CAROLINE YOON AND TESSA MISKELL

5. VISUALISING CUBIC REASONING WITH SEMIOTIC RESOURCES AND MODELLING CYCLES

ABSTRACT

Diagrams and physical manipulatives are often recommended as useful semiotic resources for visualising area and volume problems in which nonlinear reasoning is appropriate. However, the mere presence of diagrams and physical manipulatives does not guarantee students will recognise the appropriateness of nonlinear reasoning. Three case studies illustrate that the effectiveness of such semiotic resources can depend on whether they enable students to visualise, test and examine their existing incorrect mathematical approaches as they progress around the modelling cycle. Some students used diagrams and multilink blocks to test and reject incorrect linear and quadratic reasoning, whereas others who created diagrams did not use them to test their ideas, and persisted with incorrect linear or quadratic reasoning.

INTRODUCTION

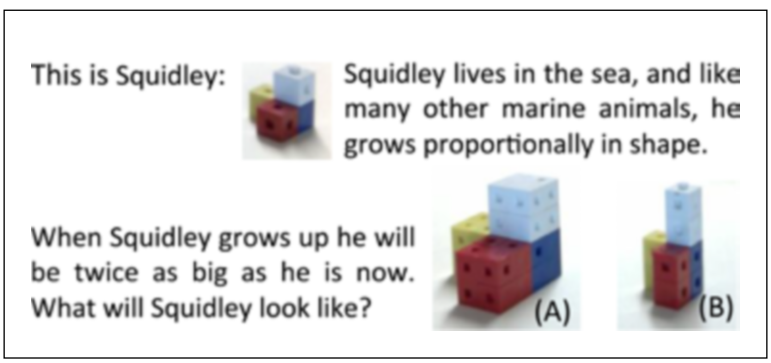

Figure 1. The Squidley warmup activity

Consider the following problem: *Baby humans' bodies are not proportional to adult humans' bodies: a baby's head is large in relation to its squat limbs, whereas an adult's head is smaller in relation to its long limbs (Thompson, 1992). In contrast, baby seahorses resemble miniature replicas of their parents, with their scaled down body part shapes in similar proportions to each other as those of adult seahorses. The following "marine animal" named Squidley (see Figure 1), was constructed*

A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 89–109.

from multifix cubes. Like seahorses, Squidley grows proportionally in shape. When he grows up, he will be twice as big as he is now. What will Squidley look like?

The answer of course, depends on what we mean by "twice as big". If we mean that Squidley's *dimensions* of length, width and height are doubled, then option A could be the answer, although option A's volume is not twice as big, but eight times bigger than young Squidley, containing 32 multifix cubes to young Squidley's four. If we mean that Squidley's *volume* is doubled, then option B could be the answer as it contains twice as many multifix cubes as young Squidley, although option B is not proportional in shape to young Squidley, having grown only in one dimension (height). In fact, neither of these two options are correct if we intend to double Squidley's volume while retaining the proportions in his shape: this would require multifix cubes with side lengths that are times bigger than those of young Squidley.

The above question highlights an important mathematical concept: that increasing the dimensions of a three-dimensional figure by a given scale factor does not yield a linearly proportional increase in volume. Instead, the change in volume obeys a cubic pattern as shown in option A above, which is $2^3 = 8$ times bigger in volume when its dimensions are multiplied by a scale factor of 2. Many students and adults find this concept counterintuitive and misapply linear reasoning, saying for example, that option A must be twice as big in volume as its dimensions have been doubled. The misapplication of linear proportional reasoning to situations where non-linear reasoning is required has been described as "the illusion of linearity" (De Bock, Verschaffel, & Janssens, 2002). It occurs in many areas of mathematics (see for example, Shaughnessey, 1992), but is particularly prevalent in problems about scaling up or down the volume (and area) of geometrical figures, where students often assume that multiplying the dimensions of a three (two) dimensional figure by a factor of x will result in a new volume (area) that is also scaled up by a factor of x, rather than x^3 (x^2) (De Bock et al., 2002; Modestou, Elia, & Gagatsis, 2008; van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005).

A common recommendation for overcoming misconceptions like the illusion of linearity is to visualise the scenario using diagrams (Polya, 1957; Schoenfeld, 1994). In this chapter, we consider how students used diagrams and other semiotic resources (van Leeuwen, 2005) within a modelling activity about the scaling up of volume of fish. The mere presence of diagrams alone did not guarantee success in identifying the correct cubic relationship. Instead, our case studies suggest that the effectiveness of diagrams and other semiotic resources depends on whether they enable students to visualise, test and examine their existing incorrect mathematical approaches as they progress around the modelling cycle.

FACTORS IN OVERCOMING THE ILLUSION OF LINEARITY IN AREA AND VOLUME

Students' difficulties with reasoning about scale and proportion in linear, area and volume problems are well documented (Lamon, 2007) and resistant to change, continuing even into adulthood (De Bock et al., 2002). Researchers have investigated the effect of three factors in overcoming the illusion of linearity, with mixed results: the use of diagrams, problem contexts and metacognitive prompts (De Bock et al., 2002; De Bock, Verschaffel, Janssens, van Dooren, & Claes, 2003; Modestou et al., 2008).

Diagrams are often credited with helping students succeed in mathematical problem solving by enabling students to discover and examine underlying relationships (Pantziara, Gagatsis, & Elia, 2009) and generate new ideas (Diezmann, 2005; Nunokawa, 2006), while reducing students' cognitive load (Gibson, 1998; Koedinger, 1994). De Bock, van Dooren, Janssens, and Verschaffel (2002) report a study in which they found only slight advantages in presenting students with diagrams in problems dealing with scaling up and down length, area and volume. Secondary school students received questions like the following:

> Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m? (p. 69)

Half of the students were allocated to a diagram treatment group, in which they also received scale drawings on grid paper for each question; the diagram accompanying the above question showed two squares representing the two pieces of land. Students in the diagram treatment performed better on questions that required non-linear reasoning (area and volume questions), but this difference was very small: success in 17% of the non-linear scaling up problems by the drawing group, compared to 13% correctness in the non-drawing group. In addition, this slightly higher performance on non-linear questions was mitigated by a similarly small but significantly lower performance on questions that required linear proportional reasoning by students who had been given diagrams, suggesting that the diagrams they presented to the students did not help students determine when non-linear reasoning was inappropriate.

In a follow up study, De Bock et al. (2003) switched from giving students ready-made diagrams to encouraging students to construct their own for a similar set of problems. Students in the diagram treatment group were given partial diagrams to complete for each problem. For example, students were given a diagram of square *Q*, and were asked to draw a scale diagram of square *R* for the following question:

> The side of square *Q* is twelve times as large as the side of square *R*. If the area of square *Q* is 1440cm^2, what's the area of square *R*? (p. 448)

Surprisingly, the students in the drawing condition performed significantly worse on the non-linear problems than those without the drawings, which De Bock et al. (2003) attributed to the very process of creating a scale drawing. They reasoned that when students drew a reduced copy of a geometrical figure, they would have measured a linear element such as its height or length, and divided that element by a linear scale factor, effectively activating a linear thought process. This could have enhanced the students' inclination toward a linear model, rather than the quadratic or cubic model that was required. De Bock et al.'s two studies highlight that diagrams are not in themselves effective or ineffective in helping students overcome the illusion of linearity as their success depends on the ability of the person viewing the diagrams to recognise the relevant structure portrayed. The diagrams in De Bock et al.'s two studies (2002, 2003) may have been ineffective in helping students overcome the illusion of linearity because the students did not know what mathematical structure to look for in the diagrams they were given or constructed.

Modelling tasks which use real world contexts have been promoted as potentially useful means for developing linear proportional reasoning (e.g., Lamon, 2007) and non-linear reasoning (Treffers, 1987). However, De Bock et al. (2003) caution that merely setting a routine problem in a real world context doesn't necessarily constitute a modelling task. De Bock et al. (2003) gave one group of students scaling up/down problems set in the context of Gulliver's Travels to the Isle of Lilliputians, a world where all lengths are 12 times as small as those in Gulliver's world, and another group solved mathematically equivalent problems presented as standard textbook formulations with no real world context. Students in the standard textbook group received questions like the one described above about the area of squares *Q* and *R*, whereas students in the Gulliver's Travels group received questions like the following:

> Gulliver's handkerchief has an area of 1296 cm^2. What's the area of a similar Lilliputian handkerchief? (p. 448)

On finding that students in the Gulliver's Travels group performed worse on the test than those in the standard textbook group, DeBock et al. (2003) reason that the Gulliver's Travels problems were simply standard textbook questions that had been "dressed up" in a real world context (Blum & Niss, 1991). They suggest that greater success may be possible with modelling tasks that require more authentic performance-based assessment, such as filling a Lilliputian's wine glass or making a Lilliputian handkerchief.

Metacognitive prompts and scaffolds are a third means for helping students overcome the illusion of linearity as they can encourage students to become more conscious of their misapplication of linear reasoning. Students are often unconscious that they are misapplying linear models whereas others knowingly apply them without realising they are not appropriate (Esteley, Villarreal, & Alagia, 2004). De Bock et al. (2002) gave one group of students a metacognitive prompt intended to provoke cognitive conflict to problems such as, "A wooden cube with an edge of 2 cm weighs 6 grams. How heavy is a wooden cube with an edge of 4 cm?" (De Bock et al., 2002, p. 71). The prompt offered two possible solutions for the students to choose between, one of which misapplied a linear model, and the other used appropriate nonlinear reasoning. For example, the two solution options accompanying the above question were (a) since the edge doubled, the weight also doubled, and (b) a cube with an edge of 4 cm will contain eight cubes with edges of 2 cm so the weight needs to be multiplied by eight. The study yielded significant, positive results but did not enable students to overcome their misconceptions completely as some students in the metacognitive treatment group continued to misapply the linear model afterwards. Moreover, the study found that students who originally applied the linear model "everywhere" started to do the same with the non-linear model, generalising it to inappropriate situations and effectively replacing one model with another.

Modestou et al. (2008) used a different metacognitive prompt to encourage students to question their spontaneous application of the linear model. Students were given sets of three questions comprising one that required nonlinear reasoning, one that required linear reasoning, and an unusual question that could have more than one correct answer. After solving all three questions, the students were asked to identify which (one) of the three questions yielded a given numerical answer. In each case, the question requiring linear reasoning was the correct match, but if students had misapplied linear reasoning to the nonlinear question, they would have obtained the same numerical answer (though incorrect). Almost half of the students who had initially misapplied a linear model to the nonlinear question ended up selecting the correct problem for the given answer, which suggests that the metacognitive prompt forced them to reconsider and correct their initial misapplication. However, a quarter of the students who misapplied a linear model selected the nonlinear question (which is incorrect), which suggests that the metacognitive prompt also led to mistakenly rejecting a correct application of the linear model.

This section has reviewed three factors (diagrams, metacognitive prompts and problem contexts) that may help students overcome the illusion of linearity. In the next section, we consider how all three factors can be incorporated into a theoretical framework based on modelling cycles.

THEORETICAL FRAMEWORK: MODELLING CYCLES AND SEMIOTIC BUNDLES

The theoretical framework used in this chapter draws on two constructs: modelling cycles (e.g., Lesh & Doerr, 2003; Niss, Blum, & Galbraith, 2007; Stillman, Galbraith, Brown, & Edwards, 2007) and semiotic bundles (Arzarello, Paola, Robutti, & Sabena, 2009). Mathematical modelling involves the complex coordination of processes that can be depicted around the modelling cycle as shown in Figure 2. The

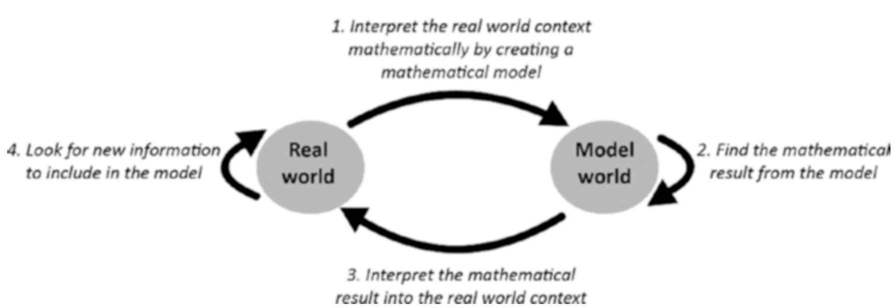

Figure 2. The modelling cycle

modelling cycle begins in the real world, where one determines which features of the real context are mathematically relevant to the problem, and incorporates these relevant features from the real world into a mathematical model. This model is then used to find a mathematical result, which is in turn interpreted back into the real world context. The fitness of the model is then assessed, and if necessary, the cycle is entered into again in pursuit of a model that incorporates more relevant information from the real world. Such cycling continues until the modeller is satisfied with the mathematical model that has been created.

The creation of the mathematical model can be regarded as the development of a semiotic bundle (Arzarello et al., 2009), which consists of signs that mathematically express relevant real world information from the problem situation. The notion of a semiotic bundle is predicated on Peirce's notion of a sign, which is something that "stands to somebody for something in some respect or capacity" (Peirce, 1931/1958, vol. 2, paragraph 228), and is defined as follows:

> A semiotic bundle is a system of signs—with Peirce's comprehensive notion of sign—that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. (Arzarello et al., 2009, p. 100)

Mathematical semiotic activity involving such signs are not necessarily confined within strict boundaries of separate modalities, but are spread across speech, inscriptions, gestures, glances and so forth (Radford, 2009; Arzarello et al., 2009). Consequently, a semiotic bundle includes not only instances of signs and sign systems, but also the coordination of and interrelationships between sign systems across multiple modalities.

Arzarello et al. (2009) often uses the term "semiotic resource" has in place of the terms "sign", "sign system", or "representation". van Leeuwen (2005) clarifies the idea of a semiotic resource as emphasising the semiotic potential of a sign or sign system:

> In social semiotics resources are signifiers, observable actions and objects that have been drawn into the domain of social communication and that have a *theoretical* semiotic potential constituted by all their past uses and all their potential uses and an *actual* semiotic potential constituted by those past uses that are known to and considered relevant by the users of the resource, and by such potential uses as might be uncovered by the users on the basis of their specific needs and interests. (p. 4)

We also adopt the term "semiotic resource" in this chapter to emphasise the semiotic potential of the diagrams and physical manipulatives students employed during problem solving to visualise mathematical structures.

The semiotic bundle approach enables us to consider the diagrams and physical manipulatives students use not as independent semiotic resources, but in relation to other inscriptions and semiotic resources that they also develop. We use both a synchronic analysis (which considers the relationships between semiotic resources activated at the same time) and a diachronic analysis (which considers the evolution of semiotic resources activated over time) (Arzarello et al., 2009) to study how students' diagrams and physical manipulatives evolve in conjunction with other semiotic resources. The evolution of a semiotic bundle over time is similar to Duval's (2006) notion of conversion, where a representational transformation involves a change in register (e.g., from graphical to algebraic), but not in the mathematical object. A number of researchers (e.g., Kaput, 1989; Thomas, 2008) have highlighted the ability to translate fluently between and sometimes within different semiotic resources as an important component of mathematical meaning making.

This theoretical framework of modelling cycles and semiotic bundles encompasses all three factors (diagrams, problem contexts and metacognitive prompts) that were previously identified in the literature as potentially productive ways of overcoming the illusion of linearity. In the first step in the modelling cycle (Figure 2), the real world context encourages students to create a mathematical model (via some semiotic bundle) that has a meaningful real world purpose. The diagrams (and other semiotic resources) that may be created to describe the mathematical model in step 2 may encourage students to test their model in step 3. And this testing and subsequent comparison of the output from the model in light of the real world context may lead students to re-examine their mathematical reasoning in steps 3 and 4. Thus, modelling activities give students the opportunity to experience the potential benefits of all three factors by going through the modelling cycle in a more holistic way than in the "dressed up" textbook problems (Blum & Niss, 1991) used in the studies by De Bock et al. (2002, 2003) and Modestou et al. (2010).

DESCRIPTION OF THE MODELLING ACTIVITY

The Snapper problem (Yoon, Radonich, & Sullivan, in press) is a modelling activity concerned with the fair division of snapper fish of different sizes. It begins with a warmup involving the Squidley question (see Figure 1) to engage students in using physical images related to scale, proportions and volume. After the warmup, students read the Snapper problem statement (see Figure 3) and work on the problem in groups of three for about 45 minutes.

The Snapper problem was designed to encourage students to overcome the illusion of linearity and to develop non-linear (in this case, cubic) models of reasoning about the fair distribution of snapper fish. Its design was influenced by six principles for designing Model-Eliciting Activities (Lesh, Hoover, Hole, Kelly, & Post, 2000), which are a type of problem noted for encouraging students to go beyond their initial, primitive ways of thinking, to develop more sophisticated mathematical interpretations of real world situations (Lesh & Doerr, 2003). The Snapper problem satisfies the *reality* principle as it is set within the realistic context of dividing up a catch of fish. It satisfies the *model construction* and *model generalisation* principles by requiring students to create a generalisable mathematical model (in this case, an argument) that can be used to solve the problem, rather than merely a single numeric solution, such as "8 small fish = 1 large fish". By giving students the physical manipulatives of multilink cubes to test out their ideas, the problem also satisfies the *self-assessment* principle. It also satisfies the *model documentation* principle as students are required to document their mathematical model in the form of a letter to Joe. Finally, the problem elegantly maps the context of a fishing trip to the need for a mathematical model about cubic reasoning with volume, so that it satisfies the *simple prototype* principle.

CLASSROOM IMPLEMENTATION AND DATA COLLECTION

We implemented the Snapper problem in four classes at a large New Zealand tertiary institution. The students in all four classes were taking an elementary mathematics course for foundation studies—that is, they were studying towards tertiary degree entrance qualification as they had not achieved the tertiary level entry requirements

The Loverich family and the Borich family went fishing together. They caught nine Snapper. Zoe Borich caught the biggest one, which was 54 cm long.

Joe Borich took the job of dividing the fish up fairly between the two families so that they had the same amount of fish each. He gave himself the big snapper as his daughter Zoe caught it, and said that it was worth two of the smaller fish (27 cm each). Peter Loverich thought that the flesh from the big fish was probably more than four times that of the smaller ones but decided not to say anything to avoid a scene.

Peter needs your help!

Your job is to work out a mathematical argument for deciding how many little fish the big fish is worth. **Write a letter** to Peter, describing your mathematical argument clearly, using diagrams if you wish. Peter wants to be able to use your argument for future fishing trips, so explain in your letter how he can make your argument work for fish of any size.

Figure 3. The problem statement for the fishing trip MEA

from secondary school. The students worked in groups of three on the Snapper problem during a 90-minute class: about 60 minutes of the time involved the students working on the problem in their groups, and the remaining 30 minutes involved students presenting their group solutions to the class. A researcher and the class lecturer were present in the classroom for each implementation. They interacted with the students to facilitate group discussion and encourage them to use physical blocks and drawings to test out their ideas but they refrained from telling that or explaining why one large fish was worth 8 small fish.

During the in-class presentations, at least one group of students in each of the four classes gave correct reasoning that showed that one fish of length 54cm was worth eight smaller fish of length 27cm with proportional dimensions. Students and the lecturer were given the opportunity to ask presenting groups questions about their solutions: some question and answer interchanges occurred in each class, but there was no in-depth whole class discussion. After the session, the students then completed written individual solutions to the Snapper problem in their own time. Forty-six students handed in their individual solutions one week after the 90-minute class. We collected the written work from the 19 groups in the four classes, the 46 individual written solutions, and researcher field notes of the student presentations. In this chapter, we present three case studies of three groups (comprising three students each) from one of the classroom implementations: Case study 1 involves Liv, Liz and Pania's group; Case study 2 involves Dee, Jan and Lea's group; Case study 3 involves Del, Lyn and Mac's group. We use both the data from their in-class group work and their individual letters to analyse their modelling cycles.

DATA ANALYSIS

Field notes and data from the students' written group work were used to construct a description of the groups' progress during the classroom implementation. We analysed the group and individual solutions to assess the effectiveness of the mathematical argument, taking into consideration the written language, diagrams, tables, numerical examples, and algebraic expressions used by students. We also

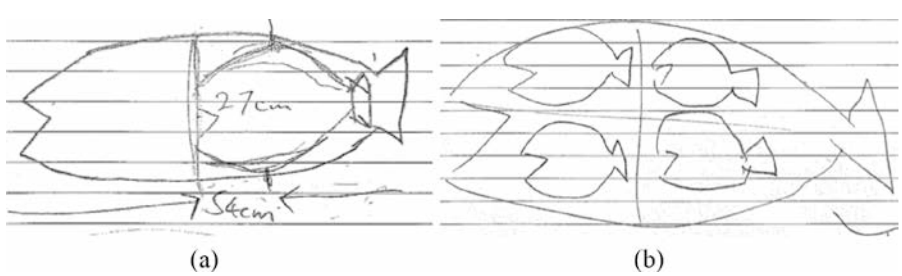

Figure 4a & 4b. Two drawings showing a linear and quadratic model drawn by Liv, Liz and Pania during in-class group work sorted the individual and group letters into three categories. The first category included those that demonstrate *no understanding* of the cubic relationship between snapper volume and linear scale factor. The solutions in this category do not use a cubic model to describe the relationship between the volume of the large fish and the small fish – instead they use either linear reasoning, saying the large fish is worth 2 small fish, or they consider some aspect of area instead of volume. The second category of letters demonstrate a *partial understanding* of the cubic relationship by correctly reporting that one large fish is worth eight small fish, but have weak or incorrect arguments to support or explain why this was so. The third category of letters demonstrate *conceptual understanding* of the cubic relationship between snapper volume and linear scale factor by articulating a convincing argument based on correct cubic reasoning for why one large fish is worth eight small fish.

For each diagram, we analysed how the students had incorporated the diagram into their written argument, the accuracy of the dimension proportions represented in the diagrams, and the mathematical understandings expressed.

RESULTS

Three case studies show how students used diagrams and physical manipulatives to visualise fish while working on the Snapper modelling activity. For each case study, we describe how the semiotic resources they created facilitated (or not) their progression around the modelling cycle during classroom groupwork and subsequent individual written work.

Case Study 1: Diagrams and Multilink Cubes Generate Multiple Modelling Cycles

During the classroom implementation, Liv, Liz and Pania initially argued that one large fish of length 54cm was indeed worth two small fish of length 27cm, and drew one long fish that was 54cm in length, with another shorter fish only 27cm in length inside it (see Figure 4a). They then realised that their diagram showed two fish that were not proportional in shape: the larger looked liked a stretched out version of the smaller, whose width was the same. This led them to revise their argument to saying that the larger fish was worth four small fish, and they drew a diagram of four small fish fitting into the area of the large fish (Figure 4b). They began writing up their solution, thinking they had found the correct solution.

They informed the researcher that they were finished, at which the researcher asked them to articulate their argument to each other using the multilink blocks. Liv Liz and Pania initially used the configuration of blocks shown in Figure 5a to show, that one large fish was worth four small fish, in accordance with the diagram they had drawn in Figure 4a. However, Pania soon noticed that the two “fish” they had constructed were not proportional in shape in the 3-dimensional representation. Pania realised in an Aha! moment (Liljedahl, 2005) that there are “two sides to each fish”, which she explained as flesh on both sides of the bones, and that they had

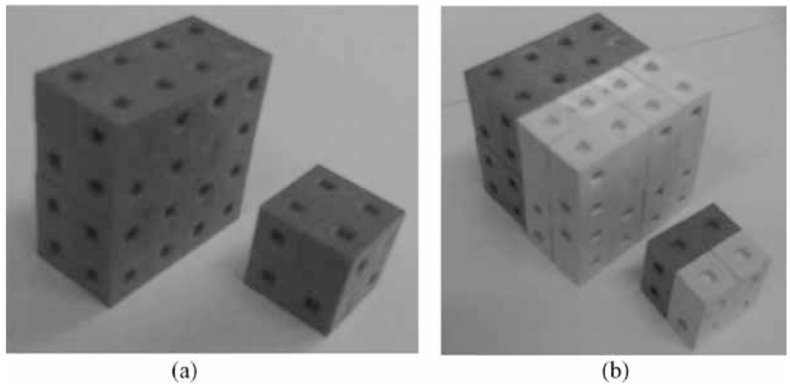

Figure 5. Two configurations of multilink cubes for representing the volume of fish used by Liv, Liz and Pania during the in-class group work

neglected to consider the thickness of the fish. This led to a new configuration of blocks shown in Figure 5b, which demonstrated that when one considers the third dimension of thickness, the volume of one large fish is worth eight small fish. In Liv, Liz and Pania's final letter, they justified their final recommendation of eight fish with the statement, "We used the cubes to help us work this out".

The diagrams and physical multilink cubes helped Liv, Liz and Pania to go through the modelling cycle more than three times, as each successive visualisation led to testing then revising their mathematical model or argument (see Figure 6). We have constructed diagrams showing the extent to which students in each case study progressed around the modelling cycle in Figures 6, 12 and 14. In each case, the students begin in the real world context of the fishing trip, and mathematise the problem by creating a model: the arrow from "real world" to "model world" indicates this process. As shown in Figure 2, progression around the modelling cycle ideally involves all four processes between and within the real and model worlds, often with multiple iterations. However, the students in our case studies did not always complete full cycles, and often only carried out a subset of these four processes. The labels on the arrows in each of the modelling cycle diagrams are numbered to indicate the chronological order of the processes.

The individual written solutions that were handed in one week later revealed that the individual students had different levels of understanding of their group's final argument. Pania's individual solution incorporated diagrams of the multilink cubes to show the three dimensions of the fish, which she then used to devise a numerical example to illustrate how to apply her mathematical argument (see Figure 7).

Liz also articulated in her individual solution that they had to take into consideration the thickness of the fish by doubling the amount of flesh on each side. She drew the following diagrams to illustrate this point (see Figure 8).

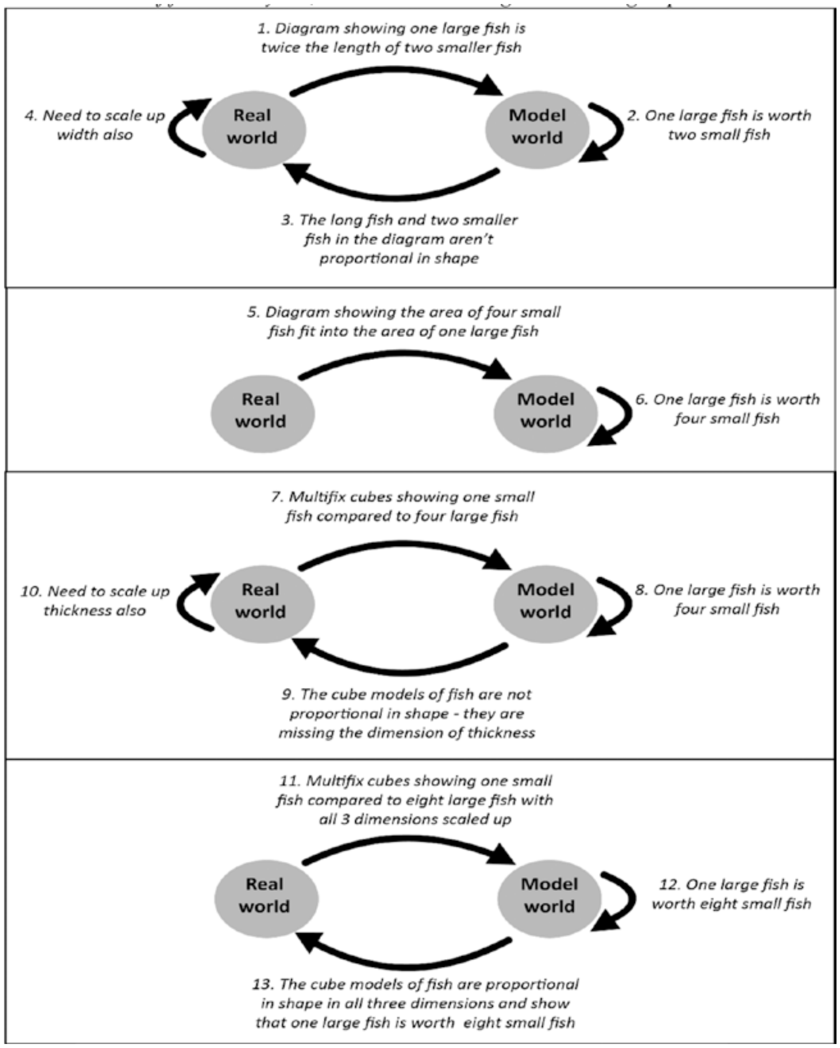

Figure 6. The modelling cycles entered into by Pania, Liv and Liz during group work

In contrast to Pania and Liz, Liv's individual solution reveals a limited understanding of the reasoning for why one large fish was worth 8 smaller fish. She drew scale diagrams of two cube configurations that were meant to represent the two fish (see Figure 9a).

Liv's diagram of the "cubes" in fact only shows a 2-dimensional representation of squares, rather than cubes. These diagrams have the correct number of squares appropriate for the argument, but do not portray the correct proportions in terms

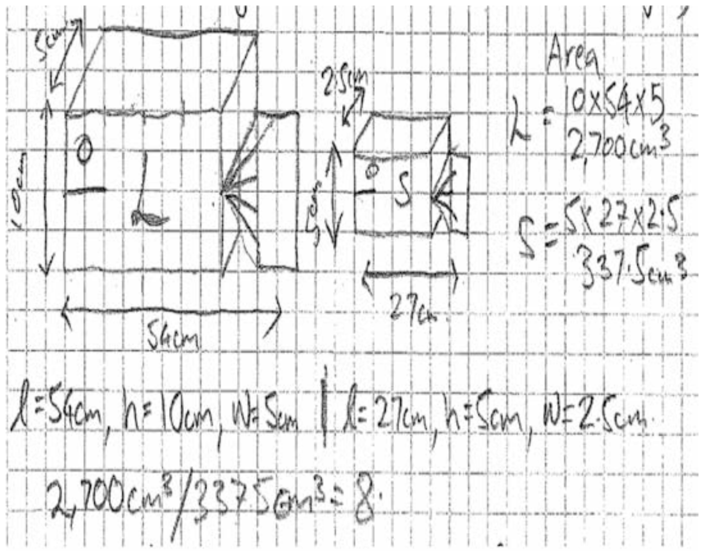

Figure 7. Pania's diagrams in her individual final letter

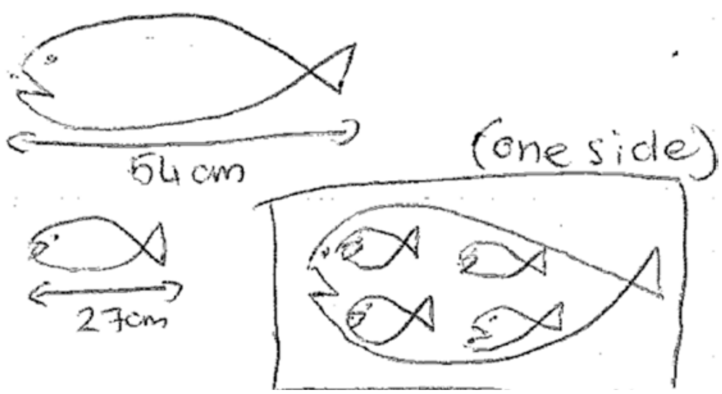

Figure 8. Liz' diagrams in her final individual letter

of shape of the collections of squares, as the diagram of the larger "fish" has doubled in width, but quadrupled in length, while the dimension of depth (which isn't shown) presumably stays the same. Figure 9b shows a redrawn version of her diagram to identify the four parts she drew using different colours more clearly. Liv's diagram suggests that she remembered the group's agreement that 8 was the mathematical result, but did not understand or remember the group's argument as to why it was so.

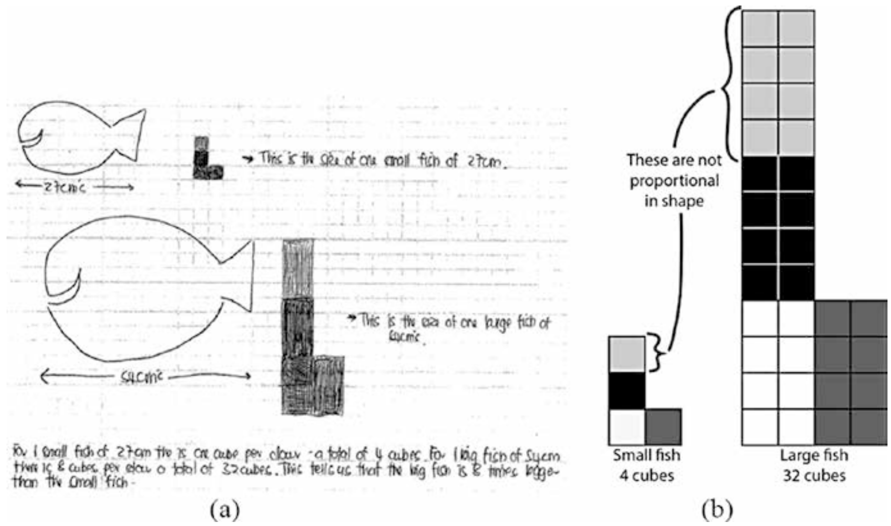

Figure 9. Diagrams (and redrawn version) in Liv's individual final letter

Case Study 2: Algebraic Equation and Diagrams Lead to Different Modelling Cycles

During the classroom session, Dee, Jan and Lea created a mathematical model (see Figure 10) that used a score combining the fish's length, width and height using additive relationships: *Score* = *Length* – (*Height* + *Width*). When they applied this score to two hypothetical fish, one whose three dimensions are double that of the other, they found that their model gave scores of 39 and 19.5 respectively, indicating that the large fish is worth twice that of the small fish.

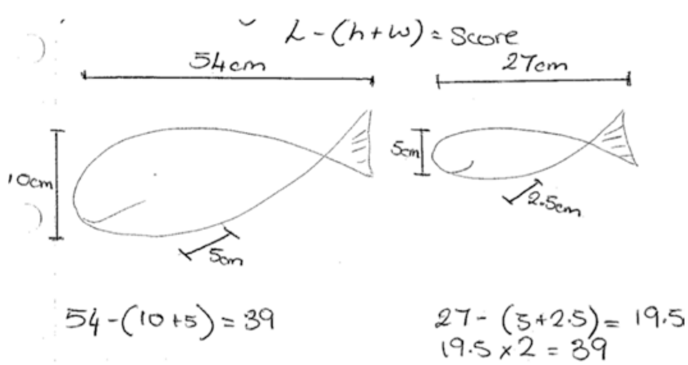

Figure 10. Excerpt from Dee, Jan and Lea's group letter written during the classroom implementation

During the group presentations at the end of the classroom presentation, Dee, Jan and Lea were exposed to three other groups' solutions that argued that one large fish is worth eight small fish in terms of volume—a result that was at odds with their solution. Dee's individual solution that was handed in one week later presented the same mathematical score as the group's, although this time, she did not demonstrate the score on hypothetical fish dimensions, nor did she communicate how many small fish the large fish was worth under this scoring system.

In contrast, Jan's individual solution was markedly different to the group's solution. She wrote that reviewing other solutions led her to believe that one large fish is worth eight small fish, and drew the diagram in Figure 11 to explain why.

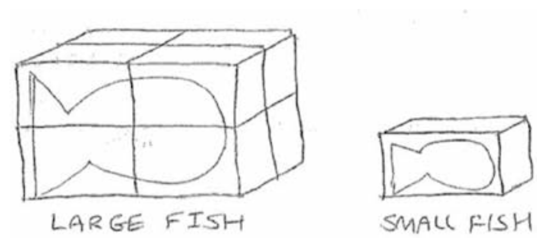

Figure 11. Diagram from Jan's individual written solution

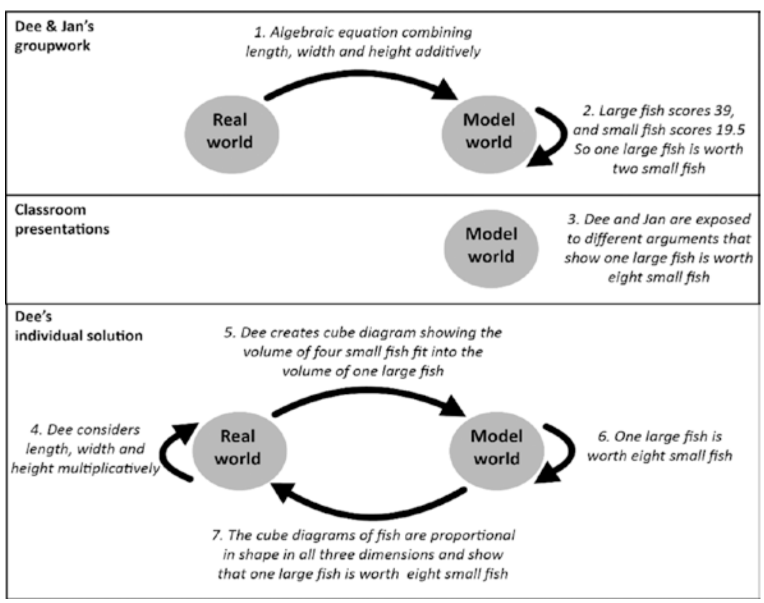

Figure 12. Dee and Jan's modelling cycles during group work and in Dee's individual solution

Note that this type of diagram, which combined the shape of the fish with the multifix cubes was not drawn by any of the other students in Jan's class, either during the in-class groupwork nor in individual written letters that were handed in one week later. Lea didn't hand in an individual solution.

Dee and Jan's individual solutions indicate different experiences in the extent to which they engage in the modelling cycle (see Figure 12).

During the classroom implementation, Dee and Jan only engaged in half of a modelling cycle, as they developed the linear score, then ran it to find a result. They were exposed to different arguments that yielded different results to their model, but only Dee used this information to revise her model and test and interpret it again.

Case study 3: Algebraic Equation and Diagram with a Quarter Modelling Cycle

During the classroom implementation, Del, Lyn and Mac's group created a mathematical argument that relied on the product of the fish girth and length: *Score* = *Girth* × *Length* (see Figure 13). They argue that multiplying the two measures (girth and length) of each fish, then dividing the product of the larger by that of the smaller will reveal how many small fish the big fish is worth.

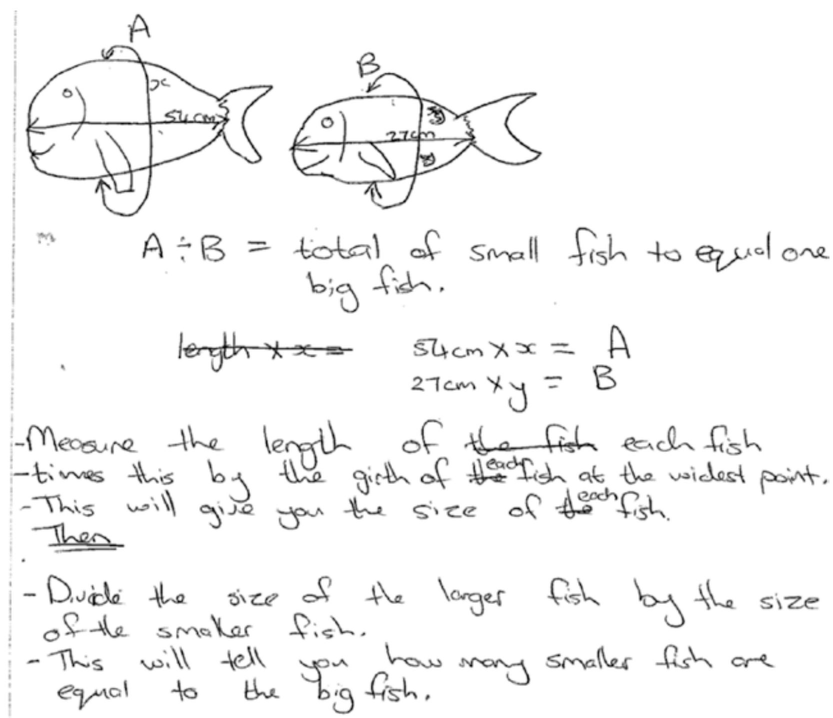

Figure 13. An excerpt from Del, Lyn and Mac's group letter

This model is not useful as it compares the *surface areas* of the two fish rather than the *volumes* of the two fish, and thereby yields the result that a fish whose dimensions are double that of a smaller fish is worth only four smaller fish. Although Del, Lyn and Mac acknowledged that this method compares surface areas, they did not test their method on any fish, and thus, did not realise that their method claims that one large fish is worth four small fish, rather than the eight small fish argued by most of their classmates.

Thus, even when they were exposed to the correct answer of eight, they did not have a point of reference to compare this amount with their own. Del, Lyn and Mac all handed in individual written letters with the same argument about *Girth* × *Height*, and none of the letters ran the model to find out how many small fish the large fish was worth. The group's modelling process can be described as undergoing only one-quarter of a modelling cycle (see Figure 14), in that they developed a mathematical model, which they visualised through diagrams and an equation, but they never went beyond this initial step.

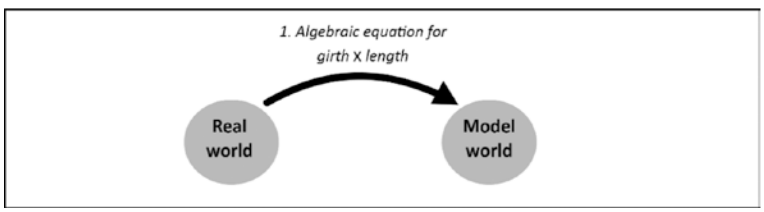

Figure 14. The quarter of a modelling cycle entered into by Del, Lyn and Mac during group work and subsequent individual written work

DISCUSSION

Liv, Liz and Pania's case study supports findings in the literature that students tend to misapply a linear model in scale problems when nonlinear reasoning is more appropriate (e.g., De Bock et al., 2002; De Bock et al., 2003; Modestou et al., 2008). Indeed, they began by assuming that one large fish was worth only two smaller fish as its length was doubled. However, it also offers a different slant into De Bock et al.'s (2002, 2003) finding that students' use of diagrams has little effect in overcoming the illusion of linearity: Liv, Liz and Pania's use of diagrams helped them test and reject their linear model, by enabling them to visualise the second dimension of width, and thereby adopt a quadratic model. Their subsequent use of the multilink cubes enabled them to test and reject a quadratic model and develop an argument for adopting cubic reasoning.

However, Del, Lyn and Mac's groupwork supports De Bock et al.'s (2002, 2003) finding: Del, Lyn and Mac used diagrams to construct their inappropriate model of *Girth* × *Length*, but they never applied this to a specific instance, so never experienced the cognitive perturbation from seeing a different result to those presented by their peers (unlike Jan in the second case study) that could have led to them revising their mathematical model. This case study emphasises that the use of diagrams to construct a model doesn't guarantee the testing of that model. In fact, an appealing diagram (especially when accompanied with an algebraic equation) may lull one into a false sense of security that one has something that "looks right", even if it is not.

The second case study of Jan and Dee adds a further insight—that experiencing cognitive perturbation from comparing one's results to different results from other models doesn't guarantee one will revise one's model. Both Jan and Dee experienced cognitive perturbation at the end of the classroom presentations, as the result from their model stated that one large fish was worth two small fish, which was at odds with most of the other group presentations that stated it was worth eight small fish. Only Jan responded to this dissonance by creating a new model; Dee's individual approach was to ignore the dissonance by removing the result (1 big fish = 2 small fish) from her letter, and simply handing in the inappropriate, untested model.

The 3-dimensional nature of the multifix cubes seemed to be most effective in helping students articulate an argument based on cubic reasoning. The multifix cubes helped Liv, Liz and Pania's group appreciate the third dimension of depth (or thickness) of the fish, which they had previously ignored in their drawings. In contrast, the 2-dimensional diagrams were often limiting in this regard, as they lend themselves to portraying two dimensions of length and width, thereby activating an area, rather than volume thought process. Jan's revised model described in her individual letter used a diagram that superimposed the 2-dimensional shape of the area of a fish onto a 3-dimensional representation of multilink cubes to reason about why eight small fish was worth one big fish. This diagram suggests that the multilink cubes used by other groups in their class presentations were particularly effective semiotic resources for helping Jan see that one large fish was not worth two small fish, but eight.

However, just like diagrams, the physical manipulatives of multifix cubes do not automatically guarantee that students will be able to perceive the 3-dimensional structure portrayed. Indeed, Liv's individual letter suggests that she remembered the presence of multifix cubes in her group's argument as to why one large fish is worth eight small fish, but she couldn't reconstruct the mathematical argument on her own. In distorting the diagram of the "blocks" to fit her assertion that there are 32 blocks in the large fish, compared to 4 blocks in the small fish, Liv reveals that she doesn't truly appreciate the impact of the the fish's third dimension (depth or thickness) on its volume. Thus, the effectiveness of physical manipulatives, like diagrams, partly lies in whether students can use them to attend to the mathematical structure that is appropriate for the problem.

Together, this trio of case studies suggests the theoretical semiotic potential of diagrams and physical manipulatives in overcoming the illusion of linearity lies in whether or not they enable students to visualise, test and examine the mathematical structures they describe in their mathematical models. In modelling terms, these semiotic resources are potentially useful if they enable one to progress through of the steps in a modelling cycle, beginning with formulating a mathematical model using the semiotic resources, running the model and examining the results, then comparing the results to information in the real world and if necessary, developing another model. Lesh et al. (2000) advocate designing activities that have some form of "self assessment", whereby students can determine for themselves whether their solution is on the right track, without having to appeal to the teacher or textbook for confirmation. Our case studies suggest that one way of fulfilling the self assessment principle may be to encourage students to construct and manipulate semiotic resources that have the semiotic potential for enabling students to visualise, test and examine their mathematical approaches.

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Caroline Yoon
Department of Mathematics
University of Auckland, New Zealand

Tessa Miskell
Department of Mathematics
University of Auckland, New Zealand