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3. CLASSROOM MATHEMATICAL ACTIVITY WHEN IT IS SEEN AS AN INTER-INTRA DOUBLE SEMIOTIC PROCESS OF INTERPRETATION

A Peircean Perspective

ABSTRACT

Semiotic reality is a fundamental part of our common reality. Where we stand in this chapter looks upon the teaching-learning of mathematics as a double semiotic process of interpretation. It takes place within the socio-mathematical semiotic reality that teachers and students inherit and jointly activate in the classroom. We argue that, during interpretation, the formation of students' mathematical conceptions and the attainment of their mathematical Concepts is constructed not only with the guidance of teachers. It also follows a progressive and corrective process of inter-intra interpretation. We emphasize that teachers' awareness of the evolving nature and refinement of their own processes of interpretation and, especially, their awareness of the interpretations that takes place in the students, is essential to maintain a collaborative and dynamic teaching-learning signifying practice. Our understanding of the Person-Object relation agrees with Vygotsky when we claim that objectification is a special case of internalization. This objectification takes place during Self-Other external activity aided by Self-Self internal activity. Taking a Peircean perspective not only puts a special emphasis on intra-placed mathematical sign-interpretant formation, but it also puts a high focus on intra-abstracting-objectification that takes place in each and every student.

INTRODUCTION

We consider the teaching-learning of mathematics to be a signifying practice, one that is framed in a complex socio-mathematical classroom that functions as an extended semiotic system. Embedded in this larger system, the discourse of teachers and students is mediated by a variety of mathematical, linguistic, and paralinguistic SIGNS. In this chapter, the word SIGN, *used only in upper case*, stands for the unified and undividable relation among the three components of the Peircean "sign".

A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 43–66.

In the classroom signifying practice, teachers and students interpret and give meaning to different kinds of socio-mathematical SIGNS. All forms of mathematical expression have intrinsic meanings and inner workings (Rotman 1988, 2000; Ernest, 2006). These expressions, significantly present in what lies ahead, also engage the subjective element of the meaning-making process of the Interpreters.

Under the lens of the Peircean triadic system of SIGNS, we look upon *classroom interpretation* as a progressive, ever changing *mental signifying process*. During this signifying process, Person *X* not only interacts with *other people* (Self-Others or Inter) but also, as we will see, when Peirce adds the third component to his more extended system of SIGNS, Person *X* also co-acts with the *Self* (Self-Self or Intra). These interpretations lead to the refining of inter-intra *cycles of objectification* that follow from intentionally constructed and highly coordinated *sign-interpretant formations.* During this meaning-making process, mathematical SIGNS are encountered in the network of socio-cultural *semiotic systems* (Wilder, 1981) and upon which mathematical semiotic systems are fully grounded.

Obvious it is that semiotic reality is significantly embedded in the natural world. Ignore it, maybe; pretend that it is not there, maybe. However, try as we will, try as we may, there is no way to make it go away. Include it we should because semiotic reality will always remain a fundamental part of our common reality. This is the same semiotic reality that teachers and students inherit and jointly activate in the classroom. Therefore, along with staying anchored to the natural world, any approach to mathematics education that does not in some way find a place for the central presence of semiotic reality is an approach that falls short, some would say far short, of its full potential.

As this chapter unfolds, it is easy to suppose that the presence of teachers and students is being ignored. This is far from being the case. In fact, there are four major layers built into the unfolding of this chapter. Whenever we start with Peirce and the topic of semiotics, this topic is so wide and so inclusive that we must start with *semiotic reality* in its far reaching and in its most general sense (G). For example, the scope of semiotic reality is so extended that we can now say that it includes the realization that people not only use SIGNS but that plants and animals also send signals (Sebeok, 1972; Deely, 1990).

(1) In the earlier part of the chapter we will focus on SIGN activity, also called semiosis, when, in general (G), each and every Person *X* makes any use of SIGNS. (2) As the chapter proceeds, we will look at semiosis as it takes place in the mathematics education community, namely, among (M)athematicians, (T)eachers, and (S)tudents. (3) Once the full scope and depth of semiotic reality is in place, the emphasis will be aimed at mathematical activity in the classroom (T and S). (4) Coming then to the primary and central goal in mathematics education, we will end by giving high focus to the *intra-abstracting-objectification* that, in some degree, takes place in each and every learner (T or S).

This chapter is divided into four sections. In the first section, when we activate the beginning part of Peirce's system, we sketch what we call a clarifying adaptation of the three main components of his triadic system of SIGNS. For us, these components are called sign-object, sign-vehicle, sign-interpretant; here also called *so*, *sv*, *si*, respectively. We also activate the beginning part just enough to call on the subcategories of each of the three components. (1) The sign-object *so* subdivides into *immediate, dynamic,* and *Real*; here called *io*, *do*, *RO*. (2) The sign-vehicle *sv* subdivides into *icon*, *index*, and *symbol*; here called *sv-icon*, *sv-index*, *sv-symbol*. (3) The sign-interpretant *si* subdivides into *intentional, effectual,* and *communicational*. After the main outline of this working frame is in place, we will look more closely at the use of only one SIGN, the use of any one SIGN in general (G).

In the second section, we introduce the use of standardized mathematical SIGNS, and we examine the central and focal role that sign-interpretant formations play in the emergence and the refinement of *mathematical conceptions*. These are the subjective formations that, in stages, will eventually approximate to the Real Object of the (M) athematicians, namely, the mathematical Concept, here called *RO(M).* We use the three components of the Peircean SIGN to unfold what happens when teachers and students progressively (a) construct their own mathematical conceptions when they *decode* standardized mathematical SIGNS and then (b) *encode* these conceptions back again *into* the given standardized mathematical SIGNS. Following from (a) and (b), teachers and students construct, re-construct, and refine their mathematical conceptions until they will be coordinated and integrated sign-objects that, at any given stage, will become their best understanding of a given *RO(M)*.

In the third section, we use Peirce's triadic SIGN to present our view of classroom interpretation. This view covers the teaching-learning of mathematics when it is seen as a *double semiotic process of interpretation*, a double process in which both teachers and students actively participate. Interpretation in the classroom is examined in terms of *inter-interpretation* and *intra-interpretation*, or what we will sometimes call *inter-intra interpretation*. Each process will be examined both as a reiteration and as a refinement of triangular cycles of objectification: (i) decoding-objectification, (ii) abstracting-objectification, and (iii) encoding-objectification.

The third section introduces an exception. This chapter is organized in terms of *inter-intra*, but in this section, intra-interpretation comes *before* inter-interpretation. It is much easier to present the separate triangles in Figure 5 *before* we introduce the two kinds of lines that interlace those triangles in Figure 6. What comes next after this section will continue in terms of *inter-intra*.

In the fourth section, we use the notion of inter-intra interpretation to call attention to a fundamental commonality that exists between Peirce and Vygotsky. It will point not only to the socio-cultural aspects of cognition but also to an important relation that exists between objectification and internalization.

PEIRCE'S TRIADIC SIGN

Historically, signs in the broadest sense were seen as mediating entities that prompt thought, that facilitate the expression of thought, and that embody original and conventional thought (Nöth, 1990). Signs themselves were believed to have intrinsic meanings, meanings that were realized when signs were translated into other signs, meanings that were *independent of the Interpreter*. Signs were thought to be *dyadic entities* constituted by signifier and signified (Saussure, 1972; Nöth, 1990; Vasco, Zellweger & Sáenz-Ludlow, 2009). Here notated as the pair (signifier, signified) or (sign-vehicle, sign-object). Note that, as indicated in Figure 1, if we start with the two components contained in the dyadic notion of sign, this leaves us with only one bidirectional relation (A), the relation between the signifier (sign-vehicle) and the signified (sign-object).

Central to the position taken by Peirce, about a century-and-a-half ago, is the key step he took when he transcended the dyadic conception of sign. He proposed that each and every sign should also have a third component, namely, what he called "interpretant" and what we will call sign-interpretant. Adding this third component extends the dyadic notion of sign to a triadic notion. We notate this triadic notion as SIGN to differentiate it from the dyadic notion of sign.

The triadic SIGN extends the dyadic sign to the part that is *intra*, to what happens *after* the mental arrival of a signifier, to what happens to the cognitive activity that takes place in the mind of an *Interpreter*. The third component, along with including the *Interpreter*, contains the world of *intra-placed* sign-interpretants. It follows that the *Interpreter*, any Person *X,* plays a double role: the role of Interpreter-Receiver who *decodes from* sign-vehicles, and the role of Interpreter-Sender who also *encodes into* standardized or idiosyncratic sign-vehicles.

In consequence, we cannot confuse the sign-interpretant with the Interpreter. The sign-interpretant is the construction that is formed in the mind of a Person *X* who is the Interpreter. In the eye of a Constructivist, the sign-interpretant is the mental construction that is formed *after* the mental arrival of a sign-vehicle. This construction has a dynamic and evolutionary formation in the mind of the Interpreter (i.e., Person *X*). Construction that emerges in the midst of Self-Self or Self-Other interaction.

Why do we come to Peirce? Because without Peirce's third component, the *intra* that exists in semiotic reality is not made a part of the dyadic notion of sign. We cannot say it more emphatically. This comes back to Figure 1. When there is no third component, there is no formal connection in "the system of signs" to both sides of the double process of interpretation. It follows that a "dyadic system of signs" falls far short of what we need. For us, in keeping with Peirce, a good system of triadic SIGNS should reach out and incorporate not only the presence of Self-Other but also the presence of Self-Self.

We notate Peirce's triadic SIGN as the triplet (sign-object, sign-vehicle, sign-interpretant) or (*so*, *sv*, *si*). This triplet could also be expressed as (signified, signifier, sign-interpretant) which is the extension of the pair (signified, signifier). To better understand this triadic notion, we call on the lower and upper levels of the tetrahedron in Figure 1. The lower level is located at the base of the tetrahedron, there showing the three components—*so*, *sv*, *si*. The upper level is located at the peak of the tetrahedron. The peak is for the *triadic unity* of these three components, the triadic SIGN. Note that, as indicated in Figure 1, when Peirce added the intra-placed sign-interpretant as a third component, he also added two new bidirectional relations among the three components: relation (B) between the sign-object and the sign-interpretant, and relation (C) between the sign-vehicle and the sign-interpretant.

Even though we support and follow Peirce's triadic system, we acknowledge that Peirce himself uses his own terminology in such a way that it sometimes leads the reader to ambiguity and confusion. This introduces both a strong precaution and a serious risk whenever we try to quote from his writings, especially given the many decades and the many stages across which he created his system. For example, he sometimes uses the word "sign" to refer not only to his triadic SIGN itself but also to the sign-vehicle component of the triplet (sign-object, sign-vehicle, sign-interpretant). More explicitly, we encapsulate the nominal ambiguity as follows: sign = SIGN = (sign-object, sign, sign-interpretant). So when reading Peirce one must pay close attention to the contextual meaning he intends.

Avoiding this ambiguity lies behind our efforts to select vocabulary that will present a clarifying adaptation of his triadic SIGN. In our notation, we refer to the triadic SIGN as follows: SIGN = (sign-object, sign-vehicle, sign-interpretant) = (*so*, *sv*, *si*). We do this by the way we label the four vertices of the tetrahedron in Figure 1. As shown at the peak vertex of the tetrahedron, the word SIGN, *used only in upper case*, stands for the unified and undividable totality that identifies the triadic relation among the components of the triplet. This tells us that the word SIGN stands for a fundamental and defining property of Peirce's semiotic system. The other three vertices in the base of the tetrahedron, sign-object, sign-vehicle, sign-interpretant, *always expressed in lower case*, refer to the three components of the triadic SIGN. Concisely, taking off from our clarifying adaptation, we will enter Peirce's system by way of the vocabulary that goes with the four vertices of the tetrahedron in Figure 1.

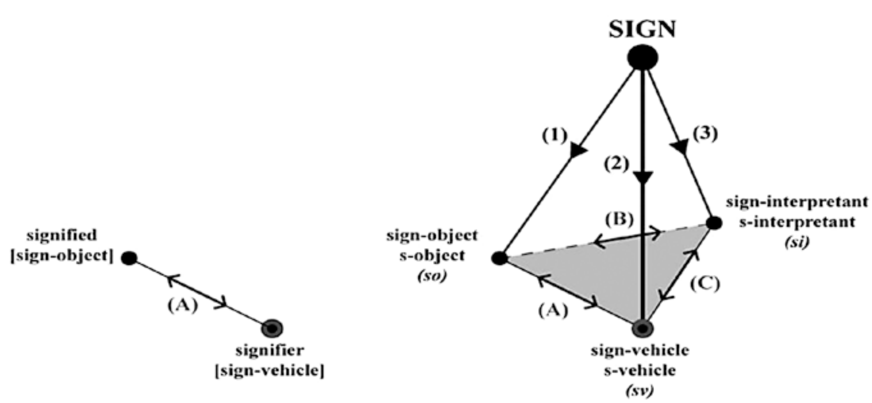


Figure 1. Dyadic and triadic conceptions of signs

Peirce defines SIGN as a *triadic relation* among its three components, a relation that determines a unified and undividable totality. He argued, on the one side, that thought can be known *between people* only by external *sign-vehicles* of some kind and, on the other side, that the only thought that Person *X* can cognize is thought that initiates the construction of *sign-interpretants*. This "one side, other side" distinction is at the heart of the inter-intra double semiotic process of interpretation. In what lies ahead, (1) "between people" will refer to the agents of Self-Others sign-interpretant formation that takes place during inter-interpretation, and (2) "a Person" will refer to an agent of Self-Self sign-interpretant formation that takes place during intra-interpretation. The same distinction will also claim center stage when we call attention to a fundamental commonality that exists between Peirce and Vygotsky.

When the intra-placed sign-interpretant is introduced as the third component, the meaning of SIGNS is located in two worlds—the world of the *intended meanings* of Senders and the world of the *interpreted meanings* of Receivers. This distinction pulls semiosis into the foreground when interpreted meanings take form, converge to, and agree with intended meanings. Such a convergence emerges mediated by SIGNS of different semiotic systems used when thinking and communicating. This tells us that the meaning of SIGNS, specifically, what the Sender *encodes into* sign-vehicles and what the Receiver *decodes from* sign-vehicles, emerges through *repeated* exchanges and *repeated* inter-intra interpretations. These exchanges and repetitions, both in the Sender and the Receiver, prompt the emergence, the construction, and the refinement of increasingly improved intra-placed sign-interpretants. What was said above does not exclude, in any way, the possibility of self-communication in which the same Person plays, alone, the roles of Sender and Receiver. This is the case of Self-Self cognitive activity.

One might think that the *sign-object* component of a SIGN is completely *encoded into* only one sign-vehicle and that it can be *decoded from* that sign-vehicle all at once. However, as we will see, three difficulties follow. (1) Just one sign-vehicle cannot completely indicate the many-sided aspects of the Real Object of a SIGN. It can only indicate at least one aspect of it. (2) Sign-interpretants prompted by a sign-vehicle and constructed at different times by Person *X* may or may not, at once, come close enough to the *intended* immediate sign-object that was *encoded into* a given sign-vehicle. (3) Sign-vehicles could function as *sv-icons*, *sv-indexes*, or *sv-symbols* depending on the contexts in which they are used and how they are interpreted in that context.

We are now ready to look more closely at only one SIGN, at any one SIGN in general (G). Peirce argues that it may be more convenient to say that, in a certain way, a sign-vehicle is determined by "a Complexus or Totality of Partial Objects" (Peirce, 1909, p. 492). He calls this Complexus or Totality of Partial Objects the Real Object of the SIGN. Here we notate it as *RO(G). RO(G)* could be material, imagined, or conceptual (whether it be conventional or idiosyncratic). The adjective "Real" in Real Object does not mean that the Object necessarily has to have a material existence in the real world. The adjective "Real" expresses the compounded comprehensiveness of a multifaceted Object.

One or more selected aspects of *RO(G)* are *offered in* and *obtainable from* the explicit form of a sign-vehicle. Thus this sign-vehicle only *presents* certain selected aspects but never *all* aspects of *RO(G)* at the same time. That is, a sign-vehicle *serves RO(G)* only when it helps to make explicit and to specify some selected aspects of it. This is to say that to comprehend *all* aspects of the *RO(G)* of a given SIGN, these aspects need to be *represented by* different sign-vehicles. As a result, Peirce conceptualizes three subcategories of the sign-object component of the SIGN: the Real Object, the immediate object, and the dynamic object. We notate these objects as *RO(G)*, *io*, and *do*, respectively.

These subcategories of the sign-object of the SIGN are described in the following paragraphs. The first paragraph is for the grounding subcategory of the sign-object. It is the target object, also called the Real Object *RO(G)*. The second paragraph is for the immediate sign-object *io*. It refers to those aspects of the Real Object that the Sender encodes into a sign-vehicle. The third paragraph is for the dynamic sign-object *do*. It refers to those aspects that the Receiver decodes after the mental arrival of the sign-vehicle.

First, the Real Object is the grounding subcategory of the sign-object component of a SIGN. The goal of the Interpreter is to make the best effort to approach the target sign-object, which is the Real Object *RO(G)*. Amid the process of interpretation, the Interpreter-Receiver generates cycles of objectification that approximate the immediate sign-object encoded into a sign-vehicle. In each cycle of objectification, the Interpreter generates sequences of sign-interpretants that will become sequences of dynamic sign-objects and that will be refined to approximate the immediate sign-object. These dynamic sign-objects are also determined by collateral successions of added experience. Peirce insists that the search for better dynamic sign-objects calls for *inquiry* and *discovery.* In the long run, the Interpreter-Receiver isolates and identifies (decodes) the aspect(s) of *RO(G)* that the Interpreter-Sender has encoded into one or more sign-vehicles.

Second, the immediate object is a subcategory of the sign-object component of a SIGN. It refers only to the aspect-object that a given sign-vehicle represents. It comes into existence only after at least one aspect of the Real Object has been selected and successfully carried into, that is, *encoded into* what will become its given sign-vehicle. In effect, the immediate sign-object refers to one or more selected *aspects* intended to represent the Real Object. Peirce argues that the immediate object is the "Object *within* the Sign [sign-vehicle]" (1977, p. 83, italics added). In other words, the immediate sign-object is the object "as the Sign [sign-vehicle] itself represents it, and whose Being is thus dependent upon the

Representation of it in the Sign [sign-vehicle]" (CP 4.536). While the immediate sign-object participates in a certain generality, it also brings specificity into focus. Thus, the immediate sign-object is a representation of some aspects of the Real Object of a SIGN and it serves to stimulate further semiosis (Corrington, 1993).

Third, the dynamic object is another subcategory of the sign-object component of a SIGN. It is constructed in the mind of the Interpreter as the product of sign-interpretants. It is always constructed *after* the mental arrival of the aspect-containing sign-vehicle. It is constructed when the Receiver makes an effort to pull out, to *decode* the immediate sign-object carried by the aspect-containing sign-vehicle. As Peirce argues, the dynamic object is the "Object *outside* the Sign [sign-vehicle]" (1977, p. 83, italics added), or that object "which, from the nature of things, the Sign [sign-vehicle] *cannot* express, which it can only *indicate* and leave the Interpreter to find out by collateral experience" (CP 8.314, italics added).

In general, under the Peircean semiotic lens, the cognitive process of Person *X* can be seen as the progressive refinement of subjective dynamic sign-objects prompted by immediate sign-objects encoded in sign-vehicles. This refinement is prompted and sustained by how Person *X* interprets aspect-containing immediate sign-objects carried by sign-vehicles. Along with constructing intra-placed sign-interpretants, Person *X*'s interpretations follow from interactions that take place both with Self-Others and within the Self-Self.

In the following section we unfold this refinement as a cognitive process that starts with beginning mathematical conceptions and that converges to mathematical Concepts.

FROM MATHEMATICAL CONCEPTIONS TO THE ATTAINMENT OF MATHEMATICAL CONCEPTS

We shift now to the lower level of the tetrahedron in Figure 1 and how it functions in mathematical conceptualization. Sign-vehicles play a primary and fundamental role in the formation and refinement of mathematical conceptions. Very much in the subjective domain (intra), these conceptions are formed during mathematical semiosis, when Person *X* decodes mathematical immediate sign-objects io's from mathematical sign-vehicles sv's. These conceptions, in stages, will eventually become the formal mathematical Concept RO(M) (the Real Object of the Mathematician M). It is during this developmental semiosis that Person *X* establishes the cognitive and epistemic aspects of the Person-Object relation.

In other words, standardized mathematical sign-vehicles serve as *mediators.* Sign-vehicles come between the other two components, between mathematical immediate sign-objects and mathematical sign-interpretants—*io(sv)si*. In fact, sign-vehicles play the role of mediating cognitive tools, which in Vygotsky's terms are called psychological tools. More specifically, (1) sign vehicles, serving as psychological tools, are *determined by* the immediate sign-objects that they carry and (2) sign-vehicles will also *determine* many possible dynamic sign-objects in the mind of the Interpreter.

It is important to note that this *twofold determination* calls for two complementary mathematical acts. The first is made when Person *X* (Interpreter-Sender) encodes a selected mathematical immediate sign-object into a selected mathematical sign-vehicle—[(*io)*](*sv*). Second is made when Person *X* (Interpreter-Receiver) *decodes* this mathematical immediate sign-object *from* that sign-vehicle to obtain a sign-intepretant from which his dynamic-object is constructed—[*(io)(sv)*] (*do)*. Note that this SIGN structure not only occupies a fundamental part of mathematical semiosis but that it is also clearly made explicit in Peirce's system of SIGNS.

The primary and fundamental role of sign-vehicles becomes even more interesting. As already mentioned, three kinds of sign-vehicles are connected to the sign-objects when they are sub-classified into *sv*-icons *sv*-index, and *sv*-symbols (1) A *sv*-icon is a sign-vehicle that bears a resemblance to its sign-object, such as the drawing of a triangle when it is taken to be the representation of the class of trilateral figures. (2) A *sv*-index has a cause-effect connection to its sign-object, such as the connection of the letter "x" to an unknown quantity. (3) A *sv*-symbol is connected to the sign-object by habit, as established by consensus, such as when ">" stands for the relation "greater than." Most sign-vehicles in mathematics belong to this subcategory. Note that, also for Peirce, the word "symbol" refers only to a subcategory of the component called sign-vehicle. This three-fold sub-classification adds to the challenge of selecting clarifying and aspect-specifying sign-vehicles.

As already mentioned, a single mathematical sign-vehicle can stand for only some of the aspects of a mathematical Concept. Therefore, different but interrelated mathematical sign-vehicles should be chosen from different standardized *systems* of mathematical SIGNS to convey a given mathematical Concept. These systems are well-established and carefully connected collections of SIGNS that extend across an extremely wide range of vocabularies, notations, algorithms, tables, graphs, diagrams, metaphors, analogies, models, arguments, proofs, etc. Even though the sign-vehicles of these systems of SIGNS were at first open to idiosyncrasy and unconventionality, they have acquired, by now, a high degree of standardization.

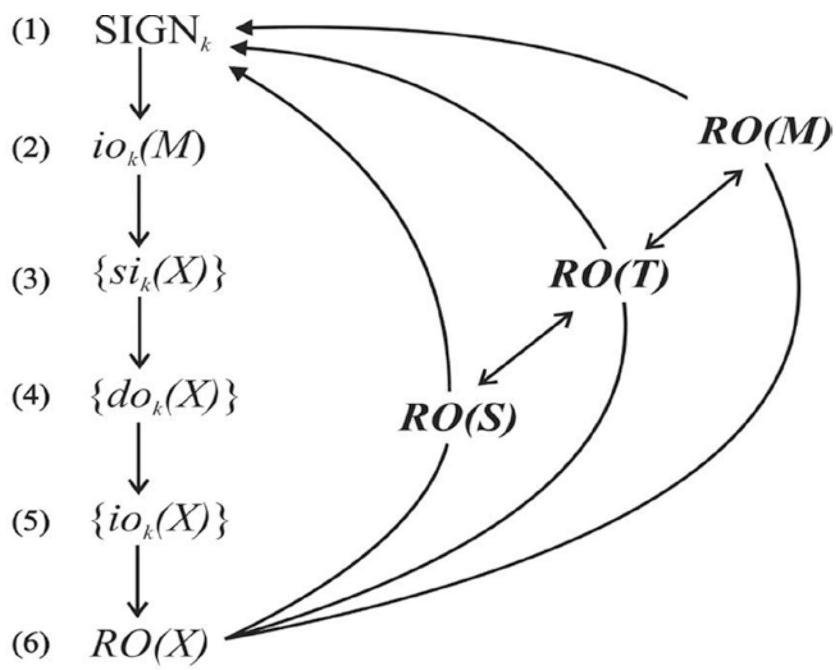
In effect, from a mathematical perspective, Person *X* interprets mathematical immediate sign-objects that have been encoded into standardized mathematical sign-vehicles. Then, he constructs and refines sign-interpretants to obtain dynamic sign-objects that will approximate the mathematical immediate sign-objects encoded by mathematicians into sign-vehicles. At each stage of this process, Person *X* makes every effort to attain the best approximation he can, at that moment, of *io(M)* and, later on, of *RO(M).* When a sign-vehicle, carrying the *io(M)*, is interpreted by Person *X*, he generates *si(X)*'s and subsequent *do(X)*'s and *io(X)*'s, which in turn will approach the *io(M)* and, later on, will converge to *RO(M),* the mathematical Concept. This process is also improved when Person *X* calls on personal collateral observations and insights based on prior mathematical knowledge and experience.

In general, the distinctions and the complementarities between the mathematical immediate sign-object as *intended* by *M* and as *interpreted* by Person *X* have implications for the mathematical semiosis of Person *X*. This activity is not only confined to the self-reference of SIGNS. It also reaches out and includes personal, inter-personal, and social experiences. These experiences may also become relevant to an ongoing semiosis even though they may be only virtually semiotic with respect to that semiosis.

Consequently, during this mathematical semiosis, sign-vehicles that carry the intended mathematical immediate sign-object of the mathematician, *io(M)*, are the sign-vehicles that prompt Person *X* to generate sign-interpretants. Some of them can become dynamic mathematical sign-objects, *do(X)*'s, that give rise to the emergence and the refinement of personal mathematical *conceptions, io(X)*'s. These conceptions will eventually isolate and identify the intended *io(M).*

As *RO(M)* is represented by different *io(M)*'s encoded into different but interrelated *sv*'s, the mathematical conceptions of Person *X* will emerge from personal interpretations. At each stage of the process of interpretation, the *decoded do(X)*'s will progressively constitute themselves into *io(X)*'s that, again, will progressively constitute themselves into a coherent unity *RO(X)*, which is the Real Object interpreted by Person *X* and taken by him as his approximation of *RO(M)*). Thus *RO(X)* is the result of a process of interpretation during which Person *X* makes every effort to approach *RO(M)*. This process will continue as long as Person *X* stays interested in increasing his mathematical understanding.

In what follows we will describe the mathematical semiosis of Person *X* in two levels. Here we need a soft warning. Since we are entering only the beginning part of Peirce's system, we do not climb into the layers of his more extended system that contains 10, 28, and 66 classes of SIGNS (Farias & Queiroz, 2003). When we do no more than stay within the scope of our working frame, it is still the case that describing the two levels will also serve as an example of how detailed this approach can become, when it is needed. Nevertheless, at first glance to a beginner, saying this much could easily be looked upon as climbing into a system of SIGNS that is too elaborate and overextended. Note, however, that the challenge is still open as to how these two levels would be described if Peirce's more extended system were activated.



(1) one mathematical representation of *RO(M)*

(2) mathematical immediate sign-object *encoded* by *M into* one sv_k

(3) *X*'s sequence of sign-interpretants induced by sv_k

(4) *X*'s dynamic sign-objects induced by the sequence of sign-interpretants

(5) *X*'s first mathematical conceptions

(6) *X*'s first approximation of *RO(M)*

Figure 2. First level of semiosis: Person X decodes only one standardized mathematical to attain his initial mathematical conceptions and his first approximation RO(X) of RO(M)

Figure 2 represents the first level of semiosis when Person *X* (*T* or *S*) uses only one SIGN and decodes its corresponding sign-vehicle. This first level is the simplest level of mathematical semiosis that appears when, in stages, Person *X* decodes only one standardized mathematical sign-vehicle sv_k. This mathematical semiosis takes place when *M* becomes an Interpreter-Sender who encodes selected aspects of a mathematical concept io_k (*M*) into one sign-vehicle sv_k and when Person *X* (*T* or *S*) becomes an Interpreter-Receiver who decodes it.

When Person *X* decodes sv_k, the $io_k(M)$ of sv_k elicits in Person *X* a sequence of sign-interpretant formations $\{si_k(M)\}$. The sub-index *k* of *si* indicates its association with sv_k. The recurrent sequences $\{si_k(X)\}$ generate recurrent sequences of mathematical dynamic sign-objects $\{do_k(X)\}$. These sequences represent the evolving subjective understanding of Person *X*. When these interrelated sequences are coordinated and integrated, they generate the sequence of interpreted mathematical immediate sign-objects $\{io_k(X)\}$, which represents the initial conceptions of Person *X* that comes to be an approximation of $io_k(M)$. Thus, when the sequence $\{io_k(X)\}$ is integrated and coordinated it constitutes itself into an *RO(X)* that Person *X* takes it to be his first approximation of *RO(M)*.

Figure 3 represents the mathematical semiosis when Person *X* uses a selected assortment of SIGNS and decodes their corresponding sign-vehicles. This second level is the more complex level of mathematical semiosis that appears when, in stages, Person *X* decodes the same RO(M) not from one but from a well chosen assortment of standardized mathematical sign-vehicles $\{sv_k\}_k$. This mathematical semiosis takes place when *M* becomes an Interpreter-Sender who encodes selected aspects of a mathematical concept $\{io_k(M)\}$ into different sign-vehicles $\{sv_k\}_k$ and when Person *X* (*T* or *S*) becomes an Interpreter-Receiver who decodes them. When Person *X* decodes the set $\{sv_k\}_k$, he generates a sequence of sequences $\{\{si_k^j(X)\}_j\}_k$ of intra-placed mathematical sign-interpretants associated with each element of the set $\{io_k(M)\}_k$. The super-index *j* of $si_k(X)$ indicates the sequence of sign-interpretants that is constructed when Person *X* decodes each sv_k.

This sequence of sequences then generates a second sequence of sequences $\{\{do_k^j(X)\}_j\}_k$ of mathematical dynamic sign-objects for each sv_k. Subsequently, this second sequence of sequences generates a more refined sequence of sequences of decoded mathematical immediate sign-objects $\{\{io_k^j(X)\}_j\}_k$. These more refined sequences constitute an improvement in the mathematical conceptions of Person *X* after every sv_k of the assortment is decoded in coordination with the others. When this latter sequence of sequences is integrated and coordinated, it converges to $\{io_k(M)\}_k$. Thus, this convergence is what, at this stage, Person *X* takes to be the best approximation *RO(X)* of *RO(M)*.

Especially important in this process is a consideration of the interpretation that takes place *between people*, when the mathematical sign-object (*RO*, *io*, or *do*) is in the mind of one Person (for example, *M*) and the mathematical sign-interpretant and mathematical dynamic sign-object (*si*, *do*) is in the mind of *another Person* (for example, *T,* or *S*). In other words, we need to consider not only what is determined in the mind of the Interpreter-Encoder (intentional sign-objects and intentional sign-interpretants) but, specially, what is also constructed in the mind of the Interpreter-Decoder (interpreted sign-objects and constructed sign-interpretants).

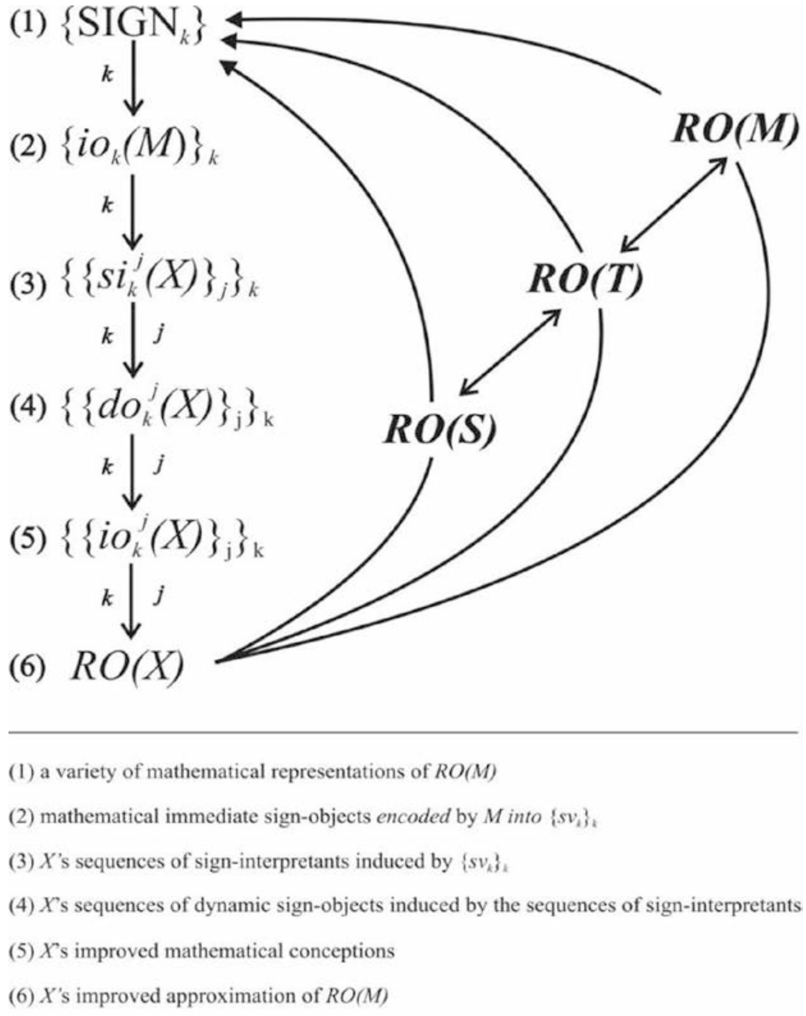


Figure 3. Second level of semiosis: Person X decodes a selected assortment of sv_k's to attain more refined mathematical conceptions $\{\{io_k^j(X)\}_j\}_k$ and, at the same time, a better approximation RO(X) of RO(M)

For communication to take place, reaching some sort of agreement (communicational sign-interpretants) is a necessary condition. In our case, standardized mathematical SIGNS will achieve their communicative function only if the agreement to be reached is whatever is expected to be *commonly understood* between Interpreter-Encoders and Interpreter-Decoders. Thus, a mathematical agreement is, in essence, the communicative invariance of mathematical SIGNS.

These are the meanings that transcend subjective interpretations, that transcend particular contexts, and that transcend any given moment in time. These are the meanings that converge to the intended meanings encoded into the second component, namely, the sign-vehicle. Even though agreement may not come in its complete totality, the classroom participants should agree, at least, on some of the essential aspects of any given mathematical Concept *RO(M)*. Aspects that are represented and carried by a set of standardized mathematical aspect-specifying sign-vehicles.

CLASSROOM MATHEMATICAL ACTIVITY

Within the Peircean semiotic approach that we have taken, we will present our view of *classroom mathematical activity* when it is seen as a *double semiotic process of interpretation*. We consider the teaching-learning of mathematics to be a complex semiotic process of interpreting standardized mathematical SIGNS, a process in which both teachers and students actively and intentionally participate.

But what is happening in the classroom is not limited to just teachers and students (T and S). Given the full presence of semiotic reality as it exists in the classroom, Person *X* could be *M*, *T*, or *S*. It is a given that *M* has gone through his own developmental stages of mathematical intra-interpretation, the stages in which *M* attains the construction of mathematical Concepts *RO(M)*. In the classroom mathematical activity, T and S also go through their own developmental stages of mathematical intra-interpretation, the stages in which T and S attain their best approximations of *RO(M)*. Even though it is obvious that *M* as a Person is almost always not physically present in the classroom, the work of *M*, namely, the selected mathematical Concepts and related sign-vehicles that point to constructions are always present. This also calls attention to the developmental stages of inter-interpretation that first go from *M* to *T, T*[*RO(M)*]. Then, ideally, they will go from *T*'s interpretation of *RO(M)* to *S, S*[T[*RO(M)*]]. Then, ideally, they will go from *S*[T[*RO(M)*]] to *T, T*[*S*[T[*RO(M)*]]]. These cycles of intra-inter interpretation among M, T, and S ground the classroom mathematical activity.

During classroom communication, sign-interpretants play an important role in the semiotic activity of both Interpreter-Sender (*M*, *T*, or *S*) and Interpreter-Receiver (*M*, *T*, or *S*). This occurs when they seek to attain some sort of consensus. Given an Interpreter-Sender with an intentional sign-interpretant in mind, what is *encoded into* a sign-vehicle is a selected mathematical immediate sign-object.

When the Interpreter-Sender *encodes* an immediate sign-object *into* one or more aspect-specifying sign-vehicles with a particular *intentional* sign-interpretant in mind, the Interpreter-Receiver is expected to *decode* it *from* the given sign-vehicles and, from this, to produce dynamic sign-interpretants and to construct dynamic sign-objects and, from them, approximations of each encoded immediate sign-object. Next, the Interpreter-Receiver becomes an Interpreter-Sender, and the cycles of semiosis will continue until some common ground—consensus or communion—is attained. In Peirce's terminology, the common ground attained by both Interpreter-Sender and Interpreter-Receiver is called a *quasimind,* a *cominterpretant,* or *a commens*.

Needless to say, intra-interpretation in the classroom coexists with inter-interpretation. They can be separated only for the purpose of analysis and description. Both occur within a semiotic reality that is not only mathematical but also social. Consequently, in the classroom, the cycles of objectification of Person *X*, both intra and inter, generate each other synergistically.

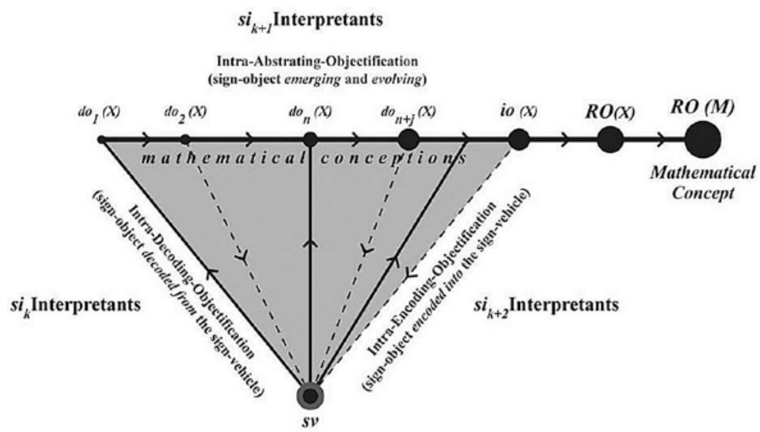


Figure 4. Triangular cycles of objectification of Person X that take place during intra-interpretation: (intra-decoding-objectification), (intra-abstracting-objectification), (intra-encoding-objectification)

Intra-Interpretation

We consider intra-interpretation to be a triangular cyclic process of objectification. Figure 4 shows the three components of the cycle: *intra-decoding-objectification*, *intra-abstracting-objectification*, and *intra-encoding-objectification*. During this process, Person *X* decodes a given standardized *sv* and constructs *si(X)*'s and *do(X)*'s to produce *io(X)*'s that are then encoded back into the same *sv or* related *sv*. Each cycle produces more refined dynamic sign-objects *do(X)*'s that are better approximations of the immediate sign-object *io(M)* initially *encoded into* a given sign-vehicle. These cycles continue, consciously or unconsciously, until Person *X* is satisfied with the construction of an *io(X)* and an *RO(X)*. Both of them will eventually converge to the mathematicians' *io(M)* and *RO(M)*.

Figure 5 shows the triangular cycles of objectification of the classroom participants—*M*, *T*, S_i and S_{i+1}—when each goes through their own triangular cycles. At this point, we also elect to describe briefly what happens in the semiotic reality of mathematicians. Mathematicians begin when they create their own mathematical conceptions by means of intra-abstracting-objectification or when they *decode* existing mathematical sign-objects from standardized mathematical sign-vehicles by means of intra-decoding-objectification. In this way, mathematicians construct their own *do*(*M*)'s and refine them so that these dynamic sign-objects cohere with the logic of broader mathematical systems. This is done through repeated intra-abstracting-objectifications, which eventually lead to the construction of new and better *RO*(*M*)'s. Finally, the mathematicians select certain aspects, *io(M)*'s, that identify, specify, and represent their *RO(M)*'s, which they then *encode into* idiosyncratic or conventional sign-vehicles that are communicated to others.

After the work of the mathematicians has been carried into the classroom, teachers together with the students always start with standardized mathematical *sv*'s. They *decode* them to generate their own mathematical dynamic sign-objects, here expressed as *do*(*T*), *do*(S_i), and *do*(S_{i+1}). In the long run, these dynamic sign-objects give rise to the formation of their mathematical conceptions. Usually, the first mathematical conceptions that are constructed by the students could be very different from what mathematicians *intended* when they *encoded* their *io(M)*'s into standardized *sv*'s for their *RO(M)*'s.

When *T*, S_i, and S_{i+1} make an effort to construct their own mathematical conceptions, they will continue to modify and refine their *interpreted io*(*T*), *io*(S_i), and *io*(S_{i+1}) so that they will converge to the *intended io(M)*. Eventually, they will construct what they consider to be their own "mathematical sign-objects" seen as their best understanding of *RO(M)* or mathematical Concept.

All of this is brought into focus by means of *triangular cycles of intra-objectification*. It tells us that the refinement, the coordination, and the integration of a sequence of *do*'s seek to isolate and make explicit the *io(M)* carried by a given standardized *sv*. Selecting different *io(M)'s* and encoding them into different *sv*'s tend to specify more general aspects of *RO(M)*.

When the classroom participants produce their own triangular cycles of intra-objectification and, consequently, their own cycles of signification, they also produce more abstract levels of intra-interpretation. Nevertheless, for us, *intra-interpretation* is nothing more than a mathematical personal process that will also be influenced by the collaborative interaction among the classroom participants. In effect, intra-interpretation keeps pace in parallel with *inter-interpretation*, which is the focus of the next section.

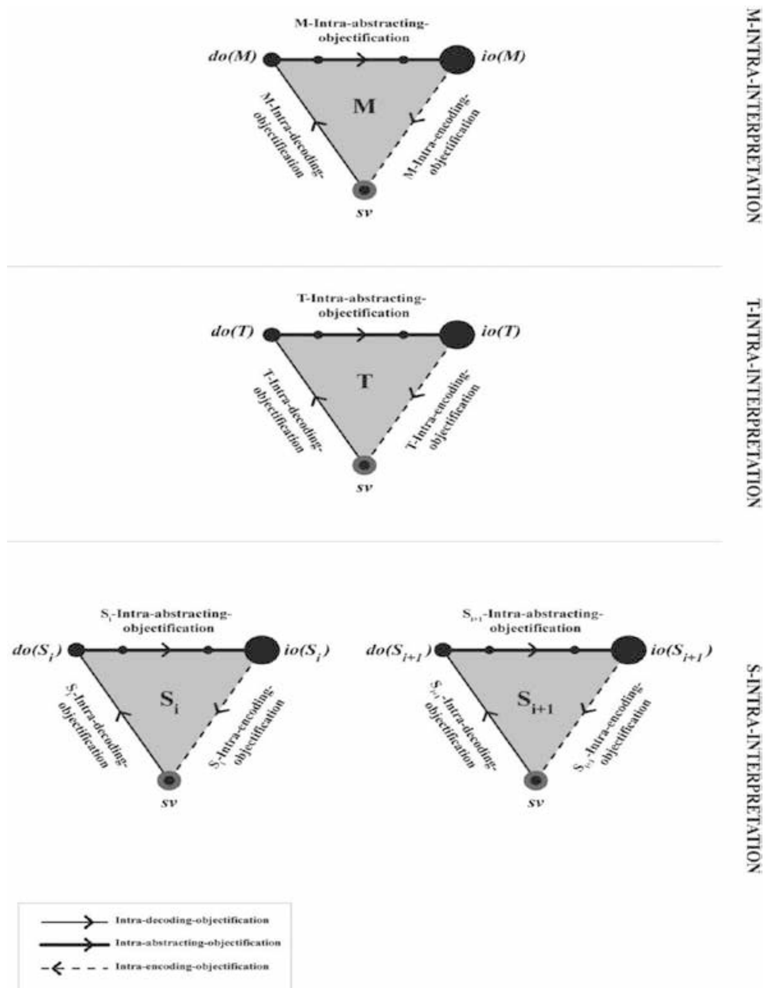


Figure 5. Intra-interpretation: Triangular cycles of intra-objectification of person X (M, T, or S) in the mathematics classroom

Inter-Interpretation

Continuing with the same format, we consider inter-interpretation to be a triangular cyclic process of *objectification*, a process aided by the presence and collaboration of others. Again, the three steps are *inter-decoding-objectification*, *intra-abstracting-objectification,* and *inter-encoding-objectification*. As before, the straight-edged triangles in Figure 6 show that the classroom participants—*M*, *T*, S_i, and S_{i+1}—when each activates their own cycles of intra-interpretation. In keeping with parallel pacing, now the curve-edged triangles of inter-interpretation connect with the straight-edged triangles of intra-interpretation. Semiotic reality is such that both sets of triangles not only coexist. They also interact synergistically. For us, the hyphen in "inter-intra" is a well placed visual sign-vehicle that stands for this synergy.

More specifically, a diagram that lays out the inter-intra connections and that indicates this synergy can be seen by following the two kinds of arrows in Figure 6. Note especially that, and this is a high focal point in our analysis, both sets of triangles have a single common side, the side with the thick horizontal edge, the side in the middle of each cycle of intra-interpretation, namely, *intra-abstracting-objectification.* Later we will look again at this thick horizontal edge. Then we will point to critical moments in the construction of sign-interpretants, construction that takes place in the socio-semiotic reality of each and every student.

In Figure 6, the teacher's inter-decoding objectification is indicated by the solid curved segment starting at the mathematicians' *sv* and ending at the upper left vertex of the teacher's triangle, *do(T)*. This decoding objectification links mathematicians *M* and teachers *T*. It is the first step in the teachers' process of inter-interpretation. The inter-decoding-objectification of the teacher is followed by own process of intra-interpretation. More specifically, T-intra-abstracting-objectification is shown by the side with the thick horizontal edge of the teacher's triangle. This objectification sustains the transformation of *do*(*T*)'s into *io*(*T*)'s. Also shown in Figures 2, 3, and 4, when the interpreted *io(X)*'s (from standardized mathematical *sv's*) are collectively coordinated and integrated, they will move more closely to approach the intended *io(M)*.

It is important to note that the T-intra-interpretation of *sv*'s is the starting point of the interaction between *T* and *S*'s. Not only are *sv*'s sub-classified into *sv*-icon, *sv*-index, and *sv*-symbol. Not only do *sv*'s play a major role when they serve as mediators that stand between sign-objects and sign-interpretants. But also the inter-action of the teacher shines significantly when skills are expressed during those moments when the same *sv*'s are first presented to the students.

When *T* conveys mathematical meanings to the students, he seeks to encode interpreted *io*(*T*)'s to match the meanings carried by standard mathematical *sv*'s. The teacher's mathematical *sv*'s are, in turn, decoded by the students—S_i- and S_{i+1}-inter-decoding-objectification—who then engage in constructing their own *do*(S_i)'s and *do*(S_{i+1})'s.

Continuing with Figure 6, the students' inter-decoding-objectifications are indicated by: (1) the solid curved segments starting at the teacher's *sv* and ending at the upper left vertices of the students' triangles, *do*(S_i) and *do*(S_{i+1}); and (2) the solid curved segments starting at the *sv*'s of the students and also ending at the upper left vertices of the students' triangles, *do*(S_i) and *do*(S_{i+1}).

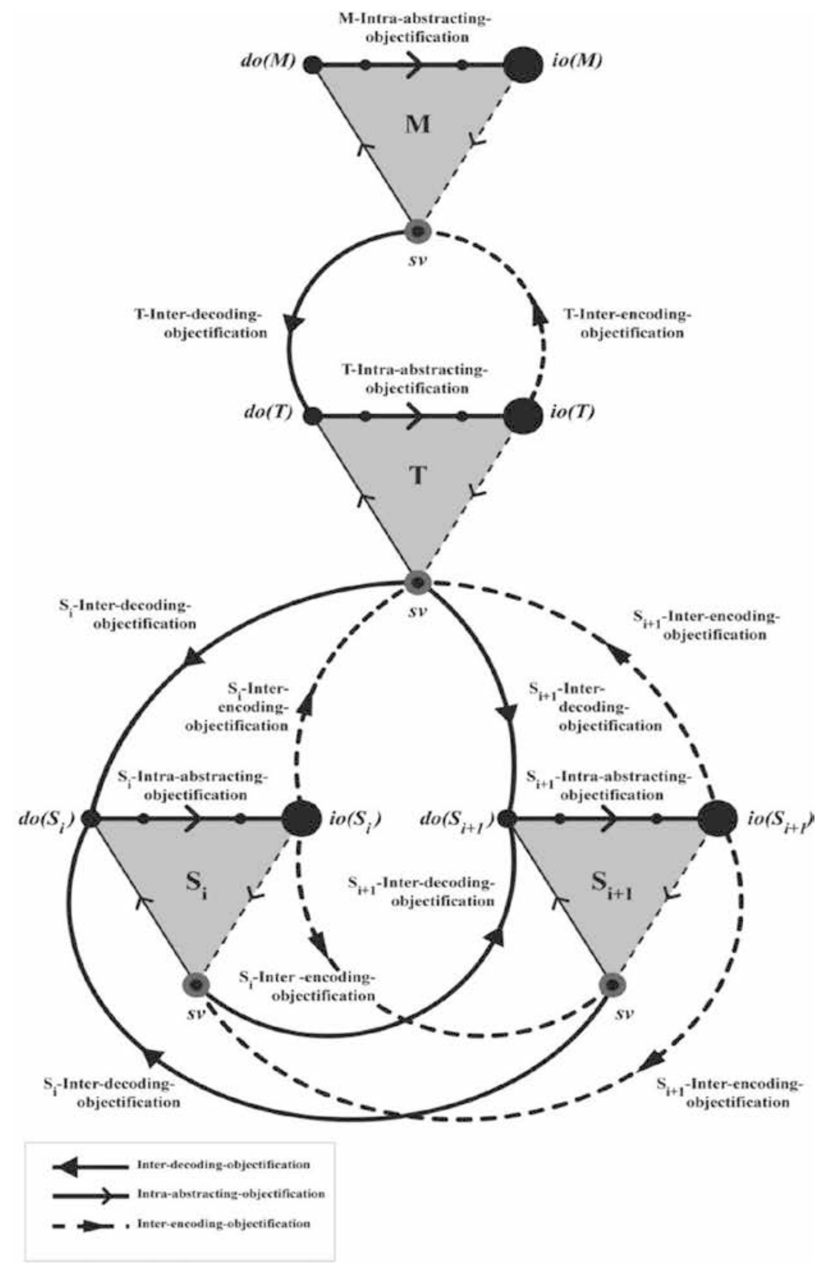


Figure 6. Inter-intra interpretation: Triangular cycles of inter-objectification and intra-objectification of Person X (M, T, or S) in the mathematics classroom

These inter-decoding-objectifications lead to S_i- and S_{i+1}-intra-abstracting-objectifications that transform $do(S_i)$ and $do(S_{i+1})$ into $io(S_i)$ and $io(S_{i+1})$. What follows is the students' inter-encoding objectifications indicated by the dashed curved segments that start at $io(S_i)$ and $io(S_{i+1})$ and end at the *sv* of either the teacher's triangle or the triangle of the other student.

Here we need a forceful alert. Look again at Figure 6 and the thick horizontal edges of the students' triangles. It is during this highly specialized mental activity, mathematically specific, that each and every student engages in intra-abstracting-objectification. Again, in inter-interpretation as in intra-interpretation, we give central attention to these high focal mental moments. Especially sensitive to the semiotic presence of the intra-placed sign-interpretants contained in Peirce's third component, it is the students' intra-abstracting-objectification that not only anchors their cycles of Self-Others *inter-interpretation* but also anchors their cycles of Self-Self *intra-interpretation*.

Consequently, in light of the double semiotic process of inter-intra-interpretation and cast within the limits of resourcefulness and ingenuity, the central and primary goal of teachers is to facilitate and sustain an ongoing *intra-abstracting-objectification* in each and every student.

PEDAGOGICAL IMPLICATIONS OF INTER-INTRA INTERPRETATION

From a Peircean perspective, we have analyzed classroom mathematical activity as a *double semiotic process of interpretation*, a process that is both inter and intra, one that, grounded in the use of standardized mathematical SIGNS, is situated not only in Self-Others but also in Self-Self.

Consequently, this view of interpretation accounts not only for the teacher's semiotic process of interpretation and not only for the students' semiotic process of interpretation. Clearly at another focal spot in our analysis, it also accounts for the teacher's interpretation of the students' interpretation. Being aware of these three parallel semiotic activities would improve not only standard teaching practice. It would also improve the learning conditions that are available to the students.

Giving special attention to the teachers' interpretation of the students' interpretation of mathematical sign-vehicles should encourage and motivate the creation, the organization, and the re-organization of instructional sequences. Such sequences ought to help students refine their inferred mathematical dynamic sign-objects and to approximate both the immediate sign-objects encoded in mathematical sign-vehicles and the Real Objects of mathematical SIGNS. The Real Objects of mathematical SIGNS—Concepts—are abstract objects apprehended by the mind through the mediation of sign-vehicles.

These sequences of mathematical objects, immediate objects encoded into and carried by sign-vehicles and dynamic objects elicited by these sign-vehicles, will allow students to experience their own learning of mathematics as an ongoing process of construction, refinement, and approximation. This is the subjective process of intra-abstracting-objectification aided by inter- and intra-decoding-objectification and by inter- and intra-encoding-objectification. These objectifications are dependent not only on the *inter-actions* among teachers and students but also on the *intra-actions* of the students within their Selves. This intentional, reciprocal, and self-reciprocal engagement of teachers and students in the interpretation of standardized mathematical sign-vehicles will not only regulate the teaching practice of teachers, but it will also regulate the learning practice of students.

VYGOTSKY, PEIRCE, INTERNALIZATION, OBJECTIFICATION, AND THE PERSON-OBJECT RELATION

As seen in sections 2 and 3, the developmental stages of mathematical inter-intra interpretation are grounded in the ongoing effort of Person *X* (T and S) to decode mathematical Concepts *RO(M)* from standardized mathematical sign-vehicles. This decoding is a deconstructive-constructive act given that sign-vehicles represent some but not all aspects of *RO(M)*. In these sections, the process of *objectification* is described in terms of *triangular cycles* that are synergistically *inter* and *intra.* These cycles are the fundamental components of the *double semiotic process of inter-intra interpretation*.

The ongoing process of interpretation comes into existence after the mental arrival of mathematical sign-vehicles *sv*. These sign-vehicles have been deliberately selected, first by *M* and then by *T*, to carry aspect-specifying mathematical immediate sign-objects *io(sv)*. These immediate sign-objects carried by sign-vehicles bring about the construction of mathematical dynamic sign-objects [*io(sv)*]*do*. These dynamic sign-objects lead to the construction of approximations of mathematical Real Objects [[*io(sv)*]*do*]*RO(T)* and [[*io(sv)*]*do*]*RO(S)* that each time will approach more closely the mathematical Concept *RO(M)*. All of this, along the way, encourages and calls forth the critical mental moments, namely, the moments that not only prompt the formation of good mathematical sign-interpretants *si* in each and every classroom participant but also prompt good contact with the socio-semiotic reality of the mathematics classroom.

The synergy between the *inter planes* and the *intra planes* of cognitive and semiotic development is not a new notion. Vygotsky clearly argued that the dialectic between the intramental and the intermental planes produces a constant evolutionary development not only in word meaning and problem-solving strategies but also in sign (i.e., sign-vehicle) use.

> We have found that sign operations appear as the result of a complex and prolonged process subject to all the basic laws of psychological evolution. *This means that sign-using activity in children is neither simply invented nor passed down from adults*; rather it arises from something that is originally not a sign operation and becomes one only after a series of qualitative transformations. (Vygotsky, 1978, pp. 45–46; italics in the original)

The above quote indicates that Vygotsky's notion of *internalization* is cast within a frame of a widely conceived semiotic reality that is socially rooted, historically developed, and based on sequential qualitative transformations. He argues that a *transformation* of an *interpersonal* process into an *intrapersonal* one is the result of a long series of developmental events. This transformation is essentially an *operation* that initially represents an *external activity* and then is *reconstructed* and begins to occur *internally*. In this process, an *interpersonal process* is *transformed* into an *intrapersonal one*.

Vygotsky (1986) also defines *internal* activity (*intra*) in terms of semiotically mediated *external* social activity (*inter*). For him, this is the key to understand what happens during the emergence and the refinement of conceptions (*intra*). According to Vygotsky, "everything internal [intra] in higher forms *was* external [inter], that is, for others it *was* what it now *is* for oneself" (Wertsch, 1985, p. 62, italics added).

The Vygotskian notion of internalization of external activity serves as an umbrella for the particular case of internalization of mathematical Concepts. These notions can also be seen, through the Peircen lens, in the double semiotic process of inter-intra interpretation. As expected, this calls for a social setting that puts inter-interpretation first in time because it makes possible what will emerge later in intra-interpretation. Inter-interpretation and intra-interpretation of standardized mathematical sign-vehicles can happen only when there is a synergistic coexistence of external activity (Self with Others) with internal activity (Self with Self) directed toward the construction, the reconstruction, and the approximation of mathematical sign-objects (immediate, dynamic, and Real) built-in mathematical SIGNS.

We can safely say that there is a fundamental commonality that exists between Peirce and Vygotsky: the notion of *internalization*. When we consider *interpretation* to be a *double semiotic process*, and in keeping with Peirce's intra-placed sign-interpretant, we can infer that *inter* and *intra* processes of interpretation are semiotically mediated, intimately interrelated, and essential to internalization. This tells us that Vygotsky's view of *internalization* is essentially not different from what we have said about *triangular cycles of objectification* based on Peirce's sub-classification of the sign-object component of the SIGN. Along with constructing intra-placed sign-interpretants, these objectifications follow from interactions that take place both with Self and Others (inter-mental) and with Self and Self (intra-mental).

Therefore it can be said that in the teaching-learning of mathematics *objectification*, within a Peircean perspective, is a special case of *internalization,* within a Vygotskian perspective. In other words, objectification is the internalization of mathematical Concepts when Person *X* attains, from first efforts to latter refinements, approximations of the Real Objects of standardized mathematical SIGNS.

Consequently, the Person-Object relation is established, from beginning to end, in the inter-intra interpretation that takes place after the mental arrival of standardized mathematical sign-vehicles. In other words, the Person-Object relation is established in the midst of the inter-intra interpretation that prompts an evolutionary cognitive development. Such development can be seen through sequential refinements, a progressive developmental transformation of subjectively interpreted dynamic sign-objects. These refinements are prompted and sustained when Person *X* interprets the aspect-containing immediate sign-objects encoded into and carried by mathematical sign-vehicles, and when Person *X* generates sign-interpretants and dynamic sign-objects that approximate not only to the intended immediate sign-objects but also to the Real Objects of mathematical SIGNS—mathematical Concepts.

CONCLUSION

Along the way, we have collected information *about* SIGNS, as we went from SIGNS in General, to SIGNS in the mathematics education community (M, T, S), and then to SIGNS in the classroom (T, S). All of this information was carried forward, as we came to the high focal mental activity that takes place in the mathematical thinking of T and S—intra-abstracting-objectification.

It is well recognized that a highly specialized and an extremely precise use of sign-vehicles is the life-blood of mathematics. As we see it, we need some vocabulary, some carefully chosen words, also called sign-vehicles, which will help us discuss both the nature of SIGNS and how they are used in mathematics. Why? Because the treatment of mathematical sign-vehicles in school mathematics is often limited to giving directions that only tell us how to exercise the proper use of mathematical sign-vehicles without taking into account the students' interpretation. Rarely, at a level above, at a meta-level, are we told anything *about* semiotic reality and *about* the nature of mathematical sign-vehicles.

To meet this need is why we come to semiotics, to Peirce, and to the tetrahedron in Figure 1, now seen as a base from which to construct a grammar at a meta-level, namely, a grammar that presents a more exact way of talking *about* SIGNS in general and *about* mathematical SIGNS in particular.

Mathematical semiosis in the classroom can be seen not only as a double semiotic process of *inter-intra interpretation*. Fundamental to its existence and standing strong within a Peircean perspective, it is also grounded in the clear presence of *semiotic reality.* This semiotic reality appears when systems of mathematical SIGNS are introduced, thereby giving rise to the emergence and refinement of mathematical conceptions, mathematical Concepts, and habits of mathematical thinking. This semiotic reality also appears in systems of classroom practice and in systems of communication that are social and cultural. These three systems are all manifestations of the functioning of living, open, dynamic systems.

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