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2. GEOMETRY, A MEANS OF ARGUMENTATION

ABSTRACT

Arguing and proving are essential elements in mathematics. To learn these skills from a mathematical point of view, elementary geometry is often used as a paradigmatic part of mathematics to demonstrate and to learn how to find an argument or how to construct a proof. This text presents several reasons why elementary geometry can be seen as a fruitful part of mathematics to learn these abilities. The chapter starts with a view on the use of geometry of ancient Greeks. Within this first part some sociological reasons are presented to which objective geometry was used in their political life. The second part of this text offers a semiotically based view on geometry. There three further reasons, all based on and motivated from the use of geometry signs, are provided. All reasons help us to understand why geometry is intimately connected to learn how to argue and how to prove.

INTRODUCTION

Educational aims currently under discussion[1] as well as publications in the didactics of mathematics consider argumentation and reasoning to be essential elements for the teaching of mathematics. Argumentation and reasoning are expected from students as early as the first level of secondary school. A significant portion of school mathematics at this level is devoted to elementary Euclidian geometry. Publications in the didactics of mathematics cite school geometry as an essential subject to develop students' processes of argumentation and proving (e.g., Graumann, 1996; Kadunz & Sträßer, 2009).

The text presented here pursues, from a semiotic position, the question of why geometry is suitable as a means of learning how to argue and how to demonstrate the validity of something. To do so, attention has to be given, in a semiotic sense, to the specific forms of construction and usage of geometry signs. From this perspective, three essential aspects are to be considered because, in the long term, they will contribute to the development of skilful argumentation. These three aspects will be supported by different theories throughout the chapter.

- The use and creation of geometric signs is closely connected to the geometric relations connected with these signs.

A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 25–42.

- In general, geometry signs rarely support algorithms as we know them from algebra or analysis. This non-support is partly substituted by the promotion of other relieving activities.
- The particular plane of a geometric configuration (which differentiates it from the two-dimensionality of e.g., an algebraic equation) results in the fact that the completion of this geometric configuration is made more difficult if only for no other reason than the particular position/situation of the geometry signs with respect to each other. The configuration hides its genesis and this has particular consequences.

The above three points outline the mathematical and didactical direction of this text. One consequence of this view of geometric signs is the almost obligatory use of argumentation or reasoning for the construction, interpretation, and reconstruction of geometric configurations. In addition, it is also important to take into account the fact that argumentation in geometry needs not only verbal but also written linguistic signs.

Denise Schmandt-Besserat (1997), in her many works as archaeologist, pursues, among other things, the question of the emergence of writing. Through the use of archaeological finds and her respective interpretations, she succeeds in seeing the origin of writing not as duplication of oral language but as a consequence of the economic and military needs of the people of Mesopotamia. Thus she demonstrates that the first writing was of a numerical nature. Is it possible to find examples of similar needs—as it were everyday needs—in order to find a *new* interpretation not only *for* the learning of school geometry but also *for* how to start it? I will argue that this question can be answered on the positive. This new start relates to the beginnings of geometric argumentation in ancient Greece. Using the three aspects of the didactic program outlined above, the origin of geometric proof and the transformation of geometry from utilitarian to scientific discipline in ancient Greece (6th cent – 4th cent B.C.) can be more readily understood.

Thus the structure of my text is predetermined. First, I will turn to the geometry of the Ancient Greeks at the time of the origin of its new use. Then, I will discuss the three essential aspects, listed above, about the nature of geometry and the development of geometric argumentation.

A PARTICULAR VIEW OF THE GEOMETRY OF (ANCIENT) GREEKS

When we consider the geometry of the Greeks, the utilitarian aspect of geometry for solving everyday problems was not emphasized even though it was the main focus in Egypt and Mesopotamia. Instead, the purpose of the Greeks was the formulation theorems and the forms of geometric reasoning. From a socio-historical point of view, which reason(s) could be given for this change of emphasis? The formulation of such reasons sheds light on the success of geometry as a guiding science in the ancient Greek culture. From a semiotic position, which respects the role of the signs in geometry, possible reasons will present themselves as conclusions as to why geometry, as an argumentative science, played an influential part in the development of Greek democracy.

Thesis: The Greeks developed their use of geometry in order to organize their democratic social order. Geometry, or more precisely, argumentation through geometric reasoning, was seen as a paradigmatic example of consequential speech. Characteristic elements in the use of the geometric signs support this.

When one reads relevant publications on the history of (Greek) mathematics (Szabo, 1969, 1994; Becker, 1975; Scriba & Schreiber, 2005), they concentrate, essentially, on a geometrically accurate presentation of the development of geometry in ancient Greek while historical citations are rather short. Mostly, philosophers' statements are inserted only at the beginning of relevant parts of an exposition to embed them in similar but finalized forms of thinking, to which geometry then also belonged. Questions as to why the Greeks' approach to geometry and its new use changed permanently, in the precise period between the 8th and the 6th centuries BC, are not posed explicitly. So the hope remains that texts which concentrate on the development of Greek thinking viewed from a position of cultural sciences will produce more results. However, at this moment, there is nothing to be found about this issue.

The collection "Early Greek Thinking" (cf. Rechenauer, 2005) contains an article entitled "On the Origin of the Written Records about Thales' Geometry" (Dührstein, 2005). It is true that specific geometric problems which are traditionally (ibid., pp. 89–90) attributed to Thales of Milet are presented. However, questions about the motives for this new view of geometry are not posed. What are discussed are questions on the existence of Thales, as well as connections between Thalesian geometry and, for example, Greek astronomy.

A similar picture is given in the book "The Knowledge of the Greek" (cf. Brunschwig, 2000) through the text "The Proof and the Idea of Science" (Lloyd, 2000). Here, geometry is discussed as a source of a specific way of thinking (ibid., pp. 240–241), but the focus of the argumentation is laid on the philosophy of Parmenides. The section "Mathematics" (cf. Knorr, 2000), in the same volume, describes the phases of Greek mathematics more as a report than an interpretation. Questions as to reasons for the emergence of these phases are not posed.

The literature shows a different side to itself when the explanations of the origin of Greek thinking consider social and political dimensions simultaneously. As an example, I refer to Jean-Pierre Vernant (1982). In the section "The Intellectual Universe of the Polis" (ibid., pp. 44–48) of his comments, Vernant reports on the origin of the processes of negotiation for the making of decisions in public space. The advancement of citizens of a polis together with their enrolment into military service was tied to the right to take part in the decision-making processes. "Within the polis the status of a soldier is at one with that of a citizen; who has a place in the military structure of a city also has it in the political organization. Thus the changes in weaponry, which occurred in the middle of the 7th century, …, create a new type of warrior, convey a new social status upon him and let his personality appear in a completely different light" (ibid., p. 58). What is hinted at here, in just a few words, correlates with the above reference to Schmandt-Besserat and her view on the emergence of mathematics from particular needs. Obviously essential changes had taken place in the world of Greek life which entailed a serious change in social behaviour. In particular, the joint participation in decision-making processes demanded previously unpractised behaviour: through discussion and objection and essentially without regard for rank (cf. Vernant, p. 61) decisions should be taken together. How had such a development come about?

In "The Birth of Science" Andre Pichot (1995) developed a plausible picture of those times. Let us follow his explanations (cf. Pichot, pp. 243–246). The ancient Greek settlement area at the time of the 8th century essentially comprised the Greek mainland, the islands in the Aegean, the coastal areas of Asia Minor, southern Italy and Sicily. The period starting from about 1200 BC to the 800BC saw times of intense migration as a result of which the density of the population decreased. Individual cities formed city states, creating small and very small kingdoms that kept to themselves. This situation lasted until the 8th century BC from when on the Greeks again actively returned to the use of writing, navigation boomed, and ceramic and metalwork flourished. It is the time of Homer and Hesiod. How did these city states organize themselves? In the ancient city states power and wealth lay in the hands of the aristocracy. In the course of time this power was distributed among the well-to-do. What was one of the reasons for this division of power? As so often is the case in the history of mankind it is the above-mentioned military factor which drove the change. The number of the people living in these cities was so small that people who did not belong to the aristocracy also had to be recruited for military missions. That also meant that the large number of weapons present through the emergence of iron manufacturing demanded a well-organized army, which required that those who were well off also joined the military. Such a partaking in military duties resulted in the person's eligibility to share in power. That power was executed by civil servants, with a council to aid them. Sovereignty laid with the people's assembly which reflected the power relationships within the military. In the year 594 BC, Solon opened up the Athens assembly to all citizen classes. Numerous reports about the nature and terror of several Greek tyrants show that such attempts at democracy were often only short-lived. Despite this we can record with Pichot that public life was reorganized anew, that constitutional laws were enacted, which also regulated and restrained power.

In consequence, these city states were confronted with the problem that the people in the assembly had, among other things, to learn the activity of amicably reaching agreement. The verb used to denote argumentation was deikmüni (show, demonstrate). It has been preserved in its Latin translation as "demonstrate". It is to be found in Plato in the shape of "apodeixis", meaning "rational argumentation", a description which clearly referred to something. The description of mathematical argumentation, as used by Aristotle in his Analytica priora, was explained in the form of syllogisms. The development of the meaning of deikmüni was, according to Lucio Russo (2005) in "The Forgotten Revolution", tied to the development of Greek democracy (in the 5th century). A further phenomenon can be confirmed at this point of time. The Greek language began to change. Whereas the passed down recordings of Old Greek from the times before the 6th century show a language exercising itself through the relating of legends, a different function now started to gain in importance. Language became the means of arranging organizational processes. With language, argumentation takes place in the form of statement and counter-statement. From where did the Greeks draw the ability to use language in such an unfamiliar manner for them? A quote by the Roman Quintillian shall point the way to a possible explanation of the reason.

> From the preceding geometry proves the following, and from the known the unknown. Do we (speakers) not do this also in speaking? Yes, does the conclusion from the preceding sentences not consist almost exclusively of syllogisms?... For if the matter demands it (the speaker) will use syllogisms or at any rate the enthymeme which is of course a rhetorical syllogism. After all, the most powerful proofs are generally called grammatikai apodeixis (providing proof by drawing); but what does discourse/statement seek more than proof? Consequently .... there is no way being an orator is possible without geometry.

(Quintillian, Instituto Oratoria, I, x§§37–38, from Russo, p. 197) This quotation by the Roman Quintillian takes me back to my initial thesis: it was geometry which had a substantial part in the development of argumentative speech. How was the emergence and development of this combination of elementary geometry and argumentation?

To answer this question let us go back in time into the 6th century BC and visit two people with very famous names. Both may be seen as representatives of a multitude of other classical surveyors of the time. Thales of Milet, known to us mainly through the theorem named after him, lived about 625–550 in the town of Milet in Ionia (today's Turkey). In varying sources, he is sometimes called a Phoenician, sometimes a Greek. In any case, he seems to have been a successful and clever businessman, who also undertook business journeys to Egypt. His numerous contacts probably took him to the area of Mesopotamia. What he imported from these two highly developed cultures was—besides highly marketable goods—knowledge of Egyptian and Mesopotamian geometry. To this body of knowledge may be counted, if you will believe the classical author Proclus Diadochus (3rd century AD), five theorems of Euclidian geometry (cf. Pichot, 1995, pp. 334–336):

1. The circle is bisected by its diameter.
2. If two straight lines intersect, the opposite angles formed are equal. (Scheitelwinkelsatz = theorem of opposite/vertical angles)
3. Angles at the base of isosceles triangles are equal.
4. A triangle is defined if the base and the base angles are given.
5. Any angle inscribed in a semicircle is a right angle.

It is impossible to determine conclusively whether these theorems were indeed imported to Greece by Thales or not. The use of such theorems which can be constructed with the simplest means is what is significant for my argument.

I will briefly return to the members of a people's assembly who were discussed above. What was their social standing and which means of shaping their lives were available to them? One can assume that they certainly did not suffer from material need. The society of ancient Greece was a slave society, despite all their achievements, in which the ruling Greeks were in possession of extremely profitable means of production. Slaves were the cheap machinery which had to carry out whatever work was required. This specific social situation enabled the Greek patricians to turn to more particular problems. The applicability of geometry was not among them in those times. The specification of reasons for the universal validity of a geometric situation was probably of greater significance. It will not be possible to give an answer as to who was first to pose this question. That it was Thales himself is doubted by the relevant authors (cf. e.g., Scriba, 2005). It is also of no relevance for my endeavor. Of essential importance, however, are the living conditions of people in ancient Greece, which provided them with the means, the motives, and the opportunity to practice geometry. The means were mostly the geometric theorems of the Egyptians and Mesopotamians, the motive was the necessity to practice arguing, and the opportunity arose from the social situation of those Greeks who saw themselves as belonging to the aristocracy.

As I mentioned two people earlier, I will add to Thales, the at least equally famous, Pythagoras of Samos, as a contrast. Similarly to Thales, Pythagoras had close contact with the geometry of the ancients, which he became acquainted with during his travels. His motives for practicing geometry as a form of argumentation were others: everything is a number was the motto of the sects of the Pythagoreans. Questions of metaphysics of a highly speculative nature were the main-spring for him and his followers in practicing mathematics and geometry. His/their reasons for engaging with geometric problems therefore had different causes.

I look back again at the sequence of means, motive, and opportunity. We can bring our argument relating to Greek geometry to a close and record geometry as a means of acquiring the means for presenting an argument. However, if we remain on such a historically-led level of argumentation an aftertaste remains. Why was geometry, in particular, which fulfilled these needs of the Greeks? Through the contacts of the Greeks with Asian culture, other means, like a board game, could have been used to acquire or practice arguing. As a first answer one could point to the successful usability of geometry. Although the Greeks, due to the work of their slaves, may have had little interest in increasing the efficiency of their daily routines, applications of geometry in seafaring would have left an impression. A further reason, which from my point of view cannot be underestimated, may be the social acceptance of geometry in Egypt and Mesopotamia. From ambitious people like the Greeks, both societies must have elicited their admiration for their intellectual achievements. And certainly geometry was one of those achievements. And let us not forget that by the 6th century BC, geometry and arithmetic had already had more than a thousand years of history (Scriba, 2005). Notwithstanding all of this, it should be possible to provide specific reasons/arguments for the success of the new way of thinking about geometry in Greece. To approach this question, I will concentrate on the usage of geometric signs, as formulated at the beginning of this essay. To do that, I will leave this historical reflection and enter into semiotic considerations.

FOR THE LEARNING OF GEOMETRY

How can we justify this new way of using geometry, which is obviously no longer used, both to structure the given physical environment and to organize thinking? In the introductory section, three possible features of geometric signs that could possible support these forms of structuring and organizing were given. In this section they will be complemented by further reasons. I would like to briefly repeat them here. Then I will present them in more detail. Finally, I will indicate possible arguments for their validity.

1. *The construction of geometrical concepts by means of visible signs is essentially determined by those relations; relations which define these terms.*
2. *In contrast to elementary arithmetic, algebra, and calculus, the signs of geometry do not support algorithmic transformations.*
3. *When we take a look at a geometric construction, this view does not show us the history of the construction.*

What arguments and references can be presented for these three claims? These arguments are presented in the following sections. The next section is about my first claim, the construction of geometrical concepts.

RELATIONS BETWEEN VISIBLE SIGNS AND THEIR ASSOCIATED GEOMETRY

The first part of my text was historically oriented. Now, I shall switch to the learning of mathematics and take the opportunity to remind the reader at a book worth reading on the didactics of teaching geometry. The title of this book is “Operative Genese der Geometrie”[2] (by Peter Bender & Alfred Schreiber, 1985) and it was published again in 2012 in the form of a reprint. Some sections of this book should be used as a first argument to back my first claim.

The aim of “Operative Genese” is the reception and implementation of a particular view of the development of science, concentrating on ideas from Hugo Dingler, a philospher from the beginning of the 20th century. From Dingler’s perspective, Bender and Schreiber developed their principle of operational concept formation (POCF) for the development of geometry. Although the prime focus of Bender and Schreiber’s book is spatial geometry and a large variety of its applications, it also contains further ideas how geometry can be used to structure parts of our everyday life. I would suggest that POCF is a tool for interpreting the relationship between signs and concepts that I mentioned earlier. I will not concentrate on the use of geometry as a structuring agent for our environment, as the authors intended, but will focus on the construction of concepts of plane geometry.

Bender and Schreiber's claims follow some constructivist positions in stating that the basic concepts of geometry do not develop by abstraction, that is, by disregarding features, but by seeing and implementing properties into "objects". These "objects" include the geometrical signs. This view of signs as realization of objects leads to material realization. Bender and Schreiber suggest that this implementation is ruled by norms, which can be seen as the operational basis for the production and use of geometry. To illustrate this, the authors present the construction of a cube. However, there is no need to focus on spatial geometry. The production of basic concepts of plane geometry (line, circle, polygon, perpendicular line, parallel line, etc.) can also be interpreted with the POCF.

They argue that "Geometrical concepts have to be constructed in an operative way; i.e., starting from some certain purposes, standards for the production are developed to meet those purposes. These standards, mostly homogeneity requirements, are implemented in procedures and guidelines for their implementation and are, therefore, the substantive basis of the corresponding terms" (Bender & Schreiber, p. 26).

Therefore the construction of concepts is determined by the production of corresponding signs. We only need to consider the instructions we offer to the student when she/he has to draw a circle or a perpendicular. Her/his drawing of the geometrical sign always obeys the intended specification. Furthermore, the successfully drawn sign reinforces the geometrical concept within the learners mind. When constructing concepts, Bender and Schreiber name this interplay between the construction of signs according to certain specifications and their successful use "operativity."[3]

However, a semiotic reason for my first claim has not been found. Which semiotic[4] terms can be used to describe the relationship between students' activity when drawing signs and the control of this drawing activity by obeying certain relations? Peirce's concept of index can be seen as an opportunity to describe the emergence of the sign (e.g., sign of a circle, sign of a parallel line etc.) as the current result of the activity of the learner when constructing such a sign.

In the conventional use of indexical signs, these signs are to refer to some desired goal. In this respect, indices refer to something that is present, or in terms of activities, this something is already done. Smoke refers to the fire, the weather-cock to the wind, and the signpost refers to a direction. An alternative point of view and thus an extension of the use of the term index, which takes this rather special property of geometric sign generation into account, is offered by a text presented by Sybille Krämer where she concentrates on the use of the word "tracks"[5] in (Krämer, 2007). Using her terminology, we can describe the interplay of production and control discussed above. How does Krämer proceed?

In a first approach, she sees the track as a sign or indication of something mainly unintentionally left in the past. The fingerprint of the burglar or the scents of the animal in the wild are examples of such tracks. When reading such a track the past meets the present. “Just as the simultaneity is the system of order of the index, the non-simultaneity is thus the ‘system of order’ of the track” (Krämer, 2007, p. 164). In this respect, tracks, traces or marks, are all indices that refer to something not (yet) visible. If I consider my first claim, I think that the socially regulated interpretation of a track, such as the “correct” use of a simple closed curve like a circle, becomes visible during the activity of producing the circle and when using the result to do some further activities. To use Krämer’s words, “tracks embody … the expectation …” (ibid., p. 166).

The reading of tracks and the associated expectation of a successful interpretation is not new in the cultural sciences field. The Italian historian Carlo Ginzberg identified this “epistemological method” in articles on art history or psychoanalysis (depth psychology) (cf. Krämer, 2007, pp. 168–170). These texts open up an initially hidden reality on the basis of traces usually in the form of incidental and unimportant details and construct their particular view of the unknown. Krämer asks whether it might be possible that “... we must recognize that the person following the track ... becomes the constructor of the referenced object, to which the track seems to refer in a ‘quasi natural way’?” (Ibid., p. 171). If we agree with Krämer, then the reading of a track becomes the construction of a sign rather than a reference to something. Geometrically speaking the track just develops and controls our own geometrical activity. In other words, referring to Krämer again (see Kramer, 2007, p. 178), the production of a geometric sign—at least when learning elementary geometry—becomes a constructive projection. With Krämer’s deliberations a first semiotic position is marked.

Can we now determine the sign more precisely with the help of Peircean theory? Helmut Pape (2007) presents the text “Footprints and Proper Names: Peirce’s theory” that takes up this question. Let’s follow his argument, which come to the conclusion, in a nutshell, that activities act as the mediator between the general and the particular.

Peirce’s use of the word index is a broad one. In a certain, generous fashion, he saw an indexical aspect implied in nearly every kind of sign (Pape, 2007, p. 41). A listing of Peirce’s use of “index” in Pape (ibid., pp. 41–42) illustrates this generosity. What all these examples have in common is their special link between signifier and signified. In this case, an index is characterized as a sign, “that refers to an object not because of a resemblance or analogy with it, also not because it is linked to the general characteristics exhibited by the object, but because there is a dynamic (including spatial) connection between the individual object on the one hand, and, on the other, the senses and the memory of the person, for whom it serves as a sign” (Peirce, 2–305, quoted in Pape, 2007, p. 43).

What can we imagine such a dynamic connection consists of? For Peirce, Pape writes, it is the effect or the strength of the indexed sign when used on the senses of the person. Here we find a perceptual dynamic in the shape of an exchange with the environment. As we will see later, it is this exchange that may determine the learning of the use of geometric signs. The peculiarity of the individual, visible, geometric sign is not "a certain indexical representation of an object. It's about individual things only insofar as they are tangible and demonstrable and can be detected within contexts and situations in a dynamic relational fashion. […] All the world and space relations of the index are determined only by their performative epistemic activation in the situation of their use" (Ibid., p. 46).

What importance can we attribute to the sensory perception of an indexical sign? Peirce thinks that the index is a link to "the senses and the memory of the person […] , for which it serves as a sign" (Peirce, 2–305, quoted in Pape, 2007, p. 47). An elaborate process of perceiving, understanding and communicating goes along with it. Pape clearly states four points (a, b, c, d), of which two (c and d) are of particular significance for my question.

a. Indices denote what is in real (existential) relations to each other – or what is presented as if it is in relations.
b. Indices orientate us cognitively towards a particular environment, which is relevant for the assessment of indexical facts.
c. All indices are regarded as signs, which are directed to a reality characterized by relations that guide our perception-based activities.
d. All statements about individual circumstances contain either implicit or explicit indices. By these indices, the hard facts of the world are involved in the use of language. This is done by speaking about individual situations and circumstances (p. 49).

With this quotation, I could now continue my own deliberation on geometry, since the two points (b) and (c) describe the construction and use of signs within elementary geometry. However, let's stay for a moment with Peirce in order to formulate a possible use of indices to describe the relationship of the particular and the general. Peirce presents here as an example, the discovery of supposedly human footprints in the sand by Robinson Crusoe, "which can be used in two ways as a sign. That footprint Robinson came across in the sand which has been carved into the granite of glory, was for him an indication (index) of a creature living on his island, and at the same time it awakened in him a symbol, the idea of man" (Peirce, 4531, quoted in Pape, 2007, p. 50). The footprint as a sign that a particular person left behind, will spark a symbolic use.

How can such a change be understood? As an example one can refer to the use of language. It is the index that adds the "meaning" to the user when we think of language as a symbol system. "… Because the external sign functions relate the general symbolizing language to present experience, they break up the structure of the description through a kind of performance. So the linguistic representations can be clearly related to an individual event, object, or an individual person." (Ibid., p. 53). Here, Pape looks to Wittgenstein and to his proposition that the construction of meaning of a word arises from its use. These indexical demonstrable relations play a central role in that process. If we were to speak without any indexical reference, our language would, in the extreme case, refer to any objects and ultimately be irrelevant.

"It would be an exaggeration to claim that we can never say what we are talking about. But in another sense, it is completely true. The meaning of the words normally depend on our tendency to associate qualities with each other and […] of our ability to recognize similarities. However, the experience is held together and is only recognizable due to forces that act on us" (Peirce, 3–419, quoted in Pape, 2007, p. 53). This mutual causality between the references to the particular, on the one hand, in the form of perceivable signs by the senses and, on the other hand, to the general rules about the use or production of signs are aspects of the signs in geometry. In this sense, these are indices which point to themselves and, at the same time, allow the universality to shine through their rule-based production. Thus, performance combines the general with the particular.

From these perspectives, which are also compatible with Dingler's "operativity,"[6] we can interpret the special role of geometry signs. During an intentional drawing activity sometimes we read relations into the construction, whether we are learning geometry as beginners or as experienced mathematicians. We draw, for example, one line parallel to another taking care that they are equidistant, i.e., that the intended relation of being parallel to one given line is specified. During this sign activity what the drawers sees "shows" the track of their current activity (Krämer) since it shows whether we are right or wrong. Tools may shorten this drawing phase but also distract us from the intended relations. This could—for example, when using dynamic geometry software (DGS)—in extreme cases lead to a complete separation of the activity (click with the mouse) from the intended relation (visible sign). It would only be when the DGS was used to vary a given straight line, and the behaviour of the drawn sign has to be interpreted again, then the connection between the visible sign and the associated relationship could be re-established (cf. Arzarello, 2002). Such an intended and target-controlled sign activity can be observed in complex configurations too. We add a sign to open up new perspectives on a geometric construction. We find this of elementary proofs in geometry, where the addition of signs is a successful strategy.

What does this mean for the initially formulated assertion that geometry is a tool of reasoning with? Looking at the comments on the use of "track", at least one thing seems to be certain. When we try to learn the basic concepts of geometry or to use them in the sense of geometry, we consider the defining relationships during the construction process. The visible trace of the signs of activity tells us whether we are drawing correctly. Thus, the perpendicular has to take this direction, because it is defined as such. If it does not take this direction then I cannot use it as a perpendicular. And we are always performing this kind of activity, at least as long as we are learning basic terms. The activity is performed because something is defined in a particular manner. The emergence of the visible has an immediate justification.

The signs activity is therefore controlled by the relationship, and the visible trace "reports" back to me if I am constructing correctly. Therefore, relationship and its visible sign are thus closely connected to one another.

GEOMETRY CANNOT "CALCULATE"

I come now to the second claim. If we look at Euclidean geometry then we recognize a lack of algorithms in contrast to, for example, elementary algebra. To put it simply, we cannot calculate in geometry. To back this claim, I would like to continue the idea of the interaction between visible geometric signs and their corresponding relations. My aim is to give reasons why the development of algorithms within elementary geometry makes little sense, but that, simultaneously, this apparent deficiency demands and causes a further specific property in geometry. This property consists in the use of definitions, theorems, or other known geometric constructions while we are drawing a geometric construction or proving a theorem.

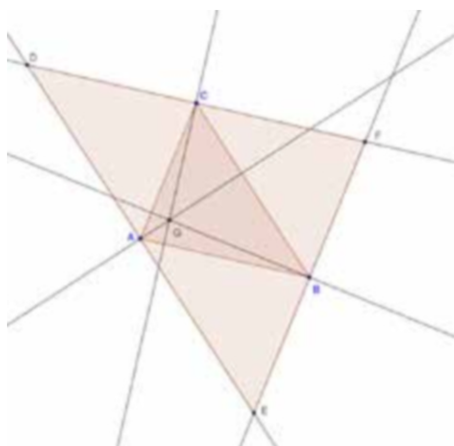

Figure 1. Orthocenter

Let us first look at elementary algebra or arithmetic, where the successful use of mathematics is rather often determined by the applications of algorithms. With the help of such algorithms we can handle parts of problems or proofs in a quasi-mechanical manner by rule based transformations. Consider, for example, an algebraic proof for Pythagoras's theorem (see Kadunz, 2000) in contrast to a proof using the signs of elementary geometry.

One reason for the success of such an algorithmic approach may lies in the two-dimensionality of the written. This two-dimensionality enables us to "walk" through a calculation line by line. Consequences for mathematics education can be found in Kadunz (2006). If we are doing geometry, then this dimensionality is increased.

The use of signs in geometry is always determined by their position with respect to each other. In this respect a construction even in plane geometry is already three-dimensional. As a consequence the reading of a finished geometric construction is often a difficult task. I will concentrate on this question within the third part of my explanations. At this point, I want to focus on the fact that within geometry algorithmic transformations are hardly feasible because of the intimate combination of geometry signs and their relations.

The only conceivable way of using a geometry signs in a new way within a drawing is to change its use intentionally. Hence the use of a sign always fulfils a specific task in a configuration. For example, let us take the proof of the orthocenter in a triangle, in which the same line can be seen as a bisector (triangle DEF) or as an altitude (triangle ABC) (cf. Figure 1). This switch is not the result of an algorithm but the consequence of a certain view of the drawing on the part of the mathematician. Ladislav Kvasz (2008) describes the impossibility of transforming by algorithms within geometry as a lack of expressiveness of the signs of elementary geometry.

What are the consequences of this apparent lack?[7] Can we gain something from this obvious lack – compared to e.g., elementary algebra? As an example let us solve the algebraic equation $(2-x)^2 = 3x+1$. After a short series of transformations based on the rules of elementary algebra, we will get the equation $x^2-7x+3 = 0$. Pupils, if practiced, recognize this expression and calculate the solution, using the formula for quadratic equations. In this sense, rule-based transformations can lead the way to configurations which remind us (sometimes) of a well-known theorem/formula.

Geometry is different. Let us consider another example. We are looking for the position of a sailing boat which can be seen from two different points on the shore joined by a given angle. A successful approach solving this task uses the application of the inscribed angle. If the pupil does not know this theorem then the likeliest solution would be the use of DGS. However, it is inconceivable that this sort of solution is proof against examination using the rules of Euclidian geometry. The very first step in the solving process requires knowledge of a geometric theorem. Such geometrical knowledge determines the path to the solution. Hardly any parts of the solution are supported by activities depending on algorithms. We always have to refer to a theorem or a definition. This is a complex activity but also presents a challenge to the pupil.

How can we describe this use of theorems and definitions? Mathematics education offers here—in addition to cognitive sciences and computer science—the notion of modules. I refer to documents about the learning of mathematics which were published from the mid-1980s onwards and more particularly to an article by Willi Dörfler (1991) with the title “The computer as a cognitive tool and cognitive medium”. In this paper Dörfler reports about the use of modules to describe the learning of mathematics. Let us take a look at some of his main arguments. Cognitive psychologists report that experts achieve their performance to a considerable extent by access to highly-structured knowledge. Within this knowledge units are directly accessible and operationally usable. For instance a chess grandmaster surveys a great variety of positions on the chessboard before making a move. Similarly experienced mathematicians can easily access numerous knowledge packages in their memory, which they then apply to different problem situations. Such packets can be algorithmic processes, for example, but also knowledge of theorems and their application. One could also say, metaphorically speaking, that experiences are transformed into modules of thinking and that these modules are knowledge in a condensed form. In my view, it is remarkable that the knowledge of a proof or some kind of inner structure is not relevant to the successful use of such a theorem.

Modules, though, are not always modules as they differ in their purpose. For example, algorithmic procedures facilitate calculations, help us to reduce our effort when solving a problem. The waiter in the restaurant calculates without knowing why the algorithm is correct or the business science student calculates the inverse of a matrix without knowing, in most cases, why the algorithm works. In this respect the application of an algorithmic procedure is a form of aid. If on the other hand we look at geometry,[8] then we regularly have to use, as already mentioned, theorems or definitions from geometry. These are different forms of modules. While algorithms relieve, theorems in geometry shorten the process of finding a solution. All we have to know in this case is the interface of the theorem—what are the conditions of applicability and what is the result of its usage. If we use these modules effectively then a proof or a calculation can become very short. In a nutshell, theorems and formulas shorten whereas algorithms relieve.

The above view of geometry reveals a characteristic feature of it. When doing geometry the access to encapsulated knowledge, theorems and definitions, in the form of modules is necessary and helpful. It can be seen as a consequence of the nature of geometric signs. One could also say that the lack of algorithms in geometry forces us to use such modules. Drawing and proving in geometry can be characterized by an extended use of these modules where nearly every step within the solution has to be backed by reasons for their use. Even on a very elementary level geometry forces us to justify our activity.

THE SECRET STORY

Geometric constructions hide their history. This again can be seen as a result of the signs used in geometry. The numerous design elements even in rather elementary drawings—e.g., the circumcenter of triangle or Euler's line—obstruct the view of the development of the construction. In contrast, when we think of elementary algebra, we can easily reproduce the 'visible' activities of a solution, because algebra is written line by line, whereas the geometric signs are superimposed. Similar to the lack of algorithms in geometry another lacks becomes visible. Geometric constructions are difficult to read. Can we profit didactically from this weakness? A certain hermeneutics which enables to successfully read a construction could be the profit. The difficult task of reconstructing a finished drawing can be facilitated by a written description of the drawing activities. This description[9] follows the rules of linear writing. Thus, the history of the drawings genesis is revealed. However, the price is the use of a second sign system. We know such descriptions from (older) textbooks or even alternatives in various DGS system, which repeat the construction at the touch of a button. I conclude my remarks on this point by referring to the Irish surveyor Oliver Byrne. In his 1847 published book "The Elements of Euclid" presented geometrical constructions or theorems primarily with the help of geometrical signs. Whenever possible, Byrne avoided labels with letters. In place of these indices, he used colours or dots hatching and the like (cf. Figure 2). Even the

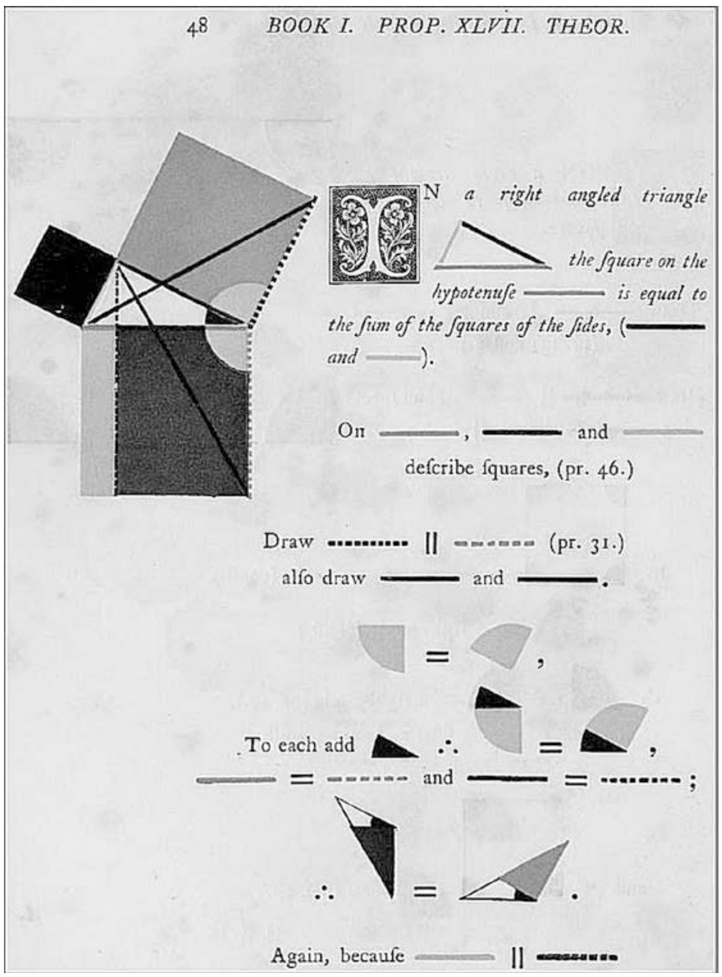

Figure 2. Pythagoras

location of parts of a structure served as an index. Thus any proof of a theorem was presented as a linear sequence of geometry signs.

Whether it is the classic written description of a construction or its repetition with the help of a DGS, the simultaneously appearing relational structure is deployed, in any case, before the observers' eyes. What was simultaneous becomes chronologically linear. The fineness of the description, the granularity, can be adapted to the learners. The interpretation or hermeneutics of a geometric design is determined by the use of signs.

SUMMARY

The considerations in part 2 presented the use of geometry signs for the learning of geometry taking into account three different but complementary aspects. Using these three perspectives reasons have been found, in addition to the historical discourse in part 1, why in ancient Greece, geometry had a special role. Geometry as a tool for reasoning and validation helped to build democratic structures. Interpreted semiotically, this is also a feature of geometry signs. At first the visible geometric sign and the corresponding geometric relations are closely linked. A semiotic interpretation of this relationship could be made through the presentation of special concept of "tracks" and especially through observing the indexical use of geometrical signs. Pupils (should) construct the signs of geometry by constant control of the visible by the corresponding geometrical relationship. The drawing activity is controlled by the geometrical relationship. These relationships are always thought along and can be used to argue the activity.

As a second point, the lack of algorithmic transformations in geometry was presented. This lack has the consequence that within geometric constructions or proofs in most cases theorems and definitions has to be used. This usage of theorems, for example, must be always justified. This is in a sharp contrast to an algorithmically oriented transformation. When working on theorems we need to give reasons why we take the next step.

As a third point the hiding of the history of a geometric construction was presented. In order to read a finished geometric construction we need to use a second sign system in addition to the geometrical one. With the help of this alternative sign system, the nonlinear set of relations is represented linearly. We can read geometry.

All these aspects of the geometry signs may have been reasons why people in ancient Greece have used geometry in order to learn how to argue. This applies in a similar manner for the learning of geometry in school.

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NOTES

1 See the educational standards for mathematics in the final year of secondary schooling (8th grade) on the website of the Austrian Federal Institute for Educational Research (https://www.bifie.at/node/49) (24th February 2014).
2 "The genesis of geometry from an operational point of view."
3 "Operativitaet"in German.
4 I focus on the semiotics of Charles S. Peirce.
5 "Spur" in German covers many different meanings. English uses many separate words for these meanings i.e., track, tracks, traces, mark, evidence, clue etc.
6 "Operative Genese".
7 If we draw a construction in geometry then a way to perform transformations can be done by using software for doing geometry (DGS) and concentrating on the drag mode. Examples can be found in publication e.g., by Reinhard Hölzl (1999) or Ferdinando Arzarello (2002).
8 Of course all parts of mathematics offer an enormous number of theorems and definitions. In this respect all we can say about the use of theorems within geometry can be said about mathematics at all. However, it is the lack of algorithms in geometry that is in the focus of my interest.
9 "Konstruktionsgang" in German.

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