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# 8. ABDUCTION IN PROVING

## *A Deconstruction of the Three Classical Proofs of "The Angles in Any Triangle Add 180°"*

> What the rigorous proof of a theorem, say the proposition about the sum of the angles in a triangle, establishes is not the truth of the proposition in question but rather a conditional insight to the effect that the proposition is certainly true provided that the postulates are true.
>
> Carl G. Hempel, Geometry an Empirical Science, 1956, p. 1637

ABSTRACT

This chapter is framed both within the Kantean notions of sensible and intellectual intuitions and within the Peircean notion of collateral knowledge and classification of inferential reasoning into abductive, inductive, and deductive. An overview of the Peircean notion of abduction is followed by a sub-classification of abductions according to Thagard and Eco. The constructive nature of the process of proving seems to involve not only deductive reasoning but also abductive reasoning. The later plays an essential role both in the anticipation of auxiliary constructions and in the construction of geometric arguments. The chapter presents a summary of Kant's classification of the proposition "the angles in any triangle add 180°" as a synthetic proposition. It also presents a deconstruction of the three classical proofs of this proposition—the Pythagorean proof, Euclid's proof, and Proclus' proof. This deconstruction discloses both the Greek analysis-synthesis method of proving and the role of abduction in the analysis phase. It also argues that the deconstruction of classical proofs has pedagogical and epistemological value in the teaching-learning of geometry.

INTRODUCTION

The proposition "the angles in any triangle add 180°" plays a fundamental role in Euclidean geometry. It appears to state a simple fact, but this simplicity hides an intrinsic complexity. Kant classifies this proposition as a synthetic proposition rather than an analytic one. There are only three classical proofs of this proposition—the Pythagorean proof, Euclid's proof, and Proclus' proof. Reading these proofs entails no difficulty. Constructing them is another matter. Reading and constructing proofs are two different processes. The main goal of this chapter is to deconstruct these

*A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 155–179.*

three classical proofs to have an insight into the role of abduction in the creation of auxiliary lines to construct valid geometric arguments. This deconstruction illustrates the analysis-synthesis method of proving employed by the Greek mathematicians. This method is in essence a working-backwards strategy effective not only for proving geometric propositions but also for problem-solving (Proclus, 1970). It illustrates that proving is a constructive process and, at the same time, it helps to answers questions often asked by students: Where do proofs come from? How do I start a proof? These questions indicate the cognitive need to have a general strategy, a heuristics to guide the proving process.

The chapter is divided into seven sections. The first presents a theoretical rationale to support the assertion that proving is a constructive process. The second argues that the production of proofs is the mind's activity rooted in observation, sensible intuition, intellectual intuition, collateral knowledge, and inferential reasoning. The third presents, in a simplified form, Peirce's classification of inferential reasoning into inductive, abductive, and deductive and focuses on the process of abduction. The fourth presents Kant's examination of the proposition "the angles in any triangle add 180°" as a synthetic proposition. The fifth presents the Greek analysis-synthesis method of proving. The sixth presents a deconstruction of the three classical proofs and an examination of the role of abduction in the creation of auxiliary constructions and geometric arguments. The last puts into perspective the essential role abductive reasoning has in the analysis phase of the analysis-synthesis method of proving. This section also brings to the fore the pedagogical value of deconstructing classical proofs to learn about the analysis-synthesis method of proving and to have them as paradigmatic illustrations of proving as a constructive process.

## PROVING AS A CONSTRUCTIVE PROCESS

Hersh (1997) synthesizes new and old philosophical perspectives of mathematics into two essential ones—the absolutist and humanist perspectives. Under the first, mathematics is seen as a system of absolute truths independent of human involvement, and mathematical proofs are seen as external and eternal only to be admired and accepted. Consequently, the purpose of proofs is to certify the admission ticket for theorems and propositions into the catalogue of absolute truths. Under the second, mathematics is seen as a system of truths that are the product of playful, consensual, social, cultural, and historical human activity.

What is the relation between these two philosophical perspectives about the nature of mathematics and the actual teaching-learning of proof and proving? The belief on either perspective is, consciously or unconsciously, transmitted from teachers to students. On the one hand, a teacher with an absolutist perspective will present students with the shortest and/or the more general proofs. These proofs are aesthetically pleasing and obvious only to those who have a holistic knowledge of the subject matter and who can appreciate their aesthetic value and conceptual significance. The role of these proofs is mathematical persuasion and the acceptance of mathematical rituals (Hersh, 1993).

On the other hand, a teacher with a humanist perspective will analyze given proofs and construct new ones with the purpose of understanding mathematical propositions and their interrelations. The humanist teacher will choose and accept more enlightening proofs and not necessarily the more general and sophisticated. For this teacher, proving is a thought experiment, an inquiry process by which and through which valid logical arguments are constructed. The role of proofs is to develop reasoning and mathematical conviction (Hersh, 1993). Research studies on proof and proving in geometry, implicitly or explicitly, support and promote the humanist perspective of mathematics (e.g., Hanna, 1989, 1995; Mariotti, Bartolini, Boero, Ferri, & Garuti, 1997; Garuti, Boero, & Lemut, 1998; Douek, 1999, 2007; Duval, 2007; Mariotti, 2007).

The humanist perspective is extended when it is acknowledged that students often experience abductive reasoning. This reasoning is often reported as the students' "Aha! moments." Abductive reasoning is at the root of the construction of conjectures and the construction of mathematical arguments. It seems that it appears at young ages in arithmetical thinking (e.g., Sáenz-Ludlow, 1997; Reid, 2002; Norton, 2009), in proving processes (e.g., Arzarello, Andriano, Olivero, & Robutti, 1998; Ferrando, 2000; Reid, 2003; Rivera, 2008), and in problem solving (e.g., Cifarelli & Sáenz-Ludlow, 1996; Cifarelli, 1999; Rivera & Becker, 2007). This type of reasoning sheds light not only on the process of proving and problem solving but also on the process of teaching and learning.

Problem solving and proving rooted in the construction of logical arguments with the purpose of understanding and convincing oneself and others was an idea advanced by the ancient Greeks (cf. Kadunz chapter on argumentation, this volume). For example, Proclus asserts that every problem and every geometric theorem contains in itself five elements: (1) the enunciation states which premises are given and the conclusion sought; (2) the specification states axioms, known theorems, and definitions; (3) the construction and machinery adds what is needed in order to draw the conclusion sought; (4) the proof deduces the truth of the conclusion from the premises; and (5) the closing returns to the enunciation, confirming what has been demonstrated (Heath, 1956, vol. I).

Polya's heuristics (1945/1973) for solving problems is in tune with Proclus' insights about the process of proving mathematical propositions: (1) understand the proposition or problem, what is given and what is asked; (2) devise a plan, construct a diagram, make an orderly list, eliminate possibilities, use direct reasoning, work backwards; (3) carry out the plan, work carefully, discard it if it did not work and choose another; and (4) look back and reflect on what worked and what did not, and on the significance of the problem in the context of other problems. When we consider proving as a particular case of problem-solving, Polya's heuristics can also be useful in the deconstruction proofs as well as in the production and reproduction of proofs.

Polya (1945/1973,1962/1985), Freudenthal (1973), and Hempel (1956) argue that *proving* is, at the same time, a process and a product. This view permeates their mathematical and pedagogical works when they motivate and guide the reader to construct logical arguments and to validate mathematical propositions. Hempel argues that proving, as a process, is essentially a conceptual analysis that discloses the assertions *concealed* in a given set of premises and the commitment one makes when they are accepted. Freudenthal argues that geometry, more than any other mathematical subject, disciplines the mind because of its closest relation to logic, and that it can only be meaningful when its relations are explored in the experiential space. For him, geometry offers opportunities to mathematize reality and to make discoveries.

In general, when problem-solving or proving, it is useful to have a heuristics, a method, a general procedure. Proclus' and Polya's heuristics are like road maps. They help to anticipate the territory and allow for the preparation of a plan to explore it. Road maps do not induce anyone to follow any major highway or any secondary road. They only insinuate different possibilities to get to the final destination. Heuristics, like road maps, only insinuate a plan of action to construct one or more arguments from which the conclusion of a proposition follows from the premises in a logical and valid manner. A heuristics may also facilitate the emergence of abductive reasoning.

Both Proclus and Polya consider the construction of geometric diagrams an essential step in the understanding geometric propositions because they unveil what is explicit or implicit in the premises. Another important step that naturally follows is the observation of geometric diagrams in order to coordinate and integrate geometric relations. Similar ideas about the observation of geometric diagrams are also expressed by Peirce and Mander. Peirce argues that "the geometer draws a diagram…and by means of observation of that diagram…he is able to synthesize and show relations between the elements which before seemed to have no necessary connection" (CP 1.383, emphasis added). Mander (1947), in his book "Logic for the Million", argues that the observation of geometric diagrams complements perception and inference to give rise to recognition and differentiation.

Both heuristics and geometric diagrams co-exist with abductive reasoning. This kind of reasoning aids the creation of conjectures, the conceptualization of auxiliary constructions, and the creation of novel ways of combining premises and collateral geometric knowledge. This is to say that heuristics, geometric diagrams, and abductive reasoning have a great epistemological value which is often not emphasized.

Actively producing a proof in contrast to passively reproducing a proof requires an insightful playing of the mind to conceptualize and re-conceptualize geometric diagrams in order to "see" geometric relations that facilitate a logical passage from the given premises to the conclusion. In the following section, we make an effort to comprehend the mind's activity in the process of proving. To do this, we borrow from the epistemological perspectives of Aquinas, Kant, and Peirce.

## MIND'S ACTIVITY IN THE PROCESS OF PROVING

Centuries ago, Thomas Aquinas (1266/2003, Summa Theologica, q. 85, a. 2) recognized that the mind performs two kinds of activity—internal and external. The internal activity is that activity that remains within a Person such as seeing with the mind's eye. In this activity, the mind formulates to itself a model of something seen or never seen before. In contrast, the external activity is that activity that passes over to a "thing" outside the mind. For example, pointing, moving, manipulating, and encoding thoughts into external representations. The internal and external activities of a Person are interrelated and the latter somewhat manifests the former. Moreover, in the interaction with others, a Person constructs and co-constructs cycles of internal-external activity in a synergistic manner.

This internal-external activity of the mind is not independent of the relation between the mind and the object of thought. According to Kant, the mind could create an object or be influenced by an object.

> Theoretically, there are two ways in which a mind, or mode of knowledge, can be directly related to an object. If the object depends upon the mind, then the mind is active with respect to [the object],…such a relation is given the title of 'intellectual intuition'. Alternatively, the mind may wait passively upon the object, and establish a relation to [the object] only in so far as [the object] affects the mind. This capacity of the mind to be affected by objects is entitled "sensibility," and the product of such affection is "sensible intuition." (Wolff, 1973, p. 73, emphasis added)

According to Kant, when the mind creates an object, this object depends on the activity of the mind (the mind is in a creative mode) and he calls this relation *intellectual intuition*. When the mind is influenced by an object, this object is received by the mind (the mind is in a receptive mode) and he calls this relation *sensible intuition*. That is, when an *object* affects the senses directly, it produces a variety of sensible intuitions—a manifold of sensations and perceptions. This manifold carries with it two kinds of elements: (i) a subjective or material element (colours, taste, hardness, etc.), which has no cognitive value; and (ii) a formal or knowledge-giving element, which is the spatiotemporal organization and ordering of sensations that facilitates the formation of perceptual judgments (Wolff, 1973). Then the internal-external activity of a Person, mathematical or not, is intimately connected with intellectual intuitions, sensible intuitions, and perceptual judgments.

For Kant, a *judgment*, in general, is an act of the intellect in which two ideas, comprehended as different, are compared for the purpose of ascertaining their agreement or disagreement (Wolff, 1973). Judgments are usually expressed in propositions composed by subject, predicate, and copula (i.e., a word or set of words that act as a connector between the subject and the predicate of the proposition).

Borrowing from Kant, Peirce argues that *perceptual judgments* on the particular and concrete contain general elements from which one can intuit general patterns, universal propositions, and principles (CP 5.180–212). Perceptual judgments, he says, are also related to the more deliberate and conscious processes of inferential reasoning, and this reasoning is continuous and carries with it the vital power of self-correction and refinement (Peirce, 1992, Vol. 1).

For Peirce, *all* knowledge is a self-corrective process of continuous refinement. He contends, following Kant, that there is nothing in the intellect that has not been first in the senses (CP 8.738). He argues that realities compel us to put some things into very close relation and others less so. But in the end, it is only the genius of the mind that takes up all those *hints of sense*, adds immensity to them, makes them *precise*, and shows them in intelligible forms of intuition of space and time (CP 1.383).

Both Kant and Peirce deeply value the epistemological power of *observation.* They consider that observation is tied to judgment, and that judgment is tied to intentionally planned reasoning. Peirce contends that any inquiry activity fully carried out by a Person is rooted in observation and perceptual judgment. For example, he argues, that when different people observe a geometric diagram, they are able to "see" different relations, some *perceived* by the senses and some *inferred* with the aid of collateral knowledge. He also considers that this collateral knowledge is a prerequisite in the apprehension and the construction of new meanings (Peirce, 1992, Vol. 2). Consequently, it can be said that geometric diagrams, observation, sensible intuitions, intellectual intuitions, collateral knowledge, and inferential reasoning (induction, deduction and abduction) are essential components in the process of proving.

There is no doubt that visual imagination, visual observation, and visual thinking play an epistemic role in the observation of geometric diagrams (Arnheim, 1969; Giaquinto, 2007). These diagrams are in essence icons of *possible* relations. They have the potential to bring to the fore logical connections between the explicitly or implicitly given in the premises of a geometric proposition and the Person's collateral geometric knowledge. These connections are essential in the conceptualization and re-conceptualization of geometric arguments to reach, in a convincing and valid manner, the conclusion of the proposition. Thus any given proof of a geometric proposition is the product of the internal-external constructive thinking process of the mind.

## PEIRCE'S CLASSIFICATION OF INFERENTIAL REASONING

Peirce, logician and mathematician himself, argues that one of the tasks of logic is the classification of inferences. He also argues that inferences and logical arguments are at the very heart of mathematical inquiry and that inferences are also at the very heart of the proving process. By *inference* he means any cognitive activity that could be internal or external, not merely conscious abstract thought (Davis, 1972).

Peirce retraces inferential reasoning from the simplest forms of sensation and perception to the most elaborated forms of semiotic activity. He considers that *each* inference draws upon former ones making logical inferences a historical process that requires *continuity* and *time* (Davis, 1972; Sheriff, 1994; Colapietro, 1989).

In general, Peirce's classification of inferential reasoning borrows from Kant's notion of perceptual and intellectual judgments. Figure 1 shows his classification of inferences into *ampliative* (*synthetic*) and into *explicative* (*analytical*). He subdivides *ampliative* reasoning into *inductive* and *abductive*, while *explicative* reasoning is classified only as *deductive.*

For centuries, inductive and deductive reasoning were known as the only forms of inferential reasoning. Less than two centuries ago, Peirce recognized a new form of reasoning that was neither inductive nor deductive. He called it *abductive reasoning*. He describes it as an inference through which and by which the mind, indirectly, comes to know the existence of an object by means of the active relation of the mind with the object (material or conceptual), relation that is based on intellectual intuition. This intellectual intuition regards "the abstract in concrete forms by the realistic *hypostatization* of relations" (CP 1.383, emphasis added).

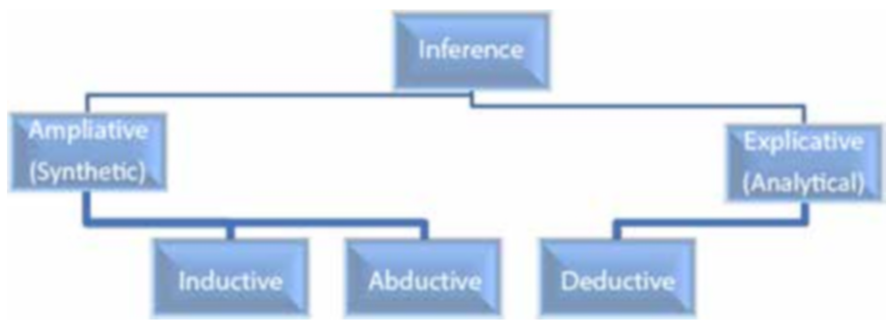



*Figure 1. Peirce's classification of inferential reasoning (diagram adapted from Peirce 1878)*

For Peirce, to interact with the world, in any way, is to make judgments of inductive, abductive, and deductive nature. Induction *evaluates* and shows that something *is* actually operative; abduction merely suggests that something *may be* (may-be or may-not-be); and deduction *explicates* and shows that something *must be* (Fann, 1970, p. 51).

Prior to Peirce's recognition of abduction as a form of inference, the meaning of 'abduction' was encoded in syllogisms in which the minor premise was *only probably true* (better known as *apagoge*). In his early work, Peirce also focused on syllogisms and on the role of the character of specific cases and classes (CP 2.508, 511). A case *S* might be a member of a class *P* and have a number of characters *M*. See Chart 1.

| *Induction* | *Abduction* | *Deduction* |
|---|---|---|
| S’ S’’ S’’’, etc. taken at random as M’s | Any M *is*, for instance, P’P’’P’’’, etc. | Any M *is* P |
| S’ S’’ S’’’, etc. *are* P | S *is* P’P’’P’’ | S *is* M . |
| :. Any M *is probably* P | :. M *is probably* P | :. S *is* P |

*Chart 1. Peirce's induction, abduction and deduction in terms of syllogisms*

Later, he describes his classification of inferential reasoning in terms of *rule*, *case*, and *result* (CP 2.623–625). Chart 2 pulls together the three types of inferences with the illustrative example given by Peirce. While in induction the *general rule* is *deduced* and in deduction the *general rule* is *given*, in abduction the *general rule* is *temporarily chosen*. In other words, abduction is the step between a fact (*case*) and its cause or origin (*general rule*). Therefore, abduction, for Peirce, is the *provisional entertainment of a rule* or *hypothesis* (that must undergo further testing) to explain that the particular *case* will follow by *deductive inference*.

| *Induction* | *Abduction* | *Deduction* |
|---|---|---|
| <u>*Case*</u><br>These beans *are* from this bag. | <u>*Rule*</u><br>*All* the beans from this bag *are* white. | <u>*Rule*</u><br>*All* the beans from this bag *are* white. |
| <u>*Result*</u><br>These beans *are* white. | <u>*Result*</u><br>These beans *are* white. | <u>*Case*</u><br>These beans *are* from this bag. |
| :. <u>*Rule*</u><br>*All* the beans from this bag *are* white. | :. <u>*Case*</u><br>These beans *are* from this bag. | :. <u>*Result*</u><br>These beans *are* white. |

*Chart 2. Peirce's inferential classification in terms of rule, case and result*

In his early work, Peirce emphasizes the differences, in logical form, between induction, abduction, and deduction. In his later work (e.g., "Lectures on Pragmatism", CP 5.14–212), he shifts the emphasis to the function satisfied for each kind of reasoning. The logical form of abduction is then reduced to

| | |
|---|---|
| *C* | *The surprising fact, C, is observed* |
| *A implies C* | *But if A were true, C would be a matter of course* |
| *A* | *Hence, there is a reason to suspect that A is true* |

At this point in time, abduction becomes, for Peirce, and “explanatory hypothesis”. Then his criteria for a good abduction comes to include, at least, that which “must explain the facts” (CP 5.197). He then argues the difference between his three forms of inference as follows: *abduction* explains the case by introducing a new rule; *induction* evaluates the consequent by comparing the conclusion drawn from it to experience; and *deduction* draws necessary conclusions from the consequent of the abduction

In recent years, philosophers and semioticians had come to the realization that not all abductions are of the same nature. Some would require a higher level of creativity and intellectual sophistication while others require a higher level of thinking to see intellectual connections. Chart 3 presents recent sub-classifications of abductive reasoning.

Thagard (1978) classifies abduction into *overcoded abduction/hypothesis* and *abduction* proper. By *overcoded abduction* he means an abduction for which *the hypothesized rule* is not a genuine creation of the mind, but rather it is automatically or semi-automatically encoded in the *case*. That is, when a Person proposes an *overcoded abduction* his effort is *in the isolation of an already encoded rule* to which the *case* is correlated. In contrast, by *abduction* (proper) he means that the Person’s effort is in the novel creation of a *rule*.

| | | | |
|---|---|---|---|
| Peirce 1878 | *ABDUCTION* Provisional hypothesis suggesting that something may-be or may-be-not | | |
| Thagard 1978 | *Overcoded Abduction* Hypothesis implicitly encoded | *Abduction (proper)* | |
| Eco 1983 | *Overcoded Abduction* Hypothesis implicitly encoded | *Undercoded Abduction* Hypothesis selected from a set of equiprobable possibilities | *Creative Abduction* Hypothesis invented *ex novo* |

*Chart 3. Thagard’s and Eco’s sub-classification of Peirce’s abduction*

Eco (1983/1988) continues Thagard’s sub-classification and further subdivides *abduction* (proper) into *undercoded abduction* and *creative abduction*. By *undercoded abduction* he means an abduction in which the Person’s effort is *in the selection of a rule* from a series of *equiprobable rules* put at his disposal by his current knowledge about the world. By *creative abduction* he means those abductions in which the Person’s effort is *in the ex-novo creation of a rule*; for example, Copernicus’ new conceptualization of the relation between the motions of the sun and the earth. These abductions are revolutionary discoveries that change established scientific paradigms (Kuhn, 1962).

In geometry, abduction, in any of its forms, plays a role in the conceptualization of auxiliary constructions, in the observation and visualisation of relations implicit in geometric diagrams, and in the conceptualization of geometric conjectures. It also plays a role in the selection, coordination, and organization of collateral knowledge to generate geometric arguments to prove geometric propositions in a logical, valid, and convincing manner.

## KANT'S ANALYSIS OF THE PROPOSITION "THE ANGLES IN ANY TRIANGLE ADD 180°"

In this section we present Kant's analysis of two geometric propositions: (1) a triangle has three sides, and (2) the sum of the angles in any triangle is 180°. His examination of these two propositions illustrates the distinction between analytic and synthetic propositions and between a priori and a posteriori propositions. We acknowledge the philosophical debate about the usefulness of this distinction in different fields of knowledge. Nonetheless, in this chapter, this differentiation brings to the fore insights into the nature of the proposition about the sum of the interior angles of any triangle. Kant's philosophical analysis provides us with a mathematical insight into the complexity imbedded in the mathematical simplicity of this fundamental proposition of plane geometry. It also sheds light onto the question often asked by students, "Where do definitions and theorems come from?" According to Kant, they come from intellectual intuitions.

Kant contends that geometry, being a branch of mathematics, contains *a priori* analytic and *a priori* synthetic truths about space and things in space (Wolff, 1973). For him, mathematical propositions are the result of judgments and intellectual intuitions *a priori* to experience. The analytic-synthetic and the *a priori-a posteriori* distinctions, combined, yield four types of propositions: analytic *a priori* and analytic *a posteriori*; synthetic *a priori* and synthetic *a posteriori*.

Kant argues that analytic propositions depend on the actual meaning of the words that constitute them. Therefore, these propositions cannot be considered *a posteriori* to experience. The predicate of an analytic proposition is inherent to the subject of that proposition. Thus, *all* analytic propositions are *a priori* since we only need to consult the meanings of the words used.

He also argues that new knowledge is possible only through synthetic *a priori* propositions. The predicate of a synthetic *a priori* proposition is *not* inherent to the subject of that proposition. Nonetheless, some knowledge can also be achieved through synthetic *a posteriori* propositions because concrete cases allow us to gain insight into the general pattern.

Kant analyzes the proposition "*a triangle has three sides*" (1) as an *a priori* analytic proposition, and the proposition "*the angles in any triangle add 180°*" (2) as an *a priori* synthetic proposition. These two propositions seem simple and straightforward to students but not so to philosophers and mathematicians. Chart 4 summarizes Kant's analysis.

Kant considers that the *truth* of propositions (1) and (2) is known prior to any physical experience. Proposition (1), he says, is by necessity analytic because it merely reveals logical relations between the meaning of the words, and its denial involves a contradiction. Proposition (2) is a synthetic universal proposition because it reveals something substantive about the character of space (Wolff, 1973). This means that the predicate of the proposition (i.e., 180°) is not inherent to the subject of the proposition (i.e., the angles in a triangle), and its denial does not result in a contradiction.

Both propositions are independent of experience in the minds of mathematicians who construct, or some would say, discover them. However, for school students of different ages, the first proposition is simply a definition to be accepted. Some students believe the truth of the second proposition only after measuring angles of triangles in the real world or after proving the proposition. Consequently, it can be said that for students, who encounter geometry for the first time, the truth of the second proposition is *a posteriori* to experience. Nonetheless, one thing is certain. Students *inherit* this *a priori* synthetic proposition from the mathematicians.

Proposition (2), whether it is synthetic *a priori* or *a posteriori*, is fundamental to Euclidean geometry. This proposition and its proof were first credited to the Pythagoreans. Later, Euclid presented a different proof—Proposition 32, Book 1 of The Elements. Even later, Proclus presented another proof in his Commentaries to the Book 1 of The Elements. In Section 6 we present a deconstruction of these classical proofs. This deconstruction not only brings forward the Greek analysis-synthesis method of proving but it also sheds light into the role of abduction in the process of proving.

Reading and understanding a given written proof of a mathematical proposition is a linear deductive process. However, one thing is to *read* and *understand* a written proof and quite another is to *produce* it. To produce a proof is to engage in a nonlinear process of thinking which interconnects abductive, inductive, and deductive inferences. This process seeks to generate, at least, one geometric argument to logically justify the conclusion. Thus a written proof is only the product of a thinking process—the process of proving.

## ANALYSIS-SYNTHESIS METHOD OF PROVING

The justification of a mathematical proposition can be done directly or indirectly. The direct method starts with the given premise P and then arrives at the conclusion Q using inferential reasoning and appropriate collateral knowledge. This method is symbolically expressed as ($P \rightarrow Q$). When done indirectly one could either use the contrapositive method or the contradiction method. The contrapositive method negates the conclusion and then arrives at the negation of the premise. This method is symbolically expressed as ($-Q \rightarrow -P$). The contradiction method starts with the acceptance of the premise P and the negation of the conclusion ($-Q$) and, from this conjunction a contradiction of a statement or principle within a mathematical

| *NATURE OF PROPOSITIONS* | *ANALYTIC PROPOSITION*<br>A proposition known to be true by knowing *only* the meanings of the words, i.e., justified by virtue of meaning. | *SYNTHETIC PROPOSITION*<br>A proposition known to be true by knowing *not only* the meanings of the words *but also* something about the world. |
|---|---|---|
| | *Proposition 1*<br>A triangle has three sides<br>(a necessary proposition) | *Proposition 2*<br>The angles in any triangle add 180º<br>(a universal proposition) |
| *A PRIORI PROPOSITION*<br>A proposition whose justification *is not* grounded in experience, but it can be validated through experience. | • A proposition whose truth value is independent of experience.<br>• A proposition necessarily true because the predicate (*three sides*) is inherent to its subject (*tri-angle or better tri-lateral*).<br>• If negated, it does result in a contradictory proposition. | • A proposition whose truth value is independent of experience.<br>• A proposition which predicate (*180º*) is not inherent to its premise (*the angles in any triangle*).<br>• If negated, it does not result in a contradictory proposition. |
| *A POSTERIORI PROPOSITION*<br>A proposition whose justification *is* grounded in experience and can be validated through experience. | | • A proposition whose truth value can be justified and validated by observation of concrete triangles. |

*Chart 4. Analytic-synthetic and a priori-a posteriori nature of propositions*

system is pursued ($C \wedge -C$). This method is symbolically expressed as $[(P \wedge -Q) \rightarrow (C \wedge -C)] \leftrightarrow [P \rightarrow Q]$.

Then it is not by chance that Mariotti, Bartolini, Boero, Ferri, and Garuti (1997) propose a system-definition of *mathematical theorems* as the triad (statement, proof, theory within which the statement makes sense). This is to say that a proof of a mathematical proposition does not happen in isolation but in the context of a system of mathematical concepts. This system contains, among other things, principles, axioms, definitions, and theorems that, in one way or another, are associated with one another (Hempel, 1956).

Producing a proof of a mathematical proposition is to produce a mathematical argument to prove that once the premise is accepted as true in a mathematical system, then the conclusion that follows need to be true in that system. There is no doubt that some mathematical arguments are more difficult to produce than others. One of the reasons is that some mathematical propositions, for example universal propositions, are stated in a single sentence (subject-verb-predicate), and the predicate is not inherent to the subject of the proposition.

They are *a priori* synthetic propositions in Kant's sense. For example, "*Prime numbers are infinite*", "$\sqrt{2}$ *is an irrational number*", or "*The angles in any triangle add 180°.*"

In order to produce a mathematical argument to prove any of these propositions, the mind is forced either: (i) to generate *ex novo* a mathematical contradictory argument, or (ii) to start "backwards" from the predicate and construct reversible *inferences* in order to arrive at some general mathematical principle and, then, reverse the inferences. This backwards method of proving was conceptualized by the Greeks and was called the *analysis-synthesis method.* Proclus (1970) contends that even the more obscure problems in mathematics can be pursued through this method. He also contends that Plato taught this method in his Academy even though it does not mean that he discovered it.

Heath (1921/1981, vol. 2) explains that the analysis-synthesis method has two well differentiated phases: (a) the backwards phase or *analysis* and (b) the forward phase or *synthesis*. The *analysis* phase traces back an acknowledged fact or principle starting from the desired conclusion. The *synthesis* phase reverses the steps of the analysis. In order to do this, each step of the chain of inferences in the analysis phase *has to be unconditionally reversible*.

> In *analysis* we assume that which is sought as if it were already *admitted*, and we inquire what it is from which this results, and again what is the antecedent cause of the later, and so on, until by so *retracing our steps* we come upon something *already known or belonging to the class of the first principles*, and such a method we call *analysis* as being solution backwards. However, in the process of *synthesis* we *reverse* the process. That is, we take as already done that which was last arrived at in the analysis and, by *arranging* in their natural order as *consequences* what *before* were *antecedents*, and *successively*

*connecting them one with the other*, we arrive finally at the *construction of what was sought.* (Heath, 1921/1981, vol. 2, p. 400, emphasis added)

The analysis-synthesis method is different from the indirect methods of proving either by contrapositive or by contradiction (Heath, 1921/1981, vol. 2). This method of proving seems to have been used in the classical proofs of the proposition "the angles in any triangle add 180°."

The Pythagorean proof of this proposition is often presented in geometry textbooks. It starts by giving the auxiliary construction and then the deductive argument follows. Reading and understanding this proof entails almost no effort because of its aesthetic simplicity. After all, the written proof (the final *product* of the proving process) is only a linear deductive organization of abductive and deductive inferences that were previously generated to construct a viable and logical geometric argument. However, the abductive nature of the auxiliary construction is anything but linear, and it is left unexplained. Thus the creative nature of the proving process is left untouched and implicit.

Constructing or producing a geometric proof, in contrast to *reading* or *re-producing* a proof, requires an active playing of the mind (internal and external) to bring into play auxiliary geometric constructions, to make geometric diagrams, and to observe and interpret them. In this process, it is also essential to bring into play appropriate *collateral geometric knowledge* to provide for the emergence of necessary logical relations from which the validity of the geometric argument follows.

The next section presents a *deconstruction* of each of the three classical written proofs of the proposition "The angles in any triangle add 180°." It is argued here that to prove this proposition the analysis-synthesis method was used by the Pythagoreans, Euclid, and Proclus. This deconstruction highlights the important role played by both abduction and collateral geometric knowledge.

## A DECONSTRUCTION OF THE THREE CLASSICAL PROOFS

It could come as a surprise that there are only three wellknown proofs of this fundamental proposition: the Pythagorean' proof, Euclid's proof, and Proclus' proof. This small number of proofs is in sharp contrast to the large number of proofs constructed for another fundamental geometric proposition "*The Pythagorean Proposition*" (Loomis, 1940/1972). This contrast between the number of proofs for each proposition points to their different nature. In the first, the predicate (180°) is not intrinsic to the subject of the proposition (the angles in any triangle). In the second, the predicate (the sum of the square of the sides of a right triangle equals the square of its hypotenuse) is intrinsic to the subject of the proposition (a right triangle).

### *Pythagorean Proof and Euclid's Proof*

There are differences and similarities between these two proofs. Figure 2 presents them side by side. The Pythagorean proof (Figure 2a) uses the parallel postulate to construct one-and-only-one auxiliary parallel line to one of the sides of the triangle which passes through the opposite vertex. Let's notice that this parallel line is external to the triangle (Proclus constructs parallel lines interior to the triangle). To construct their logical argument, the Pythagoreans choose a side (BC) and its opposite vertex (A). Then, the other two sides of the triangle (BA and CA) are re-conceptualized as transversals to the parallel lines (BC and xy). Afterwards, the congruence of the alternate interior angles formed between parallel lines and their transversals is used to show the straight angle xAy is, in fact, congruent to the angles of the triangle. Therefore, the logical conclusion is that the sum of the angles inside the triangle is the same as the measure of the straight angle—180°.

The *analysis phase* of this proof is rooted in three *overcoded abductions*: (i) the selection of a side of the given triangle and the *appropriate point* through which the parallel line should pass (the vertex opposite to the side into focus); (ii) the construction of a parallel line to form a straight angle which is known to measure 180°; and (iii) the congruence of the straight angle with the three interior angles of the triangle.

Once the side of the triangle and the vertex was chosen, the properties of the angles formed between parallel lines and transversals accounted for the relation between the measure of the sum of the interior angles of the triangle and that of the straight angle.

The *synthesis phase* is the linear and deductive organization that captures the reverse order of the argument produced in the analysis phase. The synthesis phase starts with the hypothesized construction in the analysis phase and ends up justifying the hypothesized congruence of the straight angle and the sum of the interior angles of the triangle.

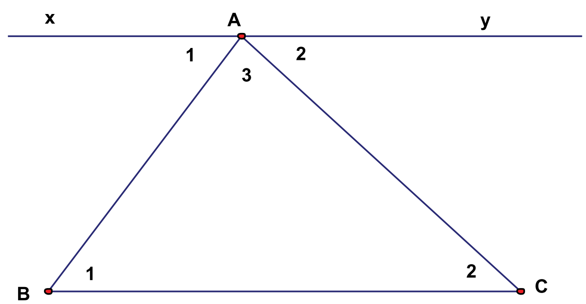
It is obvious that the written proof in Figure 2a leaves silent the creative part of the proving process—the abductive reasoning that accounted for the auxiliary construction of the parallel line and the relation between the 180° measure of the straight angle and the sum of the interior angles of the triangle.

Figure 2b presents the Euclidean proof (Proposition 32, Book 1 of The Elements). In this proposition Euclid presents not one but two propositions. The first introduces the notion of *external angles* of triangles in contrast to the notion of *interior angles*. He states that an external angle is equal to the sum of the two opposite (remote) interior angles. The second states that the sum of the interior angles of a triangle is equal to two right angles.

In the proof of this proposition, Euclid introduces two auxiliary constructions. First, he extends one of the sides of the triangle (side BC) to construct the straight angle BCD. Second, he uses the parallel postulate to construct one-and-only-one line CE parallel to side BA and passing through vertex C. He, then, re-conceptualizes sides AC and BC as transversals to the parallel lines BA and CE. Subsequently, he uses the congruence of the angles formed between parallel lines and transversals to justify the relation between the external angle (<ACD) at vertex C and the sum of the two remote interior angles (<A1 and <B1). In addition, he justifies the straight

THE PYTHAGOREAN PROOF

Prove that the sum of all three angles of any triangle is 180°.




We construct a triangle ABC and from the vertex A we construct a parallel line xAy to the side BC.
Then, $\angle A_1$ is congruent to $\angle B_1$ as alternate interior angles. Similarly, the $\angle A_2$ is congruent to $\angle C_2$ as alternate interior angles.

But $\angle A_1 + \angle A_2 + \angle A_3 = 180°$ (1) because they form a straight angle.
We proved that $\angle A_1 = \angle B_1$ and $\angle A_2 = \angle C_2$

Then we replace in the relationship (1) the congruent angles and we have $\angle B_1 + \angle C_2 + \angle A_3 = 180°$ (2)

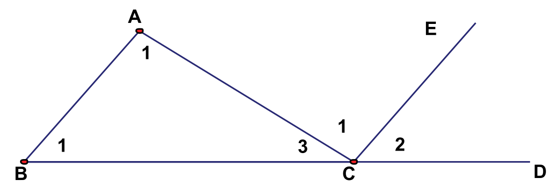
From the figure we have $\angle B_1 = \angle B$, $\angle A_3 = \angle A$, and $\angle C_2 = \angle C$

If we replace in the relationship (2) the congruent angles then we have
$\angle A + \angle B + \angle C = 180°$

*Figure 2a. Pythagorean proof*

EUCLID'S PROFF - Proposition 32 BOOK 1 in the "ELEMENTS"

In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.




Let be a triangle ABC and let one side of it BC be produced to D
Let CE be drawn through the point C parallel to the straight line AB.

Then, since AB is parallel to CE, and AC has fallen upon them, the alternate interior angles $A_1$ and $C_1$ are congruent to one another.
Thus $\angle A_1 = \angle C_1$ .

Again, since AB is parallel to CE, and the straight line BD has fallen upon them, the corresponding angles $B_1$ and $C_2$ are equal to one another. Thus $\angle B_1 = \angle C_2$
So, $\angle A_1 + \angle B_1 = \angle C_1 + \angle C_2 = \angle ACD$

Therefore, the exterior angle ACD is equal to the two interior and opposite angles $A_1$ and $B_1$.

But $\angle C_1 + \angle C_2 + \angle C_3 = 180°$ (1) because they form a straight angle.
We proved that $\angle A_1 = \angle C_1$ and $\angle B_1 = \angle C_2$

Then we replace in the relationship (1) the congruent angles and we have $\angle A_1 + \angle B_1 + \angle C_3 = 180°$ (2)

From the figure we have $\angle A_1 = \angle A$, $\angle B_1 = \angle B$, and $\angle C_3 = \angle C$ Thus,
$\angle A + \angle B + \angle C = 180°$ .

*Figure 2b. Euclid's proof*

angle (<BCD) as the sum of the external angle ACD and the vertex angle C3. With this justification, he proves that the 180° measurement of the straight angle is also the measurement of the three interior angles of the triangle.

The *analysis phase* contemplated the auxiliary constructions that aided the formation of the geometric argument. These constructions were anticipated and hypothesized by means of abductive reasoning. The first overcoded abduction was the relation between the 180° measure of the straight angle and the sum of the measures of the angles of the triangle. A subsequent overcoded abduction was the extension CD of the side BC to construct a straight angle with vertex at C. Still another overcoded abduction was the construction of line CE parallel to the side AB and justified by the parallel postulate. These auxiliary constructions and appropriate collateral knowledge (the parallel postulate and the properties of the angles formed between parallel lines and their transversals) aided the formation of the geometric argument to justify both the measure of the exterior angle and the measure of the three interior angles of the triangle.

The *synthesis phase* was the linear and deductive organization of the reverse argument nonlinearly created in the analysis phase by means of abductive reasoning. This phase starts with the construction of an exterior angle and also with the construction of a parallel line to the side AB passing through its vertex C. Then the argument follows in a deductive manner.

Again, it also goes without saying that the written proof (Figure 2b) leaves silent the creative part of the process of proving—the abductive reasoning that led the construction of a parallel line and of a straight angle congruent to the sum of the three interior angles of the triangle.

It is worthwhile to observe two details about the Euclidean proof. First, the exterior angle and its measurement as the sum of the two remote interior angles were not absolutely necessary for the argument of the proof. Observing Figure 2b, the argument could have been made as follows:

- $<BCD = 180°$ (measure of the straight angle)
  $<BCD = <C_3 + <C_1 + <C_2$.

  Then $<C_3 + <C_1 + <C_2 = 180°$ (transitivity property of equality)

- Since $<C_1 = <A_1$ (alternate interior angles between parallel lines AB and CE and the transversal AC)
  $<C_2 = <B_1$ (corresponding angles between parallel lines AB and CE and the transversal BD)

  Then, $<C_3 + <A_1 + <B_1 = 180°$

- Since $<C_3 = <C$, $<A_1 = <A$, $<B_1 = <B$, then $<C + <A + <B = 180°$.
  Then, the sum of the measures of the interior angles of any triangle is 180°

Second, instead of using the exterior angle to justify the sum of the interior angles, the argument could have been made in the reverse order. This is to say that the 180°

measure of the interior angles could have been proved first and then the measure of the exterior angle as the sum of the two remote interior angles could have ensued. Observing Figure 2b, the argument could have been made as follows:

- $<ACD + <C_3 = 180°$ (measure of the constructed straight angle)
- $<A_1 + <B_1 + <C_3 = 180°$ (the sum of the interior angles of the triangle is 180°)
- Then $<ACD + <C_3 = <A_1 + <B_1 + <C_3$ (transitivity of equality)
- $<ACD = <A_1 + <B_1$ (when equals are subtracted from equals, the remainders are equal)
- Then, the measure of an exterior angle of the triangle is the same as the sum of the measures of the two remote interior angles

The Euclidean notion of exterior angle, although not indispensable for the proof of the 180° measure of the angles inside the triangle, is an important notion that can be extended to any polygon. He not only stated the property of the sum of the *interior angles* of triangles but also the property of the *exterior angles* of triangles. In other words, he not only classified the angles of triangles into *interior* and *exterior* but also established a relation between these two kinds of angles. This creative abduction seems to have existed in the minds of the Pythagoreans. Heath (1921/1981, vol. 1) argues that we should not infer that the notion of external angle was not known to the Pythagoreans. He also asserts that more general propositions are also credited to them: (i) if $n$ is the number of sides of a polygon, then the sum of the interior angles of a polygon is equal to $(2n–4)$ right angles, and (ii) the sum of the exterior angles of any polygon is equal to 4 right angles.

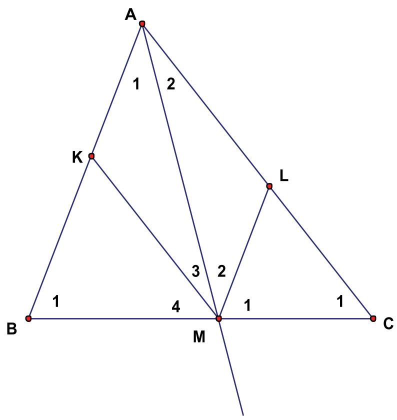
### *Proclus' Proof*

Proclus' geometric argument has some similarities and differences with the Pythagorean and the Euclidean arguments. Figure 3 presents Proclus' proof. All three arguments are similar because they are based on the measurement of the straight angle and on the construction of parallel lines. Proclus' argument is different from the other two because he makes the construction of parallel lines to two different sides of the triangle, rather than to only one side. Moreover, these lines fall inside the triangle rather than outside and they pass through an interior point on the remaining side rather than through a particular vertex.

Proclus first chooses a point M *interior* to a side of the triangle (BC) and then constructs parallel lines to the remaining sides (AB and AC). He joins M with A (opposite vertex to the side BC) and constructs line AM. He also uses point M to construct lines MK parallel to side AC and ML parallel to side AB. Then he re-conceptualizes lines AM and BC as transversals to two pairs of parallel lines (MK//AC and ML//AB). Finally, he uses the congruence of the angles formed between parallel lines and transversals to prove that the straight angle BMC (of

PPOCLUS' PROOF

Prove that the sum of all three angles of any triangle is 180°.




We construct any ray AM (M lies on the side BC). Then we construct from the point M a parallel line to the side AB that intersects the side AC at the point L.
We also construct from the point M a parallel line to the side AC that intersects the side AB at the point K.

Since ML//AB and they are intersected by the transversal BC, then $\angle B_1$ is congruent to $\angle M_1$ as corresponding angles.

Since ML//AB and they are intersected by the transversal AM, then $\angle A_1$ is congruent to $\angle M_2$as alternate interior angles.

Since MK//AC and they are intersected by the transversal BC, thent $\angle C_1$ is congruent to $\angle M_4$ as corresponding angles.

Similarly, MK//AC and they areintersected by the transversal AM, then $\angle A_2$ is congruent to $\angle M_3$ as alternate interior angles.

Therefore,

$\angle A_1 + \angle A_2 + \angle B_1 + \angle C_1 = \angle M_2 + \angle M_3 + \angle M_1 + \angle M_4$

but $\angle M_2 + \angle M_3 + \angle M1 + \angle M_4 = 180°$

Thus $\angle A_1 + \angle A_2 + \angle B_1 + \angle C_1 = 180°$

But from the figure $\angle A_1 + \angle A_2 = \angle A$, $\angle B_1 = \angle B$, and $\angle C_1 = \angle C$

Finally, $\angle A + \angle B + \angle C = 180°$

*Figure 3. Proclus' proof*

measurement 180°) is also congruent to the sum of the angles in the triangle. Fundamental to Proclus' geometric argument were the auxiliary constructions of the lines AM, ML (ML//AB), and MK (MK//AC). These auxiliary constructions were *overcoded abductions* to construct a straight angle congruent to the angles in the triangle. Appropriate collateral knowledge (the parallel postulate and the congruence of the angles formed between parallel lines and their transversals) aided in the justification of the congruence between the straight angle BMC and the angles in the triangle.

In the analysis phase of this proof, abductions, auxiliary constructions, and collateral knowledge were essential. The first overcoded abduction anticipated the possible relation between the 180° measure of the straight angle and the sum of the angles in a triangle. The second overcoded abduction was the construction of two parallel lines, through an interior point of one side, and parallel to the other two sides. The third overcoded abduction anticipated a straight angle, with vertex at the above mentioned interior point, and congruent to the angles of the triangle. The properties of the angles between parallel lines and transversals accounted for this congruence.

In the synthesis phase, the nonlinear argument produced in the analysis phase was reversed to capture the argument in a deductive manner. Thus the written proof starts with the auxiliary constructions to arrive at the conclusion sought. Without the analysis phase it would have been impossible to imagine how to start the proof and how to incorporate viable auxiliary constructions. Therefore, it goes without saying that the written proof in Figure 3 also leaves silent the creative aspect of the proving process—the abductive reasoning that allowed the emergence of auxiliary constructions.

## SUMMARY AND CONCLUSIONS

The deconstruction of the three classical proofs indicates that, given the nature of the proposition, these geometers ingeniously called upon a working backwards strategy or what they called the analysis-synthesis method. Any other method of proving would have been impossible due to the universal nature of this proposition. We argued that abductive reasoning played a fundamental role in the construction of a straight angle and the relationship between its measurement and that of the sum of the three interior angles of the triangle. This is to say that abductive reasoning played a key role in the analysis and then the synthesis phases of each proof. Chart 5 summarizes the analysis and synthesis phases that we argued were essential in the proving process of this proposition.

The *analysis phase* of each proof was based on overcoded abductions grounded in collateral knowledge (the 180° measure of straight angles and the congruence of angles between parallel lines and transversals). The first overcoded abduction was the connection between the measure of the straight angle and the sum of the interior angles of any triangle. The second overcoded abduction was the possibility of constructing a straight angle with angles that were congruent to the angles of the triangle. This construction was abductively implied from the parallel postulate. The third overcoded abduction was the actual construction of a straight angle using parallel lines and the congruence of angles formed between parallel lines and transversals.

It is important to note that an infinite number of parallel lines to one side of a triangle can be constructed due to the fact that there are an infinite number of points outside the line containing any side. Which point should be chosen? The Pythagoreans and Euclid anticipated the strategic point to be the opposite vertex to the side chosen first. Through that point they constructed a parallel line to the side into focus. Proclus anticipated the strategic point to be any point *between* the two vertices of the side first chosen (thus he excludes vertices). From that point, he constructed parallel lines to the other two sides of the triangle (forming a parallelogram); he also constituted this point into the vertex of the straight angle.

The *synthesis phase* was pursued after the *analysis phase* has produced a viable and logical geometric argument. This phase starts with the auxiliary constructions and pursues a chain of deductions by reversing their abductive reasoning in the analysis phase.

This is to say, they started with the auxiliary construction―a parallel line to an arbitrary side(s) of the triangle and passing through a particular point. What was the end goal of the construction? To form a straight angle with angles congruent to the (interior) angles of the triangle. Finally, they use the fact that the measure of the

| *Analysis phase of the proofs (overcoded abductions)* | *Synthesis phase of the proofs (deductive reasoning)* |
|---|---|
| *First overcoded abduction: Association between a geometric fact and the conclusion sought out*<br>• Could there be a connection between the 180º measure of the straight angle and the sum of the angles in a triangle? | *Auxiliary Construction (second and third abductions)*<br><br>• Construct a parallel line to one side of a triangle and passing through the opposite vertex to that side.<br>• Determine the straight angle that can be formed with angles congruent to the angles in the triangle.<br>• Construct two parallel lines to two sides of a triangle and passing through a point between the two vertices of the third side.<br>• Determine the angles between the parallel lines that are congruent to the angles in the triangle. |
| *Second and third overcoded abductions: Possible Auxiliary Constructions*<br>• Could angles congruent to the three angles in a triangle form a straight angle?<br>• Could parallel lines to one side of a triangle and through its opposite vertex guide the construction of the desired straight angle?<br>• Could parallel lines to two sides of a triangle and passing through a point interior to the third side guide the construction of the desire straight angle?<br>• Is this constructed straight angle congruent to the angles in the triangle? | *Geometric facts*<br><br>• Every straight angle measures 180º (two right angles).<br>• A straight angle, congruent to the three angles of a triangle, can be constructed. |
| *Plausible Conclusion*<br>• The sum of the three angles in a triangle *should be* 180º because a straight angle can be constructed with angles that are congruent to the three angles of any triangle. | *Conclusion*<br>• The addition of the measures of the three interior angles of any triangle *is* 180º. |

*Chart 5. Outline of the analysis-synthesis geometric argument of the three proofs*

straight angle is 180° to conclude that the sum of the interior angles of any triangle should also be 180°.

Why to deconstruct the three classical proofs of one of the most fundamental propositions of plane Euclidean geometry? Certainly it could appear to be a useless exercise. After all, the proofs are there; they are not too long; and they can be easily followed once you are given the auxiliary constructions. However, one could ask questions like "Why are these auxiliary constructions appropriate?" "Where do these auxiliary constructions come from?" or "Is there any other way to prove this proposition?"

Being aware of the very essence of the proving process also entails being aware of *how* a proof is constructed. This awareness may encourage one's mind to imitate similar thinking strategies or to generate new ones in other geometric situations. There is no wonder why some students ask, "Where do proofs come from?" How do I start a proof?" Intentionally or unintentionally, these students are asking for a method to direct their own thinking during the process of proving.

The deconstruction here presented brings to the fore a cognitive issue that seems to be also a pedagogical issue—the key role of proving in the development of mathematical thinking and mathematical understanding. A research question that could be investigated is whether or not the deconstruction of classical proofs could help students to become aware of the role of abductive reasoning in the process of proving and in mathematical thinking.

Mathematicians like Polya argue that students should not only be given proofs to be memorized but also the knowledge of "how" to go about proving. This, he says, will encourage the formation of habits of thinking and methodical work. He also encourages teachers and students to *learn by guessing* (i.e., abductive inference or hypothesis) and to learn by proving (1962/1981, vol. 2).

Hanna (1989) also contends that learners should become aware of the need to reason carefully when building, scrutinizing, and revising mathematical arguments. She asserts that proving deserves a prominent place in the curriculum "because it continues to be the central feature of mathematics itself, as the preferred method of verification, and because it is a valuable tool for promoting mathematical understanding" (1995, pp. 21–22).

The document *Principles and Standards for School Mathematics* (NCTM, 2000) advocates the teaching and the learning of geometry and, in particular, the teaching and the learning of proving in order to improve the development of students' systematic reasoning. It also advocates the teaching of geometry in such a way that allows students to explore geometric figures, to generate geometric conjectures, and to construct logical arguments and counterarguments.

More recently, the document *Common Core State Standards for Mathematics* (2012) also advocates the development of critical thinking, systematic reasoning, and habits of thinking. It argues that these are the most important competences to be developed in *all* students K-12 (Hirsch, Lappan, & Reys, 2012).

Proving, as a special type of problem-solving, is among the most powerful means to develop habits of thinking in students' minds. Memorization of proofs, alone, has less chance of developing these habits. Some proofs are great constructions (some say discoveries) done by mathematicians. These proofs should be analyzed before being memorized to serve as paradigmatic examples. However, less sophisticated proofs, like those of simpler propositions which are constructed by students themselves, will have the greatest impact on their mathematical thinking. Polya emphasizes this point in a clear and simple manner.

> Your problem may be modest, but if it challenges your curiosity and brings into play your *inventive faculties*, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime. (Polya, 1945/1973, preface of the first edition, p. v)

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