is evident.

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2.5. THE REFLECTIONS OF THE TEACHER-RESEARCHER UPON THE CREATIVITY PRINCIPLE

SUMMARY

In this unusual piece of her Teaching-Research diary, Prabhu presents a dual approach to the Creativity Principle, when she explores the creativity of herself as a teacher-researcher working on the facilitation of creativity in her classrooms. She shows through self-reflection the very process through which the creativity principle transforms the habitual teaching practices into teaching-research. She presents her main findings in that process, the concept map of the course, the narrative and the discovery method of teaching. The development of the concept map for the course, which with time acquires the general quality through the dialectical relationship between herself as learner from students, and as their teacher, becomes for her the backbone through which the meaning of mathematics is conveyed. She emphasized the narrative as the medium through which she has appropriated Koestler's triptych for mathematics as the tool with the help of which students are searching for meaning between different concepts. Finally, she describes the process through which the creativity principle enters the discovery-based method of teaching. Discovery based method signifies the classroom instruction proceeding through student inquiry into nature of mathematical concepts, which may lead to discovery of new connections between them.

The teaching-research cycle generates the creativity principle by its repeated iterations. This means that in the iterations, the teacher-researchers, while trying to match diagnosed learning needs with appropriate instructional solutions, have the opportunity to "escape more or less automatized routines of thinking and behaving" (Koestler, 1964), and to begin connecting a "previously unconnected matrix of experiences" allowing "reality to be experienced on several frames at once". Practical examples of how this transformation from habit to originality begins for me, as a teacher-researcher, is in the scrutiny that I must place on the prescribed syllabus I am expected to teach. It has been habitual to teach it the way it has been prescribed. However, when I put the table of contents of the textbook into a concept map, I begin seeing that my own mathematical knowledge begins guiding me to make rearrangements, so that the organization of topics can flow from one to another based on mathematical sense. Making sense of what is taught and making it meaningful is very important in my approach, because I have clearly recognized, as many have before me, that learning is slow at first but catches on and

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 127–131.

changes its momentum very quickly; for example, a similar observation is described by Mahavier (1999). In previous reports I described actual instances of students making sense, extrapolating meaning and discovering concepts through inquiry via activities and guided exercises that I facilitated.

The creativity principle aimed at a transformation of the habitual teaching practices of the teacher-researcher plays an essential role. Detailed illustrations of manifestations of the creativity principle are described below.

1. The creativity principle is tangible in the ways in which the materials used in the classroom begin to change; first, the concept map evolves from its original prescribed form (the official syllabus) into a natural progression of beautiful mathematics starting from just 1, and 0, and flowing as seamlessly as possible through the entire list of topics given in the syllabus. For me, the development process of the concept map is directly equivalent to the teaching process itself because one of the main factors in its evolution is the collection of questions asked by the students as well as their expressions of difficulty of certain topics. For example, many students have a difficult time with multiplication tables. To address this challenge, the instructional sequences entitled *Story of Number*, for an arithmetic course, and *Story of Number in Abstract*, for an algebra course, contain the *Prime Number Pyramid* component, a creative tool that aims to enable students to construct multiplication tables on their own.
2. The creativity principle is implicit in teaching; more precisely, the teaching process necessitates learning from students. This occurs by paying careful attention to what students say. For example, when a memorized fact is heard from a student, I ask the student "Why is it so?", and determine whether the student understands the meaning of the idea and how to support it, or is just repeating a memorized claim. This allows me to address the issue on the spot, and, sometimes, to design materials that demonstrate the meaning of the concept in question. *The Fraction Grid* is a good example of a teaching tool, built as a result of such an interaction, to address students' expressed difficulties comprehending sizes of fractions. Shifting focus from traditional one-sided teaching to the bidirectional integration of teaching and learning, the students and I begin developing mutual trust that allows and encourages students to freely express their thoughts; the classroom becomes a safe space where a "thinking partnership" is not just a phrase but is the expected form of classroom discourse.
3. Narrative becomes a useful, interesting and meaning-producing tool in the integrated teaching-learning process. Bruner, in *The Culture of Education* (1996), states that it is through narrative method that one both organizes and constitutes one's experience of the world. In describing narrative as a discourse, Bruner states, "Narrative is discourse and the primary reason for it is that there is a reason for it that distinguishes it from silence" (p. 121). The learner is drawn into a narrative communicating her or his understanding precisely at its current stage allowing further development through continuing discourse. "By using

narrative…as our organizing principle we show new learners…how to make claim of mathematical territories by populating the landscape with fictional things engaged in purposeful activity" (p. 123). Narrative became an aspect of my teaching simply by choosing to name the inquiry-based instructional sequence I was developing *Story of Number*. Shortly after, the concept map, developed through the evolution of the syllabus, was a "story" itself, conveying the natural relation of the mathematical concepts in the teaching sequence to the methods of instruction. However, there still remained the important matter of linking the narrative process to a continuing quest into a deeper learning on the part of the student. This was going to be accomplished by requesting students to write a short one page essay called *Making Sense of Fraction*, after the discussion of fractions was almost completed. This somewhat narrow written exposition led to the end-of-semester essay in which students wrote about how they make sense of numbers, in general, reflecting on how it was taught over the duration of the course, providing specific examples of how they made sense of numbers in daily life. This essay was entitled, *Making Sense of Number*. Between 2010 and 2012, when I co-instructed the course with the college's Vice-President for Student Development and Academic Librarian who has been on the teaching-research team for a long time, regularly observing the evolution of teaching strategies in real-time, our joint insight into the methodology, as teacher-researchers, led to an even broader and deeper learning experience. We found that student engagement was high eliminating any classroom affect concerns allowing me to expand the concepts taught in a typical arithmetic course to a significantly more advanced level. During the same time we found a theoretical match for our classroom discovery,—Arthur Koestler's Act of Creation. This connection suggested another novel way of bringing the creative principle into classroom teaching. Koestler posits that *Humour*, *Discovery* and *Art* are three shades of *Creativity*. Setting the goal of Discovery as the central concept, we began to use Koestler's triptychs to create another gateway into our students' thinking and meaning absorption. Using Koestler's approach of incorporating narrative into the classroom, we created our own triptychs for the purposes of enhancing the teaching-learning process in a way that allows students to take small chunks of course material and examine them deeply, making sense out of them. Students were then asked to write a few sentences about how they perceive the connections between the concepts, allowing me to analyse their understanding. Then the current triptychs were viewed by the whole class along with triptychs created by students from previous semesters providing the current students with an opportunity to rethink their own triptychs. This time the reflection was based on a library of triptychs created by over 60 students allowing the current students to recreate the stories of their own learning of the concepts. Students have continuously found this exercise very interesting and expressed interest in using triptychs in other classes such as anatomy. They claimed that it gave them greater confidence, or, that it made the concepts and their interconnectedness clearer. Our version of a triptych consists of a central column holding the relevant mathematical concept, corresponding to Koestler's placement of the "discovery" aspect. The left column contains the related instances of "humour" described by Koestler as "the back entry into the inner workshop of originality", and the right column is the contextual interpretation of the "art" aspect of the creativity principle, For Koestler, "art" is the sublimation of "discovery". In the beginning the students are provided with a triptych in with, for example, eight central topics of the course in the "discovery" column. Together, as a class, we complete two of these rows, with the entries in the "humour" and "art" columns supplied by me. However, through class discussion I can assess their grasp of how all three concepts work together within the triptych. Furthermore, having the students write a couple of sentences, using examples, describing their view of the connections, supports the discussion. The association of "simple words" is now taken to a new level illuminating the interconnectedness within a particular triple; the words have meaning linking them pairwise to each other, in addition to the presence of a single thread of meaning underlying all three. Based on the class discussion the connections are already somewhat clearer to the students, so, to contribute a few meaningful written statements, they need to inquire even further into the triptych. In this way, the environment for inquiry and for the facilitation of *bisociation* is created. *Bisociation* is the term coined by Koestler to distinguish it from association, where bisociation is the flash, or a creative leap, of insight that connects previously unrelated frames of reference allowing reality to be experienced on several frames at once. The stage for bisociation is set through the triptychs.

The application of the creativity principle in the classroom is now a useful and usable tool for both the teacher–researcher and the learners. The resulting learning process is a search for meaning and inquiry with the intent of discovery, and provides all of the new instruction aspects that transform the classroom.

A VIABLE TEACHING METHOD

To demonstrate the viability of our respective teaching approaches we must answer the following questions:

1. What are the principles of design?
2. Where do they come from?
3. What are some examples that demonstrate our approach and the application of the principles of design?

The targeted principles of design for my instruction have always been the inquiry or discovery based method of learning. However, the classroom climate had not been conducive to learning via this method, hence, over a period of continual teaching-research experimentation, a way had to be found to transform the remedial mathematics classroom into one where the discovery method could be successfully implemented. The classroom climate needed to be grounded in simultaneous attention to cognition, directions of problems, meaning of symbols, meaning of problem structures, and the synthesis of all these to achieve the students' goal to "solve a given problem."

Typically, at the end of each problem completed by a student, the instructor carried out a whole class reflection and discussion addressing the following questions:

- What were the rules employed?
  The only rule turns out to be the order of operations.
- What cognitive, or procedural, elements were used?
  The grasp of the meaning of each symbol along with the correct interpretation of the meaning in a given context was the main cognitive requirements.
- How was the structure of the problem taken into consideration?
  A solution to the problem required a careful reading and clear comprehension of the problem, followed by making sense of each symbol correctly and determining which portions of the string of symbols needed to be addressed in what priority.
- Was the solution obtained correctly?
  This consisted of checking what was done, followed by each student's individual silent reflection with the objective of identifying component errors, if any, or addressing more general misconceptions of the meaning of the problem or of the totality of the string of symbols involved.

As students, one by one, completed these problems with the instructor's facilitation, the complaints, negative affect and a general resistance in the room subsided. The following day the topic of radicals was introduced. This topic, based on the instructor's prior experience, is another commonly negatively received mathematics classroom topic. However, this time the climate was strangely different. Students were determined and intense in their thinking, and, when it was time to end the class, those students who complained earlier were continuing to work on more problems. *Discovery had arrived in the remedial algebra classroom!*

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2.6. REFLECTION AND CASE STUDY OF CREATIVE LEARNING ENVIRONMENT

INTRODUCTION

The bisociative framework of the Teaching research method supports a creative learning environment (CLE) in the classrooms. The essential goal of our teaching research team was to design research in our respective classes, which promotes creating learning environment. In the different chapters of the Unit 2, Vrunda Prabu gave several examples of CLE in her classes that she had provided to improve learning. The goal of improving the educational environment for students is characteristic of learning communities as expressed by Jaworski, "The motivating principle on which all agreed was our desire to develop better learning environments for students in mathematics at the levels of schooling in which we were associated" (Jaworski, 2008, p. 316). A common theme of the work Jaworski (2003, 2006, 2008) is that within a learning community or community of inquiry teachers through reflection upon practice undergo a shift or critical alignment (Jaworski, 2006) in which reflection upon practice leads to an attitude of being open to change. The shift from practitioner to a reflective practioner is the essence of the bisociative TR experiment in which lessons and methods are designed, tested and reflected upon by the Teacher Researcher.

We see that the central theme in improving the learning for underserved students, is Prabhu's insight of a creative learning environment which will transition the students from habits of failure to excellence. An essential component of this transition is affect, and as pointed out by Furinghetti and Morselli (2009), "purely cognitive behavior is rare in performing mathematical activity: other factors, such as affective ones play a crucial role" (p. 71). The authors Goldin (2000, 2002), DeBellis and Goldin (2006), Goldin et al. (2011) and Lomas et al. (2012) all highlight the many dimensions of affect such as student motivation, beliefs, attitudes and self-identity and its role in learning and problem solving. Yet the literature on the teacher's role in supporting positive student affect within the classroom especially within the underserved community is lacking. In Chapter 2.1, Prabhu introduces the concept of "Didactic contract, a Handshake and a Compromise" to make her class, a creative learning environment where student and the instructor have the same goal "excellence" via the participatory, dramatic intervention of *The Choice.* While in Chapter 2.2, she created an artefact a didactic tool (ex. fraction grid (FG) etc.) for the students which is a by-product of the several TR cycles and her conscious reflections

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upon it and hence presents a new concept of "particularity and generality". Her CLE is to use the fraction grid (FG) to fulfil the role of a mediator between the art and mathematics to enhance learning. "Problem-posing and Problem-solving" is another didactic tool used by Prabhu in Chapter 2.3 and in addition she team taught the class with a drama professor, manage to create an interesting and supportive learning environment. Pozman Theatre problem is a perfect example of a mediation between drama and mathematics. Chapter 2.4, she emphasized in creating a conductive learning environment, based on the principle of cognition, affect and self-regulatory learning practices to promote student's engagement in mathematics classrooms. This time she involves a counsellor and a librarian to address the affect and self-regulated learning respectively. Her teaching experiment cycle on problem solving integrates the work of Polya, Bloom and Koestler.

Keeping in mind the common theme of creative learning environment of this chapter, which was beautifully illustrated by Prabhu in the first four sections, I discuss through examples how these different ideas play roles in my classroom to improve learning and push my students towards excellence. Unlike Prabhu, I have no new outside resources added to my classroom environment. Classroom dialogues, everyday unscripted drama and guided discovery method are my main tools used to achieve the above-mentioned goal. The TR team is a tremendous source of encouragement and support for me. The often intense team discussions centred on student's participation, methodology of the lesson, discovery method etc. has given me opportunities to improve my lessons and teaching practices and help me to become a better teacher researcher. I want to extend the same wonderful opportunity to my student who needs the support system from a learning community, since many lack family support and self-esteem.

In the preceding teaching research experiments (Chapters 4.2–4.5) Czarnocha, Dias and Baker as well as Prabhu (Unit 2) the focus is on cognition and creativity, the affective component of creativity and the Aha moment is understood as causing positive affect in students (Liljedahl, 2013), also noted by Koestler (1964) in research. Following Cobb's (2011) dictum that the best place to conduct research on learning is the classroom in the following work I review my role in the classroom as a reflective practitioner to support learning with underserved often resistant students. Teaching research team suggest that research especially teacher's narratives about their classroom methodology, reflecting upon supporting and creating positive learning environments in the classroom is central to Stenhouse's vision of action research (McLaughlin et al., 2006) and in our case of teaching research experiments.

My Classroom

In a typical semester I teach at least one remedial course since majority of our freshmen students are placed into these course. In an urban community college serving minority students in the Bronx the students are mostly non-traditional and minority students. Many work full time to support children, parents. In most cases, they are first generation college students who often have a weak academic background they do not typically receive a lot of emotional support outside the classroom. Academically they often lack good study habits, discipline, and frequently they do not have a strong goal or aim in life and thus, retention is a very real concern. They often carry tremendous amounts of negative experiences with them not just in mathematics but also in their personal life. Responses such 'I hate math', 'I hate fractions', "Me and math do not get along', 'I am an early childhood education major, so I do not need math', 'when will I ever use algebra in real life' etc. are common complaints in the remedial mathematics class. The Bronx is among the poorest district in U.S. and hence student struggle with many aspect of life including finding and staying in a home environment including hunger. The instructor must address both the emotional and educational need of the student simultaneously, so the student can perform better in the class and eventually be successful in life. Before mathematics topic is taught, it is more important to teach how to be a student. Then help them to find a better student within himself or herself and finally push them towards excellence in mathematics.

Teaching Research within Creative Learning Environment

Teaching research has two essential goals: improvement in learning and working towards excellence. So every teacher researcher is a firm believer of the practice of teaching and learning where instructor and students constantly teach and learn from each other in order to achieve excellence. This can be attained in a creative learning environment: which is informal, pleasant, safe, fun and at the same time also informative.

Didactic Contract: A Handshake and a Compromise in My Class

In Chapter 2.2, the concept of Didactic contract, Handshake and a Compromise is introduced and discussed (Sarrazy & Novotná, 2013). Here I provide an illustrative classroom dialogue and narrative to explain what these concepts mean to me and how I use them to influence my class-learning environment. Usually the didactic contract begins on the first day of classes, when an instructor hands a syllabus to the student, which can be considered a contract between them. The instructor expects the students to abide by this contract. This contract may include policies on attendance and lateness, grading, classroom behaviours etc. and of course the mathematics topics. In my opinion it becomes a handshake or mutual agreement if instead of forcing the contract on them, their input is also a part of the contract. For example if class participation or going to the blackboard is expected of them, and then they should be given assurance that if they are unable to perform the task at hand, instructor or the classmates will give hint or help. Once the students feel safe and respected, the instructor is rewarded with the handshake. The handshake cannot be demanded but can be achieved by a fair act and mutual understanding where both party (instructor and students) are part of the class contract and working towards the same goal "excellence". High impact practices also calls for setting the acceptable classroom rule together, which is an excellent example of a handshake.

Any issues or situation during the class can turn into a handshake with a little give and take, a Compromise. There is no need to repeat the same method that is proven to be ineffective, instead learn from it and negotiate. For example, you are scheduled to give a test next week on Monday but students are resisting because they have other examinations or other reasons. The instructor could say "fine, I will give the examination by the end of the next week but I will cover another two sections and add them to the test". Two scenarios will play out in the class, either they will change the mind and are willing to take the examination on Monday or they will accept the latter date feeling good about themselves. It boosts their confidence and I have nothing to lose, whatever is the outcome, students are part of the decision and hence it turn out to be a handshake through compromise, a win-win situation for all. Life is full of compromises, negotiations, deals and handshakes then why not in a mathematics class.

Teaching Research Methodology: Affect

After an initial contract and handshake, one mantra all students should understand and repeat in their head is that "I will try my best because I cannot do more than that" and "the instructor will help me with all that she can do legally, and to the best of her ability." I repeat this mantra almost daily to some student during the class, I will not settle for less and neither should they.

Daily Drama Use to Create a Learning Environment in My Class

The failure of student in mathematics is all too often considered socially acceptable in U.S. In addition, our students lack study skills, have poor study habits, no family support, and plenty of financial problems, children's demands and family obligation to name a few. This creates tremendous obstacles in the learning of mathematics and in general in getting a degree. A creative and innovative approach "everyday unscripted drama" mixed with humor has the potential to transitions these habits and attitudes of failure (Koestler, 1964) to creativity and student excellence. Vrunda Prabhu has used "drama" to motivate the students. She and a drama professor joined forces to reverse the resistance of learning mathematics using scripted drama. I also believe that drama is a good thing to introduce in mathematics class, not only scripted but also unscripted, just-in-time drama.

By definition a drama is an exciting, emotional, or unexpected series of events or set of circumstances and is synonymous with incident, scene, crisis, excitement or a thrilling experience. Crisis, excitement and, incidents are part of life and everyday these circumstances take place in our life and in the life of our students. If a student has a sick child at home or parent in the hospital or is going through a domestic issue, how can they learn math if these issues are not addressed? Again if "I hate math" or "I am not good at math" attitudes are not dealt with, learning is highly unlikely. Once a student complains that he is unable to sleep well, since he had a recurring dream where he sees numbers circling, it deserves instructor's attention. Fear is a powerful force, which can play trick with one' head and numb the person to act in a responsible way. Fear by definition is an unpleasant emotion caused by the belief that someone/something is dangerous; is synonymous with panic and a panic stricken student is in no condition to learn anything. Avoiding the situation or the student is not an option for me. It is my belief that when something is unpleasant, it can be discussed, explored in order to reverse the belief or at least lessen its impact. Not trying is like burying your head in the sand and pretending it does not exist. I try to address the fear of math, personal crisis and excitement through unscripted small drama during the class.

It is human nature to become open in a safe and comfortable environment to a sympathetic ear. Let say one student comes to the class all excited and informs that it is her birthday. First of all, just because she shares with me or with the class this information is a sign of her comfort zone. I believe today she talked about her birthday, soon or in near future she will share an answer to a math problem. Well, I have two options ignore or wish her "Happy Birthday" out loud. The first option reduces her comfort zone and second option increases her comfort zone. My choice is clear, use the just in time drama to the advancement of learning by increasing comfort level of students.

Class Dialogues

a) Drama and Humour to influence student affect:

Suppose a student boldly confesses during a lesson that "I hate math". Again two choices: Ignore and use it (just in time drama) to teach something else (just in time teaching research act). I will jump in with my both feet in the water (an education act). A classroom dialogue that follows after such a statement is recorded below which addresses the fear and the resistance of learning mathematics:

Instructor: Who else hate math. (*few more hands*)
Instructor: In the scale of 1 to 10, how much?
Students: 10, 9, 5, 8 etc. (*All different level of hate*)
Instructor: Why do you hate math?
Student #1: I do not like numbers.
Student #2: I am not good at it.
Student #3: I do not need it, it is hard, and who cares?
Instructor: Do you have children? Suppose your child comes and tells you, he hates school and does not to go to school anymore. What would you say to him?
Student #3: I will tell him, no discussion, You have to go to school.

Instructor: Tough. But why does he have to attend school?
Student #3: Because it is good for him to study and get higher degrees.
Instructor: Really, You belief that, really. (*smile at him with strong eye contact*) (*The whole class burst into laughter*)
Student #3: You made your point.
Instructor: Who likes math? Can you explain why?

Instructor Note:
(*There will be few students willing to explain. That explanation can be used to show that fear or unpleasant emotion for mathematics can be conquer by studying and spending more time practicing exercises. Without imposing my idea, student stop, reflect, discuss and learn from each other. The advices from their peer are more valuable and make perfect sense to them. Transition from resistance to math learning to willingness to learn was made by appealing to the student's conscience. Through the dialog, I was able to draw a connection between research act and education act.)*

b) Drama and humour to increase student engagement:

To increase the participation during the class and facilitate active learning, I strongly encourage students to show their work on the blackboard. Sometimes students are hesitant for various reasons, either the student has no idea what to do or does not want to get embarrass for writing wrong answer. In these situations, I would sit with the student to guide him towards appropriate steps then ask to display the correct result on the blackboard. This way the student feels confident and also an important part of the class. Eventually students realize that "not going to the blackboard" is not an option but still I want them to volunteer for this task. I recall a conversation about this during the semester, which went as follows:

Instructor: who will volunteer to do the work?

Instructor Note:
(While asking that question, I was looking at a particular student called Mark, hoping he will agree. He came late and he was not working on the problem. In my opinion he needed a little push. So I decided to supply one).

Instructor: Well class today is the male day. Each male student has to do at least one exercise.
Mark: But professor yesterday was the male day since I did the work.
Instructor: Yes you are absolutely correct but I have decided that today will be the male day also, tomorrow will be the female day. Today is a good day for males.

Instructor Note:
(Immediately four male students got up and started to work on the blackboard. I realize that male vs female competition (drama) is a good idea for the future classroom. Actually the next day I did not hear any objection from the females either.

It should be made crystal-clear that if they need help, I will be there to guide them (a handshake). Once they start, they will finish it with or without my help, no one leaves unfinished task. By the end of the semester this is the class norm, if I have an eye contact with a student, he/she will automatically get up and start working on the blackboard).

c) Class dialogue to creative Learning Community

Emotional support is essential in order to improve learning. Man is a social animal and needs the support, approval of his peers, family, friend and whole community in general. Our students need this type of community especially in a remedial mathematics class, which is supportive and have best interest for that individual. A learning community that have the same fear, concern and problems and therefore it understands and support the student, where students feel safe and open about their fear and have the opportunity to bond with each other. This community of students in a proper learning environment can move towards excellence under the watchful eye of the instructor. The strength of the bond between the teacher and student depends on instructor ability to convey his/her genuine concern for the students and a strong faith on their capabilities. The learning community can be form using mutual respect, common aim, similar interest, and well-being of all individual involved. Once again everyday drama, handshake are an excellent tools towards achieving this goal.

Instructor Note
(Students are working in a group of three and a female student (Sara) walks in late. She seems upset and talking to herself or may be to her friend (June) sitting next to her but very softly. I was unable to hear even though she sat in the first row. Finally one sentence caught my attention "What is the point of all this?" Once again I face two options: address or ignore the student's behaviour. I decided to address it).

Instructor:	What is the matter?
Sara:	remember yesterday I did not come to class because I have to repeat the MRI. Now the doctor is sending me for another test. He is suspecting a tumor. Anyway why should I study, I will be sick soon.
Instructor:	Since the doctor is not sure, so it could be a good new.
Sara:	But he is suspecting something.
June:	Did he say that you may have a tumour?
Sara:	No.
June:	So why worry. Enjoy today.
Instructor:	June is correct. Start worrying, when the doctor gives you a reason.
June:	Everything will be fine, let us do the work. Take out your notebook and I will help you.
Sara:	All right if you say so.

Instructor Note:
(Both started to work on the given problem and I did not hear the concern about the tumour that day. By showing my concern for her wellbeing, I open the dialog and June was able to convince Sara to enter into a supportive learning environment. Everyday real life drama cannot be ignored in an educational setting; we must provide both emotional and intellectual need of the student for their success, especially if student lacks family support structure.)

Reflection and Analysis of Classroom Interactions as a Cornerstone of TR

One of the goal of the TR/NYCity methodology of teaching research is the improvement of learning for both instructor and students. The improvement of learning can be maximized through constant conscience reflection and discussion or interaction. A teacher who quests for excellence in teaching and learning becomes a teacher researcher by analyzing each situation carefully and encourages the students to do the same through class dialogue. Guided Discovery class dialogue enables teacher to learn about each student's level of understanding, so the Teacher can reflect and adjust the lesson to help student to cross ZPD. As the ultimate purpose of a teacher researcher is to minimize the gap between the research act and education act through reflection, analysis and making correction, i.e. the critical alignment from a teacher to a reflective practitioner of mathematics an essential tool in this transition is guided discovery in which the instructor helps students to transition from learning and to excellence by prodding them to create meaning through reflection upon and discussion of mathematics.

Case Study: Foundation for Creative Learning Environment

To influence student affect, to actually witness a transition from acceptance of failure to a motivated student requires hard work often without noticeable payback. That being said there are many success stories and each one carries within it the seeds of hope.

A pilot course in statistics for struggling students. The semester started as usual with enthusiasm to apply my new and improved ideas. One of the classes was a statistics class, a pilot class for the students who had not passed the CUNY exit from remediation exam. After several unsuccessful attempts to pass the examination, the college allowed them to take this pilot with built in tutorial i.e. a supplemental instruction class where the a peer leader is present during the entire 4.5 hours of instruction time and he/she conduct 1.5 additional hour without the instructor where group work is encouraged. Students close to graduation can be placed in this class.

I started with 15 students and after two weeks, two of the students stop coming and another student could not commit to this course because of his demanding job. Unfortunately, the students did not officially withdraw from the class, which is a very common mistake among our students. The remaining 12 students were more or less serious about passing this class and had lots of interesting personalities. Two women with college age children of their own are mother figures, one female had a chip on her shoulder hence was frequently upset with the world. The rest were students with serious math issues but were motivated to pass the class. Mark was one of my student in this class.

Mark's story. My classes are not lecture style, students are expect to participate during the lesson and display their answers on the blackboard. I try to create a learning environment where students are free to ask questions, tell me their story, jokes etc. a very informal classroom. This is a story of a student named "Mark". I knew Mark from my previous class, which was at least 2 years ago. He had repeated two different levels of remedial mathematics courses several times. He is a jolly fellow who never quite learned to be a motivated student. He comes from a broken family and is the first person in his family to attend a college. Failure is a norm and he learned to accept it gladly. He lives in the moment. During the first month of the semester he came late more than 30 minutes every day without exception. Of course he had good excuses but once inside the classroom, he participated and usually gave correct answers.

In his mind, lateness should not be a problem because he did learn something in that short period of time that he attended class. He was happy to be able to solve one or two problems, and completely refused to think about what he had missed in the beginning of the class. As a result he failed the first two tests. Even though I remind him about his lateness every day in a joking way, I realized it was not working and decided to try a different tactics.

After the class, I talked to him about his plan for passing the class. Dialogue to influence student affect- motivation"

Instructor:	Mark what is going on? Is everything okay with you?
Mark:	Yes professor, why do you ask?
Instructor:	"Do you want to pass this class?"
Mark:	Yes of course.
Instructor:	Good, what else can you do to achieve this?
Mark:	Study?
Instructor:	Great but start with coming to the class in time, okay.
Mark:	Okay professor, I will do for you.
Instructor:	Please do it for me. See you to tomorrow.

Instructor Note:
(I believe in creating a pleasant learning environment, where students are willing to listen to me and eventually learn something from me. Obviously this was not enough for Mark. I needed to show him how to be more effective, he was unable to see why it is important to be in time. In his mind, he can do math even if he is not on

time. I have to help him create a learning environment for himself which starts from the beginning of the class time. Mark is a good guy and he want to "do it for me". I decided to use this emotion to help him.

To my disappointment, he did not show up next day but following day he comes only 20 minutes late. I decided to make a big deal out of it. I express my gladness that he decided to come and one of the mother-figure students even congratulated him on being 10 minutes earlier than his usual 30 minutes late. The entire class clapped.

Actually this class was very supportive of each other, not so much in the math topics but by giving encouragement to their fellow classmates. I call all this a class drama but I tolerate it because I have their attention and as soon as the drama ends, learning begins.

I never gave up on Mark, gently and constantly I reminded him from time to time the importance of punctuality and hard work etc. I was happy to see that Mark scored 75 in the third test. He was overjoyed and once again the entire class was happy for him. Actually before the third test, he announced that he is definitely going to pass).

After Class Dialogue of student beginning the transition from acceptance of failure to motivation:

Instructor: Mark, you passed examination, great work.
Mark: Did not I tell you that I will pass?
Instructor: I know, you kept your word.

Instructor Note:
(Now Mark felt confident and this time most of class did better. After receiving their grades there was a discussion about the importance of studying and completing the work. They compare, brag the number of hours spend on studying for this test. I was the silent listener and a happy Instructor. Once again this class drama is very beneficial even though it take few minutes of the precious class time).

Once I went to the class with a plan to cover a heavy topic and could not wait to start the class on time. With only six students I taught my lesson for 10 minutes and then assigned some work for them to complete in their in class. After several minutes I requested a volunteer to display their work on the blackboard. Mark volunteered but while standing in front facing the blackboard, it seems that he is trying to listen to some voice or noise and shook his head as if he is trying to shake a fly off his face. Then he looks back at the class and turned his head again towards the board. This action was repeated at least four times. The students were alarmed and asked him several times what is happening. Meanwhile another student Joe walked in and Mark and Joe burst into laughter. The class demanded the explanation. Joe is always on time except that day, so when Mark texted him to ask his whereabouts, Joe said he is in the class and that confused Mark completely. Anyway the class had a good laugh. I was getting frustrated that this class drama was taking too much of my teaching time. After all this drama I continued the lesson and class agreed to wait additional 15 minutes to order to complete the material.

Instructor Note:
(This was the first time in my entire teaching career that it took almost 30 minutes in order to bring the whole class to a point where all were ready to learn. But once that perfect moment came, I was able to convince them to wait additional 15 minutes to finish my lesson and THEY WERE WILLING. I had their full attention. Patience and flexibility is important. Mark and three other students did work on the blackboard with the help of their classmates. It is not "what you say matters" but "how you say it" and may be "when you say it").

By the end of the second month, students were comfortable trying new things and helping each other with math topics. Before the fourth test, I gave a package of review exercises. The college was closed for five days; therefore I reminded them I will collect it on the day of the fourth test and grade, so show the work even if there are 40 multiple choice questions. Most of the student returned the package with work shown except Mark. He insisted he had all work on separate paper at home and wanted an extension. I did not give him any answer but the next day he came to class with the papers and showed me the proof that he did finish his work I took his package but told him that I would not give him full credit.

Instructor Note:
(My intention was that student will do the work, well and Mark did it, he learned something and he passed the fourth exam with a 70. I was pleased but did not want him to relax and let down his effort).

During the last week, I requested the class to complete the student evaluation for this class. Mark informed me that he had done it already and I am a great professor. I laughed and told him he can prove that I am a great professor if he passes this class with good grade (since his average grade was close to "D" but actually he thought he was failing the class). He looked at the class and said, "Just because I fail does not mean you are a bad professor". I cannot believe how comfortable he was to say that he may fail. He did work hard at the end of the semester. I administer five in-class examinations and one departmental final examination and his final grade was "C". I felt like I managed to reach him finally. The class did well, only one student failed since his job was too demanding and he did not come to the class most of the time.

TR Team comment. Persistence, patience, an interest in students and the ability to constantly push them towards their own excellence are essential components of the approach of Dias revealed in this case study of Mark whose excellence was that he passed a college level math course with a grade of C. The transition from acceptance of failure to motivated student is made difficult in the case of Mark by his self identify as a person who cannot pass mathematics.

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UNIT 3

TOOLS OF TEACHING-RESEARCH

INTRODUCTION

Vrunda Prabhu remarked at the end of Unit 2 that Teaching-Research, just like mathematics is not a spectator sport. Unit 3 develops that view in an effort to encourage colleagues, teachers of mathematics in schools and colleges to pick up the mantle of this unusually creative bisociative approach to teaching and research. Involvement in teaching-research is not particularly difficult especially taking into account again the comment of William J. Harrington, describing his work of a teacher-as-researcher in Laura R. Van Zoest (2006)

> Teachers do informal research in their classrooms all the time. We try a new lesson activity, form of evaluation, seating arrangement, grouping of students, or style of teaching. We assess, reflect, modify, and try again, as we consider the perceived consequences of changes we made.

Unit 3 is seen as the bridge between our daily "informal research" and advanced teaching-research we have been developing and presenting. The first two chapters, Chapter 3.1 on the development of teaching-research questions and Chapter 3.2 on how to approach teaching experiment (TE), emphasize informality while addressing formal concepts of teaching-research question and of the teaching experiment leading to the more formal Chapter 3.3, which addresses issues of assessment.

The focus of these three chapters is on the consistency between the development of the teaching-research question, implementation and conduct of TE, and its assessment. The discussion here tends to be informal and originating primarily in the craft knowledge of the authors. We emphasize the consistency and coherence of the relationship between three components as the expression of TE's integrity. This emphasis is especially important for practice based designs, which do not support itself by prior research (Chapter 1.1, Unit 4). We discuss the design and conduct of TE in the context of planning our next teaching experiment whose aim is to understand the process of facilitation Aha moments in a regular class of mathematics. Among the issues discussed we underline the central for us role of coordination of teaching practice with available research at the nodes of Analysis of the data and Refinements of intervention. It is based on the sense of discernment of similarities and differences.

Chapter 3.3 on Qualitative and Quantitative Assessment breaks with that informality and provides the literature background for the teaching experiment (TE) emphasizing two of its aspects, cognitive and socio-cultural. The qualitative method is followed by a quantitative assessment approach explaining this subtle process of introducing quantitative measures of concept development into classroom investigations. The next Chapters 3.4–3.9 take the reader into a different direction, that is towards three powerful pedagogical approaches, each of which can be, in the hands of teacher-researchers, the generator of Stenhouse TR acts, that is those act which are "at once an educational act and a research act": Teaching-Research interviews, Concept maps and Discovery method of teaching.

Whereas a clinical interview is a dialogue between a researcher and student, during which the researcher tries to understand student's mathematical thinking and to uncover misconceptions, the teaching-research interviews are those clinical interviews that investigate and *deal with* student misconceptions and, as a result can improve pedagogy in the teacher-researcher's class or in other classes.

In simpler words, teaching-research interviews not only investigate the state of understanding or mastery of mathematical procedures, the goal of research but also, having discovered student difficulty with a particular concept during the interview, they correct it or deal with it immediately as the expression of teacher's responsibility. It is the unification of teaching and research in the context of an interview and as such, TR interview is a Stenhouse TR act. Eric Fuchs, the leading author of Chapter 3.4 presents the development of the method and its successes in dealing with particularly challenged students of mathematics in the context of an NSF grant.

Concept maps, the theme of three independent yet closely related Chapters 3.5–3.8 have not been much discussed in Math. Education literature although their use in science and other disciplines has been extensive. Concept maps are an excellent artefacts addressing primarily conceptual understanding of mathematics. It can generate very precise Stenhouse TR acts by serving as the developmental didactic tool on one hand, and the assessment of conceptual development – on the other. Each section of the chapter throws a different didactic light upon that duality. For Prabhu Chapter 3.6, concept maps are the guides in her investigations of the structure of student ZPD while developing their schema of thinking, for Haiyue Chapter 3.7 they are the assessment tools of student conceptual development. Her survey of student appreciation of the methodology shows a significant support for concept maps as simultaneous teaching and research methodology. Roberto Catanuto Chapter 3.8 utilizes them to create the bisociative framework out of the topic of the course and of student interests in search of the "hidden analogy" between the two, which will make the subject matter closer to student heart. He finds it in three iterations.

Chapter 3.9 on the Discovery method of teaching closes Unit 3 Tools of TR. It is our main instrument with the help of which a teacher researcher can investigate authentic mathematical thinking of a student. At the same time, it (and its variants like PBL) is the closest approximation to the condition of the "untutored learning" pointed by Koestler as the environment where bisociation and its ultimate expression, Aha moment can usually take place (Chapter 1.1). Chapter 3.9 explores the background of the Discovery method, its cognitive and socio-cultural aspects and places it at the centre of Math wars and of the division between traditional and reform pedagogies. Here we see the Discovery method in terms independent of the division but in terms of the environment facilitating creativity of students and of teachers.

VRUNDA PRABHU, WILLIAM BAKER
AND BRONISLAW CZARNOCHA

3.1. HOW TO ARRIVE AT A TEACHING-RESEARCH QUESTION?

INTRODUCTION

This chapter presents three different routes to the formulation of the teaching-research (TR) questions practiced in the community of teacher-researchers of the Bronx. The differences and similarities among them are interesting. On the one hand, their natural development in the context of improving the general quality of teaching is described by Vrunda Prabhu, followed by William Baker's discussion that carries a higher level of specification in the context of a large scale teaching experiment involving many students from different sections of the course in both colleges. On the other hand, the teaching-research questions formulated by Bronislaw Czarnocha, based on a colleague's request, that address logical quantifiers, are presented in their outmost concreteness and particularity. It is interesting to note the different styles of teaching-research employed by each contributor. While Prabhu's approach focuses on the depth of student involvement in the enquiry ("*The enquiry will go as deep as the student will allow it…*") leading to the series of iteration cycles, each creating a different motivational environment such as *A Handshake*, *Didactic Contract*, *Students as Partners in Learning* and *Mathematical Creativity*,[1] Baker seems to prefer approaching the formulation of teaching-research questions starting with the methodology ("*I may have a general goal or area of interest in mind, with, perhaps, a clearly defined methodology of research, that I intend to narrow down to a yet undefined investigational focus*"). On the other hand, Czarnocha favours the formulation of a teaching approach based on the preliminary analysis of classroom observations and the questions raised by this analysis. The three contributions are communicated in a relatively informal narrative style that better illuminates the train of thought of each contributing teacher-researcher.

THE PROCESS OF ARRIVING AT THE TEACHING-RESEARCH QUESTIONS

Vrunda Prabhu

In hindsight, I recognize that as an instructor of mathematics (as a graduate assistant, or a professor), I have always had teaching-research questions; however, it is only in the process of becoming a teacher-researcher that I understood the meaning of

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 149–161.

those questions and their significance to my teaching practices and to my students' learning processes. "What are teaching-research questions?", "What is a teacher-researcher?" and "What is teaching-research?" are the key questions that I want to address.

The teaching-research questions start from quite general questions arising obviously in the teaching, or in students' exams, or other written work. These general questions get refined further and further until one arrives at the essence of the teaching-research questions. At the very essence, the teacher-researcher is in complete synchronized togetherness with the thinking of the students with whom the mathematical obstacle resides. In the process of refining the teaching-research questions from their original crude and general state, the teacher-researcher is deeply involved in the investigation of student thinking. The classroom atmosphere and the nature of the discourse at this stage is highly infused with and driven by the teacher-researcher's questioning mode. The questions within the teacher-researcher's mind include "What is it that bothers the students?", "What is not clear to them?", "What did I say that sparked that particular comment from the student?" and "When I say such and such, how exactly are the students responding?" Any response is quickly seized and acted upon to discover its origin, and, if the student is able to explain the origin, the nature of the obstacle becomes less opaque to the teacher-researcher. The teacher-researcher develops exercises, activities and conversations designed to target the specific issues that he or she recognizes as the trouble areas, and then investigates students' responses to the questioning inherent in the appropriately crafted activity, exercise or conversation. The dialog that has started between the teacher-researcher and the class, from the formation of the TR question in its first manifestation through its various stages of refinement, is a process whereby all participants, the teacher-researcher and the students, are actively learning about each other's thinking. The teacher-researcher has the active responsibility of drawing in all participants without any loss of stature among all students. The goal is to create a supporting atmosphere in the class resembling that of a graduate student seminar where both faculty and students are free to express their views in a friendly, critical, and nurturing environment, where ideas are free to be expressed without fear.

Through the above process, the teacher-researcher becomes much more aware of the various required cognitive connections that are not being made by the students and the causes for the absence of these connections. Is it a deep obstacle stemming from a misunderstanding of basic mathematics topic? Is it an issue of absence of understanding of the language of the question? Is it a chronic misinterpretation of words and phrases? Is it the habit of not expressing clearly to oneself the meaning of the mathematics presented? The teaching-research dialogue leaves very little uncovered on either the part of the teacher-researcher herself/himself or that of the participating apprentice teacher-researchers since the classroom of the teacher-researcher is gradually being transformed into a laboratory where every single person is in the process of enquiry. The enquiry will go as deep as the student will allow it, however, no student will be a silent participant, or a non-thinking member of the experiment, anymore. Each one will be confronted with the set of questions that are, in many ways, particularly applicable to their own situation and these sets of questions will have non-empty intersections with those of the others in the teaching-research team.

The teaching-research question is the beginning of the teaching experiment and the introduction to the teaching-research methodology for a person new to the field. "Learning" about teaching-research from a book or an article without engaging in the actual practice is analogous to being a "spectator in a sport". Just as we remind our students that mathematics is not a spectator sport, we must remind ourselves that teaching-research is not a spectator sport. It is a sport in which one learns strictly by doing. The teaching-research question in its most basic form is where the trip begins. Then, as we continually refine our question based on the knowledge we gain from our students, utilize what is available in the professional literature that is applicable to our situation, analyse our students' work, design and redesign our own curricular interventions, we are actively conducting the next cycle of the teaching-research methodology.

A PERSONAL NARRATIVE OF TEACHING RESEARCH FORMULATIONS

William Baker

For me, the idea of teaching-research grows through an organic relationship between teaching and educational research. By this, I mean that I rarely begin an investigation or a teaching-research cycle with a clearly defined research goal in mind. Instead, I may have a general goal or area of interest in mind, with, perhaps, a clearly defined methodology of research, that I intend to narrow down to as yet undefined investigational focus. For example, I may note a classroom interaction and how a student's comment goes against the grain of all that I am trying to accomplish, and, yet, I later ponder whether the student's spontaneous comment reveals some grain of truth that needs to be addressed in order to better reach and engage the students. I have noted how an entire class may have a personality that appreciates some aspect of a methodology and rejects or fights another, and, hence, I have had to alter my methodology of instruction midstream. Finally, reviewers' comments, especially negative ones, have inspired me to synthesize different methods of educational research and broaden my perspective that, in turn, leads to a heightened awareness of my educational practice.

The first educational teaching experiment I was involved in focused on establishing a learning community with English as a second Language (ESL) students who needed Elementary Algebra skills. At a bilingual English-Spanish institution, whose stated mission involves assisting Spanish speaking immigrants to obtain an Associate's Degree, it was noticed that many students were admitted to the college requiring an intermediate level of ESL and elementary algebra instruction. Thus, these courses were linked together by the fact that the same students were taking both courses, and, naturally, the instructors cooperated with each other in lesson and curriculum planning. The description of these teaching experiments together with their analyses can be found in Chapters 4.5 and 5.1.

Here, I am presenting the development of my teaching-research question concerned with student development and mastery of the concept of fractions. This research question grew organically out of my interest in the relationship between the use of language, especially written language, in the classroom, and mathematics proficiency that itself was a product of an earlier work. The data obtained from this teaching-research cycle showed that, for the partial tests on whole numbers, decimals, proportion and percent, the two independent variables, use of language and mathematics proficiency, were both significant and, thus, contributed significantly in predicting student performance on application problems during the comprehensive final exam. On the other hand, the written thought on the partial exam for fractions did not add significantly to understanding student performance.

The question that arose was, "Why are fractions different?" Why is it that the ability to abstract, in written form, the relevant strategy and to outline steps students need to solve a problem does not correlate with performance when the problems have a fraction while such a correlation was clearly evident for problems involving decimals, whole numbers and other non-fractional components. The explanation of Sfard and Linchevski (1994) about how understanding fractions represents a transition from operational to structural thought provides a good foundation for my observation of students' difficulties with such problems (Chapter 4.1). These authors focus on the historical transition from arithmetic to algebra, and concentrate on the operation associated with fractions, or the initial view of fractions, that is, the quotient of whole numbers, and, more generally, integers, as the "primary process" (p. 97). The structural understanding of fractions comes about through processes different and separate from those of whole numbers. As an instructor of adult students who require a transition through this cognitive divide on the way to college level mathematics, I am witness to frequent expressions of dislike, fear and dread of working with fractions resulting in the inability to solve one or two step application problems that contain fractions. This very common and familiar observation suggests that the structural complexity of the fraction concept overloads students' intuitive cognitive processes. For example, most students readily solve a rate problem such as:

Juan drove 60 miles per hour for 4 hours, how far did he drive?

Their intuition while working with a primary process involving whole numbers is, in the words of Vygotsky, "spontaneous," in the sense that it requires little conscious reflection. However, the story is quite different when students are presented with a problem like:

Juan drove 60 miles per hour (mph) for $4\frac{2}{5}$ *hours, how far did he drive?*

Given this structurally identical problem, students who get this problem incorrect frequently convert the mixed number to an improper fraction and then give up, or guess the appropriate operation. Their focus is on applying the rules ("secondary processes") relating to the fraction concept and not on the structure of the problem. Their intuitive understanding of the rate problem is temporarily misplaced. Students who get this problem correct often convert the fraction into a decimal. Let us now consider this problem:

> *Tanisha has to drive 6 miles to school. If she had already driven* $4\frac{5}{9}$ *miles how much farther is the school?*

Given a problem like the one above, these same students will again convert the mixed number into an improper fraction, and then, once again, guess the operation, usually choosing division or multiplication because these operations involve less structure than subtraction. Thus, when dealing with fractions, especially those that cannot be readily converted to a decimal, students end up choosing an operation for solving the problem depending upon which of the fraction operations is easiest as opposed to their intuitive understanding of the problem structure.

The next group of teaching-research cycles focused on statistical analyses of student solutions of exercises with fraction content in pre-algebra towards a goal of reflecting upon an existing educational model describing how students learn fractions based upon the work of Kieren (1976). In this model, the primary understanding of the fraction concept begins with partitioning, or the part-whole sub-construct, and this sub-construct is built upon to develop an understanding of fractions as ratios, operators, quotients and objects that have a placement on the number line addressing the concept of "measure."

In the first cycle of this teaching-research experiment we followed the work of Charalambous and Pitta-Pantazi (2007) who employed statistical analysis of problem sets designed to evaluate student proficiency with the sub-constructs of Kieren's model and the corresponding procedures that were added to extend this model by Behr, Lesh, Post and Silver (1983). The work of Charalambous and Pitta-Pantazi (2007) was introduced into work of the the teaching-research team in the context of the Community College Collaborative Grant (C^3IRG/CUNY) , "Fraction Grid and Fraction Domino: Investigating Effectiveness at Community Colleges of the Bronx" awarded to V. Prabhu, B. Czarnocha and James Watson in 2007. Charalambous and Pitta-Pantazi used their quantitative analysis to verify Kieren's model as well as the extensions described by Behr et al. Our work, on the other hand, verified Kieren's underlying hypothesis that the part-whole sub-construct is the foundation for the other fraction sub-constructs; it demonstrated a secondary layering of these sub-constructs. Thus, the foundational knowledge of the part-whole model was the most readily understood conceptual representation of fractions used by students to understand the other representations of fractions. Directly below this sub-construct were the ratio and operator sub-constructs. Finally, underneath the ratio and operator interpretations lie the most difficult sub-constructs of quotient and measure, the meanings of which are derived from all of the above mentioned constructs. These subtle details in our understanding of student schema were published in Baker et al. (2009) and in Baker and Czarnocha (2013).

In the second cycle we used quantitative data from the problem set of Charalambous and Pitta-Pantazi (2007) to study and interpret the Behr et al. (1983) extension of Kieren's fraction model through the lens of the procedural and conceptual knowledge divide. These results indicate that student learning builds upon previous knowledge of conceptual or procedural knowledge to learn more difficult material. In particular, foundational knowledge for learning new concepts and procedures is not restricted to conceptual knowledge and, thus, it appears that students use whatever knowledge has already been obtained when trying to solve problems containing new information. This quantitative analysis by, Baker et al. (2012) suggests that both of the two educational models outlined by Haapasalo and Kadijevich (2000) are present in student learning. The first of these views foundational conceptual knowledge as the pathway to procedural knowledge (Educational Approach), and the second, converse model, presents procedural actions as the building blocks that are reflected upon and internalized to form new conceptual knowledge (Developmental Approach). Both of these converse models have their place in student learning, depending upon the relationship between the corresponding procedural and conceptual knowledge.

At about this time we began work on a grant for problem-solving and my interest shifted to (i) how can students use their knowledge of fractions to assist in proportional reasoning, that is, problem-solving with proportions, and (ii) Vrunda Prabhu's strong support of Discovery Learning in the classroom environment founded upon Koestler's Theory of Creativity, and her belief that students' creativity should be supported to engage them in the learning process. This path shifted my attention away from the sequence in which curricula should be presented toward reflections upon my classroom learning environment. Although the concept that instructors should teach less and allow students to work and discover in the classroom was not entirely in my comfort zone, I realized that the traditional method of modelling correct problem-solving behaviour does not translate into students with good problem-solving skills. In fact, I had become increasingly convinced that only students assisted by such a methodology are already the best students. In contrast, the students at the lower, or more basic, portion of the spectrum often demonstrate a passivity to learning mathematics, and strongly prefer to simply watch the instructor or some expert tutor show them correct solutions. On the one hand, these students need much structure when left on their own with a computer software program. They can frequently be observed surfing the net behind the instructor's back; when asked to work in groups they are observed having intense social debates about life's many deep and sophisticated problems, and, yet, mathematics sadly was not on their agenda. On the other hand, they appear to have learned to be passive in mathematics courses and are reluctant to take up the pen and write notes; when given a problem to solve they may spend the entire time slowly rewriting the problem from the board rather than engage in problem-solving. Therefore, modelling and remodelling of correct problem-solving behaviour by the instructor can be seen as allowing these students to disengage from the cognitive thinking required for active participation in mathematics.

In one teaching-research cycle within the problem-solving grant, we focused on student intuition, especially related to Kieren's fraction sub-constructs, and how such intuition led to proportional reasoning skills (Doyle et al., 2016). The methodology of this research was a hybrid of both quantitative and qualitative reasoning designed to trace a statistical path for a hypothetical student learning trajectory towards proportional reasoning, and then observe and describe student reasoning within that trajectory via qualitative analyses. In this work, teaching-research that employs an effort towards the use of Koestler's Triptych with humour, guided discovery and a focus on student participation and creativity in the classroom dialogue is investigated. In these dialogues, *bisociation* is considered as a mechanism through which students make analogies, streamline and coordinate processes, promoting the development of the underlying concepts that support the associated analogy or generalization.

HOW DOES AN IDEA FOR A TEACHING-RESEARCH QUESTION ARISE?

Bronislaw Czarnocha

A colleague of mine from another community college at CUNY had sent a fascinating student response to the homework assignment in a freshman calculus course to try in my classroom. My colleague wanted, out of professional curiosity, to check how students in his class of calculus would relate to a simple logical argument. His assignment was:

Are the following arguments valid? Explain.

(a) All raining days are cloudy.
Today is not cloudy.
Today is not raining.

(b) All banana trees have green leaves.
That plant has green leaves.
That plant is a banana tree.

(c) Some students go to the beach for spring break.
I am a student.
I go to the beach for spring break.

A fascinating response of a student, call him or her S_1, is shown below this paragraph. It is fascinating because the student did not notice the internal structure of the argument; instead he or she had assessed the validity of each particular statement within each argument. Moreover, the validation discussed the empirical, everyday meaning of statements. Thus, in the first line, the student does not see "All raining days are cloudy" as part of the argument composed of three sentences but he or she views it as an individual statement whose truth has to be checked, opposed to being taken for granted as the assumption of the total argument. A similar approach is observed for all components for each of the three arguments.

My immediate question, from the teacher's point of view, was, "How can I guide that student toward the correct view of the total argument, and to assessing its validity from its internal structure?" Moreover, how am I going to achieve this in a calculus course that, by the standard design, doesn't address logical issues as part of instruction?

As a researcher I had several additional questions: Was this an isolated incident, or does it represent a common misconception? What would other student responses be, and do these responses reveal some pattern that could help me, as a teacher, to find the proper route of instruction for this particular student? Is there anything in the literature that would help me understand the nature of this initial response?

I assigned this problem using the same three arguments in my calculus class and I received yet another response of the same type. This responses, however, came from quite a good student, call him or her S_2. This suggested that I might be witnessing a more general phenomenon than simply an idiosyncratic error of a particular student.

(a) All raining days are cloudy.

- S_1: *Yes, the statement is valid; all raining days are cloudy. If it is raining there should be clouds, but that doesn't mean every cloudy days are raining.*
- S_2: *No, because it can be raining when it's sunny.*

(b) Today is not cloudy.

- S_1: *We often have no cloudy days in summer. But sometimes the weather is cloudy or raining. Today is cloudy, so statement is invalid.*
- S_2: *Yes, because today might be sunny.*

(c) Today is not raining.

- S_1: *Today is cloudy but not raining. Not every cloudy weather rains. So today is not raining. The argument is right.*
- S_2: Yes, because today might be sunny.

(a) All banana trees have green leaves.

- S_1: *Yes, this argument is true. All banana trees have green leaves. Bananas grow in places where it is summer. All trees have green leaves in summer, so banana tree has green leaves either.*
- S_2: no, some might have other leaves.

(b) That plant has green leaves.

- S_1: *Not every plant has green leaves. Our nature has colourful plants. So this statement is invalid.*
- S_2: *Yes, that statement can be true.*

(c) That plant is a banana tree.

- S_1: *We cannot say that every plant is a banana tree .We can find thousands of plants in the world. This statement is not true.*
- S_2: *The plant can be a banana tree.*

(a) Some students go to the beach in spring break.

- S_1: *Yes, this argument can be valid for some students. Spring break is kind of the beginning of summer. So some students can go to beach and enjoy their spring break.*
- S_2: *This is valid because it's possible.*

(b) I am a student.

- S_1: *I study Computer Science at College so I'm a student.*

(c) I go to the beach in spring break.

- S_1: *I like to be with my parent in vacations. Spring break is a kind of vacations for students who study college. I don't go to beach in spring break. But that doesn't mean that "I never go to beach."*

The correct approach and solution was provided by another student in my class, $\mathbf{S_3}$:

(a) All raining days are cloudy.
Today is not cloudy.
Today is not raining.

- S_3: *This statement is valid. If ALL rainy days are cloudy, then it can't be raining, if it is not cloudy.*

(b) All banana trees have green leaves.
That plant has green leaves.
That plant is a banana tree.

- S_3: *This statement is not valid. All banana trees have green leaves, but not all plants with green trees are banana trees. So a plant with green leaves can be other than banana tree.*

(c) Some students go to the beach for spring break.
I am a student.
I go to the beach for spring break.

- S_3: *This statement is not valid. Some students go to the beach in Spring break, not all students go to the beach in Spring break.*

I begin to notice what is happening as I am learning from those of my students who responded correctly. They clearly understand the distinction between "*all*" and "*some*." As student S_3 says, "If all raining days are cloudy, then it can't be raining if it is not cloudy." In other words, if a day is not cloudy then it can't be raining. This is the strength of the assertion that "All raining days are cloudy." Hence, if "all raining days are cloudy," the case mentioned by student S_2, "it can rain when it's sunny," cannot exist.

In the second argument, S_3 notices that even though we describe "*all*" banana trees, "all banana trees" are only "*some*" trees amongst all the trees with green leaves. The third argument is also resolved by understanding the difference and relationship between "*some*" and "*all*".

The observations above are very useful and are the beginning of the creation of an interesting teaching-research question. "*All*" and "*some*" are logical quantifiers. It is known that students have problems understanding their meaning, those statements that include them and the associated operations. I noticed this problem on another occasion: when I was looking at my students' ability to properly negate simple quantified statements. In fact, students appeared to have much more trouble with negating statements with the general quantifier "*all*" than with the existential one "*some*".

Now, the questions from the teacher in me to the researcher in me are "How do I instruct students S_1 and S_2 so that they can understand the arguments in the same way as student S_3? Can it be done? How does the understanding of an argument develop?" If I can answer those questions, then I can lead any student along the best path for learning the meaning and use of logical quantifiers.

The researcher in me responds, "If there is a path of learning that starts where the students S_1 and S_2 are, then it's quite probable that there should be mixed responses of students containing both understanding of the abstract internal relationships within the argument, as well as particular localized meaning of its components."

There was one response of such a mixed nature amongst the responses of my colleague's students:

(a) All raining days are cloudy.
Today is not cloudy.
Today is not raining.

- S_4: *The argument is valid. Nearly almost every day is cloudy, and if it is not cloudy, usually there is no rain. Even though there are some exceptions, we can conclude that today is not raining because it's not clouded.*

It is apparent that this student is still relying on a comparison with a real everyday weather situation. However, at the same, the student is arriving at the significance of the "*all*" quantifier with the help of simple statistical reasoning. In my opinion, the most important observation here is that S_4 sees the internal structure of the argument. It's possible that seeing this structure allows him or her to understand the importance and the power of the "All raining days are cloudy" phrase.

I believe that my instructional path has to have two steps to serve two consecutive student needs. First, a student has to arrive at an understanding of the whole structure of the argument, and, second, the student has to clearly understand the relationships within this structure to be able to draw the correct conclusion. This will be my instructional hypothesis.

Before I proceed, I need to consider whether it's possible to grasp the totality of the internal structure and, at the same time, not to understand the particular relationships between its components? Maybe, the grasp of the totality automatically forces the person toward the correct understanding.

The response of student S_5 below indicates that simply grasping the existence of the argument's structure does not lead to the desired detailed understanding of that structure:

(a) All raining days are cloudy.
Today is not cloudy.
Today is not raining.

- S_5: *These arguments are not valid because it don't have the same meaning.*

(b) All banana trees have green leaves.
That plant has green leaves.
That plant is a banana tree.

- S_5: *These arguments are valid because they are all related.*

(c) Some students go to the beach for spring break.
I am a student.
I go to the beach for spring break.

- S_5: *These arguments are valid because one argument leads to another.*

This supports my original hypothesis of the need for a two-step instruction; the first step is aimed at assisting the student in reaching the grasp of the totality of the argument, and, the second step serves to clarify the argument's inner inter-component relationships. Let us now go back to my research laboratory – my classroom.

The teaching experiment suggested by these reflections has not been realized as yet due to the curricular organization of the department's course offerings. However, several years after these notes were made, while teaching an *Introduction to College Mathematics* course, the first college-level mathematics course for liberal arts students, the difficulties with the quantifiers "*all*" and "*some*" were again revealed during the introduction to the *Logic* component of the course. In this instance the students were not only expected to understand the roles of the quantifiers but also the associated negation operation. The difficulties were further exacerbated by some language barriers with which many Hostos bilingual students, whose primary speaking language is not English, are faced with. To address the issues that Hostos students were having with logical quantifiers, the investigation titled *Negation of Universal and Existential Quantifiers Revisited* (Ye & Czarnocha, 2012) was conceived. The associated teaching-research question was composed, as always, of two components:

1. What are the students' difficulties in understanding negation of quantified propositions in the bilingual context?
2. What are the possible routes of improvement of students' understanding and mastery of the negation of quantified propositions?

NOTE

[1] Each of these motivational environments is discussed in more detail in Unit 2.

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3.2. HOW TO APPROACH A TEACHING EXPERIMENT?

TEACHING EXPERIMENT AND TR CYCLE

The general principles of TR Design, and therefore of classroom teaching experiments conducted by teacher-researchers are discussed in the next chapter as well as in Unit 4. The aim of this chapter is to discuss the type of questions and decisions that a teacher-researcher must take in the context of daily teaching and classroom research, the proverbial "nuts and bolts" issues of the design and conduct of Teaching Experment (TE).

The Teaching Experiment is imbedded into the TR Cycle. We start with diagnosis of our students knowledge and/or attitudes as the initial benchmark for our TE. Of course, every teacher knows how to construct a diagnostic test, however here we refine the standard methods a bit, so that the questions we ask can give as precise information as possible about that particular concept whose learning improvement we want to accomplish. Simlarly, in the case we are interested in affective changes, we design a survey or use that scale assessment which best corresponds to the attitudes we would like impact. Whereas designing a particular TE, we have good idea what aspects of learning we want to address in general, the diagnostic test gives us more precise information about the particular cohort of students, both as a cohort and as different individuals. Next we need to design an intervention, which we hope will efectuate learning improvement we wish to facilitate. For example, as a result of reflection while writing this book, our TR team of the Bronx, the community of teacher-researchers came to the conclusion that the next neccessary step in the design of Creative Learning Environment (CLE) is to investigate the process of facilitation of Aha moments in the classroom with the help of Koestler's theory and their possible impact upon student cognitive and affective dimensions. Diagnosis of our students' motivation and attitudes to mathematics can be obtained with the help of Motivated Strategies Learning Questionnaire (MSLQ) Prabhu used in her experiment (Chapter 2.4). The assessment of students' cognitive growth should cast a wide net because it is hard to predict before the experiment where and how the impact of the acts of creation will be registered in students' mathematical knowledge. Whatever examples we will design, we will keep them for the refinements of next iterations. They become our artefacts with the help of which we will be able to generalize our work to mathematics classrooms with other student cohorts of similar characteristics (Chapter 2.1).

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 163–170.

The question of design of intervention as well as its implementation at the next cycle is interesting and challenging. Using TR experiments to support CLE one can not cause the acts of creation that is Aha moments, one can only facilitate their occurrence; maybe one can increase their frequency. A very special feature of the Koetler's definition "*a spontaneous leap of insight which connects previously unconnected matrices of experience*" (p. 45) is that it points to *previously unconnected matrices of experience* as forming the cognitive structure of the creative environment, which we call a *bisociative framework*. We can use that hint in choosing the parts of the curriculum where such bisociative framework is already given creating higher possibility for the Aha moment. The second hint provided by that theory for us, teachers, is the word analogy" or better, "a hidden analogy" which is grasped at the Aha moment. My students need to be good at grasping analogies and that need becomes my first guiding idea for the type of mathematical examples I need to design to develop this skill. A good example are Prabhu's triptych exercises in Chapter 2.4. The third direction pointed by the discussion of bisociativity is problem solving approach because, as Baker argues in Chapter 4.1, the two unconnected matrices for a problem solver is (1) the matrix of solver's experience till encountering the problem, and (2) the matrix of the encountered problem. These two create the bisociative framework around problem solving often ripe with possibilities for Aha moments.

There is also work to do in the affective dimension. Our expectation, based on Prabhu preliminary results in Chapter 2.4 is that increased number of Aha moments in the classroom will impact positively upon students negative habits in parallel with Kostler's assertion: *The act of creation is the act of liberation. The defeat of habit by originality* (p. 96). We are counting on the positive impact of Aha moments as reported in the literature (Liljedahl, 2003) to defeat habits of the type "I am not good in mathematics "or "thinking about mathematics tires me "in other words we want our students to be able to welcome the benefit of their own creativity. That means that we have to create learning environment which also welcomes mathematical creativity. And that leads directly to the role of classroom discourse, especially in the context of the Discovery method during which we would like not only to be able to facilitate the discovery but also become more clear about the process of thinking that led to it. We see here how the cognitive/affective duality of the Aha moment (Chapter 1.1) forces the transformation of pedagogy along both dimensions. The central method in the search will be facilitation, that is creation of such a learning environment around our students that can promote their search for hidden analogies. Facilitation is the process similar to scaffolding, except that it has to be more precise so as to never limit too much student freedom of their authentic grasps, aha moments, but to focus their attention on where the search for illumination by the hidden analogy might be more effective (Chapter 4.1). We'll be using here Discovery method as the approximation to Koestler's condition of "untutored learning "when according to him, individual, subjective bisociations can take place.

Such facilitating Socrates dialogues of which many one can encounter within our TR reports become the central pedagogical tool in the classroom, and, in our experience they have been the best millieu when Aha moments have taken place in our classrooms. For a teacher-researcher, Socratic dialogues can become Stenhouse TR acts because, on one hand they, as educational acts, enable student to make progress in his/her understanding, and on the other, they can show the structure of thinking that led to that progress, as research acts. Consequently, we will have two techniques of facilitation, teaching sequences of examples and TR Socratic dialogues within the discourse of the classroom. However, we also know, we will feel the pressure of time because, of course, facilitating dialogues takes more class time that a lecture and our syllabi are already overcrowded and overstretched; one has to be very careful with the full lesson plan not to get lost in investigations of student thinking and learning without "covering" the material. Clearly one can't spend too much of the course's syllabus time so one have to be very precise in organization of teaching.One either will have to limit facilitated searches to a one class/week cycles or to focus on certain areas of the course only.

However we can see already the thread of the relationship that starts at the formulation of the TR question and through design and implementation of TE takes one straight into assessment creating at the same a conceptual consistency and coherence of the design,[1] the theme of the next section.

Internal Consistency and Coherence of TE

The central quality of the design, which assures the internal consistency of the TE is the relationship between the TR question, the nature of new intervention and methods of assessment. This relationship provides reliable answers to the research questions asked. In simple words this means that (1) what you implement during the intervention has the opportunity to answer the question and that (2) the method of assessment can assert to what degree the data, quantitative or qualitative, answer that question. Both consistency and coherence are especially important in TR practice based designs when there is no theory nor previously done research to guide the teacher-researcher.

That means, that while one designs a teaching experiment, one must decide

1. what type of intervention may respond to the TR question?
2. where could there possibly be an impact of your experimental intervention in student thinking, and then
3. one needs to design an assessment tool that would allow to measure that impact.

All of this has to be done in the context of designing the intervention itself, what kind of pedagogy will you institute, what activities and/or new assignments will you use and what impact will you be aiming for. That means that the teaching-research question and the related methodology during the intervention as well as its assessment have to be designed and developed together. A close example of the process demonstrating its step-by-step design is the LELT in Chapter 4.8.

The conjectural nature of the TR/NYCity frameworks as Design Research (Chapter 1.1, Intro Unit 4) allows for the process of mutual accommodation of these three TE components within several TR cycles. That suggests that final fit between them can be obtained sometimes after their completion, not in advance (Chapters 5.1 and 4.9). We discuss this issue below more extensively.

Priori and Posteriori Design of TE

One can approach the TE design from two ends of its duration: planning it from the beginning (priori method) or formulating it out of the revealed patterns of practice after the full teaching experiment was conducted (posteriori method). Mixed method have also been realized and observed. The priori and posteriori classification is closely related to the division of teaching experiments into exploratory and affirmative. In the exploratory design, we don't know precisely what will be and where to look for the desired changes, or more precisely, what aspect of learning will reach the improvement we hope for. Such TE designs are usually practice based, here ESL/Algebra TE of Chapter 5.1 or Calculus modelling TE of Chapter 4.9. Here the full design of the teaching experiment together with the needed precision of the TR questions reveals itself after TE is completed. On the other hand, the affirmative designs are designed to check (confirm or reject) a precise and formulated a priori hypothesis.

A clear design of the priori method in this volume is Chapter 4.10 where the strong theory-based design of integrating learning of algebra with learning of different forms of writing has been fully developed, including teaching sequences, in advance of the teaching experiment. On the other hand, Chapter 4.9 describes posteriori design for the concept of Riemann integration. In this type of TE, the teacher-researcher moves slowly, one step at a time while transforming the instruction at several critical dimensions. At each different assignment of that course to the teacher, the course becomes the subsequent iteration of that particular TE. The designs of activities as well as of pedagogy are transformed at each Analysis and Redesign nodes of the TR cycle. They are redesigned so as to better facilitate educational aim of the teacher. One of such moments occurred in Chapter 4.9 when the teacher gave up on instruction through traditional method of limits of upper and lower sums of rectangles in favour of approximating areas by sequences of upper rectangles and of trapezoids. Chapter 5.1, on the other hand describes a mixed approach, where the pedagogy of design in creating large educational spaces within classroom for the interaction of English language and algebra was not very clear beyond the close interaction of the syllabi and language. As a result, the instructors, teacher-researchers they didn't know where the possible observable impact of that interaction can be found. Consequently, the teaching-research question of that TE was incompletely formulated, and it acquired its full meaning only at the completion of the experiment and collected data. It resulted in a very interesting hypothesis of existence of the relative Zone of Proximal Development (rZPD) between ESL and algebra. Here the data must have been collected along as many as possible tracks (a wide net) of assessment to catch possible variations, which were actually caught in a long term English essay assignments of the experimental cohort. Such a TR "fishing expedition" is very exciting and involves very creative teaching but has to be conducted with the utmost attention to the impact upon students, and teacher-researcher's craft knowledge is the primary source for the ethical decisions involved in such posteriori designs.

Incorporation of Teaching Experiments (TE) or Teaching-Research Investigations (TRI) into Classroom Curriculum

One of the central questions for the TE design is how to incorporate the design into the standard curriculum of the classroom, the process of "covering" required material and, often the standard type of exams imposed either by the central authorities or a policy of the department. The process of utilizing standardized final exam in conjunction with teacher-designed semeter tests and homework for the quantitative data has been discussed by Baker in Chapter 3.3. As TR/NYCity embraces both the issue of cognitive as well as affective development, the TE design may involve the change in pedagogy from one the teacher is accustomed to, to the new approach dictated by affective requirements of that design. Earlier we discussed the example of how the cognitive/affective duality dictates the structure of the creative learning environment in which creativity of Aha moments is well received.

Thus for example, Unit 2, Creative Learning Environment describes a series of teaching experiments, whose aim was to investigate formulation of a new didactic contract focused on students' taking responsibility for their learning through their change of attitude to mathematics. Consequently, Unit 2 concentrated on the transformation of affective dimension of instruction based on facilitation of student creativity and it involved a corresponding transformation of teacher's pedagogy througout the whole course. The transformation of the pedagogy of the course was also involved in the English as a Second Language (ESL)/Mathematics teaching experiment of Chapter 5.1, however it concerned primarily the cognitive dimension of learning. The basic design of that teaching experiment required incorporation of some methods of teaching English language into mathematics classroom simultaneously with incorporation of some mathematics concepts into ESL classroom, both to enable intensive interaction between the two subjects. It also required the transformation of instruction by both participating instructors throughout the whole semester long course.

More often than not, however, classroom teaching research experiments involve certain limited components of classroom syllabus. For example, the teaching experiment investigating understanding of single quantification by students of the first college level course Introduction to College Mathematics for Liberal Arts majors (Czarnocha & Ye, 2012) – the realization of the research question formulated in Chapter 3.1 involved just one theme of the course. The change of instruction was implemented only in 3–5 of its classes. Since, however, the topic has been scheduled to be taught only during 2 class meetings, the conduct of this TE required reorganization of the larger part of the syllabus in order to free additional class time, without the negative impact on learning other topics of the course. The pressure of time forced upon the teacher-researcher by the need "to cover" the syllabus' material has been mentioned above.

Teaching and Collection of Data

Teaching during the conduct of TE in your classroom is slightly different than in a regular classroom The difference is in the increased focus both on your own didactic choices and actions as well as on student responses – ultimately you would like your hypothesis or conjecture to be fulfilled, and to the degree to which the results depend on teacher's performance, one certainly wants to impact it by increased awareness of events in the classroom. Increased awareness is also necessary as it gives us the opportunity to react if the classroom discourse and learning deviate significantly from one's aims. Such moments are very valuable in creating a bisociative framework in the classroom.

During the run of the TR Cycle the data are usually hidden in our classroom assignments, tests, quizzes and specially designed assignments to ascertain students' understanding and mastery of concepts addressed by our TE. The reliance solely on standard tests is possible but not very useful in terms of systematic improvement of learning. Ultimately, we are interested in the holistic improvement of learning of what involves both conceptual understanding and procedural mastery. However, in majority of the cases, student learning outcomes and related final exams in mathematics assess primarily procedural knowledge and to tease out the conceptual change from them is difficult, but not impossible. Chapter 4.10 presents the method designed by Baker who employed quantitative analysis, using the grades of partial exams (tests during the semester), and conceptual writing exercises together with grades on homework writing assignments as independent variables, while the results on the final – as dependent variables. To cast the wide net for assessment of cognitive learning both quantitative and qualitative analysis are needed. And that brings us to TR interviews (Chapter 3.4), one of the primary tools for the collection of qualitative data. The difference between standard clinical interviews and TR interviews is that the latter must also teach to improve learning of the subject, in agreement with the definition of TR/NYCity model. Thus TR interviews can be conducted outside of the class for example with the smart board equipment which offers the possibility of audio and video collection of data, or they can be conducted directly in the class. It's not difficult to incorporate TR interviews into classroom discourse.

In many situations these investigative dialogues occur spontaneously; they have to be noticed and their text sketched into the TR Notebook as soon as possible after the dialogue. The role of TR Notebook has been explored by Prabhu and Nunez (2008). The second method of collecting qualitative data is to analyse student responses, misconceptions with the help of a theory of learning. However, before we can use a theory of learning we have to coordinate that theory with events in our classroom. This is the moment when we can utilize JiTR consultation if our craft knowledge is not helpful. It can be used to understand a particular student misconception or to interpret larger qualitative data from the class.

JiTR coordination means that we have recognized the components of classroom situation within the possible theory and/or identifed theoretical terms of the theory within classroom work. Such a process can be a bisociative process because just at this junction we see the attempt to connect two "matrices" of experience of Koestler, the learning theory and classroom practice which were not connected till the conduct of TE. Here is the possibility of the creative moment for the teacher, the Aha moment that might reveal the hidden analogy between the two – exactly the process whose example is documented in Chapter 2.4. Once that process of identification has taken place and impacted our thinking technology (Chapter 1.1) we can use the terms suggested by the theory to orient oneself anew in the classroom. We can formulate our TR questions and analyse obtained answers already in terms of the theory, hence in a more general framework, which at the same time can guide our next steps. The same principles of JiTR apply for the analysis of the final data from TE. If we are convinced that craft knowledge doesn't give us any more guidance, then we switch to research consultation. It is good in the process of searching data base of the profession to know as precisely as possible what concept or issues we are seeking help with. The node of redesign of intervention after the first analysis of the data can become a bisociative framework when we are solving the problem of learning improvement. We are bisociating here the previous design, which gave us a limited result with a new one, unknown yet but with certain new requirements. The solution may be an Aha moment.

NOTE

[1] We see consistency as a "steadfast adherence to the same principles, course, form, etc". and coherence as "the quality of forming a unified whole" (Mirriam Webster).

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3.3. QUALITATIVE AND QUANTITATIVE ANALYSIS

INTRODUCTION

Typically, for a teaching experiment, the accepted methodology used to assess learning and conceptual development is based on one-on-one interviews with an observer prompting, recording and analysing the vocalizations of a child engaged in a problem solving activity. The teacher-researcher must extend such an assessment to a classroom environment consisting of many students, not necessarily children, engaged in problem solving. The need to analyse and interpret the conversational flow between the instructor and students as well as numerous interactions among the students distinguishes the analyses of conceptual development in classroom teaching experiments from those employed in clinical studies that commonly focus on an individual learner's solution activity, and, therefore, can be quite a daunting task. The clinical interview methodology must be expanded to include a focus on the didactic contract between the student and teacher including affective issues. Along with such qualitative approach we present a different use for a common quantitative method ANOVA in order to study and predict what can be described as students' hypothetical learning trajectories. Analysis of class lectures are typically organized around a theoretical model of how the concepts are learned and hence sequenced by the instructor. Statistical or quantitative analysis can be used to map out how a plurality of students experience relationships between concepts.

TEACHING EXPERIMENT

Qualitative Cognitive Research

Ernest (1997) notes that qualitative research has "begun to dominate research in mathematics education. Although its roots go back a long way, in mathematics education this paradigm emerged in Piagetan-style research based on clinical interview methods" (p. 22). This method for qualitative interpreting cognitive aspects of student learning in clinical interviews is based upon by analysis through the lens of a theoretical framework underlying a teaching experiment. Battista et al. (2009) refer to such qualitative research as qualitative cognition focused research which they describe as, "research that focuses on describing cognition attempts to account for individual students' and teachers' actions, reasoning, and learning" (p. 222).

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 171–178.

Glasersfeld (1995) characterizes a teaching experiment as one in which a child is given a problem and an observer converses with the child in an effort to understand the child's mental thought process during the problem solving activity. As such, this methodology is based upon conjectures about what the observer considers the child's reasoning process: "The task of characterizing someone else's concepts is necessarily a conjectural one. One cannot enter into another's head to examine what conceptual structures he or she has associated with certain words" (Glasersfeld, 1995, p. 54). In the *teaching experiment* model, Glasersfeld refers to these conjectures as inferences based upon the observer's characterization of mental operations. These inferences are conjectural in the sense that their focus is on the, "mental operations. Since these operations are never directly observable, they can only be inferred from observation" (1995, p. 70).

This model for investigating the process of learning, appealing to the principles of the scientific method, is empirical, in the sense that is based upon observation and interpretation of the results with a reasoned hypothesis gleamed from existing theories. Ultimately, the results either support and confirm existing theories or contradict them, and, as such, paves the way for new insights and understandings of existing theories. Through this procedure, as Glasersfeld states, "the resulting hypothetical model achieves a high degree of plausibility and predictive usefulness" (Glasersfeld, 1995, p. 17). As noted by Steffe and Thompson (2000) such methodology was not always accepted and did not emerge in the U.S. until about 1970 its acceptance was in part because, models of learning and methods of analysis of learning "were needed that included an account of the progress students make as a result of interactive mathematical communication" and the recognition that such methodology and models were need because, "a large chasm existed between the practice of research and the practice of teaching" (p. 270). Thus, teaching research owes much and builds upon teaching experiments while following the ideas and goal of design research i.e. reflecting upon all aspects of the classroom situation, the teacher's thoughts about students, the curricula as well as students learning process. Steffe (1991) expressed the goal of a teaching research experiment as first, "to learn the mathematical knowledge of the involved children and how they construct it" (p. 178) and second, "to formulate a model of learning" (p. 178) that explains the learning process observed. The role of the researcher was to, "make decisions about what situations to create, critical questions to ask and the types of learning to encourage" (p. 177).

Qualitative Social-Cultural Research

As educators became concerned with a focus on the mathematics classroom as a unit of study clinical interviews outside the classroom environment as a research methodology were expanded to include observation and analysis in which, everyday mathematical instruction and learning processes are central to research or as expressed by Bartolini-Bussi (2005) the didactic relationship between the teacher and the student is the unit of analysis. This type of research is termed by Battista et al.

(2009) as qualitative social-cultural research and described as a methodology that, "supplements cognitive psychology's description of individual cognition by situating individual learning (a) in the context of students …in the academic/cultural practices of schools; and (b) as occurring while classroom norms and discourse develop" (p. 223). The emergence of the classroom as a focal unit of analysis, a reoccurring theme of teaching research is based upon Cobb's (2011) dictum that there is no substitute for, "sustained, direct engagement with the phenomena under investigation" (p. 11). As opposed to cognitive psychologist in a teaching experiment who minimize the role of the instructor to one who provides tasks that provide the opportunity for student cognition, "we start with the assumption that teaching does not cause learning. Rather, teaching involves promoting the likelihood of students learning particular mathematics…An important part of the teaching process is giving tasks designed to elicit the use of certain assimilatory schemes that the students have available" (Simon et al., 2010).

Qualitative research based upon the social classroom environment focuses on the position of Vygotsky and Russian educators that, "children form scientific concepts as a result of receiving instruction in specific school subjects and that the processes of mastery can be studied only in the context of these subjects" (Cobb & Steffe, 2011, p. 24). Cobb et al. (2011) describe the personal transition from teacher researcher with a qualitative cognitive focus to a researcher that includes a social focus:

> The increasing importance that we came to attribute to the teachers' central mediating role is at odds with the way in which the teacher is back-grounded… It is apparent from the transcribed excerpts…that the teachers actions…were crucial. However, the teacher's initiatives and her responses to students are treated as ancillary to the focus on students' learning. (p. 114)

This extension of qualitative research to include the didactic contract between teacher and student expands the methodology from clinical interviews to include research studies based upon the observation and analysis of, "ordinary classes, with small or no intended intervention of the researchers in the design of the lesson" (Bartolini-Bussie, 2005, p. 299). Qualitative social-cultural research opened the research door for design research on methodology of instruction, curricula as well as qualitative research on affective issues like student and teach beliefs, insights and motivation during observation of the didactic contract playing out in a classroom. The work of Prabhu and Dias (Unit 2) in which they presents narratives of student involvement during their transition to ownership of their learning potential is an example of teaching research that is qualitative social-cultural.

Classroom Teaching Experiment

In the classroom teaching experiment the instructor plays a dual role, both as an agent of development, promoting reflection upon the solution activity, as well as an observer, noting, interpreting and analysing students' comments and participation in the class dialogue. The goal is to interpret not only an individual student's understanding through his or her comments but also the effect his or her explanations have on other students. Thus, the social aspect of class discourse involves instructor-student as well as peer-to-peer interaction. In this setting, it is not only the realizations of an individual student that are of importance but also whether and how other students in the class appear to accept and grasp one another's insights. On the one hand, insight into an individual's conceptual development is, perhaps, best understood in a one-on-one setting, as in the clinical teaching experiment. For a classroom instructor, a theoretical model explaining reflective abstraction that is reflection upon solution activity that does not incorporate the social learning environment playing out in their classroom is not useful. For a clinical researcher of classroom learning or teacher-researcher another tool at his or her disposal is statistical analysis of the relationships between information both conceptual and procedural learned by the students.

QUANTITATIVE ANALYSIS TO TEST HYPOTHETICAL MODELS

Wilkins and Norton (2009) note that teaching research involving dialogue with one or two children is frequently used to develop models of cognition and conceptual schema development that are then used to modify instructional strategies. Following Kilpatrick (2001), Wilkins and Norton advocate for quantitative analysis of the conceptual models developed in these clinical settings. They acknowledge that such quantitative research is subject to the limitations that come with trying to analyse concept development by taking into consideration a set of exercises given at one time. Thus, tests do not provide insight into the nature of student realization and the processes of concept development. They write, "Such tests are inferior to teaching experiments in building models of students' ways of operating" (p. 151). However, these limitations do not render the results invalid; indeed, as eloquently stated by Wilkins and Norton (2009), quantitative analysis is an effective and appropriate methodology for testing hypothetical models of learning that are to be used with a large number of students in the classroom situation. In this, Wilkins and Norton follow Kilpatrick (2001) insisting that, "as mathematics education researchers we are obliged to quantitatively test our hypotheses whenever possible" (p. 150). One area in which hypothetical models of learning meet classroom pedagogy is learning trajectories or hypothetical learning trajectories. The goal of such research is "creating and maintaining connections between research and curriculum development as integrated, interactive processes, using a broad range of scientific methodologies" (Clement & Santana, 2004, p. 81). A hypothetical learning trajectory (HLT) integrates conjectures based upon models of learning into classroom instruction: "An HLT consists of the goal for the students' learning, the mathematical tasks that will be used to promote student learning, and hypotheses about the process of the students' learning" (Simon & Tzur, 2004, p. 91). When the conjectures about processes of student learning involve the relationship between concepts based upon the principle of causality, as in, for example, when students build their knowledge of a new concept upon foundational conceptual knowledge of earlier concepts or similar previous knowledge, and the conclusion of such research is a trajectory through the curriculum based upon how student understanding of these concepts and their relationships is realized, then quantitative analysis based upon correlation can be an effective tool in mapping out these trajectories.

Correlation Analysis: Causality of Conceptual Knowledge

The quantitative method described below involves statistical analysis to test the principle of causality between the understandings of two concepts, or variables, X and *Y*, that have been quantified for a sufficiently large sample size. The analysis employs the straightforward statistical measures of the means of each of the two variables and the correlation between them. It is based upon the assumption that the mean scores of the two variables are significantly different, verified by a two sample *t*-test, and the principle of causality for students' understanding of *Y* as a result of understanding of X, expressed as $X \Rightarrow Y$. Such causality may be measured by demonstrating that the mean of *X* is significantly higher than that of *Y*, and that the correlation between them is both positive and significant. As noted in Baker et al. (2012). "Given a positive and significant correlation between two variables when there is a significant difference between them, it is reasonable to conclude that the variable students find easier, concept *X*, has substantial potential to be used in acquiring knowledge of the variable they find more difficult" (p. 40). Indeed, the square of the correlation coefficient gives a quantitative measure of the variation of one variable explained by the other. "$\boldsymbol{R}^2$ indicates the percent variation of *Y* explained by the variation of *X*. This will be written as $X \Rightarrow Y$ ($\boldsymbol{R}^2$%)" (p. 47). For example, two different concepts *X* and *Y*, and given that a mean difference t-test of *X* and *Y* indicates students score significantly higher on *X* than *Y* and the correlation coefficient between them is significant (every correlation is assigned a *p*-value) and, say, the regression coefficient $\boldsymbol{R}$=0.6, thus, the square of this value (0.36) represents the percent of variation in *Y* due to variation in *X*. As an instructor this can be interpreted to suggest that 36% of students proficient in employing concept *X* on an exam will be proficient employing concept *Y*. We write $X \Rightarrow Y$ (36%) in this situation. It can be useful in analysing causality that variable *Y* be tested after (in time) variable *X*, however, this is not required as long as students are statistically more proficient with variable *X* than variable *Y*.

Causality of Conceptual Knowledge: Multivariate Analysis

Many hypothetical models of learning generated through teaching experiments and educational research involve more than two variables and the extension of the principle of causality through correlation analysis is referred to as multivariate analysis and, also, a fairly straightforward, yet slightly more involved, statistical tool. In the simplest multivariate analysis there are two dependent variables, say *X* and *Y*, that are both significantly easier for students than an independent variable, say *Z*. We, however, suspect that one of these variables dominates the other, that is, to say we suspect that students who know *X* well do not substantially benefit from instruction in variable *Y*. Perhaps, there are disagreements among educational theory as to whether *Y* promotes conceptual understanding of variable *Z* independently of the presence of variable *X*. The results of the multivariate analysis will, first of all, confirm that these variables are significantly correlated with variable *Z* and, secondly, will determine whether the existence of variable *Y* adds a significant amount of predictive value to the variation of *Z* explained by variable *X*. Thus, given *X* and *Y* as independent variables that correlate significantly with *Z*, one variable may dominate, say *X*, and the other, *Y*, may not add significantly to *X*'s predictive value of *Z*, or they may work together, and in this case the presence of *Y* adds significantly to the explained variation in *Z*. In this situation, statistical analysis provides an adjusted regression coefficient for the two variables working together and we write $X + Y \Rightarrow Z$ ($\boldsymbol{R^2}$%). The following description is meant for those not familiar with this technique which we have used in several places in this book:

> To consider the effect that several independent variables have on a dependent variable a multiple linear regression analysis or analysis of variance (***ANOVA***) will be used. The ***F***-value, or ***F***-ratio, is an indicator of the strength of the relationship between the independent and dependent variables and the *p*-value determines whether the model is significant. Assuming the model is significant the adjusted $\boldsymbol{R^2}$ value determines the percent of the dependent variable explained. A comparison of the adjusted $\boldsymbol{R^2}$ value with the square of the correlation coefficient between each independent variable and the dependent variable reveals the extent to which the independent variables work together. However, more precise information is obtained from the significance or *p*-values and beta values of each independent variable. When the independent variables are all significant, they work together for predicting or explaining the dependent variable. The second indicator of how the independent variables interact is the *beta* value. The *beta* value is a measure of how much influence each independent variable has in predicting or explaining variation in the dependent variable. For example, a *beta* value of 0.5 for *X* indicates that for every unit change of a standard deviation of *X*, there is a corresponding 0.5 or 50% change of a standard deviation in *Y*. (Baker et al., 2012, p. 47)

EXAMPLES

A teaching research experiment built upon this statistical analysis is presented in Chapter 4.6, which describes how to design writing exercise in mathematics according to different theories of learning. In two separate multivariate analyses the independent variables *X*, *Y* were taken from students' scores on partial exams during the semester while the dependent variable *Z* was taken from the departmental final at the end of the semester. In the first analysis, the independent variable *X* was proficiency with procedures while the independent variable *Y* was students' score on explaining in writing how to perform these same procedures and the dependent variable *Z* was student performance on these procedures during the final exam. In the second analysis the independent variables *X* and *Y* were student performance on application problems and their written explanations of how to solve such problems (when no actual numbers were present) on partial exams during the semester while the independent variable *Z* was their performance on application problems on the final. The research question was whether written language used to express and promote student reflection upon mathematics was statistically significant in retention of their ability to solve procedures and word problems; statistical significance, in the sense that written thought (*X*) independent of student proficiency to perform procedures and application problems during the semester (*Y*), when considered alongside such proficiency contributed significantly to student performance on the final exam (*Z*). Thus, the research question concerned the relationship between written thought about mathematics and proficiency with mathematics. This statistical analysis is also used in the chapter to map out a learning trajectory through student knowledge of fractions (rational number sense) towards understanding of proportional reasoning. In this analysis, the independent variables are the various sub-concepts of the fraction concept: part-whole, operator, quotient and measure, and the dependent variable is student proficiency with proportion examples. The question in this analysis is to what extend does student understanding of these fraction sub-concepts predicts their understanding of proportions.

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3.4. TEACHING RESEARCH INTERVIEWS

INTRODUCTION

This project is based on a study assessing the effectiveness of teaching research interviews in mathematics with urban high school students. The subjects of the study were public high school students enrolled in the Mathematics and Science Partnership in New York City (MSPinNYC), a multi-year program funded by the National Science Foundation. The methodology was geared to a target population consisting of the lowest third performing in Math A Regents exams.

The researchers conducted over 40 clinical interviews with a target population consisting of the l6 lowest performing students. As a result of the intervention, the target population's average grade in the Regents exams increased by 40%, compared with only 20% for the non-target population. Through the interviews, the researchers were able to obtain insight into the students' thought processes, to pinpoint misconceptions, to develop a dialogue and to raise the students' confidence in mathematics.

The data collected from the interviews pointed to patterns of misconceptions common to groups of students. The researchers used that information to improve their own pedagogy and the pedagogy of other teachers in the program. Teaching research interviews are defined as those clinical interviews that could be used to improve pedagogy. In the rest of this document, MSPinNYC is referred to by its popular name, MSP.

Definitions

A clinical interview is a dialogue between a researcher and a student during which the researcher tries to understand the student's mathematical thinking and uncover misconceptions. Teaching research interviews are those clinical interviews that *investigate and deal* with student misconceptions and, can improve pedagogy in the researcher's class or in other classes.

Czarnocha and Prabhu theorized that in order to understand students' mathematical thinking and to deal with the students' reasoning associated with patterns of mistakes, it is necessary to conduct one-to-one *clinical interviews* with selected weak students. To the extent that the mathematical errors uncovered during the interviews result from misconceptions common to a group of students, the researchers could use the knowledge derived from the interviews to improve their

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 179–197.

own pedagogy and to avoid students' errors from becoming fossilized and turning into misconceptions. By sharing their findings with other practitioners, teacher researchers could help improve mathematical pedagogy of those practitioners. Consequently in this paper, the interviews conducted are referred to as *teaching research interviews*.

Interviews are becoming an important tool in numeracy projects because of their value in helping teachers understand student's thinking while they are working on a problem (Heirdsfeld, 2002). Heirdsfield corroborates Hunting's statement that clinical interviews "allow students to be teachers."

A teacher can learn a lot not only about a weak student's mathematical thinking, but also about a strong student's reasoning. An eighth grader interviewed by Wheatley was shown seven different problems involving arithmetic computations and applications on proportions, fractions, and geometry (Walbert, 2001). Using videotaping and interview transcripts, Wheatley found that while the student learned common procedures and how to apply them, the student failed to make connections between concepts and unknown situations. Hunting (1997) contrasts the similarities and differences of clinical interviews used as part of mathematics research vs. interviews used in mathematics classroom for assessment purposes. The interviewers should have strong interviewing skills and sound pedagogical content knowledge of mathematics. They should also know what types of questions to ask, how to answer student's questions and be capable of interpreting and making connections of students' answers (Hunting, 2002).

The interviews might be helpful in providing not only a good assessment of a selected student's knowledge, but also an understanding the student's reasoning associated with mistakes. To the extent that a student's false reasoning leads to constantly repeating the same mistake, that mistake could be a "fossilized mistake" or "misconception." By interviewing several students, the interviewers could uncover patterns of mistakes common to groups of students.

At the individual student level, the interview could be helpful in eradicating a student's fossilized errors and in leading that student on a correct path of mathematical thinking for the particular topic. An added benefit is the student's enjoyment in discovering the logic of a concept that used to be a source of confusion in the past. Naturally, when the misconception in a certain mathematical area is replaced by understanding, one would expect that in the future the student would score higher when that area of mathematics is encountered in an examination.

To better understand the MSP students' mathematical thinking, Czarnocha, Prabhu and Fuchs experimented with one-on-one teaching research interviews in the summer of 2005. Two years later, Fuchs and Menil conducted a qualitative research at Leman College with the lowest performing MSP students. Their work was guided by the following questions: Why did students solve a problem in a certain way? How did the students arrive at their results? What types of mistakes did the students make, and what were their misconceptions? (Ambrose, Nicol, Crespo, Jackobs, Moyer, & Haydar, 2003).

Not entirely to their surprise, the two researchers found that the high school students in the MSP program had the same difficulties as their college students enrolled in remedial pre-algebra and algebra classes. The theoretical underpinning of the research consists of socio-cultural theory, the sociology of emotions, elements of educational psychology and error detection and correction in mathematics.

MATH AND SCIENCE PARTNERSHIP IN NEW YORK CITY (MSPINNYC)

The Math and Science Partnership in New York City (MSPinNYC), referred in this document by its popular name, MSP, was a 13 million dollars project, budgeted for the period September 15, 2004 through August 31, 2011. The grantor of MSP was the National Science Foundation (NSF); the applicant, and the recipient of the grant was the City University of New York (CUNY). The principals under the application number 0412413 were CUNY university professors several specializing in mathematics, physics, chemistry and education: Pamela Mills, Annette Digby, Francis Gardella, Linda Bey- Curtis, William Sweeny and Vrunda Prabhu.

MSP was formed to address a number of serious problems confronting the Mathematics and science secondary education in the NYC school system: (1) shortages of mathematics and science teachers, especially experienced teachers in schools characterized by poverty and by students historically underrepresented in mathematics and science; (2) extremely low retention rates among teachers; (3) high failure rates among students who take the 8th grade mathematics exam and required-for-graduation state mathematics and science Regents examinations; (4) lack of preparedness of high school graduates for college level work; and (5) schisms and poor communication between schools, between (some) university campuses, and between science, mathematics and education faculties.

To address these problems, the MSPinNYC used the strategy of a Micro/Macro approach for reform at both the local level and system wide. At the local level, twelve hub schools were created. Each hub school served as a clinical site for teacher training, and an exemplar for excellence in mathematics and science education. The hub schools were developed by teams of college faculty and secondary teachers working closely together in a novel model for professional development to create cultures within the schools invested in teaching as a collaborative enterprise and research-driven classroom practices. Collaborative teaching teams of faculty and teachers worked during the summer with high school students who have failed a Regents exam. These teams continued through the school year with collaborative lesson development and collaborative research on classroom learning.

CUNY Schools of Education was changed to include more collaborative teaching practice and a greater reliance on the scholarship of teaching. New pipelines for recruiting talented mathematics and science undergraduates into a career of teaching were created. To create reform system wide, the MSPinNYC Macro approach included an Advisory Board of statewide policy makers, a Council of eminent scholars in mathematics and science education, and a ‘jobs-alike’ structure to bring together the leadership at the public school and college levels. Important questions of policy were raised and addressed, informed by both the scholarly and the local perspectives. The MSPinNYC Micro/Macro strategy provided a new model to approach systemic reform in large and complex systems.

The MSP teams consisted of CUNY college professors in math and science and high school teachers in these disciplines teaching at the at hub schools. The teaching research interviews discussed in this document were conducted by two mathematics professors, Eric Fuchs (EF) and Violeta Menil (VM) who each worked for three summers in the MSP project. In addition, during those years, Fuchs worked in MSP during in school year as well observing classes and collaborating with the mathematics teachers on the team. The interviews were conducted during Fuchs' and Menil's third summer on the project.

We should also mention the MSPinNYC2 project builds on the original MSPinNYC. The new program, with new partners joining most of the original partner, seeks to extend and deepen a promising program called the Peer Enabled Restructured Classroom (PERC), which was piloted during the earlier work. MSPinNYC2 is built on the premise that students could learn a lot from other students who had passed the course one year earlier, and were able to overcome the same difficulties their younger peers are faced with.

The PERC program restructures 9th grade STEM courses to have 7 or 8 Teaching Assistant Scholars facilitate group work on a daily basis. TA Scholars are average-achieving, i.e., not honors, 10th graders who passed the course and the associated required state exit examination during the previous year and are concurrently trained in a TA Scholar course led by the same teacher as the 9th grade class. Pilot studies with PERC during the MSPinNYC project suggested that the program reduces failure rates, closes achievement gaps, and improves graduation rates.

TEACHING RESEARCH INTERVIEWS

The idea for this research arose in the summer of 2005 from an initial MSP collaborative work with Czarnocha and Prabhu, who postulated that teaching research interviews might provide a better assessment of students' knowledge than the analysis of results obtained from the weekly mock Regents exams that were administered to all the students in the program. Besides helping students discover and correct their own misconceptions, Czarnocha and Prabhu postulated that the teaching interviews could help students develop metacognition in their mathematical reasoning.

In a typical elementary school mathematics class, after learning a new topic, the students are given homework in the form of worksheets to practice the concept they just learned. The teacher collects the worksheets and corrects them at home. When observing classrooms, we saw many instances of children receiving the corrected homework, look at their grade and depositing the corrected sheet in their backpacks. If the corrections were really acted upon, one would expect that on a second try, all students would get 100% or very close to it. Unfortunately, this rarely happens. Because of time constraints, teachers, who "have to cover the curriculum," rush to the next topic.

In the mind of the students, the non-corrected errors become fossilized, and ultimately translate into misconceptions of the types analyzed in this document. There is a large body of literature devoted to the topic of mathematical misconceptions and learning from errors and misconceptions (Ryan & Williams, 2007; Swinson, 1992; Swan, 2001; Bell, 1982). Recognizing that college students in developmental mathematics and college algebra make the same mistakes repeatedly, Lerch (2002) addresses methods of "unfossilization" for building new pathways to older knowledge.

Naturally, in order to correct misconceptions, one has to uncover them first. It is interesting that typical mathematics misconceptions, such as the ones discussed in this document are common to large groups of students. Hunting (1997) states that "*Understanding the mathematical workings of children's minds is now a priority for teachers, schools, and systems, as well as for academic researchers*." Hunting discusses the use of clinical interviews as an assessment strategy, and the advantage of clinical assessment methods over the traditional assessment instruments and describes the characteristics of an effective clinical interview.

The data gathered from effective clinical interviews should be used not only for "unfossilizing" interviewee's misconceptions, but to the extent that the misconceptions uncovered are characteristic to a large group of students, that data should be used to improve pedagogy and to avoid future students' misconceptions from occurring in the first place.

Thus, we define as *teaching research interviews* those clinical interviews that gather data on students' misconceptions in a way that could be used to improve particular student learning as well as general pedagogy in the classroom. We hope that research papers such as this one will be used to disseminate our and other researchers' findings about students' mathematical misconceptions and lead practitioners to an improved pedagogy that prevents misconceptions of being created in the first place.

THE RESEARCH QUESTIONS

This research was premised by three questions:

- To what extent can teaching research interviews help teachers understand their students' mathematical thinking?
- If teaching research interviews help teachers understand their students' mathematical thinking, how can they be used as a useful pedagogical tool?

- To what extent are teaching research interviews useful with low-performing students in math?

The impetus for the research came from the low retention and graduation rates of students enrolled in the associate degree programs at two CUNY Bronx-based community colleges: BCC and Hostos. As documented by the CUNY Office of Institutional Research and Assessment (OIRA), less than 22% of a typical freshmen cohort in these colleges was awarded the associate degree at the end of 6 years (OIRA, RTGI_0001).

RESEARCH PURPOSE AND ASSUMPTIONS

A primary goal of this research was to use teaching research interviews to identify and possibly correct misconceptions and areas of weaknesses in the mathematical thinking of the bottom third of MSP Math A students. A secondary goal was to determine if teaching research interviews and follow-up activities could help increase the percentage of students from the target population who passed the Regents examination.

The following assumptions guided this research study:

- It is possible to identify early in the program the bottom third of Math A students who are “doomed to fail” the Regents exam.
- Teaching research interviews can help uncover student misconceptions, areas of weaknesses and faulty mathematical thinking.
- Addressing student misconceptions and correcting these weaknesses are instrumental in raising the passing rates of Math A students in the Regents exam.

THEORETICAL FRAMEWORK

A framework for understanding teaching and learning in a democracy is provided by the intersection of three major components: knowledge of subject matter and curriculum goals, knowledge of teaching and knowledge of learners (Bransford, Darling-Hammond, & LePage, 2005). The first two components are based on Shulman’s article on the professionalization of teaching (Shulman, 1987).

The Emotional Aspect of Teaching

Based on Collins’s sociology of emotions, Tobin explains that good teaching is made up of successful interactions that are charged with positive emotions; bad teaching is made up of unsuccessful interactions that are charged with negative emotions (Tobin, 2006).

Hargreaves argues that the emotional dimension of teaching is largely ignored or underplayed by the policy makers (Hargreaves, 1998). Like painters, good teachers are born, not made!

Cultural Responsive Teaching

Research shows that "teachers' attitude and expectations, as well as their knowledge of how to incorporate the cultures, experiences, and needs of their students into their teaching, significantly influence what students learn and the quality of their learning opportunities (Banks, Cochran-Smith, Moll, Richert, Zeichner, LePage, Darling-Hammond Duffy, Andh McDonald, 2005).

Villegas and Lucas (2002) state that because of the diversity of the student population, responsible educators continuously tailor instruction to individual children in specific cultural contexts.

MATHEMATICAL DIFFICULTIES

In community colleges in the Bronx, the students' mathematics difficulties are compounded by many years of inadequate mathematics education. The researchers found many similarities in the way mathematics is approached by their community college remedial mathematics students and their MSP students. Students have difficulty with fractions and order of operations. They struggle with word problems and with mathematical logic, and they want to be shown the algorithmic way of "how to" solve a problem rather than "why." Low literary skills are also a serious impediment, particularly in word problems.

Type of Mathematical Errors

For simplicity's sake, this research project distinguishes among three types of student errors: careless errors, calculation errors and conceptual errors. Careless errors result when students rush. Calculation errors typically occur when students are tired or are working under pressure. Conceptual errors (misconceptions) result from a fundamental misunderstanding of a concept. Based on the researchers' experiences, conceptual errors are due to the reliance on memorization and algorithms, overdependence on calculators and lack of understanding.

Error Detection and Correction

Conceptual errors in mathematics are insidious. Students acquire them along the way and keep reinforcing them so that they become "fossilized". An undetected or uncorrected conceptual error becomes part of the student's mathematical construction. Without outside intervention, most students cannot correct their mathematical conceptual errors. In his Winning at Math guide, Nolting states "...it is not the fault of the students if they have not been taught how to study math. Even students taking general study skills courses are often not taught how to study and learn it" (Nolting, 2002). Students who understand and analyze their errors can capitalize on that knowledge and thus achieve a better understanding of the subject. (Borasi, 1994).

METHODOLOGY

The researchers were not privy to data on the students' prior performance. They selected the target population based on the average of the first two mock Regents tests given at the end of the first and second week of MSP in the summer of 2007. Of the 51 Math A students enrolled in the program at Lehman College campus, the researchers selected the 16 lowest scoring students. The classroom teachers corroborated our selection of the target population. The target population's average score was 44.0, compared to 54.5 for the rest of the students. The students in the target population were then assigned the codes TP1 through TP16. The test results of the target and non-target population were compiled and analyzed weekly and at the end of the program.

Conducting Teaching Research Interviews

Each student in the target population was individually interviewed several times during the summer program. The goal of the first interview was to establish a rapport with the students. The students felt comfortable enough to describe their career aspirations, college plans and attitude toward math. Some expressed their discomfort with mathematics, while others stated that they simply do not understand it. After the first interview, all the students expressed their eagerness to meet with the interviewers again.

In a typical interview, one researcher conducted the interview while the other took notes; in subsequent interviews, the researchers changed roles. Students were asked to explain *why* and *how* they solved different multiple-choice questions in the mock Regents exam that they had taken. The researchers asked the students for evidence and explanations and at times asked them to solve a similar problem. They also examined the students' work, and notes in their exam books. After identifying misconceptions, the researchers discussed the errors with the students and guided them to arrive at the correct solutions.

Interaction with Classroom Teachers and Tutors

Following the teaching research interviews, the researchers conveyed their findings to the teachers and tutors enrolled in the program. The researchers also conducted professional development sessions with the tutors during which they suggested ways of correcting students' misconceptions.

RESULTS AND FINDINGS

As a result of the intervention, the percentage of low-performing students who passed the Regents or who obtained grades for high school graduation increased significantly compared to the previous years. The target population's average grade in Math A Regents increased by 40%. This compares to a 20% increase in the average grade of the non-target population.

Below are some common student misconceptions identified though the teaching research interviews:

Order of Operations

All the students in the target population relied on the mnemonic PEMDAS (*Please Excuse My Dear Aunt Sally*) when deciding what operations to do first. Many erroneously believed they should always do addition before subtraction, since the letter A comes before the letter S in that mnemonic. Similarly, since the letter M comes before the letter D in the mnemonic PEMDAS, some students answered that $8 \div 2 \times 4 = 1$. Below is an excerpt of a teaching research interview we conducted with a student, whose code name was TP3. Appendix A contains a more complete version of this interview, as well as the interviewers' related comments.

EF: Please explain your reasoning. How much is $4 - 2 + 1$?
TP3: I was afraid you'd ask me hard questions. The answer is 3, obviously.
EF: Are you sure?
TP3: Um, let me see. Oh, now I remember. The answer is 1.
EF: There can't be two answers to that question. Is it 3 or 1?
TP3: It's definitely 1. You do the order of operations with *Please Excuse my Dear Aunt Sally* or PEMDAS. The letter A comes before S, so you add first.
EF: I see. How much is $8 \div 2 \times 4$?
TP3: You see, M comes before D in PEMDAS, so the answer is 1, right?
EF: Hmm, let's see… You are right that A comes before S in the word PEMDAS. But if you have 4 dollars in your pocket and you spend 2, how much would you have left?
TP3: I see what you're doing. I'll be left with 2 dollars, and if I were to add another 1, I'd have 3 dollars, right?
EF: Therefore $4 - 2 + 1$ ought to be equal to 3, not to 1. That means you did the subtraction first, since it came first. You did addition second, since it came second.
TP3: It makes sense with money. But in math, the rules for order of operations are set by PEMDAS—I remember this is what Ms. K. told us in grade 5, and this is what Mr. G told us in grade 10. Isn't that so?

Pedagogical implications. Mnemonics (PEMDAS, FOIL, SOHCAHTOA) provide an easy way for some students to remember *how to* perform certain operations algorithmically, without understanding the *why*, i.e. the governing concepts. We should be cognizant, however, of the dangers and implications associated with teaching with mnemonics. Examples:

- Some students use FOIL (First, Outer, Inner, Last) when multiplying two binomials precisely in the order prescribed, without understanding that the order of the terms in the resulting polynomial does not matter. These same students are lost when they have to multiply a binomial by a trinomial.
- As seen from the interview with TP3 that is provided in the appendix, teaching PEMDAS does more damage than good. We encounter college students performing operations in the wrong order (example: multiplication *before* the preceding division, or addition *before* the preceding subtraction) precisely because they rely on PEMDAS.

Operations with Signed Numbers

When asked to evaluate –4 –2, some students provided the answer 8. They "remembered" that two negatives give a positive, and confused –4 –2 with (–4) (–2).

Most students did not learn about negative numbers until the first class in pre-algebra. By that time, they were confused with many mathematics concepts. Introducing negative numbers simply added to their confusion.

Pedagogical implications. Consideration should be given to teach the negative numbers in the early grades. Children should learn to count forward and backward on a number line centered at 0, rather than starting at 1. It is easier for a child to "see" the integers on a vertical number line (emulating temperatures centered at zero, or elevators in an office building with several floors below ground level), rather than on a horizontal number line.

Division by 0

Many students correctly answered 0 to the operation 0 ÷ 6; however, they gave the same answer for the divisions 6 ÷ 0, or 0 ÷ 0.

One student explained, "When I divide six apples to two people, each one will get three apples; but, if there are zero takers, I can keep the six apples." Obviously, the student confused the quotient with the remainder.

Pedagogical implications. Rather than being taught that division by zero is "not defined", or "cannot be done", students could discover themselves *why* division by zero cannot be performed. For example, using calculators, students could be guided how to plot the results of dividing the number 8, for example, by x. The resulting hyperbola will help students understand not only why division by zero is not defined, but also understand the concept of "limit."

Operations with Fractions – Addition and Multiplication

We found that a group of interviewed student, performed fraction addition by adding numerators separately and denominators separately.

Example: $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$

Pedagogical Implications. Since we cannot add apples and oranges, we cannot add feet and inches, pounds and ounces, hours and minutes, quarters and dimes! These types of examples illustrate the need for a common denominator in fraction addition/ subtraction.

Example: How many nickels you get from one quarter and 3 dimes?

Answer: $\frac{1}{4}=\frac{5}{20}$ and $\frac{3}{10}=\frac{6}{20}$. Therefore, $\frac{5}{20}+\frac{6}{20}=\frac{11}{20}$, or 11 nickels

Operations with Fractions – Multiplication and Division

We found that many students rely on cross-multiplication as a must-do tool when working with fractions, to the point of "when in doubt, cross multiply!" Thus, they end up cross-multiplying, not only when dividing, but also when multiplying two fractions…Cross-multiplication as a way to perform fraction multiplication is a prevalent misconception among most students we interviewed.

Other students "remembered" that when working with fractions, you "always" need a common denominator. However, those students did find a common denominator not only in fraction addition or subtraction, but also unnecessarily obtained a common denominator when multiplying or dividing two fractions.

Example of a student's work: $\frac{1}{4}\times\frac{1}{5}=\frac{5}{20}\times\frac{4}{20}=\frac{20}{400}$

Pedagogical Implications. The researchers believe that cross-multiplication is useful when solving proportions or when verifying the equivalence of two fractions. The pedagogical challenge is to teach a methodology that is valid in some instances, while at the same time works to prevent misusing it in those instances when it is not valid. Would one use a knife instead of a fork when eating?

Fractions of a Number, Equivalent Fractions and Proportionality

Students struggled when trying to solve a word problem that required the use of proportions. They lacked a sense of magnitude when working with fractions and such as "If 5 workers assemble 9 computers a day, how many workers are required to assemble 27 computers in a day?"

The excerpt of the interview with TP4 illustrates the misconception associated with a fraction of a number and fraction comparison.

TP4: You're right. I got that. So since in the cafeteria they cut the pizzas in eight, each time I eat a slice I eat an eighth.

EF: Yes, that is called an eighth of a pizza; naturally, it assumes that all slices are of the same size. Let's forget about pizza at least until lunchtime, but let's stay with the fractions.
What do you think is bigger; one half or one quarter?

TP4: Half is *always* larger then a quarter. Half a dollar is 50 cents; a quarter is only 25 cents.

EF: True when you compare half a dollar with a quarter of a dollar. But would you rather have half of my salary or a quarter of Bill Gates's salary?

TP4: Come on, don't make me laugh!

EF: Precisely, so when we talk about fractions, we refer to a part of a *whole*, i.e. one third of an apple is smaller than one half of an apple that is of the same size.

TP4: I got that. This is why in class they said that you can't add, or even compare, apples with oranges.

Pedagogical Implications. From interviews with the students, we uncovered several areas of confusion and misconceptions in understanding fractions and proportionality:

- Students refer to fractions as *numbers*, rather than *parts of a whole*. Indeed, on a horizontal number line, the half is located to the right of the quarter, half of a dollar is more than a quarter of a dollar; consequently, the students wrongly concluded that one half is always larger than one quarter. The *aha!* moment for TP4 happened when one of the researchers asked him whether he would prefer half the researcher's money or a quarter of Bill Gates's money. In the same way that we teach children that 3 hours is more than 5 minutes, and that 5 is larger than 3 only when they refer to the same objects, we should get students to understand the importance of the *of what* question when dealing with fractions: one third of one hour is more than one half of one minute.
- The next step is for the students to understand, without computing, the concept of a fraction of a number: one fifth of 10 equals 2, two fifths of 10 equals 4, three fifths of 10 equals 6, and to continue the pattern by themselves without stopping at four fifths of 10. Thus, a student should understand why seven fifths of 10 equals 14, rather than multiplying 7 times 10, and than dividing 70 by 5.
- After the fraction of a number, the next concept that requires a thorough understanding is that of equivalent fractions: $\frac{1}{3}$ of a number is the same as $\frac{2}{6}$ of the same number.

Area and Perimeter

Students were shown a picture of a rectangle, with the number 3 next to one side, and the number 5 next to an adjacent side, and they were asked: "Please find the perimeter of the rectangle shown below!" Some students answered 15, some students answered 8, while some students stated that more information is needed.

Pedagogical Implications. From the interviews it appears that since the concepts of perimeter and areas might be taught at the same time, some students confuse the two. We also concluded that the students who answered 8 "remembered" that to calculate the perimeter you have to add the numbers shown. Consequently, they were adding the given numbers 5 for one side and 3 for the other side without fully understanding that calculating the perimeter requires adding the dimensions of *all* the sides of a given shape. Ultimately the students who needed more information did not know the properties of rectangles, or more generally of parallelograms (i.e. opposite sides are equal).

Generally, when two concepts, such as perimeter and area, have too much in common, it is advisable to teach them separately to avoid future misconceptions.

The Equal Sign

Since they do not see the meaning of, for example, $2+8=20\times\frac{1}{2}$, in algebra many students have difficulty solving for *x* the equation $2+c=20\times\frac{1}{2}$.

Pedagogical implications. More than 80 years ago, Renwick (1932) explained that the children's misconception concerning the equal sign is due to improper teaching of arithmetic: Students associate the equal sign with a command to perform an operation or, later on in their studies, as separating an expression from its "answer." Consequently, they do not understand what does it mean the equality between two expressions.

> Les résultats indiquent que la méthode d'enseigner l'arithmétique avait amené beaucoup des élèves à se tromper sur la fonction du signe "=" d'abord elles avaient considéré comme un commandement toute indication d'une opération, ce qui les avait empêchées de reconnaître comme formant une unité toute expression contenant une telle indication; ensuite elles avaient adopté le signe "=" comme un expédient pour séparer une expression de sa "réponse." De cela il s'ensuit que des affirmations qu'une expression en égalait une autre, étaient pour elles inintelligibles.
>
> [The results show that the method used in teaching arithmetic brought many students to confusion on the meaning of the "=" sign. Initially the students regarded the equal sign as a command to perform a certain operation; that prevented them from understanding the role played by the equal sign when dealing with algebraic expressions. Later on, the students concluded that the equal sign separates between an expression and its "answer." Consequently, the statement that one expression equals another did not make any sense].

Teaching at early age the meaning of the comparison symbols (equal, greater than, smaller than) and assuring that the students use the equal sign if and only if the left side is equal to the right side, would go a long way in avoiding the misconception associated with the role of equal sign in mathematics.

Distributivity Property

Some students equated $3(2x + 3)$ with $6x + 3$. Not understanding that that $3(2x + 3)$ is equivalent to $(2x + 3) + (2x + 3) + (2x + 3)$, they distributed the3 only to the first term in the parenthesis.

Pedagogical Implications. Students who distribute only to the first term in parenthesis do not understand the meaning of distributivity, and later, in middle school, are unable to solve linear equations. We believe that mental math in elementary school, such as multiplying by distributing, will help the students understand how to distribute correctly. For example:

$$3 \times 12 = 3(10 + 2)$$

$$3 \times 12 = 3(10 + 2)$$

$$3 \times 12 = 3(10 + 2)$$

At a more advanced level, students will understand that $3 \times 232 = 3(200 + 30 + 2)$; in the last exercise, they will realize that there are 3 groups of 200, plus thee groups of 30, plus 3 groups of 2.

FUTURE RESEARCH

The following questions should be addressed in future research:

- Should teaching research interviews be conducted in elementary school, middle school, and/or high school? There are obviously issues of manpower, compensation, scheduling.
- Should teaching research interviews be conducted by the classroom teachers themselves, or by outsiders?
- Should teaching research interviews be conducted with high-performing students as well? How should those interviews be structured?
- How could the teaching research interviews be used with remedial mathematics students in community colleges?
- What are the most efficient ways to translate the information derived from teaching research interviews into improved mathematics pedagogy?

CONCLUSIONS

Based on this project, the researchers concluded the following:

- Teaching research interviews are useful for uncovering students' misconceptions and weaknesses.
  Used properly, teaching research interviews may constitute an important pedagogical tool for the teachers.
- The power of teaching research interviews would be greatly enhanced when used in conjunction with other pedagogical tools.
- Collaboration between teachers and tutors is vital: exchanging information about individual students' strengths, weaknesses and misconceptions and discussion of joint strategies to promote error correction and enhance learning is of paramount importance.

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APPENDIX A – EXCERPTS FROM TEACHING INTERVIEWS WITH SELECTED STUDENTS

Excerpts from a Teaching Research Interview with a Student Designated as TP3

EF: Nice to meet you, TP3. How do you like math?
TP3: To tell you the truth, I don't really hate math. I don't understand it, and I didn't get much help with it.
EF: Sorry to hear that. What profession do you have in mind for your future?
TP3: Engineering or law. I hope I won't have to take more math classes for them.
EF: For engineering you will. To help you out, let's look at some simple questions, okay?
Please explain your reasoning. How much is $4 - 2 + 1$?
TP3: I was afraid you'd ask me hard questions. The answer is 3, obviously.
EF: Are you sure?
TP3: Um, let me see. Oh, now I remember. The answer is 1.
EF: There can't be two answers to that question. Is it 3 or 1?
TP3: It's definitely 1. You do the order of operations with *Please Excuse my Dear Aunt Sally* or PEMDAS. The letter A comes before S, so you add first.
EF: I see. How much is $8 \div 2 \times 4$?
TP3: You see, M comes before D in PEMDAS, so the answer is 1, right?

EF: Hmm, let's see… You are right that A comes before S in the word PEMDAS. But if you have 4 dollars in your pocket and you spend 1, how much would you have left?

TP3: I see what you're doing. I'll be left with 2 dollars, and if I were to add another 1, I'd have 3 dollars, right?

EF: Therefore $4 - 2 + 1$ ought to be equal to 3, not to 1. That means you did the subtraction first, since it came first. You did addition second, since it came second.

TP3: It makes sense with money. But in math, the rules for order of operations are set by PEMDAS – I remember this is what Ms. K told us in grade 5, and this is what Mr. G told us in grade 10. Isn't that so?

EF: We'll discuss that more next time.

Interviewer's Comments:

TP3 relies on the mnemonic PEMDAS to determine the sequence of operations. The student did not see a relationship between mathematics in school and mathematics in real life.

In a subsequent meeting with the two researchers, TP3 discovered the logic associated with the order of operations. When faced with the question $10 - 2 \times 4$, TP3 used the example of two friends going to McDonald's, eating food for \$4 each, and paying with a \$10 bill, thus resulting in \$2 change.

EF: I see that you wrote $10 - 2 \times 4 = 2$. How did you come up with the answer 2?

TP3: First, I multiplied $2 \times 4 = 8$, then I subtracted $10 - 8 = 2$. I did the multiplication first, and then the subtraction.

EF: Good job! Did you use the same acronym PEMDAS like last time?

TP3: Forget about PEMDAS! I don't need rules that work only half the time! And besides, 10 represent dollars, while 2 represents people. You can't subtract people from dollars, can you?

Excerpts from a Teaching Research Interview with a Student Designated as TP7:

VM: TP7, let's see how you evaluate the algebraic expression $ab - b^2$ when $a = 3$ and $b = -1$.
I want you to explain to me as you write the solution, okay?

TP7: Okay. First, I write the whole algebraic expression as is: $ab - b2$. Then, I replace the letters with the values assigned, and then I simplify it as follows:

$$ab - b^2 = 3(-1) - (-1)^2 = -3 + 1$$

VM: Why did you evaluate $-(-1)^2$ as 1?

TP7: Because 1 raised to any power is just 1, and two negatives equals positive, right?
VM: Yes, but I'm interested in your thinking on that.
TP7: All right, all right, I didn't get it!
VM: Could you to explain to us the meaning of $(-1)^2$.
TP7: $(-1)^2$ means -1 times -1 equals -1.
VM: Great! Now $-3 -1$ equals what?
TP7: -4 and so $ab - b^2$ is -4 and not -2.
VM: Fantastic! Now you got it right. See, I'm interested in how you're thinking about that.

Interviewer's Comments

At first, TP7 confused between $-(-1)^2$ and -1. Later, that was clarified.

Excerpts from a Teaching Research Interview with a Student Designated as TP4:

EF: Good day, TP4. Sorry to pull you out of the class.
TP4: Actually, I checked with some of my friends. They told me that they had a better time with the two of you, instead of sitting in the class.
EF: Are you telling me that you don't like the classes that much?
TP4: I'll tell you the truth, I think I first learned fractions in grade 3 or 4, and then in 5, and again in 6, 7, 8, and so on and I'm still confused a bit. I wish there were no fractions at all!
EF: That's funny, would you like to eat at lunch a whole pizza, rather than a slice or two?
TP4: You're right. I got that. So since in the cafeteria they cut the pizzas in eight, each time I eat a slice I eat an eighth.
EF: Yes, that is called an eighth of a pizza; naturally, it assumes that all slices are of the same size. Let's forget about pizza at least until lunchtime, but let's stay with the fractions.
What do you think is bigger; one half or one quarter?
TP4: Half is *always* larger then a quarter. Half a dollar is 50 cents; a quarter is only 25 cents.
EF: True when you compare half a dollar with a quarter of a dollar. But would you rather have half of my salary or a quarter of Bill Gates's salary?
TP4: Come on, don't make me laugh!
EF: Precisely, so when we talk about fractions, we refer to a part of a *whole*, i.e. one third of an apple is smaller than one half of an apple that is of the same size.

TP4: I got that. This is why in class they said that you can’t add, or even compare, apples with oranges.

EF: Precisely. Let’s cut the apple here in two equal parts, and let’s cut one of these halve in halves again. What do you get?

TP4: That one half is the same as two quarters, or if we continue further is the same as four eights.

EF: Great! But, let us not forget, that all this fractions were parts of the same apple, referred to the same whole.

Interviewer’s Comments:

By the end of the interview, TP4 was proud that he could calculate fractions of a number. He also exhibited proportional thinking, by being able to solve the problem such as “If 5 workers assemble 9 computers a day, how many workers are required to assemble 27 computers in a day?

VRUNDA PRABHU, HAIYUE JIN AND ROBERTO CATANUTO

3.5. USE OF CONCEPT MAPS IN CLASSROOM RESEARCH

INTRODUCTION

Concept map approach is a highly effective tool for the development and assessment of student conceptual understanding of mathematics and as such it belongs to the category of Stenhouse TR acts (Chapter 1.1). Teaching-research expands the utility of concept mapping to teacher's own reflection upon the coherence of the curriculum; Chapter 3.7 describes its use by a teacher-researcher as a guide in the formulation of a coherent curriculum out of the prescribed topics of remedial mathematics course, Chapter 3.6 is a full scope description of concept maps as assessment tool, while Chapter 3.8 describes use of concept maps by the teacher as the tool of finding the "common denominator" between the theme of the class and student interests.

Concept map is a graphical representation of knowledge and its construction within a particular domain. It is a network consisting of nodes and labelled lines. Nodes contain concepts, usually in boxes or circles. The relationships between concepts are indicated by connecting line segments or arrows. The labels in the lines are called linking phrases, and indicate how joined concepts are related. The linked concepts together with labels indicating the connecting phrases form a meaningfull statement. Together, this whole network of concepts and linking phrases represent student schema of knowledge relatively to the main concept to be understood mastered. Contemporary interest in concept maps dates to the work of Novak who saw them primarily as the assessment tools (Novak & Gowin, 1984). Concept maps have been extensively used in science education (Horton et al., 1993) but their usage in mathematics education has only slowly been acquiring momentum. Important in this respect was the use of concept maps in the context of modelling by Clark and Lesh (2003). The authors point out to the dual role of the concept maps as an education act and as a research act:

> using concept maps as a model-eliciting activity for teachers not only allowed teachers a tool to make sense of their own thinking but it was [also] purposeful in that teachers were designing concept maps to serve as curricular guides for student model-eliciting problems.

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 199–247.

More complete approach to the assessment of student teachers' knowledge using concept maps has been investigated by Afamasaga-Fuata'l (2009). He points out to the role of refinements through iteration in student teachers' construction of their concept maps confirming the central role of iteration in the TR/NYCity model, which has been pointed out in Chapters 1.1 and 4.3. Chapter 5.1 by Vrunda Prabhu develops and provides examples of artefact generalization through its iterated refinements with different cohorts of students. A concept map can address either an individual concept, a group of concepts or the conceptual organization of a thematic unit of the curriculum.

The concept map as the scaffolding guide both for a teacher and for a student has been one of central aspects of concept mapping for Prabhu. It guided the pathway of questions and hints which can be used by the teacher in facilitating student discovery of a particular connection between relevant concepts.She demonstrates importance of concept maps:

- as a means by which to provide students a snapshot of the big picture;
- as a way for teacher-researchers to design the problems in the instructional sequence; the structure of the concept map representation of the schema outlines the pedagogical design that will be implemented in the course;
- as an environment within which analysis of word meanings can begin and progress toward a shared understanding (Bruner, 1990).

Prabhu central interest here is in the facilitation of construction of students' schema of thinking by using the concepts maps of the full course as well as concept maps of the particular concepts. She guided herself in this work by the concept of Zone of Proximal Development created by the scientific concepts of the course concept map and the spontaneous concepts of students (Vygotsky, 1987).

The same concept maps can be also be used as a complete classroom or homework exercise in formulation of the full schema. Students might get a concept map either with empty concept boxes with individual relationships between concepts indicated or with concepts indicated but missing the relational phrases. Their task is then to find the missing concepts in the first case, and missing relationships – in the second case. The third exercise in this series might be the construction of the full concept map from a given list of concepts (Chapter 3.7). Chapter 3.7 gives a full description of the concept map as the assessment tool together with the method of assessing and grading them. The author of this chapter, Haiyue Jin, a Chinese educator from Nanging Normal University educated in Singapore has been specially invited to present the knowledge of a Singapure school of concept mapping initiated by Wong, K.Y. It is interesting to note the responses of students of the experimental cohort to the questions of the Attitude Towards Concept Map questionnaire designed by Jin and administered to all students in the cohort:

1. In which aspect do you think concept mapping is helpful?
2. Can you summarize your major achievement during this concept mapping period?

Students found that concept mapping was beneficial for review, memorization, problem solving and understanding. Aside from its use as an assessment technique, students found, responding to the question 2, that concept mapping helped with conceptual learning. They also reported that they better understood particular concepts when they were placed in a larger picture. In the discussion of results Jin comes to the realization on the basis of these student responses that concept mapping can also serve as the instructional tool during a regular class. Her results confirm our own understanding of concept maps as Stenhouse TR acts (Chapter 1.1).

Chapter 3.8 by the teacher-researcher Roberto Catanuto from a school in Switzerland presents newly formulated technique of using concept maps as medium for the "dialogue" between the teacher and students leading, in a three iterations, to the discovery of a common aspect "common denominator" between student life interests and the mathematical topic or theme of the class curriculum. Catanuto asks a central question, how can educator build an effective connection between the topic of the curriculum and interests and attitudes of students. The same question has been addressed, albeit differently, by several authors in this volume. Prabhu addresess it in Chapter 2.1 via imbedding relevant mathematics in a dramatical scene while Stoppel in Chapter 4.9. who searched, in the calculus context, for a proper modelling approach to fit student inclinations and interests finds mathematical exhibition organized by ministry of education to be the medium creating the necessary connection.

Robert Catanuto addresses it through "communication" between two concept maps. The steps described by the author bear certain analogy to the construction of the bisociative framework, consisting of MH (Mind Home) of a student representing students interests and attitudes, and of the Topic Concept Map (TCM) created by the teacher. Iterated extentions of each of the concept maps are aimed at finding the conceptual connections between the topic to be taught and the student interests, that is the "hidden analogy" introduced in Chapter 1.2.

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3.6. PICTORIAL SCAFFOLDING IN THE SCHEMA CONSTRUCTION OF CONCEPTS

INTRODUCTION

Understanding of concepts is a common theme of study across disciplines. In particular, the study of how concepts are understood is undertaken in mathematics education, mathematics, cognitive psychology, artificial intelligence, pattern recognition, and man-machine cognition. This chapter describes such an interdisciplinary approach that utilizes aspects from several disciplines in the pursuit of increasing understanding of mathematics among students in remedial classes. The concept of fractions has, unfortunately, become a topic, which students, young and not-so-young, love to hate. A large-scale study (PROMYSE, 2006), focusing on assessing students' knowledge and understanding of fractions, consisting of 200,000 students in 60 districts across Ohio and Michigan, reporting on the low passing rates among third through twelfth graders points out some of the likely sources for the troublesome situation:

- Third grade is likely the problem; little important learning in crucial areas of fractions takes place there;
- Large numbers of students are not learning crucial foundations like fraction equivalence and common denominators;
- Little more fractions-related material is learned in high school yet students are being sent on from eighth grade without adequate knowledge.

While the entirety of mathematics, as well as some more specific mathematical topics, including fractions, are intriguing puzzles, their intrigue might remain inaccessible to students given their prior negative experiences with mathematics. The preparatory deficit is very detrimental, as the PROMYSE study demonstrates:

> They are not learning enough to prepare them for the world they will face. They are not getting a chance to do all that they are capable of. In important ways, they are not making the grade even while they make their grades. (PROMYSE, 2006)

In a NSF-ROLE #0126141 study (2002–2006), *Introducing Indivisibles into Calculus Instruction*, concept maps were envisioned as assessment tools, and, this was based on the use of concept maps in science education as a research tool in the

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 203–212.

original work by Novak (1984). They are intended "as a graphical representation of the psychological structure of knowledge within the subject producing the map".

Since then, our attention has been focused on utilizing the power of concept maps, as a way of assisting learners in the construction of the schema of concepts forming a bidirectional route:

- From spontaneous concepts to scientific concepts, and
- From scientific concepts to spontaneous ones.

The effectiveness of the approach was suggested by Vygotsky (1987) who asserts that "…the development of child's spontaneous concepts proceeds upward, and the developing of his scientific concepts downward…" (p. 193) leading to the integration between the two within students' ZPD.

In this article, the use of concept maps is demonstrated in the following ways:

- as a means by which to provide students a snapshot of the big picture;
- as a way for teacher-researchers to design the problems in the instructional sequence; the structure of the concept map representation of the schema outlines the pedagogical design that will be implemented in the course;
- as an environment within which analysis of word meanings can begin and progress toward a shared understanding (Bruner, 1990).

In each case the process of meaning-making is facilitated by the use of the concept maps as pictorial scaffolding. Bruner (1990), in *Acts of Meaning*, supports this approach, and writes that organization of mathematical knowledge, which for students in college-level basic mathematics courses has been difficult, is enhanced via the pictorial scaffolding of the concept maps.

In the middle of the semester, closer to the early part, a very bright student with a disability, upon seeing the concept map shown in Figure 1 below, asked, "This is our syllabus?" After a few moments, in a very disbelieving voice, he asked again, "Are you serious? *This* is our syllabus?" When asked why he would have this question, he replied that he had never imagined a syllabus as a picture. Clichés about pictures are well-known, however, the concept map serves an important role for students. It promotes the idea that the subject of Mathematics is not overwhelming; that it is not a "bunch of stuff to be memorized". It presents Mathematics as a subject that investigates big ideas deeply rooted in us that, after being continuously questioned and scrutinized, have stood the test of time. Students and teacher-researchers together use the concept map syllabus as a tool with which they:

- Navigate their teaching throughout the semester, altering the concept map as needed, should such a need arise;
- Test their developing understanding;
- Communicate with each other in meaningful ways, across a non-intimidating medium, where words are few, and sense of what the other is trying to say can be made easier and quicker.

CONCEPT MAPS AND STUDENT SCHEMA OF THINKING

Concept maps in the TR-NYCity approach are used for the construction of schemas of concepts that either students or teacher-researchers themselves are trying to master. This two-fold use of the concept map illustrates how the TR-NYCity methodology works, from theory to practice and practice to theory—via a bidirectional route.

Vygotsky's (1987) theoretical perspective offers clear solutions. All learners possess intuitive, or "spontaneous," concept knowledge. The focus of the teaching environment is the creation, in the mind of the student, of an understanding of the mathematical, or "scientific," concepts. The latter concepts are referred to and utilized as the working knowledge. Vygotsky's theoretical viewpoint, as it is used within the TR-NYCity methodology, manifests itself as the investigation into the design of the needed scaffolding from the diagnosed spontaneous knowledge to the required scientific knowledge via the *Discovery Approach* (Czarnocha & Prabhu, 2007a).

The discovery-based development of instruction that progresses from spontaneous to scientific knowledge, with clearly articulated conventions and embedded concepts used as a common basis by the mathematical community, allows students to construct new desired concepts mathematically validated at each stage. This process ensures, via the above progression, that the students are not accidentally lead to deep misconceptions that are difficult to clarify and that leave scars that have multiple negative repercussions. An example of such a mathematical concept is that of *irrational numbers*. An important hallmark of irrational numbers was proposed by Dedekind (1901), vis-à-vis *Dedekind cuts*. Dedekind defines this cut in the following way:

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this *division* of all points into two classes, this *severing* of the straight line into two portions.

The concept is difficult, often introduced in higher mathematics and, as such, outside of the scope of most elementary mathematics books. However, many of those same elementary expositions assume Dedekind's construction and one of its important corollaries, neither of which is explicitly addressed or even stated as an assumption. Hence, in such expositions, there appears to be a jump, or a gap, in the reasoning process. A logical step is missed and, when this omission is not clarified, it creates, in the mind of the learner, a navigational difficulty towards the conclusions that follow. However, in the TR-NYCity approach, where the appropriate connections are explicitly articulated, students' difficulties with the navigation of concepts and the understanding of numbers on the real number line are significantly decreased (Prabhu & Czarnocha, 2007).

The concept map in Figure 1 below, that appears on the first page of the Instructional Sequence, *Story of Number* (Prabhu & Czarnocha, 2007), serves the purpose of providing a snapshot of the schema of the concepts addressed in the course. This concept map is repeatedly visited throughout the semester. Thus, students who are typically unaccustomed to seeing any connection between fractions, percent and decimals, now have the capability of constructing and immediately seeing their own created connections through classroom instruction. The underlying proportional reasoning inherent in the interrelated concepts takes on new meaning as a mechanism, or a tool, whose use can be extended to situations where students would have had difficulty knowing how to begin the process of attempting a solution.

Simultaneously, the concept map is revised at each iteration by the teacher-researcher to address encountered difficulties of student, and, more generally,

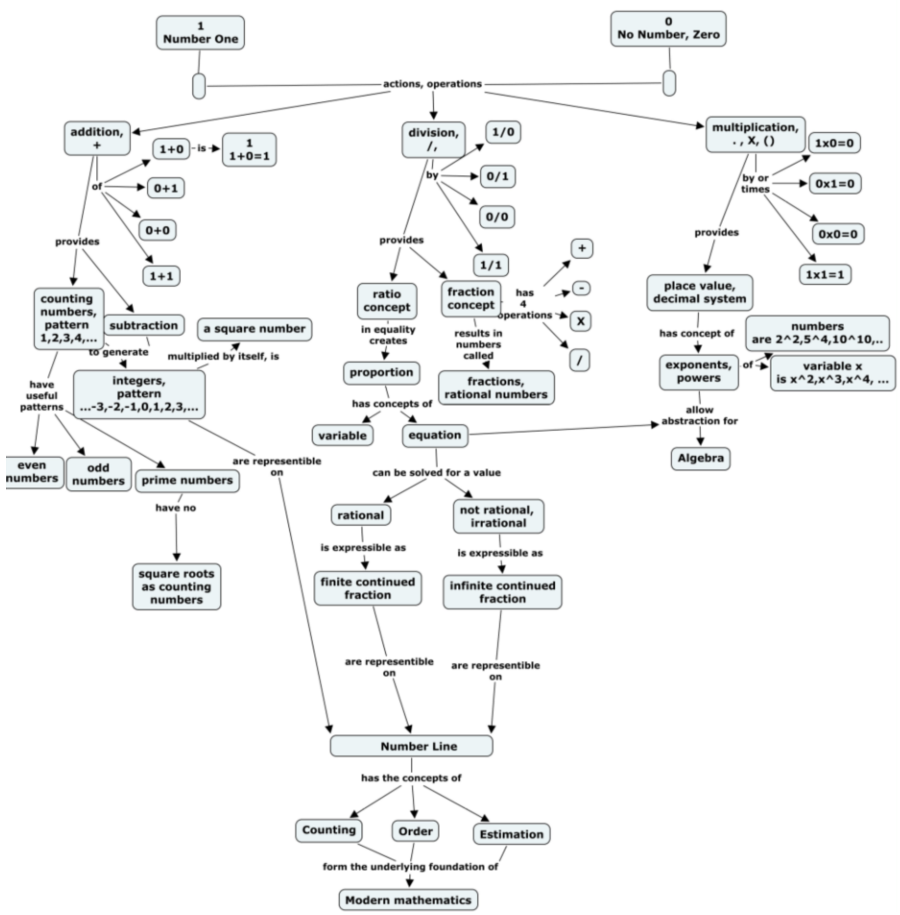

Figure 1. Concept map providing students with a snapshot of the course

by student process of learning. The concept map (Figure 2) represents the next iteration of the concept map, based on different conceptual organization. The comparison of two maps is significant: whereas the first one addresses itself to the structure of arithmetic and is therefore based on the basic concept of operations on whole numbers extended later in the course to operations on fractions, the second concept map is of arithmetic as an entry to algebra based on the concept of a number and ratio – as the comparison of numbers. This structural change of the concept map and related organization of syllabus is dictated by focusing the discussion more on mathematical relationships than operations. Thus each concept map of the particular course corresponds to a different role played by the same concepts.

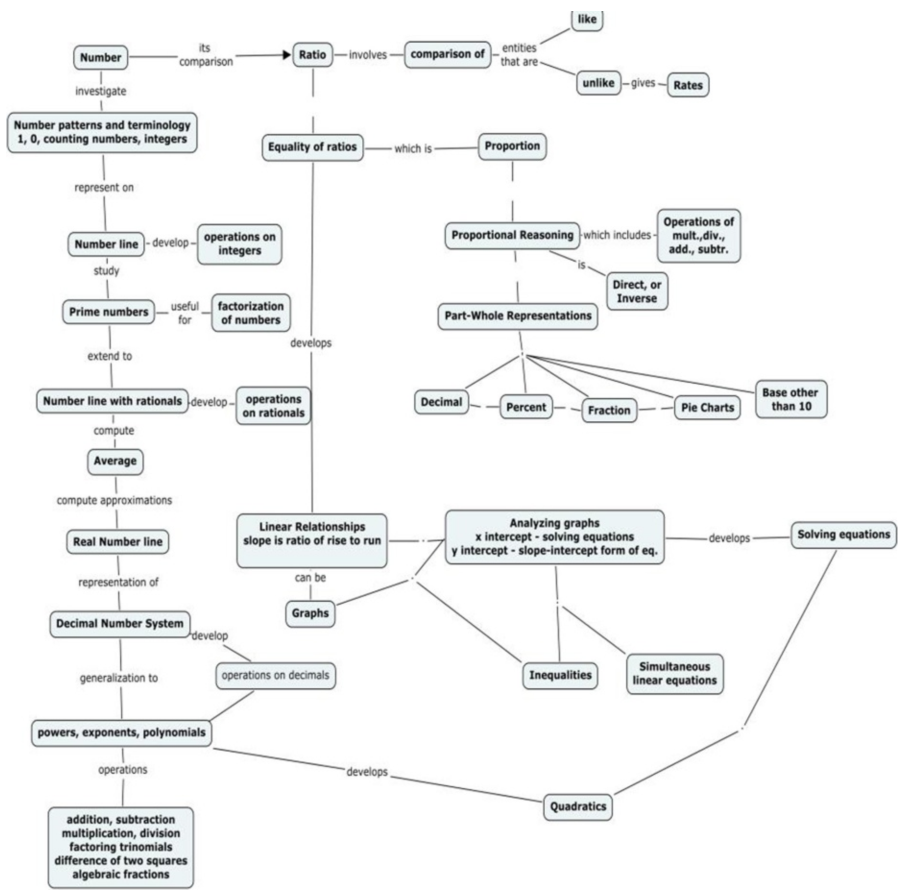

Figure 2. Second iteration of the concept map of the course

The three following concept maps in Figures 3, 4 and 5, are utilized explicitly as schema building tools, directed primarily to students, as conceptual scaffolding and are examples for the development of students' schema of thinking. According to Skemp (1987), schematic learning provides a triple advantage over rote memorization, in that we are:

1. Learning efficiently what we are currently engaged in;
2. Preparing a mental tool for applying the same approach to future learning tasks in the same field;
3. When subsequently using this tool, we are consolidating the earlier content of the schema.

Methods of Algebraization of the Concept of Ratio

In the concept map in Figure 4, the concept of the *ratio* changes its role from a basic concept in the schema of a number, as in Figure 3, to its own subschema that indicates steps of finer level approaches to algebraization of an arithmetic problem, – one of the central themes of the arithmetic/algebra divide that causes many problems for students in both middle/high school and community college settings. This concept map, serving as a guide for the development of the instructional sequence, indicates a potential need for a separate subschema for the concept of equality to be introduced, while, at the same time, re-asserting *ratio*'s central position by reappearing at the end of one of the branches of the algebraization process as the

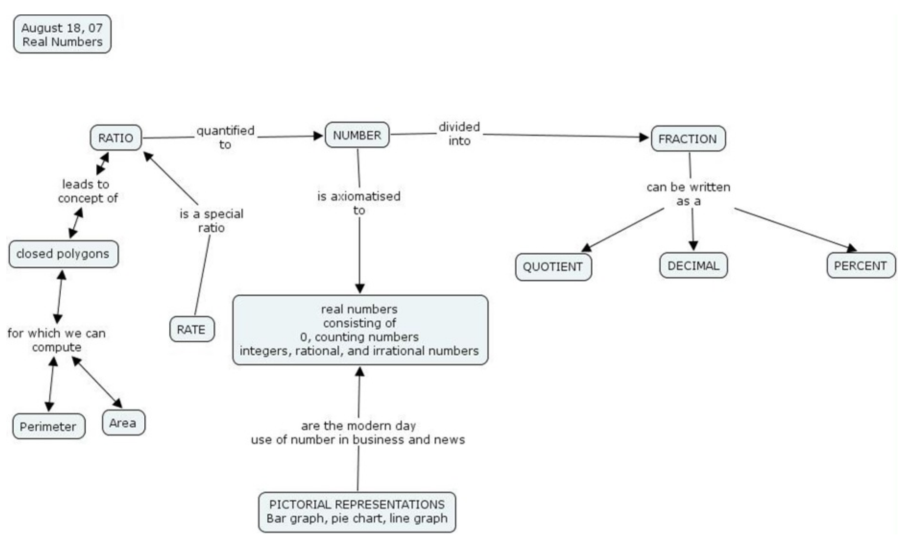

Figure 3. Sketch of the ratio concept within the general schema of a number

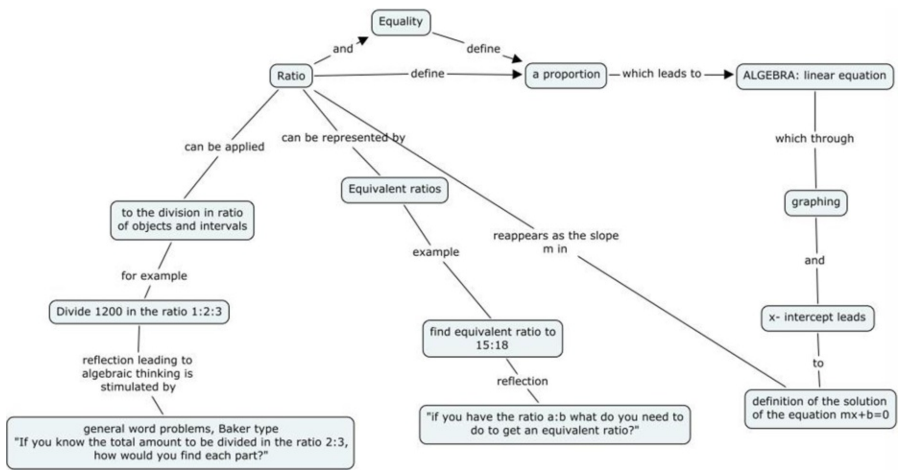

Figure 4. The subschema of the ratio concept

slope of a line (m in $mx + b = 0$) inherent to the linear equation concept. The two left branches emanating from the *ratio* concept provide a good scaffolding support for students' individual development of an algebraic angle of thinking. *Ratio* can be used both as a scientific concept whose structure is given to be explained by students, or as a facilitator of relevant spontaneous concepts that need to be developed before they can be smoothly integrated with the scientific ones. In either case, conclusions can be drawn from students' responses about the process of schema development in an individual student's mind. Understanding of the details of that development can be helpful in refining the instructional approach for the mathematics classroom.

The pictorial nature of the concept map, as illustrated above, greatly assists in accomplishing all three points described by Skemp. Moreover, Skemp outlines additional advantages of schema based learning as follows:

> A schema more than a concept, greatly reduces cognitive strain. Moreover in most mathematical schemas, all the contributory ideas are of very general application in mathematics. Time spent in acquiring them is not only of psychological value (meaning that present and future learning is easier and more lasting), but of mathematical value as well. (Skemp, 1987)

The concept maps in Figures 3, 4 and 5 are thus to be viewed as the seeds of schema formation.

The Place Value Concept

The *place value* concept map below shows the connections among three different aspects of the decimal system notation: (1) cycles of units, tens, hundreds, (2) powers of ten and (3) place value. Colour differentiation among the three concepts enhances the distinction among the concepts themselves while focusing attention on the connections between them wherever the colour is changing. This concept map is designed primarily as an instructional tool for student use around which the classroom instruction is built; it is very concrete and explicit in its content.

Another example of the usefulness of a concept map is demonstrated below, focusing on the process of working with decimal points, another dreaded and misconception-laden topic. The following concept map, in Figure 5 below, was found very useful by students in two consecutive semesters.

The *decimal fraction* concept map is designed in a similar style, showing three different aspects of the decimal fraction: (1) the decimal point, (2) decimal expansion and (3) the decimal alignment in addition and subtraction operations. General multiplicative technique of positioning the decimal point is absent since the understanding of its meaning is on a higher level of abstraction than its motion due to multiplication of the decimal fraction by powers of 10, and is to be addressed with a separate concept map.

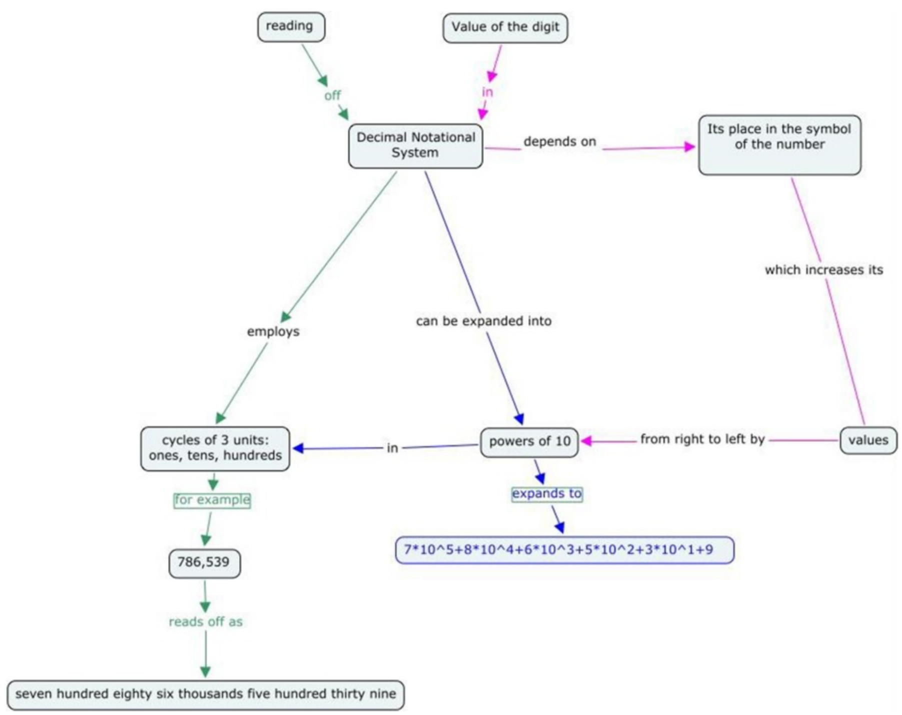

Figure 5. Place value concept map

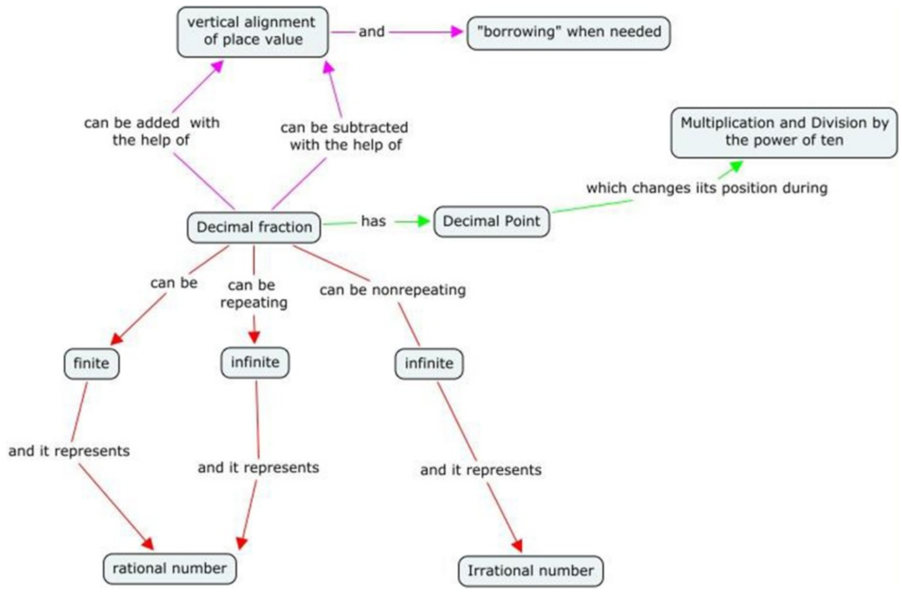

Figure 6. Decimal fraction concept map

SUMMARY

Concept maps, as used in our work, accomplish the following:

- Facilitation of learning, via the integration of theory and teaching practice in the TR-NYCity methodology of teaching-research;
- Eliciting, capturing, archiving, and using "expert" knowledge, via the cyclical creation and refinement of instructional sequences for commonly accepted difficult concepts;
- Planning instruction via the use of the instructional sequence in actual classroom teaching;
- Assessment of "deep" understanding via the periodic standard assessment instruments such as quizzes and tests, in addition to regular questioning in classroom discourse;
- Research planning via the cycles of teaching-research extending over several semesters and colleges.

ACKNOWLEDGEMENTS

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3.7. CONCEPT MAPS

Learning Through Assesment

INTRODUCTION

A *concept map* is a two-dimensional pictorial depiction of knowledge. It has been used extensively as an assessment technique of conceptual understanding, especially in science education. But its value is more than that. Students can learn during concept mapping. This chapter introduces a group of eighth grade students' reflections on their two-month experience with concept mapping. The findings indicate that the students had significant positive attitudes toward the use of concept mapping in mathematics. They generally agreed that concept mapping helped them to better understand the mathematical concepts presented in the class by clarifying the relations with other relevant concepts. Implications for teachers' adoption of concept mapping in school settings are also discussed.

For the last three decades, concept maps have been used quite extensively in educational settings as an effective technique for organizing and presenting information. Its use as an assessment tool has been explored in mathematics education as well (Afamasaga-Fuata'I, 2009; Mansfield & Happs, 1991; Williams, 1994). When concept maps are used as an assessment technique, rather than a static product, they take on a different larger role, which is of great value to mathematics educators and curriculum designers. Student-constructed concept maps achieve exactly that by shifting the focus toward the construction process and its meaning to the students. This chapter is grounded on an experimental study in which concept map construction was used as an assessment of secondary school students' conceptual understanding in mathematics. It focuses on the mapping processes rather than the mapping products.

This chapter explores observations of the activity of students' construction of concept maps and their attitudes toward concept maps as educational tools. The findings suggest that concept mapping can be a worthwhile tool in a teachers' repertoire of assessment of students' learning. Implications for the use of concept maps in classroom settings are also discussed.

LITERATURE REVIEW

A concept map is a graphical representation of knowledge within a particular domain. It is a network consisting of nodes and labelled lines. Nodes correspond

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to key terms that represent concepts. They are usually enclosed in boxes or circles. The relationships among the concepts are indicated by connecting line segments or arrows. The labels on the lines are called *linking words* or *linking phrases*, and indicate how the joined concepts are related. The linked concepts together with labels indicated along the connectors form a meaningful statement. This statement is called a *proposition*. Ruiz-Primo (2004) considered the proposition as the basic unit of meaning in a concept map, and the basic unit used to judge the validity of the conveyed relationship between any two concepts.

From its inception, in the early 1970s, concept mapping was described as an assessment technique to trace students' conceptual development (Novak, 2005). Since then, many efforts have been made toward the exploration of the concept map's use for diagnosing conceptual understanding and detecting conceptual development. Broad theories exist to support its use for capturing the attributes of an individual's knowledge structure. In cognitive psychology, it is generally agreed that human knowledge is stored in memory as information packets, or *schema* (Jonassen, Beissner, & Yacci, 1993). When learning occurs, an individual incorporates new information into his or her schema through assimilation and/or accommodation (Piaget, 1977). The balance between these two processes conveys the idea of how knowledge develops in the mind. Studies into concept formation, concept acquisition, and conceptual learning in mathematics (Sfard, 1991; Skemp, 1986) also support the pattern of relations among mathematical concepts and the equilibration processes. Thus, the concept map, with its specific features (nodes, links, linking phrases, and structure), may be viewed as an explicit representation of individuals' knowledge structure. Once the knowledge structure is represented externally, it can be assessed by others. It is generally recognized that concept maps offer an effective way to track students' learning through structural complexity and quality of propositions (Hasemann & Mansfield, 1995; Pearsall, Skipper, & Mintzes, 1997).

Different activities and applications of concept mapping can be found throughout literature. Ruiz-Primo, Shavelson, Li, and Schultz (2001) provided a systematic description of the mapping formats and characterized the tasks along a continuum, from high-directed to low-directed, according to who chooses the concepts, who links the concepts, who generates the linking phrases, and who structures the concept map. The lower the direction of the concept map-based task, the more opportunities it will have to reveal students' conceptual understanding. However, free-style mapping is too open-ended, and presents difficulties for researchers in developing a reliable scoring system since different students may provide quite different sets of concepts and relationships (Jin, 2007; Ruiz-Primo, Schultz, & Shavelson, 2001). By comparing the limitations and strengths of different mapping tasks, the experimental study reported in this chapter used a low-directed concept mapping format, with a given concept list. The concept list could guide the students to focus on a specific knowledge domain; at the same time, the students were free to make connections among the concepts and label the lines with their own words.

Mansfield and Happs (1989a, 1989b) reported on a project involving a group of eighth grade students. In this project, concept mapping was employed to probe students' understanding before and after a teaching sequence on the topic of parallel lines using a pre-/post-test design. To get the students prepared for the concept mapping task, the researchers provided a brief training. After that, the students were required to construct a map using ten concepts, related to the subject of parallel lines, in a given concept list. Though it was acknowledged that concept mapping was a difficult task for young students and a brief introduction may not be sufficient for them to construct informative concept maps, most of the eighth grade students were able to construct a concept map. The researchers could gather meaningful information about the students' conceptual understanding and conceptual development through the analysis of the concepts and propositions in the student-constructed concept maps. No detailed information about the students' concept mapping process was mentioned in the papers. But the findings encouraged further study with young students.

Afamasaga-Fuata'I (2006, 2009a, 2009b) conducted a series of case studies using concept maps to trace perspective teachers' conceptual knowledge of certain topics, in particular,—matrices and systems, length and volume, and fractions. After a period of learning a topic, the perspective teachers were required to generate a list of concepts for the topic and construct a map showing their understanding of the inter-connectedness between the concepts. After each concept map, they presented it to the class or the researcher(s). Through discussions and negotiations, the perspective teachers further revised and expanded the maps. The progressive maps were collected and then compared by the researcher. Cycles of refinements in the student teachers' concept maps were documented.

SAMPLING AND PROCEDURES

The participants of the experimental study discussed here consisted of a class of 48 eighth grade students (24 female students and 24 male students). They were selected by convenient sampling from a junior middle school in a town in the Jiangsu province, China. The students' mathematics test scores, from exams administered during their seventh and eighth school year, were collected to gauge their performance. The tests, altogether six, were all graded on a 100-point scale. Their scores were highly correlated (the correlation coefficient, *r*[1], ranged from 0.926 to 0.950, $p < 0.001$; Cronbach's α[2] = 0.989), indicating that the tests measured a common feature about the students' mathematics achievement. For each student, the mean of the student's raw test scores was taken as an indicator of his or her school mathematics achievement (SMA).

Since concept maps have not been extensively used in mathematics classrooms in China, the experimental study first trained the participant on the techniques of constructing informative concept maps. The four mathematical topics addressed were *algebraic expressions*, *equations*, *triangles*, and *quadrilaterals*. The student-constructed concept maps were analysed by considering both Novak's traditional methods (Novak & Gowin, 1984), including number of links and proposition score, and methods adopted from the Social Network Analysis,—density and numbers of incoming and outgoing links (Jin & Wong, 2013). No criterion map was produced for scoring the students' maps through comparison since Ruiz-Primo and Shavelson (1996) had found different criterion maps may lead to different conclusions.

Students' attitudes toward concept mapping were collected through a self-designed *Attitude Toward Concept Maps* (ATCM) questionnaire combined with an interview. The questionnaire was designed following Mohamed's (1993) *attitudes toward concept mapping* questionnaire in science. Some items were adopted from Kankkunen' study (2001) in which the students' opinions about concept mapping were gathered through inquiry and interviews. The questionnaire utilized a six-point Likert Scale, with the following categories: Strongly Disagree (SD), Disagree (D), Slightly Disagree (LD), Slightly Agree (LA), Agree (A), and Strongly Agree (SA). By using this six-point Likert Scale, the researcher forced the students to choose an option from either side of the agreement spectrum, not allowing a neutral response. Interviews were conducted with selected students. For the students who were not interviewed, the interview questions were assigned as an open-ended written task at the end of the study. The interview, together with the open-ended written task, made it possible to focus research attention more directly on students' process of concept mapping, and to provide additional information about the students' attitudes toward concept mapping supplementing their responses on the ATCM questionnaire.

FINDINGS

Observations of Participants' Concept Mapping

This section includes observations made during both the student training and concept mapping stages.

During the training, the students, first, treated concept mapping as a simple drawing task instead of a test of their mathematical understanding. They provided only brief and general linking phrases in their maps. For example, one of the students initially constructed a proposition that in written form reads, "A triangle may be an isosceles triangle". When asked to explain what she meant by *may be*, the student said, "when it has two equal sides, it is an isosceles triangle; when it has no equal sides, it is not an isosceles triangle." In general, when prompted, most of the students could add more links to their concept maps. This finding suggested that the students needed further training before they can construct meaningful concept maps that can be used to represent their levels of understanding. Hence, more detailed training (Jin & Wong, 2010) was provided before the concept mapping tests.

After the training, four concept mapping tests were administered, consisting of free-style mapping using ten or eleven given concepts. The tests were given on different days. Students were given 30 minutes to complete each test. It was noticed that, after the students worked on the mapping tasks for about 10 minutes, some of them put the test paper aside, indicating that they were done; while some others seemed to have been struggling with the possible connections. The researcher checked several students' maps, privately, during the test time, and inquired as to what their specific difficulties with the mapping test were. Most of them said they could not find more connections among the given concepts, but, at the same time, were unsure whether they had included all the expected connections. During one of such informal exchanges, the researcher provided a prompt: "How about exponents and like terms? Is there any relation between them?" The student thought for a while, then seemed suddenly enlightened, and started to add the link. As a result, for all four concept mapping tasks, after the students had worked on constructing their drafts of concept maps for about 25 minutes, the researcher provided four or five prompts to the class to assist with the task. Given the prompts, most students added information to their concept maps.

Examples of Well-Constructed and Poorly-Constructed Concept Maps

The correlation coefficients between the students' school mathematics achievements (SMA) and their proposition scores for the four topics ranged from 0.709 to 0.753, with $p < 0.01$, indicating that the students' ability to build informative concept maps did, to a certain extent, reflect their mathematics achievement. Focusing on the topic of triangles, Figures 1 and 2 are examples of a well-constructed concept map and a poorly-constructed concept map, respectively. Carefully comparing the two, one may gain more insight into the usefulness of student concept mapping in addressing their conceptual understanding.

Figure 1 shows that the student possessed a comprehensive understanding of the concepts. All of the eleven given concepts were involved in the map. The most inclusive concept, *triangle*, was placed at the centre with special types of triangles around it. The links from triangle to the six special types of triangles were labelled with definition-based linking phrases. The relationship between *acute-angled triangle* and *equilateral triangle* was indicated. The student noted out that an acute-angled triangle whose angles are all equal to 60° is an equilateral triangle. The relations between the other triangles were not shown in the concept map, indicating that the student may find the relationship between acute-angled triangle and equilateral triangle more obvious than the others. *Angle* was placed close to *acute-angled triangle*, *right-angled triangle,* and *obtuse-angled triangle*, since these different types of triangles are categorized using their angles. Isosceles triangle and equilateral triangle both have special properties of *median* and *midline*. This might be the reason that the student placed median and midline right below the two triangles. *Symmetry axis* was placed near *isosceles triangle*. This is reasonable since, among the special types of triangles, isosceles triangle (considering an equilateral triangle as a special case of an isosceles triangle) is the only one that always has one axis of symmetry. The student constructed 25 links for the 11 given concepts. The

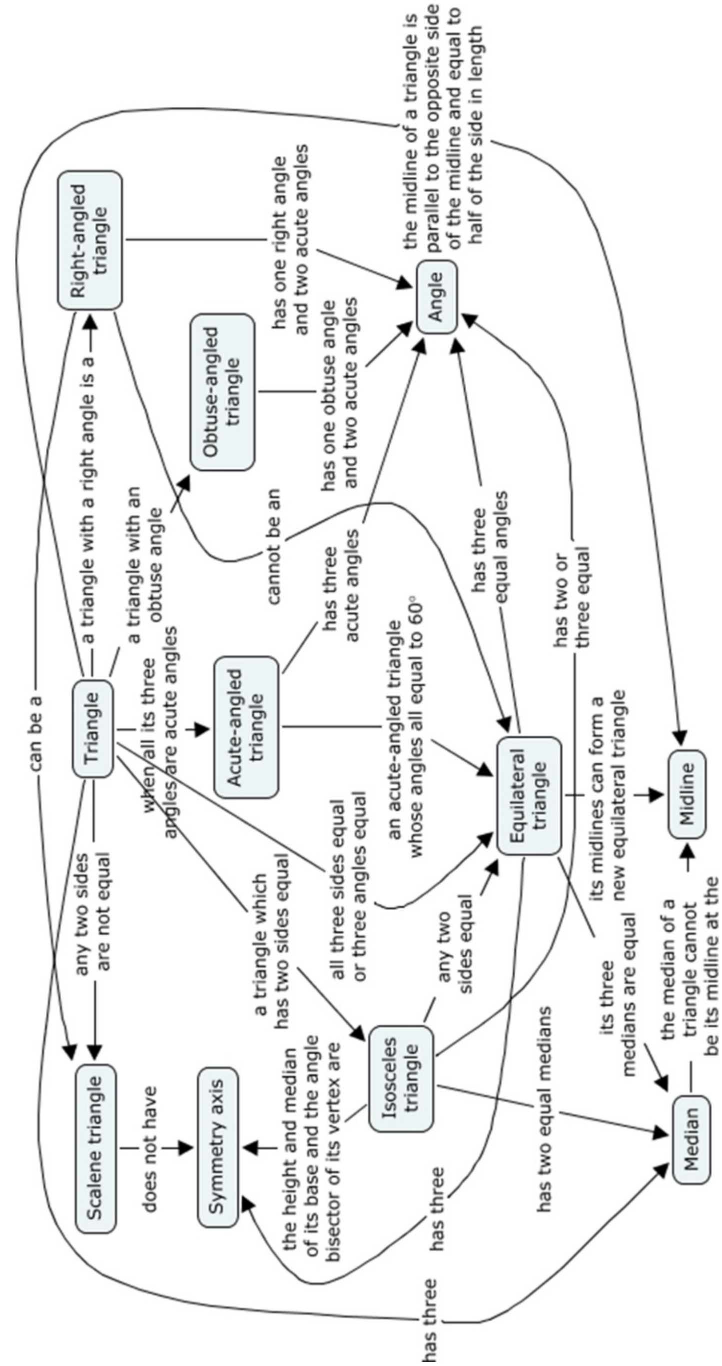

Figure 1. An example of a well-constructed concept map for triangle concepts (translated from Chinese)

number of links constructed is greater than most of the other students' concept maps. The propositions in the concept map are all correct, and some even indicated deeper insights into the relationships. Therefore, each proposition was assigned a score of 2; the entire map earned a proposition score of 2 × 25 = 50.

Figure 2 shows an example of a poorly-constructed concept map. Although it included all 11 given concepts, much fewer links were constructed in this map relative to the one in Figure 1.

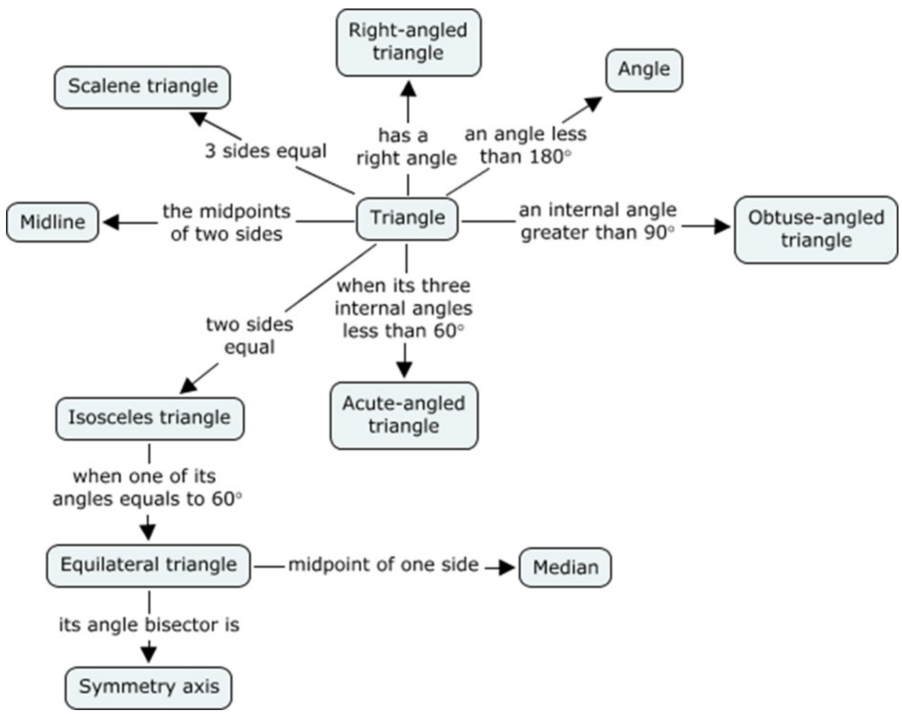

Figure 2. An example of a poorly-constructed concept map for triangle concepts (translated from Chinese)

Triangle was placed at the centre of the concept map. Seven concepts were placed around it. The links between *triangle* and the seven concepts were all labelled with definition-based linking phrases. However, no connections among the seven concepts was shown in the map. *Equilateral triangle* was placed below *isosceles triangle* since, as recognized by the student, an equilateral triangle is an isosceles triangle with a 60° angle. *Median* and *symmetry axis* were, most likely, the last two concepts added to the map. On the one hand, there seemed no space in the student's map for the two concepts to connect directly to *triangle*. On the other hand, the relationships of *median* and *symmetry axis* with equilateral triangle seemed to be the most obvious to the student. Thus, he simply linked the two concepts with the equilateral triangle. The student did not seem to have attempted to construct further connections among *median* and *symmetry axis* and other concepts in the map. It appears that the student was satisfied with his map since no concept was isolated. As a result, only ten links were constructed, each with a varying proposition score. This concept map earned a proposition score of 0 + 2 + 1 + 1 + 0 + 2 + 1 + 2 + 1 + 2 = 12. This is far less than the obtained mean class proposition score of 25.75. The linking phrases used indicate that the student does possess some understanding of the relationships; however, he did not express his ideas clearly. For example, he linked *triangle* to *midline* with the linking phrase "the midpoints of two sides." He may know that a midline is a segment connecting the midpoints of two sides of a triangle, but did not explicitly state that a midline is a *segment* in the linking phrase. In practice, some instructors may wish to reduce this strict requirement of *detailed* linking phrases.

ATCM Questionnaire

The questionnaire covered the following five aspects:

- Ease of constructing concept maps (*ease*);
- Confidence with concept mapping (*confidence*);
- Enjoyment of concept mapping (*enjoyment*);
- Usefulness of concept maps (*usefulness*), and, finally,
- Preference of using concept map for further study (*preference*).

The Cronbach's alphas of the five aspects, except for *ease,* were higher than 0.70, showing that these aspects have acceptable internal consistency. The *ease*-related items had low internal consistency (Cronbach's α = 0.339), possibly because they did not belong conceptually to the same aspect. The mean scores of the other aspects ranged from 4.16 to 5.06, showing different levels of students' agreement. The results are shown in Table 1 where the different aspects are shown in a descending order according to their means.

Table 1. Cronbach's α, means and standard deviations (SDs) of the five aspects of the Attitudes Toward Concept Mapping (ATCM) Questionnaire

Aspects	*Cronbach's α*	*Mean*	*S.D.*
Usefulness	0.724	5.06	0.20
Preference	0.867	4.37	0.20
Enjoyment	0.830	4.22	0.35
Confidence	0.724	4.16	0.88
Ease	0.339	/	/

The students shared a significantly positive view toward the *usefulness* of concept maps. All 48 students agreed that using concept maps one can clearly describe relationships among mathematical concepts. Over 90% of them indicated that constructing concept maps is, indeed, an effective assessment technique. They generally thought that concept maps appropriately reflect their understanding of mathematical ideas, and considered the maps a fair tool for gauging their conceptual achievement. Although it was found that with prompts students could add more propositions to their concept maps, from the students' perspective, concept mapping had its value as an assessment technique. The students readily admitted that they could benefit from concept mapping. For example, more than 90% of the students concurred that concept mapping is helpful for understanding mathematics concepts, and that they could see more clearly how concepts are related after building a concept map. Eight students *strongly agreed* that they are able to come up with new ideas when engaged in concept mapping. These eight students include both high and low achieving students, suggesting that students of different academic levels can benefit from concept mapping.

The *preference* aspect measures the students' preference of using concept maps in their further study of mathematics. In general, the students' responses to this aspect were positive. Majority of the students indicated that they would like to use concept maps in their further study, and hope that their teachers can use concept maps to teach mathematical concepts. However, more than 40% of the students indicated *slight* agreement only. There are two possible reasons for this hesitation. First, even though the students seemed to have a desire to use concept maps, as was evident from their appreciation of the maps' usefulness, they were uncertain whether they would continue to use this technique by themselves. Secondly, as was discovered through the informal talks, students generally viewed problem-solving as the most important issue in learning mathematics. They did not think that concept mapping is helpful for problem-solving. Among the students who indicated disagreement to the items related to *preference*, most were female students. For example, among the six students who showed disagreement to the item "I'd like to use concept map in mathematics," five were female students. This finding suggests that male students may have higher preference for further use of concept maps. The gender difference needs to be studied further.

The students indicated moderate *enjoyment* toward concept mapping. They showed the strongest disagreement on the item "I find concept mapping boring". Only 7 out of the 48 students slightly agreed with this item. More than 80% of the students agreed that concept mapping is interesting. About three quarters agreed they liked spending time on concept mapping. The students' responses to these items consistently suggest that they did enjoy concept mapping.

With respect to the *confidence* level, students seemed relatively insecure in their ability to construct and utilize concept maps. However, they admitted that given more practice they would be able to assemble better concept maps.

The ***ease*** aspect actually dealt with levels of easiness, anxiety, and time-consumption issues. Most of the students admitted that concept mapping is challenging, but more than half of them denied that they felt anxious when they were asked to construct a concept map. This is likely due to the students' awareness that their concept map scores would not be used to judge their performance in school. The students held different views on whether concept mapping is an unusually time-consuming activity. It was noted that among the eight students who chose *strongly disagree* to this item, six were male students; while, among the seven students who chose *agree* or *strongly agree* to this item, only two were male students. It appears that the female students in this study required more time to construct concept maps compared to male students. This study did not aim to study gender differences, but such differences might prove to be a fruitful area for future research; for example, to further investigate Edwards' (1993) finding that female students produced significantly more complex concept maps than male students.

In summary, the above findings about students' attitudes toward concept mapping encourage further exploration of the utility of concept maps as an assessment technique of conceptual understanding, as well as a learning strategy in mathematics.

Interview and Open-Ended Written Task

Twelve students were selected for interviews, based on their school mathematics achievement level. Six students were high-achieving and the other six were low-achieving students. The interviews were conducted one-on-one. Each interview took about 15 to 30 minutes and was audio-taped. The interview questions were provided to the remaining 36 students as an open-ended written task at the end of the study. They were given 30 minutes to write down their answers. The interviews and the open-ended written task together provided rich information about the students' thoughts and opinions about the use of concept maps as an assessment technique, a learning strategy or, even, a teaching method. Below are two of the most revealing questions posed to the students:

Question 1: *In which aspect(s) do you think concept mapping is helpful?*

- (*A*) *help with review*,
- (*B*) *help with memorization*,
- (*C*) *help with problem-solving*,
- (*D*) *support understanding of concepts*,
- (*E*) *others (please specify), or*
- (*F*) *no use at all*

Please indicate your answer by selecting one or more of the choices above, and provide your explanations below.

Question 2: *Can you summarize your major achievements during this concept mapping period?*

Table 2 below summarizes the results of students' responses to Question 1. The percentages are shown in parentheses below each choice, and typical corresponding explanations (translated from Chinese) are also provided.

As is demonstrated in Table 2, students agreed that concept mapping benefited their learning of mathematics in different ways. The students' responses to Question 2 were mostly positive. The positive statements reflected (A), (B), (C), and (D) choices of Question 1. Students indicated that concept mapping would be beneficial for review, memorization, problem-solving and understanding. Questions 1 may have influenced students' answers to Question 2; this is an unintended limitation of the design of the questions. Below are some examples of students' responses to Question 2 (translated from Chinese):

- *With concept mapping, I know I have not fully understand the concepts; concept mapping helps me to better understand the concepts learned; it is also helpful for the learning of new concepts.*
- *With concept mapping, it becomes easier to remember the concepts. It is better than learning the concepts individually.*
- *Now I know mathematical concepts are related and I know different connections among concepts; concept map helps me to organize the concepts I have learned.*
- *At first, I find concept mapping complex and it has nothing to do with problem solving. After I get familiar with it and did concept mapping for several times, I find problem solving is actually quite easy. When I encounter a complex problem which I don't know where to start, I construct a concept map in mind, go from one concept to another, then to other concepts, the problem becomes easier. I will continue to use concept map for my further study.*
- *Concept mapping is a kind of training on divergent thinking. When I see a concept, I am able to link it to other concepts. With such experience, I know to look at a problem from different perspectives.*
- *Concept mapping makes mathematics learning interesting; the concepts are not abstract and boring terms anymore.*
- *With concept mapping, I can easily find the connections between concepts. It helps me to review and consolidate the knowledge I've learned. It helps me to understand concepts and digest them. I think concept map can not only used for mathematics but other subjects as well; for example, some difficult and abstract concepts in physics. By doing so, I can easily understand the concepts. I want to construct a huge concept map in the future. It can include concepts in primary school, secondary school, and high school; while for secondary school, it covers secondary 1, 2, and 3, the same for primary school and high school. For each grade, I will put all the concepts into a concept map and put it in my space. So*

Table 2. Summary of students' responses to interview Question 1

Choice	*Explanations*
(*A*) Help with review (100%)	*Concept mapping helps me review the concepts that I have learned before.*
	Concept map is more concise than literal description.
	I used to review concepts one by one; I seldom think about their connections.
	I seldom review mathematical concepts because that is boring; but concept mapping is interesting and it does help me to recall the relevant knowledge.
	I have forgotten some of the concepts; with concept mapping, I can review and recall the knowledge.
	With concept map, I do not need to find the concepts by checking textbooks.
	With concept mapping, I can review unfamiliar concepts.
(*B*) Help with memorizing (91.7%)	*Concept map helps me to remember the concepts that I have learned.*
	Concept map helps me to remember the connections between the concepts.
	Concept mapping can help to remember new concepts.
	Concept map can help me remember easily-forgotten/unfamiliar concepts through their connections with familiar concepts.
	When connected together, we can remember the concepts better and easier.
	With detailed linking phrases, one can easily remember the concepts.
	Concept map is like a picture. It's easy to remember and it expresses the relationships clearly.
	Concept mapping impresses me by building connections by myself.

(*C*) Help with problem-solving (41.7%)	*Concept map helps me to understand/think more about the connections between concepts; with the connections, I can solve problems more quickly.*
	Once I understand the concepts in a problem, I can solve the problem easily.
	With the given conditions in the problem, I can draw a draft of concept map to help me think more about the relevant knowledge.
	Concept mapping can train diverging thinking which is good for problem solving.
	For some problems, especially geometric ones, if you don't know how to solve it, you may try concept map to find connections between concepts; you may then get the solution.
	With concept map, we can look at a problem from different perspectives; thus, can come up with different methods to solve the problem. As a result, we can solve problems more quickly.
(*D*) Support understanding of concepts (77.1%)	*Concept map helps me understand the concepts; with concept map, I can better understand the concepts through their connections with other concepts.*
	The process of concept mapping is also a process of understanding concepts.
	I find that concept map/concept mapping makes it easier for me to understand the concepts.
	I used to learn concepts mechanically and paid no attention to the connections between concepts. When I learn some new concepts, I could not understand them immediately. But with the help of concept map, I can understand them.
(*E*) Other (2.1%)	*Concept mapping arouses my motivation in learning mathematics.*
(*F*) No use (0%)	None

others can use it for study, and they can then learn mathematics, chemistry, and physics better.

Three students provided neutral to negative responses:

- *Concept mapping is just so-so. Though it did not help me too much in learning mathematics, it did help me a little bit.*
- *I do not think concept mapping is helpful for solving problems. When I try to solve a problem, the relevant concepts will come to my mind naturally. I will then solve the problem step by step. I will not think about other concepts.*
- *I don't think I will draw concept maps in most cases because it is too troublesome and time-consuming.*

The student who provided the second negative response to Question 2 above was one of the top students in the class. He had his own learning method that had proved to be effective for him, since he earned high scores on mathematics tests. Thus, he was reluctant to adopt this new technique. The other two students who gave negative responses were two academically weak students. Their mathematics teacher, who was also the teacher in charge of the class, commented that these two students were not motivated to study in general. Perhaps, this lack of motivation carried over to their opinion of the concept mapping activity, and explains their reluctance to make the effort to incorporate concept mapping into their studies.

DISCUSSION

Although, in this study, concept maps were used primarily as an assessment technique, the findings, especially the information about the students' attitudes toward concept mapping and their perceptions of the usefulness of concept maps, suggest important applications of concept maps as educational tools.

Training for concept mapping. The observations during the training and the concept mapping stages showed that, when the students were new to concept mapping, they generally had difficulty understanding the purpose of concept mapping and seemed unable to include as much information as they knew about the concepts; they needed time to get familiar with this technique. Hence, training on concept mapping solely for assessment purpose is not economical and practical for school education. Existing studies have shown that concept maps can be an effective teaching strategy. For example, in 1993, Horton and her colleagues published a meta-analysis of research concerning the effectiveness of concept mapping as an instructional tool for science teaching (Horton, McConney, Gallo, Woods, Senn, & Hamelin, 1993). The results of this meta-analysis showed that concept mapping had positive effects on both students' achievement and attitudes. Therefore, it is suggested that teachers start by introducing concept maps in classroom teaching, for example, for lesson display and information integration. During this introductory time students can gradually familiarize themselves with the attributes of concept maps and the general concept mapping process. After that, teachers can expand the function of concept mapping as an assessment tool or a learning strategy.

Students' experiences with concept mapping. After working with concept maps for two months, students generally appreciated its use in mathematics. Aside from its use as an assessment technique, over three quarters of the students indicated that concept mapping helped them with conceptual learning. For example, one of the concept mapping tests required the students to construct a map with concepts related to *equations* and *functions*. *Equations* and *functions* were taught in separate chapters in secondary schools in China but their connections were not made explicit in the students' textbooks. Most students did not rigorously reflect on the relations between them. On the concept mapping test, they were forced to think hard about the connections between the given concepts. The students reported that they better understood the concepts when they were placed within a bigger picture. During the interviews and on the written task, students also pointed out that concept maps helped them remember different mathematical concepts by clarifying their relationships to each other.

Concept mapping provides a valuable opportunity for learning through assessment, with its many unique virtues. It is a promising instructional tool for teachers who wish to elevate their students' conceptual learning and appreciation of mathematics.

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3.8. THE METHOD LEARNING ROUTES

INTRODUCTION

After the preschool age, a person must usually enter a long period of directed institutional curriculum of learning, which can span twelve years or more. A child gets into this school stage with a large range of ideas, attitudes and established ways of approaching the world, activities, peers, family and many other everyday life aspects. This background heavily affects the way he or she thinks and learns. In many situations, this intellectual heritage is almost unmodifiable (Gardner, 1993). Sadly, the encounter between this personal background and the rigor of the curriculum creates situations where the student can be classified as lazy or affected by an attention disorder. In extreme situation, he might even no longer be able to attend school at all (Levine, 2003). Many solutions addressing this problem exist (Levine, 2006), although most of them remain unexploited.

This article does not address the topic of what should be taught inside educational institutions, but it proposes a way of how a young mind may be invited to get in touch with curriculum topics chosen by the school. The proposed method mainly addresses situations where learning may be slow and difficult. The examples shown at the end of the paper deal specifically with mathematics.

As a side note, there is a large amount of scientific evidence which asserts that learning evolves better when the student is actively interested and involved in the process. The reader is referred to the famous works of Bruner (1960, 1966) and Montessori (1986) and the bibliographies therein.

The rest of this work is organized as follows: the following section explains the method proposed, its main steps and implementation. The next section shows *concept maps* collected by the author from a group of his students, along with a series of iterations through which such concept maps, constructed by both the teacher and the students, are progressively expanded. This process and its outcomes are then analysed. The final section draws certain conclusions and proposes possible directions for future research.

THE METHOD

A general learning situation can be summarized in this way: a student is about to face a new topic (where the topic may be theoretical, practical, or an ability, a mind habit or something else). Of course, this topic is not already inside the mind of the

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 229–243.

student, or it may exist there to a somewhat larger or smaller extent and deepness. In any case, an improvement is advisable, for a number or reasons, depending on the specific situation.

Let's place some starting focus questions to tackle this issue:

a. How far is the student's mind from the topic? And how can we better describe this *distance*?
b. As stated above, if we suppose the topic must be taught, how can an educator build an effective connection between the topic and the student? And, very importantly, how can we build a *practically usable* connection, not only a theoretical one? By *practically usable* we mean a learning path which is seen viable by both the educator and the student, and, mainly, by the student. Education at its core should be regarded as meaningful *especially* by the student.

Step 1: The Internal Home of a Learner

A student always starts every learning attempt moving from his/her own internal home: as addressed in a variety of research studies (Gardner, 1993; Levine, 2002) and bibliographies therein. This is what we call *internal home*,—the entire set of mind habits, memories, social and cultural heritage or traditions, personal theories about the world, other people in the world and family, things and fundamental philosophical issues a person always carries inside hidden in all of their actions. Moreover, not only do the starting conditions of students differ that way but also the learning approaches they employ to acquire new knowledge. This is famously summarized in the ground-breaking work of Gardner (1981) and, more recently, in his work of 2006. So, how can we represent this internal universe of a young student in a practical way? Usually, teachers have little time to pay attention to this fundamental problem, even when they are really aware of its importance.

Our first proposal here is that the teacher (or the educational staff as a whole) should try to represent the internal intellectual environment of the student using *concept maps* (Novak, 2010). *Concept maps* are a graph-like tool which has very good characteristics to allow for both a deep understanding of the subject studied (be it a person, an idea or whatever else) and a gentle learning curve to begin with.

Briefly, a concept map has two key features (Novak, 2006): a central concept to be analysed, and a focus question which should drive the attention both of the concept map builder and the concept map reader.

So, how to construct the concept map of the student internal home?

A widely used tool in educational settings is useful for this goal. CmapTools collects many of the features needed to depict the internal home of a learner in an easy way to be implemented. Here we propose two ways this can be accomplished, but of course many others may arise in the future:

(a) The teacher explicitly asks each student to realize a concept map which represents him or herself: this is a key step, and can also arrive after a short initial training of students to get acquainted with the tool (very easy to use, honestly). The student should be invited to place him/herself as the central concept of the map. The map can follow these sample focus questions:

- What would you like to spend most of your time on – say weekly – if you could decide your weekly planning entirely on your own? (this way of placing questions reflects the assumption, well known in pedagogy – and in everyday life – that humans learn topics better if they are more interested in them);
- Try to describe yourself showing attitudes, qualities, family, friends, and hobbies and so on; try to describe your projects about your desired future job, what kind of people you would like to be surrounded by or work with or play with or live with, etc.

Of course, these questions may heavily vary with respect to the age and other personal conditions of the students. A similar approach to the one here is depicted in Barringer (2010) where the authors suggest the following questions:

1. If you were to design your desired day, what would you be doing?
2. What parts of school are easiest for you and why?
3. What are your affinities – those things you love to do or learn about?

(b) The teacher devotes a certain amount of time to building concept maps of the students' internal homes. We think this way is slower and, moreover, has a fundamental lack: it describes the students' internal world the way the teacher sees it and not the way the students do. This is very prone to misunderstandings and might lead to failures. We understand, however, that time constraints in today educational settings are strict and so this second way can be more viable than the former.

After a while, the internal home of the student is described by a concept map. Taking advantage of the graph-like nature of concept maps, we suggest to represent them via the usual graph notation: $MH = \{V, E\}$ where *MH* stands for the *Mind's Home* of the student, *V* is the set of all the concepts the students placed in the map, and *E* is the set of all the links the student drew to connect concepts to one another. CmapTools offers an appropriate range of built-in functions which allow to group together and list concepts and linking phrases, applying the prescribed notation.

Before going any further, the teacher has to sketch a concept map of the topic he/she wants to teach, as large and detailed as possible. This will be addressed as the *Topic Concept Map* (TCM).

Step 2: The Route to the Topic, or the Learning Route

After the first step, the staff has the student's mind concept map. In a sense, the teacher has a deeper *knowledge* of the student, and this knowledge is easily and conceptually schematized.

Now what? Let's address to a frequent situation, and, also, the least desirable one: the topic to be taught is not inside the *MH* graph of the learner. Otherwise stated, in a less formal fashion, the learner does not *see* it in his/her practical and theoretical world of interests. Now, irrespective of the difficulty of the situation, the concept to be taught is, in the end, a *concept* on its own. So let's place it in the *MH* graph as an isolated node, that is, one that has no link with any other concepts sketched by the student. The second key step of the method must now be undertaken.

We recall a somehow strict similarity to the backward problem solving process described in Polya (1981): you have a mathematical problem carrying some data and you should get to a solution, which is not directly linked to those data (where directly means with just one logical step). Otherwise stated, you have to build a multi-step connection between the desired result and the actual data you have. Our analogy with the educational problem becomes clearer now: the student *MH* cannot actually be one-step connected with the topic but it might be with a multi-step process. How can these subsequent steps be built? In order to solve this issue we define here the second key concept of our method.

Taking another similarity with a well-known concept in psychology, the *Zone of Proximal Development* (ZPD) by L. Vygotsky (1986) we use here what we call the *Node of Proximal Learning* (NPL). Taking whichever of the nodes (= concepts) written in the *MH*, the student may be now asked to list topics which he/she is interested in learning, connected to those concepts inserted by him/herself.

Of course, new concepts may appropriately be proposed by the teacher him/herself to students, to check their reactions. This will eventually lead to a set *H* made up of lists of "desired" topics. The teacher now has to choose the most appropriate topics from those lists, add them to the *MH* map and see if any of those topics associate to the topic to be taught (or, stated in set like language, if the intersection between the *TCM* and the *MH* maps is now not zero).

Two situations may arise:

a. The intersection is not zero; hence the teacher and the student have built together a meaningful multistep connection between *MH* and *TCM*. The educational process begins with the very first step where the student and the teacher now agree upon;
b. The intersection is still zero, hence two steps have to be undertaken: the teacher tries to reformulate the *TCM* taking into account the $\{MH + H\}$ set he/she now has. On the other hand, the student tries to do the same with his/her $MH + H$ concept map. After that, the teacher searches again for possible intersections between $\{MH + H\}_2$ and TCM_2;
c. (a) The process goes back to (a) again, until a non-zero intersection arises. After a non-zero intersection is found between $\{MH + H\}$ and TCM_i at the *ith* step of the process, a multistep connection between the original *MH* and *TCM* is done. We would like to stress here that this connection is now meaningful, simply by construction, since it has been built both by the teacher and the student, who has enlarged his or her *MH* step-by-step until getting in touch with the borders of the *TCM*, reshaped by the teacher. The topic to be taught should make now sense to him/her, since it belongs to this expanded $\{MH + H\}$.

IMPLEMENTATION AND EXPERIMENTS

We propose here a summarizing chart where the method can be viewed in its main steps:

1. Assessment of the starting internal condition of the student/learner: this is accomplished via the generation of a concept map of the *MH* (Mind Home) of the student where his/her main personal characteristics are schematically depicted; the concept map realization is better left up to the student, since it will mirror his/her way of thinking;
2. Making the *TCM* (Topic Concept Map) of the topic to be taught by the teacher/educator: this TCM is better assigned to the teacher/educator, since it mirrors the way he/she sees the topic, and so it will eventually highlight important differences between students' and teachers' ways of thinking;
3. Searching of the conceptual route between {*MH* + *H*} and *TCM* via subsequent linking nodes (Nodes of Proximal Learning), eventually building a meaningful learning route.

Data Collected

The author worked with a group of 16 students, divided in two different classes, 9 in the first year, and 7 in the second. They were between 16 and 18 years old, in a Swiss high school, where the author is currently a teacher of Mathematics and Physics.

Initially, the students were asked to produce an original *MH*-map, about their personal interests, while the teacher decided to write a *TCM*-map about *lines*. This is a very important and wide-spanning topic across both the Math and Physics curriculum taught by the teacher. He didn't know in advance what the students' maps would have been. Hence, he decided to keep the *TCM*-map as general as possible, in order to suitably expand it to reach a connection with the several *MH*-maps.

All of the students in the first year were conveniently trained to using the IHMC's software called CmapTools, while the students of the second year were free to choose between this software and other less professional ones, because of hardware constraints. Not all of them could use laptops, but all of them use tablets daily.

The author decided that this activity should have been totally voluntary for students, meaning that they might opt out at any time they wished. This was regarded as an extremely important option, because the activity was dealing with maps about personal interests, and this was an effort to better connect students' personal lives and schools as institutions.

In the following figures, the author chose to report in detail only the entire iteration process of two students. This has been decided because their maps were optimally showing the whole learning process, and the two students actively collaborated with the author in the experiment.

Iteration 1

The following picture represents the *TCM* from the teacher.

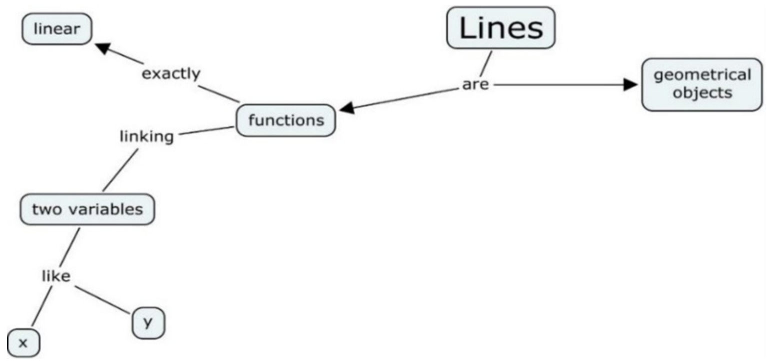


Figure 1. The lines TCM

The following pictures represent the initial MHs from the two chosen students.

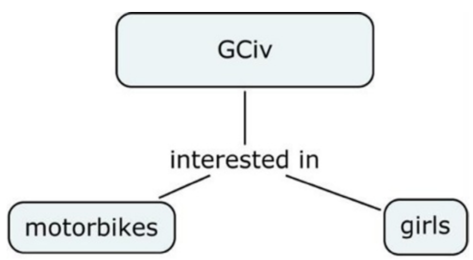


Figure 2. Student A's MH map

This map describes the interest of this student (A) for motorbikes and girls. The map is simple and plain. The author proposed the student to regard the node about motorbikes as his Node of Proximal Development (NPL), as described earlier.

The next map describes the interest of the student (B) for photography, tennis, travelling and electronics. In this case, the chosen NPL was the node about electronics (see Figure 3).

Unfortunately, the second year students decided not to carry on the activity after the first iteration step because of the proximity of the final examinations. The author chose not to force them to continue.

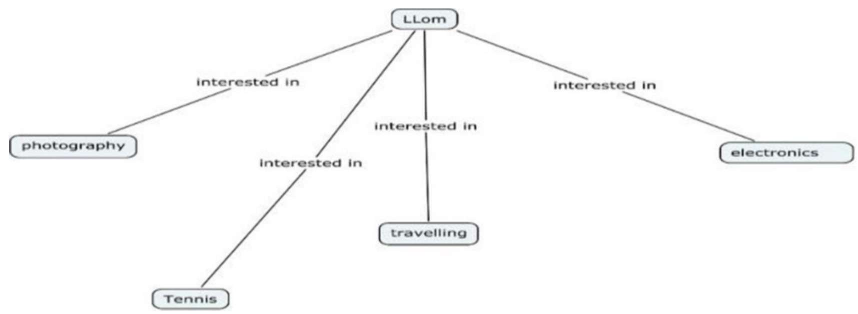
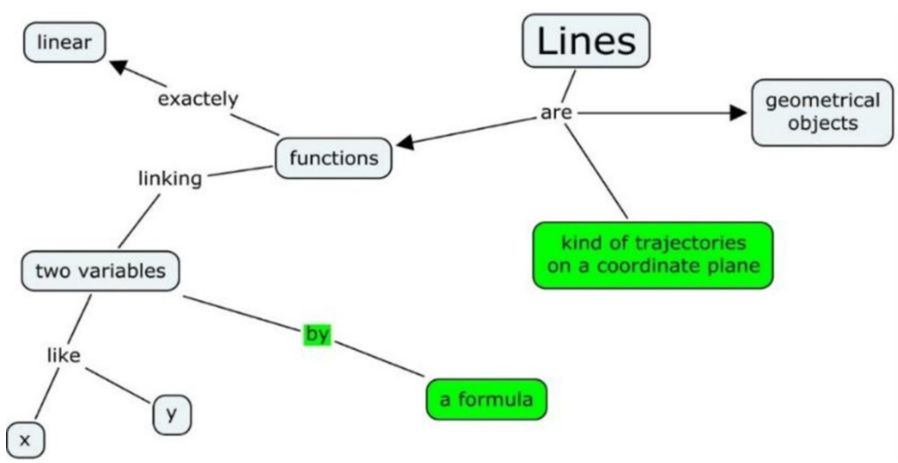


Figure 3. Student B's MH map

Iteration 2

The second iterative step followed, where the author concentrated his attention on those maps that were still not directly connected to the *TCM*. The following figure represents TCM_2 where the author expanded the map to reduce the *distance* from the *MH* of the students, in terms of nodes and links. The figure reports added nodes and links, highlighted in dark grey (green). They represent the starting traits of the desired learning routes.



Figures 4 and 5 represent the extended concept maps of the students in this step. We report here the maps of the two students, where the NPLs chosen in the first iteration have been represented in light grey (yellow), while the nodes added by the students are represented in dark grey (green).

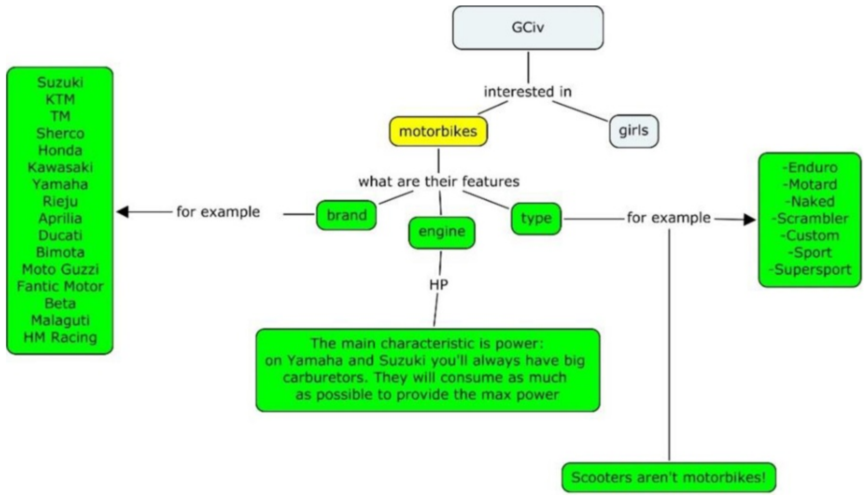


Figure 4. Student A's extended concept map

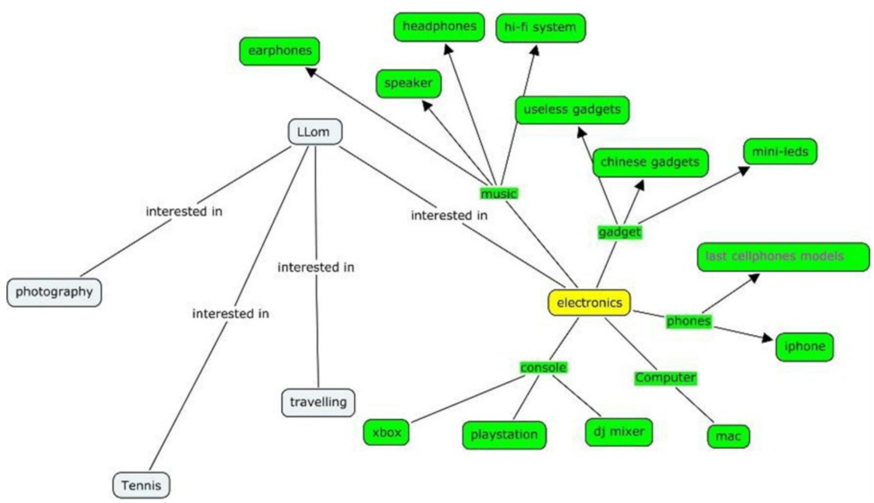


Figure 5. Student B's extended concept map

Iteration 3

We move on to the third iteration step, where the author concentrated his attention on closing the gap between his *TCM*-map and those created by the students. The following figure represents TCM_3 where the author further expanded the map to reduce the distance from the MHs of the students, in terms of nodes and links.

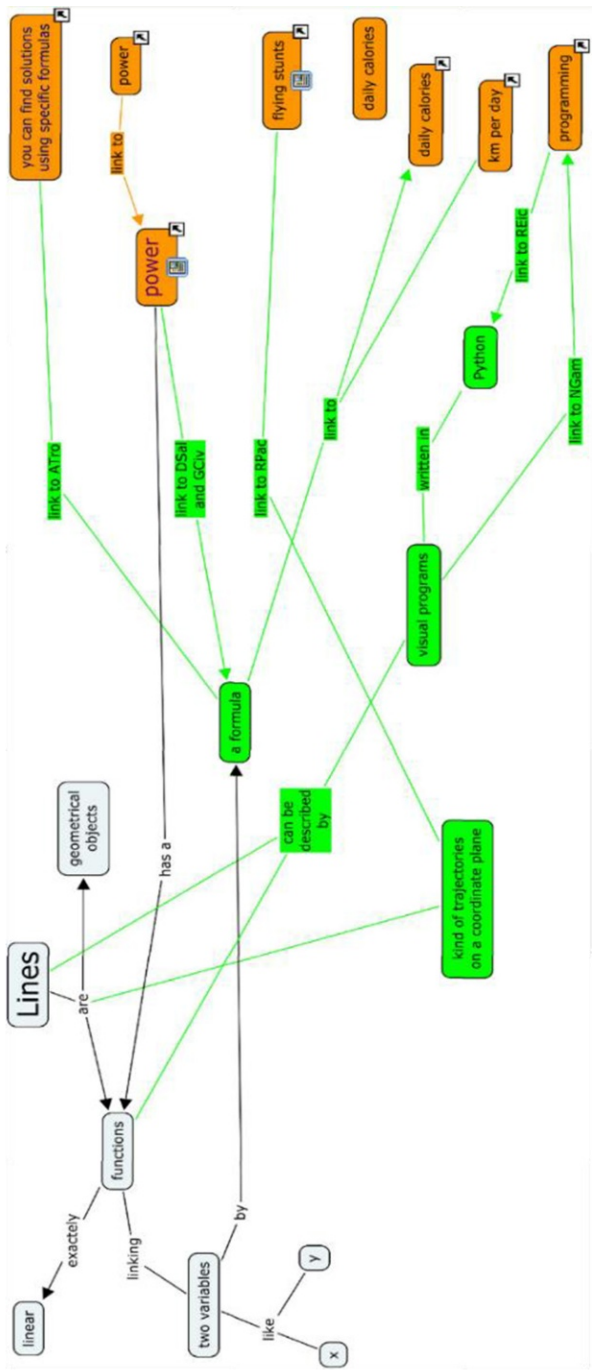


Figure 6. TCM_3 – Instructor's further expansion of the previous TCM

The figure shows added nodes and links, highlighted in dark grey (green). The links and nodes in grey (orange) are the final nodes directly connected to the students' MHs. These nodes are *common* across the *TCM*-map and the students' maps. They represent the final traits of the desired learning routes and the major practical goal of the Method.

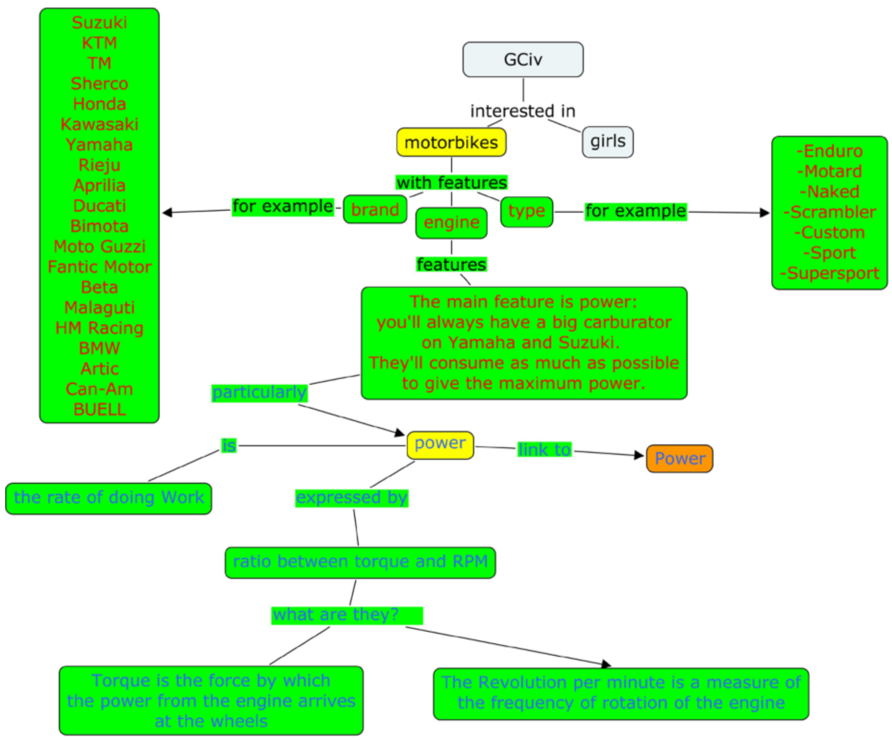


Figure 7. Student A's extended concept map linking to TCM_3

Figures 7 and 8 represent the extended concept maps of the two students in this step. All the other maps were either connected to the extended *TCM* or belonged to students that refused to proceed with the experiment.

Finally, the following chart summarizes the percent of students' maps connected to the *TCM*-maps at the *ith* iteration.

Table 1. Percent of students who achieved the required connection

	Iteration 1	*Iteration 2*	*Iteration 3*
Percent of connected maps	12.50%	31.25%	43.75%

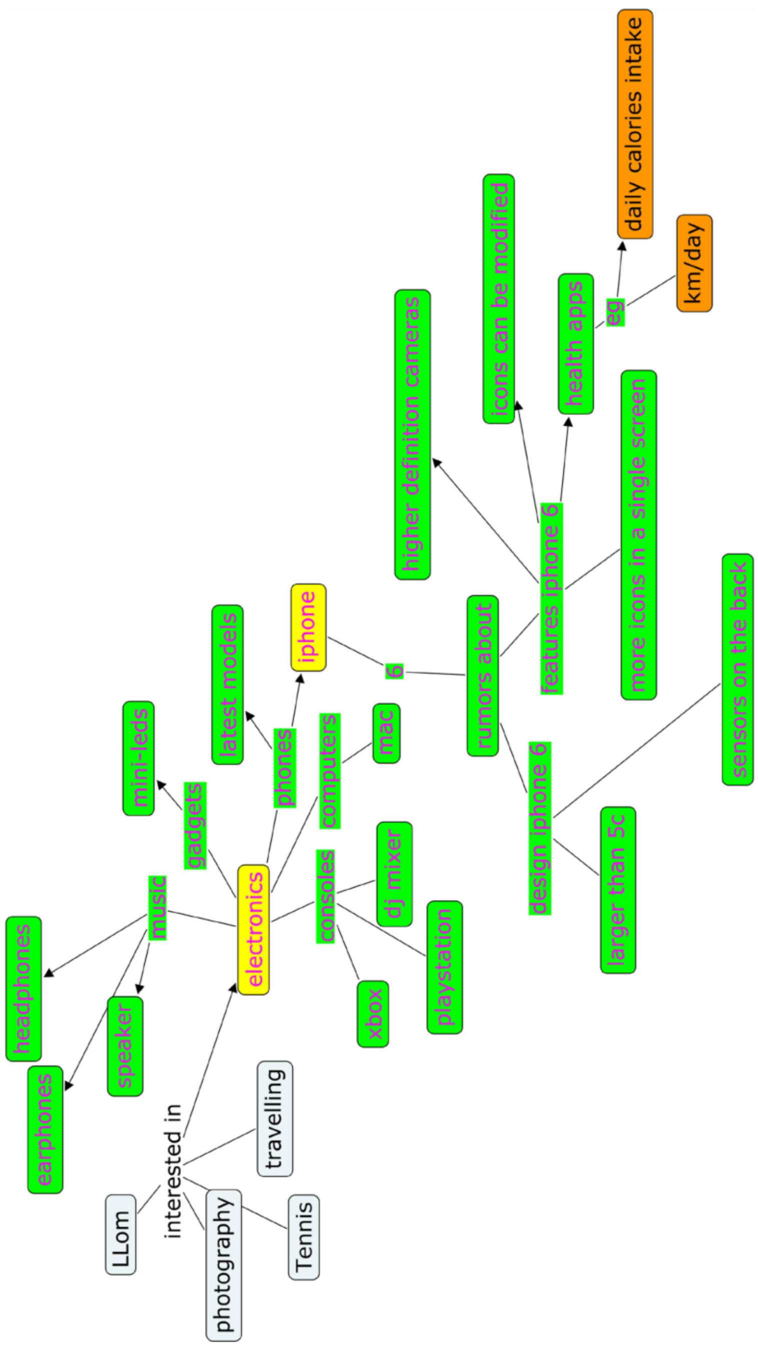


Figure 8. Student B's extended concept map linking to TCM_3

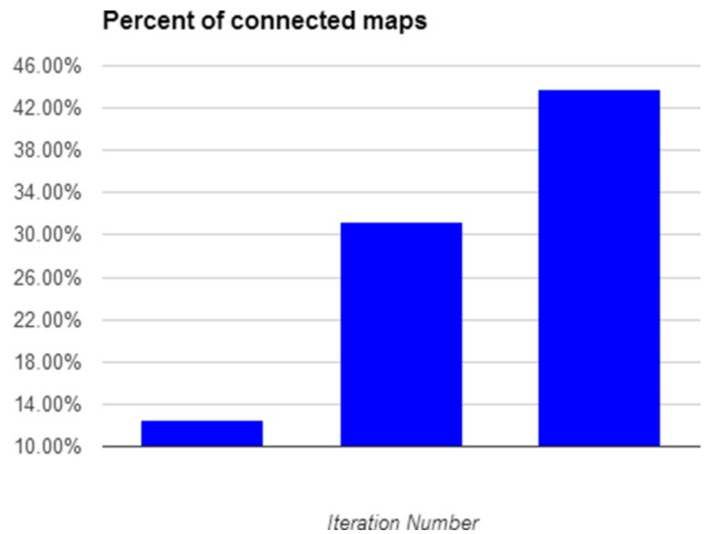


Figure 9. Bar chart of students who achieved the required connection after the ith iteration

Analysis of the Maps Collected

We can draw a few important observations about the maps built by the students and their interaction with the proposed Learning Routes method:

- The maps encourage much stronger bonds between students' personal web of interests and the topic chosen by the teacher out of the curriculum. This entails both the way the teacher could frame this part of the course and the way the students approach it. Indeed, it could be much easier to structure tests, decide which tools to use, how to divide the workload among students, when the teacher knows in such great a detail how students' interests are connected to the current topic;
- The students who were able to connect their personal maps to the *TCM* after the first iteration showed a much deeper sense of collaboration and willingness to work with the teacher on the topics discussed. They raised more questions than usual during classes, asked for more details about the topic, and most importantly decided to dig deeper into it in the near future;
- The emotional engagement of the students involved in the project was greater than usual. Even though such characteristic is somehow difficult to measure, the author certainly registered a few indicators that the engagement of those students who wanted to actively participate in the project was increasing over time. For example, they asked more questions about the topic, and spent more time around Math related topics during normal school hours; for the whole duration of the

project they asked many more questions of the teacher about mathematical topics and problems, such as the visual and numerical relations between power, torque and RPM.

Finally, we consider the following as very important observations:

- The students who had their maps connected to the *TCM* right at the first iteration step, are those who already show a good amount of interest for Math and Math related subjects or topics. The author is already carrying on some extra activities with these students, like computer programming courses;
- The students who had to go through more than one iteration step also had the chance to extend their understanding of the topics, so the author considers this learning method as really useful, at least, for increasing participation among the students involved;
- The students could not see the *TCM*-maps in advance. This was done because the author wanted to afford students maximal freedom when writing their thoughts about interests and Math related topics. On the other hand, the teacher could reshape his *TCM*-map after each iteration, in order to link it to the *MH*-maps of the students.

The author would like to add another perspective about the Learning Routes Method. The work of Czarnocha Bronislaw and Vrunda Prabhu about mathematical creativity offered the author a new insight and perspective about his own work. From Prabhu and Czarnocha (2014), we could see that the bisociative act is strongly linked to the ability of seeing the connections between previously unconnected matrices of experiences. We could regard the word *experience* here as key in the whole picture of our work. Learning at school is too often disconnected from the everyday life experiences of the student. Our work is directly aimed at building stronger connections between topics and students' lives. The bisociative act provides a new intellectual and emotional experience that fuses together previously disconnected concepts. We believe that this approach could provide a strong boost for changing students' attitudes and approaches to learning in a formal school setting.

CONCLUSIONS AND FUTURE WORK

To the best of our knowledge, this has been the first experiment carried on using the method proposed by the author. We want now to draw three main important observations:

1. Students showed much more involvement in working around Math and Math related topics. They were prone to engage in deep discussions with the teacher and provided good observations. They also faced the problems posed in Math with greater grit and stronger resilience.
2. The inner nature of the method proposed clearly invited students to dig deeper into the topics studied. They had to discover new subtopics about the main theme,

eventually building new connections between these subtopics. This, in turn led them to reach a better ability to analyse the main theme of the map.

3. The *MH + TCM* map gave the students a much clearer way of looking at personal interests and curriculum topics. They tried to make the best out of the CmapTools, exploiting especially the visual representation of knowledge promoted by this medium.

Finally, the author plans to explore in the future:

- How making explicit the visual approach to learning elicited by the author's method enhances a deeper conceptual understanding of the topics by the students. Tackling learning visually could spawn new insights into it, as already studied in many literature studies, for example the work of Ritchhart R. and Perkins D. (2008)
- How this approach might lead to deeper thinking habits in students that are interest-driven, personally relevant, and also able to spark a better metacognitive attitude in them. That means students should become more prone to think about their own thinking, seeing what they visually represent more and more meaningful;
- How students might engage in collaborative discussions about common interests, shared among two or more of them and the *TCM*-maps created by the teacher. This could lead the class to generate interest-powered groups, built upon sharing common topics in the personal *MH*-maps.

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3.9. DISCOVERY METHOD AND TEACHING-RESEARCH

INTRODUCTION

In Unit 1 we reflected upon the gap between research and practice and reviewed recent efforts to close this gap by reconceptualising mathematics education research as a design research (Unit 4). In Chapter 1.1 we began a discussion about the essential role of discovery learning in conducting teaching research, in this chapter we reflect upon discovery learning as an important tool in the effort by constructivist pedagogy to reform mathematics education. The Discovery approach to teaching relies on designing situations and using techniques which allow the student to participate in the discovery of mathematical knowledge. The main assumption behind the technique is the belief that learning can be accomplished more fruitfully if the student discovers knowledge by himself or herself rather than through direct instruction or rote learning. As noted (Unit 1) implicit in establishing a creative learning environment in the mathematics classroom is a focus on student creation of meaning for themselves i.e. guided discovery or inquiry learning. In Unit 4 we present examples of curricula material designed to be used in a discovery learning environment with rates by Czarnocha and a classroom lesson that employs guided discovery by Dias. We note that a creative learning environment with a focus on student cognition during the creation of rediscovery of mathematics is the ideal environment to observe such mechanisms of creativity and learning as bisociation and reflective abstraction (Unit 4) and in that unit Baker presents an analysis about student bisociations during a class lesson on proportional reasoning.

TRADITIONAL AND REFORM PEDAGOGY

The top down model of instruction or direct instruction is based upon the assumption that a mathematics teacher is the authority has claim to the ultimate interpretation of content knowledge which they will impart to students who will listen and learn. This approach has been rejected by constructivists who argue that the goal of classroom discourse is to promote student construction of knowledge. This necessitates a transition away from this top-down approach in which the teacher lectures to passive students. Yet, so many reform efforts are based upon a top-down approach in which teachers are supposed to passively accept a new theory inspired agenda:

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 245–252.

> Many government driven curriculum reforms, in Brittan and elsewhere, assume that the central powers can simply transmit their plans and structures to teachers who will passively absorb and then implement them in 'delivering curriculum.' (Ernest, 2010, p. 40)

MATH WARS

The so called mathematics wars were a result of the implementation in California of curricula based upon the National Conference of Mathematics Teachers (NCTM) 1989 publication of Curriculum and Evaluation Standards. These standards were themselves based upon constructivist pedagogy and sparked a controversy in the mathematical educational community (Klein, 2007; Raimi, 2006). This controversy sparked among other things duelling web-sites named 'Mathematically Correct' and 'Mathematically Sane' (Goldin, 2003). Although subsequent standards (2000) Principles and Standards for School Mathematics and (2006) Curriculum Focal Points were seen as more balanced and did much to calm the debate, the controversy over the effectiveness of constructivist pedagogy and in particular discovery learning, which is at the heart of Koestler's viewpoint that mathematics needs to be rediscovered by the student, continues. However, the controversy engendered by efforst to implement and assess discovery learning continues. Klahr and Nigam (2004) study the theoretical cornerstone of constructivist reform efforts that knowledge discovered by a student is more valuable or permanent than that taught by a teacher. They employed two groups on the extreme ends of the spectrum one with basically no teacher intervention and the other with all aspects of the class pedagogy were teacher controlled and they report that their result call into question the validity of this claim. Dean and Kuhn (2006) essentially repeat this experiment through a longer time frame and their analysis was different:

> The present study, note, does not purport to demonstrate the merits of engagement/practice methods, compared to direct instruction, with regard to efficiency of instruction. Our intention is not in establishing how fast the strategic understanding examined here can be acquired, but rather how well it can be acquired. Student in the two practice conditions spent much greater 'time on task' than those in the direct instruction condition. Given that this practice led to significant and lasting gains in strategic understanding for the majority of students, do we then need to ask whether such gains in strategic understanding could not be accomplished more quickly. (p. 395)

Discovery and Inquiry Based Learning

In understanding the failure of California's educational effort to implement constructivist pedagogy i.e. discovery learning into the classroom one can point out the lack of objective truth in the 'ism' of mathematics education (Goldin, 2003). One can also point to the lack of teacher development, e.g. student teacher preparation: "Just getting student teachers to realize this…represents a significant step forward from the naive transmission view of teaching and passive-reception view of learning many student-teachers arrive with" (Ernest, 2010, p. 40). Theoretically the argument to shape the extent of discovery learning will revolve around the nature of learning by imitation not the imitation of meaningless rules perhaps but the transition or stepping from what one knows to new knowledge attributed by Norton and D'Ambrosio (2008) to Vygotsly as opposed to the view as presented by an instructor, tutor or textbook etc, that to understand one must recreate or rediscover math for oneself. Kirschner et al. (2006) suggest that the nature of non-intervention instructional techniques, "caused a much larger cognitive load and led to poorer learning than worked-examples practice" (p. 80). Indeed these authors contend that "algebra students learned more studying worked examples of algebra than solving the equivalent problems" (p. 80). Hmelo-Silver et al. (2007) disagree with the conclusion of Kirschner et al. (2006). These authors suggest, two major flaws with Kirschner et al.'s argument. The first is a pedagogical one:

> Kirschner and colleagues have indiscriminately lumped together several distinct pedagogical approaches—constructivist, discovery, problem-based, experiential, and inquiry-based—under the category of minimally guided instruction. We argue here that at least some of these approaches, in particular, problem-based learning (PBL) and inquiry learning (IL), are not minimally guided instructional approaches but rather provide extensive scaffolding and guidance to facilitate student learning. (p. 99)

Meyer (2004) extends the scope of inquiry beyond the dynamic of all or nothing i.e. complete non-intervention or teacher explanation and re-explanation of curriculum to include guided discovery methods in an effort to study the underlying premise that: "learning is an active process in which learners are active sense makers who seek to build coherent and organized knowledge" (p. 14). Concerning the relationship of discovery learning to teaching pedagogy Meyer (2004) comments that, "although guided discovery required the most learning time, it resulted in the best performance on solving transfer problems" (p. 15) and "Children seem to learn better when they are active and when a teacher helps guide their activity in productive directions" (p. 16). Meyer (2004) suggests that: "The challenge of teaching by guided discovery is to know how much and what kind of guidance to provide and to know how to specify the desired outcome of learning" (p. 17). Meyer (2004) concludes that the ultimate measure of teaching pedagogy should be cognitive not behavioral: "Methods…should be judged not on how much doing or discussing is involved but rather on the degree to which they promote appropriate cognitive processing" (p. 17).

Laursen et al. (2014) study the effects of inquiry based learning (IBL), a guided discovery methodology with a focus on in depth rather than extent of material. Their statistical analysis revealed benefits for female students:

> This study exemplifies both the challenges and promise for implementation of IBL. We detect robust, meaningful differences in self-reported gains and attitudes among students in IBL, relative to those in non-IBL courses. Rigorous statistical modelling techniques linked student gains clearly to the pedagogy, showing that IBL benefits all students even as it levels the playing field for women, who are often under-served by college mathematics courses. (p. 415)

In Lausen et al. (2013) and Laursen (2013) the authors report benefits of IBL that are long lasting for students including minority students who are traditionally low-achievers in mathematics.

CONSTRUCTIVISM AND MATHEMATICS EDUCATION

Lesh and Sriraman (2010) note two recent, significant revolutions in mathematics education first, the 'technological revolution' and second the 'constructivist revolution.' It is to the second revolution we now turn our attention. The success of constructivist in educational literature can be evidenced not only by the attention it has received, the subsequent or related movements as well as by the extent to which mathematic educational groups such as the NCTM have supported the incorporation of constructivist based language into their standards. As Lesh and Sriraman (2010) state, "we are all constructivists" (p. 129). Thus, the success of this movement in literature leaves it open to the critique, if we are all constructivists why hasn't there been any global improvement in mathematics education? More to the point why is the gap between research and practices so hard to overcome?

Richardson (2003) ponders this question noting that the transformation of education theory into educational practice has always been difficult and "...less than satisfactory..." (p. 1623) but reflects that nature of constructivism has made that transition particularly difficult. In this view, the transition from the tradition top-down approach of classroom management to a student centred focus is difficult because there is an inherent gap between this theory, curricula and methodology to successfully implement it. Karagiorgi and Symeou (2005) suggest the reason behind this gap is that while constructivism may be the dominant theory of learning it is not nor does it say anthing about how to design instruction.

Richardson (2003) expresses the view that much of what has been used to define constructivist pedagogy is based upon what it is not, i.e. not direct instruction, not telling the answer. Thus, one role of design science or teaching research based upon constructivist philosophy is to fill in this gap between what not to do and what to do in the math classroom. Another relevant question that Richardson poses is whether student construction of meaning during the traditional lecture format should be viewed as a meaningful component of a constructivist classroom? Karagiorgi and Symeou (2005) note the pedagogical issue of what exactly is and is not constructivist methodology arises in term instructional design, i.e. what is constructivist curricula and how does one design constructivist based learning material, "Instructional designers are thus challenged to translate the philosophy of constructivism into actual practice" (p. 17).

Richardson reflects upon issues such as where does constructivist pedagogy begin, what distinguishes it from traditional lecture format. As Lesh and Sriraman (2010) suggest with the statement, we are all constructivist, she asks, does any good teacher by necessity employ constructivist pedagogy?

We also note that efforts that rely upon discovery learning have a tendency to design tasks to challenge the student and thus promote reasoning and construction of meaning. This must on the practical level be balanced against student's need to pass standardized exams. Thus, the design of constructivist pedagogy is often received with scepticism because of its focus on what is, "not being assessed on these state-wide or national standardized tests" (Richardson, 2003, p. 1629).

Guided Discovery Learning and Creativity

Koestler (1964) posits *Humour*, *Discovery* and *Art* as three shades of the creativity principle, the triptych. The method of the triptych was used by V. Prabhu "to create another entry route into thinking-making-meaning in the mathematics classroom" (Chapter 2.4). She utilized triptychs designed for the discovery of several fundamental concepts in her course. The centre "discovery" column of the triptychs contains the relevant mathematical concepts, the left column consists of related instances of "humour" described by Koestler as "the back entry into the inner workshop of originality", and the right column is the contextual interpretation of the "art" aspect of the creativity principle. The environment for inquiry is created within a classroom with humour and dialogue that supports the drama required to bring about positive student affect in an effort to prepare for engagement in cognitive inquiry and reasoning. The use of the triptychs in the mathematics classroom brings back the puzzle nature inherent in mathematics. Prabhu followed the guided discovery method outlined by Mahavier (1997) who adapts a close to pure discovery form of learning utilized by R. L. Moore (Zitarelli, 2004) for graduate level mathematics courses to a guided discovery based format that he (Mahavier) utilizes in a variety of undergraduate mathematics courses including remedial algebra.

The aim of the discovery method in the classroom is to facilitate students' authentic discoveries in mathematics. Since an authentic discovery (untutored learning) is, according to Koestler, a result of bisociation, such discovery is, naturally, a creative act. Therefore, creativity underlies the teacher-researcher's teaching as well as the students' learning in the discovery-based classroom. The bisociative acts of students during guided discovery learning in the classroom is observed and analyzed by the teacher researcher. These observations can lead to bisociative acts of the teacher-researcher in analyzing student rediscovery of mathematics. These bisociative acts of teaching and research can be seen as a unifying factor between the concept of the NYC teacher researcher model and the creative learning environment (Unit 1 and Unit 2)

Role of Instructor

The role of the teacher is to create favorable conditions for moments of discovery i.e. bisoication and learning. Hope for the success of this technique as an instructional tool is grounded in the belief that the process of discovering knowledge offers students both enjoyment and a mental exercise which will be of lasting quality. The transformation from habits of failure to excellence noted by Prabhu. In order to create such conditions, to support the process of student engagement and rediscovery of mathematics important tools for instructors are scaffolding of curricula and class dialogue. We now turn to the issue raised by Norton and D'Ambrosio (2008) of what is appropriate scaffolding and role for teachers in constructivist pedagogy. Gilles and Haynes (2011) argue that "teachers play a key role in promoting those interactional behaviors that challenge children's thinking and scaffold their learning" (p. 349). These authors note however that, "guided discovery often ends up as teacher direction" (p. 350). They review literature on teacher dialogue that suggests direct instruction of facts accounts for, "over 80% of teachers' total classroom talk" (p. 350). They study the role of the instructor in promoting student explanation and discourse and suggest that, "the greatest amount of correct and complete student explaining occurred in the classroom where the teacher did the most to elicit student explaining…by inviting students to explain and elaborate on their ideas" (p. 352). Abrahamson et al. (2012) also study the question of how instructors guide students to re-discover mathematics. They note effective teacher discourse involves probing students to justify and explain their comments and actions, make suggestions and hints for further action when students are lost, highlight central concepts or declarative information in problem situation students overlook, introduce concepts required the student does not recognize as relevant, discuss relationships between student reasoning and relevant concept, focus student attention on actions or properties of concepts and situations the student is pondering that underlie a possible strategy, in general providing the feedback necessary to solve problems and reach the upper limit of their ZPD. The instructor discourse also has a role in getting students to review their work and reflect upon underlying concepts and their relationships or principles employed to dictate actions and strategy.

Guided discovery pedagogy can be presented as a problem sequence designed to reach some didactic goal, i.e. a learning trajectory (LT) such as Czarnocha's rate sequence in Unit 4 which guides students along an LT of conceptual understanding of the rate concept. Guided discovery pedagogy can also take the design of classroom problem solving session, as a Socratic dialogue between teacher and a student(s) or as a dramatized problem situation as in the rate sequence offered by Dias in Unit 4 who employs a compare and contrast format to lead the classroom into critical thinking about the key features of a problem situation that can be used to suggest an appropriate strategy. While the class discourse by Baker in Unit 4 is designed to promote classroom bisociation between the common concepts involved in different methods of solving a proportion increase problem.

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UNIT 4

TEACHER AS THE DESIGNER OF INSTRUCTION

TR Design

INTRODUCTION

Unit 4 presents the designs of TR investigations and teaching experiments conducted by teacher-researchers in their classrooms. It represents three type of TR activity: daily classroom TR investigations (Chapters 4.2–4.5), construction of learning trajectories through iterated classroom teaching experiments (Chapters 4.6–4.8) and two teaching experiments of opposite types (Chapter 4.9) and (Chapter 4.10).

We introduce shortly the elements of Design Research practiced by the Math Education profession below as the background for the presentation of principles of the TR Design. We find the "niche" for the TR Design within the notion of the conceptual framework introduced by Eisenhardt (1991); we follow with short descriptions of particular chapters in each unit.

DESIGN RESEARCH

The terms Design Experiment, Design Research or the Science of Design are often interchangeable and they refer to the professional design in different domains of human activities. It was introduced by Ann Brown (1992), Collins (1992), and by Whittmann (1995) into research in Mathematics Education. Anne Brown had realized during her exceptional career that psychological laboratory can't provide the conditions of learning present in the complex environment of a classroom and transformed her activity as a researcher directly into that very classroom as the leading co-designer and investigator of the design in the complex classroom setting. In her own words: "As a design scientist in my field, I attempt to engineer innovative classroom environments and simultaneously conduct empirical studies of these innovations" (Brown, 1992). She provided this way one of the first prototypes of designs experiments which, theoretically generalized by Cobb et al. (2003), "entail both "engineering" particular forms of learning and systematically studying those forms of learning within the context defined by means of supporting them…" The profession has followed her lead seeing the classroom design experiments as theory based and theory producing. Paul Cobb et al. (2003) assert that Design Experiments are conducted to develop theories, not merely to empirically tune what works. Design research paradigm treats design as a strategy for developing and refining theories (Edelson, 2002). Even Gravemeyer (2009) who states that "the general goal of Design Research [is] to investigate the possibilities for educational improvement by bringing about and studying new forms of learning" hence in terms closer to the substantive quality of Stenhouse, yet he warns us that "great care has to be taken to ensure that the design experiment is based on prior research…" eliminating this way the designs anchored in prior practice. Unfortunately, the educational research profession cuts itself off by these restrictions from the source of profound knowledge contained in the tacit and intuitive craft knowledge of the teachers. The restrictions are lifted only in the context of the conceptual (as distinct from theoretical) framework introduced by Eisenheart (1991) and restated by Lester (2010) as one of three frameworks of inquiry within Math Education.

More generally, Lesh and Sriraman (2010) propose "…re-conceptualizing the field of mathematics education research as design science akin to engineering and other interdisciplinary fields…" (p. 123). The term design science is meant to distinguish investigations of artefacts, tools and conceptual systems created by humans from scientific inquiry into natural phenomena. Thus, design science or the derived notion of design research includes inquiry into thinking and learning, the artefacts or tools used to support these process i.e. the "…where, why, how and with whom curriculum materials or programs of instruction need to be modified for use in a variety of continually changing situations…" (p. 125). The object under investigation include, processes of learning, artefacts, tools and programs designed to support these processes as well as the theories and conceptual models that underlie these processes i.e. the "…complex conceptual systems that underlies the thinking of student(s), teacher(s), curriculum developers(s) or some other educational decision maker(s)." In the design research paradigm testing of tools used to promote reform pedagogy are linked to the theory and models upon which the artefact is based. As Lesh and Sriraman point out, "when the artefact is tested, so are the underlying conceptual systems" (p. 125).

Amit (2010) notes that the genesis of design research involved extending the work of mathematics educational researchers involved in teaching experiments towards two goals: "…increase the relevance of cognitive science laboratory experiments to teaching, learning and problem solving activities in real school situations… to provide stronger theoretical foundations for projects, which design educational software, courseware, or other tools and artefacts such as assessment systems" (p. 147). Wittman (1995) comments on the lack of educational research on the tools or teaching curricula involved in teaching research,

> At best teaching units have been used as more or less incidental examples in investigating and presenting theoretical ideas…Why should anyone anxious for academic respectability stoop to designing teaching and put him-or herself on one level with teachers? The answer has been clear. He or she usually wouldn't. (p. 365)

As teacher researchers we argue in favor of research on the methodology and curricula used by teachers to implement conceptual models of learning and development. We agree with Cobb assertion that one needs to continually study first-hand phenomena one wishes to learn about. We consider an analogy between (1) the creative learning environment focused on student reasoning during problem solving within the classroom and a community of inquiry conducting research on the methodology, and (2) tools or artefacts teachers employs to implement constructivist or other models of conceptual development. As one seeks the optimal didactic contract between teacher and student for learning the other seeks the optimal didactic contract between teacher and researcher to provide tools built upon theory to support such learning.

Lesh and Sriraman (2010) note that a basic tenets of design research is that inquiry should include an iteration of cycles, "in order to develop artefacts + designs that are sufficiently powerful, shareable, and re-useable, it usually necessary for designers to go through a series of design cycles in which the trial products are iteratively tested and revised…" (p. 127). This because without repetition-iterations as well as replication, the results of single study or teaching experiment conducted whether in a clinical setting or classroom will not be sufficiently generalizable to be relevant in other settings, "the challenge to solve practitioner problem ignores the fact that very few realistically complex problems are going to be solved by single isolated studies" (p. 128). A central component of teaching research is this cyclic methodology however, we point out that a good teacher is constantly adapting instruction to her/his class. To suggest that a given class is the same from one day to another that different topic cause the same level of cognitive difficulty is largely ignored in the design research position that a tool or artefact can succeed in improving education. Wittmann (1995) points out the polarity between the role of teachers and researchers while calling for the more research into design of curricula.

> The design of substantial teaching units and particularly of substantial curricula is a most difficult task that must be carried out by the experts in the field. By no means can it be left to teachers, though teachers can certainly make important contributions within the framework of design provided by experts, particularly when they are member of or in close connection with a research team…there should exist strong reservations about 'teachers centres' wherein teachers meet to make their own curriculum. (p. 365)

> This rather provocative statement is certainly not going to directly heal the divide between theoretical research and the craft of teaching yet it brings up an interesting point. As teacher researchers implementing a creative learning environment, the role of the researcher is one of producing 'meta thought' i.e. reflection upon what works and what does not. The teacher introduces the student to a structure that assists with reflection upon and the coordination of his or her intuitive thoughts. In like manner educational theory or models assist the instructor reflect upon and coordinate their thoughts about what worked

and what did not in the classroom, why and how to improve it. In one situation creativity is a result of bisociation between previous often spontaneous knowledge with a given problem situation. In the other, creativity is the result of bisociation between theory and the problem situation of establishing an environment conductive to student reasoning in the classroom. In one, the teacher role is to trace the hypothetical and actual learning trajectories of their students encouraging reflection upon the what, when, how and why of student action while in the other case the researcher is engaged in promoting meta reflections on the part of the teacher, what was she doing, how did she go about doing it and why was it important to establishing a creative learning classroom environment. This close proximity of two bisociative frameworks in teacher-student interactions create a subtle opportunity for simultaneous and coordinated "thinking together". While the student thinks about the mathematical problem at hand, the teacher thinks about student thinking while working on that problem. Thinking together moments are didactically important because they help to establish high precision of scaffolding, what can lead to the joint Aha moment of bisociation. First we have the act of the teacher who grasps the correct scaffolding step, followed by the student who grasps the teacher's hint as the "missing analogy" to solve the problem. However, since the teacher finds out about the correctness of the hint only at the moment when the student has its Aha moment realization, they can participate in the joint Aha moments of the student and the teacher.

THE TEACHING-RESEARCH DESIGN

The Teaching-Research (TR) Design is based upon two principles discussed in Chapter 1.1:

- The substantive aim to improve the process of learning in the classroom
- Bisociativity of Teaching-Research.

TR bisociative framework facilitates integration or, still better, synthesis of practice and research through instances (or sequences of instances) of Stenhouse acts which are "at once an educational act and a research acts" (Ruddock & Hopkins, 1985, p. 57). In what follows we will call them Stenhouse TR acts. The Stenhouse TR acts are the foundation stones of "thinking technology" (Chapter 1.1) within which their unity is naturally positioned. The facilitation of longer or shorter instances of Stenhouse TR acts can be reached from either teaching practice or from application of research to practice, as well as from both simultaneously.

The Principle of the Bisociative Nature of TR

- The principle of bisociativity of TR design suggests *problem solving* in the context of "inquiry based discovery method/guided discovery" as the teaching approach.

Chapter 4.1 by Bill Baker provides an extensive analysis of the bisociativity inherent in the act of problem solving, as the methodology of teaching, so that the majority of our TR designs involve teaching sequences of problems designed to reach understanding and mastery of the relevant concepts and techniques.
- The special role played by the Discovery method of teaching in the context of TR/NYCity is discussed in Chapters 1.1 and 3.9. On one hand, it provides the bisociative framework for a student, and on the other, it serves as the research instrument allowing a teacher-researcher to investigate student thinking.
- Chapter 1.2 discusses extensively the relationship of bisociation with affect exploring the implications of Koestler's remark: the creative moment is an act of liberation, of "the defeat of habit by originality". Czarnocha (2014) suggests that the cognitive/affective duality of the Aha! moment is its essential quality, explored in this volume by Prabhu in Unit 2.

This cognitive/affective duality of the Aha moment suggests a careful attention of the design to the interaction between student ZPD of Vygotsky and student Zone of Proximal Affect (Murray & Arroyo, 2002). The possibilities of expanding ZPA through facilitation of cognitive Aha moments is suggested both by empirical (Liljedahl, 2013) as well as theoretical (Koestler, 1964) observations. These possibilities are of intense interest of educators in the "underserved" communities. What are the conditions in which facilitation of bisociative creativity can overcome the impact of negative affective habits detrimental for the development of student potential? – is the fundamental question of teacher-researchers of the Bronx.

The Principle of Improvement of Learning

The principle of improvement in the TR classroom imposes additional structures to the TR Design, the most important being the *TR cycle iterations* of the design. It requires at least two iteration of TR Design process because only then the opportunity for the improvement of the original design arises, aided by JiTR results. The Analysis and Redesign node of the TR cycle offers a bisociative framework similar in its nature to problem solving bisociative framework (Chapter 4.1) In this juncture TR Design meets Anne Brown (1992) in her self-reflection on her own activity: "As a design scientist in my field, I attempt to engineer innovative classroom environments and simultaneously conduct empirical studies of these innovations".

Correspondingly, TR Design is geared to answer two connected teaching-research questions guided by the substantive quality:

- What is the effect of a given "engineered" intervention?
- What are the routes to improve upon the effect through its redesign?

The Learning Trajectories (LT) research framework which has underlined Common Core designs of instruction in UK and US pushed the number of iteration of the TR cycle into as many as are required to produce the learning trajectory well adapted to the classroom level of knowledge. The principle of improvement facilitates the bisociative quality within the process of TR Design via problem solving involved in the process of improvement by the teacher, which takes place at the Analysis and Refinement nodes of the TR cycle (see below).

The process of *adaptation of LT instruction* to the particular needs of the classroom whose essential description fills up the centre of the TR cycle (Fig.1) requires teachers to be deeply familiar with cognitive processes of learning, diagnose student state of understanding, discover its missing links and use the knowledge of research to design the corrective activities – a substantial component of daily TR activity in service of improvement of learning. Once the reflective aspect upon that classroom activity, which results in formation of new hypotheses and refinements to the design, supplements adaptive instruction, teacher obtains the entry into the full TR/NYCity process.

Science of Design and Practice of Design

According to Wittmann (1995) the practice of design by teachers existed long before the general principles of design formulated by Herb Simon came under the scrutiny of educational research. He states that "many of the best units were published in teachers' journals, not in research journals, and were hardly noticed by the research community." That implies that original components of Design Research have been formulated by practicing teachers. Therefore, in addition to the Science of Design described by Cobb et al. (2003) as: theory-based, theory producing and theory testing, of interventionist and conjecture driven nature as well utilizing extensively cycles of iterations, we can formulate the Practice of Design which originates primarily in teachers' craft knowledge and its aim is improvement of learning in the teacher's classroom and beyond, through the design and re-design of a teaching sequence or any other artefact.

Practice of Design originates with the design of additional exercises for students in need of a reinforcement of a particular concept, which are done daily by a teacher, followed by the assessment of their impact and possible refinement based on craft knowledge. The reflection upon observed, accumulated patterns and results by the same teacher leads to the formulation of an idea or guiding principles for the development of the prototype design which will be more systematically tested in the full classroom unit of relevant mathematics. It is important to underline that these initial moments of the formulation of the design depend essentially on the natural cycles of teacher's work: daily, semester long and year-long teaching cycles, when we repeat our instruction with different cohorts of students. These natural teaching cycles become, for a reflective teacher, TR cycles for the refinement of the designed prototype. Generalization of practice through the artefact generalization formulated by Prabhu in Unit 2 is one of the essential elements of the Practice of Design. The collection of designs in this volume contains three designs anchored in the Practice of Design: Unit 2 where the coordination with a theory took place during the conduct of the teaching experiment, Chapters 4.9 and 5.1 where the coordination with theory took place after the design was implemented. Those were posteriori designs introduced in Chapter 3.2. In each of the cases it was the nature of the implemented design that directed us to search for an appropriate theoretical framework.

Generally, systematic JiTR consultation and interaction starts taking place at the appropriate node of the TR cycle and leads through several iterations during which both the craft knowledge of the teachers as well as JiTR results and learning theories are involved and integrated within the produced artefact. Thus an artefact, the teaching sequence is produced, whose iterations in different classrooms provide similar level of artefact generalization as "systemic variations" in design experiment (Cobb et al., 2003).

Both Science of Design and Practice of Design relay on the iteration of TR cycles, both have "interventionist and conjectural nature". They differ in the point of entry upon the TR iteration cycle, the method by which they utilize research and in the ultimate goal of the design work. Science of Design seeks the general theory from where the improvements might be designed, while the Practice of Design seeks the learning improvement in the same or similar classrooms, while a possible general theoretical insight comes as a by-product of the improvement process. Ultimately the Science of Design and Practice of Design methodologies converge through their iteration cycles to the principles of Teaching-Research Design, which encompasses both, while creating the conceptual bridge between them.

We discussed extensively three frameworks of inquiry in Math Education: theoretical, practical and conceptual (Lester, 2010) in Chapter 1.1. Correspondingly, there are three routes through which classroom acts of the teacher can assume the unified precision of Stenhouse TR acts. Each of the routes corresponds to a particular type of TR Design.

Type A – based in practice: A teacher-researcher can start from the educational act of teaching a concept, which through the analysis of student responses and JiTR activity becomes a TR act in that it teaches and at the same time investigates student thinking during the process of learning from the educational act (while being open to an aha moment occasioned by an instance of the bisociative framework created by that simultaneity). Similarly, practice-based TR Design originates in daily work of the teacher, possibly through an artefact, which, after several cycles of iterated refinements with different cohorts makes a contact with appropriate research through JiTR method.

Type B – based on previous research: A researcher can start from the research act whose aim is to investigate certain aspect of student learning in a classroom or in clinical setting, which through, possibly cyclical "constructivist teaching experiment" of Cobb and Steffe (1983/2011) and its teaching episodes becomes a TR act (Czarnocha, 1999).

Type C – A mature teacher-researcher, whose thinking technology has had a chance to develop through several of such iterated constructivist teaching experiments, can create classroom situations, which in their inception are unified into one TR act fulfilling both roles Chapter 4.5.

TR DESIGN IN THE CONTEXT OF CHAPTERS IN UNIT 4

Unit 4 opens with the extensive discussion of Koestler's bisociation in the context of problem solving by Baker in Chapter 4.1. Its aim is to prepare the ground for understanding the creativity of the teaching sequences and methodologies of teaching in chapters of the unit. Having clarified the bisociation process he proceeds to outline the relationship of bisociation with three theoretical frameworks often utilized in Math. Education research: Piaget theory, Vygotsky and Anderson theories. Special attention should be directed towards two last sections of the chapter where the relationship between bisociation and reflective abstraction is carefully explicated showing that processes of reflective abstraction such as interiorization and constructive generalization can be realized through Aha moments of bisociation.

Chapter 4.2–4.5

The sequence of following chapters has been planned to compare different types of design related to similar themes. Chapter 4.2, the Introduction to Comparative Study of Three Approaches to Teaching of Rates, discusses three different designs pertaining to proportional reasoning and including rates, proportions and percent: Dias' practice-based design A, Baker's theoretically based design B and Czarnocha's design based both in practice and in a theory, the design type C. The introduction to the chapter provides an in depth literature background to the collection and grounds the practice-based component of their designs in an appropriate theoretical niche exemplifying JiTR method of analysis.

Chapter 4.3 presents a practice-based design, which uses guided discovery method to facilitate student understanding. Its author, Olen Dias, a reflective practitioner insists in her design on the facilitation of student reflection for every problem discussed in the class. Her methodology is problem posing/problem solving oriented through the simultaneous discussion of different problem solving techniques. Similarly to work of Vrunda Prabhu, the attention to affect is one of the strong points of the design.

Chapter 4.4 presents the attempts to incorporate bisociation-inspired methodology into the discussion of percent. Baker creates a bisociative framework composed out of percent concept and its money-wise application and is able to facilitate an Aha moment with one of the students. At the second part of the section, he is using understanding gained by that student to extend the impact of the student's insight to the whole class.

Chapter 4.5 with Czarnocha as an author has both practice – based and theory-based origins. It operates explicitly as the conceptual framework where his experience as a mathematics teacher suggests to him arithmetic/algebra integration as means to facilitate student processes of generalization, while his experience as a researcher suggest process/object duality theories (Tall, 2000) as both design and explanatory framework.

His discussion boasts of a precise Stenhouse TR act facilitated through the question

"Juan got a raise to $30/hr. If you know the number N of hours Juan works, how would you calculate his total pay now?" Its aim was to reinforce understanding of multiplication through verbalization of the procedure while investigating students' ability to cross from additive to multiplicative structure. Student responses data allowed to focus attention on those of them who had difficulty in that process.

Chapter 4.3–4.6

Present two examples of learning trajectories produced by our TR Team in response to the important question posed both by many researchers as well as designers as to whose responsibility is the design of LT's for the classroom and what is the role of research in the design. Each trajectory in the chapter has a deep connection with research yet realized in a very different manner and belongs to a different type of TR Design. We argue, on the basis of the presented designs that teacher-researchers in any educational institution can design effective LT's using JiTR method if they are provided with reassigned time from a teaching course and organizational support.

Chapter 4.7 Baker presents here a design type B of an LT as a consequence of an application and adaptation of the long line of educational research into the structure of schema of fraction. Since that research line has concerned primarily learning of children, it had to be reconceptualised for learning of adults with the help of quantitative techniques (Doyle et al., 2016). Baker presents a rational number trajectory through the many different conceptual understandings for fraction or rational number notation beginning with the part-whole conception or construct. The arrows represent relationships between the constructs of a fraction that were hypothesized by theory and verified in section 1 by quantitative analysis of student's work. In addition to investigated statistical tendencies for the design of the learning trajectory, he also provides excerpts from classroom teaching-research interviews (Chapter 3.4), which support the design. The teaching-research interviews were conducted during the implementation of the second iteration of the TR cycle.

Chapter 4.8 Czarnocha presentation of the LELT (Linear Equations' Learning Trajectory) is in the context of the further discussion on the differences between research and teaching-research, which in his opinion, disappears with subsequent iterations. He sees the essential trade off from the research as the basis of the design to JiTR approach within the design as one of the necessary conditions for teachers' buy-in for the incorporation of research into classroom teaching. LELT is the TR Design type A. The need for and the first idea for the learning trajectory on the subject, which originated during reflection upon the persistence of student errors in the question: “Solve for x in terms of y”, was based on teacher’s craft knowledge of teaching. The original design was triangulated with Confrey et al. (2010). A careful reader will notice two JiTR moments during two iterations.

Czarnocha’s LT is composed of different approaches whose aim is students’ reflective abstraction upon the processes involved in solving linear equations. These different methods or teaching sequences are incorporated into the design of LT through subsequent iterations of a teaching experiment. Under the second iteration are translations from verbal statements which require algebraic expressions to readily solve; such translations are at the heart of the transition from an arithmetical or operational understanding to a structural one. One might solve a statement such as: 4 more than a number is 12 through reversal of the arithmetical process of addition. Yet translation of this statement into algebra and employing algebraic manipulations requires less cognitive load. This problem sequence employs language to assist students understand the initial stage of the variable concept as a solution to a statement (single variable linear equation). In the first iteration Czarnocha presents a problem sequence that requires students to generalize a simple linear equation with one variable: $ax + b = c$ (solve for x) to those of the form: $ax + b = cx + d$ (solve for x) as well as to linear equations in two variables: $ax + by = c$ (solve for y in terms of x). This abstraction of the process of solving a linear equation promotes schema development through the intermediate stage. In the third iteration the solver is asked to compare different equations and different solution strategies for solving a simple one-variable linear equation.

Chapter 4.9

The search for effective “real world problems” approach is the theme of Chapter 4.9. It is the “purest” practice-based design of the collection, TR design type A whose complete structure appears only after the successful third experimental iteration. Table 2 of the chapter shows 4 different dimensions characteristic for the complex design: technological medium, structure of classroom activities, connection to reality dimension and the approach to Riemann integration. Classroom activity and reality connection constituted two central dimensions which required longest time of detailed experimentation.

The series of three iteration of the design is addressing the question of formulating effective instruction for modelling a graph of the function obtained from a “real world” situation. Lehrer and Shauble (2003) assert that modelling emphasizes the connections between mathematics and science (or between mathematics and “real world”), which are usually regarded as separate objects. Clearly, referring back to the definition of bisociation and of the bisociative framework, it follows that modelling is a bisociative activity within the bisociative framework created by two traditionally separate domains of mathematics and “real world”. Therefore, the pathway traversed by the teacher-researcher in this chapter represents the search for the best learning environment for the bisociative nature of the mathematics and science of calculus to fully reveal itself to students in the context of Riemann sums. What's interesting, at the last iteration the teacher adds a very strong motivational element: the public Gallery exhibition of mathematics and the posters prepared by teams of students. The teacher realized, similarly as Prabhu in Unit 2 that apart from cognitive and technological issues, affective ones need to be addressed with equal force.

The difficulties with computer based media Computational Algebra Systems (CAS) or Graphic Calculator (GC) during modelling that took significant period of time for the teacher to find the solution for are familiar and have been recognized by Johnson and Lesh (2003). In particular, they emphasize difficulties with coding and organizing relevant information in forms that are recognizable by the computer. Hannes Stoppel solves this difficulty by the hint cards with relevant algorithms, interestingly, however, only after a short period during which he wanted to eliminate technology completely from the classroom. While he consulted several Tall papers encouraging use of technology in calculus classes (Tall, 2002, 2003), he came up with a "compromise" of JiTT solution providing algorithmic hint cards.

From the cognitive point of view, however, the most interesting is the "dialog" between successive student cohorts of different iterations with the teacher concerning tools of modelling the area under the curve: two sets of rectangles of Upper and Lower series which theoretically define the Riemann integral or two different modes, rectangles of one type and trapezoids. It was only at the third iteration, when the trajectory of student thinking vis-à-vis this concept revealed itself to the teacher through spontaneous questions of students: first consider different types of series to show the invariance of the limit, and only then to modify the type of the series you take.

From the social point of view that is individual-versus-group work, the reflective teacher-researcher apprentice arrived at well balanced and well-reasoned organization of learning in accordance with the similarity of the level of thinking in the class. As a result, there were several working groups as well as individuals working alone in the classroom. The third iteration had the highest passing rate with everyone passing with the grade 4–6 in the scale 1–6.

Whereas we don't know whether students had Aha moments within that bisociative framework, we know it was a fully satisfying class both to students and the teacher. Its components were: strong mathematically related motivation, streamlining the use of computer based devices (CAS or GC) with carefully designed scaffolding, if needed, a cognitive pathway that's close to student thinking and the work organization which takes into account the variety of levels of thinking in the classroom.

Clearly, the report is of a master teacher for whom substantive aim to improve the learning of the integral in his classroom pushed him into unknown terrain in the search for successful teaching. What was the nature of his success?

Chapter 4.10

In this section Baker and Czarnocha presents TR Design of the type B addressing one of the most important domains of TR activity, the relationship between learning mathematics and learning language. The experience of teaching in a multilingual environment made the teacher-researchers particularly sensitive to the role of language in learning mathematics. The hypothesis of the teaching experiment Arithmetic/Algebra was that use of writing as an explanatory tool in mathematics will raise student level of abstraction in the context of problem solving. Our expectations were based on Vygotsky's concept of ZPD, which was constructed as a generalization of three different in content but similar in structure, learning transitions: from verbal to written language, from native to foreign language and from arithmetic to algebra (Vygotsky, 1987, p. 196). The similarity of transitions suggested integration of an appropriate cognitive development theory with the corresponding development of written exercises producing at the same time a teaching sequence based on the integration. The first iteration confirmed the hypothesis, the second iteration made it more precise. The chapter suggests a natural question: could the impact be in reverse? Could mathematics writing improve writing skills per se? Chapter 5.1 from the next unit provides the answer to that question.

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4.1. KOESTLER'S THEORY AS A FOUNDATION FOR PROBLEM-SOLVING

INTRODUCTION

Steffe suggests that researchers in teaching experiments should act as instructors: "A distinguishing characteristic of the technique is that the researcher acts as a teacher" (Steffe, 1991, p. 177). The implementation of constructivist pedagogy requires that students must be encouraged and guided to construct knowledge, to discover and create meaning of mathematics themselves. To be successful in this endeavour clearly students need to act in the classroom, we suggest that for research about teaching to be effective the instructor needs to be an active participant in the research and ideally a synthesis of these roles occurs. In this unit we present lessons and lesson plans that demonstrate how teachers can design methodology that integrates research and craft practice in the bisociative framework of teaching research.

The theoretical foundation to support much of our teaching research experiments specifically those in this chapter is taken from; the work of Koestler on creativity, the material on concept development from Vygotsky as well as research inspired by Piaget (Sfard, Dubinsky) while the work of Anderson will provide a basis for the cognitive psychology driven approach to problem-solving. In addition the work of Glaserfeld and Cifarelli will provide insights from problem solving schema development based upon Piaget.

We have noted (Chapter 1.2) how mathematics educators such as Sriramen et al. (2011) and Leikin and Pitta-Pantazi (2013) have called for adapting the genius view of creativity in pure mathematic research to include the processes of students rediscovering mathematics for themselves. This would extend the study of creativity into research in mathematics education. We also note that (Chamberlin, 2013) and Shiriki (2010) consider the need for the study of creativity in mathematics education to include not only gifted students but also students who may be viewed as resistant to mathematics as well as the underserved populations of students who all too frequently do not perform well in mathematics.

In an effort to adapt Koestler's insights on creativity into the daily working of the mathematical classroom we follow the synthesis of creativity and schema development due to Glaserfeld (1998). Glaserfeld's thesis is that creativity represents an integral component in the construction of any new knowledge. This thesis of Glaserfeld based upon the theories of Peirce provides us with an avenue to connect

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 267–286.

Koestler's 'biscociative' mechanism of creativity and the Piaget based mechanism of learning 'reflective abstraction' to analyze how students construct knowledge.

BISOCIATION

Koestler (1964) describes the main mechanism of creativity in terms of a code and a *matrix,* or a frame of reference. These terms are defined broadly and used by Koestler with a great amount of flexibility. He writes, "I use the term *matrix* to denote any ability, habit, or skill, any pattern of ordered behaviour governed by a *code* or fixed rules" (p. 38). He uses the same definition later, substituting the phrase *pattern of activity* in place of *pattern of ordered behaviour*. The encompassing nature of these phrases allows one to include most processes used in the mathematics classroom, the caveat being that there must be some underlying order to the patterned activity. Indeed, as Koestler states, "all coherent thinking is equivalent to playing a game according to a set of rules" (p. 39). It follows that the term *matrix* can be applied to all coherent, logical or rule-based thought processes employed by an individual learning mathematics:

> The *matrix* is the pattern before you, representing the ensemble of permissible moves. The *code* which governs the *matrix*…is the fixed invariable factor in a skill or habit; the *matrix* its variable aspect. The two words do not refer to different entities; they refer to different aspects of the same activity. (Koestler, 1964, p. 40)

Creativity occurs when an individual observes an analogy between two or more previously unrelated frames of reference: "I have coined the term *bisociation* in order to make a distinction between the routine skills of thinking on a *single plane* as it were, and the creative act, which … always operates on *more than one plane*" (Koestler, 1964, p. 36). Thus, for Koestler (1964), *bisociation* represents a, "spontaneous flash of insight …which connects previously unconnected matrices of experience" (p. 45).

The creative process of bisociation, that is, the "transfer of the train of thought from one matrix to another governed by a different logic or code" (p. 95) is also used by Koestler to describe original inventions. For example, when Gutenberg fused together two matrices to invent the printing press, "the bisociation of the wine-press and seal, when added together, became the letter-press." (p. 123). Koestler, indeed, extends the range of these terms beyond problem-solving activities or creative inventions, to cover entire studies of science, citing such examples as Kepler's creative discovery of the laws of planetary motion resulting from his attempt at the, "synthesis of astronomy and physics which during the preceding two thousand years had developed on separate lines" (p. 124).

The focus of the Act of Creation Theory is on the bisociative leap of insight, that is, an *Aha!* moment, or a *moment of understanding*—a phenomenon that can be observed amongst the general population. It is our thesis that bisociation can be used to draw insights into the creativity of all students in a classroom setting. Thus, Koestler's theory can be seen as the foundation for understanding educational research on creativity and insights into learning i.e. the rediscovery of mathematics in the general classroom.

Learning Environment for Bisociation

Koestler would allow that bisociation is the essence of untutored learning: "Minor, subjective bisociative processes do occur on all levels, and are the main vehicle of untutored learning" (p. 658). This statement demonstrates two important viewpoints of Koestler that: (1) bisociation is an essential mechanism in the learning process, and (2) the subjective learning environment must allow for and/or approximate the conditions of "untutored learning". We believe, as did Koestler, that students cannot engage in mathematical problem-solving until they, in some sense, discover mathematics for themselves. This leads us to explore the mechanism of bisociation as foundational to learning mathematics, ideally within a creative learning environment that guides or supports the discovery process of students (Chapter 3.9).

For Koestler, bisociation explains the individual's use of analogies in learning and discovery. The distinction between the *Eureka* moment of originality and routine thinking is in the degree of novelty or unexpectedness of the analogy used. He writes, "one of the basic mechanisms of the *Eureka* moment is the discovery of a hidden analogy; but hiddenness is a matter of degrees. How hidden is a hidden analogy" (p. 653).

As teacher researchers with the goal of improving learning through a discovery based methodology and integrating Koestler's work into our classrooms we now study bisociation separating the mechanism itself from the affective experience in order to employ and analyze it in classroom practice. We also integrate Koestler's sense of an untutored learning with the work of Vygotsky who makes the classroom learning environment pivotal. This allows us to speak of bisociation within a mathematics classroom in which a methodology of guided discovery but not necessarily completely untutored learning is employed. Koestler, Vygotsky, cognitive theorist and theories of learning influenced by Piaget all consider problem solving as the underlying environment for learning. Thus, we extend our focus from the work of Koestler and Vygotsky to include insights on problem-solving and learning from the work of Anderson (cognitive theory), Von Glaserfeld, Sfard, Dubinsky and others (Piaget based theories of concept and schema development).

The Mechanism of Bisociation

In his 2012 work, Berthold describes Koestler's concept of bisociation as a *foundation* for the study of computational creativity, where he separates the transcendental *Aha!* moment from the mechanism of bisociation, that is, the simultaneous synthesis of the matrices (Berthold, 2012). Berthold describes several instances of bisociation: "The most natural type of bisociation is represented by a concept linking two domains" (p. 3). In this case, the concept is understood as being represented in both domains simultaneously. Berthold notes that a small *subset* of the concepts (as opposed to an *individual* concept) contained in separate matrices can also be viewed as linking the matrices or domains simultaneously. A third type of bisociation is a link formed by a structure between concepts that can be represented in two matrices: "In both domains two subsets of concepts can be identified that share a structural similarity" (p. 5). One might imagine such a situation as an isomorphism of these structures. In this approach, the transfer of concept(s) or structure(s) of concept(s) between any two matrices or independent frames of reference is an example of bisociation.

Subjectivity in Bisociation

The work of Berthold distinguishes the mechanism of bisociation from the requirement that the individual experiences an *Eureka* moment based solely upon their own cognitive effort. An issue that arises in applying this mechanism of bisociation to learning is how independent (objectively) do the matrices have to be. This is especially relevant with students experiencing difficulty learning mathematics. Koestler considers the situation subjectively, that is, entirely dependent upon the level of the individual understanding: "Thus, the degree of independence of the matrices … which combine in the solution of a problem can only be judged with reference to the subject's mental organization" (p. 657). Koestler makes an inverse relationship between the subjects' previous knowledge in solving a problem with the amount of learning or creativity required. His view is expressed in his distinction between an *exercise of understanding,* that is, an association of a problem situation to a familiar habitual matrix of thought, and *progress in understanding*, when the solver does not recall any familiar matrix or schema to interpret the information given in the problem. We review Koestler's distinction between association and bisociation, as well as relevant literature on schema development, in order to situate this dynamic within the context of Piaget's concepts of assimilation and accommodation. At present we note that Koestler's distinction between habitual associations and creative bisociations rests upon the degree of consciousness in discovering the hidden analogy: "Among the criteria which distinguish originality from routine are the level of consciousness on which the search is conducted, the type of guidance on which the subject relies and the nature of the obstacle" (p. 654). The subjective nature of bisociation, as dependent upon the mental organization and proficiency of the learner, leads us to review literature on stages of learning. We also note that Koestler's inclusion of level of consciousness is a unifying factor with (1) Vygotsky's theory of concept development in which *reflective consciousness* is the distinguishing factor between spontaneous and scientific concepts, and (2) models based upon the work of Piaget in which the mechanism of concept development is *reflective abstraction*.

The Hidden Analogy in Problem-Solving

If one were to look up the word analogy, it will be most commonly defined as a 'correspondence in some aspect between otherwise dissimilar things.' As commented, for Koestler, the dissimilar criteria is subjective, that is, dependent upon the level of understanding of the problem solver. When a solver recognizes the given problem information as analogous to a known schema, the association leads to routine, habitual thought—an exercise in understanding. In contrast, when no such analogous schema comes to mind the solver begins to search through their mental collection of operators-matrices that in some way may have some correspondence to the present situation. This process of finding a hidden analogy is described by Koestler as:

> bring[ing] successive perceptual or conceptual analyser-codes to bear on the problem; to try out whether the problem will match this type … the subject looks for a clue the nature of which he does not know, except that it should be a clue … a link to a type of problem familiar to him…he must try out one frame after another to look at the object before this nose, until he finds the frame into which it fits, i.e. until the problem presents some familiar aspect-which is then perceived as an analogy with past experience and allows him to come to grips with it. (pp. 653–654)

CREATIVE DISCOVERY AND CONCEPT DEVELOPMENT WITHIN PROBLEM-SOLVING

The Problem-Solving Environment

Koestler's theoretical foundation for creativity and, to a wider extent, learning itself, is based upon problem-solving: "Problem solving is a gap between the initial situation and the target … It means firstly, searching for a matrix, a skill which will bridge the gap" (pp. 650–651). Problem-solving leads to progress as opposed to merely an exercises in understanding when,

> the situation resembles in some respects other situations encountered in the past, yet contains new features or complexities which make it impossible to solve the problem by the same rules of the game which were applied to those past situations. When this occurs we say the situation is blocked. (p. 119)

Vygotsky (1986), like Koestler and Anderson, situates learning, or more specifically, concept development, with the framework of problem-solving. He writes that, "concept formation…is an aim directed process…for the process to begin, a problem must arise that cannot be solved otherwise than through the formation of a new concept. Although problem-solving is one of largest branches of mathematics education, much of what is understood as *problem-solving* comes from cognitive psychology (Lester, 1983). The cognitive theorist Anderson (1996) asserts that, "all cognitive activities are fundamentally problem solving in nature … human cognition always purposeful, directed to achieving goals, and to removing obstacles to those goals" (p. 237). Anderson (1996) lists three traits that characterize problem-solving: (1) goal directedness, (2) sub-goal decomposition, and (3) operator selection, where "the term *operator* refers to an action that will transform the problem state into another problem state. The solution of the problem is a sequence of these known operators" (p. 238).

Anderson and Koestler both base their models, in part, on the earlier work of the psychologists Kohler with chimpanzees in captivity. Kohler noticed that primates who played with sticks and reached out of their cages for food would eventually learn to use sticks as a tool. For Koestler, this spontaneous synthesis is an example of bisociation to extend their reach. Anderson describes this behaviour in terms of cognition, using the stick (*the operator*) to extend one's reach (*obtaining the goal*) (pp. 237–244). Anderson's use of the term *operator* can be viewed as equivalent to an individual's use of their internal matrices to formulate an action during problem-solving. By this we mean that Koestler understood his framework as describing routine problem-solving as well as creative activity, "the hum-drum routine of planning and problem-solving in daily life" (p. 651). However, Koestler's use of the matrix includes more than just a skill or action: "I shall use the word matrix to denote any ability, habit or skill, any pattern of ordered behaviour governed by a *code* of fixed rules" (p. 38). Anderson purports that an individual acquires operators to obtain a goal in one of three manners. The first is through direct instruction, the second is by discovery, and the third is using analogy to previous solutions (p. 244). Thus, for Anderson, there is a sharp distinction between analogy and discovery used in what he calls 'insight' problems.

The unification of creativity and concept development, and indeed, all cognition within the framework of a problem-solving environment leads us to simultaneously consider the work of Vygotsky, Piaget and Anderson and authors of concept development theories based on the work of Piaget.

Vygotsky: Conscious Reflection and Scientific Concepts

Vygotsky's framework focuses on the development of structural or scientific concepts, such as those of algebra generalized from the arithmetical concepts that arise spontaneously. This transition is marked by an individual's ability to reflect upon one's thinking process. He writes, "in spontaneous concepts the child is not conscious of them because his attention is always centred on the object to which the concept refers, never on the act of thought itself" (Vygotsky, 1986). In this reference Vygotsky's use of the word *consciousness* indicates the awareness of one's thought process, "we use consciousness to denote awareness of the activity of the mind – the consciousness of being conscious" (Vygotsky, 1986, p. 171). This meta-consciousness, or reflection upon one's thought process, in Vygotsky's framework comes into existence simultaneously with scientific concepts, "reflective consciousness comes to the child through the portals of scientific concepts" (Vygotsky, 1986, p. 171). Furthermore, these scientific concepts are inherently embedded in a schema or hierarchy of concepts, "a concept can become subject to conscious control only when it is part of a system" (p. 171). Thus, the learner's development, as they reflect upon their actions during algebraic and mathematical reasoning, is dependent upon formation of schema which for Vygotsky (in contrast to Koestler) is built within a cooperative relationship between the teacher and the learner.

Vygotsky: Role of Education in Concept Development – Zone of Proximal Development

In contrast to Koestler, Vygotsky (1986) has a strong focus on the role of education in concept-schema development:

> Instruction is one of the principal sources of the schoolchild's scientific concepts and is also a powerful force in directing their evolution. (p. 157)
>
> School instruction induces the generalizing kind of perception and thus plays a decisive role in making the child conscious of his own mental processes. Scientific concepts, with their hierarchical system of interrelationships, seem to be the medium within which awareness and mastery first develop. (p. 171)

This consciousness as suggested by Vygotsky arises with adolescence, during the middle school years of education when students are required to, and many struggle to, learn fractions, and proportional and algebraic reasoning. Koestler (1964) considers such consciousness as essential to originality and creativity in the search for an operator-matrix to solve a problem:

> The search for the appropriate matrix is never quite random…among the criteria which distinguish originality from routine are the level of consciousness on which the search is conducted. (Koestler, p. 654)

The importance of education in Vygotsky's framework is due in part to his observation that when presented with direct instruction assistance some students, "could with cooperation solve problems … while other could not" (Vygotsky, 1986, p. 187). Vygotsky labels this phenomena the child's Z*one of Proximal Development* (ZPD), defining it as "the discrepancy between the child's mental age and the level he reaches in solving problems with assistance" (p. 187). The role of education in Vygotsky's framework is to present problems on the upper structural level of the individual's ZPD, and then provide them with guidance in reaching that goal. To the constructivist like Koestler, who believes that learning only has meaning when the individual reinvents or relives the process of discovery, Vygotsky would counter that guided learning, even when students follow instruction without grasping the essential processes, is valuable within the individual's ZPD. He states that "to imitate, it is necessary to possess the means of stepping from something one knows to something new" (Vygotsky, 1986, p. 187).

Our own teaching-research experience, as facilitators of 'guided inquiry leading to discovery' method allows us to find the space of freedom within which, we can, in the context of a student's ZPD, eliminate the possibility of imitation and approximate the conditions of "untutored learning" necessary for Koestler's bisociation as the tool to "reinvent or to relive" the process of discovery.

Progress in Understanding

Koestler (1964) characterizes *progress in understanding* as a process that results in, "the formulation of new codes through the modification and integration of existing codes by…empirical induction, abstraction and discrimination, bisociation" (p. 619). In contrast to progress in understanding, Koestler asserts that an, "exercise or application of understanding … becomes an act of subsuming the particular event under the codes formed by past experience" (Koestler, 1964, p. 619).

We outline several distinguishing features that Koestler draws between association, as an exercise in understanding, and bisociation, as progress in understanding.

- *Consciousness:* Association may involve conscious thought and reflection; however, bisociation is marked not only by such thought but also by intuition led discovery. In the discovery classroom this guidance is supplied by the instructor, leading the student to the upper limit of their ZPD. For Koestler, such guidance must come from the individual's intuition: "a further criterion of the creative act was that it involves several levels of consciousness" (1964, p. 658).
- *Independent Matrices and Novelty of Task:* Association occurs, "within the confines of a given matrix," while, in contrast, "bisociation involves independent matrices" (Koestler, 1964, p. 658). We have noted that this criterion is subjective,— matrices that are separate for students at one level, may have already been synthesized for more advanced solvers. For Koestler, progress in understanding occurs when the solver encounters a new or novel task. In this framework, when a solver is faced with such a novel situation in which there does not exist any associated matrix, then he or she begins searching for a hidden analogy, a matrix with some analogous aspect to both, that will give some meaning to the situation, allowing them to come to grips with it.

Thus, Koestler allows that the problem information itself is a matrix, a reality that may be confusing for the solver, and the hidden analogy is the other independent matrix. While it does happen that two matrices are synthesized as in the development of the printing press, progress in understanding, the creative bisociative act includes the discovery of a previously hidden analogy and the synthesis of past experiences with new problem situations in such a manner as to give meaning to the present situation.

Associations occur when the solver recognizes the task as familiar and leads to, what Koestler would refer to as, the *condensation of learning into habit*. He writes: "The matrices that pattern our perceptions, thoughts and activities are condensations of learning into habit" (Koestler, 1964, p. 44). Anderson describes this process in terms of the solver's ability to associate the declarative problem information with an appropriate operator-matrix. With practice this process becomes a more automatic procedural response with less reliance upon the conceptual problem information. Anderson refers to this as *proceduralization*, which, he believes, requires less cognitive load that is, "the procedure becomes more and more automated and rapid" (Anderson, 1996, pp. 274–275). For Koestler (1964) this transition is an anathema to the creative process:

> When life presents us with a problem it will be attacked in accordance with the code of rules which enabled us to deal with similar problems in the past … when the same task is encountered under relatively unchanging conditions in a monotonous environment, the responses will become stereotyped, flexible skills will degenerate into rigid patterns, and the person will more resemble an automaton, governed by fixed habits. (pp. 118–119)

SCHEMA THEORY: ASSIMILATION AND ACCOMODATION

For Koestler (1964) the end result of learning is construction of matrices or schema to make sense of our environment: "We learn by assimilating experiences and grouping them into ordered schemata, into stable patterns of unity in variety. They enable us to cope with events and situations by applying the rules of the game appropriate to them" (p. 44).

Steele and Johanning (2004) note two types of learning commonly discussed by schema theorists—assimilation and accommodation. They consider assimilation to be a type of generalization when one extends an existing cognitive structure without significant change while accommodation requires reconstruction of an existing cognitive structure. Norman and Rumelhart (1976) consider the situation when existing schema serve as the basis for new ones leaving basic structure or relationships unchanged a process they refer to as '*schema tuning*'; in Koestler's terminology, the code of the matrix remains basically unchanged, and this process does not result in the creation of new matrix/schema. In contrast, more creative forms of accommodation occur, "when existing memory structures are not adequate to account for new knowledge, then new structures are required, either by erecting new schemata specifically designed for the troublesome information or by modifying (tuning) old ones" (Norman & Rumelhart, 1976, p. 45). As Cifarelli notes, when one's previous or "current knowledge results in obstacles, contradictions surprises" (Cifarelli, 1998, p. 241).

Von Glasersfeld (1989), reviewing the work of Piaget, cooments that:

[The] construction of a scheme … consists of three parts:

1. Recognition of a certain situation…
2. Association of a specific activity with that kind of item…
3. Expectation of a certain result (p. 65).

The ability of a student to recognize a previous schema that relates to the situations is viewed by math educators as the first (recognition) stage in the development of problem-solving (Cifareli, 1998). Educators frequently employ repetition leading to internalization and thus the ability for recognition of problem schema. This is often accompanied by problem solving sequences with increasing difficulty or level of structure and abstraction (Steele & Johanning, 2004) e.g. Czarnocha (4.6 Chapter 4.5). The question that arises from a constructivist pedagogical view is when does repetition lead to rote memorization?

Von Glaserfeld (1989) points out that accommodation may begin when solution activity or the choice of an operator does not bring about the expected result. More precisely, he states that such an obstacle, "generates a perturbation and *disappointment,* and perturbation is one of the conditions that sets the stage for cognitive change" (p. 127). In this view the essence of constructivist pedagogy and creativity within education begins with perturbation or disappointment i.e. the lack of ability of the solver to recognize or associate a previous matrix-schema to a given situation.

Creativity, Reflection and Abstraction

Von Glaserfeld (1989b) postulates that the ability to search for patterns, regulations, groupings and rhythms is innate, "the ability and, beyond that, the tendency to establish recurrences in the flow of experiences; this, in turn, entails at least two capabilities, remembering and retrieving (re-representing) experiences, and the ability to make comparisons and judgements of similarity and difference" (Von Glasersfeld, 1989a, p. 128). This innate ability underlies Von Glaserfeld's argument that creativity is involved in the learning process, when the learner is presented with a cognitive conflict, reviews a past situation or a pattern from past situatiosn and then abstracts a rule or code from this pattern that can be used in the present situation. This abstraction process for Von Glaserfeld is a creative and in our view comparable to Koestler's description of the discovery of the hidden matrix. It is considered by Von Glaserfeld as a central mechanism in the learning process. As such is comparable (in intent if not content) to Simon et al. (2004) description of Piaget's notion of reflective abstraction based upon an individual's reflection upon their solution activity and the abstraction of underlying relationships:

> Thus, within each subset of the records of experience (positive or negative results), the learners mental comparison of the records allows for recognition

> of patterns, that is, abstraction of the relationship between activity and effect. Because both the activity and the effect are embodiments of available conceptions, the abstracted activity-effect relationship involves a coordination of conceptions … Note the activity and the effect are conception-based mental activities, our interpretation of Piaget's notion of coordination of actions. (pp. 319–320)

The view expressed by Simon et al. (2004) that learning is a result of reflection upon solution activity and abstraction of relationships suggests that Koestler and Von Glaserfeld's description of creativity would be included as such activity. We note however that, the focus of Simon et al. (2004) description of reflective abstraction is on Von Glaserfeld's third stage of schema formation that is, when the solver compares the results of his/her solution activity with their expected outcome. In contrast, Koestler's bisociation as the discovery of a hidden analogy focuses on the first stage i.e. recognition of an appropriate matrix yet both would be included as reflection of solution activity and both require abstraction of underlying principles and hence both can be considered as reflective abstraction when defined loosely as, "a means of classifying and characterizing problem solving activity" (Tracy Goodson-Epsy, 1998, p. 226).

Piaget: Reflective Abstraction

Piaget's mechanism for conceptual development is called *reflective abstraction* and, as used in mathematics education is associated with conscious reflection and abstraction of solution activity (Simon et al., 2004). Piaget focuses his attention on the individual's reflection upon and abstraction of processes or actions rather than concepts. According to Piaget, construction of schema or cognitive change:

> proceeds from the subject's actions and operations, according to two processes that are necessarily associated: (1) a projection onto a higher level (for example, of representations) of what is derived from the lower level (for example, an action system), and (2) reflection, which reconstructs and reorganizes, within a larger system, what is transferred by projection. (Piaget & Garcia, 1989, p. 2)

Piaget refers to these two processes as *constructive generalization* and *reflective abstraction*. He clarifies their role in the construction schema:

> First, projection essentially establishes correspondences at the next higher level, associating the old contents that can be integrated within the initial structure but permitting it to be generalized. Second, these first organizations also lead to the discovery of related contents, which may not be directly assimilated into the earlier structure. This makes it necessary to transform that structure by means of a constructive process until it becomes integrated within a larger, and

> therefore partially novel, structure. This mode of construction, by *reflective abstraction* and *constructive generalization*, repeats itself indefinitely, at each successive level so that cognitive development is the result of the interaction of a single mechanism. (Piaget & Garcia, 1989, pp. 2–3)

Piaget's description is perhaps influenced by his intention to explain individual and historical development as parallel to one another and thus it has a one directional approach in which a lower less structured plan is always projected into a higher structured one and then coordination occurs to re-organize or constructively generalize the schema. In contrast Koestler's mechanism of bisociation allows for the synthesis of two matrices without consideration of a higher or lesser structure. Furthermore, this description by Piaget and Garcia appears to imply a certain time period for coordination to re-organize one's schema while Koestler's bisociation is instantaneous. However, both descriptions involve two planes of reference that are related to one another presumably by concepts that upon reflection results in abstraction of relationship or codes that allow for progress in understanding or accommodation of one's existing schema.

Dubinsky (1991) studies Piaget's mechanisms of reflective abstraction with respect to actions and processes within the process/object duality of concept development. Dubinsky considers and names specific forms of reflective abstraction: *interiorization*, *coordination*, *generalization*, *reversibility* and *encapsulation*. Interiorization involves the internalization of processes or actions, while reversibility involves reflection upon the inverse of a known process. The strength of this important contribution by Dubinsky is that specific types of reflective abstraction are named and spelled out which assists the instructor-researcher in designing curricula and methodology to encourage and promote such thought in the mathematics classroom. However, for our conceptual framework this effort to categorize specific instances of reflective abstraction removes the focus from problem solving activity in which creativity and learning occur. That is, Koestler's main concern of how the solver recognizes or an appropriate matrix and the similar issue raised by Piaget of how one schema is used to constructively generalize another are not directly addressed.

Bisociation and Reflective Abstraction

Within Koestler's problem-solving framework, when the solver encounters a novel problem and cannot directly associate a relevant or analogous matrix, he or she begins searching for a matrix-operator based upon some aspect of the problem information. If, during the coordination of problem information with previous matrices, the solver realizes that the concepts and structures of a simpler, more concrete, or more readily understood matrix can be generalized to the current situation, bisociation is achieved. At this phase, in the solver's mind, the better understood concept structure exists simultaneously in both frames of reference, and the stage for schema formation, or *constructive generalization*, is set. One may view the problem situation as an emerging matrix being structured by the bisociative action of the generalization.

Like Simon et al. (2004), we consider the comparative process between the mental records (matrices) of thought with the problem information as *coordination*, employing Koestler's understanding of this mental activity; that is, one during which the solver is searching through existing matrices, comparing each to the declarative information presented by the problem to be solved. The term *coordination*, in this context describes this flux of comparisons in the solver's mind, in his or her attempt to find a match between some known matrix and some aspect, of the problem structure. This search sets the stage for the bisociative realization that previously unrelated concepts and structures can, indeed, be associated with the problem at hand. This, coordination prepares the way for bisociation which, in turn, provides meaning to the solver's choice of operator. We note that coordination continues after an operator is chosen, as the solver compares his or her memory of how the operator correlated with the current situation and, consequently, how the current situation is connected to the ultimate problem goal, Simon et al (2004).

In the event that no effective operator-matrix can be identified, the solver, with external (tutor, instructor, or peer) or intuitive guidance, may revisit a similar but simpler problem situation, invoking a matrix that is appropriate for a *simpler*, more concrete or better understood problem situation that resembles the given situation. This initial or primary matrix is then projected (generalized) into the current problem situation. In this process, the concepts/structure of the initial matrix is bisociated with the existing problem situation (emerging matrix), a direct result of which is an operator or relevant action for the solver. *Constructive generalization* occurs when the actions directed by the solvers understanding of the initial matrix are coordinated with the problem information and goal.

PROCESS-CONCEPT DUALITY

The transition from spontaneous to scientific concepts in the work of Vygotsky has a parallel in theories influenced by the work of Piaget in which there is a transition from a process to an object understanding. This feature based upon Piaget's insight that, with sufficient exposure and use of "actions" or processes an individual will internalize these processes enabling them to reflect upon and thus treat what was essentially procedural knowledge as "thematized objects of though" which can then be assimilated into an individual's schema (Tall et al., 2000, p. 224).

Tall et al. (2000) attributes educational focus on process/object duality to the work of Dubinsky and Sfard: "The transformation of a process into an object took new impetus in the work of Dubinsky and Sfard" (p. 234). Although we utilize the work of both these authors, the work of Sfard is more focused on the transition from arithmetic to algebra.

Operational to Structural

Operational to Structural: History and Individual Development of Algebra Sfard and Linchevski (1994) like Piaget and Garcia (1989) trace out the transition from operational to structural thinking in both the historical development of arithmetic and algebra as well as through the lens of individual growth in mathematics. Thus, they operate under the thesis that individual growth in mathematical understanding parallels the historical development i.e. an operational view precedes a structural understanding.

More specifically Sfard and Linchevski (1994) trace the growth of a concept from an operational understanding of arithmetic, algebra and mathematics as computational actions, procedures, and algorithms to a structural understanding in which these processes are themselves considered existing, real objects that can themselves be acted upon. From an operational viewpoint an algebraic expression such as $5x - 7$ would be viewed as an algorithm containing: an unknown quantity ‘x’ and two procedures or actions to be performed once the value of the unknown variable has been revealed. From a structural viewpoint this expression is an object named a binomial, more generally a polynomial, for which there are rules that govern how such objects can be combined. Sfard and Linchevski (1994) considers that the operational view precedes the structural view and that the capacity of structural conception is a distinct advantage for the solver able to make this transition: “What happens in such a transition may be compared to what takes place when a person carrying many things in her hands decides to put all the load in a bag” (p. 94).

Student growth in mathematics can be explained as a transition from a spontaneous understanding of concepts or operational understanding of actions and processes to a more scientific and structural understanding. In this light the instructor’s role can be to guide them through their ZPD from where their understanding is to a more structural level. As we have noted Vygotksy viewed the development of scientific concepts and its associated structure as developed in an educational setting. The student’s ability to assimilate new information or accommodate new structurtal growth is dependent upon where they are in this developmental process e.g. Dias (4.2 Chapter 4.3). For this reason Piaget and Garcia, Sfard, Vygotsky as well as cognitive psychologists all postulate stages of learning or concept development that are useful in classify and interpreting student experience in the mathematics classroom.

STAGES OF CONCEPT-SCHEMA DEVELOPMENT

Piaget and Garcia: Intra Stage

Piaget and Garcia characterize the initial or intra stage by, “a concentration on repeatable actions or a correct operation…These remain isolated and analyzed and comprehended only in terms of their properties taken individually and irrespective of others" (p. 175, Piaget & Garica, 1989). These authors characterize subjects in this stage as having limited ability to make deductions based upon sometimes faulty internal matrices-processes and inability to coordinate these process, "an absence of coordination…and the presence of errors…as well as gaps in the consequences subjects are able to deduce (p. 174). In short, "without the capacity to inset it in a system of conditions or consequences, which would extend its application" (p. 175).

Sfard (1991) would characterize the first stage as beginning with learner's who need to carry out one-step at a time actions using pencil and paper and ending with an internalized process. She borrows the term "internalization" from Piaget to describe an internalized process as one that, "can be carried out through mental representations" (p. 18).

Challenges for Constructivist Pedagogy with Weak Problem Solvers

Less proficient solver often chose operator based upon whatever operator was used in the previous problem a phenomena cognitive psychologists refer to as "temporal proximity" (Anderson, 1995). When the choice of operator is based upon declarative problem information (concepts) their rational can be hazy, and they become easily, "surprised by complications that might occur" (Goodson-Epsy, 1998, p. 224). At this level of understanding the solver has only a vague sense about what analogous matrix may be appropriate. In such a situation their previous understanding of schema provides an unstable source for: choosing, planning or evaluating an action or operator(s). Their choice of an operator-action is tenuous, one step at a time, without understanding to guide their actions and choices. Thus, when an operator does not immediately work i.e. at the first sign of difficulty they become discouraged and give up i.e. admit defeat. Their lack of confidence and interest prevents them from searching for a different strategy or a different operator which in turn limits their lack of ability to reflect upon, "elaborate, re-organize and re-conceptualize their solution activity while engaged in mathematical problem solving" (Cifarelli, 1998, p. 241).

The paradox for constructivist pedagogy is that philosophically it believes students only learn or create knowledge when they must accommodate new information or demonstrate progress in understanding i.e. in situations where solvers must, modify an existing representations, matrix or schema and yet it precisely such situations that weak solvers experience the most anxiety.

Intermediate 'Inter' Stage

Piaget and Garcia (1989) characterize the inter stage by, "the capacity to deduce from an initial operation, once it is fully understood, others which are implied by it or to coordinate it with other similar ones" (p. 174). At this level actions or "operations become organized into systems…which include certain transformations of the operations themselves" (p. 174). Thus, one distinguishing mark of this stage is the reflective abstraction referred to by Dubinsky (1991) as 'coordination' of processes another is 'reversibility.' Piaget and Garcia (1989) refer to these abstracting processes as transformation and consider 'negation' or 'reciprocities' as, "two possible forms of reversibility that is the common characteristic of all operations and their compositions, hence that essentially mark the progress between intra and inter-operational systems (p. 177).

However, the intermediate stage is still limited to a within structure coordination of processes, "but there are still fairly limiting restrictions on the compositions possible…a strictly step-by-step approach" (Piaget & Garcia, 1989, p. 176).

At this stage of reflection and abstraction solvers can, "reflect upon their potential solution activity and generate anticipations about its results without the need to actually carry out the particular actions with paper-and-pencil" (Cifarelli, 1998, p. 246). More detail is added to this basic theme by Sfard (1991) who notes the leaner "becomes more and more capable of thinking about a given process as a whole without feeling an urge to go into details…the leaner would refer to the process in terms of input and output." As a result of this reflection and abstraction this stage is characterized by, "combining, making comparisons and generalizing…growing easiness to alternate between different representations of the concept" (p. 19).

The intermediate stage of concept development for Vygotsky is 'complex thinking' this stage is marked by relationships between concepts which are 'coherent and objective' and yet in which, "ideas are based on experience and associations rather than logic or a system but the learner is able to abstract actual attributes of the idea" (Berger, 2004, p. 11). Berger (2004) identifies many different types of errors students make that fit the different types of association solvers make between declarative or conceptual problem information and appropriate operators:

- Surface Associations in which a keyword, or other part of a problem is focused on to the exclusion of other more relevant concepts and thus an inappropriate operator is chosen.
- Errors due to complementary concepts in which the solver recognizes the appropriate matrix and yet the matrix involves complementary or related concepts that the student cannot distinguish between or properly identify.
- Associations switching errors when the solver employs a matrix-operator instead of its inverse that is required. Lamon (2007) refers to such errors as "non-conservation of operation,"
- Incomplete association errors occur when the solver makes a required association to a matrix-operator but, he/she applies it incorrectly due to lack of coordination with other concepts-processes.

A Paradigmatic Example

In this example finding the domain of f(x) = $\sqrt{x}$ was understood by the student but they had difficulty finding the domain of the function $\sqrt{x+3}$

Note that it is the spontaneous responses of the student from which the teacher-researcher creates/determines the next set of questions, thus, balancing two frames of reference, his/her own mathematical knowledge and the direction taken by the student. Thus this example of a TR act balances pedagogy influenced by constructivist theory that students need to construct, discover or create their own meaning with student responses.

The student prompted by the teacher-researcher's questions, balance their understanding-matrix for finding the domain of f(x) = $\sqrt{x}$ with the new problem situation.

The problem starts with the function $f(x) = \sqrt{x+3}$. The teacher asked the students during the review: "Can all real values of x be used for the domain of the function $\sqrt{x+3}$?"

Student (**S**): (**1**) "No, negative x's cannot be used." (The student habitually confuses the general rule, which states that for the function $\sqrt{x}$ only non-negative values can be used as the domain of definition, with the particular application of this rule to $\sqrt{x+3}$.)

Teacher (**T**): (**2**) "How about $x = –5$?"
S: (**3**) "No good."
T: (**4**) "How about $x = –4$?"
S: (**5**) "No good either."
T: (**6**) "How about $x = –3$?"
Student, after a minute of thought: (**7**) "It works here."
T: (**8**) "How about $x = –2$?"
S: (**9**) "It works here too."
A moment later the **student** adds: (**10**) "Those x's which are smaller than –3 can't be used here." (***Elimination of the habit through original creative generalization.***)
T: (**11**) "How about $g(x) = \sqrt{x-1}$?"
Student, after a minute of thought: (**12**) "Smaller than 1 can't be used."
T: (**13**) "In that case, how about $h(x) = \sqrt{x-a}$?"
S: (**14**) "Smaller than a cannot be used."(***Second creative generalization***)

Analysis

The student makes an incomplete association from the matrix of finding the domain of a function such as f(x) = $\sqrt{x}$ to this problem situation which involves a transformation of the previous situation (the argument within the square root, x+3)

In lines (6) and (8) the instructor employs concrete counter examples to provide a *perturbation,* or a *catalyst*, for cognitive conflict and change. "…perturbation is one of the conditions that set the stage for cognitive change" (Von Glasersfeld, 1989a, p. 127).

In lines (**6**) – (**9**) the student reflects upon the results of the solution activity. Through the comparison of the results (records) they abstract a pattern,—"the learners' mental comparisons of the records allows for recognition of patterns" (Simon et al., 2004).

Evidence of the abstraction of the principle that the transformation of the argument x+3 beneath the square root shifts the domain 3 to the left is given in line (10). Thus, this realization can be seen as reflective abstraction in the sense of Simon et al. (2004). In that the solution activity of substitution was projected into and coordinated with the solver's knowledge of the domain of the square root function it is also an example of reflective abstraction according to Piaget and Garcia (1989). In that solution activity through substitution with the students matrix or schema of evaluating $x - a$ is coordinated or integrated by the student with the student's understanding (matrix) of the domain of the square root function their realization in line (10) can be viewed as bisociative in nature.

In lines (**11**) and (**12**), the perturbation brought about by the teacher's questions, leads the student to enter the second stage of the Piaget and Garcia's Triad. The student understood that the previously learned matrix or domain concept of radical functions, with proper modifications, extended to this example. They student was then able to reflect upon this pattern and abstract a general structural relationship in line (**14**). Thus, the student's understanding of finding the domain of a square root has undergone constructively generalization to accommodate transformation providing evidence of the structural understanding noted by Sfard (1991) and the third stage of the Triad (Piaget & Garcia, 1989).

The TR act of the teacher manifests itself in the scaffolding which led the student to the cognitive conflict between the two frames of reference. In the first case, the data driven results obtained through the matrix-process of substitution was synthesized with their limited matrix of the possible domain of a radical function. This bisociation, and the resulting abstraction, led to a more complete understanding of the possible domain for specific functions. This represents a transition from the first to the second stage of the Triad. Continuation of this questioning process led to further creative moments of understanding, in which the student was able to synthesize their understanding of the domain for two separate special cases of radical functions. This bisociation, and the resulting abstraction into structural understanding (line (**14**)), suggests that the student had crossed the ZPD from the second to the third stage of the Triad relatively to the concept of the domain of the square root function.

Trans Stage: In the final trans stage of schema development is characterized by the ability to not only coordinate processes but also synthesize them into new structures: "There are not only transformations, but also syntheses between them, leading the way to the building of new structures" (Piaget & Garcia, 1989, p. 178).

Sfard (1991) describes structural thought within problem solving as the ability to understand the structure, the conceptual relationships inherent in a problem solving schema as opposed to a one-step at a time sequential approach:

> the operationally conceived information, although absolutely indispensible and seemingly sufficient for problem-solving, cannot be easily processed. This kind of information can only be store d in unstructured, sequential cognitive schema, which are inadequate for the rather modest dimensions of the human working memory. (p. 28)

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4.2 COMPARATIVE STUDY OF THREE APPROACHES TO TEACHING RATES

SUMMARY

This chapter has a special aim and organization. It presents the comparative study of three different approaches to the same theme of proportional reasoning, in particular of the concept of the rate. The idea for the study is related to the Chinese Keli lesson study method (Huang & Bao, 2006), whose one of the approaches is the observation of two classes taught by different instructors and presenting different approaches to the same theme (CTRAS 5, 2013) followed by the discussion comparing the instructional approaches. Both instructors as well as the observing team of teachers participates in, usually heated, discussions. The chapter starts with the review of literature background for the theme of proportional reasoning as well as with short characterization of each of the approaches taken by instructors. Remarks of the members of the team follow.

INTRODUCTION: RATES AND PROPORTIONAL REASONING

Norton and D'Ambrosio (2008) discuss the role of the teacher in assisting students cross their zone of proximal development (ZPD) within the context of a teaching experiment, which, following Steffe and Thompson (2000) they take to be one in which the teacher must continually establish student understanding of content and adjusting pedagogy to best suit this understanding.

Norton and D'Ambrosio consider that the essence of the teacher's role is to serve as a consciousness that assists in directing the student until he/she can direct their own actions during problem solving. Scaffolding is one of the most important tools the teacher can use to guide discovery learning. Mariotti (2009) describes this role as one in which the teacher uses tools or artefacts to mediate between the student's understanding and their potential i.e. to develop genuine understanding that is linked to the artefact but able to be generalized and thus detached from the artefact.

Thus, the artefact, scaffolding or methodology used by the teacher provides external assistance until the learner moves towards internalization and self-help. The role of the teacher involves not only setting an appropriate task for the students but also directing their attention to appropriate matrices that can be used in the

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present situation i.e. to lead them towards bisociation that is the discovery of a previously hidden analogy. When this does not work, the teacher can simplify the task, perhaps choosing a simpler problem in which the underlying concepts and problem structure are clearer more readily grasped. On the other hand if the problem involves the coordination of several processes the teacher may lessen the cognitive demand by leading the student one step at a time. In answer to Richardson (2003) question about what constitutes constructivist pedagogy, Norton and D'Ambrosio (2008) consider on the one hand a classroom setting in which the teacher sets the goals then encourages student activity to reach these goals while filling in gaps i.e. maintaining conscious control over the goals and the activity; on the other hand, they postulate a setting involving group work in which the students negotiate goals and sub-goals without guidance or minimal guidance by the instructor. If the essence of constructivist pedagogy is student construction of knowledge these authors note that Vygotsky (1997) would make a distinction between intelligent and conscious imitation of students that allows for construction of meaning as opposed to rote training that, "… results in meaningless … habits" (p. 221). In this they are similar in nature with Koestler's distinction between an exercise in understanding i.e. Piaget's notion of assimilation and Koestler's progress in understanding i.e. Piaget's notion of accommodation. In this dynamic it is not the cognitive difficulty of the material that dictates creativity and learning of the individual per se but rather the sole valid measure is the quality of the reflection. In this sense a constructivist classroom is essentially distinguished by effort to promote conscious reflection during problem solving as opposed to efficiency through repetition i.e. what Anderson (1995) would refer to it as "proceduralization."

Huang and Bao (2006) struggle with the question posed by Richardson (2003); of how to promote constructivist pedagogy. Their approach builds upon action research, case studies in which educational experts gather to reflect and analyse how to assist, "… teachers not only to develop an understanding of mathematical tasks and how the cognitive demands evolve during a lesson, but also to develop the skill of critical reflection on their own practice…" (p. 280). These authors note that it is naive to think that a professional development workshop i.e. top-down approach can transform a teacher into an expert who is capable of reform pedagogy. Huang and Bao approach for teacher development employed experts involved in a community of inquiry undergoing cycles of reflection and refinement on teacher lessons as oppose to expert's lessons i.e. design science.

In the following work three teacher researchers Czarnocha, Dias and Baker follow the inspiration of Huang and Bao (2006) and review each other's lessons and methodology in order to understand the role of the instructor within a constructivist pedagogical framework or, if you will, acting as a community of inquiry into the promotion of a creative learning environment. We review and critique lesson plan as teacher researchers with the main focus on the conceptual framework(s) involved.

PROPORTIONAL REASONING

The three lessons all involve applications of ratio, rates etc. The setting within educational research for understanding students' difficulties with rates, ratios are typically referred to as proportional reasoning. For Piaget, proportional reasoning marked the beginning of formal operational thought because it involved second order thinking about the relationship between two ratios (Inhelder & Piaget, 1958). Proportional reasoning has been described as a foundation or core of algebra and higher mathematics (Berk et al., 2009; Lo & Watanabe, 1997). Despite the importance of proportional reasoning in subsequent math courses, educators point out that, "… a high percentage of college students fail to manifest formal operational performance on the appropriate tasks" (Adi & Pulos, 1980, p. 150). Lamon (2007) affirms that the lack of ability to reason proportionally is widespread when she notes, "a sense of urgency about the consistent failure of students and adults to reason proportionally … my own estimate is that more than 90% of adults do not reason proportionally …" (p. 637).

Informal Proportional Reasoning

The transition from unstructured spontaneous arithmetical thought to structural algebraic thought within the contexts of ratios and rates has also been described as a transition from intuitive to formal proportional reasoning. Mathematical educators lament that students intuitive reasoning tends to be disregarded in mathematics classrooms instead of built upon (Fernandez et al., 2010; Christou & Philippou, 2002; Singh, 2000).

Multiplicative Conceptual Field

Proficiency with multiplication is also used by educators to describe the transition from informal reasoning about proportions, frequently built upon an additive approach called building up to a more functional or multiplicative approach (Caddle & Brizuela, 2011; Fernandez et al., 2010).

The "multiplicative conceptual field" of Vergnaud was designed as a holistic approach to analyse the many concepts in mathematics that relate to proportional reasoning, "a situation cannot be analysed with the help of just one concept; at least several concepts are necessary" (Vergnaud, 1994, p. 46). For Vergnaud the advantage of a comprehensive model to study proportional reasoning is the ability it affords researchers to analyse the connections between topics that would otherwise be treated as separate, "It is difficult and sometimes absurd to study separately the acquisition of interconnected concepts… it would be misleading to separate studies on multiplication, division, fraction, ratio, rational number… they are not mathematically independent of one another…" (Vergnaurd, 1983, p. 127).

Vergnaud's model also contains the intuitive or qualitative reasoning that is typically associated by students with outside the classroom activities as it seeks to understand and, "to account for the knowledge contained in most ordinary actions… the fact that this knowledge is intuitive and widely implicit must not hide the fact that we need mathematical concepts and theorems to analyse it" (Vergnaud, 1994, p. 44). Vergnaud's concept of a multiplicative conceptual field underlies the learning trajectory of Baker (Chapter 4.6).

Czarnocha Rate Sequence

Czarnocha's rate sequence is an example of a designed instructional task i.e. an artefact used as the instructional foundation to implement a methodology of discovery learning. Thus, Czarnocha would employ this sequence as a scaffolding tool in which the student would be given minimal guidance working in small groups, individually perhaps as a take home assignment. The teaching experiment has had several cycles and Czarnocha outlines the first two in this section.

The conceptual model Czarnocha uses is based upon the constructivist theories of Piaget and assumes a process/object duality (Tall et al., 2000). This model is based upon the work of APOS (action-process-object-schema) theory (Czarnocha et al., 1999; Arnon et al., 2013; Sfard, 1991, 1992). These theories view learning through the lens of a cycle that begins with procedures acting upon existing concept knowledge. These procedures are then internalized into processes which themselves become concepts or objects of thought to be acted upon and ultimately assimilated into the learner's schema. The effect of this process upon a learner's schema is described by Sfard (1991) as a transformation that begins with an, "…unstructured, sequential cognitive schemata, which are inadequate for the rather modest dimensions of the human working memory…" (p. 26) and ends with a hierarchical structure of concepts with relationships. The APOS model predicts that as concepts or object become assimilated into an individual's schema, the organization-coordination or structuring of these concepts leads to good problem solving skills. Sfard and Linchevski (1994) study the transition from arithmetical to algebraic thought from both an individual and historical development, and the rate sequence of Czarnocha is designed to encourage the transition of student thought from an unstructured arithmetical nature to a structured algebraic one. This rate sequence begins with encouraging students to use their innate ability to recognize patterns (Glaserfeld, 1998) and through repetition and reflection upon their action to internalize their knowledge i.e. the ability to reflect upon, verbalize and generalize to other relevant problem situations. The transition along the process/object duality continues in this conceptual model with the ability to condense their internalized knowledge of what operation is required with what concepts to symbolic form as required in the sequence.

The methodology that Czarnocha employed was discovery learning with minimal guidance is the natural fit in constructivism. The scaffolding of the exercises is designed through repetitions and pattern recognition to promote internalization of the process of multiplication to find the total primary amount i.e. distance given rate and secondary amount and then to find the rate through division of the primary by the secondary amount etc… Sfard (1991) and Dubinsky (1991) would refer to this internalization of knowledge of a process as "interiorization."

Dias Rate Sequence

As previously noted Cobb (2011) attaches much importance to "…sustained, direct engagement with the phenomena under investigation… be it young children's understanding… the collective mathematical learning of the teacher and students in a classroom, or the learning of a professional teaching community" (p. 11). In this situation we analyse the class dialogue of a lesson plan of a rate sequence as designed and implemented by Dias in her classroom. The artefact or object of design research is not simply the sequence of problems presented but includes the methodology of the lesson presentation during the class i.e. the teacher guided class dialogue. "Whether or not one considers all learning of mathematics to be discourse-based, analysis of discourse-including genres of speech and the inherent argumentation, in mathematics classrooms, is an important area in which there is growing literature and interest" (Presmeg, 2003, p. 132).

Dias considers the role of the instructor to be a mediator between students' understanding and the structural understanding that the instructor believes they are capable and required to attain, i.e. the upper level of the students' ZPD. Her lesson plan requires active participant through scaffolding class dialogue in order to encourage students' cognitive growth which is marked or accentuated when the student's intuitive frame of reference meets the instructors' structural frame of reference, at which point bisociation between underlying concepts may occur.

Dias' would agree with the Treffinger's (1995) assessment that "traditional views of steps and stages for students to follow in solving problems must be re-examined… we must call into question the prescriptive, step-by-step lockstep for problem solving… an effective framework must be flexible and dynamic" p. 309. Her insistence that students participate at every stage of the class dialogue is founded upon her belief one the one hand that, one cannot demonstrate a prescribed solution path and on the other that, "creativity can be expressed among all people" and that "people can function creatively, while being productive to different levels or degrees of accomplishment or significance" (Treffinger, 1995, p. 302). In particular Dias agrees with Prabhu's assertion that student creativity is essential to the transformation of resistant students, especially those in underserved populations, from habits of the acceptance of failure to the self-confidence required to take responsibility for one's learning.

Dias's lesson demonstrates repeated attempts to encourage active participation by students to reflect upon solution activity. More specifically, after providing a problem-situation i.e. the goal, Dias leads students to suggest appropriate solution activity and then to reflect upon how such activity may reach the goal (Simon et al., 2004). The scaffolding dialogue Dias employs to stimulate reflection upon solution activity is based upon understanding existing conceptual knowledge or declarative problem information and comparing/contrasting problem types. The focus on existing conceptual information and its relationship to appropriate strategy synthesize theoretical aspects of problem solving and conceptual development in order to assist student with the Cifarelli's (1998) first stage of problem solving i.e. 'recognition.' In which the solver can "…recognize characteristics of a previously solved problem in a new situation and believe that one can do again what one did before" (Goodson-Espy, 1998, p. 224). Class dialogue that focuses on comparative analysis between problem structures can be seen as critical thinking – the basic essence of meta-strategic thought about what strategies one employs (Kuhn, 1999). Bailin (1987) and Paul and Elder (2008) point out such critical thinking can pave the way for creative thought. In the dialogue Dias encourages students to verbalize their synthesis between their conceptual understanding of the concept of the rates and time with the required operations. Such critical analysis paves the way for reflection that hopefully leads the learner to, "…abstract a relationship between their activity and its effect" (Simon et al., 2004, p. 320). This abstraction will hopefully serve as, "…a means of classifying and characterizing problem solving activities" (Goodson-Espy, 1998).

Proportional Reasoning with Percent Lesson: Baker

The proportional (percent) reasoning lesson plan of Baker, like that of Dias is based on class dialogues but also on the proper utilization of the unexpected teaching moments supplied by the students comments. The instructor believes that the dialogue is critical in motivating and focusing student reflection upon problem solving activity.

In this student-centred lesson, student comments dictate the flow of the lecture. When a student offers a solution strategy the instructor presents it to the class. However, when another student chooses not to follow the reasoning of the first, immediately his misconceptions or limitations are used as a teaching moment. Then instructor follows the alternate solution path provided by the second student and solution activities are discussed in the class in detail. The purpose of this activity is first and foremost to encourage and to show appreciation for student's creativity and secondly to show alternate solution paths towards the same goal. The students are seldom shown multiple routes for the same destination which in turn promotes the misconception that the correct and the only answer to the problem must be memorized, which leads to math anxiety. After all multiple solutions are discussed and displayed, students engage in the "compare and contrast" activity to achieve higher level understanding.

Finally these methods are compared based upon the relationship between the underlying complementary concepts of: original amount, increased amount, increase, and percent increase.

This methodology is based upon the belief that it is critical that students take ownership of the problem process. In order to accomplish student participation in the classroom, the teacher must be flexible with different ideas and dialogues. He/she must adapt to their understanding, as expressed through their comments, questions and suggestions. The reader may note that in this dialogue Baker is acting as a researcher in the teaching experiment in that he prompts Bella to explain her thought processes that were based on part-whole construct. He interprets her answer theoretically by placing her onto the initial stage of the concept development and he acts as a teacher in creating a dialog among their peers to explain how a part-whole construct could be used to solve that problem – a Stenhouse TR act.

TR TEAM COMPARISON OF METHODOLGY

The lesson plan of Czarnocha relies heavily upon student discovery with minimal instruction or student-instructor dialogue. The transition from intuitive arithmetical to structural algebraic thought is encouraged through a step-by-step sequence of problems, in which students are first expected to use their innate ability to recognize patterns in order to internalize the three relationships between the complementary concepts of rate, with the primary and secondary unit quantities. These are done separately as three cases i.e. multiplication of rate with the secondary unit amount to obtain the primary quantity. The sequences employ increasing levels of abstraction as the student transitions from the need to perform the action required to written explanations of what needs to be done.

The teaching sequence of examples given by Czarnocha is designed to promote the transition from an operational or arithmetical thinking to a structural (Sfard, 1991) understanding of the relationships between the complementary rate concepts i.e. the formulas. This lesson sequence first encourages “interiorization” of a particular rate formula and then challenges the student towards encapsulation or reification through comparison of the different representations of the rate formulas.

In other words the coordination of these schema (formula representations) is designed to transition students to the second and third stage of the Piaget and Garcia Triad (1989) in which the transformations or schemes are coordinated into structures i.e the rate formulas are understood as being three faces of the same structure. We note that the design of Czarnocha’s teaching sequence of rate problems has a dual objective in addition to promoting conceptual development and that is to provide the teacher researcher information on the students’ thinking process.

The lesson of Dias is based upon strong student-instructor dialogue to rediscover mathematics that requires students to transition along their ZPD towards a structural understanding. The guided-dialogues also help the students to rediscover their own mathematics inherent capability and use this knowledge to the present situation. By making the students aware of their comfort zone in mathematics and its strength is a powerful tool, which can be used to reduce math anxiety. The goal is for the students to be able to recognize problem situations and associate these situations with strategies, operators or processes that will lead them closer to their goal. The underlying conceptual or declarative problem knowledge is highlighted in this dialogue as the features the solver needs to reflect upon in order to discern their action. Both Baker and Dias use class dialogues as discovery tools.

Sfard (1991) describes an unstructured schema as sequentially linked together with a focus on step-by step operations to be carried out. In much the way Anderson (1995) would describe, production rules as encoded, "…crystallized problem-solving operators as condition-action rules" (p. 249). That is a sequence of conditional if such and such a situation linked with then perform such an such an operation. The difficulty for a problem solver without conceptual or structural understanding is described by Sfard (1991) as "…the operationally conceived information, although absolutely indispensible and seemingly sufficient for problem-solving can only be stored in unstructured, sequential schemata, which are inadequate for the rather modest dimension of human working memory. Consequently, the purely operational ideas must be processed in a piecemeal, cumbersome manner… in the sequential cognitive schema there is hardly a place for assimilation of new knowledge, or what is called meaningful learning" (p. 26).

We note that, the teaching sequence of Czarnocha has an extremely detailed step-by-step scaffolding of exercises focusing on one step solutions for each of the three strategies or three representations of the rate formula. Thus, Czarnocha's sequence does not require students to switch between formulas, comparing or coordinating these schemes in multi-step problems until late in the exercise set. The objective is to promote reflective abstraction on the relationships between the three complementary rate concepts, in order to promote structural schema development in the sense of Sfard. In contrast the lesson plan of Dias requires students to compare different schema or formula representations and to coordinate between these different schemas in multi-step within eight exercises. The intent of the lesson plan of Dias is to encourage student reflection between problems of similar and different structure (schema) using concepts as the distinguishing features of recognition. The class reflection was on solution activity with the goal to promote student's ability to recognize such problem situation

In the lesson of Baker the students are asked to reflect upon their solution strategy as well as the thoughts and comments of other students and the relationships between different methods. Dias employs multiple strategies in her lesson, in particular the two sided number-line to provide a visual of rates i.e. the coordination of two distinct units or dimension (distance and time) in solving the rate problems as proportional reasoning. However, the focus of the lesson presented is on the comparisons of different types of rate problems. In Baker's lesson the focus in not on comparison of distinct problem types but instead on comparison of different methods to solve the same problem.

The instructor's role is to find teaching moments that build upon student comments and questions. That is, to build upon student understanding to present an appropriate strategy and to compare this strategy with strategies presented by other students. Baker's lessons demonstrate a teacher acting as a researcher in that mistakes by a student are capitalized on, the instructor as a researcher in a teaching experiment prompts Bella to expand upon her part-whole knowledge and integrate it into this problem situation. Although Bella is unable to do so the teacher-researcher (acting like a teacher) solicits the correct response from another student Ashley that solves the problem in a method consistent with Bella's part-whole construct. Acting as both a teacher and researcher (TR act), Baker then tries to get the students to understand the relationship between these methods. Chris has a bisociative insight as she understands the percent increase (and the 4000 base) are the same in both methods and thus are equivalent – a hidden analogy. In this lesson, a two-sided number line provides a visual diagram to explain Chris' insight that the different methods are essential the same to the remainder of the students in the class.

The lesson plans of both Dias and Baker are designed to encourage student engagement and participation in the classroom. The lesson of Dias has a focus on 'student recognition' of problem types (schema) in the terminology of Koestler student bisociation between the declarative information (problem situation) and the appropriate operator-matrix or schema. The objective is to transition students from their level of understanding to the structural level required to solve such problems i.e. to cross their ZPD as rapidly as possible.

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4.3. RATE AND PROPORTION TEACHING SEQUENCE

INTRODUCTION

Years of teaching remedial mathematics has taught me that traditional, or lecture, method do not always work with our students. Modelling a solution repeatedly and expecting the student to mimic without understanding doesn't help either. When a question involving rates and proportions is asked, I am confident that a few students will be able to give the correct answer. In this situation, an instructor writes the answer on the blackboard, the students copy the answer and believe they understand, that is, they perceive that the displayed answer makes perfect sense to them, thus, they move on to the next question. Due to a lack of discussion and reflection in this learning environment, students struggle to understand the concepts and, instead, attempt to memorize the steps. They get frustrated easily when they cannot solve the next series of questions. Students' participation is limited in this situation. The "Aha!" moment of realization and/or the "act of creation" moment are neither noticeable nor appreciated.

I wanted to increase class participation and the students' involvement during the development of the lesson. During our research on problem solving, I decided that class dialogues and disclosure within the lesson will serve as a vital tool to engage students in reasoning about mathematics. Through constant discussion, students will reflect on their thinking process and identify previous knowledge of related concepts needed for the current problem. Comparative analysis, generalization and pattern recognition are essential components of class disclosure that allows students to associate the current problem information with an operator/matrix. The goal of this thought process: recognition of the problem concepts and features, coordinating these problem concepts with appropriate operator using previous knowledge or forming a new association with their intuitive knowledge of whole numbers (generalization), then using this new found idea to solve the problem through estimation of the outcome and other processes of constructive generalization; that is, creating an atmosphere or a classroom culture of a conscious reflection on the problem activity (Glasersfeld, 1989). The coordination of the mental record/matrices with the past experience, such as mastery of whole numbers and related operation, and the problem information leads the students into the realization that they can project/map or generalize the matrix into the present situation in a way that gives meaning to their choice of operators during problem solving. Koestler considers this new realization or connection a hidden analogy. My aim is to stimulate schema/matrix formation

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by first, encouraging them to coordinate problem features with past knowledge (matrix) that leads to the selection of the appropriate operator and, second, to guide them to reflect upon solution activity involving their chosen operator comparing the expected result with the result obtained and the problem goal.

My lesson utilizes the guided discovery method of learning through class dialogue with tight sequencing of problems to minimize confusion due to incorrect choice of operator. The student's most common error will be computational, which can be easily corrected by other students. During the lesson, not much reflection is done on the actual answer. Unlike most text books' approaches, that direct the students to work with structurally similar problems to find the appropriate code through pattern recognition, I prefer to mix different types of problems to help students coordinate a given problem with the correct and specific code. The sequencing of the questions is very important; students compare and contrast the subsequent problems with the previous ones, and develop new schema or operator using analogy to their previous knowledge, such as a solid familiarity with operations on whole numbers (Anderson, p. 204). As the problem structure becomes harder and involves higher level of cognitive understanding, coordination of different problem codes (more abstract, multi-level) with difficult/complex and abstract issues is required in order to form a schema.

In this lesson, participants are expected to actively participate in the discussion that eventually unveils the goal of the lesson,—the concepts of *rate* and *proportion*. The goal is to enable the students to formulate a schema or a structural identity through pattern recognition, comparison and contrast and intuitive knowledge. I involve conscious reflection (Vygotsky, 1986; Piaget & Garcia, 1991) and the process of "abstraction and discernment" to help students organize their thoughts and foundational knowledge to come up with a structure for the current task. These steps involve reaching the upper level of their Z*one of Proximal Development* (ZPD) through reflection upon solution activity that adds meaning to the learning process.

This methodology is heavily influenced by Koestler's work on creativity and Glasersfeld's work (1989, 1995) on how students construct schema via *constructivism*. Glasersfeld (1989) notes that "construction of a scheme… consists of three parts – recognition of a certain situation, association of a specific activity and the expectation of a certain result." Thus the class discussion I present circulates around these three steps using guided discovery. I prefer to proceed slowly, breaking a sample problem into smaller parts, in order to discuss the approach to each step in the solution sequence. First, students identify the problem features, then, building on their initial/basic knowledge of whole numbers and the associated operations, new vocabulary and concepts are introduced as needed. Students are expected to rely on their intuitive knowledge, looking for any patterns, decide on an operator, reflect upon the outcome or results of the chosen operator, and, through this process, abstract an appropriate schema.

I then want to see whether the students can apply their emerging schema in a slightly different setting using a sequence of correct exercises. Students play a vital part in structuring the lesson while I control the flow and the direction of the discussion. I do not allow the students to wonder off and strive to keep them on topic. The following lesson is designed to be presented over a period of two consecutive sessions.

At the end of each exercise, I present a visual *Vertical Line* method to set up proportions and to coordinate the processes and concepts involved in the proportions, since students are very comfortable with solving proportions. This helps students coordinate the two units (the two different matrices), and, eventually, obtain a more formal and structural understanding of proportions.

Below is a little clip of class dialogue where the students are engaged in this reflective discussion.

RATE AND PROPORTION: FIRST ITERATION

Exercise #1: ***Juan is making $24/hour as a carpenter. At this rate how much will he earn in 40 hours?***

Class Discussion:

Instructor (I): "What does *at this rate* mean?"
Student (S): "Same rate as $24 per hour."
I: "So, *rate* means how much he makes?"
S: "Yes."
I: "Minor change; here, rate is *how much he makes in one hour*. This is actually called *unit rate*, because the word *per* means one unit,—hour, in this case. So, if you are driving your car at 60 mph. Is this a unit rate?"
S: "Yes."
I: "Great! Now, what is given and what do we have to find?"
S: "In one hour he makes $24, and we have to find how much he makes for 40 hours."
I: "So the answer will be in dollars? Yes or no?"
S: "Yes."
I: "Which of the following operations will NOT give the answer: $\times, -, \div$?"
S: "Subtraction and division."
I: "Excellent! Justify your answer."
S: "Subtraction and division will give a smaller number."
I: "In other words, you are saying that the correct answer will be a big number. Great! Give me the answer."
S: "960."

Instructor's Notes:
This class dialogue is focused on the first step of schema formation (Glasersfeld, 1989),—understanding problem information and relating it to previous knowledge (rate-matrix to choose an operator). Estimation and student's intuitive knowledge of whole numbers (rates and operations) are used to select the operator.

I: "960 bananas? Then it is wrong answer."
S: "Professor, you know. It is in dollars. OK. It is 960 dollars."
I: "Alright; now it makes sense. Let's see: we had *unit rate* and *time* and we performed *multiplication*. Can you please write the rule in your own words? And, later, at the end of the class, I will collect your rules."

TR Team Reflections:

In this first example the mediating dialogue serves to focus students on the underlying rate concept. Specifically the questions, "What does, at this rate mean" and "What operation will not give you the answer" highlight not only on the rate concepts but also its relationship to the operation chosen. Dias hopes that by such conscious reflection students will internalize this relation and enable them to generalize it to a new problem situation.

Instructor's Notes:
After the class discussion on each exercise, I also show my students a vertical line method, which is visual and more mathematically structured and straight forward. Although not much discussions or reflections are done here, visual presentation is the right side of the triptych, an artistic point of view (Koestler, 1964). It also reaffirms their new schema.

Table 1. Alternate method: Proportion using the vertical line method

Vertical Method	Alternate Discussion
Hours | Dollars Write as proportion	Since 1 hour corresponds to \$24
40 — ? $\frac{40}{1} = \frac{x}{24}$	Hour 1 → \$24
Cross multiply 960 = x	2 → \$48
1 — 24	10 → \$240
Answer:	40 → \$240 ×4 = \$ 960
He would make \$960 in 40 hours	

Exercise #2. ***Jose is driving at the average rate of 75 mph (maybe he is late for something) for*** $3\frac{1}{5}$ ***hours before he stops for a rest. How far has he travelled?***

Class Discussion:

Instructor's Notes:
Exercise #2 tests the ability of the students to apply the previous emerging schema to a similar but slightly more difficult problem due to the added structure of a fraction in the problem information.

I: "Is this similar to problem #1 (see the previous problem presented)?"
S: "No! This one is hard. It has a mixed number."

Instructor's Notes:
As expected, the students do not see the connection between the problem structures; this is due to the presence of the mixed number which makes their intuitive knowledge of whole numbers more difficult to access and utilize.

I: "Hmm. What was my question? Did I ask whether this is hard or easy?"
S: (laugh)

Instructor's Notes:
Little humour goes a long way, left side of the triptych.

I: "I see the issue. So, let's assume for a minute that it is just 3 hours instead of $3\frac{1}{5}$. Can you read the revised problem with *3 hours*?"
S: "Jose is driving at the average rate of 75 mph for 3 hours before he stops for rest. How far has he travelled?"

Instructor's Notes:
I use students' intuitive knowledge of whole numbers to illustrate the structural similarity with the previous rate exercise.

I: "Now, tell me if this version is similar to problem #1."
S: "It seems similar."
I: "Similar? How?"
S: "We have to find the total distance, and, in the previous one, we had to find the total price."

Instructor's Notes:
They relate to total amount, not to unit rate. They may or may not understand the rate concept, or are unable to verbalize it. Although I am not sure what the case is, I decide to move on.

I: "Excellent! So what should be done in order to find the solution?"
S: "Multiply, like in the previous exercise."

Instructor's Notes:
The class has made the association between the two problems, understanding both of them as multiplication to find the total. The class dialogue has helped them coordinate the problem information with their emerging rate matrix.

I: "OK. Multiply 75 with $3\frac{1}{5}$."
S: "Change into improper fraction."

I: "And we must find LCD. Yes or no?"
S (several): "No! No! We do not need LCD."
I: "Great!! Just checking if you are awake. Great. Give me the answer. Can you please write your answer in the blackboard?"

$$75 \times 3\tfrac{1}{5} = \frac{75}{1} \times \frac{16}{5} = \frac{1200}{5} = 240$$

240 what? Hours? No.
Answer: 240 miles

Figure 1. Student's Solution

I: "If you notice a pattern in the solution process for the last two exercises, please construct and write down the formula."

Instructor's Notes:
Verbalization of the solution activity, with a focus on the why, and appropriate terminology in the choice of operator is designed to promote schema formation.

Eventually, a student will volunteer and write the following on the blackboard:

Total Value = Unit Rate × Time

As usual, I will show the class several alternative approaches.

Table 2. Alternate Solutions for Exercise #2

Alternate Solution 1	Alternate Solution 2	Alternate Solution 3
Each bar represents 75 miles, so $\frac{1}{5}$ represents 15 miles. 75 miles 75 miles 75 miles 15 | | | | 75 + 75 + 75 + 15 = 240 miles for $3\frac{1}{5}$ hours	$3\frac{1}{5}$ means 3 hours and $\frac{1}{5}$ of the hour. Find the distance for 3 hours and then find the distance for $\frac{1}{5}$ of an hour, and add. $75 \times 3\frac{1}{5}$ $= 75 \times 3 + 75 \times \frac{1}{5}$ $= 225 + 15$ $= 240$ Answer: 240 miles	Two different units, hours and miles, are placed each on one side of the vertical line to create the proportion. Hours | Miles $3\frac{1}{5}$ — x 1 — 75 $\frac{3\frac{1}{5}}{1} = \frac{x}{75}$ ↓ *cross-multiplication* ↓ $3\frac{1}{5} \times 75 = x$ ↓ 240 miles = x

TR Team Reflections:

The second example is designed to test the hypothesis that a focus on the underlying concepts and the sequential order of the problems will allow students to recognize the structural similarity between them despite the increased cognitive demand added by the mixed number. Sfard and Linchevski. (1994) hypothesize that it is the structural complexity of fractions as distinct from whole numbers that make them difficult for students to work with. In this second problem student understanding is superficial as they do not realize the structural similarity between the first and second problem i.e. they both require multiplication of the rate with the time. In this teaching research act Dias realizes that the students have not internalized the connection between the concepts of rate and time with the operation of multiplication, i.e. the connection remains within the domain of their intuitive reasoning and they have not experienced sufficient conscious reflection upon this relationship to make the bisociation. Thus, Dias adapts the instruction by simplifying the cognitive demand i.e. by replacing the mixed number $3\frac{1}{5}$ hours with the whole number 3 hours. The students then realize the structural relationship between the problems as one of finding the total distance or total dollars and with prompting they vocalize the associated operation of multiplication.

Dias' questioning about what operation and her insistence that students express the relationship between rates, time and total distance of price in written form or formula can be seen as an attempt to assist students generalize and abstract their spontaneous or intuitive understanding of the relationship between rate, time and the total distance or price into a more structural (Vygotksy, 1997).

Exercise #3: ***Henry treated his 15 friends for his birthday party. He spent a total of $330 for the dinner. On average, how much did he spend on each person?***

Class Discussion:

I: "Is this similar to exercises #1 and #2?"
S: "No, unit price is not given."

Instructor's Notes*: Note the discernment,—students do recognize the difference.*

I: "Then what is given?"
S: "Total money and the total number of people."
I: "So, to find the missing *unit rate* what operation can we use?"
S: "Divide… Divide 330 by 15… Divide 15 by 330. No. That makes no sense."

Instructor's Notes:
This is a very common error. However, the students can correct themselves; intuitive knowledge of whole numbers and division help the decision. My goal is to guide them from intuitive choice of operators with whole numbers to verbalization, that

is, the use of correct terminology in words, and, thus, towards a higher level of abstraction. Further class discussion unveils the problem features. I am trying to model problem solving behaviour.

Class Discussion Continued:

I: "OK. But why?"
S: "You cannot divide 15 by 330."
S: "No, you have to divide total amount into 15 people."
I: "Excellent. So, what is the answer?"
S: "22… 22 dollars."
I: "Well, what is the formula for the unit rate?"
S: "Divide total cost by number of people."

Instructor writes on the blackboard:

$$\text{Unit Rate} = \frac{\text{Total Cost}}{\text{Total Number of People}}$$

Instructor's Notes:
Using an appropriate sequence of exercises is very important. The sequence encourages student reflection during the coordination of information with previous knowledge as they compare the current problem situation with previous ones. It allows students to move toward a higher abstraction level.

Table 3. Alternate Solutions for Exercise #3

Student's Solution	**Alternate Solution 1 (using proportion)**	**Alternate Solution 2:**
Unit rate $= \frac{330}{15}$ = 22 dollars per person	Dollars | Number of People 330 — 15 x — 1 $\frac{330}{x} = \frac{15}{1}$ *cross* multiplication 15x = 330 dividing both sides by 15 x = 22 dollars	One whole thing is worth \$330; divide into 3 pieces each worth \$110; then divide this \$110 piece into 5 parts, resulting in each part being worth \$22. \$330 \$110 | | \$110 **\$22** | | | | | | | | | | | | | | Answer: \$22 per person

Exercise #4: ***Alda's friend told her that she paid a total of $25 for 15 boxes of the same cereal. How much did she pay for each box?***

Instructor's Notes:
In this exercise, I want to see if the students can understand the similarity with the problem code from the previous exercise in a slightly more difficult situation; note that the result of this division is not a whole number.

Class Discussion:

I: "Can you solve this?"
S: "Yes. Yes. Divide."

Instructor's Notes:
They readily make the connection; hence they understand the problem code (division) for the unit rate. However, they will experience difficulty with the actual calculation process.

I: "Great! After you get the answer, explain how would you know whether it is correct or not."
S: "Multiply unit rate by 15. The answer should be 25."

Table 4. Solutions for Exercise #4

Student's Solution:	**Instructor's Approach:**
Unit rate = $= \frac{25}{15} = 1.6666\ldots = 1.\overline{6} \approx 1.67$ Unit rate ≈ \$1.67 Check: Total Cost = 1.67 × 15 = 25.05 Since we rounded up, this gives little more than \$25	To resolve any confusion about division, we can use the proportion method. # of Boxes | Cost 15 | 25 1 | x $\frac{15}{1} = \frac{25}{x}$ $15x = 25$ $x = \frac{25}{15}$ $x = 1.6666\ldots$ $x \approx \$1.67$

TR Team Reflections:

In exercise 3 and 4 Dias presents an inverse problem situation in which students must divide units to find the rate. Both examples are given within the domain of whole numbers and the students readily grasp the relationship between the declarative problem information and the required strategy of division to find the rate. As in problem 1 and 2 Dias ends the mini-sequence with a structural formula. The alternate methods Dias presents are divided into methods the students commonly present when asked to show their work on the board and as such show hypothetical learning trajectories that students may follow on their own initiative within this rate lesson(s). The instructor alternative method utilizes a table to help students organize the problem information into a pattern or format in which they can apply proportional reasoning. This method is frequently accompanied by a visual two sided vertical number line and emphasis on the visual table and number line is a trade-mark of Dias. Sfard (1991) would claim that visualization supports a structural understanding of mathematics, "Mental images, being compact and integrative, seem to support the structural concepts…Visualization, therefore makes abstract ideas more tangible, and encourages treating them almost as if they were material entities" (p. 5). Leaving aside the issue (Sfard, 1991; Presmeg, 2006) of whether visualization assists in abstraction and generalization we note that Koestler (1964) considered visualization to be the language of creativity within mathematics for many eminent mathematicians and physicists.

Exercise #5: ***Alda wants to buy cereal and is looking for a sale. One supermarket is selling 3 cereal boxes for $5.25. If she decides to buy 12 boxes, how much does she have to pay?***

Instructor's Notes: *This exercise has much harder cognitive level and my hunch is that students will need a significant amount of help leading them to the answer.*

Class Discussion:

I: "Is this problem similar to either #1, #2 (total value), #3 or #4 (unit rate)?"
S: "Somewhat. There are three numbers in the problem."

Instructor's Notes:
As expected, students are lost.

I: "I mean, can you use the above formulas?"
S: "No, the price of three is given instead of unit price. Time is not given."
I: "Great! This is just called *rate.* As you have mentioned that there is no *unit rate*. Is it possible to somehow find it with the given information?"
S: may be.
I: "I am waiting."
S: "Divide the price by three."
I: "Is that the final answer?"
S: **"**No. We have to find the price of 12 boxes**."**
I: "Okay, work with somebody and find the solution. Two steps: first find the unit rate and, then, the total price. Remember that since this problem is about price and quantity and we have to find the total cost, time is *not needed*. Previously, we had

total distance = distance per hour × time in numbers of hours. Now we have *total cost = cost per box × number of boxes*, and, hence, a similar exercise."

Instructor's Notes:
Students face tremendous difficulty to associate and coordinate two different processes. Before they start to feel frustrated, I decided to give the steps to them.
After some discussion, the class arrives at the formula:

$$\textit{Unit Rate} = \frac{\textit{Total Value}}{\textit{Number of Boxes}}$$

Instructor:
"Let us summarize. Can you please write the formulas you have just learned?"
Student writes the following on the board:

$$\textit{Total Value} = \textit{Unit Rate} \times \textit{Number of Hours}$$

$$\textit{Unit Rate} = \frac{\textit{Total Value}}{\textit{Number of Boxes}}$$

Instructor:
"Notice that in each problem we have *total value*, *unit rate* and *number of items* (number of hours, number of boxes, etc.). Please re-write the formulas using these, more general, terms."

$$\textit{Total Value} = \textit{Unit Rate} \times \textit{Number of Items}$$

$$\textit{Unit Rate} = \frac{\textit{Total Value}}{\textit{Number of Boxes}}$$

Instructor's Notes:
Writing and rewriting the formulas is a helpful exercise for assisting the abstraction of viable solutions and is good for long term memory. This activity is an extra push in order to reach the goal of internalizing the concept that promotes students to enter the third and structural stage of concept development.

Instructor continues: "What is the formula used in the vertical line method?" One student immediately replies, "Is this a trick question? There is no formula." Instructor responds, "Just checking if you all are with me."

Instructor's Notes:
At this stage, students have the formulas and steps for the questions. They will work in groups and after some time, one student will write the answer on the blackboard.

Table 5. Solutions for Exercise #5

Student's Work	**Alternate Solution 1 (by another student):**	**Alternate Solution 2 (by instructor, using a proportion):**
Cost of one box $= \frac{5.25}{3}$ $= 1.75$ Total cost = 1.75 × 12 = $21	3 boxes → $5.25 3 boxes → $5.25 3 boxes → $5.25 3 boxes → $5.25 (*using addition or multiplication*) ――――― 12 boxes → $21.00	Write proportion # of Boxes | Cost 12 — x 3 — 5.25 $\frac{12}{3} = \frac{x}{5.25}$ (cross multiplication) 12 × 5.25 = 3x 63 = 3x 21 = x Answer: $21 for 12 boxes

TR Team Reflections:

In exercise 5 the students need Dias to prompts them in order to recognize the division rate structure from exercise 3 and 4 and project this into problem 5. This projection of an existing scheme into a new problem situation followed by coordination is an example of "reflective abstraction" (Piaget & Garcia, 1991). In this exercise there are two critical dialogue prompts or scaffolding questions the first asks, is it possible to find the rate with the given information? This prompt focuses student attention on the necessity of finding the unit rate the structure of this problem that most resembles exercises 3 and 4. The second prompt occurs immediately after. "Is that the final answer?" With this prompt the students immediately realize they need to find the second rate and compare (coordinate) these rates to determine the answer. The students ability to immediately vocalize the required steps despite not being able to do so without prompting suggests they are in what Tzur (2007) refers to as the participatory stage, "A mathematical understanding that depends upon being prompted for the activity at issue" (p. 277).

Exercise #6: ***A nurse makes $35 per hour and she receives a check of $402.50 before any tax deductions. How many hours did she work to earn this money?***

Comment: Students recognize quickly that this is a division problem but correctly building the quotient is a more challenging task. After allowing some group discussion time, students' solutions are written on the board for examination by the whole class.

The resulting formula is expressed by the students and validated by the instructor:

$$Quantity\ (\#\ of\ Items) = \frac{Total\ Value}{Value\ of\ One} = \frac{Total\ Value}{Unit\ Rate}$$

Table 6. Student Solutions for Exercise #6

Student Solution 1	Student Solution 2
In 1 hour → \$35 2 hours → \$70 (double) 10 hours → \$350 (\$70 × 5) 11 hours → 350 + 35 = 385 12 hours → 385 + 35 = 420 She works between 11 and 12 hours. 402.50 – 385 = 17.50 Guess how many hours result \$17.50? Hard work!	Hours | Money — Proportion x | 402.50 1 | 35 $\frac{x}{1} = \frac{402.5}{35}$ cross-multiplication is trivial and unnecessary here $x = 11.5$ or $11\frac{1}{2}$ Answer: 11.5 hours

Exercise #7: ***An apprentice bricklayer earns \$18 in $\frac{1}{2}$ of an hour. How many hours does he have to work to make \$288?***

One student provides an incorrect solution: “Divide 288 by 18, obtaining $\frac{288}{18}$ = 16 hours.”

I: “Can we check the answer?”
S: “Yes, it is correct since 18 × 16 = 288 dollars.”

Instructor’s Notes:
The students have learned the code associated with the problem. They can coordinate the operator but are unable to recognize and use that code-schema when a mixed

Table 7. Solutions for Exercise #7

Student Solution	Alternate Method (using proportion)
½ hour → \$18 1 hour → \$36 Number of Hours = $\frac{288}{36}$ = 8 hours	\$ | # of Hours 288 | x 18 | $\frac{1}{2}$ $\frac{288}{18} = \frac{x}{\frac{1}{2}}$ $\frac{1}{2} \times 288 = 18x$ $144 = 18x$ $8 = x$ *Note: No need for unit rate!*

number is present in the problem information. They can't assimilate the fraction knowledge with the current schema. This suggests their rate schema is transitioning from the spontaneous-intuition to a more structural understanding.

I: "But why are you multiplying 16 by 18, 18 is not the unit price? *Unit* means 1, that is, 1 hour."
S: "Okay, then we do not have unit rate."
I: "If you get paid $18 dollars for ½ hour, how much many you will make in 1 hour?"
S: "$36"

Instructor's Notes:
It is easy to understand money and the fraction ½. Proportional reasoning (intermediate stage) without a formal proportion works well.

I: "Is this called *unit rate*?"
S: "Yes!"

Instructor's Notes:
At this cognitive level, it is helpful to remind the students that this is not a unit rate problem, and that this means that they first have to find the unit rate, and then find the number of hours.

I, the instructor, summarize our findings, and compile and place all derived formulas on the board for reference.

Table 8. Summary of Rate Formulas

Total Value = Unit Rate × Quantity	$\text{Unit Rate} = \frac{\text{Total Value}}{\text{Quantity}}$	$\text{Quantity} = \frac{\text{Total Value}}{\text{Unit Rate}}$

Exercise #8: ***A typist typed 40 wpm during the first session. During the second session the typist typed faster at the rate of 60 wpm for 30 minutes. All together she typed 2600 words. How long was the first session?***

Class Discussion:

Instructor's Notes:
After it was explained to the students that wpm stands for word per minute, I first ask them to figure out how many words are typed in one minute. I instruct them to note that there are two sessions mentioned here and request that they write the information given for each session in two different columns. This exercise requires inverse algebraic reasoning, a much higher level of cognitive maturity; hence I used

the visual representation (artistic side of the triptych) for the problem information. The table helps to organize their thoughts and problem features, so operator selection process will be easier.

Table 9. Preliminary Student Work for Exercise #8

Session 1	Session 2
Unit Rate = 40 wpm Time = *x* Number of words = ?	Unit Rate = 60 wpm Time = 30 minutes Number of words = ?

Instructor (I):
"Can you find the number of words typed during the first and the second session each?"

Student (S):
"Pretty sure; and it is 60 × 30 = 1800 words."

Instructor's Notes:
Student has no problem using the correct code to select the operation and calculate the missing number. Now the output of this action has actually become the input of the other, making the question complex. So, I am guiding the class through all of the steps.

I: "Now we know everything about session 2, is this the final answer?"
S: "No. We have to find the time for first session."
I: "To find the time, we must know the other two values. The unit rate is 40 wpm; is the number of words = 1800 or 2600?"
S: "2600."
I: "Please read again to make sure."
S: "2600 words in total,—session 1 and 2 combined. Subtract 1800 from 2600, which is 800 words."

Instructor's Notes:
In this lesson the student first used the concept of unit rate, and the time to find the total words. Then the student realize that part + part = whole and the inverse operation (subtraction) to find 800 words. The coordination of these two matrices can be viewed as a bisociative moment of realization for the class guided by the instructor.

I: "So, 800 words is the answer?"
S: "No. We have to find time."
I: "Fine, give me the final answer."

Table 10. Solutions for Exercise #8

Student Solution	Alternate Solution
$\frac{800}{40} = 20$ Note: $\frac{800}{40}$ is not same as $\frac{40}{800}$. Answer: 20 minutes	Words | Minutes 800 — x 40 — 1 Proportion: $\frac{800}{40} = \frac{x}{1}$ $20 = x$ Answer: First session lasted 20 minutes

TR Team Reflections:

In exercise 8 the students again need to associate a previous problem structure and then coordinate information to obtain the final solution. Dias again helps the students organize the information with a table and the students readily associate the correct operation-multiplication yet they need assistance and only after several prompts are they able to grasp the role of the given problem information 2600 words with the result of the multiplication 1800 and how to employ (coordinate) this information to find the solution.

Instructor's Notes (including a plan for the second iteration):
Class was active, most students were involved. For the second iteration, after each type of question, I will ask the students to come up with their own similar exercises which they will solve as a group before moving to the next concept. Each group will present their answers on the blackboard. This activity will improve understanding and reinforce higher order thinking. It takes the student from the surface procedural knowledge to a more complex internalized understanding.

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4.4. PROPORTIONAL REASONING AND PERCENT

INTRODUCTION

Schoenfeld (1992) notes that constructivism rooted in Piaget is grounded in the belief that "…learning proceeds through construction not absorption" (p. 340). He surmises that teacher beliefs directly influence their pedagogy, he notes a teacher who believed that, "…mathematics is fixed and predetermined, as dictated by the physical world" (p. 349) would not tend to focus on the process of learning and construction of knowledge but rather regard math as, "…a finished product to be assimilated." The importance of objective truth in mathematics education was raised by radical constructivism.

In the first iteration of teaching-research on problem-solving, Baker employed a traditional lecture format but with a strong focus on problem-solving. Thus, problems were posed in a step-by-step fashion with frequent repetition, sequencing similarly structured problems as a means of concept scaffolding. Thus, the lesson format would have been much the same as presented by Dias, and the intent as in both Czarnocha and Dias was to transition students as rapidly as possible from a spontaneous-arithmetical understanding i.e. their intuitive reasoning with whole numbers to a structural-algebraic one.

The guiding principle was to assist students cross their ZPD from a spontaneous understanding of simple patterns toward more complex algebraic thought processes. The instructor's role was to present students with a goal, suggest solution activity when needed and ultimately make sure the students experienced a structural understanding i.e. objective truth. The essence of constructivism pedagogy was that as an instructor I believed student participation in problem solving was critical, so on the one hand student engagement was essential on the other hand the transformation to algebraic solution strategy was a must.

TRANSION FROM TRADITIONAL TO REFORM

In a traditional methodology the instructor presents solution activity in depth and tends to avoid diversions and the possibility of cognitive conflict, by focusing on one solution method, typically the structural i.e. the most condensed and abstracted process to efficiently solve any structurally similar problem. The emphasis on one method results in a lack of comparative analysis between different possible methods

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 317–328.

of solving problems, fostering the common students' view that, for any given problem in mathematics, there is only one correct method of solution, or, at least, a single proper one that is approved by the teacher. As a result, students tend to accept that the best path to *success* is to focus on the collection of rules and steps presented in the classroom.

In my view this lack of reflection did not appear to promote any progress in understanding, evidenced by, what is frequently referred to as, the *next day effect*,— students did not recognize similar problem structures during the following class session. Furthermore, the strong focus on one method appeared to discourage students from sharing other methods, especially those based upon their intuition, limiting the class discourse.

BISOCIATIVE TEACHING RESEARCH MOMENT

One day, I presented an exercise similar to the second exercise in Dias rate sequence (Jose drives 75 mph for $3\frac{1}{5}$ hours, how far does he drive). I was hoping and expecting that a student would understand the need for multiplication. However, the class was silent until a student suggested an answer similar to alternate solution 1. That is they separated the whole number multiplying 75 times 3 and the fraction of the rate, one-fifth of 75 and added the results.

Initially I was rather annoyed the student had presented an alternate method that might confuse other weaker students but I came to believe that to assist students cross their ZPD it is not important to focus on where they should be i.e. the structural understanding of multiplication of a rate with time, rather the primary focus should be build upon where they are. The essence of constructivism pedagogy is for the instructor to present material at the students' level and engage them with reasoning at their level.

The structural understanding which previously was a must for each problem presented remained the goal but I have come to the view that presenting a finished process to students does not help them understand or cross their ZPD as much as promoting reasoning at their level. This has led to a transformation from pedagogy designed to show students a method of solving a problem to a class methodology designed to explore student solutions and difficulties.

Instead of being concerned about the confusion caused by different or alternate solutions a central component of this methodology is to compare and contrast different student approaches in the classroom.

In the lesson presented the reader may note that the teacher does not initiate any fixed method of solution yet goes out of his way to develop a solution strategy based upon the foundational part-whole understanding of fractions that answers student difficulties with another student's solution strategy. Both solution strategies are fully developed with prompting by the instructor so that students can create meaning along different learning trajectories.

This iteration of a problem-solving teaching-research cycle is based on the belief by Baker that no matter how many times an instructor models correct problem-solving behaviour, students do not internalize the solution process; they must *discover* and create meaning for themselves. Progress in understanding, the transition from the initial to the intermediate stage of concept development, and beyond, requires student reflection on solution activity. Applying Koestler's theory to the framework of problem-solving, this reflection involves coordination of matrices of past experience with given problem information that, by means of intuition, leads to the bisociative *Aha* moment, where a previously hidden matrix is understood as analogous. The combination of reflection upon solution activity and the resulting cognition is referred to, by Piaget (1989), as *reflective abstraction,* and is seen by constructivists as the foundation for learning at the abstract level of mental thought, and the eventual genuine schema development. Glasersfeld notes that, "…whatever results the reflection upon these mental processes produces, are then called *reflective abstraction*… The material, from which these abstractions are formed, consists of operations that the thinking subject itself performs and reflects upon" (1995, p. 69). Within this framework, the desired progress in understanding still necessitates Koestler's bisociation through recognition of hidden analogies and re-appropriated association. However, it also includes reflection upon the expected results of the newly recognized operator-matrix with the actual results obtained.

The presented set of dialogues denotes edited versions of actual class discussions. They exemplify how ordinary moments of students' realizations during class discourse, led by the instructor, along with peer guidance, can be viewed not only as bisociative acts on the part of each individual student, but also as significant moments of understanding influencing the attitude of the entire class and contributing to a positive learning environment. Such conclusions about student behaviour and development, stemming from the instructor's intuition and supported by discussions among the members of the teaching-research team, are at best conjectural and only approximate reality.

> One might say that assuming something as *given* or not is exclusively the subject's business. Hence, at best an observer can make educated guesses, taking into account – as does any experienced diagnostician – several indications collected over an extended period of observation. (Glasersfeld, 1995, p. 17)

The class methodology, or the pedagogy, to promote reflection on solution activity involves a focus on student errors through peer engagement and the encouragement of alternate solutions. After a student's question or incorrect answer provides insight into the underlying conceptual misunderstanding, the teacher's questions are designed to induce cognitive conflict among these underlying concepts, leading the student, with the assistance of peers, to the correct solution. The student's questions and vocalization of reasoning provide a measure of their level of concept development by revealing the associations they apply. The reflection that occurs during the solution activity is designed to help students cross the ZPD by clarifying and abstracting the underlying concepts, enabling the student to discover and create meaning for the notation, rules and procedures, with they are expected to be proficient (Vygotsky, 1997).

The reflection and abstraction upon these underlying concepts can often be described in terms of the processes of *constructive generalization* and *reflective abstraction,* such as *reversal of inverses* and *coordination* (Dubinsky, 1991). The bisociative moments of understanding can be viewed as providing meaning to these reflective processes, which allow the solver to *condense* the sequential collection of operators-matrices into a structural matrix-schema (Sfard, 1991).

COMPARATIVE ANALYSIS AND BISOCIATOIN: BASE ANDPERCENT-INCREASE

In this session the instructor (**T**) is going over *percent increase* which had been briefly discussed in a previous class by the tutor who used a particular method one student was not comfortable with. The instructor compares two methods of solving this problem. For students whose reflection and abstraction allow, the common concepts of the base and the percent-increase in both methods are bisociated between these methods.

Problem: ***A car dealer pays $4000 for a used car and sells it for $5000. Find the percent increase or, equivalently, the percent profit.***

Class Discussion.

T: How should we do this problem?
(Silence)

T: OK, how do we set this problem up? What formula do we use?

Ray: Last time we used $\frac{part}{total} = \frac{n}{100}$.

T: OK, so how do we fit the problem information $4000 and$5000 into this formula?

Ray: We put the 5000 on top and the 4000 on bottom.

T: OK, anyone, is this correct? $\frac{5000}{4000} = \frac{n}{100}$? Does everyone agree?

Bella: No, the 5000 should go on the bottom

T: Why?

Bella: Because the 4000 is a part of the 5000

T: Does everyone agree is 4000 the part of the base 5000?
(Silence)

In this situation a lack of understanding of the complementary concepts of *amount* and *base* has led Bella to a surface association (Berger, 2004) in which her conceptualization of the *part-whole* representation of a fraction does not allow for a situation where the amount is larger than the base thus, she appears to be in the initial stages of concept development. The teacher seizes on this misrepresentation for a learning moment.

T: Alright guys, what are the correct-mathematical terms for percent problems? (Silence)
T: OK, which of these two numbers did we say was the base, the 4000 or the 5000?
Ray: The 4000.
T: Why?
Ray: Because it came first, it is where you start.
T: Correct, the base is the original or starting point, the one that comes first in a time sequence. So in this problem the $4000 came first; it is the base, and then it was increased to $5000.
Chris: Is the base always the smaller of the two numbers?
T: In percent increase, yes, because it starts at some point and increases; but in percent decrease it's the other way around.

Ray, although his language is not mathematical can re-represent, or re-construct from memory (Cifarelli, 1989), an appropriate solution activity. Thus, he is in the intermediate stage of problem solving demonstrating correct techniques (imitation) and an ability to verbalize, not yet necessarily using correct math vocabulary for this *pseudo-concept* level.

Chris's question reveals that she is formulating a matrix-schema for percent increase questions: she understands there are two complementary concept-quantities and, with the instructor's guidance, she begins to discern between them, to identify which goes where.

T: Let's go back to the question raised by Bella: If $4000 is the base and goes on the bottom then what is the part, or amount, that goes on the top?

Instructor's Notes:
At this point I realized that Ray understands his method but that Bella is at an initial stage focused on the need for the problem to be interpreted, though the part-whole construct of a fraction i.e. the numerator must be less than and a part of the denominator. I intend to lead the class through an appropriate method based upon a part out of a whole construct and thus address Bella's concern. However, I first need the class to focus in the relationships between the complementary concepts of base and amount before they can relate these to the fractional conceptualization of part-whole.

Ray: The $5000.
T: Well, Bella does not understand this, and I can see her point,—how can the part be larger than the base?

Ray: But the tutor did the problem like this, using the larger as the part.

T: Well yes, you can do the problem like this but we had better not call the \$5000 the part, it would have to have another name which maybe we would call the increased amount while the \$4000 is the original amount or base. Let's hold off on the \$5000 for a minute and answer the question raised by Bella what is the part or more correctly the amount of increase?

Ashley: The part would be the \$1000.

Bella: Why is it not the \$4000?

Ashley: Because the \$1000 is the increase

T: So we start at the base of \$4000 then the amount of increase from this base is \$1000 and thus we use these numbers in the formula to find the percent increase.

The instructor then writes:

$$\frac{n}{100} = \frac{1000}{4000}$$

T: So in this method that Ashley is using where is the 5000? Why do we not see it?

Ashley: We use it to get the 1000.

T: Correct

Ashley with the instructor's direct question (scaffolding) has realized that the amount of increase \$1000 (analogous to the part conception) can be used with the base of \$4000. Ashley's realization was made with such little effort that it resembles more of an association, or assimilation, of information, that is, a recognition of this information, "…as an instance of something known" (Glasersfeld, 1995, p. 62).

Ashley's realization, as progress in understanding or an association (assimilation) made as a result of answering the instructor's direct question, focused the attention of the class on this previously hidden association. It had a bisociative effect on the classroom environment. That said, not all students understood Ashley's association between the \$1000 and the partial amount of increase, as evidenced by Bella's subsequent question,

Bella: "Why is it not the \$4000?"

Thus, Bella, unlike Ashley and many other students in the class who appeared to understand this association, is struggling to coordinate her *part-whole* conceptualization of a fraction with the complementary concepts of *amount* and *base*. It appears that this coordination is taking her out of her comfort zone in which the problem information must fit neatly into an existing proportional matrix without undue analysis.

Ray: But the tutor did it using the 5000 over 4000.

T: Yes, we can use this method also; however, understand that 5000 is not the amount of increase, it's what we might call the new increased amount.

Instructor's Note:
It appears that Ray's difficulty is not so much a lack of understanding of Ashley's association; instead, he does not see its relevance and his understanding is sufficient for him. It is important for him that I demonstrate his method to validate it.

T (to class): Let's work out both methods together and compare them. First, in the method that Ray does, we place the 5000 on top we have (writes on the board):

$$\frac{n}{100} = \frac{5000}{4000}$$

T: In this case, 5000 is called the increased amount and 4000 is called the original amount, or base, and we are finding what percent of the original 4000 is 5000. Cross multiplying, we get (writes on board):

$$n = \frac{500000}{4000} = 125\% \Rightarrow 5000 \text{ is } 125\% \text{ of } 4000$$

T: In the second method, using the amount of increase of 1000, on top we have (writes on board):

$$\frac{n}{100} = \frac{1000}{4000}$$

T: In this case, 1000 is the increase from original amount, or base, of 4000 and we are finding the percent increase. Cross-multiplying we have (on board):

$$n = \frac{500000}{4000} = 125\% \Rightarrow 1000 \text{ represents a } 25\% \text{ from } 4000$$

T: So, now are these answers the same or not?
Chris: Yes, they are the same, because with the 125% we need to subtract the 100% to get the answer, and then they are both 25%
T: Good; does everyone understand this?
(Silence)

Chris's realization that the two approaches yield the same result is a bisociative moment for her, as the 25% exists as a concept simultaneously in both planes-matrices, or methods of solution. In contrast to Ashley's realization that $1000 was equivalent to the partial amount of increase, and, therefore, could be coordinated with the percent increase, which most of the class understood, the class does not appear to grasp Chris' realization.

The teacher employs a visual to help students understand Chris' realization
T: Let's see this problem solution through the number line:

$$\frac{n}{100} = \frac{5000}{4000}$$

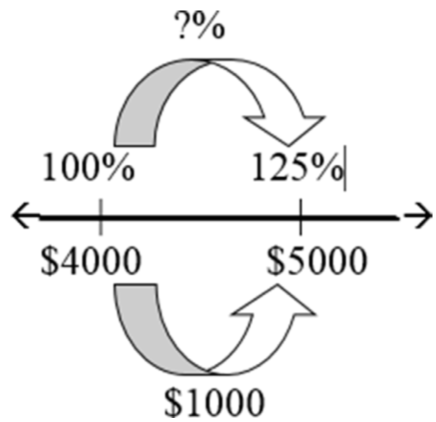

Figure 3. Visualization of the Percent-Proportion Relationship

T: Guys we agree that when we increase $1000 from the $4000 this is a 25% increase, this is the bottom part of the diagram. Okay?
(Silence)

T: In the method used by Ray we start at the 100% or $4000 and increase to the $5000 which is 125% as represented in the top part of this diagram. Then we subtract to find what percent increase, which is what?

Chris: 25%.

T: Yes the difference or percent increase is 125%–100% = 25%.

T: Class its important note that either way using Ray's method or the 1000 increase over the 4000 method we get the same percent increase of 25%!

The visual provides students with insight into where the proportions used in each method came from, how they are related to one another, and why a proportional percent can be improper. Thus, the image allows the student to coordinate the concepts of *dollars* and *percent* with each other, and, more abstractly, the concepts of *base* (100% as the original base $4000), *the increased amount* (125% as the percent increase $5000) and *the amount of increase* (25% as the percent of increase). With this diagram, Chris understood.

Most of the students understand these two methods as completely different frames of reference, preferring one or the other. However, Chris, with the assistance of the visuals, has abstracted and, thus, bisociated the common underlying concepts of the *base* and *percent-increase*, and understands these as analogous methods. She understands that the $1000 of increase from the base of $4000 corresponds to a 25% increase. We use the term *bisociation* as a mechanism for the transfer of analogous concepts between (synthesized, or integrated into) two frames of reference, and, thus, existing simultaneously in both planes of thought.

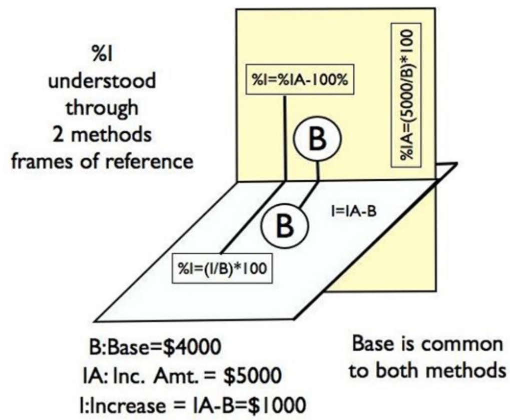

Figure 4. Chris' Bisociation: Base $4000 and percent increase 25%

T: Look guys, in one method we first subtract to find the 1000 and then use this over 4000 in the formula and then cross multiply, in the other method we first use 5000 in the formula over 4000 and cross multiply and then subtract. Either way, we need two steps,—a subtraction and the proportion.

Most, but not all, students nod their head. The verbal comparison of the two methods, as essentially the same two-step process with interchanged steps, leads to the third stage of learning. At this level, the individual can verbalize not only the solution activity but compare and contrast different methods and, thus, begin to classify these different problem activities; this is the *trans*-stage, as referred to by Piaget and Garcia (1989). This abstraction of problem characteristics and concepts in a matrix is an essential component in progress in understanding, and facilitates the solver's ability to recognize and associate similar problem situations with this matrix.

Analysis:

Ray understands a method presented by a tutor which some members of the class have not seen. He explained this method using a part-whole analogy for setting up a percent proportion but when he then employed the increased amount over the original base amount he confused Bella whose part-whole construct knowledge is limited to proper fractions in which a part is a subset of a whole. Bella appears to be at an initial or beginning level of proportional reasoning as her incorrect i.e. surface association of 4000 as the part and 5000 as the base does not appear to be corrected even with Ashley's explanation that 1000 is the part because it is the increase. Ashley unlike Bella engages in reflective abstraction as she is able to project her part-whole construct into this problem situation and associate the difference between the quantities given with the part 1000 as directed by the need to find the percent increase. Chris makes the bisociative realization that two steps (solve a proportional and subtract) are required to find the percent increase in either method i.e. the concept of the based 4000 and percent increase 25% are same in each method.

TR Team Reflection:

How can one say that Chris' realization that the two methods obtained the same answer was a bisocication? Was there an 'Aha' moment? The instructor Baker confirms that her affect i.e. as she had a spontaneous insight that was noticeable. The discussion continued grappling with the issue of whether and to what extend the mechanism of bisociation (simultaneous existence of concepts in two frames of reference) can be separated from the affective-illuminative experience known as the Eureka moment. A resulting hypothesis of this discussion for classroom pedagogy centred on creativity within guided discovery is that, the mechanism of bisociation is necessary for the affective experience but the affective experience may not necessarily accompany the mechanism of bisociation.

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4.5. RATE TEACHING SEQUENCE

INTRODUCTION

The teaching experiment *Integrated Course of Arithmetic and Algebra* has been designed in response to the challenges that freshmen community college students experience while learning algebra. As the recent *New York Times* article "Is Algebra Necessary?*"* (July 28, 2012) demonstrates, the seriousness of the challenges is formidable, both at CUNY and nationwide. The study conducted at the City University of New York (CUNY) in 2005 (Akst, 2005) revealed that among those students who start their mathematics developmental sequence with arithmetic only 37% pass the subsequent course in developmental algebra. From this it may be inferred that the central source of their difficulties is what is known as the *arithmetic/algebra divide* (Filoy & Trojano, 1985, 1989), which explains the general hurdles students encounter in the transition from arithmetic to algebra. Around 75–80% of incoming freshmen to CUNY community colleges are experiencing these challenges. The central argument for the success of the teaching experiment in accomplishing students' understanding and mastery of algebra is based on creating very close connections between the two domains throughout the syllabus of the course. The connection bridges are either teacher-student classroom dialogues or problem teaching sequences, whose solutions facilitate student development of mathematical thinking from concrete arithmetic representations to general algebraic formulations. The Rate Teaching Sequence, that is a sequence of problems addressing understanding and mastery of the rate concept, is one such bridging technique within the syllabus that leads to a progression from working with elementary numerical computations to manipulating symbolic algebraic formulae through generalizations of the former. The Rate Teaching Sequence was originally designed for the elementary arithmetic course. However, its didactic usefulness became apparent in the context of the integrated arithmetic/algebra course where it allowed the concept of proportional reasoning to be recognized as a gateway to algebra.

The integrated course syllabus is intense since it combines two courses into one. The course meets four times per week for one hour and fifteen minutes. The intensity of learning requires that special cohort of developmental mathematics students chosen among freshmen who can sustain the heightened degree of effort. The aim of the teaching experiment is to assess the effectiveness of the new syllabus and the associated pedagogy as well as to determine an adequate cohort of students

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 329–342.

by choosing optimal values of the required placement test (see below) results in arithmetic and algebra, a test taken by all students entering CUNY. The ultimate goal of the research is to find, through several iterations, the proper student sub-cohort for whom the final teaching sequences bring about the best results. After establishing the correct benchmarks at this level the process can be applied to standard sections in the college while simultaneously incorporating pedagogies needed to address the challenges for the remaining students, outside of the established cohort.

This chapter describes one of the three threads similar to the Rates Teaching Sequence as it appears in its first iteration followed by a fully redesigned Rates Teaching Sequence for the second iteration.

PRINCIPLES OF THE DESIGN

The theoretical aspect of the design of the teaching sequence has been based on what is known as the *process/object* class of Piagetian theories of conceptual mathematical development (Tall et al., 2000), in particular, on Sfard's *reification* theory (Sfard, 1992) and Dubinsky's *Action-Process-Object-Schema* (APOS) theory (1991, 2001). Sfard (1992) sees the transition between operational and structural understanding of the related concepts (as in the transition between arithmetic and algebra) as proceeding in three steps:

> First there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self-contained whole will emerge, and finally an ability to view this new entity as a permanent object must be acquired. These three components of concept development will be called interiorization, condensation and reification (Sfard, pp. 64–65).

A similar yet different characterization of that process is provided by Dubinsky's APOS theory as described below by Tall:

> A step-by-step action becomes conceptualized as a total process [and] is encapsulated as a mental object… the final part of the APOS structure occurs when actions, processes and objects… are organized into structures, which we refer to as schemas (Tall et al., 2000).

The process through which a schema becomes a cognitive object is called *thematization* (Piaget & Garcia, 1987) that is, involving the schema in different problem situations, so that all its components and transformations between them are clearly perceived and assimilated.

The Design: Three Strategies to Solve Rates Problems (Iteration 1)[1]

Three concepts: **R** – Unit Rate; **T** – Total Amount, **N** – Number of Units.
Strategy #1 (R, N given; T unknown)

Juan is making $24/hour as a carpenter**.**

1. ***How much, in total, will he make in 1, 2, 7, N hours?***

Table 1. ***R****,* ***N*** *given and* ***T*** *unknown*

Total Amount **T** ($)	Calculations with the Unit Rate **R** = 24 ($/hour)	Number of Hours **N**
		1
		2
		7
		N

Thinking Reflection questions:
Recall the steps of the calculations you made above; thoughtfully look into the numbers in the table and answer following questions:

2. ***If you know the number N of hours Juan works, how would you calculate his total pay?***

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Now look back into last two problems, compare the steps of calculations and answer the question:

3. ***If the total pay is T, the rate in $/hour is R, and hours of work are N, how would you write the correct general formula governing this problem?***

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Application Exercises

To the student: *Read every problem carefully and decide which strategy you will use to solve it.*

Exercise #1. It costs Lovell $2400 for 12 credits at a community college. Find the cost per credit that he is paying.

Exercise #2. If Jorge jogs 25 km in 2.5 hours, what speed was he jogging with?

Exercise #3. The deer runs along the path in the forest, which has a distance of 1000 yards. If the animal runs with the speed of 125 yards per second, how many seconds will it take for the deer to get to the forest? Note that here the student may develop the formula

$$N_{umber\ of\ units} = \frac{T_{otal}}{R_{ate}} \tag{1}$$

Exercise #4. Travis goes roller blade skating at a rate of $8\frac{1}{4}$ miles per hour for $\frac{2}{3}$ of an hour. How far does he go?

Exercise #5. Sheila's pick-up truck gets 16 miles per gallon of gasoline, at this rate how far can she drive with $5\frac{3}{4}$ gallons of gasoline?

Exercise #6. Juana typed 225 words in $7\frac{1}{2}$ minutes. How many words did she type per minute?

Exercise #7. If you knew the rate in words per minute that Tanya was typing her Humanities term paper and you knew how many minutes she typed, how would you find the number of words she typed?

Exercise #8. Juan typed a paper 640 words long with the speed of 40 words per minute. How many minutes did that typing take?

DISCUSSION OF THE DESIGN (IMPLEMENTED FALL 2012)

Note the correspondence between the learning theory and the design of the assignment:

1. Computational exercises in the table are the process "performed on familiar objects" of Sfard's theory (or actions of APOS);
2. Note that the reflective question #2, asking for a verbal explanation, investigates the student's ability to recognize the process of multiplication of the rate by a number into a unified "more compact self-contained whole" interiorizing the process;
3. The ability to respond to the reflective question #3 checks to what degree does the concept of the rate becomes the structural component of the elementary schema within which the following formula is constructed:

$$T_{otal} = N_{umber\ of\ units} \times R_{ate} \tag{2}$$

Several natural questions immediately arise during this thought sequence:
Question 1: If you claim that that this is the Rate Teaching Sequence, why don't you start with the formula, which defines it?

$$R_{ate} = \frac{T_{otal}}{N_{umber\ of\ units}} \tag{3}$$

Answer: Multiplication is simpler for our students than division, both conceptually and procedurally. That fact suggested the presentation of all formulas, so that the process of development takes place along each of them separately to be later unified into one formula. Moreover, since the concept of the rate appears each time in the context of appropriate formula, the process of encapsulation of that concept might take place simultaneously with the thematization of the whole schema. The investigation of that process will be extended during the second iteration to include "thinking aloud" interviews with chosen students to be conducted outside of the class.
Question 2: Isn't it the case that the teaching sequence presented does not in any way suggest the unification or generalization of all three formulas, (1), (2) and (3), into one?
Answer: That's true. This apparent shortcoming and several other features of the teaching sequence will be refined and re-thought before the 2nd iteration next semester

ANALYSIS OF RESULTS AND RESEARCH OBSERVATIONS (AFTER THE FIRST ITERATION)

The data of the first iteration consists of the following:

- collected Rate and Ratio assignment from 14 students;
- scores on the Algebra Final of 19 students;
- scores and work on the Arithmetic Final Exam of 19 students.

The main research question for the series of TR cycles of this teaching experiment is "What is the most effective Arithmetic/Algebra curriculum and associated pedagogy for significantly increasing student understanding and mastery of algebra at the college level?" Naturally, the complete answer to that question will take several cycles, since at each particular cycle only some of the main question's sub-components are addressed. In agreement with the definition of the TR-NYC model, many of the results are applicable specifically for a classroom environment where the teaching experiment is conducted. However, a significant amount of the research addresses more general issues and can be applied to all community colleges within CUNY as well as, perhaps, student populations in many other urban learning centres in the country. The analysis of the data led to the following research observations:

1. Use of theories of conceptual development in the design of student activities was very useful and showed that **(a)** there were students at each of the three levels of concept development, and **(b)** adaptive instruction strategies to help students reach the full understanding of the rate concept could be determined on the basis of their initial placement within the three levels. The two students placed at the first level of concepts development need strong reinforcement in the *variable as a generalization of a number* sub-concept, while those students placed in the

transition between levels benefit from *reinforcement of multiplication as repeated addition* sub-concept:

- Two students who, instead of replying with the majority answer of "**$24 × *N***," used the additive structure stating that "you have to add them all up." It seems that those two students are at the yet unfinished process of progressing from the condensation level to the second level of understanding. The absence of student understanding of multiplication as repeated addition creates a cognitive obstacle for the creation of the "compact self-contained whole" needed here, accordingly to Sfard, 1992, to progress along the developmental trajectory;
- Two other students displayed even more elementary mathematical behaviour responding to the same questions. Instead of reasoning with the variable ***N*** to answer the question, they assumed a concrete value (***N*** = 13) and calculated the total based on this value. For them, the concept of the rate didn't progress beyond the first stage of development of using it only in concrete, elementary cases;
- Additional relevant information was provided by students' results in the departmentally designed final exam, the results of which could be divided into standard three subgroups: the top 20%, the middle 40%, and the bottom 40% who failed the exam. The two students who originally placed in the first level of development failed the exam, while the two students who placed along the transition between the first and second level passed the exam, scoring within the second subgroup of the test results' distribution.

2. COMPASS is an untimed, multiple-choice, computer-based test, consisting of four parts, designed to measure students' knowledge of a number of topics in mathematics with the intention of proper placement. Placement into CUNY's required basic math courses is based on results of the first two parts of the test, numerical skills/pre-algebra (M1) and algebra (M2). Numerical skills/pre-algebra (arithmetic) questions range from basic math concepts and skills (integers, fractions, and decimals) to the knowledge and skills that are required in an entry-level algebra course (absolute values, percentages, and exponents). The algebra items are questions from elementary and intermediate algebra (equations, polynomials, formula manipulations, and algebraic expressions). At the time of this teaching experiment, the passing scores for M1 and M2 were 35 and 40, respectively. Based on the researchers' professional experience, the cohort of students participating in the first TR cycle consisted of those who scored higher than 24 on M1. At the end of the semester, after the final results were tallied and combined with the instructor's semester assessment, it became clear that the majority of those 40% of the students who failed the final exam were not sufficiently prepared to handle the intensity of learning. That led to the question of how to characterize this sub-cohort in terms of their original performance on the arithmetic and algebra placement tests. The analysis of the patterns provided an answer, which will be utilized in the second cycle. The placement prerequisite

scores were established to be as follows (where M1 and M2 are the arithmetic and algebra placement scores, respectively):

$$\mathbf{M1} > 24 \text{ and } \mathbf{M1} - 9 < \mathbf{M2} < \mathbf{M1} + 9 \text{ and } \mathbf{M2} < 40$$

The general observation of the placement scores of participants revealed that there was a smaller difference between the **M1** and **M2** scores among those who passed than those who did not. It was decided to limit that difference to no more than 9, leaving students with a double-digit difference outside of the sub-cohort. These empirical observations might be supported by the hypothesis that it is easier to cross the arithmetic/algebra bridge if the difference in knowledge of the two domains is not large. The second TR cycle will test this hypothesis.

3. The collection of the results below might be significant beyond the confines of the classroom where the teaching experiment was conducted. This set of observations from the results of the multi-part assignment, outlined immediately following this paragraph, is extremely important for the design of the syllabus for the course. The results of the assignment compared with the results of the Ratio Teaching Sequence (not discussed here but designed along similar principles) show that students have much less difficulty in arithmetic and algebraic problem solving incorporated into the sequences than with the development of conceptual understanding facilitated by their design. If indeed, the arithmetic and algebraic problem solving skills of our students are better developed than their generalization skills and, if this observation is confirmed in future iterations, then we may have to consider a significant change of the whole syllabus. The original design was based on the idea of algebra as a generalization of arithmetic (**arithmetic → algebra**). However, it may be useful to design the instruction based on the inverse process (**algebra → arithmetic**), which sees arithmetic as a particularization of algebra. This point of view is the basis of the designs in the Davydow school of thought in Russia (Schmittau & Morris, 2004). The significance of such a change of instruction on the level of developmental mathematics would, if successful, impact the instruction far beyond the confines of the current experimental classroom.

RE-DESIGN: RATES TEACHING SEQUENCE (SECOND ITERATION, SPRING AND FALL 2013)

The Three Strategies to Solve Rates Problems (2nd iteration)

Three concepts: **R** – Unit Rate; **T** – Total Amount, **N** – Number of Units.
Strategy #1 (R, N given; T unknown)
Strategy #2 (R, T given; N unknown)
Strategy 3 (T, N given; R unknown)

The teaching sequences for each of the strategies are similar in design to the first iteration. To focus attention we will just present strategy #3 followed by the building of the schema of the rate formula.

Strategy #3 (calculation of rates, that is, T, N given and R unknown)

This strategy applies whenever we want to find the unknown rate **R** and we know both the total amount **T** and the number of units **N**. Solve the problems below and write the answers in the provided table.

3.1 ***You are earning $1000 in 4 weeks. What's your unit rate in $/week? (How much do you make per week?)***

____

____

3.2 ***A worker is making $2400 in 3 months. How much is he getting per month?***

____

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3.3 ***Juanita is earning D dollars in 5 hours. How would you calculate her wage in $/hour?***

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____

3.4 ***Marco, the salesman, sold S shirts for $200. How would you calculate the cost of 1 shirt?***

____

____

3.5 Now look back at the last four problems. Compare the steps in the calculations and answer the question: ***If the total is T, the rate is R, and the number of units is N, how would you write the correct general formula governing these problems?***

Table 4. ***T****,* ***N*** *given and* ***R*** *unknown*

Total **T** ($)	Unit Rate **R**. Show your calculations and write your answers with the correct units of the unit rate	Number of units **N**
$1000	…$/week	4 weeks
$ 2400	…$/…	3 months
$ *D*	…$/…	5 hours
$ 200	…$/shirt	*S* shirts

3.6 (Solve in *Table 5* below).

Instructor's Notes: *Table 5 generalizes the strategy to obtain different categories of rates by the same technique.*

(a) ***A runner runs 8 km in 2 minutes. What's his unit rate in km/min?***

(b) ***A car trip of 250 miles used 5 gallons of gas. How many miles/gallon did this car use for the trip?***

(c) ***Registering for classes you paid $1200 for C credits. Describe in words how would you calculate the cost of one credit?***

(d) ***A student can type a 540 word essay in M minutes. How would you find the rate, in words per minute, at which the student types?***

Instructor's Notes: *Questions 3.7–3.9 aims at developing student problem solving schema through the strategy choice method.*

3.7 Compare your method of calculation while using the **Strategy #1, Strategy #2** and **Strategy #3. What is similar and what is different in all of them?**

Table 5. Generalization of the strategy to different units

Total **T**	Unit rate **R**. Show your calculations and the units of **R**	Number of units **N**
8 km		2 min
250 miles		5 gallons
$1200		*C* credits
540 words		*M* minutes

3.8 When would you apply each of the three strategies?

____

____

3.9 Fill in the missing cells in the table below; solve the following problems by applying the correct strategy **(#1, #2 or #3).**

Table 6. Strategy Choice

Unit rate **R**	Total **T**	Number of units **N**	# of the strategy
7 ft./min		12 minutes	
	650 typed words	25 minutes	
12.5 miles/hr.	250 miles		
	5000 miles	250 gallons	
	192 oz.	12 lb.	
3 feet/yard		11 yards	

Instructor's Notes: *The series **A, B, C** develop student use of decimals and fractions in each of the contexts – standard challenge for students in remedial mathematics*. We will omit these exercises for the clarity of exposition

Instructor's Notes: *The aim of the following section is to develop mastery in transformation of the basic rate formula followed by its application to more complex situations. It is expected that students will encapsulate their understanding of rates as the schema of the single formula.*

Synthesis of the Three Strategies into One Formula:

$$T_{otal} = N_{umber\,of\,units} \times R_{ate}$$

Write your previous answers/formulas to the questions:

Strategy #1 ***1.4*** ____

Strategy #2 ***2.5*** **(a)** ____

Strategy #3 ***3.5*** ____

Fill out the table:

Table 7. Algebraic transformations of the formula

Transformation of strategies	Word description. What will you do to transform the formula?	Computation	Target, final formula/strategy
1→2?			
1→3?			
2→1?			
2→3?			
3→1?			
3→2?			

Applications of the Rate Formulas to More Complex Problems

- I make \$50/hour during the first 8 hours of work. I received an hourly increase for the next 5 hours of work. All together I received \$725 for 14 hours of work. What was my hourly pay increase for the last 5 hours of work? What was my percent increase?
- A typist typed 40 words/min during the first session of typing. During the second session of typing, which lasted 30 min, the typist typed 60 words/min. Altogether she typed 2600 words. How long was the first session?
- David and his wife are driving from work to David's parents' house. The trip takes 2 hours and 30 minutes if David drives 60 mph. He knows that there is a good diner at exit 24, which is located at about 75% of the distance from work to his parents' house. About how far is it to the diner from their jobs?

RESULTS AND ANALYSIS AFTER THE 2ND ITERATION

The 2nd iteration strongly confirmed the difficulty along the 2nd step of the development, that is, in making clear the relationship between the numerical series of examples and its generalization to the formula. Students, in general, were competent in numerical computations as well as in strategy choice exercises, thus, they had knowledge on both sides of the arithmetic/algebra transition but the transition itself was not clear: the success rate for the first cohort (Spring 2013) was a relatively low 31% (4 out of 13 students), while the second cohort (Fall 2013) demonstrated a similarly low success rate of 27% (5 out of 18 students). The approach those students took in the last problem of *Table 2* exercise ***1.1***, "What is Juan's total pay if the rate is \$24/hour and he works ***N*** hours?" was to assume a certain value for ***N*** and calculate the total with that assumed value. These students did **Exercises 2.3 and 2.4** similarly. It is not clear for students how to integrate operations on a number with operations on a letter. This lack of clarity undermined student problem solving capacity. For example, a student who has no problem in the application of **Strategy #3** in the numeric case of ***3.1 and 3.2*** but exhibits difficulty in exercises ***3.3 and 3.4***, mixing a variable with the number, responds with $\frac{5}{D}$ (instead of $\frac{D}{5}$) and, in ***3.4***, the written answer reads "multiply # of shirts [S] by \$200" instead of \$ $\frac{200}{S}$ following the technique in ***3.1 and 3.2.***

For some of those students verbal expression of the computation was more natural than the symbolic, indicating a verbal/symbolic divide – the subject of Bruner's theory of learning (Bruner, 1977). Not a single student was able to address independently the problems in *Table 7* that involved operations on formulas. It could be that the measuring instrument, *Table 6,* is not well understood and needs refinement. It also may indicate that, although they mastered the developmental aspect of the variable, they could not deal with the "scientific" level of the concepts indicated by mathematical operations on them, quite possibly because of the weakness in dealing with formula (1) shown on page 3. That means that students' mastery of the "scientific" level has to be addressed individually as part of the sequence so that an additional small series of problems developing this aspect must be added to the sequence to be implemented in the next iteration.

The transitional difficulty of students is interesting because it can be viewed through several different lenses in addition to the developmental one, offered by Piaget and Garcia (1989), as well as by Vygotsky (1987). In particular, Duval (2000) points to essential learning difficulties where two different representational systems, such as numbers and variables, interact. Students have to coordinate two meanings, the meaning of the number and the meaning of the variable – the generalization of the number, where the meaning of \$ $\frac{200}{5}$ shirts has to be coordinated with the meaning of \$ $\frac{200}{S}$ shirts. At the same time, the work of Koestler (1964) discussed in Chapter 1.3, suggests that the presence of such two different matrices of thought creates the favourable condition for facilitation of *bisociation*, that is, the creative leap of insight. It may be the case that general difficulties in the transition between two representational systems are due to the absence of the efforts towards creativity of both students and teachers. Clearly, the design of a sequence that addresses effectively the transitional issue between numbers and variables is the task for the next iteration. Looking at the whole Rates Sequence, this author's hope is that, after dealing with the above-mentioned issue along with the issue of algebraic transformation of formulas, the sequence will be ready for testing in a larger number of sections of the course.

TR Team Reflection:

Initially the team discussion focused on design of the teaching sequence: It was noted that, the teaching sequence demonstrate a slight lack of notational clarity and even internal consistency that might confuse a casual reader. For example the notation in table such as (1→2) may not be understood as clearly as writing out D=R×T and R = $\frac{D}{T}$ also if the ultimate goal in Table 7 is to have students manipulate formulas then such a level of understanding should be required in all preceding tables. Thus, Tables 1 through 5 might have additional and final row that requires the coordination of three variables instead of only two.

The TR team discussion also focused on student reaction, in particular the relationship between the visual diagrams and the exercises that required a transition between language and algebra to promote structural algebraic thought. It was noted that one reason students may have experienced difficulty with 3.3 was that it was presented before Table 4 which provided a visual pattern of the schema. This is in contrast to the first two sequence sets in iteration 2 involving Tables 2 and 3 as well the first iteration set all of which the table was first. This conjecture is that a visual pattern provides the scaffolding needed by the student to associate the correct operator-matrix or schema. If visual pattern recognition plays a significant role in this sequence then language translation may even be secondary to this factor in promoting schema development. This hypothesis supports the importance that Koestler would place on imagery in creativity as well as Von Glaserfeld's thesis that pattern recognition is innate to the human mind. We note that recognition of visual patterns may not necessarily be considered as creativity by either Koestler or Von Glaserfeld rather as intuition unless the solver demonstrates conscious reflection. As Von Glaserfeld points out creativity occurs when the solver generalizes the principle. In this sense Czarnocha's exercise set provides excellent material for creative generalization of underlying relationships of the rate schema.

NOTE

1 The full design contains three such teaching sub-sequences, each for one of three representation of the formula $T_{otal} = R_{ate} \times N_{umber\ of\ units}$

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4.6. TWO LEARNING TRAJECTORIES

INTRODUCTION

In design research instructional design and research are interrelated, "the design of classroom learning environment serves as the context for research and, conversely, ongoing and retroactive analyses are conducted in order to inform the improvement of design" (Cobb et al., 2011, p. 75). The design of mathematical tasks or learning sequences of exercise and their analysis is a central component of research in mathematics education and ties together learning trajectories and design research. The design and assessment of sequences of mathematical tasks, and classroom methodology during their implementation, is a foundation for promoting discovery learning and a creative learning environment.

Simon and Tzur (2004) describe the importance of mathematical tasks in the classroom as, an important point of contact between the teacher and student and as playing a, "key role in the effectiveness of mathematical instruction" (p. 92). Two factors that guide the formulation of mathematical tasks, or teaching sequences is first, they must challenge students to engage in progress in understanding not simply exercise in understanding i.e. they should not focus on rote or routine tasks, "if students are challenged at an appropriate level they develop their cognitive abilities and engage in rich mathematical conversations" (Simon & Tzur, 2004, p. 92) second, there should be engagement of students in the concepts to be learned. In the teaching sequence of Czarnocha (Chapter 4.5) as well as the written exercises of Baker and Czarnocha (Chapter 4.10) the transition from exercise to progress in understanding is promoted by increasing abstraction from arithmetical exercises to algebraic thought through the use of language i.e. asking the solver 'how would you solve such and such' without numerical values the goal is to create a situation that requires reflection upon what one does instead of action. The goal of these exercises is to provide an opportunity for internalization of one's actions.

Learning Trajectory

Simon and Tzur (2004) expand on the concept of a teaching sequence to formulate that of learning trajectories. For these authors a hypothetical learning trajectory (HLT) contains three essential components, first a goal for student learning, second a sequence of mathematical tasks and third a hypothesis about the process of student learning. They note that, "the teacher's goal for student learning provides direction

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for the other components, the selection of learning tasks and the hypothesis about the process of learning are interdependent" (p. 93). In like manner, Clements et al. (2011) define learning trajectories with a,

> description of children's thinking and learning in a specific domain and a related, conjecture route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (p. 129)

For Simon and Tzur (2004) a hypothetical learning trajectory should have:

1. A goal for student learning –conceptual development
2. A sequence of mathematical activities that will be used to promote student learning towards this goal.
3. A hypothesis about the process of student learning-conceptual development based upon learning theory.

Roots of Design of Instructional Sequences

Gravemeijer (2004) notes that theories for instructional design such as Gagne's Principle of Instructional design that focused on how experts solved problems were in vogue in the 1960' and 1970's but faded from interest in part due to the rise of the constructivism with its focus on understanding student's thought process:

> The instructional design principles of the 1960's and 1970's do not fit reform mathematics instruction. The main problem is that the older design principles take as their point of departure the sophisticated knowledge and strategies of experts to construe learning hierarchies. Following a task analysis approach, the performance of experts is taken apart and laid out in small steps, and a learning hierarchy is constituted that describes what steps are prerequisites and in what order these steps should be acquired. (p. 106)

According to Gravemeijer (2004) (social) constructivism has resulted in a focus on the social classroom learning environment:

> The reform pedagogy is elaborated in terms of classroom culture, social norms mathematical discourse, mathematical community, and a stress on inquiry and problematizing…it could be necessary to draw the attention to the curriculum counterpart of this innovative pedagogy. (p. 106)

An important center for development of learning sequences was the Freudenthal Institute "The design research at the Freudenthal Institute grew out of the desire to develop mathematics education which corresponded with Freudenthal's ideal of mathematics as an human activity…That is to say, students should experience the process by which new mathematics is learned as a reinvention process in which they themselves play an active role" (Gravemeijer, 2004, pp. 108–109).

Cobb et al. (2011) commenting on the work at the Freundenthal Institute with learning trajectories notes there was a strong focus on the learning of individual students:

> The focus of the designers at the Freudenthal Institute…appeared to be on the mathematical learning of individual students…However, the acknowledged diversity in students' reasoning tended to fade in the background when they outlined long-term learning trajectories that constituted the rationale for instructional sequence. (p. 80)

Despite this effort these Cobb et al. (2011) note with some regret this work has been largely overlooked in part because of a perception that a learning trajectory requires all students to learn in the same manner:

> It is fair to say that our conceptualization of hypothetical and actual learning trajectories in terms of evolving mathematical practice has had remarkably little impact in mathematics education research…Several mathematics educators have critiqued our conceptualization of learning trajectories foe implying that all students in a classroom should follow a single learning path…this was an unfortunate interpretation given that our primary reason for recasting the notion of a learning trajectory in collectivist terms was to take into account the diversity in students mathematical reasoning. (p. 81)

A central thesis of design research and/or teaching research is that reflection upon student learning on sequences of tasks in the classroom by teachers and communities of teachers and researchers (TR team) will lead to improvement of teaching and the learning process. In other words creative bisociative acts of both teaching – research will provide the impetus for what Jaworski (2006) would call a 'critical alignment'. In this scenario the role of the teacher is co-participate with the students as the lesson unfolds, managed by instructor–led class discourse and/or a learning sequence of tasks. Reflection upon the class lecture by the TR team and analysis rooted in theory assist the teacher develop and grow as a professional. Much of the work of Prabhu and Dias in Unit 2 and the previous work on rates and proportion of Czarnocha, Baker and Dias in Unit 4 can be understood in this light. The resulting conclusions, assumptions and hypothesis that develop can be referred to as a local theory of instruction i.e. what has worked in our setting.

Theory of Teaching and Learning Trajectories

Sztajn et al. (2012) comment on the lack of connection that may exist between theories of learning and teaching and the lack of theories of teaching: "Theories of learning can develop with no necessary connection to teaching, and theories of teaching are far common than their learning counterparts" (p. 147). The understanding of the

TR team about what works and why, analyzed through the lens of learning theories provides hypotheses and theories of effective instruction.

Gravemeijer (2004) comments that such local theories of instruction are critical in promoting task centred reform pedagogy:

> Externally developed local instruction theories are indispensible for reform mathematics education. It is unfair to expect teachers to invent hypothetical learning trajectories without any means of support. In addition, it can be argued that without them, the chances to reconcile openness toward students' own contributions and aiming for given end goals are very slim. (p. 108)

For Gravemeijer a local instructional theory must address two concerns first, how students are to use and interact with the activities or tools introduced and second a sense of how students conceptual develop is to proceed i.e. a theoretical framework. Inherent in this work is that reflection by the teacher and/or TR team is central to improvement in learning and teaching thus, the teacher or instructor must be comfortable with the methodology they employ in the class to implement the learning trajectory.

Baroody et al. (2004) note a range of beliefs about classroom methodology:

At one end of the direct to indirect continuum is the traditional skills approach. Consistent with a dualistic philosophy, a teacher in this approach serves as the authoritative source of knowledge and uses direct instruction and practices to impart the correct procedure. The aim of such an approach is the mastery of basic skills. At the other end of the continuum is the laissez-faire problem-solving approach. This is a process-orientated approach in that the aim is to develop mathematical thinking: learning content is secondary and incidental. As its underlying philosophy is extreme relativism – a teacher neither imposes solution procedures nor provides feedback on the correctness of solutions.

The first of the two intermediate approaches is the conceptual approach, the aim of which is mastery of basic skills with understanding. A teacher can use for example, highly structured guided discovery learning to lead students in a predetermined direction. Consistent with a pluralistic philosophy; teachers can tolerate even encourage alternate procedures, but they ultimately ensure the standard procedure is adopted. The second intermediate approach is the investigation approach. As a blend of the conceptual and problem-solving approaches, its aims are mastery of basic skills, conceptual learning, and mathematical thinking. The investigative approach, then is characterized by both meaningful and inquiry-based instruction, and by purposeful learning and practice. That is, a teacher uses worthwhile tasks to create a need to explore and use mathematics. As this approach is based on a philosophy of instrumentalism, teachers are concerned about students' understanding and promote the use of any relatively efficient and effective procedure as opposed to a predetermined or standard one.

In this range of beliefs, those who hold to the traditional view often emphasize the need to complete the syllabus and to expose students to the full range of topics.

The emphasis on student learning and their thought process becomes more and more pronounced as one move through this spectrum.

Learning Trajectories and Curricula

Another use of learning trajectories is curricula development. Clements et al. (2011) comment and the relatively low performance of student in the U.S. on mathematics especially for children in "low rescores communities" (p. 128). They note the lack of research studies on curricula and call learning trajectory research based upon learning theories. In the work of Confrey et al. (2010) learning trajectories are taken as "researcher-conjectured" as opposed to teacher conjectured paths of learning. Furthermore, unlike the focus on individual student paths to cross their ZPD such researcher conjectured curricula have been employed to develop state-wide even nation-wide curricula in the U.S. referred to as the common core standards for matheamtics (Confrey et al., 2014). The use of researcher-conjectured pathways to develop state wide and nation-wide curricula runs counter to the emphasis on reflection upon individual student learning. As noted by Cobb et al. (2011) those who employ such methods can be criticized as insisting that all student must learn at the same level and pace.

The chapter on Learning Trajectories has several planes of discourse. It addresses fundamental student obstacles in high school mathematics education, the transition between arithmetic and algebra: proportional reasoning and linear equations. It is the summary of our efforts in this interphase (Chapters 2.4, 4.2, 4.10 and 5.1), which at the same time is one of the central student obstacles in our remedial sequence. The difficulties on that interphase are serious. Lamon (2007) affirms that the lack of ability to reason proportionally is widespread when she notes "… *a sense of urgency about the consistent failure of students and adults to reason proportionally … My own estimate is that more than 90% of adults do not reason proportionally …*" (p. 637). On the other hand, Hacker (2012), in response to the challenges students have with algebra, suggests elimination of the subject from the curriculum: "*Of course, people should learn basic numerical skills: decimals, ratios and estimating, sharpened by a good grounding in arithmetic*". But algebra, the generalization of arithmetic is not necessary for workers, since "*a definitive analysis by the Georgetown Center on Education and the Workforce forecasts that in the decade ahead a mere 5 percent of entry-level workers will need to be proficient in algebra or above*". NYT, 7/22/12.

The chapter provides the answer to the question "*Whose responsibility is it to construct learning trajectories?*" asked by Steffe (2004, p. 130). We concur with Clements and Sarama (2004) who note, "that learning trajectories could and should be re-conceptualized or created by small groups or individual teachers, so that they are based on more intimate knowledge of the particular students involved…" (p. 85) and the two discussed trajectories are an example of such a process. We do believe that the framework of Learning Trajectories is extremely useful to teachers of mathematics because it let them bring forth all of their craft teaching, often intuitive experience to the table – learning trajectories can be seen from the side of practice as research-based teaching sequences. Use of JiTR approach in the context of natural for teachers' iteration cycles, helps to assure the conceptual precision and coherence of teacher – produced learning trajectories.

While both trajectories address the interphase between arithmetic and algebra yet their designs are very different. The Number Sense/Proportional Reasoning trajectory design is of the type B, a research-based design showing at the same time the degree to which a standard Math Ed research can be conducted in classroom context by a community of teacher-researchers, who at the same time develop teaching sequences based on that research. Similarly to the design in the next chapter, this LT contains several distinct pathways depending on the cognitive knowledge of the student. Here we have an example of the quantitative research being supported by the qualitative analysis of classroom TR Interviews (Chapter 3.4). Note, that each TR interview sequence terminates only when involved students understand those very issues the instructor inquired about in the interview process.

The TR Design of the Linear Equations trajectory is C; it is based both on the craft knowledge of the teacher and preliminary research of Confrey and Maloney (June 2010). The important aspect of this section is presentation of the JiTR approach formulated in Chapter 1.1, when the algebra research by Mexican researchers served as Just-in-Time-Research consultation. It played important role in demonstrating that the learning trajectory designed by the instructor on the basis of his craft knowledge contained in reality three different trajectories, which together formed the basis for the whole design. The choice of the particular sub-trajectory was left to the teacher and it depended on teacher's assessment of students' knowledge.

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4.7. LEARNING TRAJECTORY

Rational Numbers Sense to Proportional Reasoning

INTRODUCTION

This section is intended to demonstrate how statistical, or quantitative, analysis can be used, alongside qualitative investigations of discourse, to study hypothetical trajectories and relationships between conceptual knowledge as students' progress through *rational number sense* toward *proportional reasoning*. This chapter expands upon an earlier project whose results have been published in Doyle et al. (2016). Here, we include both statistical analyses of the data collected at that time and qualitative interviews used to determine a path, or a learning trajectory, moving through the rational number sub-constructs proposed by Kieren (1976) to proportional reasoning – a hypothesis suggested recently by Lamon (2007).

PROPORTIONAL REASONING

Community Colleges and Remedial Mathematics

Many community colleges (two-year or junior colleges) in the United States have an open admission policy and, as a result, give entrance/placement exams to assess students' language and mathematics skills in order to determine college readiness in these disciplines. Students who place below college-level mathematics are commonly enrolled in remedial classes that review pre-algebra and elementary algebraic mathematics. Due to low retention and passing rates, these courses can serve as a barrier to students' desired proper college education (Hagedorn, Siadat, Fogelo, Nora, & Pascarella, 1999).

Proportional Reasoning: Transition from Pre-Algebraic to Algebraic Thoughts

The transition from spontaneous arithmetical thought to the more structured reasoning required in algebra, as highlighted by Vygotsky (1986) and Sfard and Linchevski (1994), is frequently linked to student mastery of proportions and, yet, many college students fail to manifest effective formal proportional reasoning (Adi & Pulos, 1980). Proportional reasoning has been described as a foundational core of algebra and higher mathematics (Berk, Taber, Gorowarand, & Poetzl, 2009; Lo & Watanabe, 1997). It is considered a prerequisite for success in science,

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mathematics and nursing courses. However, college students often demonstrate an inconsistent ability for such reasoning (Thornton & Filler, 1981; Garfield & Ahlgren, 1988; Hoyles, Noss, & Pozzi, 2001). Lamon (2007) affirms that the lack of ability to reason proportionally is widespread when she notes, " a sense of urgency about the consistent failure of students and adults to reason proportionally … My own estimate is that more than 90% of adults do not reason proportionally" (p. 637). In the remedial pre-algebra course, proportions are covered after, and are related to the concepts of *ratio* and *rate*. This sequential presentation often leads students to view these topics as different entities despite the emphasis that educators place on the connections between them (Streefland, 1984, 1985; Lachance & Confrey, 2002; Behr, Lesh, & Post, 1992). Furthermore, many educators believe there is a strong link between proportional reasoning and the broader concept of *rational number sense* (fractions, operations on fractions and their applications) that could be utilized for effective instruction in mathematics (Shield & Dole, 2002; Behr, Harel, Post, & Lesh, 1992).

The Kieren Model

Kieren proposed that the concept of a fraction could be viewed as the composition of five related but distinct sub-constructs. In this model, understanding of the primary *part-whole* sub-construct is the foundation for the four secondary sub-constructs of *ratio*, *operator*, *quotient* and *measure*.

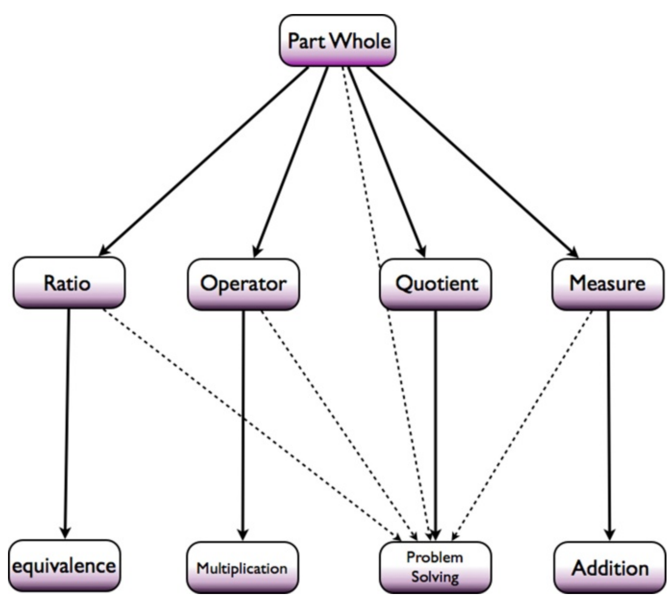

Figure 1. Recreation of the Model of Behr et al. (1992, p. 100)

Thus, in Kieren's model, *ratio* is considered as a sub-construct of fractions. An extension of this model to the corresponding fraction operations and problem solving was developed by Behr et al. (1983). For the sake of simplicity, this will be referred to as the *Kieren-Behr model*.

Kieren's Sub-Constructs as the Foundation for Proportional Reasoning: Lamon

Adjiage and Pluvinage (2007) used quantitative analysis with equivalent forms of Kieren's sub-constructs to investigate their role in proportional reasoning of middle school children. In a longitudinal study of students from grades 3 to 6, Lamon (2007) also used Kieren's sub-constructs as a foundation, or a focal point, of instruction aimed at developing rational number sense and proportional reasoning. Thus, she followed an educational approach based on the Behr et al. (1983) extension of sub-construct knowledge to proportional reasoning:

> My hypothesis is that proportions arise in the study of rational numbers ... and that as one develops rational number sense through the various experiences with many personalities of the rational numbers, one learns to reason proportionally. (Lamon, 2007, p. 640)

Lamon's (2007) research focus included determining the connections between these sub-constructs, and whether students' competency related to these sub-constructs facilitates proportional reasoning. She concluded that these different sub-constructs are, "essential characteristics of the rational numbers, but they are inextricably connected ... the measure sub-construct seemed to be the strongest because it connected most naturally with the others" (p. 659). Regarding the question of effectiveness of the sub-constructs for developing proportional reasoning, Lamon concludes that, "incorporating rational number interpretations other than the traditional part-whole meaning, will, in itself, be insufficient to facilitate meaningful learning" (p. 660).

Lamon's assertion that the rational number sub-constructs are the basis of proportional reasoning led Doyle et al. (2016) to attempt to verify this assertion using both quantitative Wilkins and Norton (2009, 2010) and qualitative methodologies. Lamon's statement that these sub-constructs are linked together, and that it is insufficient to develop proportional reasoning through a focus on only a single sub-construct, *part-whole*, for example, as well as her understanding that *measure* plays a crucial role in the development of proportional reasoning, provides the impetus for the present review of the data used by Doyle et al. (2016). The objective is to discern whether there is a pathway that student can practically follow, through the rational number sub-constructs leading to proportional reasoning. Specifically, what role does the *measure* sub-construct play in this hypothetical learning trajectory? We turn to the statistical analysis begun in Doyle et al. (2016) that uses multivariate linear regression techniques (ANOVA), and expand upon the results focusing on the possible learning trajectory through the rational number sub-constructs. However, before we do so, we give a brief review of the rational number sub-constructs and the exercises that are used to evaluate their understanding; for more information, we refer the reader to Charalambous and Pitta-Pantazi (2007) and Baker et al. (2009, 2012).

Rational Number Sub-Constructs

The definition and exercises used to evaluate the mastery of the fraction sub-constructs are, for the most part, akin to those used by Charalambous and Pitta-Pantazi (2007). However, exercises were included from the adult curriculum for the *ratio* and *operator* sub-constructs to ensure that the exercise sets were appropriately challenging for adult students. The complete sets of exercises and the results of principal factor analysis (Cronbach's alpha) and reliability tests (Cramer, Post, & delMas, 2002), as well as the Kaiser-Meyer-Olkin measure of sampling adequacy are presented in the appendix.

The *part-whole* sub-construct employs the symbol notation p/q to represent the partitioning of a whole entity, either continuously or discretely, into *q* equal parts and, then, taking *p* out of the total number of *q* shares of the entity. The *part-whole* sub-construct is used as a foundation for developing rational number sense in the mathematics curricula, and generates much of the language used for describing fractions (Behr et al., 1983). The mastery of the *part-whole* sub-construct was evaluated through two related exercise sets. The first set involved translating pictorial representations of *part-whole* relationships into symbolic numerical notation. The second set involved understanding written statements describing the process of taking parts of a whole.

The *ratio* interpretation of the symbol notation p/q involves a comparison between two quantities, *p* and *q*, and, as such, it is seen as a comparative index of the relative magnitudes of two numbers (Behr et al., 1983). Thus, when *p* and *q* represent two parts of a quantity, the *ratio* can be considered as separate from the other sub-constructs, and can be represented as a single numerical value. However, if *p* is taken to be a part of a total quantity *q*, the *ratio* and *part-whole* notation overlap with one another (Clark, Berenson, & Cavey, 2003). The *ratio* sub-construct used by Charalambous and Pitta-Pantazi (2007) involved three sets of exercises: the first set required students to represent part-part and part-whole relationships in fraction notation, the second involved the *rate* concept, and the third entailed comparisons of two ratios. Three exercises were included from the adult curriculum that required students to re-write a given ratio in simplest terms.

The *operator* concept is associated with applying a function to a quantity, that is, the process of taking a fraction of some given quantity. Thus, the *operator* sub-construct interpretation of p/q involves multiplication, or, more precisely, multiplication, or expansion, by *p* and division, or contraction, by *q* (Behr et al., 1992). Exercises used to evaluate the mastery of the *operator* sub-construct included three sets of exercises used by Charalambous and Pitta-Pantazi (2007).

The first set contained *input-output machines* in which the output is a fractional amount of the input quantity. The second set contained language statements that related the process of taking a fraction p/q of a quantity with multiplication by *p* and division by *q*. The third set involved taking a fraction of a quantity. Several iterations of this last type were added to this set of exercises. Charalambous and Pitta-Pantazi consider the *measure* concept to be interconnected with the relative size of a number. In particular, *measure* is evaluated through an understanding of partitions of the unit interval into segments that are then joined to represent the measure. In this sense, *measure* involves an application of the *part-whole* concept to determine the placement of the fraction p/q on an interval with designated units, by dividing the unit interval into *q* equal parts and aligning *p* such parts to obtain the location. Thus, the *measure* sub-construct can readily extend the *part-whole* process to include improper fractions. The *measure* sub-construct is a source of some contradiction and lack of clarity. Behr et al. (1983, 1992) tend to consider *measure* as an extension of part-whole. Lamon (2007) agrees, indeed she considers measure as the most central sub-construct because it is so connected to the others. However, the work of Charalambous and Pitta-Pantazi (2007) with children and Baker et al. (2009) with adults, both reveal that the *measure* sub-construct is the most difficult for students, and, that, out of all the secondary rational number sub-constructs, it correlated the least with the *part-whole*. Determining the relationship of the sub-constructs to *measure* is, therefore, an important objective in this study.

The *quotient* sub-construct interprets the symbol p/q through the dual interpretations associated with partitive and quotative division (Behr et al., 1992). Exercises supporting both interpretations were taken from (Charalambous & Pitta-Pantazi, 2007). The exercises used to evaluate mastery of proportional reasoning were not taken from the work of Charalambous and Pitta-Pantazi. Instead, they were taken from *ratio*, *rate* and *proportion* exercises that an adult student is typically required to be able to solve.

REVIEW OF ANALYSIS OF VARIATIONS (ANOVA) WITH MULTIPLE VARIABLES

While statistical correlations are sufficient to test the relationship between two variables, to consider the effect that several independent variables have on a dependent variable, a multiple linear regression analysis or analysis of variance (ANOVA) must be used. The *F*-value, or *F*-ratio, is an indicator of the strength of the relationship between the independent and dependent variables, and the *p*-value determines whether the model is significant. Assuming the model is significant the *adjusted* r^2 value determines the percent of the variation in the dependent variable that can be explained by the independent variable(s). A comparison of the *adjusted* r^2 value with the square of the correlation coefficient *r* between each independent variable and the dependent variable reveals the extent to which the independent variables work together. However, more precise information is obtained from the significance communicated by *p*-values and *beta* values of each independent variable. When the independent variables are all significant, they work together in predicting, or explaining, the dependent variable. The second indicator of how the independent variables interact is the *beta* value. The *beta* value is a measure of how much influence each independent variable has in predicting or explaining variation in the dependent variable. For example, a *beta* value of 0.5 for *X* indicates that for every unit change of a standard deviation of *X*, there is a corresponding 0.5 or 50% change of a standard deviation in *Y*.

QUANTITATIVE RESULTS OF DOYLE ET AL. (2016)

Quantitative data was collected by six professors of mathematics acting as teaching researchers at two community colleges in the Bronx. The case set consisted of 334 adult remedial students solving sets of problems that evaluated their knowledge of the Kieren sub-constructs as well as proportional reasoning. These results demonstrated that the *part-whole* construct was the most readily accessible sub-construct, at 74%, followed by *ratio,* at 67%, *operator* at 62%, *quotient* at 55%, and, finally, *measure* at 49%; *measure* was the most difficult. The exercise set evaluating mastery of proportional reasoning was more difficult that any of the sub-constructs, at 29%. It was also determined that all sub-constructs and the *proportional reasoning* exercise set correlated significantly with one another, with *p-value* < 0.01. Thus, it made sense to ask whether the sub-constructs predicted, or explained, *proportional reasoning*. The results confirmed this hypothesis with the exception of the *part-whole* sub-construct.

Quantitative Analysis of Rational Number Sub-Constructs

The results of Doyle et al. (2016) also suggest that knowledge of *part-whole*, *operator* and *ratio* may be used to promote the understanding of *quotient*, and that, in turn, knowledge of all four constructs, – *part-whole, operator*, *ratio* and *quotient,* may be used to promote knowledge of *measure*. If verified, this would provide quantitative evidence of a trajectory through the rational number sub-constructs. Accordingly, a multiple regression analysis with *part-whole*, *operator* and *ratio* as independent variables, and *quotient* as the dependent variable, was conducted using the same student data of Doyle et al. (2016). The analysis with these three independent variables resulted with an *F*-value of 30.4, with $p < 0.001$, with *adjusted* $r^2 = 0.256$. These values indicate the presence of a significant model that predicts 25.6% of the variation in the mastery of *quotient*. This is substantially more than that explained by any of these variables alone. The significance and beta values indicate that *ratio*, with $\beta = 0.27$ and $p < 0.001$, is very significant and influential in explaining quotient. *Operator* is also significant but much less influential with $\beta = 0.19$ and $p = 0.004$. *Part-whole* is slightly less significant and influential than *operator*, with $\beta = 0.17$ and $p = 0.006$.

Quotient and Measure

In order to extend this pathway to *measure* as well as investigate Lamon's understanding of *measure* as the central construct in developing proportional reasoning through rational number sense, an analysis of variation (ANOVA) with *part-whole*, *ratio*, *operator* and *quotient* as independent variables was conducted to predict *measure*. A backwards multiple regression analysis eliminated the *operator* construct.

These results suggest that the pathway through the rational number sub-constructs to proportional reasoning begins with *part-whole*. It then appears to split into two paths,—one proceeds through *operator*, the other through *ratio* leading to *quotient* and, then, *measure*.

This multiple regression analysis, with *part-whole*, *ratio* and *quotient* as the three independent variables, and *measure* as the dependent variable, with N = 334 resulted in an F-value of 38.8, $p < 0.001$ and the *adjusted* $r^2 = 0.306$. These resulting values indicate a significant model that explains 30.6% of the variation in *measure*. The relatively high percent of *measure* explained by this multiple regression model with adults is in stark contrast to the small amount of *measure* explained for children, at 4.84%, obtained in the study of Charalambous and Pitta-Pantazi (2007, p. 308).

Table 1. Beta (β) and Significance (p) values for the measure sub-construct (N = 334)

Predictor variable	*Beta*	*p-value*
Part-whole	0.19	$p = 0.001$
Ratio	0.25	$p < 0.001$
Quotient	0.27	$p < 0.001$

Thus, as suggested by Figure 1, *part-whole* is the primary, or foundational, concept in the rational number sense trajectory. Then, there is a secondary level containing *ratio* and *operator*. At this second level the path splits with one track proceeding through *operator*. This track has little influence in predicting student competency with *quotient* and none in predicting *measure*. In contrast, the alternative track through *ratio* is very influential in predicting student competency with *quotient* and *measure*. These results verify Lamon's statement that *ratio* is more powerful than *operator*, at least, in predicting *quotient* and *measure*. It follows that the *ratio* concept should be introduced early in the curriculum because it is readily learned and more accessible to students, and should be used to assist student mastery of *quotient* and *measure*.

Formal Proportional Reasoning

Doyle et al. (2016) use multiple linear regression to predict formal proportional reasoning however there focus was wider and thus we recreate the essence of their work here focused more narrowly on only the rational number sub-constructs as independent variables to predict or explain student ability with proportional reasoning. A backwards linear regression eliminated part-whole a result that agreed with the analysis of Doyle et al. (2016). The resulting multiple regression model with operator, quotient and measure as independent variables and 334 students to predict proportional reasoning yielded, $F(3,257) = 39.6$, $p < 0.001$ with adjusted r-square value of 0.375 reveal a highly significant model that explains 37.5% of the variance in proportional reasoning.

Table 2. Mean scores and standard deviations of the four sub-constructs (N = 334)

Sub-construct	*Beta*	*p-value*
Operator	0.20	$p = 0.001$
Ratio	0.24	$p < 0.001$
Quotient	0.17	$p = 0.004$
Measure	0.20	$p = 0.001$

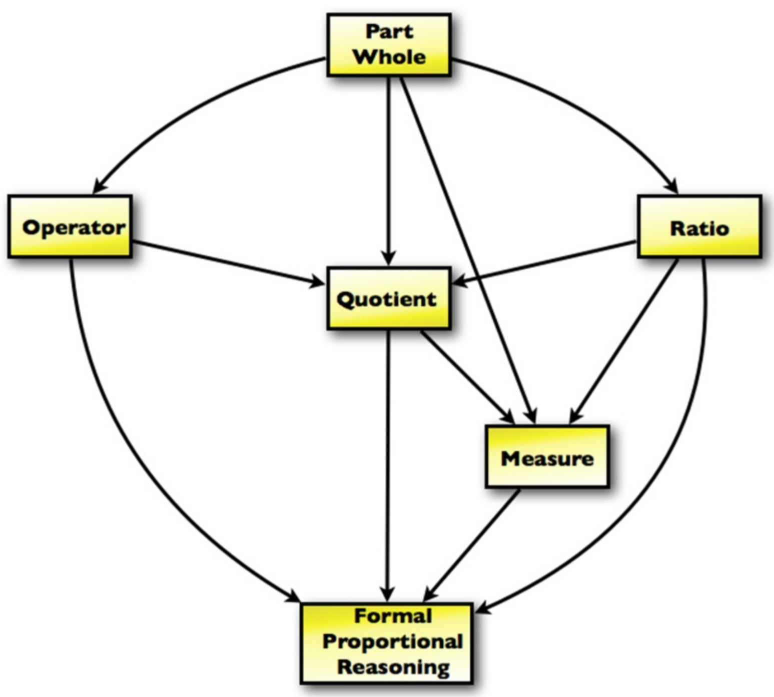

Figure 2. The trajectory structure: Rational number sub-constructs to proportional reasoning

The *beta* values in Table 2 show that *ratio*, *operator*, *quotient* and *measure* are all influential and significant factors in explaining students' formal *proportional reasoning.*

In Figure 2, the sub-constructs are arranged according to their mean score, and the pathways are those given through the correlations and statistical analysis reported in Tables 1 and 2. The foundational sub-construct of *part-whole* is influential in explaining student performance in the other more difficult sub-constructs but *not proportional reasoning*. Note that this result contradicts Figure 1. In Figure 2, the trajectory splits. One path proceeds through *operator*; this pathway does not have much influence in predicting the other rational number sub-constructs but is influential in explaining *proportional reasoning*. The other path proceeds through *ratio*; this path is influential in predicting *quotient* and *measures, and, subsequently, each of these connected sub-constructs,—ratio, quotient and measure,—are* influential in predicting student competency with *proportional reasoning*.

QUALITATIVE ANALYSIS

The quantitative analysis provides evidence of students' learning pathways through the rational number sub-constructs toward mastery of proportional reasoning. Next, transcripts of small group sessions and classroom discourse are reviewed to gain further insight into the two trajectories strongly suggested by the above statistical analysis. The interpretation of these dialogues is accomplished using the theoretical foundations of problem-solving and creativity developed in earlier chapters. We review it for the readers benefit.

Theoretical Model for Proportional Reasoning within Problem-Solving

The theoretical problem-solving model developed in Chapter 1.3 relates concept development and problem-solving; we summaries it here for the reader's benefit. According to Koestler (1964) a solver compares the given problem information to matrices of past experience and engages in selective attention to determine whether any problem information that does not fit the analogous matrix is miscellaneous or not. Students in the early stage of concept development are able to engage in this process of recognition (Cifarelli, 1998) as long as similar problems have been presented within the class lecture, or, as cognitive theorist would say, within short-term memory. The ability to compare problem information with matrices of past experience from long-term memory requires coordination of several matrices of past experience with the problem information and the re-representation of past experience (Glasersfeld, 1995). Rather than simply recognizing that presented information matches a matrix at hand, the ability to coordinate problem information with past matrices of experience and to recreate the essential structure of an analogous, or best-fit, matrix to solve the current problem is heavily dependent upon the solver's understanding of the concepts that exist in the code of the matrix. In other words, such coordination is strongly contingent on the solver's understanding of the structure of the problem-solving schemes. This model suggests a strong relationship between problem-solving and concept development, especially, in regard to an individual's choice of strategy. A solver at a higher level of concept development will be capable of coordinating the problem information with a wider range of past matrices because he or she will understand which concepts, codes and structures may or may not be relevant to the given situation. Furthermore, since the choice of strategy is based more on reasoning about the problem information using the learner's concept base, and less so on temporal proximity, copying, hunches or guess work, when the chosen matrix does not work, there is a greater likelihood that he or she will make real progress in understanding through synthesis and coordination of existing processes to find a proper solution strategy. In this model, when faced with a situation where no analogous matrix that fits the situation is immediately apparent or the chosen best-fit matrix does not lead to the desired goal, concepts play an important role in both the choice of strategy as well as the synthesis and coordination of matrices.

Rational Number Sub-Constructs and Proportional Reasoning: Qualitative Analysis

The first transcript was recorded early in the semester, before proportions were formally introduced. It conveys the discussion during a small group session during which the group leader (GL) attempts to arrive at the understanding of *proportional reasoning* appealing to the *ratio* and *rate* concepts. There were three students but only two gave responses. The two students are referred to using the pseudonyms Laura and Fran.

Problem 1: *If 2 items cost $5.00 then how much will five items cost?*

GL: What do we do first?
Laura: If 2 items cost $5 then 4 items cost $10.
GL: Good how did you get this?
Laura: I added.
GL: Okay, what do you do next?

After some waiting while the group is silent, **GL** continues: Can anyone else tell me what to do next?

Laura: I then add $2.50 to get the answer … (working it out) … $12.50!
GL: Very good, where did you get the $2.50 from?

The subsequent short period of silence indicates that Laura cannot verbalize her understanding.

GL: Does anyone know? (Turning to other members of group)
Fran: Well, 1 item is $2.50.
GL: How do you know that?
Fran: I took half.
GL: Good, so we take half to find the cost of how many items?
Fran: It's the price of 1 item.
GL: Did anyone do it a different way? (Looking around at the work) What about you, Fran? How did you do this problem?
Fran: Well, like I said, I know that 1 item is $2.50 so I multiplied by 5 to get the answer.
GL: Okay, good! Laura, why did she multiply by 5?
Laura: I'm not sure.
GL: Fran, can you explain why?
Fran: Because you want the cost of 5 items.

The additive approach employed by Laura represents foundational knowledge of addition of whole numbers employed in the *ratio* setting. Its use is spontaneous and intuitive for her; it is immediate and without much long thought. She then synthesized this process with the multiplicative process of taking one half of a quantity. The coordination of the *ratio* and *operator* concepts to solve this proportional reasoning problem demonstrates the use of two trajectories in Figure 2. Both lead to *proportional reasoning*, with one emanating from *ratio* and the other from *operator*. *Taking half a quantity* is an intuitive example of the *operator* construct and is unique among fractions, in that most solvers have a spontaneous understanding of this concept. Pitkethly and Hunting (1996) also note that, in their studies that, "knowledge of one half ... was quite robust" (p. 11).

Laura's coordination of the *additive ratio* concept with the *operator* concept of *half of a quantity* is intuitive and spontaneous as evidenced by her inability to vocalize why she had chosen these procedures. Thus, Laura is transitioning from the first level of concept development, where her reasoning is spontaneous and immediate but it is difficult to convey, and to, therefore build upon. Fran is able to vocalize her reasoning that is based on the concept of a *ratio* for the task of *finding half of a quantity*. Fran understands this problem through a multiplicative construction (Vergnaud, 1983), which synthesizes the *operator* concept by evaluating five times a quantity with the *ratio* concept. This multiplicative conception of proportional reasoning uses the same two pathways in Figure 2 as Laura but is more informed and advanced. Indeed, the establishment of proportional reasoning is frequently said to begin with the transition from additive to multiplicative reasoning (Karplus, Pulos, & Stage, 1983; Singh, 2001; Behr et al., 1992; Lamon, 1993).

The second example is taken from a class discourse on multiplication of fractions and introduces the *operator* sub-construct. The students involved were given the pseudonyms Irene, James and Alison.

Problem 2: *In a company with 360 employees respond to a survey, how many responded?*

Teacher (**T**): Can anyone tell me what the first step is?

Irene: We multiply by 360.
T: Good, how do you multiply these numbers?
Irene: I wrote 360 over 1 and then multiplied by 4 to get 1440 and then divided by 9 to get 160.
T: Very good, can anyone besides Irene tell me why we multiply times 360?
(A short period of silence)
T: Okay, if we take of some number, what operation does this involve?
(Another short period of silence)
T: Okay, if we take half of some number, what operation does this involve?
James: Dividing by 2
T: Okay, so taking th would involve what operation?
James: Dividing by 9.

The instructor senses that the class is not really following the dialogue. He draws a circle with 9 equal parts to visually represent the division process as partitioning.

T: Okay class, if the entire circle is 360 who can tell me what each part is?
Allison: Each part is 40.
T: Good, and how many parts would we need for ?
Allison: We take four of them.
T: Yes, and how much is this?
Allison: It would be 160.

Irene recognizes the *operator* conception in the problem of taking a fraction of a quantity, and readily relates this to a multiplicative approach. However, neither she nor any of her classmates can vocalize a reason. In Irene's case, and for many students in this intermediate stage, this phenomenon may be an illustration of the discrepancy between procedural proficiency and ability to verbalize one's thoughts. Glasersfeld (1995) alludes to this interpretation:

> The fact that conscious, conceptualized knowledge of a given situation developmentally lags behind the knowledge of how to act in the situation, is commonplace ... it is analogous to the temporal lag of the ability to re-present a given item relative to the ability to recognize it. (p. 106)

If this explanation is correct (the instructor certainly believed it was valid in Irene's case) and Irene is able to recognize the need for multiplication but has not conceptualized this *operator* process enough to vocalize her reasoning then she may experience difficulty recreating this process. As the rest of the class does not appear to grasp Irene's reasoning and she cannot convey it, the instructor generalizes the intuitive *operator* concept of *taking one half of a quantity*. James follows this generalization but the instructor senses that the rest of the class is too silent and not engaged. As noted by Pitkethly and Hunting, (1996) the generalization of the *operator* concept of *one half* to other fractions can be difficult for students, in part, because computing *half* of a quantity is so intuitive, that students often prefer to view fractions only through iterations of this process and, thus, ignoring or avoiding the cognitive demand of fractions that cannot be represented in this manner:

> The act of halving by subdividing is a scheme ... called algorithmic halving. This powerful strategy inhibits ... schemes to create fractions that have odd number denominators. (p. 11)

The instructor then employs the *part-whole* conceptualization to give meaning to the *operator* process of taking four ninths of a quantity; a strategy easily understood and followed by Allison and most other students. This presents a two-fold aspect of the *operator* construct,—one based on the foundational *part-whole* concept which is represented by the arrow from *part-whole* to *operator* in Figure 2, while the other is functional, in which the numerator and denominator are associated with separate operations. This line of thought contributes to the arrow from *operator* to *proportional reasoning* in Figure 2.

The final example is a transcript of a discussion that took place in a small group session addressing ways of finding quotients through equipartitioning (partitive division) and counting the number of groups that can be formed with a given number of objects (quotative division). This group session lead by a group leader (GL) was recorded with three students,—Julia, Allison and Gina, when the class was covering multiplication and division of fractions.

Problem 3: *If 6 ¾ lb. of meat is divided into packages and each package weighs ¾ lb. How many packages are there?*

GL: Did anyone get this?
Julia: I got 9 packages.
GL: Good, how did you do it?
Julia: I took 6 pieces and then took ¾ from each piece. This left 6 pieces each with a ¼ lb remaining. These make 2 more for a total of 8.
Allison: Where did you get the 9 from?
Julia: From the remaining ¾.
GL: Remember there were 6 ¾ lbs. and we used only the 6, this left ¾ lb. which is one more package.
Allison: Okay.
GL: Did anyone do it differently?
Gina: I did it by counting ¾ lb. is 1 package, then 1 ½ lb. is 2 packages. Thus 3 lb. is 4 packages and finally 6 lbs is 8 packages and the remaining ¾ makes a total of 9.

Gina points to the number line she has made with the help of the group leader (as shown in Figure 3 below)

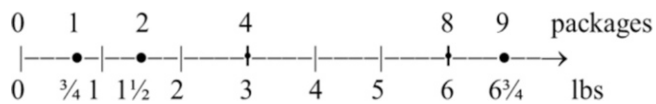

Figure 3. Gina's Illustration of her approach to the solution of problem 3

Allison: How did you get 3 lb. are 4 packages?
Gina: I doubled them.
GL: And you did the same to obtain the 6 lbs. makes 8 packages correct?
Gina: Yes.
GL: Allison how did you do it?
Allison: I divided.
GL: Okay why?
Allison: Because it says to divide it up into packages!

Julia's work is a good example of quotative division as she begins with the 6 ¾ lbs and begins to partition this quantity these into packages. As this approach had never been presented in class, it represented a spontaneous and creative synthesis of her understanding of the problem requirement to divide 6 ¾ lbs into packages with her *part-whole* understanding evidenced by her comprehension of what it means to take ¾ of a whole, and coordinating this with her ability to keep track of remainders. Thus, referring to Figure 2, Julia's reasoning illustrates the trajectory from *part-whole* to *quotient* followed by the path from *quotient* to *proportional reasoning*.

Gina, on the other hand, used an additive approach that is conceptually more readily understood; however, as the addition involved fractions, she employed the use of a number line to assist her in reaching a whole number *ratio* and proceed from there. Gina explained her work with the help of a number line following the method explained earlier by the instructor in the context of a different problem. The use of the number line to establish a relationship between the iterative processes of building up a fractional ratio to obtain a ratio of whole numbers, and proceeding with the simpler whole number ratio to get the final solution, helped Gina visualize the conceptual interactions and understand the relationship between the *ratio* and *quotient* sub-constructs.

Thus, the iterative process, beginning with the unit amount of ¾ lb. per package, followed by either the building up of units, as Gina did, or applying repeated subtraction, as Julia did, is essential for understanding division. This class discourse illustrates how the additive *building up* process underlies the relationships between *ratio*, *quotient* and *measure* in the process of *proportional reasoning* illustrated in Figure 2.

CONSIDERATIONS

In Figure 1, the *measure* sub-construct is connected to the *additive* structure, yet in Figure 2, *measure* is closely related to *ratio* and *quotient*, both of which can be understood as part of a *multiplicative* structure. This raises the question of whether *measure* plays a dual role in both the additive and multiplicative structures of rational numbers,—a hypothesis supported by the class discourse in which the additive *building up* process connected the *measure*, *ratio* and *quotient* sub-constructs. Vergnaud (1983, 1994) introduced the concept of a multiplicative conceptual field as a holistic approach to the development of proportional and formal reasoning in mathematics. Lamon has commented that the encompassing nature of this concept has made it difficult to employ in mathematics educational research. We propose that Figure 2 may serve as a basis to map out the multiplicative conceptual field.

In the mathematics curriculum for adults reviewing pre-algebra, the sequence leading up to proportions includes whole numbers, fractions, decimals, and then ratios and rates. This sequential presentation often leads students to view these topics as different entities despite the emphasis that educators place on the connections between them (Streefland, 1984, 1985; Lachance & Confrey, 2002; Behr, Lesh, & Post, 1992). The relationships shown by Figure 2 strongly support the interconnectedness of the rational number sub-constructs in promoting proportional reasoning.

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APPENDIX

Exercise Sets
Ratio[1]

Component 1
#1.) In a History class there are 2 male to every 3 female students, use fraction notation to write the ratio of male to female students in the class.*
#2.) In a History class there are 2 male to every 3 female students, use fraction notation to write the ratio of female to male students in the History class.*
#3.) Use fraction notation to write the ratio of female to total students in the History class.*

Component 2
#4.) Write the ratio 4 to 36 in simplest terms.* †
#5.) Write the ratio 48 to 16 in simplest terms.* †
#6.) Write the ratio 0.8 to 4 in simplest terms. †

Component 3
Juan and María are making lemonade. Given the following recipes whose lemonade is going to be sweeter?
#7.) Juan uses 2 spoons of sugar for every 5 glasses of lemonade. María uses 1 spoon of sugar for every 7 glasses of lemonade.*
#8.) Juan uses 2 spoons of sugar for every 5 glasses of lemonade. María uses 4 spoon of sugar for every 8 glasses of lemonade.*

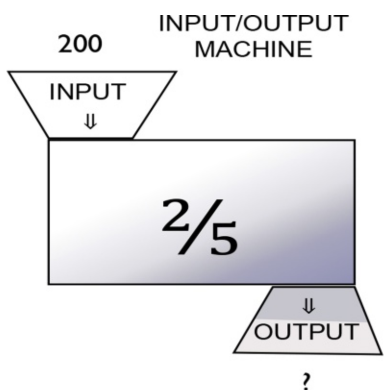
Component 4
Jose jogs each morning before work. Determine which of the following days he jogged at a faster rate. Please choose from the answers (a) Monday, (b) Tuesday or (c) Not able to determine from information given.

#9.) On Tuesday he jogged a longer distance than he did on Monday. On both days he jogged exactly the same amount of time.*
#10.) On Tuesday he jogged a shorter distance than he did on Monday. On both days he jogged exactly the same amount of time.*
#11.) On Tuesday he jogged a shorter distance than he did on Monday. On Tuesday he jogged less time.

Operator[2]

Component 1
There were three exercises that evaluated the *operator* concept through functional input-output boxes.

#1.) The following diagram represents a machine that outputs 2/5 of the input number. If the input number is 200 then what is the output number?*



#2.) An input-output machine has outputs that are 1/5 of the input. If the input number is 480 then what is the output number?*
#3.) An input-output machine has output that is 1/5 of the input. If the output is 200, then find the input. †

Component 2
#4.) Taking 2/5 of a number is the same as dividing the number by 5 and multiplying this result by 2; True or False? *
#5.) If we divide a number by six and multiply by twenty-four this the same as multiplying by the fraction1/4; True/False? *

Component 3

#6.) A recipe calls for 1 ½ cup of flour. Which of the following expresses the amount of flour required for of this recipe? *†

a) 3/2 ÷ 1/3
b) ½ ÷ 1/3
c) 3/2 × 1/3
d) 1 ½ – 1/3
e) not given

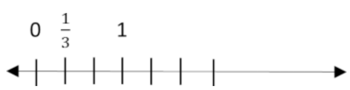
#7.) Find 4/5 of 7/8 of 40,000. *†
#8.) Find 3/5 of 5/8 of 4,000. †
#9.) Find half of hours.

***Measure*[3]**

Component 1

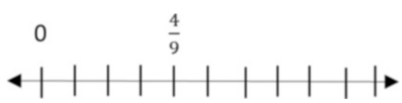
Locate the following numbers on the number line:

#1.) 1/6 #2.) 4/3 #3.) 5/6



Component 2

#4.) Locate the number "1" on the number line below:



#5.) Locate the number "1" on the number line below:

***Quotient*[4]**

Component 1

#1.) Three pizzas are shared equally among four students what fraction of a pizza will each receive? *
#2.) It takes ¾ kg of apples to make one pie. How many pies can be made using 20 kg? *

Component 2

#3.) A beach 3/7 miles long is divided into 6 equal parts. How long is each part? *

#4.) Two pizzas were shared equally among a group of students. If each student received 2/5 of a pizza then how many students were there?

#5.) If 3 pizzas are shared evenly among seven girls while 1 pizza is shared evenly among three boys. Who gets more pizza, a girl or boy? *

Part-Whole[5]

Component 1

#1.) Given a picture of four triangles and five circles; the question is what fraction of the objects are triangles?

#2.) Given a picture of a circle with 2 out of 5 equal parts shaded; the question is what fraction of the circle is shaded? **

#3.) Given figure composed of seven squares, three of which are shaded; the question is what fraction of the squares are shaded? **

#4.) Given five equivalent objects three of which are circled; the question is what fraction of the objects are circled?

Component 2

#5.) Given a rectangle composed of six equivalent squares one of which is shaded; the question is what fraction of the squares are shaded? **

#6.) Given four equivalent objects one of which are circled; the question is what fraction of the objects are circled? **

#7.) Given a rectangle composed of six equivalent squares four of which are shaded; the question is what fraction of the squares are shaded? **

#8.) Given a figure composed four equivalent objects three of which are circled: the question is what fraction of the objects is circled? **

#9.) Given a figure with sixteen equivalent objects four of which are circled; the question is what fraction of the objects are circled? **

#10.) Given a rectangle with 24 equivalent squares four of which are shaded; the question is what fraction of the squares are shaded? **

Component 3

#11.) The fraction 2/3 corresponds to taking a chocolate bar, dividing it into three equal parts and taking two of these parts. True or False? **

#12.) The fraction 2/3 corresponds to taking a set of objects dividing it into three equal parts and taking two of them. True or False? **

Formal Proportional Reasoning [6]

Component 1

#1.) Hank drove 500 miles in 8 1/3 miles, what was his average speed or rate in miles per hour? **

#2.) If ¾ cup of coleslaw contains 120 calories. How many calories are there in 2/5 cup? **

#3.) If the ratio of is and b is 4200 then find the value of a. **

#4.) If 2/5 of 4000 is equal to ¼ of some number then find the number. **

#5.) If 0.5 ml of medicine are mixed with 2 ml of water to form a solution then what is the ratio of drug to water in simplest terms? **

Component 2

#6.) Which of the following fractions is closest to 1? **
(A) 2/3 (B) ¾ (C) 4/5 (D) 5/6

#7.) Circle the smallest fraction: **
(A) 2/11 (B) 3/13 (C) 4/23 (D) 5/6

* ***Commonality with other exercises in the same set is more than 0.5.***
** ***The commonality value was at least 0.4***.
† ***Added by present authors and not found in (Charalambous and Pitta-Pantazi, (2007)***

NOTES

1 The Kaiser-Meyer-Olkin measure of sampling adequacy for these 11 questions was 0.64 thus these 4 components are accurate with 67% of the variation explained by these four components. The Cronbach's alpha value for these 11 exercises was 0.67 thus the ***ratio*** exercise set is reliable.

2 Kaiser-Meyer-Olkin measure of sampling adequacy 0.672 with 57% of the variation explained by these three components. Cronbach's alpha 0.645 thus ***operator*** is a reliable set of exercises.

3 Kaiser-Meyer-Olkin measure of sampling adequacy 0.621 with 63.3% of the variation explained by these three components. Cronbach's alpha 0.618 thus ***measure*** is a reliable set of exercises. Commonality was at least 0.45 or above for all exercises.

4 The Kaiser-Meyer-Olkin measure of sampling adequacy was 0.621; 53.2% of the variation was explained by these two components. Cronbach's alpha was 0.49 thus ***quotient*** was not a reliable set of exercises.

5 The Kaiser-Meyer-Olkin measure of sampling adequacy was 0.821 and 59.3% of the variation was explained by these three components. The Cronbach's alpha value was 0.79 thus ***part-whole*** is a very reliable set of exercises.

6 The Kaiser-Meyer-Olkin measure of sampling adequacy was 0.75 and 51.3% of the variation was explained by these two components. The Cronbach's alpha value was 0.70 thus the ***formal proportional reasoning*** exercises formed a very reliable set of exercises.

BRONISLAW CZARNOCHA

4.8. LEARNING TRAJECTORY

Linear Equations

INTRODUCTION

Iteration has emerged as one of more important methodological processes within the environment of evidence-based Common Core standards. Its importance increases together with the goal to formulate effective student learning trajectories, that is, those theoretical pathways of learning mathematical concepts that come closest to actual student learning. The following definitions of "iteration" from Merriam-Webster and Oxford dictionaries refer explicitly to the successive approximations to a desired solution of the problem. The Merriam-Webster Dictionary defines "iteration" as "a procedure in which repetition of a sequence of operations yields results successively closer to a desired result." The Oxford English Dictionary provides a similar definition emphasizing the term's mathematical undertones: "A repetition of a mathematical or computational procedure applied to the result of a previous application, typically as a means of obtaining successively closer approximations to the solution of a problem."

Educational research needs iteration in order to formulate, refine and tune learning trajectories from a collection of fragmented and diverse research results concerning the concepts in question. For example, Confrey's formulation of the "equi-partitioning learning trajectory" relies on 600 different research pieces (Confrey, 2010). To transform such a large amount of research results into a smooth working teaching sequence facilitating student understanding and mastery of a given concept requires the successive approximation approach to revamp, change and improve the components of the teaching sequence while at the same time creating smooth connections between them.

The iteration methodology used by teachers in the construction of effective teaching sequences is very natural because of the cyclical nature of the teacher's workload assignments (Wittmann, 1999). Teachers can, and often do, teach the same course from one semester to another, or from one academic year to another, creating an environment in which any teaching sequence of a given concept can be iteratively refined over several application cycles. The integration of this natural cycle of work with the teaching-research cycle (TR cycle) discussed in the Chapter 1 creates an extremely powerful methodological tool tailor-made to address the complex question of *learning trajectories*.

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 373–393.

Often, when the current authors' work based on the TR cycle is presented to an audience of educational researchers, the most common question is "What is the difference between your cycle and the design research cycles?"

The difference is subtly profound. The standard design research cycle as well as the APOS theoretical framework cycle (Asiala et al., 1996), that served as the formative basis of the TR cycle created by the current authors, initiate from theoretical models, infer theoretical results, and, then, apply these models to the classroom setting. The TR cycle, on the other hand, starts most often from practice in a particular classroom setting, and its aim is the improvement of learning and related teaching in the very same classroom, and beyond. The theory here is a by-product of iterated practice, and it's not the main objective. Although seemingly insignificant, this change of the starting position for iterated investigation results in significant changes in the research methodologies. Table 1 presents a sample side by side comparison between the methods, aims and results of standard academic research versus the classroom-driven TR model.

LEARNING TRAJECTORIES

The concept of a *Learning Trajectory* has acquired recently new importance as the organizing principle of the new Common Core Standards in Mathematics (CPRE, 2011). There are several definitions of a "learning trajectory" within the research profession (Baker et al., 2012) indicating that the concept didn't yet "condense" (Sfard, 1992) sufficiently in its development. Therefore, one has a certain amount of freedom in focusing one's own investigation on different aspects of the construct. For the purpose of this work the authors adopt Clements' definition:

The learning trajectory (LT) of a particular mathematical concept consists of three components:

- A specific mathematical goal,
- A developmental path along which students' thinking and comprehension develops and,
- A set of instructional activities that help students move along that path (Clements & Sarama, 2009).

The idea of LTs has a wide range of applications. It can be an excellent assessment tool precisely informing the teacher about the successful pathways of mathematical thinking of his or her students as well as about their weaknesses. At the same time, it can serve as a tool, a map or a guide constructed, preferably, by the teacher and for the teacher, providing information about possible trajectories for learning improvement strategies, asked for explicitly by the designers of the approach (Figure 1, Center, Daro et al., 2011). Active implementation of the LT framework in the development of curriculum facilitates intense discussions about the effectiveness of the relationship between abstract research and practicing teachers toward the support of the Common Core effort. "Whose responsibility

Table 1. Comparison of standard academic research and the TR model

Standard research (Design-Based Research) model	*TR model (TR-NYC model)*
Theory-driven:	*Practice-driven:*
"Design-based research can contribute to theoretical understanding of learning in complex settings" (Sandoval, p. 00). Each of the articles by Sandoval, Tabak, and Joseph reveal how the design of complex interventions is an explicitly theory-driven activity.	Teaching-research is grounded in the craft knowledge of teachers that provides the initial source and motivation for classroom research; it leads to the design-based practice and, the primary aim is the improvement of learning in the classroom and beyond.
"In addition, the design of innovations enables us to create learning conditions that learning theory suggests are productive, but that are not commonly practiced or are not well understood" (Author, 0000)	The design of innovation enables the teacher-researcher to establish a creative learning environment based on teacher's craft knowledge that improves learning in the classroom and transforms students' habits (such as misconceptions) into student originality (Koestler, 1964). Learning theories are used as needed to support teachers' craft knowledge. (Prabhu & Czarnocha, 2006)
Cobb and Steffe (1983) assert that the interest of a researcher during the teaching experiment in the classroom is "in hypothesizing what the child <u>might</u> learn and finding [as a teacher] ways and means of fostering that learning".	"…the interest of a teacher-researcher is to formulate ways and means to foster what a child <u>needs to</u> learn in order to reach a particular moment of discovery or to master a particular concept of the curriculum (Czarnocha, 1999)". Since, however, "such moments occur only within students' autonomous cognitive structures, the [constructivist] teacher has to investigate these structures during a particular instructional sequence [in order to be of help to the students]. In this capacity, he or she acts as a researcher" (Prabhu & Czarnocha, 2007)
Articulating, refining and validating is an "iterative process of research synthesis and empirical investigations involving" many types of evidence: Step 1. Meta-research of the concept to create the prototype; Step 2. Iterative refinement of the prototype. (Confrey, 2010)	Use of iteration in the TR-NYC model: Step 1. Process of iteration starting with the first iteration designed on the basis of teaching practice. Step 2. Incorporation of research results as needed in between consecutive iterations. It is the concept of iteration of the design from semester to semester together with the related refinement that can allow for the immediate implementation of the naturally relevant research results illuminating the current classroom situation and providing further insight into the design of appropriate sets of assignments.

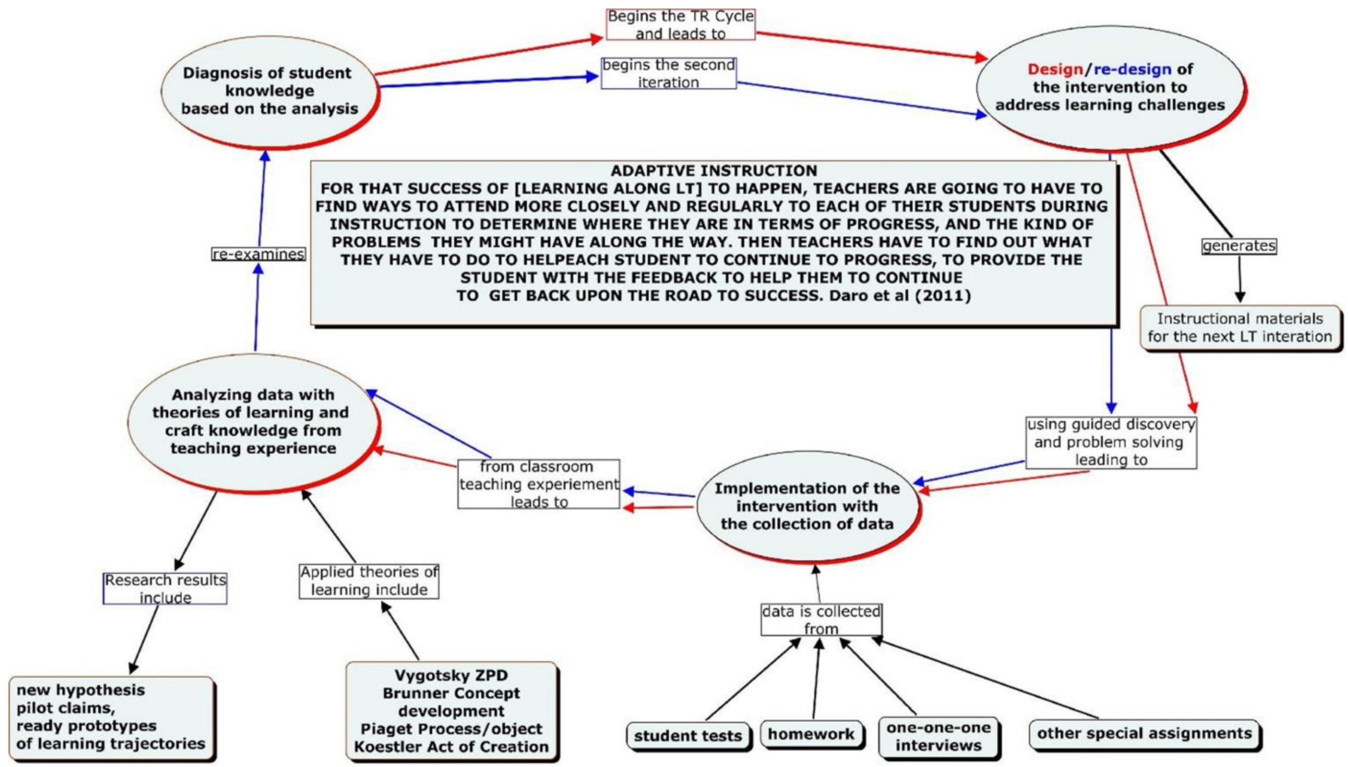


Figure 1. Teaching-research cycle

is it to construct learning trajectories?" asks Steffe (2004, p. 130). Battista (2004, p. 188) states, "to implement instruction that genuinely and effectively supports student construction of mathematical meaning and competence teachers must not only understand cognition-based research on students' learning, they must also be able to use that knowledge to determine and monitor the development of their own students' reasoning." Empson (2011) adds a layer of complexity to the current research on learning and invites one to think seriously about how to support teachers to incorporate knowledge of children's learning into their purposeful decision-making about instruction. Clements and Sarama (2004, p. 85) note, "that learning trajectories could and should be re-conceptualized or created by small groups or individual teachers, so that they are based on more intimate knowledge of the particular students involved…"

Thus, in agreement with Kieran, "it is [only] the teacher who can affect to the greatest extent the achievement of one of the main purposes of the research enterprise, that is, the improvement of students' learning of mathematics" (Kieran et al., 2013). Therefore, the search is on for the most effective routes of joining educational research with classroom teaching (Kieran et al., 2013). Kieran also addresses the variety of differences shared by researchers and teachers that make collaboration challenging (Kieran et al., 2013). It makes sense, therefore, to focus on what is common between researchers and teachers involved in classroom teaching-research. Our assertion is that the concept of iteration as a component of the research methodology is common to both.

THE METHOD OF ITERATION

This presentation is focused primarily on the methodological aspects of the proposed route of research/teaching integration showing an essential methodological trade off necessary (though not sufficient) for teachers' buy-in in the LT approach. The discussion describes the method of iteration for learning trajectories during the process of their research-based construction (Confrey & Maloney, 2010). The TR cycle of the TR-NYC model (Czarnocha & Prabhu, 2006) is the theoretical framework within which iteration is effectuated in classroom teaching-research. Two consecutive examples of the process are presented for the Learning Trajectory for Linear Equations (LTLE) under construction in the context of the Integrated Arithmetic/Algebra Course Teaching-Experiment being conducted at present at an urban community college.

The desired goal is the sequence of instructional problems and strategies that produces the most optimal effective understanding and mastery of the relevant mathematics (linear equations, in this case) in the classroom. Each new iteration of the teaching sequence is produced at the analysis of the data node of the TR Cycle through its major or minor refinement. The refinement may consist in the change of component strategies, their sequencing or the changes in learning environment. The changes are suggested by the analysis of learning in the previous cycle, the craft knowledge of the teacher-researcher as well as through the relevant research results.

The Iteration Trade-Off

Since, generally, every teacher has an option of teaching the same course every semester to a new cohort of students, the TR cycle allows for the continuous process of classroom investigations of the same research question during consecutive semesters or academic years. The TR-NYC model asserts that two such consecutive cycles constitute a single unit of activity explicitly aimed at the improvement of learning (Czarnocha & Maj, 2008). Two cycles are needed to enable the refinement of the particular LT from one iteration to the next. A methodology for construction and validation of a learning trajectory had been thoroughly described by Confrey and Maloney (2010) in the case of the Equi-partitioning Learning Trajectory. According to Confrey and Maloney, articulating, refining and validating is an "iterative process of research synthesis and empirical investigations involving" many types of evidence. Their research sequence starts with the significant research effort in the design of the first prototype. The iterative process is the second step of the research.

Within the TR-NYC model, the iteration becomes the primary methodological tool, while the initial learning trajectory is designed more on the basis of the teaching craft knowledge of the mathematics teacher than on the basis of the relevant research results. The fine tuning of the learning trajectory to the needs of the student cohort through the incorporation of the research knowledge into the design process takes place during the consecutive iteration phases while fulfilling the requirements of adaptive instruction (Daro et al., 2011). It is the concept of iteration of the design from semester to semester together with the related refinement that can produce relevant research results illuminating the classroom situation or providing help in the design of an appropriate set of assignments.

Thus the initial theoretical period of gathering available research required for standard research is not necessary for the classroom teacher-researcher designing learning trajectories because it can be transformed into its "just-in-time" utilization at each refinement node of the TR cycle. The "just-in-time" manifestation occurs along the iteration cycle. This change of emphasis in the role of research as the starting point of investigation to its "just-in-time" consultation is one of the necessary conditions for the incorporation of research into classroom practice.

ADAPTIVE INSTRUCTION

The process of iterative refinement of the teaching sequence associated with a given learning trajectory introduces, in a natural manner, a new type of instruction that adapts itself to students' state of knowledge. It's a promising concept in that it has an application to every student in the class and, thus, it ideally accounts for learning for all students. The process of adaptive instruction outlined by Daro (2011) corresponds to nodes of the TR cycle. For example, "the determination where students are in their progress and the kind of problems they might have along the way" (Daro et al., 2011) corresponds to the Diagnosis node of the TR cycle; "finding out what to do to help students to continue to progress" (Daro et al., 2011) corresponds to Design/Redesign node of the intervention to address learning challenges; providing "students with the feedback to help them to get back upon the road to success" (Daro et al., 2011) corresponds to the Data Analysis nodes followed by the Diagnosis node again, and next the Redesign node. Thus, if there is a need to help students with their immediate problems, the TR cycle may be traversed a couple of times within one class. The paradigmatic example in Chapter 4.1 is a good illustration of several TR cycles taking place within a short classroom dialogue lasting only several minutes. This unity of research investigation and adaptive teaching is possible through the development of thinking technology within the practice of the teacher-researcher touched upon in Chapter 4.1.

CONSTRUCTION OF A LEARNING TRAJECTORY

The construction of a learning trajectory for linear equations through three iterations, demonstrated below, provides an illustrative example of the method.

The Learning Trajectory for Linear Equations (LTLE) has been designed on the basis of algebra classroom teaching craft mathematical knowledge of the teachers and triangulated with the Learning Trajectories Display of the Common Core State Mathematics Standards developed by Confrey et al. (July 2010). The design of LTLE is the adaptive response to the observed challenges of students with the following problem:

Solve for y in terms of x:

$$3x - 2y = 6 \tag{1}$$

Students' recorded solution:

$$\begin{array}{r} 3x - 2y = 6 \\ -3x \qquad\quad \\ \text{- - - - - - - - - - - - - -} \\ -2y = 6 \\ y = -3 \end{array} \tag{2}$$

The First Iteration LTLE, pictured in Figure 2, was designed to respond specifically to student difficulties described above. It outlines the necessary prerequisite and sequential knowledge to understand the central concept "solve for x in terms of y" as well as new concepts dependent on that understanding. The concept map is designed in the environment of the Institute for Human and

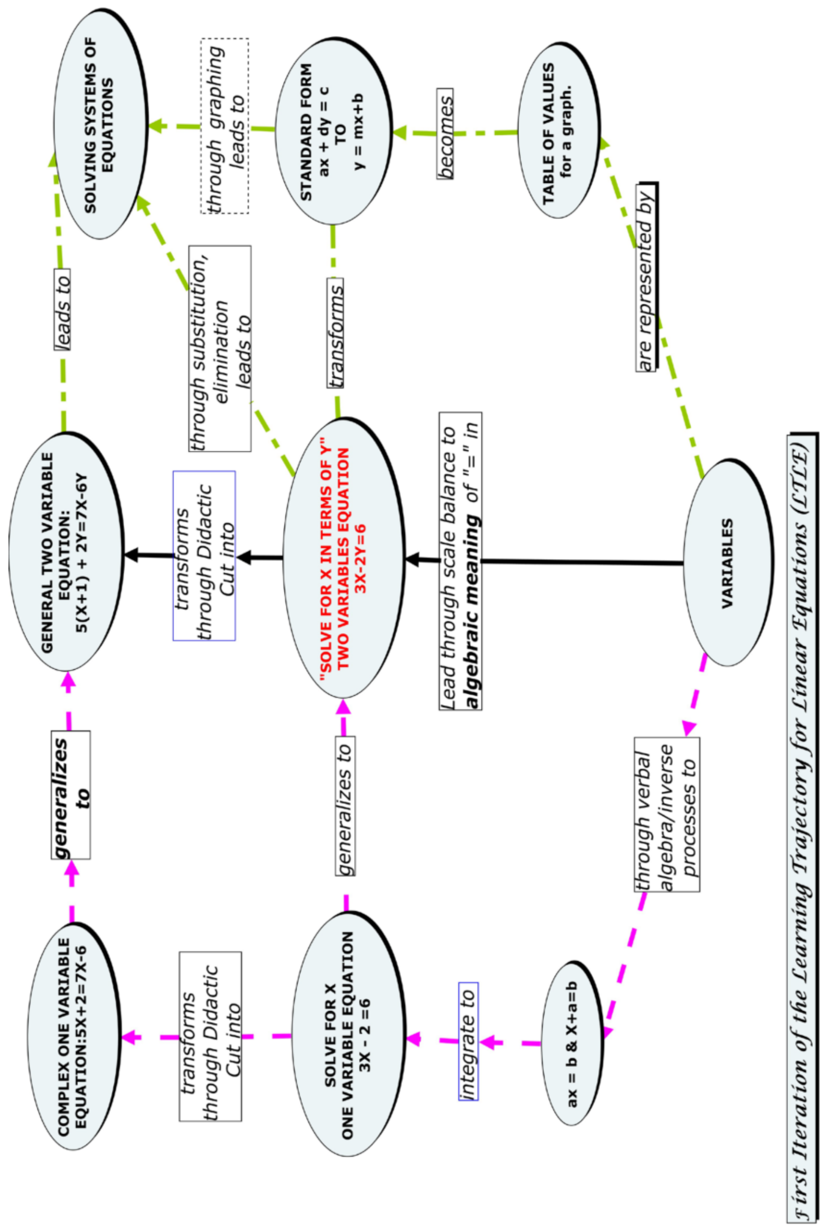


Figure 2. First iteration concept map demonstrating the three component trajectories associated with the learning trajectory for linear equations

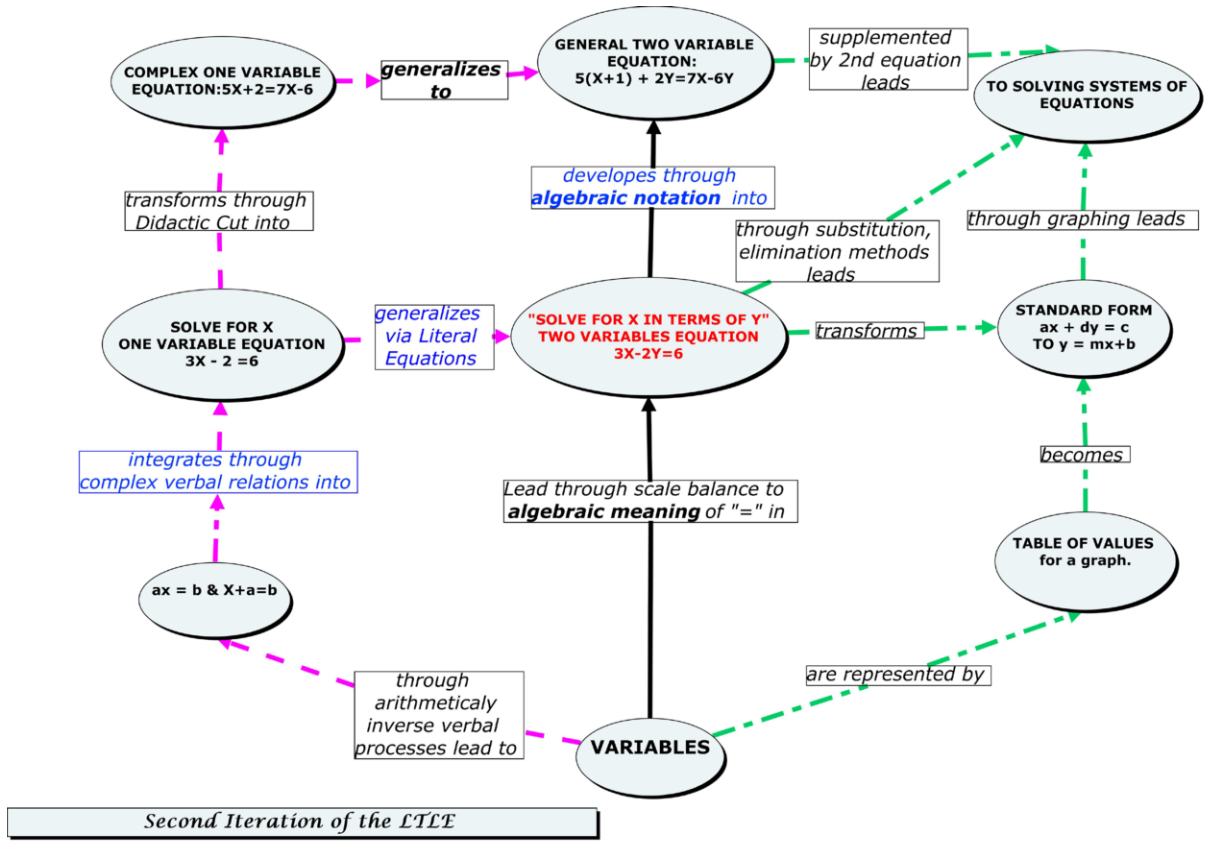


Figure 3. The revised concept map: Second iteration of the learning trajectory for linear equations

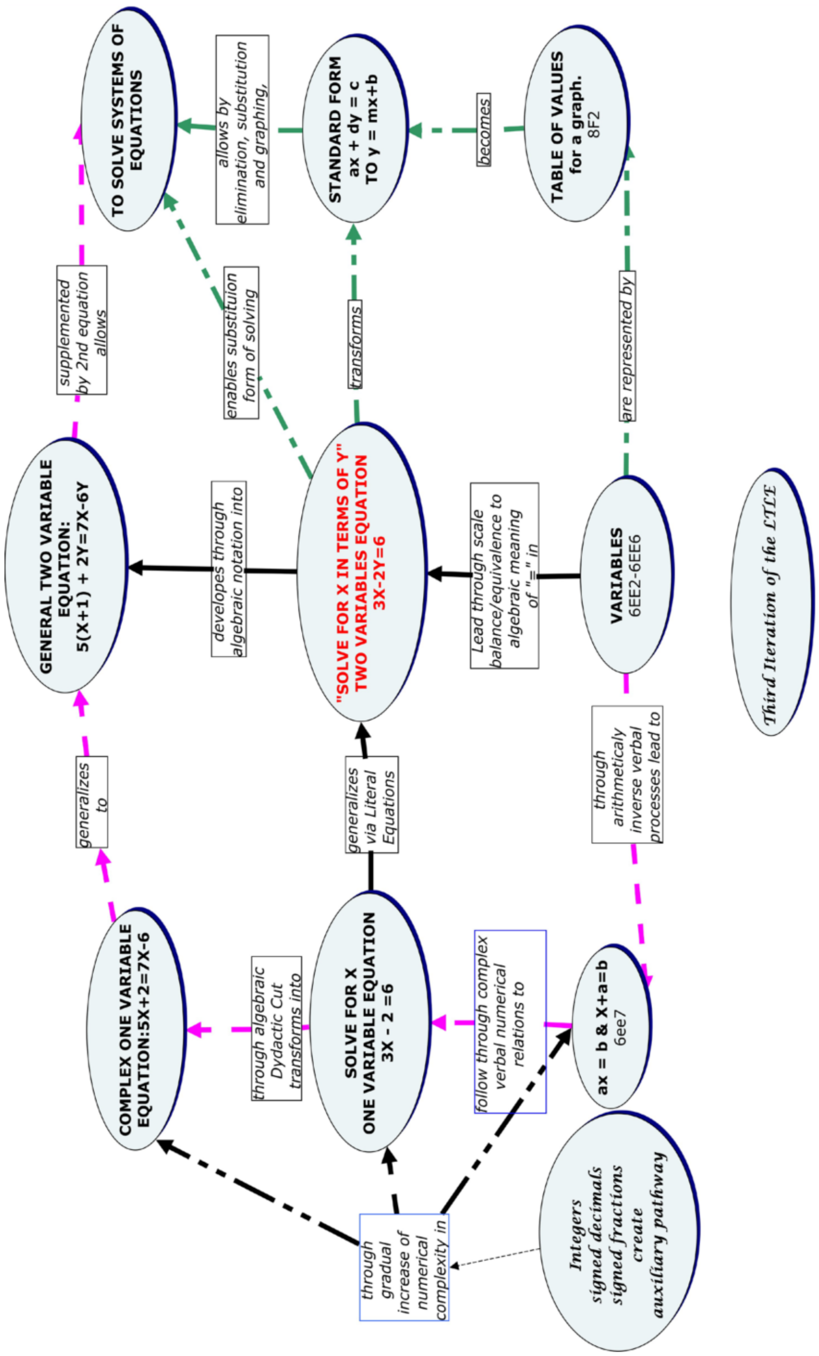


Figure 4. The revised concept map for the third iteration

Machine Cognition CmapTools at http://cmap.ihmc.us/. The oval shaped components represent the concepts, or mathematical objects, that are joined by propositions describing relationships between them. The concepts "solve for x" and "solve for x in terms of y" represent encapsulated or reified procedures.

Teaching-Research Diagnosis

The reasons for the erroneous solution include (a) absence of awareness of the functional relationship between the variables x and y, evidenced by transforming the problem to a simpler equation with one unknown leading to (b) misapplication of the variable as a specific unknown, (c) the absence of understanding the algebraic meaning of the equality symbol "=" evidenced by adding "$-3x$" to one side of the equation only, and, finally, as it was demonstrated by the teacher-researcher Vrunda Prabhu, (d) careless reading. The LTLE consists, therefore, of three separate but connected learning trajectories of (i) the variable as an unknown (broken arrow - - - - -▸ (pink) in Figure 2 above), (ii) the variable as a general number (black in Figure 2 above) and (iii) the variable in a functional relationship (broken arrow –-·–-▸ (green) in Figure 2 above) (Ursini & Trigueros, 2011).

The three component trajectories of the LTLE just discussed are shown in different colours on the first iteration concept map above (see Figure 2). The pink - - - - -▸ one leads along the process of generalization, from a formally similar equation in one variable to a corresponding equation in two variables. This trajectory is useful if the class has mastered solving simple one variable equations. Otherwise, the second trajectory, shown in –-·–-▸ (green), is available via the graphing component of the schema, that connects the challenge of the problem with its foundations within the concept of a variable, meaning of equality and the functional relationship between x and y. The cognitive fragility of the left upper rectangle in the concept map is well-known in the literature. Filloy and Trojano, for example, observe that the increase of algebraic content along the pink vertical arrow intersecting this rectangle is a serious problem for students because the solution of the more complex target equation departs from that of simpler equations such as $4x + 2 = 6$ (Filloy & Trojano, 1989; Ursini & Trigueros, 2009). The simpler linear equations enjoy more accessible arithmetic interpretations. Filloy and Trojano (1989) coined the term "Didactic Cut" to refer to the associated cognitive step. The two horizontal pathways indicate abstraction from and the generalization of a one-variable equation to a two-variable equation – an arduous process according to many investigations focused on problems that students have with generalization as they begin to study algebra in middle school. Most studies conclude that generalization is a difficult obstacle for the majority of these students (Bell & Malone, 1993; Arzarello et al., 1994; Bednarz & Janvier, 1994; Radford & Grenier, 1996; Bolea et al., 1998a, 1998b). The alternative graphing trajectory, shown in dark grey (green), develops the concept of "solving for y in terms of x" through transformation of a standard form of an equation into a known functional relationship $y = mx + b$.

The third component trajectory, shown in black, joins the concept of the variable as an unknown to the discovered difficulty along the theme of algebraic equality "=" through a series of "scale balance" type of problems. The assumed equilibrium of the scale in such problems is the metaphor for algebraic equality "=". The possibility of distinguishing three different learning progressions within the concept map demonstrates the versatility of such an integrated concept map/learning trajectory for classroom teachers and its usefulness in addressing diverse learners. According to (Ursini & Trigueros, 2009), the best, flexible development of the schema of the variable is to engage, in coordination, the three subschema: (1) variable as a specific unknown, (2) variable as a general number, (3) variable in a functional relationship. This implies the use of all component trajectories, because all three sub-schema are involved in the problem.

Instructional Sequences for the First Iteration

Here, we provide two small instructional sequences, which were used in the design of the first iteration.

We begin with the Teaching Sequence of Mathematical Activities that are meant to propel a student along the pink trajectory of generalization. The trajectory uses a "writing mathematics approach" to increase the meta-cognition and reflection upon the methods of solution. The aim of this sequence is to lead the student in the direction of development of generalization from a simple equation in one variable to the corresponding equation in two variables. The idea is to focus student's attention on the similarity of the solution procedure for one variable to the solution procedure for the task of "solving for y".

Problem 1

Solve for x. As you solve write every step you make in the solution. Look at the three descriptions, collect similar actions in the three examples and write them as one set of steps that apply to all three problems.

(1a) $2x + 7 = 15$
(1b) $-4x + 8 = -28$
(1c) $5x - 3 = 12$
My general set of steps is ____
____
____

Problem 2

Look at the following three examples that are similar but different from the previous set, and solve for *x* ***in terms of*** *y* ***by applying your general set of steps from Problem 1 to these three equations. Write your steps carefully and keep careful track of their order.***

(2a) $2x + y = 15$
(2b) $–4x + y = –28$
(2c) $5x – y = 12$

Problem 3

Now, solve for y in terms of x (note the change of the instruction) by applying your general set of steps to these three equations. Write your steps carefully and keep careful track of their order.
(3a) $2x + y = 15$
(3b) $–4x + y = –28$
(3c) $5x – y = 12$

Write the general description of steps for the instruction "Solve for y in terms of x" ______________________________
____

Problem 4

Solve for y in terms of x:

(4a) $4x + 2y = 12$
(4b) $6x – 3y = 15$
(4c) $–2x + 3y = 15$
(4d) $–2x + 3y = 15$

What is the critical computational difference between the last two and the first two problems?

____
____
____

Instructor's Notes: *The role of Problem 1 is to introduce the solution procedure for a simple and familiar case that consists of subtraction of a number from both sides followed by the division of the result. The role of the Problem 2 is to expose students to the variation in the procedure when an integer from the Problem 1 set is changed into the second variable, y. Problem 3 changes the task from "solving for x" to "solving for y"; students are expected to transfer the procedure from Problem 1 and Problem 2 accounting for the change. In the second iteration, problems (3b) and (3c) were changed from* $–4x + y = –28$ *to* $–4x + 2y = –28$, *and from* $5x – y = 12$ *to* $5x – 2y = 12$, *respectively. The aim of that change was to incorporate the division by the numerical coefficient of the variable y. Two examples of the type are needed to indicate the difference between answers using only integers and those using fractions. Fractions are one of the main obstacles students experience en route* *to algebraic thinking.*

Using the Scale Balance Manipulative: Reinforcing the Meaning of the Algebraic "=" and Extending the Method across the Didactic Cut (Filloy & Trojano, 1989)

The details of the teaching sequence meant to develop the idea of algebraic equivalence are presented here.

A) Solve the equation by removing weights from the scale in such a way so that the scale remains balanced (at an equilibrium). Describe the steps you are taking to keep the scale balanced.
B) Solve the equation algebraically by the Equivalence Principle.
C) What other equivalent equations can you make out of this one?

____

____

D) Solve for x:
$0.75x + 0.5 = 2$
E) Solve for x:

$$\frac{1}{3}x + \frac{2}{3} = \frac{5}{3}$$

The Didactic Cut
A) Solve the equation by switching the weights from one side to another in such a way so that the scale remains balanced (at an equilibrium). Describe the steps you are taking to keep the scale balanced.
B) Solve the equation algebraically by the Equivalence Principle.

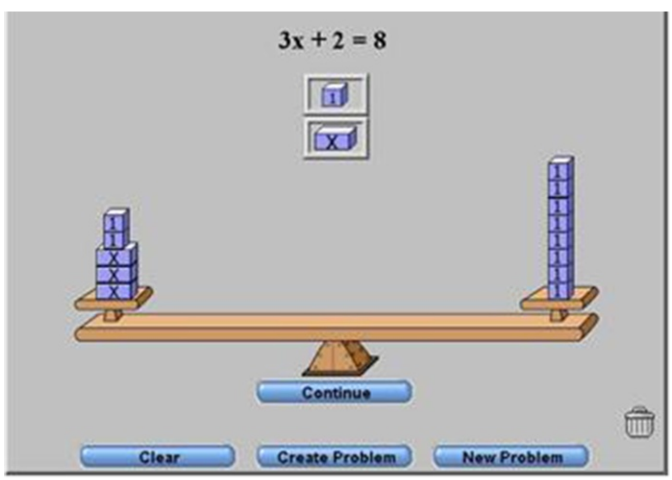


Figure 5. The scale balance manipulative I

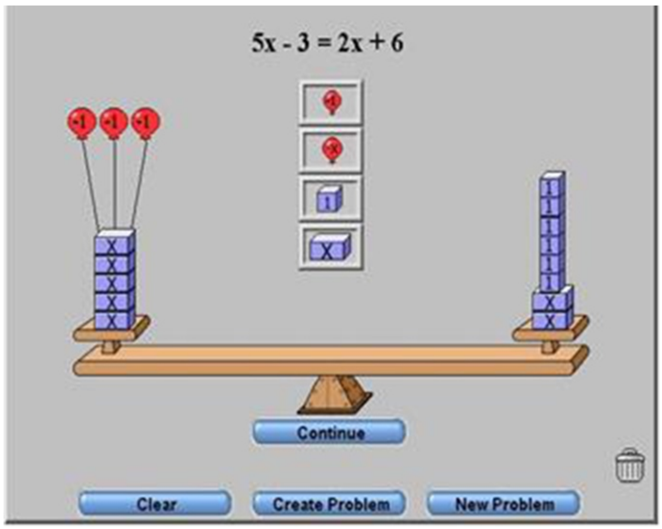


Figure 6. The Scale Balance Manipulative II

C) What other equivalent equations can you make out of this one?

____
____
____

D) Solve for x:
$5.2x - 3.6 = 2.2x + 6.4$

E) Solve for x:

$$\frac{3}{4}x + \frac{3}{8} = \frac{5}{4}x - \frac{5}{8}$$

Instructor's Notes: *Each of the Scale Balance problems starts from the concrete problem that can be solved by changing the weights while keeping the balance at equilibrium followed by the request to solve the same problem algebraically. Description of the steps is intended as the transition to algebraic operations followed by the reinforcement of the Equivalence Principle. Finally, the practice of technique is extended to decimal and fractional numerical coefficients, a well-known Achilles heel of remedial students of mathematics.*

The Second Iteration

The teaching experiment leading to the second iteration had been conducted during the fall 2012 semester at Hostos CC. Analysis of the results of the implementation of the first iteration along with observed student difficulties suggested the following needs:

1. A development of an auxiliary trajectory of algebraic notation;
2. An increase in the complexity of numerical coefficients from integers to signed decimals and signed fractions;
3. A much stronger emphasis on the discovery of numerical relations, and
4. Introduction of literal equations as the scaffold for the procedure "solve for *x* in terms of *y*".

The refinements (1), (3) and (4) are indicated in blue in the Second Iteration concept map (see Figure 3). The need to emphasize numerical relations as the background for algebraic problem-solving suggested a new point of view for the entire curriculum of the Arithmetic/Algebra course. Until this moment the curriculum was based solely on the generalization/particularization relationships between arithmetic and algebra. The new point of view has been provided by the discussion of the curriculum of V. Davydov (Jean Schmittau & Anne Morris, 2004), that takes mathematical relation as the foundation of the approach. The curriculum of the course then became a composition of two principles: generalization (algebraic expressions, polynomials, rational functions) and algebraic relation underlying theory of equations and functional relationships.

Example of Exercises, Which Focus Attention on the Numerical Relationships

The design follows the idea that a process and its inverse reinforce the reflective abstraction, and, hence, the development of the concept; in this case, the concept of the numerical relationships.

Problem1. Translate the verbal statement into an algebraic one:

(1a.) Twice a number is equal to 16 □ ______________________________

(1b.) 0.5 of a number is equal to 10 □ ______________________________

(1c.) Twice the number increased by 5 is equal to 11 □ ____________________

(1d.) The negative of twice the number decreased by 8 is equal to negative 4 □ ____

Problem 2. Express the relations between indicated pairs of numbers verbally:

- Two numbers are related additively if they are related by addition "+"
- Two numbers are related multiplicatively if they are related by multiplication "×"
- Two numbers are related additively and multiplicatively if both addition "+" and multiplication "×" are involved.

(2a) What is the additive relation between the numbers 4 and 15?

(2b) What is the multiplicative relation between the numbers 4 and 15?

(2c) What is the additive relation between the numbers –4 and 15?
(2d) What is the multiplicative relation between the numbers –4 and 15?
(2e) What is the additive relation between the numbers –4 and 15?

Instructor's Notes: *Note that the two problems above are "quasi" inverse processes of each other*: (i) *verbal statement* - - -▸ *algebraic relation, and* (ii) *numerical relation* - - -▸ *verbal relationship. In addition, the second iteration contained a component addressing "literal equations" as a scaffold for the "solve for y" task.*

The Third Iteration

The central improvement for the third iteration was to significantly increase the impact of the "algebraic relations" approach. This resulted in grounding the whole lower half of the trajectory in algebraic problem-solving (see Figure 4). This, in turn, leads up to the algebraic solution methods of systems of simple equations with two unknowns. Inclusion of Davydov's ideas is an example of "just-in-time" employment of new learning theory and related research results. After this basis has been established, the instruction along the upper half of the trajectory readily follows. The "scale balance" manipulative had been taken away for two reasons:

- It didn't make much of an impact on student understanding of the equivalence principle;
- The public software is not sufficiently developed to imitate the algebraic procedure of solving such equations.

Instead, a small algebraic teaching sequence had been designed employing, once again, the process and its inverse method. It is presented below.

Problem 1. Decide which of the pairs of equations below are equivalent and explain the reasons for your decisions?

(1a) E1: $x – 5 = 3$ E2: $x – 5 = 3$

___

___

(1b) E1: $x – 5 = 3$ E2: $x + 2 = 11$

___

___

(1c) E1: $x – 5 = 3$ E2: $x = 8$

___

___

(1d) E1: $3x = 9$ E2: $6x = 12$

____

____

(1e) E1: $3x = 9$ E2: $9x = 27$

____

____

Problem 2. Each of the two columns below contains a triplet of equations. Is the first equation in each column equivalent to that column's last equation? Explain the reasons for your answers

(A)	(B)
$2x – 6 = 12$	$2x – 6 = 12$
$2x = 18$	$4x –2 = 24$
$x = 9$	$4x = 36$

Conclusion: In order to solve the equation of the type $ax + b = c$ we need to _____

____

____

____

Instructor's Notes: *The problems above require use of the equivalence principle to decide whether the pairs of equations are equivalent. This way the role of the principle is clarified and then it can be applied in the context of a standard set of problems where the principle is used to obtain solutions.*

CONCLUSION

This chapter presents a work in progress. Our aim here has been to demonstrate the process of constructing a formal learning trajectory and to show that a teacher in the classroom can accomplish it. The assessment was primarily done through class observation, results and difficulties of students in their homework assignments and tests. As soon as we arrive at the learning trajectory we are intuitively satisfied with, we will establish more precise assessment measurements and extend their application to other sections of the course led by different instructors. The presence of the teaching-research community in the school described in the Unit 5 is central in the process of tuning and applying the trajectory beyond the initial classroom.

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HANNES STOPPEL

4.9. CALCULUS

Searching for a "Real World" Approach

SUMMARY

The theme of the chapter, modelling real world problems in freshman calculus were discussed in the Introduction to Unit 4. Here we would like to explore the research question of the chapter, what is the nature of the master teacher's success in reaching his successful design. We will trace the development of the teacher's thinking in the chapter from cycle to cycle to identify patterns of the changes.

Stage 1 (2005) Introduction of verbal problem reality together with given modelling analytic function.

The teacher is changing his approach from traditional lecture style at the urging of students who don't see connection between involved concepts, question the usefulness of the integral in favor of approximation techniques by rectangles or trapezoids, and ultimately ask, how one would find the best function if its analytical form is not given. That remark motivates the teacher to design next approach, which is an elementary modelling problem with elements of the discovery method.

Stage 2 (2008) The second iteration of the course focuses on the first attempts at modelling the function from the graph. That step naturally implies Discovery method.

Stage 3 (2011) However, it is only at the third iteration when the transformation of the Learning Environment along several dimensions of the design makes it into a cognitive unit of thought, which convinces students that Riemann integration makes sense.

Multidimensionality of instructor's efforts in the third stage is impressive: motivational gallery tour/poster preparation, diversifying the designed problems into modelling from the graph and modelling from the numerical data, new work organization in the classroom creatively and successfully addressing different students' levels of preparation, JiTT hints card with GC algorithms.

The persistence of student difficulties with understanding the role of Riemann construction forces the teacher (1) to abandon the quest for understanding lower and upper sums in favor of rectangle and trapezoids methods in the third iteration of the course. He realizes that the new prescribed curriculum in the state of Westfalia in Germany has limited the mathematics exposition in calculus solely to continuous functions, which do not allow for the full argument justifying

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 395–424.

necessity of lower and upper Riemann sums. He decides the concept of common limit between rectangle and trapezoid series to be the central goal of classrooms investigations. It's only when students' own reflection suggests a question, what would happen if one takes upper rectangles instead of lower ones, that the question of upper and lower sums is revisited with full collaboration and understanding by students.

The process of development through three stages with the third stage, usually significantly different from the first two in scope and generality of the effort, have been noted by (Czarnocha, 2013) and characterized as elementary learning trajectories, PG Triples. PG triples are simple manifestations of PG triad (Chapter 4.1).

Answering the posed research question concerning the nature of the successful design through practice, we could say, it should happen in three steps, which make up the simplest process of learning based on the triad of concept development by Piaget and Garcia (1987). As the second source of the teacher's success one needs to point to a very tight relationship between students' sense of interest and involvement in learning the concept. Each subsequent design in the series is motivated primarily by teacher's reaction to student comments to the previous iteration. Such a close relationship can substitute, in the hands of a master teacher, a more precise but less intuitive formal assessment tools. Finally, one needs to direct readers' attention to the multitude of teaching obstacles created by the design of curricula by the central authorities of a German state, such as Nordrhein-Westfallen, which was gracefully dealt with by the instructor.

INTRODUCTION

For a long time, the question of how to design an effective introduction to integral calculus has been raised in mathematics education. In the past ten years, the author has taught four calculus courses at the 12th high school grade level in Germany. Many different aspects of each course were documented and analysed. For every successive iteration of the course, revisions of concepts and content, based on the observations made during the preceding course, were implemented. In this article, all four courses will be described and analysed. Weaknesses and strengths of the different approaches will be discussed in detail; changes and adaptations will be examined and evaluated. The observations strongly suggest that, in order to make students fully understand the meaning of the integral, it pays to choose a non-mathematical detour when introducing the topic. Additionally, we will show how graphic calculators (GC) and computer algebra systems (CAS) can support this aim.

The introduction to integral calculus almost always takes place in higher education. In Germany, integral calculus is studied during the last two years of high school providing students with the necessary qualifications to attend university. The calculation of areas between two graphs or between a graph and the *x*-axis can be done before formally introducing integrals, with the help of GC or CAS (Stoppel, 2002; Stoppel, 2006; Stoppel, 2010; Ministerium für Schule und Weiterbildung des Landes Nordrhein-Westfalen[1], 2008). However, integral calculus demands more. The deeper related mathematical background, that is, the connection between differentiation and integration expressed in the fundamental theorem of calculus, is expected to be understood; yet the questions of how to introduce the integral and how to approach the fundamental theorem at the high school level remain unrequited.

Usually, a sequence of integration lessons begins with calculating definite integrals, and then addresses the concept of the anti– derivative (Schmidt, Körner, & Lergenmüller, 2011, Section 4.1; Bigalke & Köhler, 2011, pp. 192–197; Brandt & Reinelt, 2007, pp. 154–156). There exist many different approaches to such a sequence. One popular way to introduce integration is by calculating series of upper and lower Riemann sums. Since series have been eliminated from the standard high school curriculum in Germany several years ago, we need to look elsewhere. The curricula of North Rhine-Westphalia still include upper and lower sums, but textbooks discuss them in separate disjoint places (Schmidt, Körner, & Lergenmüller, 2011, pp. 159, 160; Bigalke & Köhler, 2011, pp. 192–197; Brandt & Reinelt, 2007, pp. 157–161). Nonetheless, these topics are still viewed as proper qualifications intended to generally prepare students for university, independent of which subjects they choose to study later. As high school curricula diverge from university expectations, and these inconsistencies increase as years go by, universities will have to adapt their approaches to reflect students' abilities. This especially applies to the introduction of basic concepts such as integral calculus.

This study is based on the development of lesson sequences of introduction to the definite integral in four different twelfth grade high school classes from 2003 to 2011. Digital instruments are used in high school mathematics classes in different ways (Kultusminister Konferenz, 2009, p. 5). From 2003 to 2011, the curricula that incorporate the usage of digital media like GCs and CAS have significantly changed (current issue Ministerium für Schule und Weiterbildung des Landes Nordrhein-Westfalen, 2013, pp. 22–26). To be able to fully comprehend the development of epistemic actions (Hershkowitz, Schwarz, & Dreyfus, 2001) we tried to keep much of the structure of the topic of introduction to integral calculus as intact as possible.

Graphic Calculators (GC) and Computational Algebra Systems (CAS) were used because of their potential to bypass tedious calculations, when appropriate, and emphasize conceptual understanding instead. Furthermore, the employment of tools like GCs or CAS is prescribed in North Rhine-Westphalia in grades ten to twelve (Ministerium für Schule und Weiterbildung des Landes Nordrhein-Westfalen, 2013) and is supported by Kultusminister Konferenz (2012).

Another development lays emphasis on the role of modelling as a general aim of mathematics education, and focuses on teaching methodology. This idea developed out of the cooperative work, as will be most evident in the last iteration of the course.

The main lesson aspects analysed here are:

- Form of work
- Paths towards the definite integral
- Definitions of the integral, exactness from mathematical point of view
- Usage of media
- Duration

During and at the end of the sequence of lessons the above aspects were analysed via feedback from students during and after the introduction as well as instructor observations noted while planning the next version of the course.

The educational goals of the lesson sequence consist of a workable understanding of the following ideas:

- Definition and basics of the definite integral
- Theory and use of limits
- Approximations to areas between graphs of functions and the x-axis by geometrical figures (rectangles and trapezoids); and, that these approximations get better as the number of rectangles/trapezoids increases
- Calculations of areas reduced to areas above or below the x-axis
- The relationship and differences between arithmetic and graphical meaning of the integral
- The rigorous definition of the definite integral from the mathematical point of view

The approaches to the introduction of the definite integral were different in each of the four iterations. The first of these was used in 2003, and was motivated by a mathematical question about the calculation of the area between the graph of a function and the x-axis, and began with a study of linear functions followed by polynomials of degree two. Next, upper and lower Riemann sums were calculated for continuous, piecewise monotonic functions, leading to the Riemann integral.

In 2005, the introduction was driven by an application and modelling exercise. The modelling problems were presented along with the functions being used and challenged the students to use the given functions properly in their approximations of the area. The first approximation of the area between the graph of a function and the x-axis was *discovered* by students through exchange of ideas and lead to the sum:

$$\sum_{i=1}^{n} f\left(a+i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)$$

The definition of the integral was reduced to differentiable functions.

A study about the introduction to integral calculus is discussed in Thompson, Byerley, and Hatfield (2013). The study began with purely mathematical exercises stressing calculations as well as modelling exercises emphasizing applications of the same type used by us in the 2003 and 2005 sequences. The study reported that, sometimes, the students used computers for simulations. The authors recommended that curricula should be changed to allow for an appropriate use of CAS.

The introduction in 2008 started with a modelling exercise based on a picture without given measurement. The students had to determine the function from descriptions, and develop the procedure for the calculation of the area themselves. Subsequently, the upper and the lower sums were introduced, and the rest of the introduction was analogous to that of 2005.

After a detailed evaluation of the three iterations before, the integral was introduced in a quite different way in 2011. The students started with different topics in working groups. Every group had a look at an application of integral calculus where one will start with approximations and optimize results with integral calculus. The topics required different competencies, so that the students could be challenged in relationship to their capabilities associated with mathematics. After a variety of student-motivated approaches, the students found approximations of areas using rectangles and trapezoids, and, eventually, upper and lower sums. The students *themselves*, guided by the instructor, constructed the steps needed to evaluate the integral. Although the definition of the integral was the same as in the 2008 sequence, students were exposed to examples of continuous but not differentiable functions.

DETAILED LESSON SEQUENCES: FIRST ITERATION – 2003

Description

The introduction to integral calculus in 2003 took three periods of 45 minutes using the *classical* style, that is, by calculating upper and lower Riemann sums leading to the Riemann integral. As is typical, equidistant partitions of the intervals of integration were used and simple case integrals were evaluated (Stoppel, 2002, Chapter 5).

In the beginning, the students were lead to the problem of calculating the area between the graph of a function and the x-axis. Then, an intuitive mathematical question about a graph of a function came up. After a few hints, the students found an approximation using rectangles below, or partly above, the graph of the function as can be seen in Figure 1.

During the next few exercises students were finding difficulties in calculating sums of the areas.

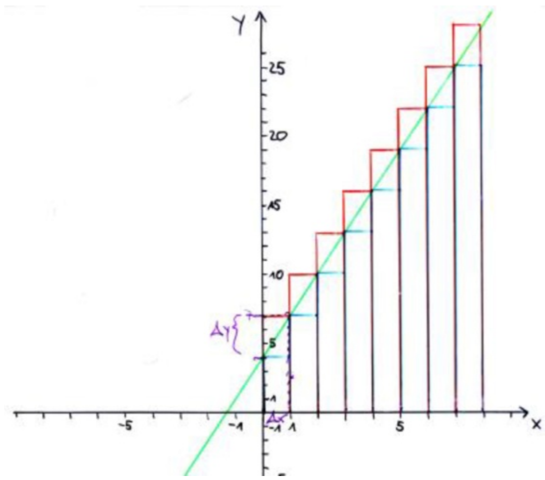

Figure 1. Upper and lower sums for various subdivisions of a linear function over the interval [0, 2]

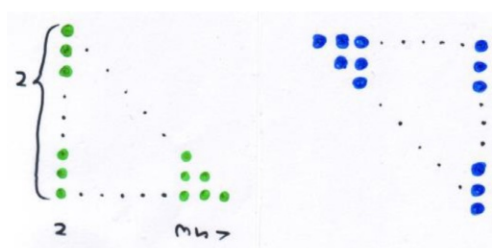

Figure 2. Transparency of an idea of the proof of $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

After a few geometrical considerations, such as those in Figure 2, and a proof of the equation $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by induction, they found an equation for calculating the sum of areas of n rectangles above or below the graph of a function. This led to the following definition:

Definition: *Let f:* $[a, b]$ ¦ **R** *be a continuous, piecewise monotone function. A partition of the interval* $[a, b]$ *is given by*

$$Z_n := \left\{ a, a + \frac{b-a}{n}, \ldots, a + (n-1)\left(\frac{b-a}{n}\right) \right\}$$

Furthermore, let M_i be the maximum and m_i be the minimum of f(x) on the i^{th} interval of the partition. Then, we call

$$U_n := \sum_{i=1}^{n} M_i \left(\frac{b-a}{n} \right)$$

the upper Riemann sum of f(x) relative to the partition Z_n and

$$L_n := \sum_{i=1}^{n} m_i \left(\frac{b-a}{n} \right)$$

the lower Riemann sum of f(x) relative to the partition Z_n.

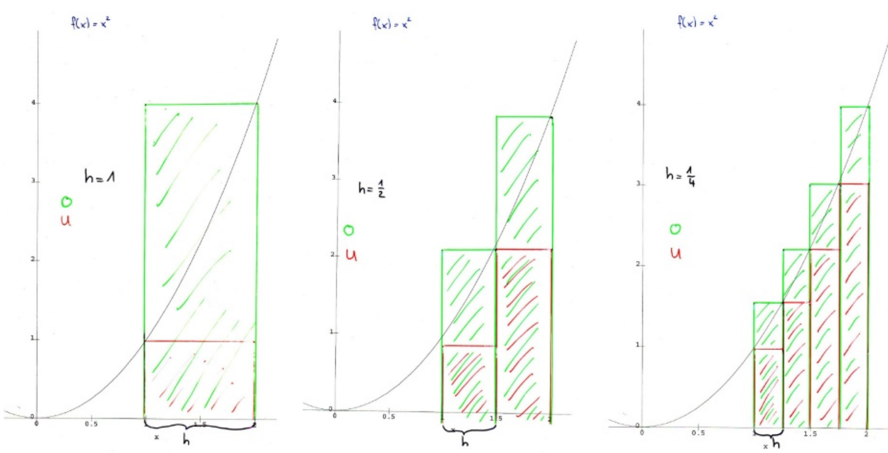

Figure 3. Upper and lower sums for various subdivisions of polynomial with degree 2 for the interval [1, 2]

The following exercise was used as an example.

Exercise 1. *The function f is given by* $f(x) = 3x + 4$. *Calculate the upper sums U_n, the lower sums L_n, and the limits $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} U_n$ for the intervals*
(***a***) [0, 2], and (***b***) [0, *b*]. Note: we present here the calculations for the interval [0,2].
***Solution*:**

(***a***) Take $I = [0, 2]$, then

$$Z_n = \left\{ 0, \frac{2}{n}, 2 \bullet \frac{2}{n}, \ldots, (n-1) \bullet \frac{2}{n}, 2 \right\}$$

It follows, that

$$f\left(i\cdot\frac{2}{n}\right)=3\cdot i\cdot\frac{2}{n}+4=\frac{6}{n}\cdot i+4$$

And then

$$U_n=\sum_{i=1}^{n}f\left(i\cdot\frac{2}{n}\right)\cdot\frac{2}{n}=\sum_{i=1}^{n}\left(\frac{6}{n}\cdot i+4\right)\cdot\frac{2}{n}=$$

$$=\sum_{i=1}^{n}\frac{6}{n}\cdot i\cdot\frac{2}{n}+\sum_{i=1}^{n}4\cdot\frac{2}{n}=\frac{12}{n^2}\cdot\sum_{i=1}^{n}i+\frac{8}{n}\sum_{i=1}^{n}1=$$

$$=\frac{12}{n^2}\cdot\frac{n(n+1)}{2}+\frac{8}{n}\cdot n=\frac{6(n+1)}{n}+8$$

$$L_n=\sum_{i=0}^{n-1}f\left(i\cdot\frac{2}{n}\right)\cdot\frac{2}{n}=\sum_{i=0}^{n-1}n\left(\frac{6}{n}\cdot i+4\right)\cdot\frac{2}{n}=$$

$$=\sum_{i=0}^{n-1}\frac{6}{n}\cdot i\cdot\frac{2}{n}+\sum_{i=0}^{n-1}4\cdot\frac{2}{n}=\frac{12}{n^2}\cdot\sum_{i=0}^{n-1}\frac{12}{n^2}+\frac{8}{n}\sum_{i=0}^{n-1}1=$$

$$=\frac{12}{n^2}\cdot\frac{(n-1)n}{2}+\frac{8}{n}\cdot n=\frac{6(n-1)}{n}+8$$

Then, the limits are $\lim_{n\to\infty}U_n=14=\lim_{n\to\infty}L_n$.
The students then used the *Maple* CAS for calculations of upper and lower sums, applying an environment where one needs to enter the function and the interval parameters of limits and partition. They worked in small groups and had to calculate upper and lower sums for several more complicated functions while having the opportunity to get hints from the teacher. The exercises, along with *Maple*, allowed for and encouraged students to evaluate definite integrals without any formulas. The students arrived at the following definition:

Definition. *A continuous, piecewise mo notone function f*: [*a*, *b* → *R*] *is called Riemann-integrable over* [*a*, *b*], *if*

$$\lim_{n\to\infty}U_n=\lim_{n\to\infty}L_n$$

for any upper sum and any lower sum. If it exists, the limit is called the definite integral of f over $[a, b]$ *and is denoted by*

$$\int_a^b f(x)\,dx$$

Students had to calculate more examples, some with rational functions, using *Maple*, before we reached the Fundamental Theorem of Calculus.

Observations and Remarks from Students

In 2003, the series of lessons was a purely mathematical introduction to the definite integral; the relation to applications was only briefly discussed in the very beginning. Only a few students understood the correlations between the CAS file and the mathematical background of the calculations.

It was difficult to convince the students of the importance of the formulas used to calculate upper sums and lower sums, because it was quite easy to evaluate using the CAS. They needed to be adverted.

The derivation of the definite integral was too theoretical and the students missed connections with the applications. Therefore, the sense of the integral was vague to the students, as was evident from students' questions. Even more examples could not change the default attitudes of the students.

Through feedback, the students informed the instructor that the distinction between the upper sum and the lower sum appeared senseless, because $\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$ every time. Some of them did not realize the importance of U_n and L_n, and were unable to grasp why one needs both of them. In high school in Germany, we stress that a function *f* is integrable if upper sums and lower sums for equidistant partitions of the intervals converge to the same value. Moreover, the students' interest in Riemann sums was lowered when integration of polynomials was introduced. As was mandated in the curriculum, one does not have to use functions that are not integrable. In addition, because of the restriction to equidistant partitions of the interval for integration only a small class of functions is given for practice.

During the proof of $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ the students were diverted from the main goal of the lesson calculations. During the calculations of upper sums and lower sums with the CAS some students still needed hints from the instructor.

DETAILED LESSON SEQUENCES: SECOND ITERATION – 2005

Description

The introduction to integral was revised for this course, and the focus was more grounded in reality utilizing modelling. To this end, the course started the introduction to integral calculus with a verbal "real-world" exercise, for which the need for a mathematical procedure was clearly visible. During these teaching units the students only used a simple calculator.

The introduction took three 45-minute lessons. The first exercise was presented by a student teacher.

Exercise 1. *The owner of a piece of land needs to sell a piece of it. To calculate how much money he should be asking for, he needed to measure the area first. However, this proved to be difficult because, even though it has three straight perpendicular edges, the fourth edge is along the coast of a stream, the shape of which can be very well described by the function* $f(x) = x^3 + 2x^2 + 3$. *The price of a square meter was about 220 €. How much should the owner expect to earn with his area? (Refer to Figure 4)*

The task includes an application. The students could only approximately determinate the area, because they did not know the integral yet. Thus, everybody exchanged some ideas for the solution, and they formulated three different approaches:

(I) Both local extremes of the graph should be connected by a line segment. Then, the area of the resulting trapezoid should be calculated, as in Figure 5. The coordinates of the extremes can be only approximately determined.

(II) The best possible horizontal line should be drawn through the inflection point. This new horizontal line, the given horizontal edge (along the x-axis), along with the two vertical lines that pass through the local extrema now form a rectangle. The area of this rectangle can be approximated and will be approximately equal to the area of the irregularly shaped region. This solution is shown in Figure 6.

(III) The third approximation was almost a combination of (I) and (II). The interval along the horizontal edge opposite the stream (along the x-axis), between the x-coordinates of the extrema, is to be partitioned into 6 parts of the same length (which appears to be 0.25 in this case), and for every subinterval, the areas of the small rectangles with height $f(0.25k)$, for $k = 1, 2,\ldots, 6$, should be calculated. Afterwards, the sum of these areas should be added. See Figure 7.

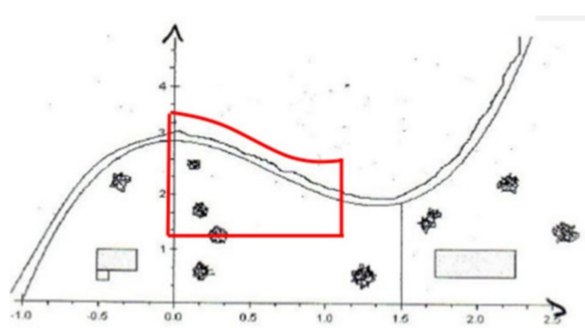

Figure 4. A sketch of the layout of the owner's land, from Exercise 1, that is to be sold

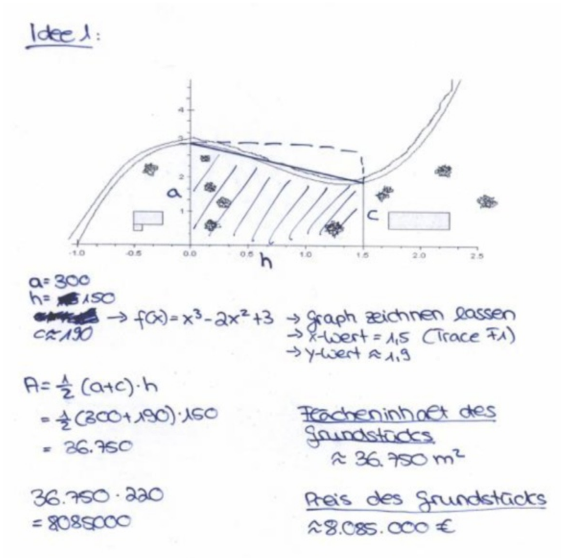

Figure 5. The trapezoid approach of students' proposal I

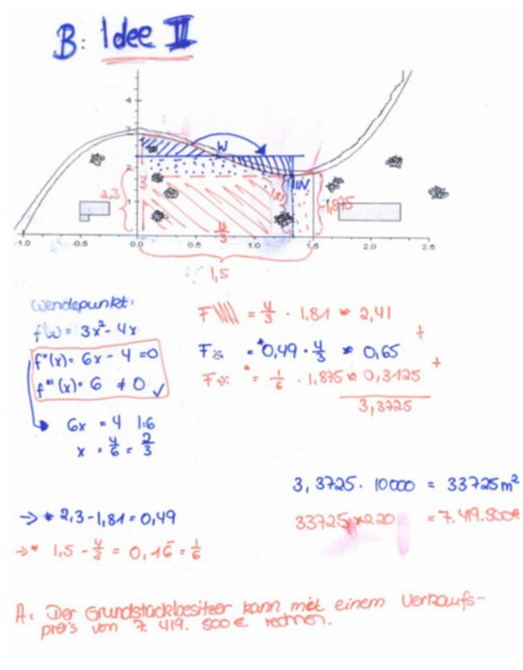

Figure 6. The inflection point rectangle approach of students' proposal II

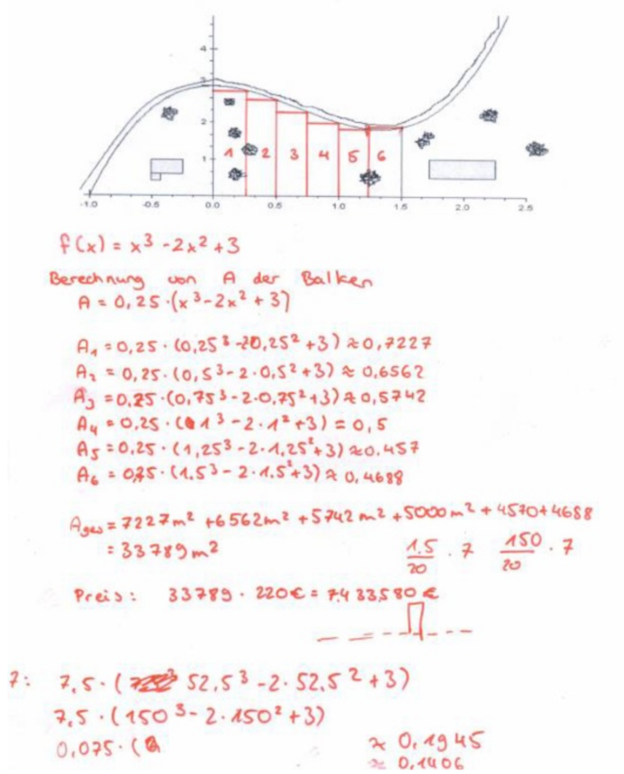

Figure 7. The sum of sub-rectangles approach of students' proposal III

Each idea was explored by the group that came up with it. Then, each of the three groups presented its results. After some discussion, the students recognized that version (III) would be the best one since the interval [0, 1.5] can be easily divided into a larger number of parts, and a better approximation for the area between the graph of $f(x)$ and the x-axis can be obtained. Following this logic, they continued working with version (III).

Now, using method III, through pattern recognition, students were able to develop a formula for approximating the area between the graph of f and the x-axis that was easily generalizable for an arbitrary large number of subintervals. Next, the students attained better approximations of the area by making finer partitions with the aid of a calculator. As the subintervals used became smaller and smaller, students felt ready to formulate a more general algorithm to approximate the area for any interval. This line of reasoning lead to the definition of the definite integral. The convergence of the sum of areas of the sub-rectangles as $\Delta x \to 0$ was assumed. This definition of the integral relied on the convergence of the series below, since it was now defined as the limit, as $\Delta x \rightarrow 0$, of the sum

$$\sum_{i=1}^{n} f\left(a + i \circ \left(\frac{b-a}{n}\right)\right) \circ \left(\frac{b-a}{n}\right)$$

Later in the course, more complicated but differentiable functions were used, and the formal definition of the definite integral was established as it appears below:

Definition. *Let* $f\colon [a,\ b] \rightarrow R$ *be a differentiable function, let* $h = \dfrac{b-a}{n}$ *and let* $A_n = \sum_{i=1}^{n} f\left(a + i\,h\right) h$ *be the sum of the areas of the n rectangles based along the x-axis over the interval* $[a, b]$ *with widths h and heights* $f(a + i\ h)$. *Then*,

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(a + i\,h\right) h$$

is called the integral of the function f over the interval $[a, b]$ *and is denoted by*

$$\int_a^b f\left(x\right) dx$$

Finally, several different numerical integrals were evaluated using the definition, whereby one arrived at the Fundamental Theorem of Calculus.

Observations and Remarks from Students

In 2005, the introduction to integral calculus started with a mathematical modelling example. Inside his feedback one of the students pointed out that no one ever explained how one would be able to find he function in question. Furthermore, the figure in the example did not correspond to reality, so that from the students' point of view, the connection between the integral and "natural problem" was not comprehensible. One of the arguments was, that the area might be determined almost exactly with sensible rectangles or trapezoids. A function is not able to describe a bank exactly and, certainly, not much better than rectangles or trapezoids. So why does anybody need the integral?

As the students remarked during a discussion of the introduction to integral calculus and their comprehension, the introduction did not really correspond to any real situation that might be used in real life. Because of that their motivation for finding the solutions to given exercises and to understand the mathematical background was quite reduced.

DETAILED LESSON SEQUENCES: THIRD ITERATION – 2008

Description

The introduction took two teaching units of 45 minutes. In contrast to 2005, the course used a GC. Unlike the 2003 iteration of the course, we took extra care to avoid problems concerning the usage of CAS or GC. The introduction began with a modelling exercise. In contrast to 2005, the lessons started with a picture including an object with labels and some measurements (see Figure 9), but without the usage of any coordinate system. Hence, the first step was to re-envision this exercise within a useful coordinate system. The students were given the following instructions:

Exercise 1. *The local government of the city of Weyhe needs help from a competent person. The owner of a property on the bank of Donuper pond would like to sell the area for 28 Euros per square meter. He and the city of Borgen have to agree on the buying price. Both of them want to measure the area using division into rectangles. Try to determine the area of the shaded region using your GC. Keep in mind that both parties must attain their wished and a compromise is an option.*

Each student had to suggest a coordinate system and make an educated assumption about the function that describes the shape of the edge of the bank (polynomial of degree two). After several trials and hints from the teacher, many students were able to identify a graph of a function with the shape of the bank edge. They assumed that the bank might be described by a polynomial of degree two. This lead to a system

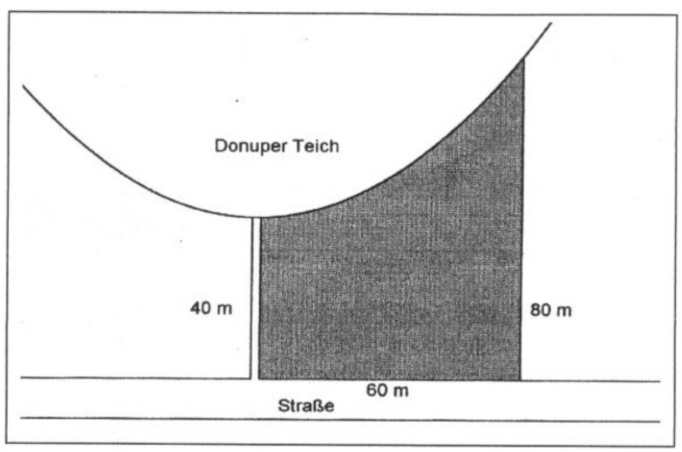

Figure 8. The figure for Exercise 1 above; students are to evaluate the area of the shaded region

of linear equations (SLE), and the students tried to solve the SLE themselves by hand or using their GC. One group of students, using rectangles, approximated the area above the graph of the function. Another group of the students used rectangles lying completely under the graph of the function. A third group used rectangles such that the graph of the function intersected each rectangle in the middle of the upper horizontal side. The students declared the third version as too difficult and neglected it. Afterwards, referring to an earlier introduced similar formula, they derived a general formula for calculation of upper sums and lower sums.

The students continued to solve more sum calculation exercises using their GCs. The use of a GC has not been crucial up until this point. Its only usefulness lied in its quick computation power. Most students did not have the required knowledge of a GC to suggest a more efficient use such as the creation of an algorithm. I shared my algorithm with the students.

Based on several calculation of different lower and upper sums for a fixed interval [*a*, *b*] under finer and finer partitions the students reached the hypothesis that

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$$

Therefore, the students decided to work with upper sums only and, after several more examples, constructed the formula below:

$$A_n = \sum_{i=1}^{n} f\left(a + i\,h\right) \cdot h \text{, with } h = \frac{b-a}{n}$$

for the calculation of areas. This, in turn, led to the definition of the integral.

Observations and Remarks from Students

Some students had difficulties operating the GCs, although they had solved similar exercises before, while others solved the exercise quickly and presented their results. They clearly saw that upper sums and lower sums converge to the same limit. On the other hand, some students did not see the reason for doing both types of sums assuming that their equality is a given (One has to consider both upper sums and lower sums, or to restrict to one of them, if "simple" functions are used or only a graph is given).

In their responses about the quality of the lesson, students stated that the introduction was done too quickly and felt rushed and insufficient to understand the mathematical foundations and appreciate the theoretical background. In their opinion, the leap from example to understanding had been too short and inadequate; and more examples would have been helpful. They also conveyed that the use of the GC occurred too early and its usefulness was not clear. The GC has not been used

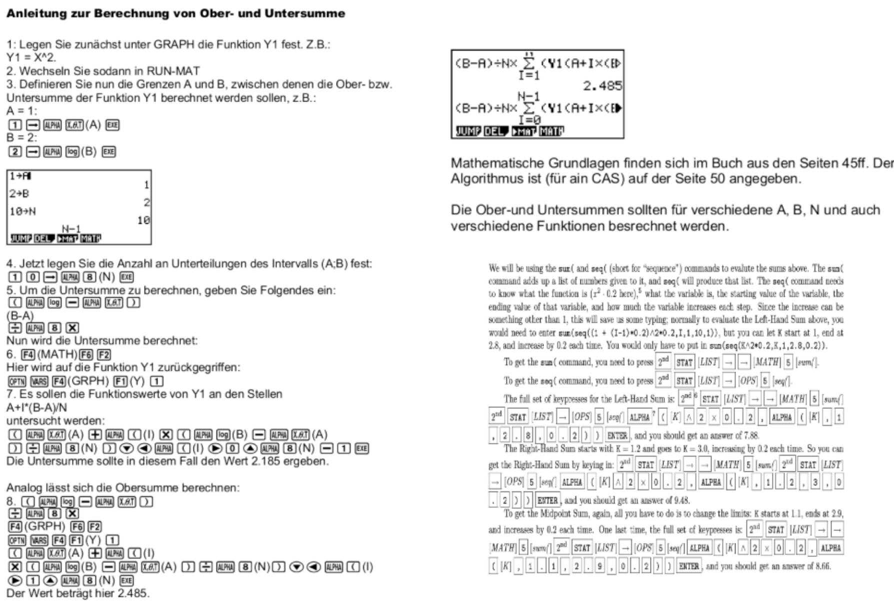

Figure 9. GC Instructions for calculating upper sums and lower sums
(with a similar guide for TI-83 in English)

prior to this lesson, and even now, they felt that it was unnecessary and, at times, created more difficulties rather than assistance. The structure of the manual input of the algorithm into the GC was not clear to the students. They felt side-tracked by the input of the algorithm into their GC from the exercise sheet and did not recognize the connection to the upper/lower sum concept. They lost the connection to the starting point of the exercise.

In 2003, the problem with using GCs or CAS was slightly different. Students, most of whom did not have the skills to utilize a CAS, felt distracted by that technical part of the lesson and did not see the relationship between the CAS input and the concept of areas. The ability to simply provide students with a ready-made algorithm solved the problem of the lack of programing skills. However, a large chunk of time was still lost. For the next iteration, GCs or CAS will not be used at all in the introduction to the definite integral lesson. If an instructor decides to use one or the other, a more detailed lesson on the basics of the technology, as it relates to calculus, is necessary.

One of the goals set after the 2005 iteration was to improve students' sense of the connection of the starting example and *reality*. Therefore, only the measures of the bank were given without a functional description. At the beginning, the students were divided into small working groups. Some of the students had a hard time, or no ideas at all, in determining the necessary function for the description of the bank.

Others insisted that is was necessary to determine this function. Many students had struggles developing an approach to finding this function. During the review, the students remarked that most of the difficulties were due to a lack of confidence and a deficiency of appropriate mathematical background. Some of them had problems recognizing the shape of the bank as a polynomial function of degree two. They suggested that the function used in such a first example should be more elementary and, thus, easier for them to recognize; for example, a linear function would convey the idea of the definite integral equally well.

Like in 2005, some of the students argued that, for this *real-world problem*, the calculation of the area only needed to be approximate, and that it was possible to accomplish this easier with figures like rectangles. Other students had an opposing view, and argued that the calculation was much easier, and, certainly, faster, with a function rather than with a large number of figures whose areas had to be computed separately and then added. These students appreciated the intended power of the integral for a problem such as this, and favoured the integral as the most efficient tool. It is worth mentioning that, while a similar argument came up amongst the students in 2005, almost all rejected the benefit of using a function to model the shape rather than a collection of differently sized squares and rectangles, claiming that the function approach is not more efficient and/or accurate.

DETAILED LESSON SEQUENCES: FOURTH ITERATION – 2011

In this instance, the introduction sequence consumed the largest amount of time relative to all of its previous iterations, with five 45-minute lessons. Unlike in the previous years, instead of aiming to arrive at the concept of upper and lower sums as the ultimate central tool, this time, rectangles and trapezoids were emphasized and heavily utilized for the approximation of the area between the graph of a function and the x-axis. This approach was more visual, familiar and intuitive for the students than the Riemann sums. This approach did not require the students to distinguish between upper and lower sums; this is an unnecessary task for the classes of functions addressed in the curricula of German secondary schools. Furthermore, the introduction took place in different student working groups with exercises of different levels, with different duty cycles within the working groups, phases of different styles and media usage for internal differentiation.

Because of the more complex structure of the lessons and an incorporation of many new ideas, compared to the earlier introductions to the integral described here, the following section is quite a bit longer and is subdivided into more subsections to clearly describe the different moving parts.

Development of Foundations for the Final Iteration

The introduction to the definite integral began with group work with a following *gallery tour* created by the Ministerium für Schule und Weiterbildung des Landes

Nordrhein-Westfalia (2007, pp. 87–121). These exercise motivate the conceptual development of the definite integral from a very different point of view. To illustrate the flow of the ideas involved, two examples are presented below along with the corresponding student solutions.

Exercise 1 – The Spirometer. *To visualize the wave representing the rate of airflow during the breathing process of a human subject physiologists use a spirometer. The subject inhales and exhales into a mouthpiece connected to a device that measures the difference in air pressure which, in turn, is used to measure the rate of airflow. An example of such a curve is shown in the following figure:*

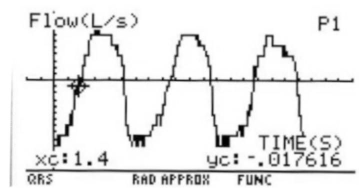

Figure 10. An example of a graphic output of a spirometer.

The following figure shows an idealized example of such a spirometer output.

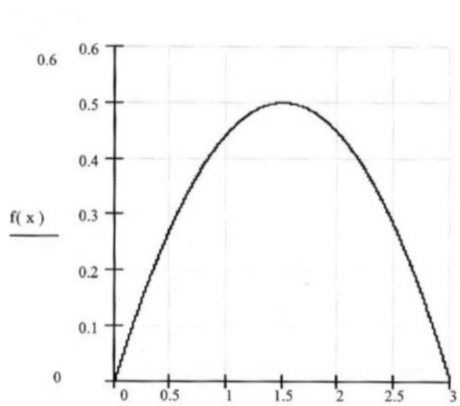

Figure 11. An idealized graph of the rate of airflow during a single inhalation phase

The above curve represents a single inhalation phase of a person at rest. This airflow rate curve can be approximated by a function $f(x)$ measured in liters per second.

- *Use the given graph of the function $f(x)$ to describe the inhalation action in as much detail as possible.*

- *Create a sketch of the graph of the function $L(x)$ that describes the amount of air inhaled during a single phase, with $L(x)$ measured in liters. Explain your procedure (Hint: When does the quantity of the inhaled air increases the most?).*
- *Make your best possible approximation of the amount of air inhaled by the human subject during the first three seconds.*
- *Is it possible to use the immediately preceding result to sketch a more accurate graph of $L(x)$?*
- *Use the graph to find the explicit form of the rate function $f(x)$ and calculate its derivative. Interpret the meaning of the derivative in the context of this exercise.*
- *On three separate sets of axes, lined up vertically, **(i)** sketch $f(x)$ in the center, **(ii)** sketch $f'(x)$ below, and, in the uppermost space, **(iii)** sketch a new function $F(x)$ such that $F'(x) = f(x)$ and $F(0) = 0$*

Exercise 2 – Amalgam. *Ms. Schulze found out that dental amalgam fillings contain elemental mercury, and release low levels of mercury vapour that can be inhaled. High levels of mercury vapour exposure are associated with adverse effects in the brain and the kidneys. She wants to know whether her actual filling should be removed, so she is seeking more information about this topic. Some dentists believe that the levels of mercury present in the body as a result of amalgam fillings are constant and are independent of the size and the number of such fillings, while others hypothesize that these mercury levels are dependent on the numbers of amalgam fillings that a person has. In a scientific study, levels of mercury (***Hg***) concentration in the urine of patient* $\boldsymbol{P_1}$ *were measured before and during the first six months immediately following the removal of all of the patient's mercury fillings, and recorded in a table below:*

Table 1. Mercury concentration for patient $\boldsymbol{P_1}$

Number of days	0	2	30	60	90	120	150	180
Hg in µg/day	3.5	3.2	2.4	1.8	1.2	0.8	0.5	0.4

These results were then compared to the same type of measurements obtained from a different patient, $\boldsymbol{P_2}$, *whose fillings have not been removed. The level of mercury concentration in the second patient was almost constant at 3.5* µ*g/day.*

- *Illustrate the data graphically.*
- *Sketch the best fit line for the data from* $\boldsymbol{P_1}$.
- *How much total mercury was excreted by each of the two patients at the end of the six-month period?*
- *Is there any way to obtain a more accurate result for the total amount of* **Hg** *excreted? Use different processes and display formats for your approximations.*

Within the presentations in the gallery tour the students came up with two different ideas for the calculation of the area between the graph of a function and the *x*-axis,—(i) using rectangles and (ii) using trapezoids. The gallery tour inspired students to formulate ideas that they further developed on their own afterwards. The teacher did not provide students with quick answers or formulas, nor did they expect him to. The instructor was able to utilize and build upon students' original thoughts, naturally leading to the formulation of the concept of the definite integral.

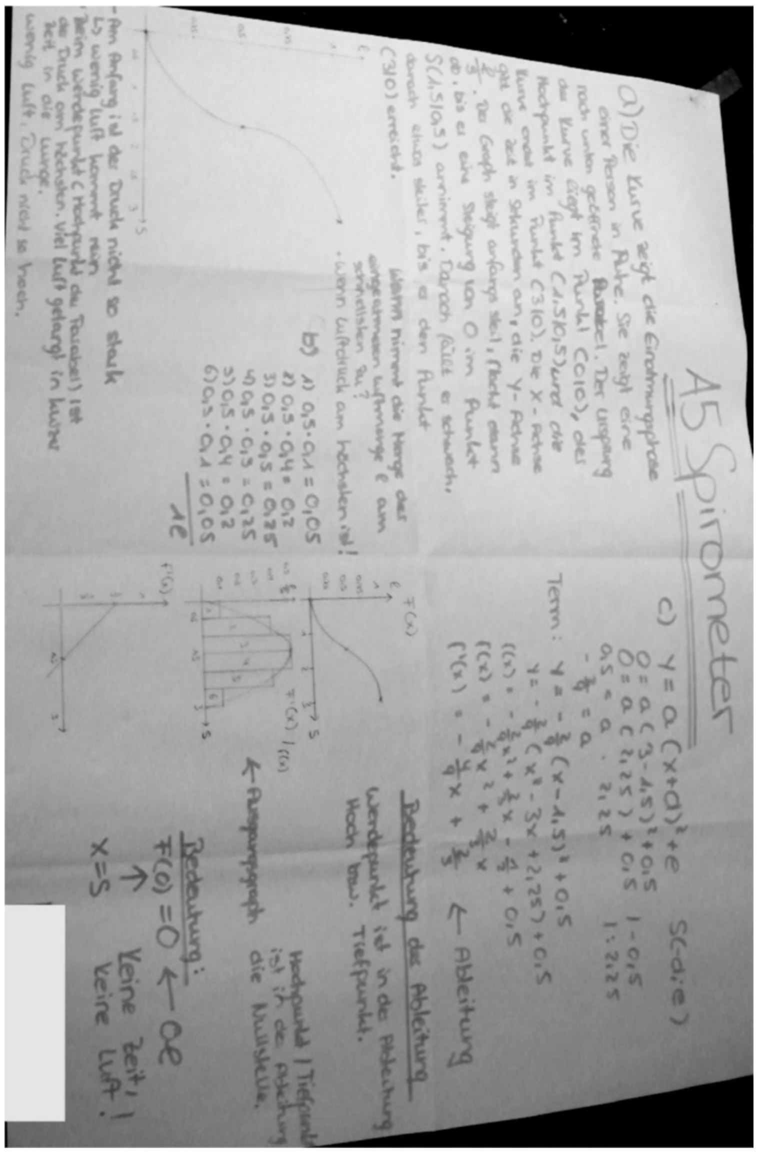

Figure 12. The solution poster for Exercise 1

The solution posters created by the working groups are shown on the following page (see Figures 12 – 15).

After the gallery tour and a discussion of their results, the students had to calculate areas between a graph of a function and the *x*-axis, presenting their methodology and results on transparencies. They were given the graph of the function

$$f(x) = \frac{1}{2}\left(x - \frac{1}{2}\right)^3 + \frac{1}{2},$$

defined over the interval [0, 8] (see Figures 16 and 17). The students were not given the explicit definition of this particular function and were working only with its graph. They had to divide the interval into two, four and eight parts with the same base, then draw the appropriate rectangles or trapezoids, and, finally, reading the functions values from the given graphs, had to approximate the areas in question. Each of the two approaches was used by one half of the class. Students worked in groups of two for approximately twenty minutes. Two resulting transparencies are shown in Figures 16 and 17.

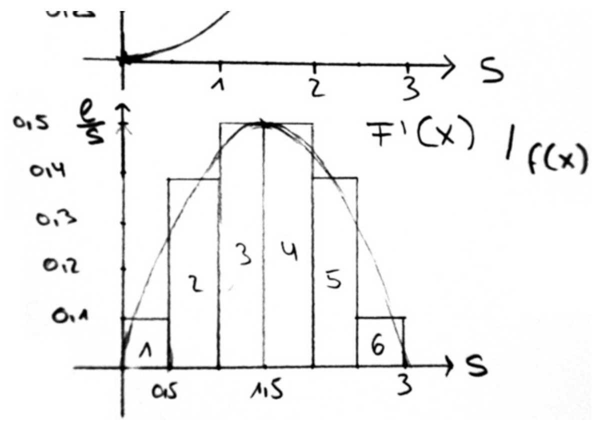

Figure 13. A closer look at the key student construction in the solution poster for Exercise 1

During the presentations, the students recognized that the sums of the areas of rectangles and trapezoids seemed to converge to the same value. Since it would be too tedious to verify their hypotheses by hand, they used a calculator to confirm their predictions.

Calculation of Areas

In groups of two or three, the students continued practicing approximating areas under the graphs of functions. One half of the class used trapezoids, as is prescribed

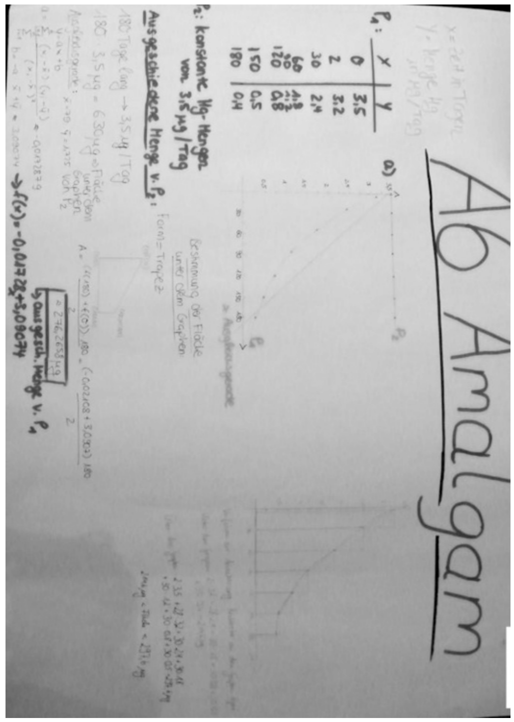

Figure 14. The solution poster for Exercise 2

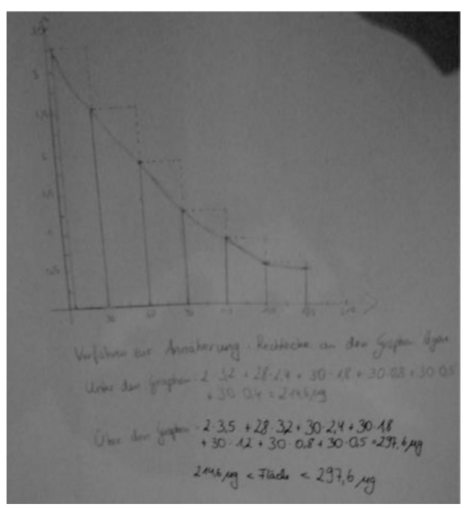

Figure 15. A closer look at the key student construction in the solution poster for Exercise 2

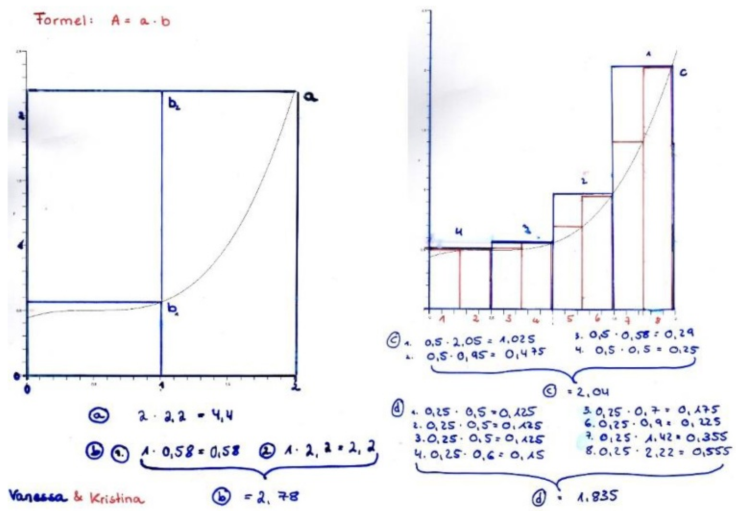

Figure 16. A worked out example of approximating the area between the function $f(x)=\frac{1}{2}\left(x-\frac{1}{2}\right)^3+\frac{1}{2}s$ and the x-axis over the interval [0, 8] using rectangles as presented by students on a transparency

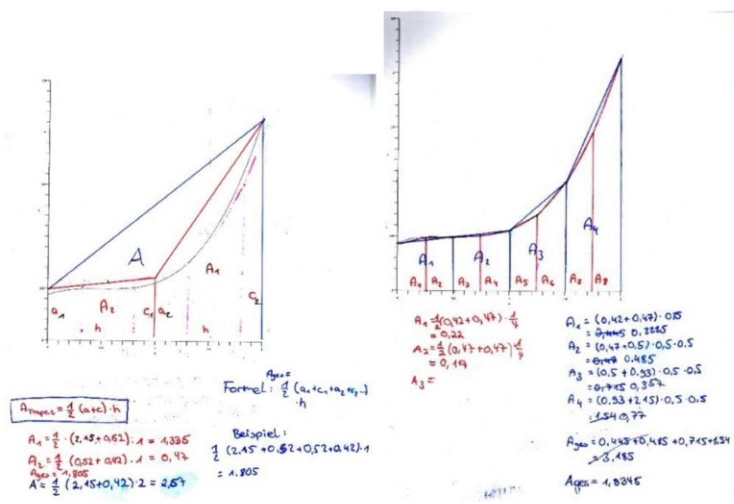

Figure 17. A worked out example of approximating the area between the function $f(x)=\frac{1}{2}\left(x-\frac{1}{2}\right)^3+\frac{1}{2}$ and the x-axis over the interval [0, 8] using trapezoids as presented by students on a transparency

in Exercise A, and the second half used rectangles, as is prescribed in Exercise B, and, now, with the added aid of GCs.

Exercise A. *Using the same function whose graph was used before, now knowing that the function used is defined as* $f: R \rightarrow R$ *with* $f(x) = \frac{1}{2}\left(x - \frac{1}{2}\right)^3 + \frac{1}{2}$*, approximate the area between* $f(x)$ *and the x-axis over the interval* [0, 2]*, using trapezoids.*

Exercise B. *Using the same function whose graph was used before, now knowing that the function used is defined as* $f: R \rightarrow R$ *with* $f(x) = \frac{1}{2}\left(x - \frac{1}{2}\right)^3 + \frac{1}{2}$*, approximate the area between* $f(x)$ *and the x-axis over the interval* [0, 2], *using rectangles.*

Students were given *hint cards*, outlining the step-by-step calculator procedures, to assist them in working through the assigned exercises. The *hint cards* were designed specifically for the method of dividing the interval [0, 2] into four equal parts. From several previous examples, the students were already familiar with the necessary skills for using GCs to calculate sums.

After exploring their results using finer partitions of the given interval, the students noted that the sums of areas of the relevant rectangles and trapezoids seemed to approach the same limiting value. This reinforced their assumption that the limit of the approximations of the area between the x-axis and the graph of $f(x)$ is independent of the types of figures used as well as the kinds of interval partitions, even if the subintervals are not equal. To check their assumption students used their GCs to approximate areas for different functions and different intervals. These exercises were not time consuming since, when working with GCs, they only needed to change the function definitions and the boundaries of the interval, and did not have to repeatedly input the entire general formula. Eventually, students formulated a general procedure for calculating the area between the graphs of a function $f(x)$ over an interval I, where $f(x) > 0$ for all $x \in I$, using smaller and smaller rectangles and trapezoids, eventually arriving at the formulation of the limit as $\Delta x \rightarrow 0$. Their conjectures were further reinforced by more different examples.

In basic courses, the types of functions studied are limited to differentiable ones, and one does not need to distinguish between upper and lower Riemann sums. However, one of the students asked:

> Wie wäre es denn, wenn die Rechtecke nicht über, sondern unter dem Funktionsgraphen liegen würden?
>
> [[What would happen if the rectangles would be drawn with the top sides above the graph instead of below?]]

This question led to the discussion of upper and lower sums, and further GC experiments. The experiment began with an example of a function $f(x)$ defined on $[a, b]$. Students were permitted to use prepared *hint cards* for calculating upper and lower sums using their GCs shown below. After several more calculations, students recognized that, for polynomial and trigonometric functions, the upper and lower sums appeared to have the same limits.

Reinforced by their results using upper and lower sums the class arrived at the definition of the definite integral analogous to the one described in the 2008 class.

Observations and Remarks from Students

In 2011, the current author was the instructor for the entire duration of the course. Furthermore, he had taught the same group of students during the previous year, and, thus, was able to divide the students into working groups with members of similar strengths, and distributed exercises that he found appropriate for a particular working group. In 2005, many students noted that the functions used for the exercises were visible on the worksheets and suggested that they should not be included, while others did not notice them at all, and the rest simply didn't utilize them. The division into working groups was not criticised by the students, most of whom embraced the group work.

In 2011, the passage into rigorous mathematics was more careful and measured. We started with an illuminating presentation of area approximation exercises relying on a variety of methods developed by students themselves during their group work. Most of the students recognized the similarities in the approaches to finding areas under curves on their own, and, eventually, fully understood the reliability of the rectangle/trapezoid areas accumulation methodology for approximating areas between the x-axis and the graph of a given function. Aided by the functionalities of the GCs, students were able to calculate the areas almost exactly and arrived at the concept of the definite integral based on their own observations and ideas.

The use of application examples and the freedom to develop different approaches themselves at the onset of the lesson sequence was praised by the students who then stated that this approach greatly contributed to their understanding of the topic. Even after revealing that $f(x)=\frac{1}{2}\left(x-\frac{1}{2}\right)^3+\frac{1}{2}$, all of the students remained motivated throughout the development of the rigorous mathematical models. Students explained that they did not lose the connection to reality due to the motivation established by the initial *real world* examples. Some of the stronger and more vocal students remarked that they would have been distracted and side-tracked by an increased number of applications.

Beginning with the approximate calculation of areas using rectangles and trapezoids up to the introduction to upper and lower Riemann sums, almost every student was able to follow the topic. During the subsequent lessons, some of the students expressed having difficulties, just as they lost their sense of imagination. They emphasized the effectiveness of graphs in helping them visualize the theoretical aspects of the definite integral and to understand the meaning of the associated calculations.

At the end of the entire course more class discussions took place. When asked about any difficulties or challenges they may have experienced utilizing their GCs during the lessons covering the introduction to the calculation of upper/lower Riemann sums, students said that they appreciated the visualizations and *by hand* calculations investigated during the *gallery tour*. They stated that they could transition smoothly to approximating areas under graphs of explicitly defined functions using a larger number of rectangles and trapezoids with the help of GCs, fully grasping the ideas behind the computations of the necessary limits, instead of blindly using the utilities of the GCs.

REFLECTIONS

Detailed lessons addressing the calculative approximations of integrals are not presently included in the high school curricula designed in North Rhine-Westphalia (NRW) or in Kultusminister Konferenz (2012), so students never need to perform, and, thus, understand such approximations of integrals by sums; see (Hijab, 2011), for example. In some federal states, like NRW, sequences are no longer included in the curricula, so that it is impossible to properly introduce differentiation and integration fully. As a result, the types of functions studied are limited. This does allow instructors to teach approximate calculation of integrals using rectangles and trapezoids over regular partitions (equal subdivisions) of the interval, and this material is already included in existing textbooks (Schmidt, Körner, & Lergenmüller, 2011; Bigalke & Köhler, 2011; Brandt & Reinelt, 2007). Note that, in all iterations here, this topic was explicitly taught.

Replacing the Riemann sum approach by geometric approximations in 2005 showed that the geometric method was more accessible to students. Leading the lesson sequence with approximating areas using upper and lower Riemann sums appeared to be much more difficult for students, and, since the functions used are relatively simple, one does not need to distinguish between the two, possibly causing more confusion; the geometric approach, which is valid for this class of functions, is simpler for students. At this level of calculus, any types of functions for which the convergence of upper or lower sums are not sufficient are not studied.

The inclusion of mobile GCs in 2008 and 2011 turned out to be more effective compared to the use of stationary CAS, since the students were more readily able to use GCs, the class could concentrate more on the mathematical details of the exercises and lectures, instead of allocating time specifically for mastering the CAS. One argument that the students mentioned against the use of CAS in 2003 was the “invisibility” of the encyclopaedia of syntax for the commands.

One of the big differences between the 2005 and the 2008 iterations is that, in the latter, the functions used in the introduction were *not* explicitly defined, and students relied on values they *read* from the graphs. Recall that in 2005 students referred to the explicitly defined functions as “unbelievable” or “irrelevant.” In contrast, in

2008, they had to *find* the appropriate functions that could be used to *model the applications*.

In 2008, the required manual input of GC commands appeared to be strange to the students. To address this, in 2011, a separate discussion of the tedious details and syntax of GC commands was substituted by *hint cards*. The students did not explicitly comment on the use of the hint cards, but there appeared to be no difficulties concerning the use of GCs. One could argue that detailed discussions concentrating on the syntax of GCs divert the students' attention away from the intended learning goals of the exercises. The inclusion of hint cards seemed to be a great and effective alternative for dealing with any possible syntax issues students might have with GCs, allowing them to concentrate on the mathematical content of the exercises.

Another clear problem, present in 2003, 2005 and 2008, was the presence of skill gaps among different students within a given group during the group activities. This inequality within groups presented a challenge in addressing individual questions or misunderstandings of every student in the class. This issue was addressed in 2011.

In 2008, the GCs were used too early. The role of GCs was not meaningful for some students; some struggled with operating GCs altogether. This frustration stifled students' engagement and understanding during the shift from the mathematical theory to the actual calculations of Riemann sums. Since the calculations relied on their ability to use GCs properly, those students who did not master this step, almost completely, lost the affiliation between the approximate calculations of sums and the idea of the integral.

Upon reflection and based on students' remarks, the increase in time dedicated to the introduction of the definite integral in 2011 was justified and worthwhile. If students' ability to remember details of a topic long after its presentation is increased then it is worthwhile to take more time and care during the initial introduction of the topic, reducing the need to repeatedly review the topic as the course progresses.

Carefully modifying the structure of the lesson sequences and reflecting on the outcomes over the four iterations described here allowed me to hypothesize about the effects of didactical changes within each sequence.

For example, *efficient* use of CAS or GCs allowed students to maintain focus on the main conceptual goals of the lessons by not getting lost in long and, sometimes, tedious calculations. This benefit was improved with the implementation of *hint cards*. Even though instructional media varied greatly over the four iterations, students had no problems adapting to them in 2011. In my opinion, instructors, or program directors, should carefully plan the role, and scope of use, of different media used in the classroom well in advance of implementation,—several years, perhaps. It is crucial for students to get accustomed to new and different media, such as GCs or CAS.

Another significant observation, based on instructor's reflections and student comments, is that increasing the amount of time spent on the introductory lesson sequence has a positive effect.

Poster presentations appear to improve student motivation and participation, especially by incorporating group work into the classroom. Group work, in general, seems to foster higher student attentiveness and engagement. In 2011, this was amplified by the necessity of every student to present the results of their group to other working groups. By appropriating themselves to working groups, with the teacher's guidance, taking into account the students' level of mathematical ability and difficulty levels of the contents, students were able to choose suitable exercises.

See the summary table (Table 2) on the next page for a side-to-side comparison.

Table 2. Comparison of the four iterations of the definite integral lesson sequence

Year	*2003*	*2005*	*2008*	*2011*
# of Lessons	*3*	*3*	*2*	*5*
Media	CAS, transparency, board	calculator, transparency, board	GC, transparency, board	GC, poster, transparency, hint cards
Classroom activity structure	single person working, two people working	two people working, group work	group work	single person working, two people working, group work, gallery tour
Types of media exercises	students got completed files for the CAS and only had to run them	students calculated for small *n* with their calculator	students calculated for bigger *n* with their GC	same as 2008
Approximate calculation	calculation of upper sums and lower sums in every example	calculation of sums of numbers or areas of trapezoids or rectangles	same as 2005	same as 2005
Connection to application	beginning only, without repeated connections to *reality*	little relation to *reality*, constructed (not real) connection to *reality*	small relation to *reality*, not credible simplification	relation to *reality* from different points of view, credible because started from *real life* examples
Type of integral	Riemann upper and lower sum	rectangular, trapezoidal and with $\lim_{n\to\infty} A_n$ and $\sum_{i=1}^{n} f(a+i\cdot h)\cdot h$	upper sum, lower sum, restriction to type 2005	same as 2005

NOTE

[1] Ministry of Education and Training of North Rhine-Westphalia,—the school or the Ministry of Education of the German state of North Rhine-Westphalia and one of ten ministries of the North Rhine-Westphalian government.

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4.10. FROM ARITHMETIC TO ALGEBRA

A Sequence of Theory-Based Tasks

SUMMARY

The chapter deals with language in the mathematics classrooms, especially with mathematics remedial classrooms. It presents an unusual example of integrating two independent theories, the Sfard (1991) theory of reification and Shepard (1993)/ Shuell (1990) integrated theory of cognitive development and writing categories. In the first cycle it's the TR Design Type A, from Practice; it uses problem types designed through practice and supports itself by a standard yet simple statistical analysis. However, in the 2nd cycle TR Design become of the type B, theoretically-based on the results of the first cycle. In the first cycle we find the evidence of the actual impact of language enhanced instruction upon student final exam achievement.

The aim of the second cycle was to refine the intervention by aligning it closer to the theoretical pathway of concept development accordingly to process/object theories of conceptual mathematical development (Davis et al., 2000). That entailed coordinating writing exercises with a theory of mathematical development. The basic information about writing categories and about the stages of the reification or encapsulation closes the paper. However, the central piece of the paper is the second cycle in which the stages of development are used to explain the written exercises in the Appendix.

The ultimate result of the study demonstrated that indeed written assignments have a significant impact upon problem solving and retention of algebraic knowledge. Chapter 5.1 asks exactly the reverse question namely, whether written assignments in mathematics can impact learning of English.

INTRODUCTION

This chapter presents an instructional sequence that is the first and second iterations of a teaching-research study on the effect of written thought to assist students transitioning from pre-algebra to algebra at Hostos Community College of the City University of New York (CUNY). This instructional sequence is aimed at community college students enrolled in remedial courses whose purpose is to help students reach the level of college mathematics. The adult students (18+ years of age) at HCC are about 75% female, 90% minorities; about 25% have never earned

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 425–443.

a high school diploma, instead they have a General Education Diploma (GED) obtained by passing a state exam. Well designed use of language in our mathematics classrooms may offer help in grasping and in retention of these elementary yet challenging concepts.

The Case That: Written Thought Promotes Conceptual Thought in Mathematics

Mathematical educators who employ written thought to reflect upon problem solving activity often invoke the theoretical work of Vygotsky (Puagalee, 2001, 2004; Bicer et al., 2013; Adams, 2010). Pugalee (2004) asserts that, "Vygotsky held that writing involved deliberate analytical action on the part of the producer. Written words require the writer to maximally compact inner speech so that it is fully understandable, thus making necessary the deliberate structuring of a web of meaning" (p. 28). Specifically, Vygotsky (1997) draws parallels between the conscious reflection required in the written expression of one's thoughts as opposed to spontaneous verbal thought and the structural or scientific conceptual thought required in algebra as opposed to the spontaneous thought required in arithmetic. Bell and Bell (1985) note, "Public dissatisfaction with the educational system's ability to deal with basic skills instruction has become a crisis of such magnitude that it affects all levels of education and defies easy solution." (p. 211) They call for integration of expository writing into the math class writing to learn mathematics (WTLM) as one method of dealing with this crisis. Thus, they agree with Vygotksy's thesis that, "Expository writing can become a 'mode of learning' that directly affects a student's command of the subject matter" (p. 213). Aspinwall and Miller (1997) express their dissatisfaction with the lack of student conceptual understanding in calculus; "This lack of conceptual mathematical understanding in calculus is pervasive in this country. Students are moving through college mathematics curricula with little comprehension of concepts fundamental to mathematical thinking. College students memorize formulas, algorithms and procedures in order to solve routine problems and obtain correct answers" (p. 253).

These authors advocate for WTLM in developing, "students' abilities to reflect on and speculate about subject matter" (p. 256). Educators who believe in writing as a mode of learning to address student difficulties with problem solving noticed that it promote the process of conscious reflection and encourages students to, "analyse, compare and contrast and synthesize relevant information" (Bicer et al., 2013, p. 364).

First Cycle

The first cycle bordered on action research in that the instructor Baker was testing (statistically) the hypothesis that written mathematical thought considered as evidence of conceptual knowledge would be independent of procedural knowledge in predicting how students would perform on a departmental final exam. This hypothesis came from the observation that students who performed well on computational problems during the semester often appeared to forget everything by the final exam. The goal as an instructor was to improve retention of student knowledge. Although there was no clear theoretical framework to support the written exercises during the first cycle there was a research goal in understanding the relationship between written conceptual thought and procedural thought during the semester and its effects on retention of student knowledge on the final exam. Thus the research focus was to weigh in on Powell and Lopez (1989) statement that despite the reasonableness of the claim that writing improves concept development in mathematics there has been, "...little evidence of students' concept development or increased mathematical maturity has been proffered despite the reasonableness of this assertion" (p. 160).

Instructional Methodology

The pre-algebra course has a strong focus on computational modelling of procedural knowledge, and although many of the students do well enough to pass during the semester, they perform poorly on the departmental final exam. While the major area of concern was that of application problem-solving, students also displayed difficulties with retention of procedural knowledge. These writing exercises were designed to encourage the reflection and abstraction of underlying conceptual knowledge and principles that could guide student decisions during problem solving. They were short and relatively easy to implement following the advice of Meier and Rishel (1998) who note that, "keep in the first time we assign a writing project may well be the first time our students have been asked to write in a mathematics course" (p. 7).

The comprehensive departmental final exam did not contain any writing exercises, however, the partial exams contained written exercises, procedural problems and application problems.

TEACHING EXPERIMENTS OF FIRST CYCLE

In the first cycle, written thought on partial exams or homework given during the semester was found to be a statistically significant factor in predicting student performance as measured by the comprehensive final examination. Furthermore, this written thought was independent of students' procedural knowledge (assessed during partial exams during the semester) in predicting their performance on the comprehensive final exam (Baker & Czarnocha, 2002, 2008). The writing exercises of the first cycle were based on authors' craft knowledge.

Example 1. *Compare and contrast how to multiply a decimal number by a power of ten with how you divide by a power of ten.*

Example 2. *Compare and contrast the greatest common factors of two numbers with the least common multiples of these two numbers.*

Example 3. *Michelle and her friends buy the supplies for her church charity car wash, which charges $10 per car. If you know the number of cars they wash and the amount they spend on supplies how would you find the profit that they makes for the charity?*

The statistical analysis involved correlation techniques and multivariate (ANOVA) analysis, with written conceptual thought and procedural proficiency as independent variables to predict or determine student proficiency with first procedural knowledge and second problem solving ability on the final exam. The first hypothesis being that written conceptual thought (being of an independent mode of learning and engagement) would assist in retention of procedural knowledge. For the second ANOVA with the same independent variable and dependent variable problem solving the hypothesis is that, these two types of knowledge would again work together yet independently in the schema formation required to retain problem solving ability.

When using two independent variables to predict a dependent variable, there are two important criteria to be considered,—first, both independent variables should demonstrate a statistically significant level of correlation with the dependent variable, and, second, the correlation between the two independent variables cannot be too high otherwise one variable will typically dominate and make insignificant the effect of the other. In this study, the departmental final exam, which tests procedural knowledge and ability for problem-solving (application or word problems) was separated into the procedural and problem solving components and these components were used as the dependent variable. The procedural component of the partial exams was one independent variable. The other independent variable was writing scores, obtained either from the partial exam or from homework assignments.

Resulting Data and Analysis

We analyse learning as taking place over the semester, using the scores for writing or procedural knowledge on the partial exams throughout the semester as independent variables. Written homework assignments were collected, graded and the results collected for two semesters, while written exercises on partial exams were recorded for three semesters.

When the data for the entire three semesters was combined (N = 117), the correlation between writing and the procedural component on the partial exams, as well as the procedural component of the final exam are listed in Table 1.

Table 2 lists the relevant correlations when procedural knowledge and written conceptual knowledge on the partial exams are compared to the application problem or problem-solving component of the final exam.

Table 1. Total correlation results: Writing vs. procedural component (All correlations were significant at the 0.01 level)

	Procedural on partials	*Writing on partials*	*Procedural on final*
Procedural on partial	1.0	0.855	0.683
Writing on partials	0.855	1.0	0.526
Procedural on final	0.683	0.526	1.0

Table 2. Total correlation results: Writing, procedural components and problem-solving (All correlations were significant at the 0.01 level)

	Procedural on partials	*Writing on partials*	*Procedural on final*
Procedural on partial	1.0	0.855	0.603
Writing on partials	0.855	1.0	0.580
Procedural on final	0.603	0.580	1.0

Clearly, written thought on the partial exams satisfies the first criteria for success since it correlates well with the procedural and problem-solving component of the final exam however, it also correlates even higher with the procedural knowledge on the partial exams. Multivariate analysis revealed that written thought on the partial exam was not independent of procedural knowledge on the partial exam in assisting students retain this knowledge on the final. In contrast, such written thought was significant at the 0.05 level ($p = 0.04$) in assisting students in solving the application problems on the final exam. The ***R***-value for the multivariate model with writing and procedural knowledge on the partial exams to determine problem solving on the final was $\boldsymbol{R} = 0.622$. This represents a 6.4% increase in retention ($\Delta \boldsymbol{R}^2$) of demonstrated ability for problem-solving on the final exam over the use of only procedural knowledge on the partial exams $\boldsymbol{R} = 0.603$ (Table 2). The formula used to determine the decimal equivalent for the percent increase due to use of two variables $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ instead of only $\boldsymbol{R_2}$ is

$$\Delta \boldsymbol{R}^2 = \frac{R_1^2 - R_2^2}{R_2^2}$$

Thus, written thought was statistically independent of procedural knowledge on the partial exams in assisting students in solving application problems but not for procedural proficiency on the final exam.

For the two semesters_in which written homework assignments were given (N = 71), the correlation between these writing scores, the procedural component of the partials and the procedural component of the final exam are listed in Table 3.

Table 3. Correlation results for two semesters?? with writing homework assignments: Writing, procedural components and problem-solving (All correlations were significant at the 0.01 level)

	Procedural on partials	*Writing on HW*	*Procedural on final*
Procedural on partial	1.0	0.650	0.688
Writing on HW	0.650	1.0	0.547
Procedural on final	0.688	0.547	1.0

Again, written homework assignments correlated well with procedural knowledge on the final exam. Multivariate analysis revealed that such written thought was independent of procedural knowledge on the partial exams ($p = 0.024$) at the 0.05 level. The ***R***-value for this multivariate model was ***R*** = 0.715, which represents an increase of 8% in the procedural knowledge retained on the final exam over the amount when only procedural knowledge on the partial exams is used ***R*** = 0.688 (see Table 3).

CONCLUSION

The written thought on the homework (example 1 and 2) was more in depth than that on the partial exams, the extended time allowed students to reflect more, and thus, we consider these results a more accurate measurement of student capability with written conceptual thought. In this light, the results for Table 3 indicate that written conceptual thought underlying procedural knowledge can be independent from procedural knowledge and that it can assist student in retaining such procedural knowledge for the final exam presumably the result of a deeper more integrated conceptual understanding.

The results of Table 2 suggest that written conceptual thought which included expressions of operator choice and coordination of steps to solve a problem (Exercise 3) as an independent variable is separate from and adds to student performance in solving application problems on the final. Thus, the ability to deal with and perform on computational-procedural problems is independent of the ability to explain underlying concepts and one's choice of an operator during problem situations in predicting problem solving ability during the final exam. This suggests that written conceptual knowledge was helpful in demonstrating problem solving schema development. That written conceptual thought about operator choice (metacognition) would assist in predicting schema formation is in agreement with the work of Pugalee (2004) who noted that student who wrote about their solution activity during problem solving tended to be more efficient than those who did not or who did so verbally. "Students who wrote about their problem solving processes produced correct solutions at a statistically higher rate than when using think-aloud processes" (p. 43). In particular, the written expression of thought was noticeable in promoting ‘statements’ by the students that were classified as metacognitive-orientation that is involving, “understanding how information in the problem relates to the problem-solving tasks” (p. 40) as well as statements about “performing local goals, monitoring goals and redirecting...” (p. 41). Pugalee’s thesis that written conceptual thought focuses students on metacognitive thought processes fits with our data results. If so this would appear to validate Vygotksy’s assertion that written thought promotes scientific concept development over verbal speech.

In contrast, the results of Porter and Masingila (2000) spread a ray of caution; as their results show that students who engaged in classroom dialogue of a conceptual nature did not perform statistically different than those who expressed their conceptual thoughts in writing. “If students who engage in non-writing activities that focus on concepts and involve discussion can achieve the same level of conceptual and procedural understanding as students who use WTLM activities and discussion, then mathematics instructors have a viable alternative to using writing activities” (p. 174). It would appear that the quality of the conceptual dialogue and the written expression of one’s thoughts is of tantamount importance. As noted by Bessé and Faulconer (2008) WTLM may not necessarily result in higher scores on standardized tests but will focus student attention to connections between mathematical objects and procedures. Our results suggest that while student assessment scores may not increase, such written conceptual thought is independent of procedural skill and helpful in promoting such test scores, it does not say students cannot engage in meaningful conceptual through class dialogue or other means

DESIGN EXPERIMENTS OF THE SECOND CYCLE

We decided to refine our previous approach by leaving the support of our craft knowledge and utilizing the reification theory of concept development Sfard (1991, 1992).

The writing exercises used in the second cycle were based upon the research work of Shepard (1993) who synthesized the writing categories of Brittan et al. (1975) with cognitive stages of development due to Shuell (1990). As Shield and Galbraith (1998) note, “Shepard maintained that the development of understanding can be stimulated by moving students in to more demanding writing tasks” (p. 30).

In this second cycle, a sequence of instructional tasks that include written, geometrical and computationally based exercises (Baker & Czarnocha, 2002, 2008) were developed, through a synthesis of the stages of concept development via the lens of process/object duality presented by Sfard (1991, 1992), Sfard and Linchevski (1994).with the work of Sheppard. The goal being to bring the stages of Sfard which are representative of research on the transition from arithmetical to algebraic thought with the phases of learning due to the cognitive theorist Shuell in the work of Sheppard.

MODELS OF DEVELOPMENT BASED UPON PIAGET

Davis et al. (2000) review several models of learning based upon and beginning with the work of Piaget, who focused on how actions or procedural knowledge become *thematized* objects of thought or concepts. According to the authors, these models all share a process/object duality in which learning mathematics is viewed as a cycle that begins with procedural knowledge acting on existing conceptual knowledge; followed by reflection that leads to clarification of procedural knowledge and the underlying concepts. This cycle ends when the process becomes integrated into the learner's schema, and in this object form can itself be acted upon. This process of the assimilation of procedural knowledge into one's conceptual schema, is called "the *encapsulation* (or *reification*) of a process into a mental object" (Davis et al., 2000).

The terms *encapsulation* and *reification* are used by Dubinsky and Sfard, respectively, to express the final stage, or object level understanding, of a learner in their thoughtful and separate interpretations of Piaget's process/object duality. As Davis et al. (2000) wrote, "The transformation of a process into an object took new impetus with the work of Dubinsky and Sfard" (p. 224).

In the model of Sfard, this process/object duality involves a transition from an *operational* to a *structural* understanding. The distinction between *operational* and *structural* is expressed by Davis et al., when they classify the "evaluation of an expression such as $2x + 3$ for a numerical value of x as operational and the manipulation of such an expression as structural" (1999, p. 233).

In this article, we employ the model of Sfard because its focus on an individual's transition from an arithmetical (*operational*) to algebraic (*structural*) understanding closely parallels the intent of our instructional sequence. For Sfard (1991, 1992), this transition is a three-step process, which begins with a procedural oriented stage called *interiorization*, and continues with the more abstract stages of *condensation* and *reification*.

TRANSITION FROM ARITHMETICAL TO ALGEBRAIC THOUGHT

First stage: Interiorization – Operational Understanding

The term *interiorization*, which is originally due to Piaget, is described by Sfard in terms of reflection upon procedural knowledge—a procedure is interiorized when it "can be carried out through mental representations, and in order to be considered analysed and compared it needs no longer to be actually performed" (Sfard, 1991, p. 18). Dubinsky analysed the work of Piaget and the role of interiorization as well as other forms *reflective abstraction* in learning mathematics. He, similarly, described interiorization in terms of the internalization of actions or procedures:

> First, an action must be interiorized. As we have said, this means that some interior construction is made relating to the action. An interiorized action is a

process. Interiorization allows one to be conscious of an action to reflect upon it and combine it with other actions. (Dubinsky, 1991, p. 107)

Tasks designed for students in the initial interiorization stage follow this sequence: (1) learning a new arithmetical process or action, followed by (2) applications that will require them to perform this action, and, as they become proficient, (3) they will be asked to reflect upon this process.

Intermediate Stage: Condensation

Condensation, the intermediate stage of development, is described by Sfard:

> Condensation is a period of squeezing lengthy sequences of operations into more manageable units. At this stage a person is more and more capable of thinking about a process as a whole without feeling an urge to go into details … a learner would refer to the process in terms of input-output relations rather than by indicating any operation. (Sfard, 1991, p. 19)

In describing the effect of condensation Sfard also notes, "Combining processes, making comparisons and generalizations becomes much easier" (Sfard, 1991, p. 19).

Dubinsky mentions several types of *reflective abstraction* that an individual employs after the initial *interiorization* stage to conceptualize processes: first, generalization of a process or schema,—"when a subject learns to apply an existing schema to a wider collection of phenomena, then we say that the schema has been generalized" (Dubinsky, 1991, p. 101); second, coordination of processes,—"two processes can be coordinated to form a new process" (Dubinsky, 1991, p. 104) and, third, reversal or inverse of a process, for example,—"subtraction and division of an equation" (Dubinsky, 1991, p. 105).

Tasks designed for students in this middle stage will require them to generalize the arithmetical processes they have learned to a more abstract algebraic setting that involves simple algebraic expressions and/or the ability to coordinate previous knowledge, whether conceptual or procedural, with the newly learned process.

Final Stage: Encapsulation – Reification: Structural Understanding

The term *encapsulation* is described by Dubinsky (close to the spirit of Piaget) as the "conversion of a dynamic process into a static object" (Dubinsky, 1991, p. 101). The effect of *reification* is described by Sfard as "convert[ing] the already condensed process into an object-like entity" (Sfard, 1992, pp. 64–65).

The final stage, in which the learner has arrived at a structural understanding in their transition from arithmetical to algebraic thought, is described by Davis et al., "… in algebra the symbols are now algebraic expressions" (as opposed to numerical values). They continue, "The symbols themselves can be manipulated algebraically and a finite number of such manipulations can be used to solve linear and quadratic equations" (Davis et al., 2000, p. 236). Thus, the transition from arithmetical to algebraic thought begins with performing and interiorizing arithmetical operations, generalizing these into algebraic expressions, and, then in the final stage, an individual is able to use such expressions efficiently in problem-solving.

> Operations at one level (in this case algebraic formulae as generalized arithmetical operations) become objects of thought at a higher level (algebraic expressions) which can themselves be manipulated. (Davis et al., 2000, p. 239)

The ability of an individual to demonstrate problem-solving skills, and its link to schema development in structural understanding, are highlighted by Sfard, "Unstructured and sequential cognitive schema … [which are] … inadequate for the rather modest dimensions of human working memory" are restructured and organized, "by turning sequential aggregates into hierarchical structures" (Sfard, 1991, pp. 26–27).

The tasks designed for the final stage will involve the use of variables and manipulation of algebraic terms in solving application problems. Furthermore, students are asked to consciously reflect on the process of combining and sequencing these manipulations when solving algebraic problems. This conscious reflection will demonstrate students' understanding of the hierarchical structure of the problem solving schema.

THEORETICAL FOUNDATION FOR WRITING EXERCISES

Writing Exercises

The development and implementation of writing exercises in this study follows, as in the previous cycle, Shepard's philosophy on the relationship between written thought and conceptual development as eloquently set forth by Shield and Galbraith (1999) "Shepard maintained that the development of understanding can be stimulated by moving the student into more demanding writing tasks" (p. 30). Shepard uses the writing categories of Britton (Britton et al., 1975). The three Shepard-Britton writing categories that were used in this study to transition students out of early phases of learning and develop their conceptual thought are: *generalized narrative*, *low-level analogic* and *analogic*.

Shepard/Britton Writing Categories

Generalized narrative. In the generalized narrative category a students is "tied to concrete events but begins to detect a pattern … begins to see generalizations." In this category, Shepard recommends that students be asked to "explain definitions or procedures in one's own words" (Shepard, 1993, p. 290).

Low level analogic – LLA. Unlike the initial generalized narrative stage, students can begin to understand new material beyond the examples used in the classroom. As one progresses in the intermediate stage the ability to recognize, explain and apply knowledge in more varied situations becomes possible. LLA phase requires students to use information in a more purely generalized manner. However, they do not recognize the logical, overriding structure inherent in the subject matter. Writing exercises in the LLA category required students to "explain how to solve a problem, or given an incorrect worked problem, student explains what was done wrong" (Shepard, 1993, p. 290).

Analogic writing category. In the analogic writing category students begin to make generalizations and to organize their knowledge into a schema. Shepard recommends that analogic exercises ask students to "explain how concepts are related, i.e. similar or dissimilar" (Shepard, 1993, p. 290).

INTEGRATION OF SHEPARD/SHUELL COGNITIVE PHASES WITH SFARD'S MODEL

In his article, Shepard (1993, p. 290) matches the writing categories of Britton et al. with the three cognitive learning phases described to Shuell (Shuell, 1990). We review the work of Shepard with the goal of integrating these cognitive learning phases with the stages of development of Sfard's model. This will serve as a foundation for the instructional task sequence (including both written and computational exercises) used for transitioning students from an operational to structural understanding of algebra.

The Shepard-Shuell model of learning characterizes the initial phase as follows,—"the individual perceives the facts, terms and concepts being presented as isolated pieces of information. The learner does not see the organizing structure of this new knowledge and has little personal knowledge with which to relate it" (Shepard, 1993, p. 288).

The initial phase of the Shepard-Shuell model focuses on the lack of connections between new terms or definitions and previous knowledge. In contrast, the initial *interiorization* stage of Piagetian models focuses on an individual's ability to reflect upon a procedure. Because conscious reflection upon new knowledge, whether a procedure or definition, is facilitated by (one might say begins with) the learner making relationships to previous knowledge, the initial stage/phase in both models describe a situation in which an individual is faced with new knowledge that has not been assimilated into their schema. Thus, we designate the initial phase of Shepard-Shuell as equivalent to the first, or the *interiorization*, stage in Sfard's models.

In the intermediate learning phase of the Shepard-Shuell model:

> [The] learner begins to see relationships between (the previously isolated facts and terms and hence, begins to form a more meaningful internal structure. The learner is also becoming less dependent on specific, concrete examples as their conceptual knowledge becomes more abstract. (Shepard, 1993, p. 288)

Both Shepard and Sfard describe the intermediate stage as one in which the learner becomes less dependent upon specific examples and can generalize more readily to an abstract understanding of the knowledge presented. Thus, both demonstrate an individual that has begun to assimilate the new material into an appropriate schema and, hence, we designate the intermediate phase as equivalent to the middle stage of *condensation*.

The final phase of the Shepard-Shuell model is one in which new knowledge has become integrated into an existing schema; it can be readily accessed and efficiently used in problem-solving. While this terminal phase matches well with the final structural level of understanding in which an individual can efficiently employ and manipulate algebraic variables in a problem-solving environment, the advanced tautological writing category that Shepard matches with final phase is deemed inappropriate for the tasks presented. For one reason, the tautological writing category requires reflection beyond the domain of a one-semester pre-algebra/algebra mathematics course: "producing a new system or method for solving certain categories of problems" (Shepard, 1993, p. 290). Furthermore, the goal of these tasks was to encourage the student's ability to engage in algebraic thought, that is, the manipulation of algebraic expressions; although writing exercises may assist students in approaching this goal, the insistence that students reply in written language to questions at this level may hinder their ability to manipulate algebraic expressions. For these reasons, the authors replace the tautological writing category with an analogic transitional category; at this level, when asked to explain how they would manipulate variables in solving application problems, students' answers may involve written (analogic) explanations, or they may transition to algebraic formulae. In such a case, written thought can be viewed as a tool, or vehicle, that carried the student to the desired destination; having arrived at the ability to engage in algebraic or structural thought they are engaging in the language of algebra, and the use of written language is no longer required.

Integrated Framework of Shepard-Shuell Cognitive Phases and Piaget-based Stages of Learning: Transition from Operational to Structural Thought

The primary objective of this article is to present an instructional sequence of tasks that begins at the operational level of understanding appropriate for students with an arithmetical thought process, and transitions to a structural, more algebraic, understanding of mathematics. This transition is marked by the use of *reflective abstraction*: *generalization*, *coordination* and *reversal of process* (Dubinsky, 1991), as well as the integration of newly learned processes with previous knowledge.

At the end of this transition the students' structural understanding is measured by their proficiency to manipulate variables while solving application

APPENDIX-: INSTRUCTUIONAL SEQUENCE OF RATIO TASKS

The following tasks were designed to be done by pre-algebra mathematics students at a community college after a reading passage (see appendix for sample reading passage) and/or classroom lecture on ratios, the objective is to transition through the interiorization and condensation phases by promoting conscious reflection directly on processes and related processes/conceptual knowledge and thus attain to a more structural understanding.

Many of the following exercises involve an understanding between the part-part relation characteristic for the ratio and the part-whole relation (also used in ratio) characteristic of a fraction (presumed to be previous knowledge). Through the use of the operation of addition and the reverse process of subtraction these exercises assist in building and making connections between the ratio and fraction schema. Language is used to abstract these processes, strengthen students' conceptual understanding and provide an entry way into algebra through written reflection that assists the transition from numerical to algebraic processes. A pictorial representation of this coordination between schemas can be viewed in concept map form in Figure 1.

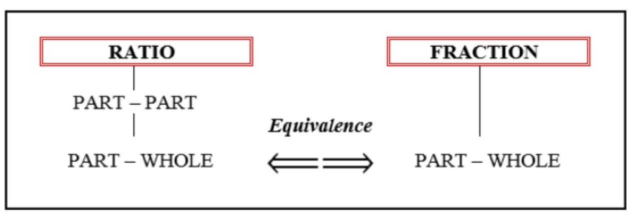


Figure 1. Concept Map for Ratio, Fraction, Part-Part and Part-Whole

We employ the following objectives:

1. Actions applied to determine ratios; the operations of addition and subtraction applied to relate part-whole concepts involved in ratios and fractions.
2. Language used to abstract concept of ratios and equivalency to fractions
3. Use of variables in representing ratios and fractions

Part I: Early Interiorization, Interiorization and Early Condensation Stages

Writing Exercises – General Narrative

1. What is a *ratio*?
2. When is a ratio in *simplest* form?

(EI-Stage/Critical Reading/Translation between comparative language and symbolic mathematical statements)[1]

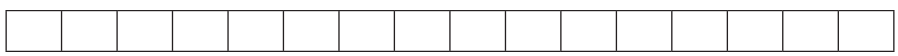
3. Edwin's age is triple the age of his younger sister Ericka. Write the ratio of Ericka's age to Edwin's age. *(EI-stage)*

Writing Exercises – LLA

1. Marica wrote a ratio of 7 female to 5 male students as 7/5 = 1 2/5. Explain what she did wrong. In the above situation, with 7 female and 5 male students, Sandy wrote the fraction of students that are female as 7/5. What did she do wrong? *(Critical reading to explain what was done wrong/early interiorization stage)*

Geometry Exercise

Exercise 1.) Divide the following 16 boxes into two parts that are in a ratio of 3 to 5.



(Interiorization -stage: student must work with concept of total boxes and process of forming ratio of two parts from this total)

Procedural and Application Exercises

Exercise 2.)
Edison has 120 pencils, 50 green pencils, 40 yellow and the rest blue.

- 2a.) Find the ratio of blue to green in simplest terms. (*IandEC-stage*)
- 2b.) Find the ratio of yellow to total in simplest terms.(*I-stage*)
- 2c.)What fraction of the pencils are yellow? (*IandEC-stage*)

Exercise 3.) If a team won 28 out of the total of 36 games played, find the ratio of the games won to the games lost in simplest terms. (*Interiorization Early Condensation -stage*)

(*Interiorization Phase: In a specific situation, the student was asked to coordinate and sequence the actions/processes of, finding the ratio and using reverse operation of subtraction to find part-whole concepts in ratios. Students also asked to understand the equivalence of the part-whole concept in the ratio and fraction schema.*)

Exercise 4.) Two sisters Sheyna and Kelly are measured by their father. Sheyna is 3 feet tall while Kelly is 30 inches tall. What is the ratio of Sheyna's to Kelly's height in simplest terms?

(*Exercises 4 Interiorization Phase: Critical understanding of ratios as involving quantities with the same units in specific situation—conversion of units required.*)

Susan has to take two subways and a bus to get to work. The first train takes half an hour, the second 5 minutes and the bus takes one quarter of an hour.

- (6a) What is the ratio of her time on the first subway train to the second train in simplest terms?
- (6b) What is the ratio of her time on the two subway trains to the time spend on the bus in simplest terms?
- (6c) What is the ratio of her time spend on the first train to her total commute time in simplest terms?
- (6d) What fraction of her total commute does Susan spend on the train?

(Exercise 6: Interiorization Phase: Critical understanding of ratios as involving quantities with the same units. Student asked to coordinate and sequence the actions/processes of: finding the ratio, using operation of addition to find part-whole concepts in ratios. Students also asked to understand the equivalence of the part-whole concept in the ratio and fraction schema. All in specific situation)

Exercise 7.) Franco and his younger sister Francis are to share amongst themselves a bag of pieces of candy. Franco being older is to get 4 pieces of candy for every 3 pieces his younger sister receives. If Franco receives 12 pieces how many does his sister receive?

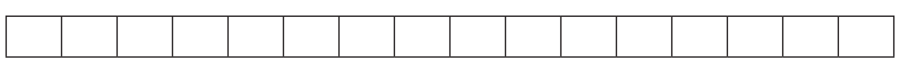
(Student must demonstrate pre-proportional reasoning by coordinating given numerical ratio with real life ratio situation to find unknown value; interiorization--early coordination stage)

Writing Exercises – Lower to Middle level analogic in interiorization and Early Condensation stage

1. Joanna was given a classroom ratio of 15 male to 25 female students. When asked to find the fraction of students that were female she wrote 5/3. What did she do wrong?
   (error made in a specific situation that involves the equivalence of the part-whole concept in the ratio and fraction schema. Student asked to explain the error.)

Geometry Exercise

1. Two sisters Francis and Debbie are sharing 16 bags of Halloween candy, Debbie being the older is getting twice as much as Francis. The grid below has 16 squares that represent the 16 bags of candy; divide these up into two parts in a ratio of (1:2) the smaller part for Francis and the larger twice the smaller for Debbie. How many bags does each girl receive?



(*Application problem in which student must coordinate, translation between language and mathematics and the process of dividing squares into two parts in ratio of (1:2); Interiorization and early condensation stage*)

Part II: Condensation Stage

Procedural Exercises

1. Three brothers Ali, Baba and Paul share a sum of money in a ratio of 7:8:9. If Baba receives $1624 how much does Paul receive?
   (*Student must disregard extraneous information and demonstrate pre-proportional reasoning by coordinating given numerical ratio with real life ratio situation to find unknown value*)

Writing Exercises – Analogic

1. Stephanie knows the number of female and male students in a mathematics class. Describe how she would find the ratio of female to total students.
   (*Analogic /Condensation-stage, in an abstract situation, the student is asked to explain, how to coordinate and sequence the actions/processes of, finding the ratio and use operation of addition to find part-whole concepts in ratios*)
2. Professor Skinner knows the number of students in Melville High and the number of male students how would he find the fraction of students that are female?
   (*Middle Leveel analogic in an abstract situation, the student is asked to explain, how to coordinate and the reverse operation of subtraction to find part-whole concepts and apply this in finding the finding a fraction*)
3. Sarah has a total of x pencils in a case. If she gives out 12 of these pencils how would she find the ratio of pencils given out to those remaining?
   (*Analogic/Condensation -stage, student must demonstrate encapsulation of the total concept into the variable x. Student must explain how to use the reverse operation of subtraction and given numerical part 12, to find the remaining part concept in ratio form. Alternatively, they may express the remaining part algebraically*.)
4. Sharon has 25 pens, she gives x of these to Wendy, how can you find the ratio of pens Wendy has to those remaining?
   (*Analogic/C-stage, student must demonstrate encapsulation of the part concept into the variable x. Student must explain in words or algebraically how to find, the part remaining using the given numerical total 25 and reverse operation of subtraction and express answer in ratio form.)*
5. Josephina has x blue and y pink candies, when asked to write the fraction of candies that are pink she writes y/x. Explain what she did wrong.
   (*Analogic/Condensation with some evidence of encapsulation of: part-part ratio concepts into variables. Student must explain error made by lack of addition to*

convert part-part ratio concept into part-whole fraction concept. Situation is totally abstract with only variables but because it requires student to explain what was done wrong instead of generating the answer it is considered CandEE-stage) *Translation between comparative language and symbolic mathematical statements.*

1. Jorge's age is twice Rick's age, Debbie's age is three times Rick's age. Write the ratio of Debbie's age to Jorge's age.
2. James is two-thirds the age of his brother Charlie. Which of the following represents the ages of James to Charlie?

(A) 2:3 (B) 2:1 (C) 3:1 (D) 3:2 (E) Not Given

(*Condensation phase – coordination of language with same process applied twice or existing fractional schema in specific situation.)*

Part III: Encapsulation Stage

Exercise 1.) There are x blue and y yellow marbles in a bag.
(1a) Write the ratio of yellow to total. (*CandEE-stage*)
(1b) Write the fraction of yellow marbles. (*E-stage*)
(1c) Write the fraction of blue marbles. (*E-stage*)
(1d)How would you determine which fraction was larger? (*Anal./E-stage*)
(*Analogic, encapsulation of part-part ratio concepts into variables; students must use process addition on variables to convert them into part-whole concept in ratio form and coordinate with its equivalent part-whole concept in fraction schema*)

Exercise 2.) Jorge and his younger brother Juan are sharing money. If Jorge gets twice as much as his younger brother Juan, who receives \$$x$ and together they have a total of T dollars, which equation below can be used to find x?

(A) $T = 2x$ (B) $T + x = 2x$ (C) $T = 3x$
(D) $T = x$ (E) None of these

(coordination of processes in abstract setting that requires use variables and manipulation of these variables – encapsulation)

Exercise 3.) Kelly-Ann has three times as much as her sister who has x dollars. Together they have a total SUM of money. Describe how you can find x.

Exercise 4.) Francis and Franklyn are sharing a total T candies. Francis receives three candies for every 2 that Franklyn receives, describe how you would find the amount of candy each one receives.

(Encapsulation -stage/Analogic-Transitional; students combine and integrate processes in abstract setting that requires use variable terms and manipulation of

these terms, as well as conscious reflection upon this process. Student's answers may be written thought with variable terms or algebraic formulae.)

NOTE

[1] Italicized instructor comments will follow certain exercises/sets of exercises in parenthesis, for the remainder of this section.

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UNIT 5

TEACHING RESEARCH COMMUNITIES

INTRODUCTION: LEARNING COMMUNITIES

We have spoken of the traditional view of mathematics in the classroom in which the teacher is the authoritarian voice and students – passive recipients of his/her knowledge, and the alternate or reform view exposed by constructivists, as well as Koestler and most prominent mathematical educational institutions in which the focus is on the role of students as active participants in the construction of knowledge. Yet the reform movement in mathematics inspired by this view has led to controversy, even to math wars. It has been suggested that the lack of constructivist pedagogy to accompany theory has led to difficulty with teachers implementing it is in their classroom i.e. lack of clarity of the role of scaffolding and direct instruction in a discovery or constructivist mathematics classroom. Another claim is the resistance of teachers and students to change who have been predominantly exposed to the traditional methodology. Another reason which we now explore is that mathematics educational research to date has been itself grounded in a traditional methodology that discourages teacher participation and thus provides minimal incentive for substantive change. Towards this goal, we review mathematics educational research on learning communities or communities of inquiry which typically involve mathematics educational researchers collaborating with practicing teachers on an equal level e.g. teaching—research. We note that, learning communities may also be organized solely by teachers or teachers and administrator e.g. action research and students working together to become teachers e.g. study groups.

TRADITIONAL AND REFORM PEDAGOGY

In the traditional view of mathematics education research the educator provides theoretical foundation for reform or ideas for an area of study which the teacher implements in their classroom while the researcher reflects upon and then publishes the results. The teacher is seen as part of the experimental environment and thus a separation between research and practice is observed; "…in much mathematics education research, teachers are viewed as recipients, and sometimes even as means to generate or disseminate knowledge, thus conserving a distinctive gap between research and practice." (Kieran et al. 2013, p. 361) This traditional separation or gap between research and practice expresses a reality of the very different environments of theoretical research and classroom practice. Current research emphasizing collaborative efforts to study the mathematics classrooms i.e. the unit of analysis of educational research point out that closing this gap will potentially benefit not only teachers but educational researchers as well. "In most cases the worlds of teachers and researchers differ greatly, even if there are also cases where they work together so closely that the traditional roles begin to blur…The major question is: How can mathematics education research have an impact on mathematics classrooms, on students' learning abilities, beliefs and interests? And how can researchers benefit from the rich body of knowledge and subjective theories that teachers have? (Kieran et al., 2013, p. 362).

THEMES OF LEARNING COMMUNITY RESEARCH

One theme of such literature is that action research, teacher-researcher efforts or collaborative efforts between educators and teachers is needed to close the gap between research and practice because substantial changes or implementation of constructivist or reform pedagogy requires that teachers become active participants in implementation of educational theory in their classroom. "With new research on educational change small and large, the important role of the participants in enabling, shaping and maintaining change processes has become more and more recognized…It is not surprising that action research is seen as one lever to better practices in mathematics. Action research promises to support the change of the most important change agent, to ground change locally where change is necessary, and to bring about personal growth that affords the retention of pursued changes" (Benke et al., 2006, p. 283). This viewpoint is reflected in language expressing the need for teachers to be active participants or key stakeholders in educational research (Kieran et al., 2013). Jaworski (2006) expresses this view in her statement, "Theories help us to analyse or explain, but they do not provide recipes for action, rarely do they provide direct guidance for practice" (p. 188).

Despite the growing awareness of the need for such research Lerman and Zehetmeier (2008) note "There are few examples of either face-to-face or cross-school networks research when compared to the rest of the body of research on mathematics teacher education" (p. 149). The statement that for the most part education research continue the focus on educators reviewing work of teachers practice as separate non collaborating agents within the mathematics community is substantiated by the review work of Adler et al. (2005) who state that, "Or focused analysis of papers in JMTE and JRME and in PME proceedings between 1999 and 2005 forcefully bears out this claim. Of articles representing research that focuses on teacher education. 90% of JMTE articles, 82% of PME and 72% of JRME articles were of this type" (p. 371). According to Krainer (2014) despite the attention and sense of importance attached to such collaborative efforts the situation is not significantly improving. "Despite the efforts and continuous claims of how important teacher-researcher collaboration role is, teachers are most often seen as more or less passive recipients of researchers' knowledge production and sometimes as a means to produce knowledge" (p. 49). In this view, teams of action researchers, teacher researchers or collaborative team efforts of researchers and teachers are necessary if reform pedagogy is to be implemented. "In the past decades, we have seen the call for and implementation of reforms of education in many educational systems around the world…change will always fail until we find some way of developing infrastructure and processes that engage teachers…" (Benke et al., 2008, p. 288).

Another themes is that learning communities involving teams of teachers reflecting upon their practice (action research), or teams of teachers and educators (e.g. communities of inquiry) or in our case teachers who are also reflecting on our own classroom experience educators (teacher-researchers) is that the critical process of reflection upon one's own practice is ideally suited for social analysis by a group or team of peers. One might go so far as to say that a social setting is essentially a requirement for most individuals i.e. teachers to engage in substantial reflection upon their practice. Reflection upon teaching practice, i.e. the classroom lesson experience as the unit of analysis, is a common perhaps defining feature of research on learning communities. We note that while student reflection upon the mathematical process has been extensively studied as leading to reification (encapsulation) to objects the same cannot be said for reflection leading to reification or teaching processes leading to objects or artefacts that can be used by teachers in their practice. "A further important element of any conception of action research is the notion of reflection…" (Benke et al., p. 285). The goal for the teacher is to develop a "reflective practice as a practice in which a practitioner is engaged in a constant conversation with his or her problem situation" (p. 286).

Another theme that emerges in the study of learning communities is that collaboration is essential for closing the gap between research and practice in order to make significant change in teacher attitudes. Just like the didactic contract between teacher and student in which student affect is a central factor in promoting participation in classroom discourse, the didactic contract between the researcher and the teacher is a necessity for a realignment of the role of the teacher to that of a co-participant in the inquiry process. The teacher needs to make a transformation from acceptance of student failure to problem situation observer and with collaborative partners in reflection to understand what is occurring in the classroom and come to a decision about what changes they think should and could be implemented. This transformation like that of the student has the goal of the teacher arriving at place where their attitude towards their craft is one of an educator who inquires as well as a teacher that imparts knowledge, nourishes and supports their students. Jaworski (2006) speaks of a "critical alignment" between the normal desirable state to one of inquiry and reflection. The normal desirable state is that rewarded and encouraged by the community of practice in whatever school environment the teacher works in. As Cobb and McLain (2010) point out teachers identifies and actions are to some extent a reflection of the, "…institutional settings of the schools and the districts within which they work…the communities of practice within a school district whose enterprises are concerned with teaching and learning." (p. 207). As noted by Jaworski (2006) these communities of practice result in normal desirable states that in which, "…participation here looks more like a perpetuation of the practice…an alignment that lacks a critical dimension" (p. 191). A critical alignment in contrast, "…includes some sense of teachers critiquing and trying to develop, improve or enhance the status quo (p. 191).

We have noted that difficulties for teachers with a traditional methodology to implement constructivist pedagogy includes the lack of a recipe or formula on how this is to be done. For example, the role of scaffolding in a lesson, too much promotes rote meaningless copying while an insufficient amount may result in student involvement in discussion without significant math content. For example, Sherin (2002) reports of a teacher who, "…found that was relatively easy for her to get students talking and sharing their ideas about mathematics. However, it was quite another matter to understand, from the teachers' point of view what to do with these ideas…in order to facilitate the discourse effectively" (p. 208). While reflection with colleagues upon one's classroom methodology and experiences are a central and necessary component of learning communities, changes in one's teaching methods especially those obtained in a community of practice can be difficult. As Schoenfeld (2006) comments "…teachers, like other professionals, develop a particular type of perception common to their profession" (p. 483). This perception or beliefs and habits do not readily change especially when not supported by existing communities of practice. In the words of Jaworski (2006) "…the significance of normal desirable states is just that they are desirable within the social practices in which they have been developed. It is hard to operate against such practices or to challenge them…" (p. 191). Although "Community building and networking represent the core factors fostering sustainable impact of professional development programmes" (Lerman & Zehetmeir, 2008, p. 149), one factor frequently noted that works against involvement in learning and action research is the time factor, "Action research requires a lot of time and energy from teachers" (Benke et al., 2008, p. 197).

Kieran et al. (2013) review case studies of various learning communities around the world, from teams of teachers doing cycles of action research involving planning, implementation, reflection and revision in Japan, to an adaptation of this in China involving a collaboration of an educational export with teachers to programs in U.S. and Norway in which the educators finance and set up a collaborative effort between researchers and educators. The example in Norway is taken from the work of Jaworski in which educators are referred to as those involved in the study of didactics. "Didacticians who are teacher-educators work with practicing or prospective teachers to enable a transformation of theoretical ideas and research findings into modes of teaching that are informed by theory and research. Here we see transformative work at two levels one between didacticians and teachers and one between teachers and students" (Jaworski & Huang, 2014, p. 174).

TR TEAM OF THE BRONX – THE COMMUNITY OF TR PRACTICE

The community of teacher researchers working with TR/NYCity model suggests an answer to the issues besieging Math Education profession mentioned above. The “critical alignment “of Jaworski is made explicit in the substantive bisociativity of the TR/NYCity conceptual framework (Chapter 1.1), which leads to the new unit of methodological analysis: Stenhouse TR acts.

In response to Kieran et al.’s (2013 p. 362) questions: How can mathematics education research have an impact on mathematics classrooms, on students’ learning abilities, beliefs and interests? And how can researchers benefit from the rich body of knowledge and subjective theories that teachers have? – we propose the integration of the two on the local classroom level with the help of JiTR method, motivated by the substantial quality of TR/NYCity methodology with the central research benefits arriving as by-products of the improvement effort. In simile to the work of Vrunda Prabhu who brought Koestler’s theory of the Act of Creation to the attention of teacher-researchers through her attempts to create a TR community around her classroom of remedial arithmetic. However, it is our conviction that this process of integration must go beyond the Action Research as well as beyond the researcher-teacher collaboration to reach balanced bisociative teaching-research centred around Stenhouse TR acts of the classroom teacher – the central actor in this process.

The community of the TR team of the Bronx has been in existence since 1997/1998 when the first teaching-research experiment Algebra/ESL was conducted at Hostos CC, which investigated the impact of algebra learning on learning of English as a Second Language (Chapter 5.1) below. It is significant for the future development of the community that this first teaching experiment was conducted on the interphase of the bisociative framework created by Elementary Algebra and Intermediate ESL, generally two domains of knowledge that don’t have much connection with each other. The teaching experiment uncovered the hidden analogy of this bisociative framework to be coherence of thought and its written expression. Our interest in the mathematics/language interphase continues; Chapters 4.10, 5.1 and 5.2 develop this theme a bit further.

Mathematics and Language are at present two main academic challenges undermining the access to higher education among the students from “underserved population”, which underlie their Achievement gap. Achievement gap refers to the observed, persistent disparity of educational measures between the performance of groups of students, especially groups defined by socioeconomic status (SES), race/ethnicity and gender. The results of PISA 2012, for the first time segregated in accordance with occupations of parents shows the wide gaps between children of professionals and managers on one hand and the children of parents with “elementary” professions (according to OECD classifications (PISA in Focus 36) in many participating countries of Europe, Asia and Americas. Unfortunately, there is a general consensus that Math. Education Research community has been ineffective for dealing with manifestation of the achievement gap, not only in US but also in other advanced countries of Europe and Asia. Thus the effective techniques and approaches developed in the Bronx, one of the four most underserved areas in US may address needs of student in similar socio/economic/ethnic environments as for example in Tamil Nadu, India (Chapters 2.2 and 5.3).

The importance of the teaching-research community of practice which has existed in the Bronx community colleges of CUNY has been indicated in many chapters of the volume as "thinking aside" comments mostly. However closer reflection upon instances of documented here interactions between different members of the TR team of the Bronx suggests that the central benefit of the community's collaborative effort is in the mutual "cross-fertilization" of ideas grounded in members' different theoretical views and practical experiences. That interaction together with JiTR method is the background of thinking technology, which leads to Stenhouse TR acts (Chapter 1.1). It also is the source for the creation of the language of the community composed of shared meanings. At the same time, within that common way of thinking each member develops its own theme in that shared space, which create interesting conceptual undercurrents throughout the discourse of the volume.

The examples of work of the TR community we encounter in Chapter 2.4, three teachers simultaneously leading the class of remedial arithmetic, as well as in Chapters 4.2–4.5 and 5.2, Czarnocha, Dias and Baker collaborate as a team of teacher researchers reviewing the rate sequence of Czarnocha and the lesson plans of Dias on rates as well as the proportion lesson given by Baker. This is an example of the teaching research modelled on the Chinese teaching research Keli method. The three professors acting as teachers develop and teach the same material in accordance with their understanding of educational theory. Then the group reflects upon the outcome and provided feedback for one another. After such a cycle of implementation, reflection and revision, the results are analysed through the lens of a conceptual framework built upon creativity and learning theories.

Prabhu argued forcefully for a creative learning environment to support student transition from habits of failure to excellence. The learning community formed around its nucleus of Czarnocha, Prabhu, Dias and Baker proved to be a fertile environment for cycles of teaching research i.e. implementation, assessment-reflection and refinement or revision ideas of what worked and what did not. This has led to an understanding of the importance of having an attitude of inquiry, a willing to observe, reflect with others and being open to change that is at the essence of both a teacher and a researcher, as well as a teacher-researcher in establishing a creative learning environment in one's classroom and with one's colleagues.

The following two chapters paint the picture of TR community expanding beyond its birthplace that is mathematics into Mathematics/ESL interphase as well as into triple-phase of Mathematics/English/Freshman seminar. Both show the impact of mathematics learning upon, traditionally, outside domains.

Chapter 5.1 belongs to the series of teaching-research reports from the mathematics/English interphase – an important bisociative framework, which however, in the context of bilinguality of our student population, is also very political. The theme of ESL and Spanish/English bilinguality became more difficult to study and to impact given the political environment in NYC since the turn of the century.

Algebra/ESL teaching experiment brought forth quite unexpected yet statistically significant impact of learning algebra upon learning English as a Second Language. The impact was noted in the long term essay writing by the experimental cohort, which had shown remarkable increase of coherence and cohesiveness in students' final drafts over drafts of similar thematically essays written by previous cohorts of the language instructor. The increase of coherence was mediated by the 16% increase of all connectors and subordinating clauses. The role of these particles of language is to form connections beween diferent facts and concepts. Formation of connections between ideas results in the increase of coherence of a piece of writing. To understand the process more clearly we bring back a significant result of Chapter 4.10 which agreed with observation of Pugalee (2004) thesis that written conceptual thought focuses students on metacognitive thought processes. Together with observation of Bossé and Faulconer (2008) that writing in mathematics may not necessarily result in higher scores on standardized tests *but will focus student attention to connections between mathematical objects and procedures*, we are led to the following conjecture.

Since the schema of algebraic concepts, that is the network of relationships between them is much more precise than the schema organization of language on the level of Basic Competence of Cummins (1980), working in language with mathematical concepts lends itself to rapid syntactical development, meaningful because based on mathematical understanding. That development manifested on the level of connectors allows the student to organize her thinking in writing with increased coherence and cohesiveness of the written expression. We conjecture here the existence of algebra ZPD which is on a higher, conceptually, level than ZPD of student's syntactical component of language development, which together form *relative ZPD (rZPD)* between algebra and language. The conjecture is in agreement with Krashen's *input hypothesis*, which states that learners progress in their knowledge of the language when they comprehend language input that is slightly more advanced than their current level. Krashen called this level of input "i+1", where "i" is the language input and "+1" is the next stage of language acquisition (Krashen, 1977).

We conjecture also existence of a reverse relationship as well that is the situation when language development is at a higher level, relatively, than the development of the relevant mathematical concept of the student. Then mathematics writing will be of significant help in the process of understanding of mathematics. While contemporary research literature certainly confirms the usefulness of writing for the development of mathematical thinking its understanding in terms of interaction between ZPD's of different nature is missing.

Chapter 5.2 expands the Mathematics/English language TR framework to Mathematics/English/Freshman seminar framework. It's interesting in several aspects:

- Its theme, Part of Whole is borrowed from the definition of a fraction and properly generalized to the whole learning community so that its meaning is found as much in the developmental English as in the Freshman seminar.
- This two cycle TE approached from socio-cultural perspective is a very good example of developing the ability to generalize across different domains – a possible bisociative framework.
- It reflects upon the development of meaning of education for the Bronx student as an underserved student population.
- It brings in concerns and methods Vrunda Prabhu was addressing in Unit 2, and in particular, the didactic contract she introduced in her repertoire of motivational means.

The chapter deepens our understanding presented in Chapter 1.3 of the nature of difficulties underserved student population has been experiencing in fitting into the educational system as well as the role TR/NYCity model can play in overcoming them.

The last two chapters closing the unit address professional development of TR communities in two different socio/ethnical domains.

Chapter 5.3 reports from professional development of teachers of mathematics in Dalit villages of Tamil Nadu, India.

The report contains the information of a new concept within TR practice, that of teaching-action-research (TAR). TAR is the integration of classroom teaching-research with action research addressing the issues in the village community which negatively affect learning of students in their schools. It was formulated and implemented in several TR cycles at the request of local grassroots organizers and it provides a new model of community involvement in education of its child. The second direction of professional development was directed to women, section Focus: Women Tamil Nadu reports from the application of TR/NYCity model to literacy campaign, who were embarrassed to have to sign their different documents with a thumb. The impact upon self-assertion of women was dramatic: "Now we know we can draw. We like your methods."

The chapter strongly relates to and provides evidence for Vrunda Prabhu's considerations of artefact generalization so that it could be applied in classrooms beyond its initial development in Chapter 2.1.

Chapter 5.4 investigates process of development of the teacher-researcher. It is based on the international project supported by the Socrates Comenius 2.1 grant of the European Commission in 2005–2008, called Professional Development of Teacher-Researchers. It represents a direction for systematic involvement of teachers into teaching-research practice. However, the methodology used in PDTR has to be refined for its next cycle to account for its new bisociative nature and the methods of facilitation of Aha! moments.

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BRONISLAW CZARNOCHA

5.1. ALGEBRA/ENGLISH AS A SECOND LANGUAGE (ESL) TEACHING EXPERIMENT

INTRODUCTION

A teaching experiment in correlating the instruction of courses in Elementary Algebra and Intermediate ESL is described whose results suggest a measurable transfer of thought organization from algebraic thinking into written natural English. It is shown that a proper context to situate this new effect is (1) the Zone of Proximal Development (ZPD) of L. Vygotsky and (2) a new concept of the "Relative ZPD" characterizing the relationship between the ZPD of arithmetic/algebra and ZPD of native/foreign language. The chapter is the amplification of the Czarnocha and Prabhu (2002) paper with the same title.

The relationship between teaching mathematics and English as a Second Language (ESL) has been the topic of many papers and presentations (Anderson, 1982; Birken, 1989; Connolly, 1989; Luria & Yudovich, 1971). Yet the literature and research on the subject suffer from several shortcomings. First, the majority of the research deals with the role of language in learning mathematics, leaving the reciprocal relationship, that addresses the influence of learning mathematics on the development of language, almost totally unexplored. Furthermore, while several benefits of writing as an instructional tool in teaching mathematics, such as better understanding of conceptual relationships (Birken, 1989) or the facilitation of "personal ownership" of knowledge (Connolly, 1989; Mett, 1998) have been proposed, there has been, until recently, little evidence to explicitly demonstrate these benefits (Powell & Lopez, 1989). Finally, there is a relative absence of theoretical considerations that could provide a context in which to properly situate the reciprocal relationship between the development of mathematical understanding and mastery of language.

ESL related literature presents us with more or less the same situation and focuses on the role of the mathematics instructor as "a teacher of the language needed to learn mathematical concepts and skills" (Cummins, 1980). The methodology of classroom practice based on this principle was formulated in (Dale & Cuevas, 1987). An important theoretical distinction in the area of second language acquisition has been introduced by Cummins in (Cummins, 1980), who asserted that the process of language acquisition has at least two distinct levels: (1) the Basic Interpersonal

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 455–473.

Language Competency (BILC) level of everyday use, and (2) the Cognitive Academic Language Proficiency (CALP) level.

This presentation addresses the shortcomings listed above. A brief discussion of certain ideas of Vygotsky in (Vygotsky, 1986) outlines a context in which the relationship between mathematics and language can be situated. This is followed by a new and interesting result obtained during a teaching experiment at CUNY's Hostos Community College, in which an Elementary Algebra course was pedagogically linked with an ESL course. The findings of this experiment suggest a potentially powerful influence of mathematical reasoning on the development of descriptive writing.

THEORETICAL BACKGROUND

The existing literature contains sporadic hints about the relationship between mathematical understanding and the acquisition of language. Recognizing the similarities between writing skills and problem-solving skills, as pointed out by Kenyon in (Kenyon, 1989) this relationship can be appreciated by the necessity of mastery of a common set of problem-solving strategies. This point of view, that doesn't take into account the peculiarities of each of the disciplines, is strongly supported by Anderson's Adaptive Control of Thought theory (Anderson, 1982).

A point of view that gives justice to the richness of relationships between thought and language can be found in some early works of Vygotsky (Vygotsky, 1986). Following Vygotsky, thought and language exist in a "reciprocal relationship of development" (Kozulin, 1986). Vygotsky writes, "Communication presupposes generalization … and generalization … becomes possible in the course of communication" (Vygotsky, 1986). In other words, in order to communicate, we need to think; and in order to think, we need to communicate. Such a view opens, in a very natural way, the possibility that thought, in our case, more specifically, mathematical thought, has the potential to shape natural language. One of the ways through which this process can take place is across the Zone of Proximal Development (ZPD) (Vygotsky, 1986).

The ZPD arises in Vygotsky's theory through his distinction between spontaneous and scientific concepts. It represents the depth to which an individual student can develop, with expert help, his or her spontaneous concepts concerning a particular task or problem, as opposed to his ability to do it alone.

Valsiner had noted that the development of the ZPD can be fostered even further if the environment is structured in a way that leads the student to use elements that are new and yet unfamiliar, but are reachable from his or her ZPD (Valsiner, 1993). One of the essential characteristics of the upper level of the ZPD, as compared to the level of the corresponding spontaneous concepts, is its higher degree of systemic structure. In the experiment explored here, the abstract character of elementary algebra had created exactly that type of ZPD with respect to the "spontaneous" level of natural English.

EXPERIMENTAL REALIZATION

To confirm Vygotsky's highly dialectical view one would need to clearly detect the presence of two different directions of developmental progression: the acquisition of the English language under the influence of mathematical thinking, and the acquisition of mathematical understanding under the influence of a sufficient grasp of the English language. While the main topic of the current discussion is the first of the two directions, we note that the importance of the second has been confirmed, for the first time, in a recent experiment by Wahlberg (Wahlberg, 1998). Measuring the level of students' understanding of calculus when assisted by a systemic incorporation of essay writing, she observed a substantially higher increase in the experimental group as compared to the control group.

ELEMENTARY ALGEBRA/INTERMEDIATE ESL TEACHING EXPERIMENT

The general goal of the ESL sequence at Hostos is to develop what Cummins calls the Cognitive Academic Language Proficiency (Cummins, 1980), and, what Vygotsky calls the language of "scientific concepts". Our experiment had two goals: to see how far algebra can help in that process, and to investigate the cognitive correlation in the acquisition of both. More precisely, the questions of the teaching experiment stated above were translated into the following goals:

- to formulate a series of tested instructional strategies for teaching English and improving critical thinking skills within both the mathematics and ESL courses;
- to analyse the degree to which the syllabi of the content courses need to be modified to incorporate language instruction;
- outline major problems encountered during the interdisciplinary collaboration;
- understand the learning process of English acquisition through the teaching of math in the context of our student population;
- to identify a series of hypotheses concerning the cognitive relationship between learning English and mathematics by students whose primary language is not English.

METHODOLOGY

A group of seventeen students was enrolled in an intermediate ESL class and in a remedial Elementary Algebra class taught in English. In the previous semester, these students passed the second lowest level ESL course as well as the first remedial mathematics course (Basic Arithmetic). The Algebra class was the only class they were taking in English, and, thus, constituted their only exposure to academic English. Although the classes were separate, the communication between the instructors was frequent and substantive, involving weekly meetings, exchange of materials, and mutual class visits. The methodology of the experiment was based on two assumptions. First, since we were interested in the influence of the algebraic language upon the natural one, we needed to verbalize the symbolic algebraic language to the highest possible degree. That meant we needed to make the symbolic notation of algebra explicit in speech and/or writing – to verbalize the procedural steps and the content of algebraic thinking. Second, these elements, having been made explicit in their algebraic context, needed to be transferred into the context of the ESL class, both on the semantic and the grammatical level. As a result, student discussions in the Algebra class often, by design, involved a level of academic discourse somewhat above the students' capacity at the given time. We hypothesized that it is this increased level of student effort to communicate the comparatively abstract mathematical ideas in English is at the root of their eventual linguistic improvement. At the same time, the ESL class deliberately involved discussions of the linguistic peculiarities of algebraic language, such as the role of word order and sentence structure with the aim of improving their mathematical reasoning skills. Below are examples of specific instructional strategies in both classes. A special attention was paid to the careful observation of cognitive difficulties experienced by the students during the actual process of learning. This was a way to reflect upon, improve, and increase our understanding of how teaching and learning takes place. In particular, we were interested in understanding the details of the developmental learning process in the context of collaborative instruction of mathematics and English. One of our main goals in using this methodology was to create a profile of a teacher-researcher paradigm at Hostos Community College through which:

- The teacher becomes engaged in critical reflection and experimentation of his/her own instructional practices,
- The instructor's teaching practices are critically evaluated in relation to the available theoretical research, and
- The theoretical research is critically applied in the instructional context for further questioning and development.

We believed that this new profile would help us gather data and more successfully understand the challenges concerning the acquisition of English through content areas. Furthermore, it would allow for the formation of a supportive intellectual structure to address and to solve other present and future pedagogical issues such as shortening the remediation time of students enrolled in Mathematics and English courses.

NEW ALGEBRA INSTRUCTIONAL STRATEGIES

Verbalization of Algebraic Procedures

Example – Solving linear equations:

Solve $2x + 5 - 5 = 9 - 5$ for x.

Solution	Steps (to be explicitly written by students)
$2x = 4$ $\frac{2x}{2} = \frac{4}{2}$ $x = 2$	*First, I add –5 to both sides of the equation in order to eliminate the +5 on the left side.* *Second, I cancel the opposite numbers and add the like terms.* *Third, I divide both sides by 2 in order to have X alone.* *The answer is X = 2*

Explication of Algebraic Symbolism Through Writing Paragraphs

Example:

(a) Write a paragraph explaining the difference between 3×5 and 5×3. What does it mean to you that 5×3 = 3×5?
(b) What is the difference in the meaning of the equality symbol in the following two expressions?

$$3\times16 = 48 \text{ and } x + 5 = 12$$

Analysis of Algebraic Rules and Principles

Example:

(a) Compare the rule for the addition of signed numbers with different signs with the rule for the multiplication of signed numbers with different signs.
(b) Write a paragraph addressed to a fellow student, who missed a couple of classes, explaining how to solve the problem below. Clearly verbalize to him/her the order of steps in the procedure, warning against any possible errors and reminding him/her of the rules, which justify your steps in the solution.

Simplify

$-2[2(3x - 5y) -3(y - x)] -4(2x + 3y)$

Readings and Linguistically Adjusted Word Problems

One of the most important problems that became evident was the absence of sufficient readings about mathematics suitable for our students. To address this, a series of special word problems were developed. They were made out of paragraphs from stories read in the ESL class. Additional explanatory paragraphs and questions about mathematics itself were $2x + 1 = 5$ developed to assist student learning. For instance:

- What does it mean to you that $3×5 = 5×3$?
- How do you differentiate the operation of subtraction from addition?
- What is the difference in the meaning of the equality sign in $2^2 \times 3 = 12$ and $2x + 1 = 5$?

These paragraphs of explanations were motivated by the advice from the mathematics educators L. Steffe and Smock (1975), in their work entitled *Model for Learning and Teaching Mathematics*. These authors proposed that a "…carefully arranged interplay between spoken words which symbolize a mathematical concept and the set of actions performed in the process of constructing a tangible representation of the concept should be mandated … A mathematical vocabulary should be developed… to explicate and provide embodiments for the concept."

Example 1 – A literary word problem from a paragraph from *The Pearl* by John Steinbeck:

> Kino awakened in the near dark. The stars still shone and the day had drawn only a pale wash of light in the lower sky to the east. The roosters had been crowing for some time, and the early pigs were already beginning their ceaseless turning of twigs and bits of wood to see whether anything to eat had been overlooked. Outside the brush house in the tuna clump, a covey of little birds chattered and flurried with their wings.
>
> Kino's eyes opened, and he looked first at the lightening rectangle that was the door and then he looked at the hanging box where Coyotito slept. His eyes wandered again to the rectangle of the door, to its familiar elongated shape. They doors were much shorter in width than in the height. Kino knew their dimensions by heart because it was him who made the doors when he and his wife, Juana, moved in here. The height was exactly three times the width, which made it a tall and narrow entrance. Sometimes, though rarely, Juana would cover the entrance with her long blue shawl whose lengths of four sides added to 32 units. The shawl fit exactly the opening of the door. He turned his head to Juana who lay beside him on the mat, the blue head shawl over her nose and over her breasts and around the small of her back. What were the dimensions of the shawl?

Example 2 – An explanatory paragraph utilizing a house cleaning analogy to demonstrate commutativity:

> The product of your work at home is the cleanliness of your house. If you do it carefully and with thought, you know that some actions must be taken before other actions. For example, you *have to* dust the surfaces before you clean the floor. Otherwise, you will have to clean your floor twice. This means that the action *dust the surfaces* and the action *clean the floor* are not commutative. The order in which you do them matters for the efficiency of your work. On the other hand, if you have two bedrooms, both coming out into the hall, it doesn't matter which you clean first. The action *clean the first bedroom* commutes with the action *clean the second room*. Cleaning the two bedrooms is like multiplication. It doesn't matter which number (or bedroom) comes first.

PHILOSOPHY OF TEACHING ESL BY THE ESL INSTRUCTOR – A NARRATIVE

As a reaction to both behaviouristic approaches to language teaching and to the traditional way I learned English in Spain, I, like many educators, argue that the teaching of linguistic forms and structures by themselves is inadequate in learning how to communicate in the target language. It is well known that we use language primarily to interact with one another in meaningful contexts. Language is the most precise means that we have to express feelings and opinions and to share information and experiences with each other. Therefore, I believe that the fundamental and final goal of second language acquisition is to become communicatively competent. That is to say, learners should be able to express and negotiate meanings with one another within everyday situations and social and academic environments.

I teach English holistically in my class. This means that I do not use the English language to teach isolated grammatical forms or structures. I use the English language to share and communicate interesting, challenging, and rich content, which I make sure relates to the students' own experiences. Through this content, students become interested in participating and learn language in meaningful context. Then, they are able to learn specific structures and vocabulary in context, as well as to communicate with one another successfully. I integrate the four language skills: reading, listening, speaking, and writing, and I try to provide as many different learning modalities (text, graphics, pictures, sound, video, computer technology) as possible so that every student can choose the modality he/she feels most comfortable with. I also provide many opportunities for oral and written production, make sure students get enough challenging input in listening and reading, and try to make them feel relaxed in the class. I encourage students to think about how they best learn as I want them to become aware of their individual learning processes and to take responsibility for their own learning. Content, meaning, and fluency of ideas and thoughts always come first in my class. Accuracy and correctness come afterwards. The integration of algebra in my ESL class has simply reinforced and empowered my beliefs in meaningful, content-oriented ESL teaching and learning.

New ESL Instructional Strategies

The goal of the mathematics related ESL exercises was to extend the meaning and application of algebraic words and concepts into natural English. In order to integrate the mathematical vocabulary and structures in meaningful contexts in my ESL class, I used the following five teaching strategies throughout the semester:

Strategy #1 Written Math Summaries with Activities

In the first place, I spent a considerable amount of time reading the textbook for Elementary Algebra and I familiarized myself with the mathematical concepts and terminology. Later, I wrote short summaries of each of the units covered in the math class syllabus. They included:

1. Addition of signed numbers
2. Subtraction of signed numbers
3. Multiplication and exponent power
4. Evaluating variable expressions and grouping symbols
5. Solving linear equations
6. Word problems
7. The rules of exponents
8. The rules of negative exponents
9. Addition and subtraction of polynomials
10. Multiplication of polynomials
11. Factorization
12. Simplification of algebraic fractions

These summaries were primarily used in my ESL class as reading assignments. They included reading comprehension questions, oral activities for group work or class discussions, and short writing assignments. In creating these summaries and the accompanying activities, I had three goals in mind: (1) to simplify the language structures in the math textbook and make them more accessible to ESL students, (2) to integrate the specific grammatical structures taught in the intermediate ESL course into the context of mathematics, and, finally, (3) to insure that the four language skills (reading, listening, speaking and writing) were all integrated into this collaborative teaching effort. These summaries were developed during the summer of 1997 and were not part of the research carried out during the spring of 1997. Below is an example of such a summary presented as a reading exercise on the topic of addition of signed numbers:

> In algebra, we talk about numbers in different terms. For instance, *integers* are numbers such as 1, 2, 3, 0, –1, –2, –3, etc. They can also be written as fractions. For example, $3 = \frac{3}{1}$ *Rational numbers* can be expressed in decimal forms such

as 1.5, 0.5, 3.55, etc. or as fractions such as $\frac{5}{7}$, $\frac{3}{4}$, $-\frac{3}{4}$, $-\frac{7}{5}$, $\frac{3}{1}$, etc., as long as the denominator is not zero. Therefore, rational numbers can be written as one integer divided by another integer. Since integers can also be written as fractions, we can say that all integers are rational numbers. However, not all rational numbers are integers.

Integers and rational numbers can either be *positive* or *negative* numbers. Zero ("0") is at the middle of the *number line*. Positive numbers are to the right of zero on the line and negative numbers are to the left of zero on the line. When you owe money, your money is represented as negative numbers. When you make or earn money, you have a positive amount of money. Also, when you read the thermometer, the temperature below zero is expressed as a negative number. The temperature above zero is expressed as a positive number. Positive and negative numbers are *opposite* numbers. This means that they have the same *magnitude*, but different signs. They can be represented on different parts of the line.

When we add two signed numbers with the same sign, we add the magnitudes of the numbers and we keep the common sign in the answer. For instance, you have $100 (or +100) in the bank. Then, you deposit $30 (or +30) more. You'll have a total of $130 (+130) in the bank. This money is positive because it is yours. How about the opposite? After that, you borrow $100 from your father (–100) because you need to buy some books. Then, you borrow $30 (–30) more from a friend because you want to go to the movies on Saturday evening. You'll end up owing $130 (–130). This money is negative because you have to give it back. It does not belong to you.

When you add two signed numbers with a different sign, the rule is to find the difference between the larger magnitude and the smaller. Then, the sign of the number having the larger magnitude will be kept in the answer. For example, you make a deposit of $150 (+150) into your bank account one afternoon. Then, you go shopping and spend $50 (–50). This money is paid with your credit card and is directly taken from your bank account. You'll end up having $100 (+100) left.

Reading Comprehension Questions

Please answer the following questions in complete sentences:

1. What is the difference between an integer and a rational number?
2. What is the difference between a positive and a negative number?
3. What do you need to do in order to add two signed numbers with the same sign?
4. What do you need to do in order to add two signed numbers with a different sign?

Activities for Group Work or Class Discussion

1. Make a list of everyday situations where you can use positive or negative numbers (For example, reading the temperature).
2. Create a real life situation where you have to add three signed numbers with a different sign. Solve the problem for the situation.

Topics for Writing

Please write a clear and well-developed paragraph on each of the following topics. Provide specific examples where appropriate.

1. Discuss why all integers are rational numbers, but not all rational numbers are integers.
2. Discuss when the temperature on a thermometer can be expressed as a positive number and when it can be expressed as a negative number.

By reading and discussing the math summaries and by doing the corresponding activities, the ESL students had an opportunity to use the language of mathematics in oral and written form in the ESL class. This was particularly beneficial to my students because it gave them extra practice on manipulating math concepts and on talking and thinking mathematically. They were forced to talk and think more abstractly. I feel that this practice strengthened and enhanced the students' critical thinking processes.

Strategy #2 Mathematics Vocabulary in Non-Mathematical Contexts

My goal was to integrate the math vocabulary in my ESL class expanding it into other linguistic contexts or domains. Once the students had grasped the vocabulary in its mathematical context, I tried to pull out those words and phrases from that context, showing the students that these words were also used in other everyday situations. I did this by creating short passages for reading or listening activities where these vocabulary words were used. At the same time, I tried to integrate passages with the grammatical structures that had to be covered in the intermediate ESL course. I also provided lists of sentences in which these words were used differently, with other meanings. And finally, I created ample opportunities for students to create their own sentences, using these words in oral and written production. Below is an example of such a reading comprehension passage, addressing the concepts of simple present tense and new vocabulary:

> Mary counts her money five times every day. Today she realizes she has only ten dollars and sixty-two cents. That is all. She wants to buy a birthday present for her sister Anne, but she thinks she cannot buy much with the money she has. Finally, she decides that she is going to borrow some money from her aunt.

How much money is she going to borrow? She asks her aunt, “Could you lend me ten dollars?” Mary tells her aunt that she wants to buy a beautiful present, which costs $25 for her sister. Mary’s aunt immediately asks her, “What do you do with the money you earn every week? Do you spend it all? Don’t you save any?” Mary explains to her aunt that she doesn’t make much money at her job. She says that she earns very little each week. She feels embarrassed and says that living in NY is expensive. She says that she has a lot of expenses every month. “I have to pay so many bills like the rent, the phone, and the electricity bill every month and then I need money for food and clothes. Sometimes I can’t make ends meet.” Finally, Mary’s aunt agrees to lend her $10. Mary counts her money again. How much money does she have now? And how much money does she owe to her aunt now?

1. Underline the verbs in the passage that are in the Simple Present Tense.
2. Explain how the Simple Present Tense is used in affirmative and negative sentences. Give an example of an affirmative and negative sentence.
3. When do we use the Simple Present Tense in English?

Highlighted Vocabulary Words

Lend/Borrow/Earn/Save/Spend/Owe/Own

Borrow: We *borrow* books from the library every month.
He *borrows* a pen from me every day.
He has a loan to pay back because he *borrowed* a considerable amount of money from the bank.

Lend: She *lent* me her copy of the article to read.
He always *lends* us his car when ours breaks down.
Could you *lend* me your pen for a second, please?

Earn: He *earns* a lot of respect every time he speaks in public.
He *earned* $3000 by writing short stories.
They all *earned* their bachelor’s degree last year.

Save: We should all learn to *save*.
We will *save* time if we drive.
Please *save* me! I am in danger!

Spend: He *spent* the afternoon reading a newspaper.
They *spend* too much money on clothes every year.
He *spent* three years in prison.

Owe: He *owes* me a favour.
They *owe* her $20 for her work.
They *owe* loyalty to their country.

Own: Who *owns* that beautiful house over there?
He *owns* a lot of property.
He always drives the car that he *owns*.

Simple Present Tense Vocabulary Practice

Answer the following questions about yourself using these vocabulary words:
Lend/Borrow/Earn/Save/Spend/Owe/Own

1. How often do you borrow money from someone?
2. How many science degrees do you own?
3. Why do we save money?
4. What are some things besides money that you save?
5. How do you spend your free time?
6. Do you spend a lot of money on your summer vacation every year?
7. Do you know anybody who owes a lot of money?
8. Approximately, how much does this person owe?
9. Is it important for you to earn a lot of money? Why?
10. Do you owe favours to some people? What kind of favours?
11. What is the best way to save money?

Listening Comprehension: Simple Present Tense and New Vocabulary

Answer the following questions based on what you heard and understood. Remember to respond with a complete sentence.

1. Why does Pepe go to the Dominican Republic?
2. How does Pepe pay for his airplane ticket?
3. How much money does Pepe spend on shopping and eating out?
4. What does Pepe realize when he gets back to New York?
5. What problem does Pepe have with his computer printer?
6. What is the best advice you can give to Pepe so that he can solve his problem?

The questions above refer to the following transcript:

Pepe goes on vacation to the Dominican Republic and spends $600 on his airplane ticket. He buys his ticket on the phone and adds this amount of money to his credit card. He already owes $200 on this credit card. As soon as he gets to Santo Domingo, he goes shopping and eats out with his friends. He pays all these bills with his credit card again. He spends approximately $250. He spends all this money with his credit card, and his debt increases.

When he gets to New York, he realizes that he has charged too much money on his credit card. He thinks he will not be able to pay back all the money he owes at once. He tries to save some money when he receives his next paycheck, but he realizes that the money he makes is just enough to pay for his rent, his utility bills, and his monthly food expenses. All of a sudden, his computer printer breaks down and he is needs to buy a new printer, which costs $350. He has to use his credit card again because he doesn't have enough cash.

Pepe is in trouble because he has spent more money than he makes and he does not know how he can pay back his debt. Could you think of a good solution to help him?

Strategy #3. Expressing Similar Mathematical Concept in Different Ways in English

Example (Solving Linear Equations)

Instructions: *Fill in the blanks with the appropriate word*. (The words in parentheses are not given to the students and appear as blanks)

$$x + 16 = 20$$

- *Sixteen (added) to an (unknown) number (is) twenty.*
- *If an (unknown) number is (increased) by sixteen, the (result) is twenty.*
- *The (sum) of an (unknown) number (and) sixteen (equals) twenty.*
- *Sixteen (more) than an (unknown) number is (equal) to twenty.*

These exercises allowed students to understand how a specific Math idea or concept could be phrased in many different ways in English. This provoked opportunities for students to think and internalize Basic English sentence structure.

Strategy #4. Word Order Exercises

Since Math sentences sometimes tend to be very complex and their word order is quite fixed, a lot of exercises were done on word order and sentence structure in the Math context. This allowed for serious thinking about what was said and how it was phrased. Students learned that the way concepts and numbers are put together in Mathematical language is essential for the understanding of that particular operation. As a consequence of paying so much attention to word order in Math, students became very sensitive to word order in English. For instance, after learning the logical order and steps required in order to apply math operations, students were better able to organize their writing in English.

Instructions: *Put the following words and expressions (and punctuation marks) into the right order.*

1. *temperature / in / by / +10 / The / decreases / evening / the / degrees*
2. *the numbers / multiply / He / the parentheses / the number / needs to / in the / outside of / the parentheses / by*
3. *much / How / you / your sister / to / money / owe / do / ?*
4. *makes / of / John / same / as / do / money / every month / the / I / amount*
5. *If / perform / the / using / you / numbers 12 and 4 / you'll / multiplication / the number 48 / get*

Serious thinking and discussion about what was said and how it was phrased accompanied these exercises. Students learned that the way concepts and numbers are put together in the algebraic language is essential for understanding algebraic operations. By paying attention to the word order in algebra, students became sensitive to word order in English.

Strategy #5. Editing Exercises

Finally, another important teaching strategy I used was the creation of editing exercises, based on students' writing about math concepts. This was extremely valuable because it allowed for meaningful class discussions about the math concepts as well as about the use of certain grammatical items and vocabulary expressions. Students' ideas were compared and contrasted, and little by little the students became more aware, and more sensitive to what and how they were expressing themselves in math and English.

Instructions: *Please correct the following paragraphs, written by different students, not only for correct ideas but also for mistakes involving any of the grammar rules studied so far.*

$(x^a)^b$ — **Student Statement:** *I think that raise a power to another, I need to put the variable and then multiply the exponent.*

$(xy)^a$ — **Student Statement 1:** *When you have two variable in the parenthesis and you raise to any power. You have to multiply each variable with same power.*

— **Student Statement 2:** *I need to get each variable to the power separately. I need to multiply the variable with the raise power.*

— **Student Statement 3:** *It is when you have two variable raising to a power. I solve this raising separately each variable to the power. For example, I do this because I multiply the variable by the exponent.*

ASSESSMENT

A term-long (6-week) essay, written on a word processor, with 3 drafts discussed with the instructor was used as an assessment tool. The topic was *In Between Two*

Cultures; the students were supposed to compare and contrast their life experience in the Dominican Republic and in New York City.

Data Collection and Analysis

As has been stated above, the teaching experiment had two goals: to use algebra to help in the development of natural English, and to investigate the possibility of a cognitive relationship between the acquisition of both. Vygotsky suggests such a possibility when he asserts: "…one might say that the knowledge of the foreign language stands to that of the native one in the same way as knowledge of algebra stands to knowledge of arithmetic… There are serious grounds for believing that similar relations do exist between spontaneous and academic concepts" (Vygotsky, p. 160).

For the purpose of the present discussion, the main tool of analysis were the term-long essays on the topic *In Between Two Cultures* that the students wrote in the course of the semester. The process of writing was important because:

> Written speech assumes much slower, repeated mediating analysis and synthesis, which makes it not only possible to develop the required thought, but even to revert to its earlier stages, thus transforming the sequential chain of connections in a simultaneous, self-reviewing structure. Written speech thus represents a new and powerful instrument of thought. (Luria & Yudovich, 1971)

To assess the changes in the students' written mastery of English, we first used the holistic assessment of the ESL instructor – a standard way of judging student essays in English courses. Next, we translated this judgment into syntactic components. Finally, we compared these with the corresponding components in the essays of a control group. As our control group, we chose a past class taught by the same ESL instructor. The topic of the essay of the control group was *Our Family Conflicts*. This topic was judged to be the closest in meaning to the topic *In Between the Two Cultures*, assigned to the experimental group.

The judgment of the ESL instructor after reading all of the essays of the experimental group was that they were more cohesive. As cohesiveness is closely related to the use of conjunctions—words such as "because", "yet", "although", etc., all the conjunctions used by all students in their essay were categorized, counted, and averaged by the total number of submitted pages by all students in each of the two groups (44 in the experimental and 45 in the control). The results were compared with the corresponding numbers from the control group. Our conclusion was that, on average, there was a 16% increase in the number of connectors and subordinating clauses in the essays of the experimental group. This confirmed the ESL instructor's assessment that the term-long essays of the experimental group were more cohesive than the essays of the students who did not participate in the instructional link under discussion.

*Examples of conjunctions for the different categories and types for Table 1:
Subordinating Conjunctions:
Time: when (*the most common one)* / as soon as / before / after / etc.
Cause: because (*the most common one*) / as / etc.
Purpose: in order that / so that / etc.
Condition: if / unless / etc.
Contrast: although / even though / etc.
Place: wherever / etc.
Transitional Words or Connectors:
Time: first / second / finally / etc.
Cause: therefore / as a result / in consequence / because of this / etc.
Contrast: however / nevertheless / on the contrary / etc.
Addition: in addition / moreover / also / etc.

Table 1. Comparison of the use of connectors in student essays

*Category**	*Type**	*Experimental (Raw #/ Average per page)*	*Control*	*% Difference*
Time	**Total**	228/**5.18**	217/**4.82**	**+ 7%**
	Subordination (When)	114/**2.59**	106/**2.36**	**+ 10%**
	Others	38/**0.86**	53/**1.18**	**– 27%**
	Connectors	66/**1.50**	56/**1.24**	**+ 21%**
Cause	**Total**	181/**4.11**	152/**3.38**	**+ 22%**
	Subordination (Because)	140/**3.18**	118/**2.62**	**+ 21%**
	Others	20/**0.45**	20/**0.44**	**+ 2%**
	Connectors	21/**0.48**	14/**0.31**	**+ 53%**
Purpose	**Total**	29/**0.66**	20/**0.44**	**+ 48%**
Condition	Total	21/**0.48**	28/**0.62**	**- 23%**
Contrast	Total	29/**0.66**	22/**0.49**	**+ 35%**
	Subordination	14/**0.32**	18/**0.40**	**– 20%**
	Connectors	13/**0.30**	4/**0.09**	**+ 232%**
Place	**Total**	19/**0.43**	3/**0.07**	**+ 548%**
Addition	**Total**	46/**1.05**	38/**0.84**	**+ 24%**
TOTAL		553/**12.57**	489/**10.87**	**+ 16%**

Using a one-tailed proportion difference hypothesis test (refer to Table 3), the above data shows that the passing rate for the experimental group was statistically significantly greater than the passing rate for the control group with a ***p-value less than*** **0.005**. As is shown, there was a 23% average increase in the passing rates

Table 2. Passing rates in the intermediate ESL classes taught by the ESL professor over a period of eight semesters

Semester	*Withdrawn*	*Failed*	*Passed*	*Passing rate (as a % of those who completed the course)*
Spring 94	0	6	23	79%
Fall 94	1	19	12	67%
Spring 95	1	19	12	38%
Fall 95	2	13	16	55%
Spring 96	2	12	17	59%
Fall 96	7	9	10	53%
Totals/Average	**13**	**78**	**90**	**53.6%**
Spring 97 (LINKED)	3	3	11	79%
Fall 97 (LINKED)	0	5	15	75%
Totals/Average	**3**	**8**	**26**	**76.5%**
% INCREASE (LINKED – CONVENTIONAL)				**+ 23%**

Table 3. The results of a two sample proportion test using MINITAB

Sample	X (PASS)	N (TOTAL)	Sample Prop. (RATE)
1	26	34	**0.764706**
2	90	168	**0.535714**

Difference = p (1) – p (2)
Estimate for difference: **0.228992**
95% lower bound for difference: 0.0936272
Test for difference = 0 (vs > 0): Z = 2.78 **P-Value = 0.003**

in the linked ESL classes. In a class of 20 students, a 23% increase in the linked ESL class passing rate translates to an average of 4.6 more students passing the class. Throughout the experiment we kept track of the students' performance in their respective mathematics courses and did not observe any changes in the related passing rates. The stability of the passing rates in the corresponding math courses indicates that the linking of Algebra and Intermediate ESL was helpful in teaching English without jeopardizing the standards of the mathematics course.

These measurements provided an independent confirmation of the holistic assessment that the essays written under the influence of algebraic thinking were more cohesive and more thoughtfully written. Despite the novelty of this observation one should not be surprised by it. Algebra, as an abstract area, depends heavily on the relationship between different concepts, ideas and mental actions. Connectors and subordinating clauses are those particles of language that are used to express the relationship between ideas, events or facts; these are words such as "because", "in order to", "finally", "if…then…" They are used to express cause and effect relationships, conditions, reasons, and contrast; thus, they seem to be closely related to what is called a critical (or analytical) mode of thinking. Correct use of connectors determines the organization of ideas within an essay. The increase in the (correct) use of these linguistic tools meant that there was an increase in the number of relationships between ideas, resulting in better conceptual organization expressed by our students in their writing, making it more cohesive. The dense and complex mathematical relationships addressed in the Algebra course, when translated into natural language with the help of connectors, were able to penetrate the simpler language of descriptive writing, and the correlation of the ESL syllabus with the Algebra course induced an increase in the level of thinking effected by the ZPD.

The description of the experiment and its results is based on The Final Report of the ESL/Elementary Algebra Teaching Experiment: Mathematics and Natural Language Acquisition supported by the New Visions Program Grant of CUNY, July 1998 – Merce Pujol, Bronislaw Czarnocha

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5.2. "*JUST TELL US THE FORMULA!*" CO-CONSTRUCTING THE RELEVANCE OF EDUCATION IN FRESHMAN LEARNING COMMUNITIES

INTRODUCTION

Students and teachers alike bring their own individual perceptions of the relevance, intent and proper delivery of education, and the classroom provides a space to negotiate these diverse perceptions. The teaching-research paradigm allows teachers to prescribe certain interventions and improve student performance by systematically integrating these interventions over the course of an academic semester. We argue that students who find relevance in their education are more confident in approaching the problem-solving process and are, ultimately, more successful in all their courses. In this article we explore *how* education becomes relevant for students in two successive learning community cohorts that link three developmental courses: Freshman Orientation and Career Development (OCD), Developmental English, and Elementary Algebra. We achieve this type of student investment by infusing our teaching practices with a socio-cultural approach theoretically informed by Vygotsky, Bruner, Brousseau, Piaget and Garcia, and Sen.

As teachers, we view our teaching practice as an opportunity to create an environment conducive to learning in order to expand our students' intellectual horizons. We concur with Bruner's notion that learning is a process and not an end product; this foundational belief necessitates a constant continuous integration of concepts, readily filling out any existing gaps in notional schemas (Bruner, 1971). As teachers, we are interested in engaging our students and encouraging them to take part in the learning process. Our students, however, often come to our classes with the expectation that teachers are there to *give* them knowledge—subscribing to Freire's "banking model" of education (Freire, 1973). Students demand, "Just tell us how to do the problem!" and claim, "I want to get an A in this class." This conflict of perspectives between teachers and their students creates an ongoing tension in the classroom that must be constantly mutually negotiated in order to co-construct a shared understanding of the relevance of education. We observe that students who learn to view their education as a process that will continue even after the semester ends are more able to deepen their understanding of how education

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 475–486.

can positively affect their lives; this, in turn, impacts their willingness to take risks as well as to make and learn from their mistakes. Here we examine our experiences of making education relevant to students in our freshman learning community at Bronx Community College (BCC), elaborating on the specific challenges we each encounter and the ways in which we have each chosen to address them.

THEORIZING THE RELEVANCE OF EDUCATION: HOW OUR LEARNING COMMUNITY REFLECTS A LARGER SOCIO-CULTURAL MILIEU

Bruner (1996) argues that, "what is done in school must be seen in the broader context of what society intends education to accomplish"; indeed, he explains that "culture shapes the mind and provides the toolkit by which individuals construct worlds and their conceptions of themselves and their powers." Our learning community, a microcosm of the *real world*, is distinct from regular classrooms of roughly thirty students who get to know each other gradually over the course of a semester; rather, our learning community is similar to a small pocket of society that interacts closely with each other. As teachers, we bear witness to the impact of societal norms as well as the effects of public school culture on our students' attitudes towards their education and their resulting academic performance. Our students participate in but do not actually create the educational systems; as teachers, we attempt to create a learning environment that reflects the goals of a quality education, that is, where students fearlessly engage in problem-solving and, under the watchful guidance of their teacher-researchers, develop into confident and independent thinkers. We attempt, in particular, to make the relevance of education transparent to our students. Aronowitz (2008) notes that "the large number of community colleges whose *mission* is now almost exclusively confined to preparing trained workers for the corporations with whom they have developed close relationships." Bruner (1971), however, emphasizes the personal development of the learner rather than the corporations who need the trained workers; he writes, "The imparting of knowledge should be approached from the standpoint of equipping students with the skills that will enable them to achieve personal significance in their lives."

We also wonder if our college students' perceived inability to articulate the relevance of their respective educations may be a learned trait from years of public schooling. Manzo (2008) reports that:

> Educators are still searching for the right blend of academic and developmental strategies for middle school students. More than a decade after a prominent group of middle-grades reformers set out to infuse higher academic standards into what critics deemed the touchy-feely world of middle schools, many teachers are still grappling with ways to motivate students to excel intellectually while helping them adapt to the dramatic physical and emotional changes that come with puberty. That mix of rigor, relevance and responsiveness, experts say, is crucial for guiding students, particularly those most at risk of dropping out, on

> the path to high school graduation and later success. Too many schools serving sixth through ninth graders, however, have yet to find the right prescription for keeping those youngsters engaged at a time when their growing curiosity, independence, and need for the acceptance of their peers may lead them to act out or zone out in school. (Manzo, 2008)

Given the existing academic structure within which students operate, where the *value of their work* may not have been made explicitly evident to them, we wonder about the impact of learning in the college classroom. In his dissertation, *The Role of Utility Value in the Development of Interest and Achievement*, Hulleman "evaluates whether helping students see the value in their coursework contributes to interest and achievement" (Hulleman, 2007). The current classroom situation faced by instructors, as described by Hulleman, would resonate with most instructors: "interest in school tends to decrease over time, with students with lower competence beliefs reporting lower interest and motivation than students with higher competence beliefs" (Jacobs, Lanza, Osgood, Eccles, & Wigfield, 2002; Lepper, Corpus, & Iyengar, 2005). Thus, declining interest could also mean declining performance and commitment to their academic advancement. Hulleman points to the need for students to transition from situational interest towards developing individual interests. While we agree with Hulleman about the close connection between motivation and cognition, we claim that even with the intense efforts of instructors in creating a learning environment in which students can develop the needed individual interests, the instructor's efforts are made difficult by the accepted societal norms and those that have become long-standing habits, such as the absence of independent work and counter-productive behaviours like listening to music, texting, talking, and eating in the classroom.

How can we navigate our students' learning as it currently exists towards the norm of the profession? We employ theoretical support from a variety of sources; Vygotsky's socio-cultural approach and Bruner's approach both support our instructional design (Vygotsky, 1986; Bruner, 1971, 1996). We developed our instructional materials using scaffolding. This maximizes the students' passage from the *spontaneous* concepts the learners bring to the classroom with the *scientific* concepts with which the learners are expected to work. In the classroom learning environment, we create the support needed for the learners' swift and efficacious passage through the Zone of Proximal Development. In the mathematics class, for example, this results in new instructional materials being produced that, in coordination with the concrete/enactive – iconic – symbolic phases of learning, also integrates exercises of written thought to allow language to act as a further aid in the development of students' comprehension of concepts.

Second, how can we effectively hold students accountable for their own learning? We introduced Brousseau's didactic contract into the teaching-research cycles in the fall of 2007 (Brousseau, 1997); incorporating the objective of a game has been helpful in determining and making explicit to students how significant and crucial their participation is to their performance. We also have students monitor their individual progress via Self-Assessment Reports due at the end of the semester. In classroom language, the didactic contract is called a *handshake*, and we seek the *handshake* repeatedly over the course of the semester as a means of attaining excellence.

Third, how do we connect the individual interests of students with the mutual goals of teaching and learning? We focused on the common theme of *Part of Whole* in our fall 2008 learning community. The assignments in each of our three classes covered a range of activities and were geared toward building a schema. The foci, both in the various disciplines and within the same course, ranged to cover the aspects that were diagnosed as missing or in need of amelioration, creating opportunities to see the details of the structure present in all three classes. For example, in the mathematics class, students were shown how to factor an expression that was a difference of two squares. Then, in the first-year OCD course (see Example 1), students closely examined the connection between the submission of an unreadable term paper and the resulting grade. The overarching theme of *Part of Whole* was also personalized as a means of encouraging students to engage in their own creative interests so that they could see themselves as *part of a whole*. Over the course of the semester, each student developed a term paper that incorporated the following five writing exercises: (1) a short introduction of the student to the class; (2) a short essay on the college theme *I Am BCC*; (3) a students' perspective of the research results from the Ohio and Michigan Mathematics and Science Partnership that reported on the low passing rates among third through twelfth graders on a fractions test; (4) a piece on *My Interests*; and (5) a commentary about the federal document, *Tough Choice or Tough Times*, which also provided a venue to inform students' of the mathematical requirements in the various jobs and fields, and to create an additional interest, perhaps, motivated by the attraction of a well-paying job (if the latter was an attraction).

How do we pay attention to the details inherent in the structures of English, Mathematics and OCD classes? How, for example, do we learn to see the fraction we represent as a part relative to the whole? The Part-Whole relationship linking the three courses of the learning community arises from a prior diagnosis of a weak schema (Skemp, 1987) among students in developmental classes. In the sequential nature of mathematics, where building a schema is an essential part of the developmental process, and given that this is not a demonstrated proficiency for our students, the development of the schema takes the form of paying attention to the parts that comprise the whole totality of the different course contexts.

CHALLENGING OBSTACLES: GAUGING THE RELEVANCE OF EDUCATION IN OUR LEARNING COMMUNITY

The authors of *Beyond the Banality of the Mathematics Classroom: Teaching Situations as Objects of Research* point to the advantage of being anchored in theoretical frameworks that allow the classroom to be treated as a unit of analysis. In our learning community, we have chosen to ground our pedagogy in a socio-cultural approach intended to improve learning. Through the creation of the appropriate learning environment and guided by a didactic contract, students create a natural space for the classroom situations to be studied as units of analysis over the course of the semester and across different courses. With the objective of collectively and individually improving learning, students create a shared understanding of their own relevance of education by reflecting upon classroom practices.

In our learning community, we, as community college professors, are simultaneously teachers and researchers who are attempting to change the teaching-learning situation in the classroom. Utilizing the daily classroom situations as units of analysis, our team works across classes with students who need help. Our learning community constitutes a cohort of students who are enrolled in three linked courses: Orientation and Career Development, Developmental English and Developmental Mathematics. Our students' standardized test scores are weak. Many students occasionally will admit that they just marked the answers to be finished with the test; however, on in-class assessment it is clear that the mathematical gaps in our students' backgrounds are quite severe for many. Given the host of compounded difficulties, our teaching-research project naturally includes all the known ways of addressing our students' diagnosed learning difficulties. We clearly distinguish our use of the term *learning difficulties* from its most prevalent use. The abilities of our students are diagnosed as being very high; however, the motivation and desire for learning is seen to be only sporadically present. The wide disparity of difficulties to be addressed are echoed by a range of students' statement like:

- Why should I learn math? I hate it, and I have never been good at it.
- Just tell me what to do—what is the formula?
- Is there not an easy way, a shortcut?
- I don't get it. I get it in class, but then when I go home, I don't know how to start
- Am I even on the right track?

To understand the significance of giving due consideration to steadily working with students on realizing the importance of the relevance of education, it is easiest to understand the classroom dynamic from the standpoint of our three participating learning community courses: First-Year Seminar, English, and Mathematics. The theme we selected, *Parts of a Whole*, helps in creating an understanding that freshmen year is the first stage leading to the whole identity of becoming a professional. With this in mind, students understand that a personal transformation will occur during their educational journey. Students are expected to embrace the changes that will occur in behaviour, actions, and thinking; ultimately, these changes occur for the betterment of each individual student. In this classroom environment, we challenge students to leave their former unprofessional and counter-productive behaviours in the past. College is the pivotal time where these young adults will begin certain practices that will shape who they become in the future. In other words, *practices* they adopt now will become *habits* in the future. The following scenarios exemplify how this can happen.

Example 1

In one incident, for a class assignment, students were expected to research a career and submit a five-page paper for a grade. After collecting the papers on the assigned due date the instructor returned to the office to begin grading. Thirty-minutes into the task one student's paper caught his attention. Let us call him or her Student A. Student A's paper followed the parameters as outlined in the syllabus, but when the professor arrived to page 3 something was different. It appeared the printer he used began to run out of ink. Regardless of this obstacle, Student A continued to use the fading ink cartridge and submitted the paper. The professor's initial reaction was to continue reading the paper hoping there was improvement in print quality in subsequent paragraphs. After stumbling through numerous sentences and squinting at the various paragraphs, the professor became frustrated and ceased the extra effort. Apparently the student did not exert the extra effort to find another printer. The professor subsequently formulated a 50% grade (acknowledging the 50% of the paper that was legible), with a lengthy note indicating the frustrations in the paper's legibility and extending an offer for the student to resubmit the paper for a better grade. Given this same scenario with a resume and a potential employer, more than likely Student A would not have been given a second chance by an employer to resubmit a better quality document. This example supports the belief that students should always uphold high standards in all they do during their developmental collegiate years. Again, these experiences are the training grounds for how students will present themselves to future faculty as well as potential employers.

Example 2 (of the difficulty faculty faces in forming new successful habits with students)

Students typically are concerned with their assignment grade and fail to notice how they can become better students. As a community college instructor, it is not uncommon for faculty to teach multiple sections during a typical semester. As such, many instructors are responsible for over 120 students distributed throughout 4 sections. Using the research paper as a class assignment, students are expected to complete a 3–5 page paper investigating and communicating different aspects of the journey towards their future occupational choice, and pose and answer related detailed questions. Amongst all of the sections, approximately 100 papers were submitted at the same time, during the week following midterms. As a former professional student, instructors understand first-hand the importance of feedback and students perceive instructors who grade papers with no comments as lazy and uninvested. To this end, we all made every effort to offer input or feedback on our students' papers, and in a timely manner. In addition, such information helps us remember each paper clearer should a student contest a grade. After the daunting task of commenting on each paper, we distribute the papers in class for students to only watch them bypass all comments and target the last page for the grade. Unfortunately, many students, not learning from their trivial mistakes or our suggestions to have a peer proofread their work, are only interested to know if they passed or failed. Students do not realize that they are doing themselves a disservice by ignoring our comments – a practice that would be just as detrimental in any of their future academic and professional endeavours. In conclusion, we would hinder our students' educational growth by not offering comments, and encourage them to refer to our suggestions by providing the incentive to earn a higher grade for those students who resubmit their work with corrections.

Example 3

"Am I on the right track?" A student's need for affirmation of his/her work from the teacher might perhaps be common at the beginning of the course; however, there is a pocket of students who may be unable to work without the outside confirmation for longer. A student might want to repeatedly have his or her work checked. It might be so intense that in the beginning of the essay the students may only have typed up the heading and title before asking their first question, "Am I even on the right track?" In a math class, during board work, he or she may have only written the equation to be solved and immediately ask, "Am I even on the right track?" The student's work may also demonstrate an unclear connection between procedures and concepts. The conceptual writing exercises that take place in the computer lab demonstrate that the students who are in the habit of asking "Tell me what to do next" have to extricate themselves from this habit. In our experiment, one class has been shifted one day a week to accommodate students and allow them to receive more individual attention from the instructor. This certainly needs to be addressed. The learning environment should be sufficient enough to support the student in figuring out how to approach the problem at hand.

Example 4 (of the motivational approach used to reinforce the successful habits of the future)

After a particularly trying day, the instructor asked students to take a few minutes to respond to the following poem, entitled *Indivisible Excellence*[1]:

If my Excellence be Me
Indivisible from the rest of Me,
What competition does it have?
Headphones, earphones, cell phones,
Chatting, texting, sleeping,
Eating, day-dreaming,

Asking questions just answered,
Is this the competition of Me to Me?
Am I to win the battle of Me with Me?
Is there to be a Handshake in this class?

There were ten student responses from a class of twelve students (two were absent). Of the ten students, the answer to the final question was provided by five students. Two students responded to the individual questions in the poem and provided a response to the whole poem. One student responded only to individual questions in the poem without responding to the whole. Their responses are below:

1. Yes. If I was to be more focused and determined the handshake can happen. I am just a little forgetful about the math I used to know;
2. This is an example of what goes on in our class. If one has a part of whole that shows excellences when speak or works but their other part of them that shows on the outside has habits such as chatting, texting, eat, then the two has to come to a freedom to accomplish good work;
3. I believe there shouldn't be a *handshake* because one has to be responsible as an adult to be organized and on top of things;
4. Yes, because texting and day-dreaming takes over what I do in class. I could win a battle of me with me just by doing the opposite;
5. I think yes, because every person have its own responsibility to do and act like they know they are supposed to behave;
6. I like the statement professor. There should be a handshake;
7. (Answers written next to the last three questions in the poem above) (**i**) Yes, (**ii**) Yes, (**iii**) Yes, because student need to know some class mates and study together or form study groups;
8. (Answers written next to the last three questions in the poem above) (**i**) No, (**ii**) Yes, (**iii**) I think there should be a handshake but we need to put effort in learning. Do all assignments and don't let any distractors catch your attention;
9. No, there wouldn't be the competition of me to me. If it was to be there wouldn't be no negative to it. There will always be positive. Me to me would be a job well done. In fact me to me shouldn't be competition. Me to me is supposed to be parallel to each other not perpendicular. Yes, there WILL be a handshake to this class;
10. (Answer written next to line 3 in the above poem) I have no competition. I just try and do the best that I can in everything I do. (Answers written next to the last three lines in the poem) (**i**) No, (**ii**) Yes, (**iii**) Not each person is accountable for his/her work.

Example 5

Students were administered the Motivated Strategies for Learning Questionnaire (Duncan et al., 2005). There are two parts to the test consisting of a total of 81 questions. Part A asks 31 questions pertaining to Motivation and Part B asks questions pertaining to Learning Strategies. In response to the question: "I make sure I keep up with weekly readings and assignments for this course," six students answered "Yes" and three answered "No". The remaining three students were absent. These results demonstrate a clear connection between the students' motivation and how they perceive their cognition.

Example 6

A student's hostility can be transformed. This transformation can only be attained by repeated, deliberate and clear verbalizations of the instructor's persistence and faith in students' work. Students' hostility, especially toward mathematics, is often directed at the instructor; this is common knowledge for any mathematics instructor with several years of teaching experience. Over the past five years, utilizing the cyclic teaching-research methodology in our mathematics classrooms, we have singled out the *classroom environment* as the primary platform to counter the obstacles for learning diagnosed in the first phase of the ongoing teaching experiment, helping it evolve each semester. In the Basic Mathematics courses, students have had prior exposure to the topics in question; however, the level of hostility is still somewhat tremendous. The need for rapid intervention in student *morale* affecting performance is significant and urgent.

As the semester progresses, we observe that students and teachers, in the process of adjusting to each other, begin to learn from each other. We hope that by the end of the semester, both students and instructors accomplish the goals of a quality education. The learning that occurs over the course of the semester illuminates the widely varying views of students on the absence of relevance of education that are both startling, disturbing and necessitate a change on the part of students and the instructors. However, this change, which must be facilitated by the instructor, is a slow and long process, and is affected continually and significantly not only by classroom dynamics but also by factors outside the classroom, such as juggling family and work responsibilities as well as managing personal and economic stress. A constant question we struggle with addresses the way we communicate the relevance of education to our students, who seem not to see eye-to-eye with us. The students' intelligence and talent, so openly evident to the instructor, are disregarded by participating students, resulting in several conflicts whose negotiation is the integral part of the creation of a successful learning environment. At the end of the semester, the hope is that there is a profound common understanding of the relevance of education that is shared by both students and instructors.

CONCLUSION

We believe that students who experience a sense of absence of the relevance of education may not care about passing the course or, in extreme cases, even plan to fail. As learning community instructors, we attempt to bring about a modicum of success through our concerted, regular efforts towards developing students' belief in their own excellence; however, the success always leaves the desire for a greater success and, hence, the constant effort persists until the very end. By the repeated efforts of the instructional team, students are forced to face a cognitive conflict in which their fight/participation in their own failure is made explicit publicly (and sometimes not publicly), and they have the opportunity to take action upon the fear of mathematics or academia in general or other non-voiced, non-exhibited fears that prevent success. The responsibility of teaching and learning is inextricably connected and shared by students and instructors, and both groups are influenced by the existing societal environment. Lesch (2007) explores how "the structure of schools might be changed so that students in their formative years are able to learn in a manner that allows them to be more creative." As Aronowitz states, education is a *public good* and absence of relevance of education among a significant student body is a matter of concern for the young minds and the society in which they mature. Aggressive and invested action towards fuelling a change that brings relevance back to education is, thus, a prerogative of benefit to everyone.

It is clear how the relevance of education is being developed in our classes; indeed, it is realized in the refusal of the instructors to give up in the face of students' fierce resistance to learning and in the persistence of the instructional team to find ways to extend the possible partnership that has shown signs of promise, even if the progress is only fleeting at moments. However, each student has demonstrated their *excellence,* in no uncertain terms, at least once over the course of the semester in one of their three classes. This *excellence* has been diagnosed in clear but intangible terms and is stifled by the ever-present question mark students attach to the relevance of education, often failing to view the learning community they belong to as a microcosm of the "real world." Our teamwork provides a wall of support for both instructors and students who are learners, struggling for self-expression, on their own terms, within the often clashing cultures of youth and academia. We observe that the repeated expression of public, tangible and non-judgmental support instils in students the expectation that they can meet the instructors' standards and that excellence is within their reach. Such a realization creates some small but nevertheless positive steps in the direction of the attempted didactic contract or handshake.

Students' absenteeism or, when they are present, the lack of focus and attention, along with an impulsive need for instant gratification that may come from "getting the solution" to a problem without the careful thought processes that require sustained concentration and active participation are built up over a long period of schooling and disinterest in their own work. Youth culture, where calling one another "loser" as a friendly chide, lends itself to pride in failure, and often awards academic failure as an achievement. To fail is to be rebellious. Students are rebelling against the system, and their rebellion is directed at the teacher in front of the classroom, yet it is without venom, without desire to hurt. The teacher is the personification of the system. The students, especially the younger ones, want to *enjoy* and live in the moment. They strive to enjoy every minute. Hence, for them, every moment is joyful. For the instructor in the classroom, these responses can be deeply disturbing while at the same time, the physical and mental exertion is tremendous. As a result, teacher burnout can be intense. Our learning community was a class of only twelve students. One student, particularly engaged in *enjoyment*, asked the instructor: "Are we a fun class, Miss?" The instructor had replied to enquire, "What were you like in school?" He answered: "Yes, and there were 30 of us. The teacher didn't have a prayer." All of this was said with ease, absolutely no harshness or anger, just statement of facts. Learning by necessity requires silence. Especially in mathematics, learning requires a chain of thoughts to be built and followed, and the silence to allow for that train of thought. This silence is absent in youth culture and is taken up instead by the ubiquitous noise of texting, chatting, listening to music and other sensory distractions.

The relevance of education, seeing oneself as a part of a larger whole community, and working on one's interests with a sense of belonging, are all concepts closely connected to both a quality education and a competition with oneself. As teacher-researchers, we have the opportunity within a span of one semester to challenge our students' long-held beliefs in their own shortcomings, and to inject a love of mathematics, English, and learning, in general, through the acts of writing and thinking together. The meaning that emerges may not be tangible to the spectator at the end of the semester, but, nevertheless, the joint effort creates meaningful partnerships that enhance the problem-solving process in mathematics and sciences, enhance creativity in language arts, promote the desire to learn and resonate with our students throughout their academic careers.

NOTE

[1] This poem by Vrunda Prabhu had been included in the published collections: Vrunda Prabhu, *Chosen Poems: Is there a Trick to Happiness?* Vrunda Prabhu – The Teacher-Researcher of Life-in-Truth.

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5.3. PROFESSIONAL DEVELOPMENT OF TEACHER-RESEARCHERS (PDTR) IN TAMIL NADU, INDIA

Focus on Women in Community-Based Schools of Tamil Nadu

INTRODUCTION

We report on our investigation of the applications of the TR/NYCity model of mathematics teaching-research approach to teaching and learning in Tamil Nadu, India. The TR/NYCity methodology and the design experiment were presented during epiSTEMe-1 (Czarnocha & Prabhu, 2004). Immediately after the epiSTEMe-1 presentation, outlining the outcomes of the methodology for the integration of practice and theory, the request for such an integration was initiated by Senthil Babu (then, present in the audience):

> The teaching research aspect will be focused on the teacher volunteers who are in charge of the night schools. These volunteers need to be trained in localized content development of the curriculum of the night schools, aware of the necessity of keeping the contents alive in relation to the political-economic dimensions of the village; be trained in areas that will enable them to intervene in day to day events of the village.

Babu stated his vision:

> We need action research programmes concerning pedagogy,—but not merely concerning on teaching children better but how the night school as a total institution in a rural setting can become viable by incorporating the educational needs of the agricultural labourers and the fish and boat workers. The night school needs to be seen in totality not as fragmented centre of concerns for fragmented ideas. We need to create institutions that will sustain themselves, and spur out waves of change from there out to the big bad world.

His counterpart from Sathyamangalam, Karuppuswami, the head of the READ NGO, informed us last January, that the community is not interested anymore solely in the technical/mathematical knowledge of the teacher; now, a strong need arises for a new profile of the teacher, that of a community motivator and a social entrepreneur,—a teacher, who can work along the cognitive development of children

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 487–498.

in the classroom towards the establishment and transformation of the community surrounding the school. These are new ideas, originating directly from India, that offer an exciting opportunity of self-transformation of the community, where intellectual and emotional needs of the community's children are one of the basic principles and aims of this transformation.

It turned out that many of the villages of the community-based night schools in question were situated in the path of the tsunami in Nagapattinam and Cuddalore region.

Reflection upon the requested task reveals a powerful vision where the TR methods of classroom enquiry into the process of teaching and learning mathematics, are integrated with the Action Research activities in the communities housing the night schools. The vision allows to ponder on the possibilities of social transformation of communities that is in agreement with the learning needs of the children of the communities. Consequently a natural teaching-research question arises:

> *What is the methodological route to smoothly integrate mathematics teaching-research focused on the improvement of mathematics teaching and learning with Action Research aimed at improving the socio-cultural and economic well-being of the community?*

The present report describes the organization of the pilot teaching experiment in three different rural communities of Tamil Nadu, and its three stages: (1) *Exploratory*, (2) *Workshop #1 – Mathematics and its pedagogy*, (3) *Workshop #2 – Mathematics and Psycho-Social Action Research*. The report describes the preliminary results of mathematics workshops as well as a possible answer to the stated research question.

ORGANIZATION

Exploratory Phase

A grassroots community effort over several years had developed an infrastructure of community-based night schools aimed at changing the significant difference in the expected and actual academic performance of the children in the government-run public schools. The teachers of the community-based schools are volunteers, with first-hand experience with both the social and educational circumstances of the children. The expressed need was the strengthening of their own mathematical and pedagogical expertise. The teacher-researchers familiar only with academic environments needed to understand the new experimental territory and to adjust their methods to the existing needs. This was accomplished in the exploratory phase by:

- Visits to several community-based night schools observing, taking field notes;
- Conversation with teachers about the difficulties they faced;
- Conversations with the coordinators.

Workshop #1 – Mathematics and Its Pedagogy

Three mathematical concepts were selected for the first teaching-research workshop: Signed Numbers, Fractions, and Beginning Algebra. The three concepts formed the basis for exchange of mathematical knowledge and pedagogy. All three topics were hands-on, manipulative-based; the teaching of the content was interactive, engaging the participants to discover, and to reflect upon (a) how they would use the method in their own class, (b) what problems did they foresee, and (c) what problems might be addressed through the approach. Some participants were able to experiment in their own night schools with members of the team as observers. Data collected from these mini-teaching experiments was helpful in further targeting the methods to the needs. The participants, aware of long-term teaching difficulties experienced in the classroom, questioned the efficacy of the methods to increase the rate of the process of learning, and the teaching-research team, aware of the knowledge of the field could (i) directly, through the manipulatives, demonstrate how this efficacy could occur, and (ii) through the theoretical linkage between the work of Piaget (Piaget & Inhelder, 1958), Bruner (1966) and Vygotsky (1987), demonstrate the theoretical foundation on which the increase in the rate of learning was based.

Workshop #2 – Mathematics and Psycho-Social Action Research

The second workshop alternated the techniques of psycho-social action research with TR/NYCity to (i) provide an exposure of psychosocial methods, (ii) continue the content and pedagogy exchange, and (iii) underscore the importance of teachers' own roles as teacher-researchers. What emerged was an integration, Teaching-Action-Research (T-A-R), of the psycho-social and the mathematics teaching-research methods, which is theoretically grounded in the concrete-iconic-symbolic developmental theory of concept formation (Bruner, 1966).

We continue to present new paths along which school/community integration can be accomplished in order to facilitate the main dream, answering the following central questions:

> *How to bring up the children we encounter in the communities so that the whole enthusiasm, brightness and smartness stays with them into the teens, twenties, thirties, fifties, sixties and beyond? What would have to be the qualities of the teachers to be able to keep that flame of children alive and developing? What do teachers need to know? What attitude and knowledge do they need*

> *in order to maximize the intellectual potential of the children? How village communities would need to be organized to make sure that the Spark of the Child grows unfettered to its full brightness? What are basic components of the required community transformation?*

Learning Community of the Future is the village community with its community-based schools that develop along parallel, mutually reinforcing tracks to assure the development of intellectual and emotional potential of its children as well as the development of the village's social capital to their respective maxima.

FOCUS: WOMEN OF THE COMMUNITIES

Montessori-for-Mothers (M-f-M)

Every community of the T-A-R project always had a contingent of very active women. In some sites in the tsunami-affected region, mothers stood at the fence of the school for hours watching the education of their children, cheering when one's child answered and encouraging those children that shied away. In October 2006, the women of the Salem and Erode communities expressed an even stronger interest, via their wish to educate their own children. Given that the community-based school is manned by one or two young teachers, and the number of children is large, with large age range, it would be better if the teacher(s) could focus their attention on the school-age children, and the younger ones could receive the needed educational nurturing from the parents at home. The idea of Montessori-for-Mothers (M-f-M), as the first bridge between the school and the community, was then born.

The T-A-R approach to Montessori-for-Mothers has the dual purpose of reaching women and children simultaneously through addressing the following question:

> *How can the mother become a critical thinker in the process of creating an appropriate learning environment for her young child?*

Any *Action Research* project works along the cycle of design, implementation, data analysis and re-design of the approach. The initial design of the M-f-M approach was envisioned to have three parts for the mother and one part for the child. In its first implementation, in January 2007, at four sites across the Salem and Erode districts of Tamil Nadu the following three parts were included:

- A discussion of three case studies of women across India who had been successful in using their own personal strength to combat oppression;
- Documenting their own life-map to begin the process of critical reflection upon their own lives;
- Design of Montessori materials: Tamil letters, words, numbers and colour cubes to enable the child and mother to begin *reading the world through the word*.

The third part sets up the environment of learning for the child, and all three parts are meant to motivate the mother to create the required environment, while she herself learns.

The actual implementation of the M-f-M workshops at the four different sites has yielded an evolved new structure that could not have been envisioned prior to implementation, and much more robust.

Given the harsh living conditions, the woman may not see many truths in her lifetime, and the spark of her child is the *Truth* she has definitely seen, and does not doubt. It is what she wants to preserve. It became apparent in every visit to every night/community-based school that the woman is crying out desperately for help in every which way she can. On the last visit it became clear that she is a quick learner – and she knows her mind.

"Teach me your methods."

"Now I know I can draw."

While each such remark can be very difficult to digest and reconcile to academics, it is the existing reality. Three readings drawn from the real world of success stories where women had overcome their constant neglect and abuse via standing up for themselves, did not have the impact as had been anticipated by the teaching-action-research team. The women of all the communities are ready and, in fact, hungry for education, for a decent life for themselves and their children. The neglect and abuse is a commonplace occurrence; the women were not affected by the narration of the three case studies. Instead, women listened and very carefully, but did not have many comments and asked from all of it exactly what they needed.

The Tamil alphabet was created by Ranganathan, an artist from Sathyamangalam. The felt fabric for several sets was provided, and together, in January 2007, the first movable Tamil alphabet was created by him. The women took immense labour over learning. Many said that they did not yet know, some guessed; they laughed and it was a game. Slowly, as each group kept exchanging the letters (there were 104 women at four separate sites), the fear dropped, and they were recognizing the felt letters, feeling them, seeing them, speaking them. The group reinforced their courage to try, even when wrong, and they continued until they were happy, just wanting to get another new letter to learn. PalaniSamy, a grassroots organizer, a man, was actively involved in the creation of the environment that resulted. His desire to embrace the new with passion, energy, drive and the absence of the "me" was apparent as he moved from group to group, encouraging, joining in the game and prodding where needed. PalaniSamy as a model grassroots organizer was, perhaps, the key for the success of this particular site. The women liked the methods.

At the fourth site, when the women had been given paper and pencils to "just draw," they had sat quiet for a long time looking around. Then, one by one they said, "We do not know how to draw." Urged by the teaching-action-research team, they slowly began. What emerged was comparable to the descriptions by Paulo Freire in

Figure 1. Working with Tamil alphabet

(Freire, 1998). One by one they got up to describe their drawing to the group and they said, "Now we know we can draw, we like your methods!"

Figure 2. Now we know we can draw, we like your methods!

It came as a big surprise, but, in retrospect, it makes perfect sense. They use their hands all the time, however, it is not for writing. They do draw rangoli; though, that is with powder. It can be seen made outside the houses, and this had prompted the thought that drawing would be liked by the women. In fact, at one tsunami shelter, a severely traumatized woman who did not speak, after much coaxing from the man and other women in the room, began drawing, and very methodically drew the grid before completing the rangoli. She definitely preferred the iconic representation. However, given one's years of conditioning, the team's teacher-researchers still had not made the connection that drawing on paper with pencil would be difficult. It was the women's remarks, one after the other, from several in the fourth group, that clarified, for certain, the landmark they had carried out that day. The re-design cycle includes roles that the participating women saw and found for themselves.

From the success of the three-pronged approach to M-f-M described above, along with community involvement, a larger project originated, addressing the following concerns:

- Older illiterate women in all sites and, especially, the fourth semi-urban site, directly expressed their desire to have such a program fully functional. They said that they felt humiliated by having to use a thumb impression, not being able to sign or read their own name, among other things. Furthermore, they noted, "We have seen our youth with certificates on which the caste brought about a rejection, so their interest was not in certificates."
- Grandmothers who are unable to work stated they would like to set up little sites in their villages and take care of village children.
- Young mothers were seen as the potential beneficiaries of the program, however, the turnout was not restricted to the imagined age group. The intended beneficiaries were very excited about the project. In Ambedkar Nagar, Maheshwari, a young woman, a mother of two, a full-time agricultural labourer, who, with her 12 years of education, has been eager to take on the task of running a Montessori school for the children of the community. She has agreed, through her own initiative, to start a day Montessori Children's Hut and all materials required for her were purchased by the teacher-action-researcher.

The organizers are Tamil speakers and members of the community and are predominantly (all except one) men. When not in Tamil Nadu, the T-A-R team has only the e-mail contact with grass-root organizers as the means of communication with emerging M-f-M groups.

SELF-HELP GROUPS

The existing self-help groups (SHGs) in most communities became the way to disseminate the M-f-M-derived needs. In particular, Alamelu, a young woman with a Bachelor's degree from the Salem district, who for the past several years has been actively working setting up self-help groups, became the engine of the M-f-M. The SHGs organized by Alamelu are special. The women are actively working toward freeing themselves of any bonded labour, and are finding schemes to be self-employed. In one particular village the situation is striking. The story is inspiring. Some of the women in the communities with the help of Alamelu organized a self-help group. Soon the other women formed two more self-help groups. While only the first SHG existed, the other women were supportive but reluctant to form their own. Six months and repeated visits later, Alamelu had all women in their own self-help groups. The stories of the self-employed women of the self-help groups are fascinating and inspiring. What are the effects of the women of the self-employed self-help groups upon the community?

1. There are no child labourers.
2. All children attend school.
3. The men have stopped their drinking habits.

It is the women of these self-employed self-help groups that provided the answer to the teaching-action-research question set by the community organizers,—how do we get back our self-respect?

The women were vocal in their sense of self. Self-respect means to not be afraid all the time. "At one time if a car had driven into our village late at night, we would have been afraid. Now let 10 cars drive in and we will not be afraid." The sense of strength and self; the simplicity of the open camaraderie toward each other, toward the children and men of their own communities and to the visiting people are evident, as is their clear voices. Tesoriero (2006) confirms the extent of empowerment which in generally is reached by women of SHGs. An older woman laments spontaneously, "I sent my own children to be labourers. What did I know, I was illiterate too." The self-employment created by the self-help groups generated the needed stability in their own lives to question and reflect upon their own actions, to be non- judgmental and supportive of new ideas. In the design of the Community of the Future, anchored on the community school, SHGs become one of the main supports towards unifying and integrating the life of the emerging community. Both Tesoriero (2006) and Agarwal (2007) point out to SHGs' involvement and capacities to make positive changed in the village community matters leads to wide acceptance and support by men, and the whole community.

CONCLUSION

The aim of the teaching-action-research team to assist in finding solutions that were organic was accomplished within the first cycle of the work. Whereas we don't have yet the final answers to the main questions concerning the necessary qualities of teachers working both as M-f-M educators of the mothers, teachers in the classrooms and the community organizers, we have the initial postulates, which together with the cyclic principle of the T-A-R methodology give justifiable promise to arrive at the sufficient knowledge of those conditions. The cyclic methodology of T-A-R, inherited here from *Action Research,* assures that after every cycle (analysis of the problem _ design and implementation _ collection of the data and their assessment _ analysis and refinement _ …), the implemented process can be improved or refined, new methods can be put into practice so that after several cycles one can arrive at the satisfactory conditions for the community transformation and school learning. One can postulate therefore that the T-A-R teacher working in the community has to

1. have knowledge about the principles of individual and community development ;
2. start teaching women through Montessori techniques which are deeply integrated with the daily activities and artefacts, women use in their life (not pencil and paper, but the ground and rangoli);
3. guide herself/himself by the learning theories that postulate development at the very concrete level leading to iconic level (Bruner theory);
4. avoid too early connection with other similar cases of women emancipation.

NEXT STEPS

The organic seeds have to be spread and the grassroots organizers along with the teaching-action-research team have to find the needed next steps. (Menaka Roy, 2004) suggests that the empowerment of communities takes place during the literacy campaign when literacy is linked to the development activities. The Teaching-Action-Research approach in Tamil Nadu rural communities has yielded a possible initial structure for the development of the Community of the Future: a Community-based school, women Self –Help Group and Montessori-for-Mothers program.

Communities of the Future is a living – in-action idea, initially born out of the experience of Tamil Nadu's grass root organizers of rural Dalit communities, and of the Bronx mathematics teachers researchers. Their common central idea underlying the collaboration has been to build upon one principle, the primacy of the well-being of children in Dalit communities. On that basis, the new profile of the teacher had been established as the teacher who on one hand is attentive and competent relatively to the intellectual and emotional development of the child in the school, and on the other to the cultural, socio-economic needs of the community and its development. Such a teacher had been called the Teacher-Community Motivator by Arunthatiyars. The lens through which the teacher might be looking, to be in agreement with the main principle could be: how should the community be organized so that children in the school reach their true maximum?

At the next meeting/workshop a new idea was tried out with success, namely, Montessori for Mothers (M-f-M) workshop. The methodology of M-f-M assures that children intellectual needs are taken care from the early moments of life by the mother who becomes an organizer of knowledge for her child. It is expected that through practicing the method, the mother will acquire critical skills and consciousness, which will enable them to make appropriate changes in their families and communities. Community schools might be the places where the Montessori knowledge will be anchored.

Hence the first foundations of the Community of a Future are composed out of:

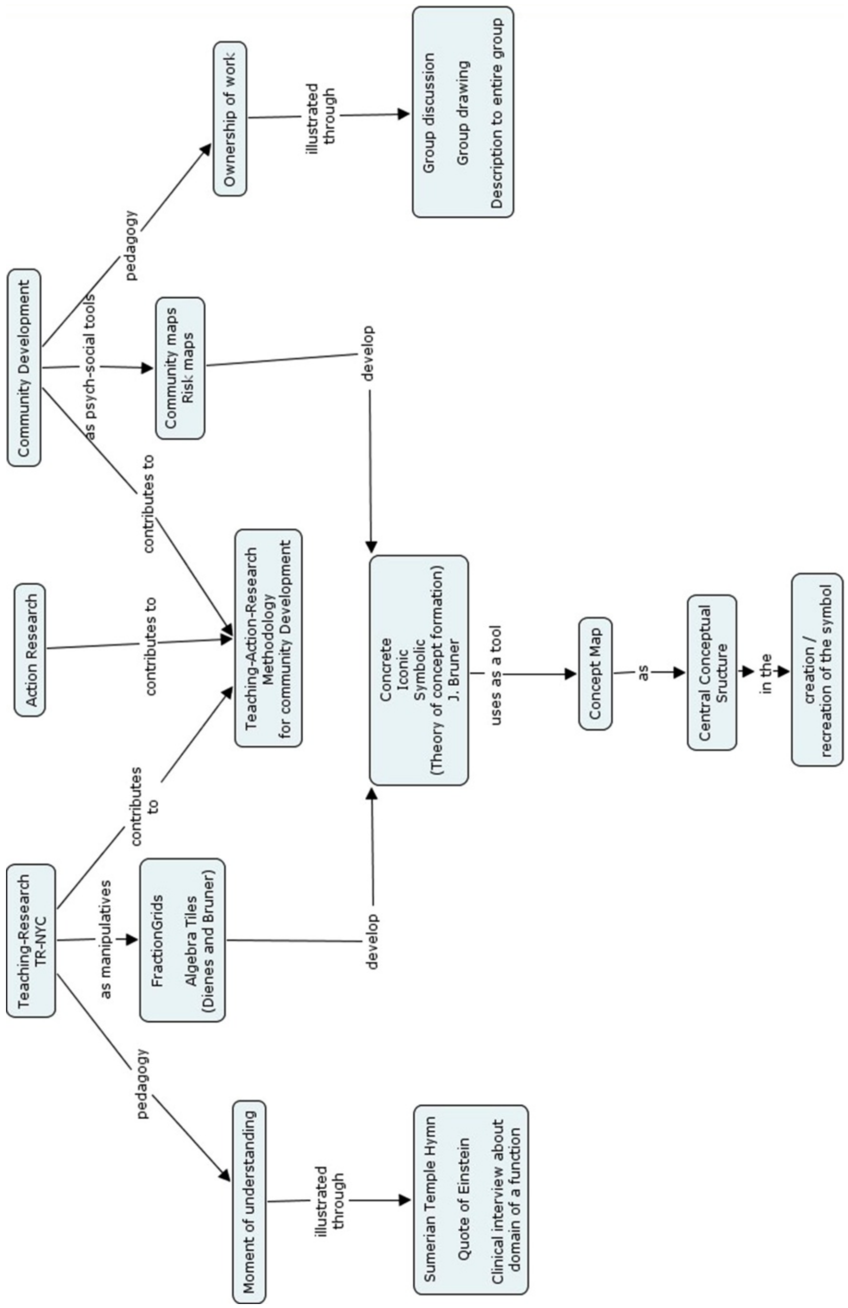

Figure 3. Development of teaching-action-research in the Tamil Nadu rural community

- Community night school,
- Community of the night school,
- Mothers of the communities practicing Montessori Method.

Since the teachers in the school are in the central position, their education has to be very precise and adequately wide. It needs to involve the following themes:

1. Mathematics and mathematics teaching
2. Child and adolescent development
3. Montessori Method development and training
4. Analysis, Synthesis and Development of the communities
5. Teaching-Action-Research practice.

A new component had been added to the design of the Community of the Future,—Self-Help Groups. Those amongst them which have reached self-independence that is bought themselves out of bondage and created means of support may have reached the consciousness to make them the underlying support for the emerging communities through placing in their utmost attention the life of community schools, possibly as the Mother Council Members, and place it with equal care to the life of Montessori Mothers. Consequently, SHG can become a complementary element to the teacher of the community school, which is envisioned as extremely aware and knowledgeable individual. Although literature materials are still scanty, the few that are available point out to the unusual volume of social capital residing in women's SHGs, which can be utilized both in community development and in creating good education for community's children.

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5.4. PDTR: THE DEVELOPMENT OF A TEACHER-RESEARCHER

Formulation of a Hypothesis

INTRODUCTION

This final chapter is devoted to an essential question for any, so to say, emerging profession: can we actually describe and characterize the development that leads a teacher to become a teacher-researcher? Many sub-questions suggest themselves by this essential inquiry: Does this development depend on the particular approach to the craft of teaching-research? Can we effectively describe a profile of teacher-researcher (TR) in order to actually promote it? And, more specifically, can that development be institutionalized as a professional development of teacher-researchers?

Before facing the main issue, let us first try to devote some reflections to the last question regarding institutionalization. Indeed, the problem of developing a professional figure of teacher–researcher can be viewed from the research side—that means that we observe the phenomenon of some teachers that are able to do more and better than others, and try to understand why and how this happens, and, in particular, which conditions favour such improvement of their abilities. But it can be viewed also from the policy-makers' side. In this sense it becomes a part of the general issue of teacher training, and can be described as a sort of special goal to be pursued—that of preparing *gifted* teachers. Our special interest is for mathematics teachers but it can be said that the same applies to every teacher. Moreover, an important distinction occurs between future and in-service teachers.

Of course, when things are viewed from the policy-makers' side, a central issue is always that of expenses (in several senses), so that any discourse has necessarily to compare outcomes with costs.

Generally speaking, when an investment is made in favour of gifted people, a more or less equivalent amount of resources is subtracted from other objectives; in our case, from the care of *normal* or *poor* people. This can certainly be a good choice, but has to be justified.

The question becomes – why we turn our attention to gifted people? Possible answers are:

a. because it is easier;
b. because it is more rewarding;

B. Czarnocha et al. (Eds.), The Creative Enterprise of Mathematics Teaching Research, 499–518.

c. because it allows to directly raise the average value of some indicator of the whole population;
d. because it is a first step toward a more general goal concerning the whole population;
e. because it is a side goal with respect to the principal one, and we believe that it is reachable while the other is not.

We, the authors, are not trying to impose, or, even, suggest which of the above answers we instinctively support; our aim is to specify aspects that have to be taken into account, and to follow a path of logic and rationale to an informed opinion. One might argue, for example, that answers like (a) and (b) are to be considered inadequate to justify a discourse of institutionalization for teacher-researchers, while answer (c) is very unlikely. Usually, what happens is that when a subgroup of a population obtains high scores, then the average value remains stable or even decreases, and, in both cases, the global result is anything but desirable, since, in a phenomenon like the one we are speaking about, what really matters is an increase in the average value and perhaps a reduction in the variance. Answer (e) is realistic, but it can be considered as provisional and, therefore, requires continuous revisions of the policy. This leaves answer (d) as the most practical, interesting, and potentially persuasive. It can be said that it is the implicit motivation that lurks behind any serious attempt to promote the diffusion and the enhancement of teacher-researchers. We do believe that it should be a true priority to try to make the claim suggested by answer (d) as explicit as it possible, specifying in what sense a teacher-researcher can help the advancement of the whole class of teachers.

Coming back to our main issue, we propose a model for the development of the teacher-researcher based upon four different principal sets of data:

1. The implemented design for the Professional Development of Teacher-Researcher (PDTR) realized by the EU Project supported by the Socrates grant awarded to the University of Rzeszow in Poland during 2005–2008;
2. The set of responses of the 26 participants of the PDTR to a questionnaire that was designed, distributed and collected in 2013;
3. Some excerpts from a previous case study (Mellone, 2011) focused on one of the teachers participating in the PDTR project;
4. Chapter 3.1 of the current book titled How to Arrive at a Teaching-Research Question?

PRINCIPLES OF DESIGN

From 2005 to 2008, seven different teaching-research teams from five different countries participated in the Krygowska PDTR (Professional Development of Teacher-Researchers) Project (supported by the grant Socrates Comenius 2.1 Program No. 226685 –CP-1-2005-1-PL-Comenius-C21): Hungary (Debrecen), Italy (Modena and Naples), Poland (Krakow/Rzeszow and Siedlce), Portugal (Lisbon) and Spain (Barcelona). Each team included between 10–15 mathematics teachers from different teaching levels, whose work was facilitated by the team coordinator, a mathematics educator from the participating university, and a team of mentors, made up mostly of Ph.D. students in mathematics education. The project as a whole employed three teaching-research experts who rotated among the teams, facilitating workshops, and one mathematics expert to coordinate the content of the associated mathematics courses.

The design of the PDTR was based on several characterizations of teaching-research formulated in Czarnocha (2002) and Czarnocha and Prabhu (2006), ultimately integrated into the definition and structure of the TR/NYCity model, as described in Chapter 1.1:

i) Research-teaching teams composed of researchers and teachers or instructors analyse the problem in question with the help of the available educational knowledge base and the empirical database;
ii) On the basis of the review of literature, an analysis of errors and intuitive knowledge of instructors, the research questions and the hypothetical developmental model concerning concepts in question is formed and the instruction is designed in the teams;
iii) The introduction of the formulated pedagogy into instruction and its process of refinement take place through the instruction/analysis cycles, that are performed consecutively in order to better fit to the real ecological dynamics of the classes involved;
iv) After participation in at least two such experiments, apprentices will be guided to design and to perform a teaching experiment on their own, with the simultaneous goals to improve their classroom practice and to contribute to the general educational knowledge base.

Czarnocha (2002) asserts that:

> Professional Development of Teacher-Researchers is based on the careful composition of ideas centred on *Action Research* (Lewin, 1946) with the ideas centred around the concept of the *Teaching Experiment* of the Vygotskian school in Russia, where it "grew out of the need to study changes occurring in mental structures under the influence of instruction" (Vygotsky, 1962). From *Action Research* we take its focus on *the improvement of classroom practice* and its cyclical instruction-analysis methodology, and from Vygotsky's *Teaching Experiment* model, we take the idea of the *large-scale experimental design based on a theory of learning and involving many sites – different classrooms.* …The proposed methodology provides the organizational structure within which the *research investigation, classroom instruction and teaching-research professional development* combine into a mutually supporting whole. The main tool of the professional development here is the large scale teaching experiment designed on the basis of a general hypothesis and conducted by

> the research-teaching teams composed of the experienced teacher-researchers and interested and motivated teachers. The teaching experiment is designed to address a specific pedagogical difficulty, which is widely manifested among the given student population; it is conducted in the properly chosen sample of classrooms in a school, college or a school district. Apprentices of teaching-research are the teachers who, together with their classes, participate in the experiment.

The PDTR project had two main cycles, each lasting one year. Each such cycle pair began with the introductory cycle whose main goal was to coordinate different TR methodologies of different teams with the aims and design of the whole project. This first main cycle was organized around individual classroom teaching experiments conducted by individual teachers mentored by the team coordinators and TR experts. The aim of the first main cycle of individual teaching experiments was to address the difficulties of students with the types of problems proposed by PISA international test, which at the time of the project has just gone through its first two mathematically focused cycles: 2000 and 2003. All participating teams were from the countries, whose results placed them below, or at the average of, the PISA test. All of the participating country teams reported significant educational difficulties, the understanding and resolution of which was the aim of the first main cycle, in accord with the general aim of the TR/NYCity methodology,—"to improve learning in the classroom, and beyond". Thus, the first cycle was governed by the principles of *Action Research* as one of the components incorporated into the formulation of the TR/NYCity model. Each individual classroom investigation followed the *TR cycle* process described in Chapter 1.1. Furthermore, each investigation was composed of at least two such cycles to enable the systematic reflection upon the results and the refinement of the proposed interventions. Naturally, during the intervening year TR apprentices were participating in the systematic seminars devoted to the practice and theory of teaching-research where the conduct of teaching experiments was discussed, analysed and refined. Detailed examinations of the tasks of the different teams and the accompanying TR seminars revealed a clear common thread among the various practical tasks and the themes of the theoretical seminars,—a natural need and inclination towards a *just-in-time* approach of introducing and utilizing research results. For example, during the project teams' individual 7th meetings, when the task in front of the TR apprentice was to design the diagnostic tests and analyse student errors, the corresponding TR seminar was focused on the process of error analysis; similarly, when the task for the each individual team's 8th meeting was the design of the new instructional approaches addressing discovered pupils' errors, the corresponding TR seminar was focused on different types of instructional strategies such as mathematical writing, inquiry method, short clinical interviews as well as other pedagogical techniques. The teaching designs and sequences that were developed during project team meetings were put into practice during the time between them. In addition, apart from the TR seminars, the participants held study team meetings that were devoted to more informal discussions, analysis of results obtained in the classroom, and planning of future actions.

The results of the individual teaching experiments were presented at the general project meetings that took place in the summer; the TR reports from the PISA centred investigation were collected in the first of the two books ultimately produced by the project,—Stefan Turnau (Ed), *Handbook of Mathematics Teaching Improvement: Professional Practices that Address PISA*, University of Rzeszow, 2008.

The second main cycle was envisioned as a large scale teaching experiment focused on the common teaching-research question formulated during the first cycle and was meant to involve all of the teams. However, due to the methodological difficulties attributed to the coordination of different cultural approaches practiced by different schools of teaching-research, such as the very specific approaches of the Italian School of TR, the Lisbon School, and the Barcelona School, with those of the TR/NYCity model, the second cycle was not fully implemented and was limited to an Italian-Hungarian collaboration (Navara, Malara, & Ambrus, 2010). Instead, the different project teams addressed their national concerns, and the findings were reported in the second book produced by the project,—Bronislaw Czarnocha (Ed.), *Handbook of Mathematics Teaching-Research: Teaching Experiment—A Tool for Teacher-Researchers*, University of Rzeszow, 2008. The conduct of the project and the difficulties encountered as seen by the Italian team are described in (Malara & Tortora, 2009).

THE SECOND ASSESSMENT

A new questionnaire, aimed at the assessment of the project's overall experience was designed in 2013 and sent to all participating teachers. The aim of the questionnaire was to create a database of results that would allow one to make an assessment of the whole project, addressing more details and reaching well beyond the scope of the final grant project report. Twenty-six of teachers involved in the project responded, and the analysis of their answers is one of the foundations for the model of teacher-researcher development proposed here.

Since the teaching experiment is the basic tool of a teacher-researcher, in the TR/NYCity model, we assume *the presence of the concept of such a teaching experiment* within the potential teacher-researcher's individual activity as one of two criteria for establishing a teacher-researcher profile. The second criterion is the explicit acknowledgement of the public aspect of teaching-research either through the publication of the research, or teaching-research, reports, or a book. The reference to either of these two criteria in the responses to the questionnaire will be taken as evidence of entering the teacher-researcher profile.

Although, we must acknowledge that there is a certain dissonance between the two criteria: there are teacher-researchers amongst the participants who have publicly published articles, either alone or together with university-based researchers that contain reflections rather than *descriptions and/or analyses of teaching experiments* that are necessary to fulfil the second condition. There are also teacher-researchers who have done, or are planning to perform teaching experiments and, thus, fulfil only the first condition, with the expectation of fulfilling the second by an eventual publication.

We also acknowledge that the full hypothesis concerning the development of the teacher-researcher is formulated by the integration of information from the questionnaire data with the information provided in Chapters 3.1 & 3.2.

The presented profile and criteria differ significantly from those currently present in the teaching-research literature; in particular, in one of the best accounts of the TR developmental process by B. Jaworski in *Mathematics Teacher Research: Process, Practice and the Development of Teaching* (Jaworski, 1998). Jaworski, similarly to the more recent work of Herbert-Eisenman and Cirillo (2009), look upon TR activity primarily through the reflection upon practice motivated by the *Reflective Practitioner* framework introduced by Schön (1983). Whereas, of course, the process of reflection has been present in the work of teachers participating in PDTR, the questionnaire responses suggest that it is but one of the stages in reaching the development of the full profile of the teacher-researcher in accord with the TR/NYCity methodology characterized by the concept of the *teaching experiment*. This means that, although teaching experiment presupposes, almost by definition, the reflection upon practice, neither the reflection itself, nor the reflection-in-action discussed by Jaworski does necessarily lead to the concept of the teaching experiment.[1]

This fact suggests, of course, the question of the nature of the necessary conditions for a teaching-experiment as well as a clear characterization of the transition between the reflective practice and teaching experiment. The analysis of the questionnaire responses, presented later in this chapter, provides the basis for the hypothetical answer, that is, can be visualized via the following developmental path:

Master Teacher (MT)
⇩ ⇩ ⇩
Reflective Practitioner (RP)
⇩ ⇩ ⇩
Teacher-Researcher (TR)

The last transition is reaching the beginning of the formation of the full development of a Teacher-Researcher. A more detailed exposition outlining the specifics of the full formation of the TR profile is presented in Chapters 1.1, 3.1 & 3.2.

The concept of the developmental sequence "**MT ⇨ RP ⇨ TR**" has been motivated by an analogy to the developmental Piaget and Garcia *Triad* formulated by the authors in (Piaget & Garcia, 1987) and touched upon in this volume in Chapter 4.1. The *Triad* of Piaget and Garcia is a mechanism of thinking leading to concept formation formulated on the basis of the thorough comparative analysis of the development of physical and mathematical ideas throughout history of science, on the one hand, and the psychogenetic development of these concepts in a child, on the other (Piaget & Garcia, 1989). It is defined as the passage through *intra-operational, inter-operational* and *trans-operational* stages:

> *Intra-operational* stages are characterized by intra-operational relations, which manifest themselves in forms that can be isolated… *Inter-operational* stage is characterized by correspondences and transformations among the forms that can be isolated at previous levels…The *trans-operational* stages are characterized by the evolution of structures whose internal relationships correspond to inter-operational transformations.

The natural environment of the Piaget-Garcia *Triad* is, of course, the development of mathematical and physical concepts; however, it was also successfully applied in the analysis of the heroic fairy tales (Czarnocha, 2013), and in the analysis of the development of revolutionary consciousness (Czarnocha, 2014). Here we use it to model the different profiles of:

i) the Master Teacher who is able to incorporate the new methodologies and teaching techniques into classroom without explicitly mentioning the process of reflection and/or teaching experiment;
ii) the Reflective Practitioner who reflects upon his/own practice;
iii) the Teacher Researcher who thinks in terms of the teaching experiment and/or public presentation of TR results.

The distinction of the Teacher-Researcher profile is also motivated by the vision of a teacher as *resonance mediator* (Guidoni, Iannece, & Tortora, 2005), that is, one who is involved in the careful task of creating resonance between students' cognitive needs and insights and the epistemological features of the mathematical content, the object of the teaching experiment. As highlighted in (Malara & Tortora, 2009), teachers are influenced by important factors such as knowledge, beliefs and emotions. One of the goals of the PDTR project was to make the teacher become more and more aware of all these components, and empower him or her with the ability to change them, if and when necessary, driven by a deep understanding of pedagogical ideas and theories related to mathematics education. For this reason, as described in the first section of this paper, special sessions were organized during the project in order to introduce teachers to selected literature samples. These include papers focused on didactic-methodological aspects, on classroom practices, but also epistemological studies and mathematics oriented papers. One of the core idea of the PDTR was to try to favour the structural aspects of the mathematical contents, in Shulman's sense (1986). For this reason, many working sessions involved teachers of all school levels. Indeed, in our view, the longitudinal way of perceiving mathematics curriculum could support teachers in moving from the specific mathematical content to teach toward both its roots and its possible developments and ramifications.

METHODOLOGY

1. The set of TR apprentice responses was coded anonymously, stripped of names and national team membership.
2. Each response was searched for the excerpts which describe the responder's conceptions concerning his/her understanding of the effect of PDTR and/or TR process, according to three stages: that of a master teacher, a reflective practitioner and a teacher-researcher who thinks in terms of the teaching experiment and/or public presentation of TR results. Recall that a master teacher is seen as a person who was able to incorporate the new methodologies and teaching techniques into the classroom without explicitly mentioning the process of reflection and/or teaching experiment.

Master Teacher Typical Responses

- (MT1) Thanks to the PDTR experience, I have acquired a didactical style based on classroom discussions, the collective construction of meanings and concepts and the gradual shared formalization of their representation. …In particular, I have learnt to pay more attention to both the language used in the classroom and the relationship between different communication codes.
- (MT2) Based on my PDTR experience I prepare myself on my lessons more carefully (seeking for effective introductory examples, detailed lesson-plans with notes, reflecting on lessons, modifying my teaching ideas)

Reflective Practitioner Typical Responses

- (RP1) Now I am able to reflect on and to critically consider the teaching job and I face my doubts without fear.
- (RP2) I have developed the awareness that the construction of a free and trusting teacher–student relationship (where the teacher has no prejudice toward students and the students are not afraid of expressing their thoughts) allows students' more solid learning and also their acquisition of high level competencies

Teacher-Researcher Typical Responses

- (TR1) Besides I am introducing the teaching experiment concerning calculating with the help of a soroban (a Japanese abacus of Chinese derivation).
- (TR2) I am considering the need to investigate the effects of e-textbooks since the plans of the Ministry of Education is to introduce such textbooks widely. The question is to investigate positive and negative effects of such an approach.

As we explained in the previous section, here, we understand the *intra* stage as the stage of the Master Teacher, where often separate strategies and techniques have been incorporated into the teacher's repertoire without, necessarily, reflecting upon their development; the *inter* stage – as the stage where the reflection upon classroom practice can connect different instances and processes of teaching; and the TR stage, when the teaching experiment integrates the previous activities into the transcending structure of the mature complete classroom experimental framework.

The Evidence of Stages and Transitions between Them

The evidence of the transitions characteristic of the proposed developmental sequence are found in the fragments containing both stages of the transition.

Master Teacher ⇨ Reflective Practitioner:

- (MT¦RP 1) I usually find myself focusing more than before on students' answers and behaviours, searching a way to give a constructive sense to their way of behaving; but above all I pay attention to how my way of working on students can influence them and their thinking processes.
- (MT¦RP 2) Working in PDTR empowered me, increased my belief in myself. I am more courageous in speaking my mind. I treat problems investigated during PDTR as the long term projects for interested students.

Reflective Practitioner ⇨ Teacher-Researcher:

- (RP¦TR 1) The program showed me how to be a teacher and do classroom research at the same time and I think it proved this way of researching to be effective. My teaching has become more creative and I feel that I am often reflecting on what happened in the classroom. If something went well I try to remember and explain the reason for success. If something didn't go as well, then I try to look for the reasons and try to find ways of improving.
- (RP¦TR 2) In these year I published a scientific paper with D. Iannece: D. Iannece, P. Romano "What does it mean to have a *scientific approach* for a teacher? A reflection" in *Proceedings of the 5th International Colloquium on the Didactics of Mathematics*, vol. II, ed. by M. Kourkoulos and C. Tzanakis, Rethymnon, Greece, pp. 409–419, ISBN 978-960-87898-3, and last year I contributed to a scientific paper with Pr. R. Tortora and Dr. M. Mellone in which we proposed an analysis of some interesting solving processes of 9-grade students engaged in an arithmetical task. It originates as an item of a national test that has proven to be very critical for Italian students…In those years I learned how much important is the reflection about the learning processes of my students and the reflection about my own activity. I tried to involve some of my colleagues at school, but they don't really want to be involved. I've changed my ideas about the relationship between teacher and students and I am much more relaxed in the classroom, so that also my students really enjoy studying math.

In the process of classification of reached stages, those TR practitioners who were found on the Master Teacher level without the comments about reflective practice were placed in the Master Teacher stage; those who were found on the Reflective Practitioner level but without comments about the teaching experiment were placed on the Reflective Practitioner level, and those who were found on the Teacher-Researcher level were placed accordingly. Altogether, among the 26 respondents, we have found 9 on the Teacher-Researcher level, 9 on the Reflective Practitioner level, and 4 on the Master Teacher level. Some respondents could not be placed on either level indicating that the application and development of the new profile is limited by the amount of organizational work required from the teacher. Partial responses from these 4 participants are shown below:

> In my opinion beside the normal teaching work it is very hard to make classroom researches. In the average classes we have only 3 lessons in a week, there is no time to try out new ideas. (Anonymous Respondent 1)

> I think the combination of two elements, teaching and researching, is very effective for pupils and teachers but, at the same time, too tiring and difficult. The main problem in my opinion is the organization of school in Italy and perhaps in other countries. The innovative methodologies should be a goal of the whole scholastic system. Teaching maths through a research action requires a system that can support and help teachers. Instead the teachers are alone dealing with parents, colleagues, and headmasters. (Anonymous Respondent 2)

> I don't think there can be effective teaching without assessment. However, school practice is very unsupportive for such a work. We have to cover the material in short number of days; if he doesn't he has problems with the director, parents, weak exam etc. There is no time for the search for effective methods to teach and its research. (Anonymous Respondent 3)

> It is a difficult problem. Combining teaching and researching may work, but you need much more time to do some special work with students. And the main problem is, that we have no time enough: students have to learn certain things that are in the curriculum, and teachers have no time enough research their own activity in everyday hurry. (Anonymous Respondent 4)

REFLECTIVE PRACTICE _ TEACHING EXPERIMENT TRANSITION

At this point, the teacher profiles proposed and the data presented have shown interesting pictures of the international group of teachers involved in PDTR a few years after the end of the project. What we are going to see now is that the profiles presented and, most of all, the development among these profiles can actually describe the path that leads a teacher to become a teacher-researcher. Here we refer to other data, coming from a previous case study, of which one of the teachers involved in the PDTR was an object (Mellone, 2011).

Many different indicators, reflections and observations of this teacher were collected during the years of the PDTR. It is really interesting to notice that during an interview conducted with her at the beginning of the project, this teacher describes herself as an innovative teacher. She had already preferred an innovative approach in the management of her lessons by favouring interaction among students. Moreover, in the same interview, she explained how she usually allowed her students to work together in small groups and involved them in general discussions where they could speak openly and confront and discuss their mathematical methods and solutions. In this direction, the description that the teacher gave of herself at the beginning of the project fits very well with the Master Teacher profile, as proposed here. Indeed, we can argue that, well before the project, she was already able to incorporate new methodologies and teaching techniques into the classroom, but at that time she didn't explicitly mention any particular reflections on her own practice, neither did she refer to any design for a teaching experiment.

Written during the project, exactly after one year from its beginning, we can read from her diary:

> …My usual approach to Algebra was sustained by the unconscious hypothesis that instrumental understanding could generate relational understanding [...] structural thought is natural, but not spontaneous, so it is substantially a cultural acquisition and requires appropriate teaching mediation. I think that not only the teacher should be aware of these things, but she should use them by building environments that allow learners to develop conscious shifts between the two complementary kinds of thought.

At that time the teachers involved in the project were provided with some mathematics education research literature. She was referring particularly to Sfard (1991). In her words we can see how, from a starting reflection on the theoretical model studied, she moved to reflect on the teaching practice and on the actions to perform in the classroom setting in order to promote what she called the *structural thought*. These, along with other diary entries made by this participant (for more diary entries see Mellone, (2011) allow us to characterize her as a Reflective Practitioner at that point in time. Something changed later, as we can read from her diary:

> Last year I proposed some investigation activities on natural numbers to look for properties and regularities, but I didn't know exactly in which directions they would work. So I built the path step by step following the curriculum and pupils' reactions. This year I have had the opportunity to rearrange systematically these activities and to utilize their potentialities. I have always thought of myself as a good teacher, but now I understand that this is a different way of being a teacher because I'm pursuing different goals.

The need and desire to better design her teaching actions is the support for another important step toward a well-organized teaching experiment. Indeed, in the expression *rearrange systematically* we can recognize the wish to organize the educational environment of her classroom in a more framed way making her teaching actions explicitly consistent with the new goals she wants to pursue. Finally, some years after the end of the PDTR we can read from her questionnaire:

> In these years I published a scientific paper with Pr. D. Iannece: D. Iannece, P. Romano "What does it mean to have a *scientific approach* for a teacher? A reflection" in *Proceedings of the 5th International Colloquium on the Didactics of Mathematics*, vol. II, ed. by M. Kourkoulos and C. Tzanakis, Rethymnon, Greece, pp. 409–419, ISBN 978-960-87898-3, and last year I contributed to a scientific paper with Pr. R. Tortora and Dr. M. Mellone in which we proposed an analysis of some interesting solving processes of 9-grade students engaged in an arithmetical task. It originates as an item of a national test that has proven to be very critical for Italian students…In those years I learned how much important is the reflection about the learning processes of my students and the reflection about my own activity. I tried to involve some of my colleagues at school, but they don't really want to be involved. I've changed my ideas about the relationship between teacher and students and I am much more relaxed in the classroom, so that also my students really enjoy studying math.

The reference to her scientific papers, as well as to her organization of the teaching experiment using the items from the national test, frames her professional figure as a Teacher-Researcher.

THE DEVELOPMENT OF RESEARCH QUESTIONS REVISITED

All three authors of Chapter 3.1 portray the characteristics of mature reflection upon classroom practice that have been incorporated into daily teaching practice. We can also recognize the process of incorporating and integrating theories of learning or research results into that reflection whose further refinement will lead to the formation of a *thinking technology*,[2] as defined in Chapter 1.1, when research results and methodologies are smoothly integrated in the context of reflective practice with the craft knowledge of the teacher. The process of reflection on practice motivated by the aim of improvement of learning focuses on the formulation of the teaching-research question. This is a long process with an interesting structure. It is the process during which methodologies of research are incorporated into the classroom with the purpose of clarifying and refining the teaching-research question. Vrunda Prabhu incorporated the methods of clinical interviews into the structure of classroom dialogues which enabled her to investigate her students' thinking and, on that basis, introduced changes in the curriculum. Bill Baker does statistical analysis on his partial exams during the semesters, compares it with data on the final exam and from the obtained data notices that everything works well except for fractions that leads him to engage in quite an extensive research program. Bronislaw Czarnocha replicates colleague's set of logical arguments in his classroom, compares the responses and starts drawing first conclusions about the structure of the possible teaching experiment. This suggests that, before a formal teaching-research experiment is conceptualized and proposed with the full coordination between the teaching-research questions and methods of assessment, an informal teaching experiment or other research techniques are being used as teaching methods to clarify and refine it, naturally promoting the development of the *thinking technology* amongst the practitioners. However, we stress that a further important characteristic of a teacher-researcher is his/her ability to autonomously implement teaching research experiments and to communicate them by means of scientific papers. A teacher-researcher is something very different from an educational researcher; indeed, for a researcher the choice of the theoretical framework is something that comes from research needs, while a teacher-researcher, having always in mind his/her responsibility as an educator of human beings, uses a theoretical framework according to his/her beliefs about the educational process and the mathematical content. In other words, the teacher-researcher walks on the boundary, selecting, time after time, his/her priorities between research and educational responsibility.

In lieu of the above paragraph, the following questions arises naturally,—At what moment does the process of refining the informal teaching research questions culminate in a proper teaching experiment?

Theoretically, the answer is obvious,—it's at that moment when the teaching-research question has the clarity that can be communicated, measured and assessed. In practice, usually, something external or spontaneous triggers that process; it might be bisociation between theoretical knowledge refined through practice and the environment.

This bisociation, followed by the conduct of the designed teaching experiment is the completion of the development of the TR/NYCity profile of teaching-research.

In case of Vrunda Prabhu, it was meeting the new VP for Student Development with inclination towards teaching mathematics and knowledge about motivation and self-directed learning methods that sparked the formal collaborative teaching experiment. In the case of Bill Baker, it was his role as a researcher on the Prabhu, Czarnocha (2007) C^3IRG 4 CUNY grant, while in the case of Bronislaw Czarnocha, in his Algebra-ESL teaching experiment, it was the presence of the new Dean of Academic Affairs, Dr. Tony Baez, the only occupier of the OAA office at Hostos Community College who appreciated teaching-research and helped to spur the successful application of the grant supporting the teaching experiment.

NOTES

1. Teaching Experiment is seen in the TR/NYCity as distinct from the Design Experiment; the second one responding primarily to the needs of educational research, while the first one – to the needs and experience of teachers and teacher-researchers (see Chapter 1.1).
2. Thinking technology describes the culmination of the thinking processes suggested by research and integrated within teaching-research into classroom practice (see also Chapter 1.1).

[3] There was a significant change of the focus of the organization of the PDTR after the second year. Where until now each of the TR apprentice was designing individual teaching experiments in their classrooms whose goal was to improve certain of the competences or type of difficulties students have demonstrated, during the third year they are participating in the large scale teaching experiment designed collaboratively by the team, addressing a common issue, a common difficulty in their classroom. The overall goal of the large scale TE is similar as before, to formulate the innovative instruction, which improves students' performance on PISA-like tests. However, the team has chosen a certain aspect of the problem and the individual teaching experiments performed in each of the classrooms are the components of the large research questions of the team. It is hoped that the set of research questions of all teams address the full scope of the project. Clear and effective communication within each team and across teams will be essential in this part. The role of the coordinator and of the management team will be to coordinate the results, questions and comments between different teams leading to the mutual reinforcement of results while taking into account the national specificities and cultural differences. The themes of the TR seminar will concern more general aspects and their coordination with the craft knowledge of teachers. Concurrent Research teams (previously Study Groups) meetings will have as their first priority, watching over the efficient conduct of the individual and team's teaching experiments. Monitoring continues through the reporting system outlined in Section 5 of the Project Description. The members of the teams schedule the time for the preparation of research papers and presentations as well as for the preparation of TR peer-to-peer workshops for their peers.

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APPENDIX A

The Second Assessment Questionnaire

"Dear PDTR participant:

Five years have elapsed since the end of the Krygowska PDTR project. We would like to know how you regard today your experience in that project. This is the reason of our sending to you this questionnaire, hoping that you find the time and the energy to respond in a most sincere way. Please do it up to the end of March and send back to us by e-mail. You may respond in English or, if you feel not comfortable in doing so, in your mother language.

QUESTIONNAIRE

1. Since the end of PDTR, in September 2008, what have been your main professional and academic activities? (Have you been teaching in a school? Have you been involved in teacher education? Did you assume other professional roles? Have you finished or are about to finish an academic degree? Have you participated in other projects? Have you published research or professional papers?)
2. In your view, your experience in PDTR had any impact that you notice on your professional or academic activity? Did you notice any changes in your teaching and/or in your relationship with other teachers? Have you been working alone, or continued to cooperate with other members of PDTR or new colleagues? Can you document such an impact with some episodes or other aspects of your professional or academic life?
3. The main thrust of PDTR was that teachers could be actively involved in doing classroom research and that, doing it, they could become better teachers, while at the same time improving student learning of mathematics. Do you think that these two elements – teaching and researching – may be combined together as PDTR did, may be combined better in other ways, or will be always very difficult to combine? Should one strive to combine them? Please, justify your response.
4. After the PDTR project have you been involved in other initiatives (regional, national, European, or global) for the improvement of classroom work and above all for teachers development through action research methodology?
5. Please report on an incident in your classroom that you find interesting enough to be examined, diagnosed, accounted for in the follow-up teaching, and discussed with others. Tell also of the action you took afterwards, if any. Have you realized new classroom teaching experiments? Please tell us about them and describe some of its episodes you consider important in your experience.

APPENDIX B

Table 1A. Detailed Outline and Schedule of the PDTR Project

YEAR 2 – Main Cycle – Phase 1 – Introduction to TR through the design and assessment – Emphasis on teaching experiments in individual classrooms					
Stage in life of project	***Outputs / Achievements***	***Activities***	***Start and end date of activities***	***Partners/ persons involved***	***Time input (person/days or person/months)***
Project Meeting (PM) #7 All Sites	1. Administration of the diagnostic test 2. Design of first innovative measures and methodologies 3. Development of understanding of student errors amongst teachers 4. Presentation of evaluation methods for the change of students' attitude towards learning mathematics	1. Administration of the PISA diagnostic test. 2. **TR seminar #6**: Analysis of Student Errors in mathematics. Introduction to the task design and teaching experiments. 3. **Study Teams:** designing innovative tasks in all class levels 4. Mathematics, English intensive courses 5. Community building, dissemination: meetings with parents of involved students	Middle Sept 5 days	81	405
PM #8 All Sites	1. Second design of the innovative instruction to be implemented till the meeting #9 2. Detail descriptions of the strategies used, their successes and failures. 3. Redesign of the strategies. 4. Collection, translation of agreed upon strategies for PISA handbook and interactive WEB site	1. **TR seminar # 7**: different TR instructional strategies; inquiry teaching, use of written language, inquiry technique open problem solving, clinical interviews with students 2. **Study teams**: TR discussions: successes and failures of the first teaching experiments, their re-design, incorporation of new techniques into instruction and task design. ***Evaluation of teaching experiments*** 3. Mathematics and English intensive course. Translation of new instructional strategies into English	Oct/Nov	81	405
PM #9 Meeting of the management team All Sites	1. Refinement of instruction based on the theories of learning. 2. Collection of the second set of data on English/Math relationship in the process of learning 3. Design of clinical interviews to be held with students 4. Video meeting of the management team.	1. **TR seminar # 8**: Theories of learning and their manifestation/application to classroom instruction. Impact of the organization of the classroom discourse upon individual learning 2. **Study Teams** TR discussions: refinement of the instructional design; design of interviews with students 3. Mathematics and English intensive course, Exams. 4. Evaluation Seminar: ***Evaluation methods for PDTR project***	Management Strict Barcelona 2 days Middle Dec	12 81	24 405
PM #10	1. Design of the second generation teaching-experiments with sound assessment methods. 2.Collection, translation of agreed upon strategies for PISA handbook and interactive WEB site ***Self-evaluation of participants***	1.**TR seminar #9** : Qualitative and Quantitative classroom assessment, different method of assessments and their relationship to standard exams and to new methodologies 2. **Study Teams**: design of the second generation of micro-teaching experiment by individual teachers 3. Mathematics and English intensive courses	Jan/Feb	81	405

(Continued)

Table 1B. (Continued)

Stage in life of project	***Outputs / Achievements***	***Activities***	***Start and end date of activities***	***Partners/ persons involved***	***Time input (person/days or person/months)***
PM #11	1.Redesign of the TE with the appropriate assessment techniques 2.Collection of the data for English/mathematics TE 3. Collection, translation of agreed upon strategies for PISA handbook and interactive WEB site	1.**TR seminar #10** Discussion on the incorporation of theory approaches into classroom instruction; pertinent research literature 2. **Study teams**: reinforcing TE with a sound assessment procedures. Outline of questions for the year's summary. 3. Mathematics and English courses. Collection of the data for the English/mathematics TE	Mid-March	81	405
PM #12 Management team – preparation for the 2nd annual meeting	1.Ready presentation of each team to the 2nd Annual Meeting of TR Project 2. Formulation of the team's research question for the next cycle. 3. Collection, translation of agreed upon strategies for PISA handbook and interactive WEB site. ***Evaluation of the team work***	1. **TR-seminar #11**: Year's Summary – Analysis of the processes of learning in the classroom from TR perspective. What have we learned? 2. **Study teams**: final discussions on the second series of TE's , leading to the formulation of the team's research question for the next cycle 3. Planning the national team presentation during the 2nd Annual Meeting 4. English workshop: translation of the Handbook into English 5. Mathematics course	Management Beginning May	12 81	24 405
PM #13	1. Formulation of the First Draft of PISA 2000 Mathematics Improvement Handbook, TR Handbook and WEB site design. 2. Redesign of the products 3. Formulation of the final research questions for each team 4. Final designs of large scale teaching experiments. 5. Plans for English Advanced yearly workshops ***Data for intercultural comparison of cognitive victories and challenges.***	**1. Three seminars**: TR #12, Mathematics, Invited Guest 2. Presentation of the developed instruction by the national teams in English. Discussion of results, in general and comparison across their cultural environments, assessment techniques. Five **General sessions** 3. Critical discussion of research questions to be addressed by the collaborative research teams 4. General design and critical discussion of designs for collaborative Teaching Experiments for the next cycle. 5. Concurrent dissemination for invited other education stake-holders, school administration, parents, Professional Development Centers 6. Three Specialized team meetings : presentations of first draft : PISA Improvement Handbook, TR Handbook, WEB design, evaluation 7. Two social events + reflection time. ***Formative evaluation of the second year-summary***	End of June 4 days	81	324

Table 1C. (Continued)[3]

2007/2008 – Phase 2 – PDTR – Year 3 – Implementation. Large scale teaching experiments of the Teaching-Research national teams, assessment of the innovative instruction.					
Stage in life of project	***Outputs / Achievements***	***Activities***	***Start and end date of activities***	***Partners/ persons involved***	***Time input (person/days or person/months)***
PM#14	1. Ready design of classroom TE's coordinated with the large scale design	**TR Seminar 13**: Design of the large scale TE's **Research Team**: Tuning the large scale design with the individual classroom TEs. Underlying question: what have we learned about 8 mathematics competencies of PISA in the context of the PDTR experiment English Intensive	Mid-Sept. 5 days	81	405
PM#15	1. Formulation of research themes and design of the presentations. 2. Redesign and collection of the products for the outputs.	**TR seminar 14**: study of the pertinent research literature and its coordination with teachers' craft knowledge. Use of the research results in classrooms. **Research team**: Discussions leading to formulation of themes for the research papers of participants. Reports on the conduct of TEs English Intensive	Oct/Nov 5 days	81	405
PM#16 Meeting of the Management team- planning for the finalization of the project	1. Design of language rich instruction in mathematics	**TR seminar 15**: The role of language in the didactics of mathematics. .The results of mathematics/language TE of PDTR **Research teams**: design of instruction using different language approaches, possible refinements to the conducted TEs English Intensive	Mid-Dec 5 days Management team, 2 days	81 12	405 24
PM#17	1. Formulation of the conclusions from the assessment of the large TE. 2. Collection, translation of agreed upon strategies for PISA handbook and interactive WEB site 3. Early draft of the national Handbook	**TR seminar 16**: Advanced assessment techniques. Assessment of the stage of the large scale of the Research Team. Decision on which particular issues need further experimental refinement **Research teams**: working on the final text of the Handbook English Intensive	Jan/Feb 5 days	81	405
PM#18	1. Formulation of the assessment of the large scale teaching experiment of the team. 2. Initial preparation for the national presentation in the summer.	**TR Seminar 17**: Advanced review of TR teaching techniques: problem solving, inquiry technique, organization of the discourse, use of language. Discussions of the large scale assessment **Research team**: assessment of the refinements in the classroom TEs work on the presentations and workshops English Intensive	Mid-March 5 days	81	405

(Continued)

Table 1D. (Continued)

Stage in life of project	*Outputs / Achievements*	*Activities*	*Start and end date of activities*	*Partners/ persons involved*	*Time input (person/days or person/months)*
PM #19 Meeting of the management team. Final decision of the outcomes of the PDTR, preparation of the 3rd Annual meeting	1. Final draft of the Handbook 2. Completed preparation of the presentation of the team. 3. Completion of the products: national PISA handbooks, web-sites, TR Handbook. 4. Final version of teacher-to-teacher workshops	**TR seminar 18**: Preparation for the presentation of results to the 3rd Annual meeting of the project. **Research teams**: finalization of research presentation of individual TR's Dissemination Presentation to all national stakeholders. English Intensive	April/May 5 days Management 2 days	81 12	405 24
PM #20 3rd Annual Meeting	Completion of the project and discharge of the responsibilities, distribution of the graduation diploma	1. Teams presentation, 2. PDTR graduation diplomas 3. Discussion upon the final products. Opening of the full Interactive Project Web – site. 4. Celebration of achievement; Plans for future projects 5. Dissemination event with invited stakeholders 6. **Seminar**: Relevant mathematics	June 3 days	81	324

BRONISLAW CZARNOCHA, WILLIAM BAKER AND OLEN DIAS

EPILOGUE

We want to express the hope, as parting words, that the reader has enjoyed the volume as much as we enjoyed writing, designing it and reflecting upon the knowledge it has brought to us. The realization that TR can be a bisociative framework characterized by the enhanced creativity explained to us significant number of new hypotheses and theories brought forward in the volume as byproducts of the TR methodology: bisociation of Koestler as the medium of reflective abstraction of Piaget (Chapter 4.1), hypothesis of the relative ZPD (rZPD) (Chapter 5.1) generalization within practice through the artefact refinements (Chapter 2.3), design of learning trajectories from classroom practice (Chapter 4.6), bisociative principle of the Creative Learning Environments (Chapter 2.4) and many others.

It often happens that authors, while designing the book with certain themes as its focus, they have to leave many new ideas out of that focus, in their initial raw, undeveloped state. To complete them is often beyond the scope of the book's design so that to achieve the completion they have to be formulated, presented or published outside, in professional conferences and their proceedings, often simultaneously with writing the book itself. This happened also to us and starting from the PME38, where the research report by Vrunda Prabhu and Bronislaw Czarnocha (2014) suggested Koestler's bisociation as the new definition of creativity in mathematics education, a stream of dozen of presentations and publications by our TR Team has been produced and appeared till now. We list them below.

Among the explored topics below we want to direct attention to the computer creativity domain based on mechanism of bisociation, the cognitive-affective duality of the Aha!Moment and the role of bisociation in the precise timing of the Aha!Moment.

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William Baker, Mathematics Department, Chair, Hostos CC, CUNY, NYC. William's interest is with applied research or teaching at the intersection of creativity and learning theory with adult students who struggle with mathematics. Thus, as a researcher he is am interested in how students learn or create meaning and as an instructor how to promote their thinking and engagement in meaningful and relevant thought during problem solving. He likes to work with a group that is within a learning community and has enjoyed working with his colleagues in the research team of Czarnocha, Prabhu and Dias, and collaborating as both instructors, researchers and authors in this book.

Roberto Catanuto, Mathematics, Physics Teacher Everest Academy Lugano, Switzerland. Roberto has been helping people to learn across a wide spectrum of age, grades and subjects, for the last 15 years.

In public and private sectors, schools, after-school centers, 1-to-1 tutoring, competitions and exhibitions, his core goals are summarized as: "you want to learn and I'm here to create an environment where you can thrive and grow your personal abilities".

He has helped children learning Robotics and Programming, young students learning Mathematics and Physics, young adults learning basics of ICT tools and finally adults learning how to deal with modern technological instruments and how their children use them.

His first interest is teaching students how to learn thinking Mathematically, which is more than learning Mathematics to pass an exam. It's a life essential skill.

His main interests can be summarized as: differentiated learning, Mathematics Learning and Computational Thinking, collaborative learning, constructionism, the maker movement.

He teaches Mathematics and Robotics at Everest Academy (Switzerland) from 2011. He has led Robotics laboratories for children from 2005, in Italy and Switzerland. He holds a post-doc and a Ph.D. in Applied Mathematics from University of Catania, Italy, and a Master in Physics from the same University.
He has attended various online courses for teaching and learning with top programs from around the world.

Bronislaw Czarnocha, Department of Mathematics, Hostos CC, CUNY, NYC. Czarnocha is a quantum theoretical physicist turned mathematics educator. His physics background helped him to visualize mathematics classroom as the laboratory for teaching experiments. Strong believer in the Aha!Moment-based teaching in mathematics classrooms, especially among marginalized populations of students, he investigates the conditions of their successful facilitation during the instruction. Concerned with bringing mathematics to "the mases" he taught algebra on the streets of NYC during Occupy Wall Street movement.

Olen Dias, Mathematics, Hostos CC, CUNY, Assistant Chair. Olen Dias is a mathematician with the background in computable matrices, a PhD graduate from CUNY. She has participated in several NSF grant including MSPinNYC where she developed interest in designing mathematics approach for the students of South Bronx, in particular, in the context of peer leaders methodology. She has organized a team of teacher-researchers at Hostos CC and introduced ideas of the Lesson Study to community college faculty.

Ruslan Flek, Assistant Professor of Mathematics and Director of Quantitative Reasoning. New School for Social Research, NYC. Dr. Ross Flek received his B.S. in Mathematics from Hunter College of CUNY and his Ph.D. in Mathematics from the CUNY Graduate Center. While completing his doctoral work he served as an adjunct lecturer at Hunter College and a math learning center supervisor. He received his Ph.D. in 2009, and had served as an Assistant Professor of Mathematics at Hostos Community College (CUNY) prior to joining the Interdisciplinary Science Department at the Lang College of the New School University. Over a span of almost two decades he has had the opportunity to experiment with different pedagogical approaches and styles. He briefly summarizes his teaching philosophy below:

> I strongly believe in active student participation during course lectures. It keeps the students engaged and provides me with immediate feedback about their understanding of the topic at hand. Whenever possible I try to incorporate my research interests into the courses I am teaching, which is possible for courses ranging from Basic Mathematics to Calculus and Differential Equations. I do assign homework problems which students are expected to hand in. From

experience, I have observed that even short hand-in assignments are quite beneficial for students' overall understanding of the material. In addition, these assignments serve as a good measure of students' grasp of the necessary concepts…

Eric Fuchs, Metropolitan College of New York, Dept. of Science and Mathematics Education.

Eric was born in Bucharest, Romania, and received his early education there until age 16. He then completed his education in Israel, Canada and the United States. Currently, he is an assistant professor in the Master of Science in Education program at Metropolitan College in New York. He is also adjunct assistant professor in the math department at Bronx Community College, where he has been working for over fifteen years. His primary interest is teaching developmental mathematics. He uses technology for assessment, teaching and learning; He also employs a socio-cultural methodology (peer-tutoring and co-teaching) to improve pedagogy and student learning.

Ted N. Ingram, Associate Professor in the Department of General Counseling at Bronx Community College (B.C.C.) of the City University of New York (C.U.N.Y.). Dr. Ingram earned a bachelor's degree in Spanish at the State University of New York, College at Albany, a master's in higher education at Rowan University, and a doctorate in higher education administration from Indiana University. Dr. Ingram works primarily with incoming students. Since 2007, he has been preparing freshmen students with basic survival skills in the area of time management, career goals, study skills and other tools to navigate the higher education landscape. Dr. Ingram maintains a presence at professional conferences related to higher education, presenting on issues that reflect his research interests: underrepresented students experiences in higher education, African American male achievement gap, and the recruitment and retention of students of color in community colleges. Ingram has published his research in several book chapters, continually referees journal outlets, and other scholarly sources. In 2014, Dr. Ingram received the CUNY Chancellor's Research Fellowship for conducting studies to improve academic performance among community college students. Ingram's recent book, *Exploring Issues of Diversity within HBCUs*, highlights the varied experiences of students and faculty at HBCU. Professor Ingram is currently working on a book (with Professor James Coaxum, III) on supporting African American men in community colleges, under contract with Information Age. In the midst of his teaching and scholarship, he remains actively engaged in campus life and leads a venerable campus-wide faculty and staff organization. He is president of the illustrious Kappa Xi Lambda Chapter of Alpha Phi Alpha Fraternity, Inc.

Hayue Jin, lecturer in the Department of Education Science, Nanjing Normal University, China. Hayue Jine received her PhD from National Institute of Education, Nanyang Technological University, Singapore. Her research interests include educational assessment, teacher education and international comparison of mathematics curriculum.

Maria Mellone, Associate Professor in Mathematics Education at Dipartimento di Matematica e Applicazioni ”R. Caccioppoli”, University Federico II of Naples, Italy. Maria Mellone’s main research interests regard mathematics teacher education, early algebra and cultural issues concerning mathematics didactical practices.

Howard Pflanzer, playwright, poet and Adjunct Professor of English at Bronx Community College/CUNY. Howard Pflanzer created, adapted, and directed *Walt Whitman Opera,* a hybrid poetic performance work with music by Constance Cooper, undergroundzero festival (2014). *Living with History: Camus, Sartre, De Beauvoir* premiered at Medicine Show Theatre (2011) as did *On the Border* (2007), his play about Walter Benjamin. Fulbright Scholar, theatre (spring 2003) in India where he directed the world premiere of *The Terrorist.* MFA Yale School of Drama. Awards: NYFA Playwriting Fellowship, two ASCAP Awards, a Puffin Foundation grant and co-winner of an NEA Media Arts grant for the opera, *Dream Beach* (with Michael Sahl). Plays and musicals performed at La MaMa, (*The House of Nancy Dunn* with Steve Weisberg/ Andy Craft), Playwrights Horizons, Symphony Space, Medicine Show, Kraine Theater (*Cocaine Dreams*), The Living Theatre, 2011 Malta International Theatre Festival (*Alien*, collaboration with Teatr Palmera Eldritcha) and broadcast over WNYC and WBAI FM. Playwriting Residencies: Fundacion Valparaiso, VCCA, and the Ragdale Foundation. *Dead Birds or Avian Blues* published by Fly By Night Press (2011).

Vrunda Prabhu, Bronx Community College, CUNY, NYC, 1961-2013. Prabhu was a born educator of powerful imagination. In her childhood she organized herself a classroom out of square marble floor tiles in her home, in Bombay. Twenty years later she became a Montessori mother introducing the Montessori principles straight into the upbringing of her infant son, what became the basis for an NSF grant proposal, Montessori for Mothers, another twenty years later. She was mathematician, PhD, in one of the most classical branches of mathematics that is in point set topology with interest in the concept of "imbedding", in particular in Čech imbeddings of Hausdorf Spaces; she strongly doubted the correctness of the Dedekind axiom. Vrunda's work on Koestler's creativity of Aha!Moments is at the basis of this volume. She was devoted student of yoga and non-duality philosophy of Ramana Majarshi and Sri Nisargadatta. A poet.

Elizabeth Smith, English Department, Bronx CC, CUNY. Dr. H. Elizabeth Smith earned her Ed.D in English Education at Teachers College, Columbia University in 2004 and has been teaching Composition, Literature and Developmental Writing in the English Department at Bronx Community College (CUNY) since 2003. Her pedagogical research interests include curriculum development, multi-culturalism, portfolio assessment incorporating multiple measures, and scaffolding complicated assignments for optimum student success.

Edme Soho, Assistant Professor, Hostos CC, CUNY. Edme Soho is an Assistant Professor of Mathematics at Hostos CC. He is originally from Benin (West Africa) and did a part of his undergraduate work at Montclair State University in NJ, and his graduate work at the Arizona State University in Tempe, AZ. Generally, he is interested in the application of mathematics to solve real world problems arising in the life, social, and physical sciences. His mathematical research uses systems of differential (and sometimes integral) equations to model the spread of epidemics, dynamical systems in immune-epidemiology, dynamics of infectious diseases, population dynamics. Specifically, he has studied the dynamics within or between pathogen-host, and explored how the variability in immune system response within an endemic environment affects an individual vulnerability.

Hannes Stoppel (hannes.stoppel@uni-muenster.de), teacher for mathematics, physics and computer science at high school level in Germany.

Hannes Stoppel studied mathematics and physics from 1989 to 1994. He trained as a high school teacher for these subjects from 1994 to 1996. Together with Birgit Griese he wrote a book about linear algebra at university level, which was published in 1998 and is right now in its 8th edition. Until 2013 he worked as a teacher for mathematics, physics and computer science at a high school in Germany. Because of his interest in mathematics education, particularly in the usage of Graphic Calculators (GC) and Computer Algebra Systems (CAS) in mathematics education, he started his own research. In 2001 he wrote a mathematics book for students at the transition from high school to university, with special emphasis on the usage of CAS. From 2004 on Hannes devoted himself to probability theory and statistics in high school, with reference to the possibilities of CAS and GC, and wrote material for teachers, including advanced training courses for teachers in probability and statistics. Since 2012 Hannes has been working in mathematics education at the University of Münster.

Hannes has not lost sight of physics, though – especially of particle physics and cosmology – and joined working groups of high school teachers, at universities and at CERN research centre in Geneva, Switzerland, designing and implementing material for high school education over several years, published in 2016.

Over the last 15 years, Hannes has worked with gifted students in projects and research concerning mathematics, physics and computer science. At University of Münster, he uses his courses to combine lessons and research. Recently, he has started exploring student's competences and beliefs as well as their development, which particularly led to research results concerning creativity. The conclusions were published in 2013.

Roberto Tortora, Associate Professor in Math Education at University "Federico II" of Naples, Italy. Tortora gives courses for Math students and for prospective primary school teachers, and postgraduate courses for future secondary school teachers.

He is author of more than 70 publications. His research interests include Mathematical Logic, Foundations of Mathematics and Mathematical Education, with particular emphasis on Teacher Education.

In some papers a theoretical 'resonance' model (supported also by results from Neurosciences) is proposed, both of the cognitive process and of the role of the teacher as 'resonance mediator'. More recently, Tortora's interests focus on epistemological, linguistic and cognitive aspects of basic mathematical topics, with applications to teacher education. In particular, some studies investigate how ambiguities and errors of students can be seen as opportunities and resources to build new knowledge.

James Watson, Head of Technology Services and an Assistant Professor in the Library at Bronx Community College, City University of New York.

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