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10. THE IMPORTANCE OF ABDUCTIVE REASONING IN MATHEMATICAL PROBLEM SOLVING

ABSTRACT

Charles Sanders Peirce (1839–1914) made a distinction between formal and informal reasoning, and argued that the formal reasoning processes of induction and deduction were not sufficient to explain those instances when individuals entertain new ideas to explain surprising facts. Peirce asserted the existence of another kind of reasoning, abduction, through which the individual generates a novel hypothesis to account for or explain surprising facts under consideration. The hypothesis represents an initial explanation that is both plausible, in the sense that it is the best explanation under the circumstances, and also provisional in the sense that it is open to further exploration. While research in mathematics learning has acknowledged the importance of hypothetical reasoning, few studies have identified the prominent role that Peirces's theory of abductive reasoning may play in problem solving, and fewer still have acknowledged how we as educators might help nurture and support abductions that our students make. This chapter addresses two key questions. (1) Why is it important that our students be able to make abductions when they solve mathematics problems? (2) How should educators help students develop reasoning habits that include abductive reasoning?

INTRODUCTION

Accounts of mathematics learning have long acknowledged the need for learners to develop autonomous cognitive activity, with particular emphasis on the learner's ability to initiate and sustain productive patterns of reasoning in mathematical problem solving situations (Burton, 1984; Cobb, 1988; NCTM, 2000; Schoenfeld, 1985). Nevertheless, explanations of problem solving have often focused on the application of objective strategies and processes, providing little explanation of the subjective actions solvers often generate prior to introducing formal algorithmic procedures into their actions. For example, cognitive models of problem solving (Reed, 1999), while useful in providing microscopic analyses of cognitive processes, have been challenged because "they fail to recognize the need to place cognitive functioning in a broader perspective that takes into account aspects such as affect, motivation, attitudes, beliefs and intuitions, as well as social and cultural factors" (Verschaffel & Greer, 2003, p. 62). In particular, they seldom address aspects of the

A. Sáenz-Ludlow & G. Kadunz (Eds.), Semiotics as a Tool for Learning Mathematics, 209–225.

solver's idiosyncratic reasoning activity such as the solver's selfgeneration of novel hypotheses, intuitions, and conjectures, even though these processes have been documented as crucial tools through which mathematicians ply their craft and thus are goals in the teaching of mathematics (Anderson, 1995; Burton, 1984; Carlson & Bloom, 2005; Mason, 1995; National Council of Teachers of Mathematics, 2000; Schoenfeld, 1985). Moreover, several researchers have documented that subjective actions play an important role in mathematics learning and have called for additional studies to examine the novel actions of learners (Cai, Moyer, & Laughlin, 1998; Cifarelli, 1998; Mason, 1995; Reid, 2003; Rivera, 2008; Sáenz-Ludlow & Walgamuth, 1998).

The chapter begins by developing a rationale for how Peirce's theory can be considered to examine problem solving processes. The second part of the chapter summarizes the mathematics education research that has been conducted on abduction. The third part examines the episodes of a college student solving a mathematics problem that involved a visual array, documenting and explaining the important role that abduction played in her solution. The fourth part discusses instructional implications for mathematics education.

ABDUCTIVE REASONING IN MATHEMATICS EDUCATION

Charles Sanders Peirce (1839–1914) made a distinction between formal and informal reasoning, and argued that the formal reasoning processes of induction and deduction were not sufficient to explain those instances when individuals entertain new ideas to explain surprising facts. Peirce asserted the existence of another kind of reasoning, *abduction*, through which the individual generates a novel hypothesis to account for surprising facts under consideration (Fann, 1970). The hypothesis represents an initial explanation that is both plausible, in the sense that it is the best explanation under the circumstances, and also provisional in the sense that it is open to further exploration. In contrast, Peirce viewed *deduction* as a process that explicates and clarifies hypotheses, deducing from them the necessary consequences; and *induction* as a process through which hypotheses are explored and tested for their explanatory merit and usefulness (CP 7.202–207; CP 8.209).[1] According to Peirce, abduction is the only logical operation which introduces new ideas (CP 5.171).

Peirce's theory of hypothesisbased reasoning is helpful to explain how learners develop plausible explanations to address 'surprising situations' they find themselves faced with. This chapter thus takes to heart Cobb's (1988) assertion that solvers actively construct new knowledge in problem solving situations when "their current knowledge results in obstacles, contradictions, or surprises" (Cobb, 1988, p. 92). Hence, genuine problem solving situations can be viewed as opportunities for problem solvers to reason abductively as they generate problem solutions.

The view that abduction may play an important role in mathematics learning and problem solving is not new. Abduction has been mentioned within various theoretical perspectives. For example, von Glasersfeld (1998) described abductions as accommodations that help stimulate and structure the learner's novel actions. According to von Glasersfeld, "abduction appears in accommodations of action schemes on the sensorimotor level as well as in subsequent levels of concrete and formal mental operations", calling them "the mainspring of creativity" (p. 9). Hence, a focus on the learner's abductions in problem solving situations may help provide an explanation for the formation and modification of the learner's schemes.

The idea that the solution of a problem may involve hypothesisbased reasoning of the type theorized by Peirce is useful if one adopts a constructivist broadbased view of problem solving in which solvers continually buildup their mathematical knowledge. For example, while solving a problem, the solver might experience an unanticipated difficulty that requires further reformulation of the original problem. Silver referred to this process as *withinsolution* problem posing (Silver, 1994) where the essence of the problem, as viewed through the eyes of the solver, has undergone a change and must be reformulated in order for the solver to proceed. The solver may reformulate the original problem as a collection of several 'smaller problems' that can be addressed and solved individually, and then organize his/her actions accordingly 'to break the problem up'. In this example, the solver's reformulation derives from their changing perceptions of what is problematic and awareness of the need to reorganize their goals and purposes for action. Hence, the reformulation indicates a plausible yet provisional action on the part of the solver to solve the original problem. If the solver's reformulation is hypothesized-based and has as its goal the explanation of some aspect of the problematic that requires further investigation, then the reformulation may involve abductive reasoning.

The work of Polya (1945) is consistent with the view that problem solvers may engage in abductive or hypothesis-based reasoning while in the course of solving a problem. Specifically, Polya identified heuristic reasoning as "reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem" (Polya, 1945, p. 113). Further, Polya cited the usefulness of varying the problem when solvers fail to achieve progress towards their goals because the solvers' consideration of new questions serves to "unfold untried possibilities of contact with our previous knowledge" (Polya, 1945, p. 210). Hence, solvers who engage in hypothesis-based reasoning are: (1) cautious in their reflections about appropriate courses of action to carry out; (2) always looking to monitor the usefulness of the activity they plan to carry out; and, (3) willing to adopt a new perspective of the problem situation when their progress is impeded.

A good example of students demonstrating abductive reasoning in solving a problem is found in Reid (2003). Reid illustrated how two students, Jason and Sofia, solved the Handshake Problem (determine the number of handshakes exchanged from among n individuals) by reformulating the problem to examine a particular case. From this particular case, Jason then hypothesized a rule to solve the general case. Specifically, the students solved the problem for the case of six people, first using a diagram to count the number of handshakes (15) and then finding that they could get the solution by summing the numbers from 1 through 5 (Figure 1).

$1+2+3+4+5 = 15$ handshakes

Figure 1. The students' diagrams (adapted from Reid, 2003)

Then Jason and Sofia tried to solve the problem for N=26 people. Jason got the correct answer of 325 using his calculator to compute the sum 1 + 2 + …+ 24 + 25 = 325. Sofia claimed to "know an easier way", noticing that the sum could be computed more efficiently by first grouping numbers that sum to 26, and made a diagram to sum the numbers (Figure 2).

$$1+2+3+\ldots+23+24+25$$

Figure 2: Sofia's grouping strategy

By grouping the numbers in this way, Sofia is demonstrating the Gauss method for summing n consecutive integers. Sofia reasoned that there were 13 such sums, an assertion that is incorrect. There are actually only 12 such sums, totalling 312, and a middle number of 13, so that the total sum is 325.

Sofia: So it is. $26 * \frac{26}{2} = 338$. (Reid, 2003, p. 6)

Sofia's calculation, $26 * \frac{26}{2} = 338$ was incorrect. However, Jason focused on her result of 338, comparing it to what he knew to be the correct answer, 325.

Jason: That can't be right. But you were close.

Jason: Maybe it's the number times half the number, hmm subtract half the number?

Sofia: You lost me.

Jason: Because that would work, 338 subtract 13, which is half of 26, is right.

Jason's use of the word "maybe" indicates the beginning of a hypothesis about a more general rule, ("Maybe it's the number times half the number, hmm subtract half the number?"). His use of the word "because" suggest the beginning of an explanation of why Sofia's calculation was close ("Because that would work, 338 subtract 13, which is half of 26, is right.")

According to Reid, Jason used abductive reasoning to arrive at the general rule

$$h(n) = n \times \left(\frac{n}{2}\right) - \frac{n}{2} \tag{1}$$

> [The number of handshakes is] the number [of people] times half the number, subtract half the number. (Reid, 2003, p. 6)

From the specific case:

> Because that would work, [the number of handshakes for 26 people is] 338 subtract 13, which is half of 26, is right. (Reid, 2003, p. 6)

In other words, Jason hypothesized a general rule that helped explain how Sofia's result was "close". He verified the rule in the specific case (by revising Sofia's calculation accordingly) and then tested the rule for other cases.

The preceding example, while showing that abduction may play an important role in problem solving situations, also indicates the intricacies of assessing abductions as examples that fit with Peirce's definition, a point that has been echoed by Mason (1995). According to Reid, the difficulty lies in the fact that Peirce focused on different aspects of abduction at different times in his writings. Hence, trying to identify precisely the particular components to Peirce's theory can be challenging. For the example provided, "The abduction is used (as the later Peirce would suggest) to explore (in finding a formula) and to explain (why Sofia's method gave an answer that was close)" (Reid, 2003, p. 6).

The following section will elaborate on these challenges and also summarizes the various ways that abduction has been interpreted by researchers in mathematics education. This discussion will help provide further context and rationale for studying the role of abduction in mathematical problem solving.

STUDIES OF ABDUCTION IN MATHEMATICS EDUCATION RESEARCH

Reid (2003) examined the writings of Peirce and noted how he emphasized different aspects of abduction at different times. Reid found that Peirce focused on the logical form of abduction in his earlier writings (CP 2.508; 2.623), emphasizing syllogisms and the role of characters of specific cases and classes to summarize the process. Reid then documented how in his later writings (CP 5.197), Peirce emphasized abductive reasoning in terms of the purposes and needs satisfied by the reasoning, thereby providing a more elaborate description of how abductions, though provisional, explain the surprising facts under consideration. The two characterizations of abduction proposed by Reid (2003) provide a useful lens through which to view the research that has been conducted.

Some of the studies of abduction within in geometry microworlds such as Cabri and Geometer Sketchpad (Arzarello, Olivero, Paola, & Robutti, 2002; Baccaglini-Frank, 2009) exemplify the first category of abduction as described by Reid. These studies focus on the logical form of abduction, considering abduction as a logical modality that supports the development of conjectures (Hoffman, 1999).[2] For example, Arzarello et al. (2002) examined dragging practices in the Cabri geometry environment and how, through continued feedback, they support the solver's emerging conjectures about the problem being solved. In this context, abduction is viewed as a logical operation that mediates a hierarchy of various dragging routines and thus "rules the transition" in cognitive focus that occurs when the solver moves between actual experiences (exploringconjecturing) and emerging theoretical ideas (proving results) (p. 67). While Arzarello et al. focused their attention on subjects' use of dragging schemes during the development of conjectures, Baccaglini-Frank (2009) documented how the subjects' use of particular dragging schemes induced patterns of abductive reasoning, thus suggesting a source of how abductions originate in the Cabri environment.

Studies that fit Reid's second category include those that focus on the structure of abductions and its role in inquirybased activity (Rivera, 2008; Ferrando, 2006). For example, Rivera (2008) characterized *complete* abductions as those hypotheses that undergo a series of developmental transformations that eventually result in generalized rules. Similarly, Ferrando (2006) characterized students' learning of calculus concepts in terms of abductive cycles of reasoning.

This second set of studies appear more useful to interpreting Peirce's theory to examine problem solving since they focus on how individuals form and transform their actions as needed while solving a problem. In particular, the abduction is viewed as a source for generating and organizing the exploration that follows. In this way the individual modifies his or her solution activity so that subsequent explorations become opportunities to develop new goals that reformulate the original problem. The individual can then express (or carry out) this reformulation to examine particular cases.

The comments above suggest that being aware of abduction in the context of problem solving enables a focus on the evolving structure of one's activity as he or she elaborates and extrapolates his or her ideas. Designing studies that focus the individual on these structuring processes would seem to provide a means to examine not only the interconnections among the individual's abductions but also among his or her inductions and deductions. According to Peirce, abductions interconnect with deductions and inductions. Once the explanatory hypothesis has been generated the individual must develop and formulate the hypothesis so that it can be tested (CP 7.202–207; CP 8.209). This is the deductive phase, which may involve slight modification of the original hypothesis through clarification and refinement, to render it testable (CP 7.202–207; CP 8.209). Once the hypothesis has been conformed, the hypothesis can then be tested through further action to determine its usefulness (CP 7.202–207; CP 8.209). This is the induction phase, the result of which places a degree of acceptance on the hypothesis.

Viewing a problem solver's actions under the lens of Peirce's theory of abduction may provide a useful framework with which we might be able to clarify and make better sense of the seemingly meandering actions that solvers sometimes demonstrate. However, we need to be careful in adopting only one point of view. There are many views of hypothesisbased reasoning, not all of which are compatible with Peirce's definition of abduction. For example, Magnani (2009) argues for the inclusion of nonexplanatory hypotheses in his definition of abduction. Hypotheses and conjectures made by individuals have always been acknowledged as important processes in problem solving. It is thus important to keep in mind Peirce's notion of abduction and its interconnections with inductive and deductive reasoning as a powerful theoretical lens through which we can view the problem solving activity of individuals in a coherent manner.

The following section examines the episodes of a college student solving a mathematics problem that involved a visual array of numbers. The analysis focused on the student's solution activity from initial problem formulation through eventual solution, highlighting episodes where she appeared to demonstrate abductive reasoning.

PROBLEM SOLVING INTERVIEWS

Sarah came from a graduate class in Mathematics Education taught by the researcher, at a southern university in the United States. Observing college students solving mathematics problems has proven to be an effective way of modelling the processes of problem solving (Carlson & Bloom, 2005; Cifarelli, 1998; Schoenfeld, 1985). In addition, studying the problem solving of graduate students can be useful in explaining a developmental range of actions (Carlson & Bloom, 2005; Cifarelli & Cai, 2005). Observing such solution activity is important to capture in view of the broad range of processes that appear to encompass abductive reasoning.

Sarah was interviewed by the researcher on 3 occasions during the semester. During the interviews, she solved a variety of word problems while 'thinking aloud'. Sarah worked individually in solving the problems and was given as much time as she wished to complete the tasks. Interviews were videotaped for subsequent analysis.

Sarah's Solution of the Number Array Task

During the second interview, Sarah solved the Number Array task (Figure 3). The Number Array task is discussed extensively in Becker and Shimada (1997), including a detailed description of typical patterns students will see in the array. Samples of the various mathematical relationships students typically construct are provided in Table 1.

Find as many relationships as possible among the numbers

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 3. Number array task

Table 1. Samples of relationships constructed by students solving the number array task

1. All numbers on the left-to-right diagonal are squares (1, 4, 9., …, 100)
Relationships about the spatial arrangement of numbers.
2. The numbers are symmetrically arranged about the left-to-right diagonal numbers
Relationships about the sums of numbers.
3. Sum of numbers in any row is a multiple of 55
4. Sum of two numbers in a row or column that located symmetrically about a pivot number is two times the pivot number.
Relationships about the products of numbers.
5. The number in the m^{th} row and n^{th} column is $m \times n$
6. For any rectangle or square array, the products of the end numbers are equal.
7. For any square array, the products of the numbers on the two diagonals are equal

Sarah began by focusing on simple relationships that had to do with the symmetry of the numbers. Sarah explored several of the fairly simple patterns such as those drawing from the symmetry of the arrangement of numbers, and simple arithmetic relationships. For example, she noticed that any entry in the table can be found by multiplying the row number by the column number, relationship #5 in Table 1 (e.g., $12=3\times4$). In addition, in any 2×2 block, the product of the diagonal entries are equal, and that the result holds true for any square block, $N\times N$, $N>2$.

After identifying several additional simple patterns, she focused on finding more mathematically sophisticated relationships. Episodes of her verbal statements are presented to refer to and support the assertions made by the researcher. (Italicized comments within the episodes indicate inferences of the researcher regarding the nonverbal gestures made by the student.)

> *Sarah:* Let's see … (long reflection) … I was wondering about those square numbers on the diagonal going from left to right (points to the sequence 1, 4, 9, 16, …, 81, 100). They seem to relate to the dimension of the square blocks, … I don't know, … Maybe they relate to the sums of these blocks I had earlier (points to the 2×2, 3×3, 4×4 blocks). So, let's check it.

Sarah proceeded to examine the sum of the entries of each NxN block that contained the square numbers on the diagonal (Figure 4). From her analysis she developed an informal method to find the sums of the entries of the NxN blocks going down the main diagonal (Figure 5).

> *Sarah:* So, for a 1×1, I get a sum of 1 *(points to the sequence of square numbers on the diagonal)*. For a 2×2 *(points to block [1, 2 : 2, 4])*,[3] I get a sum of 9 … but what happened to 4? It appears to have been *skipped*! *(several seconds of reflection)*. Okay, let me try this, I will write down the sequence of squares of all numbers, all in a row *(writes the following sequence of square numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225)*. So, the first number, 1, tells the sum of the very first matrix, a 1×1. And the first 2×2 has a sum of 9. …. So, I *skipped* over 4 to get the next sum *(crosses out the 4 in the sequence)*, going from 1×1 to a 2×2, a sum of 9. The 4 gets *skipped*? Interesting!

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 4. Examples of 2×2 and 5×5 blocks on the diagonal

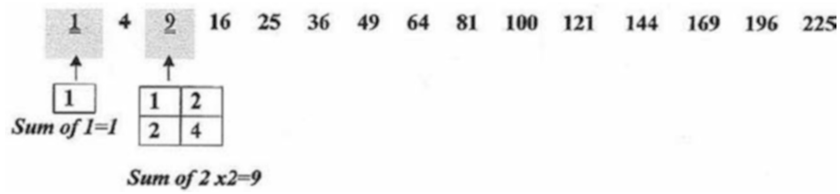


Figure 5. Sarah's skipping to find sums of block entries

With her actions, Sarah sensed a new problem to solve – she thinks that there could be a relationship between the sequence of square numbers on the diagonal of the array and the successive sums of the entries of N×N blocks. Sarah was able to continue her 'skip' method to generate the sequence of sums of the entries of all NxN blocks.

Sarah: So, for the first 3×3 *(points to [1, 2, 3 : 2, 4, 6 : 3, 6, 9])*, I already did this over here, so it is 36. So, in going from the 1×1 to the 2×2 to the 3×3, we go from 1, to 9, to 36 – so we skipped over the 16 and the 25 *(she crosses out the 16 and 25 in the square number sequence)*, a skip of 2 in this sequence!! So, okay, if this is true, then it looks like we will skip over the next 3 square numbers, and that should tell us the sum for a 4×4 should be equal to 100 *(crosses out the 49, 64, 81 in the square number sequence)* – that is what I have over here!! Cool! So, for a 5×5, we skip over the next 4 numbers in the sequence, *(points to the sequence 121, 144, 169, 196)* and get 225 – yes, I got that one earlier for the 5×5. (Figure 6)

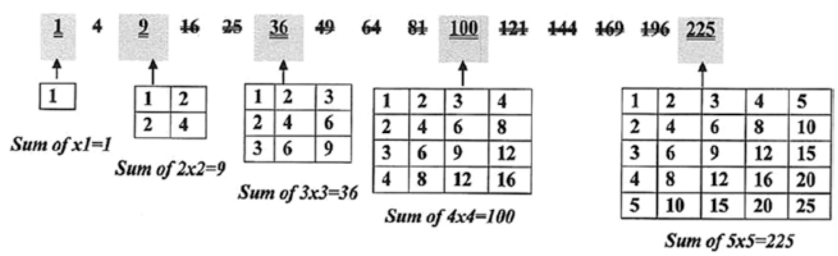


Figure 6. Sarah's skipping to find sums of block entries

Sarah then looked to make sense of her method with some further exploration.

Sarah: I wonder why this skipping works? Let's see it another way, for the 6×6, we add the entries in the rows to get 21+42+…+126 = 21(1+2+3+4+5+6) = 21×21 = 441. Do we get 441 by skipping the next 5 in the square sequence? *(Sarah extended her original sequence beyond 225, crossed out* *the corresponding 'skips,' and got a result of 441 as the next number in the sequence)* (Figure 7). But also, I notice that 21 over here *(points to the factored form 21• (1+2+3+4+5+6))* is the sum of the first 6 numbers in that first row. Yes!

In the last statement she makes, Sarah noticed that the sum of the row entries is the sum of the numbers from 1 through 6. She then makes a projection in her thinking to a general case:

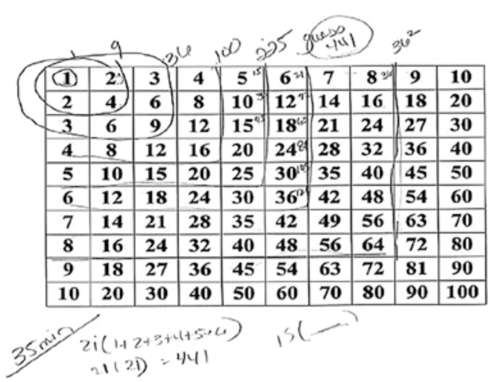
Sarah: So to find the sum of these N×N blocks, I bet you just need to look at the sum of 1 to N and then square that total to get the sum.

This is the first evidence that Sarah had made an abduction, that she had hypothesized the calculation she had carried out for the 6×6 block could be generalized to N×N blocks. However, the abduction appeared to have its source in her earlier comments:

Sarah: Let's see it another way, for the 6×6, we add the entries in the rows to get 21+42+…+126 = 21(1+2+3+4+5+6) = 21×21 = 441. Do we get 441 by skipping the next 5 numbers in the square sequence?

So Sarah had a sense of the general in the particular and her hypothesis about summing the numbers from 1 to N resulted from her deductions made by reflecting on the results of her factoring of the sums:

Sarah: I notice that 21 *(points to product 21• (1+2+3+4+5+6))* is the sum of the first 6 numbers in that first row. Yes!



1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 7. Sarah's diagram of her computation of sums in a 6×6 block

This enabled Sarah to re-state her hypothesis: In order to find the sum of entries in an N×N block, she needed to sum the numbers from 1 through N, and square the result. Sarah then looked to test her hypothesis on an 8×8 block (Figure 8).

1	2	3	4	5	6	7	8	36
2	4	6	8	10	12	14	16	
3	6	9	12	15	18	21	24	
4	8	12	16	20	24	28	32	
5	10	15	20	25	30	35	40	
6	12	18	24	30	36	42	48	
7	14	21	28	35	42	49	56	
8	16	24	32	40	48	56	64	

Skipping in the sequence: Finding the sum for the 8x8 block

225	~~**256**~~	~~**289**~~	~~**324**~~	~~**361**~~	~~**400**~~	**441**	**Skip next 6 for 7x7 case**	**784**	**Skip next 7 for 8x8 case**	**1296**
15^2	16^2	17^2	18^2	19^2	20^2	21^2		28^2		36^2

Figure 8. Sarah's computation of the sum for the 8×8 block

> *Sarah:* Let's try a big one, say 8×8. So, I guess that it would be …. 1+2+… +8 = 36, I don't know why I am adding these individual numbers since I know that the sum is (8×9)/2, and then I take 36^2? So that comes out to be … 1296. And does it check with my skipping over here? Let's see, so for 8×8, I first skip 6 over 21 to get 28^2 for 7×7, and then skip 7 more to get the one for 8×8, … so 7 more is 35, and the next one is 36! So my algorithm seems to work! The algorithm is pretty efficient for larger numbers, beyond all of these *(pointing to the array)* – how about a 100×100 grid! – But I thought that the skipping relationship was pretty cool!

DISCUSSION

This chapter addressed two questions. (1) Why is it important that our students be able to make abductions when they solve problems? (2) How should educators help students develop reasoning habits that include abductive reasoning?

The results help provide an answer to the first question. Sarah developed an informal method to find the sum of entries in NxN blocks and then transformed her method into a more general method that both explained the results for the particular cases she had solved and also could be used to solve the problem for larger values of N extending beyond the array. Sarah's solution activity is important for the following reasons. First, Sarah's development of her informal method to compute the sums made use of a metaphor (Saenz-Ludlow, 2004), 'skipping', that named and explained her method for finding sums of entries in the various blocks, by 'skipping' through a sequence of square numbers. With these idiosyncratic actions, she had constructed an informal method. She verified that the method appeared to work for other cases that could be generated from the array. This finding is consistent with research that identifies informal methods as playing a prominent role in the development of formal algorithms (Cai, Moyer, & McLaughlin, 1998; Sáenz-Ludlow, 1995).

Second, abduction played a prominent role in her actions, and came about from her goal to explain why the 'skipping' method worked for computing the particular sums. With her abduction Sarah hypothesized a general method (rule), that then explained not only the particular cases within the array that she had already verified with 'skipping', but that could be used to compute cases that went beyond the actual array (e.g., "how about a 100×100?"). Specifically, her subsequent development of the general method involved her first making a subtle shift in her attention from validation and verification of the 'skipping' method for blocks of dimension 2×2, 3×3 and 4×4, to efficacy considerations (why it appeared to work for the cases she generated). Exploring issues of efficacy for one's problem solving actions is an important though under-utilized activity in most instructional settings. In Sarah's case, this exploration with a view to explain the usefulness of her actions made possible her abduction. Her goal to examine her action in a new light provided for her an opportunity to unfold the process, and relate her informal 'skipping' method to operations on the row and column numbers. Her re-writing of the sum of row entries into factored form $21+42+\ldots+126 = 21(1+2+3+4+5+6) = 21\times21 = 441$ appeared to be the first indication of her abduction, hypothesizing that the results of applying her 'skipping' could alternatively be found by operating on the row and column numbers. Her reflection on the factored form to conclude that the sum of the numbers in parentheses represented the sum of the column numbers in the particular row indicated that she had made a deduction because these actions led her to state the hypothesis in a form that made possible further testing ("I bet you just need to look at the sum of 1 to N and then square that total to get the sum."). In this way, she was able to generalize her method from skipping within a simple sequence to a formal algorithm that was more efficient for finding the sums of entries in N×N blocks beyond the 10×10 array. She proceeded to test her hypothesis (the rule) for cases she could verify (with 'skipping') within the 10 × 10 array.

Sarah's abduction appeared to be an example of a creative abduction (cf. Sáenz-Ludlow chapter on abduction in proving, this volume; Eco, 1983) for the following reasons. First, her abduction of the general rule did not draw from consideration from among several equally probable hypotheses; rather, her hypothesis drew from her creative actions performed by reformulating her original problem of finding sums of entries blocks, to determining why the particular 'skipping' method worked. Second, while Sarah's abduction was based on her stated goal to determine why the "skipping' worked, she was quick to value the efficiency of the general rule over the 'skipping' method. Sarah's consideration of efficiency in making her hypothesis would appear to be an example of a type of 'aesthetic value' that is a basis on which creative abductions are formulated (Eco, 1983).

TEACHING AND LEARNING IMPLICATIONS

The results do not suggest an easy answer to the second question and must be treated with care in making particular recommendations for the teaching and learning of mathematics in K-12. It may be useful to reformulate the question as two separate related questions: 1. Can abductive reasoning be taught explicitly? and 2. Do certain kinds of tasks induce the solver's use of abductive reasoning?

Can Abductive Reasoning be Taught Explicitly?

The question of whether abductive reasoning can be taught explicitly is not easy to answer. Sarah demonstrated conceptual growth in her problem solving because she was able to selfgenerate and selfregulate most all of her solution activity with little prompting, skills that many students in K-12 find difficult to develop. Moreover, as Sinclair (2006) has remarked, abductions, with their air of uncertainty, can be risky for students to make in K-12 mathematics classrooms because it leaves them vulnerable to ridicule by peers (N. Sinclair, personal communication, 2006). However, there are some recommendations that might be useful.

Promote reflection and discussion in classroom discourse. Abductions can occur only if the student has a secure sense of his or her role as a problem solver and is not afraid to express their ideas even if they may be incorrect. In order for students to become secure in their role as a mathematical problem solver, they must be provided with ample problem-solving opportunities that enable them to explore their understandings. Ferrando (2006) voiced the concern that students are often unwilling to explore the mathematics problems they are faced with and more often than not, give up working on a problem if they do not see an immediate strategy to pursue. Hence, we must carefully listen to students and observe what they do rather than conduct classroom activities based on our expectations of what we think they will say and do. Due to large class sizes, it is difficult for teachers to engage in the lengthy discussions represented in the interviews conducted in the study. However, more one-on-one communication can be facilitated using small group problem solving that invites students to share their thoughts about both the decisions they make and difficulties they face while solving problems. This in turn provides teachers with opportunities to respond to the problems and questions that students formulate.

Encourage proactive agency in problem solving. Students must not only be able to develop ideas about the problems they face, they must be willing to present and defend them in classroom discussions. Sarah viewed herself as in control and aggressively switched course whenever unexpected problems arose. Instructional activities that allow students opportunities to share and defend their ideas for solving particular problems prior to actual solving help develop self-advocacy in students and contribute to a proactive sense of agency.

Do certain kinds of tasks induce abductive reasoning? While Sarah performed well with the Number Array task and all of the other non-traditional task that she solved in other interviews, we must be careful in concluding that abductions can be stimulated through the use of particular tasks and problems. For example, one approach that has gained prominence in recent years involves the use of 'open ended' tasks to stimulate problem posing and solving (Becker & Shimada, 1997). The results suggest that our focus should be on the students' mathematical thinking and learning, and helping them to open up and explore their own interpretations of mathematical situations. The ideas generated by Sarah were their own, self-generated to help her 'make sense' of the situations she faced, and seen by them as plausible explanations of the problems. While it is true that Sarah's solution of the Number Array task involved problem posing and solving in an unfamiliar context, her initial ideas evolved into conceptually rich ideas that included new problem formulations and re-formulations, and conjectures about how potential solution activity would work out. In this way, she developed novel structures for her solution actions as she saw fit. In other words, the external structure of the task was less important for Sarah than her evolving of structure within her actions. These results suggest that while there can be a degree of novelty designed into the tasks we give students, the greater need is for mathematics educators to broaden their view of problem solving as learning opportunities and incorporate problem solving tasks that provide abundant posing and solving opportunities to our students so that they stretch and broaden their understandings as they solve problems.

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NOTES

1. Some of the Citations of Peirce that appear in this chapter are taken from *The Collected Paper*, Volumes 1–6, edited by Charles Hartshorne and Paul Weiss, Cambridge, Massachusetts, 1931–1935; and volumes 7–8 edited by Arthur Burks, Cambridge, Massachusetts, 1958. The standard format for citing Peirce has been used. For example, CP 5.172 refers to Volume 5 of *The Collected Papers*, paragraph 172.
2. Since Hoffman (1999) argued that there is no logic of abduction in the sense of syllogistic logic when it comes to the generation of hypotheses, and that "logic" should be understood in the broader sense of "methodology", these studies might be better described as studies that involve the methodological understanding of abduction.

[3] A bracket notation is used to list the top to bottom rows of the block being considered. For example, the 2×2 is indicated by the sequence [1, 2 : 2, 4] and a 3×3 block is indicated by the sequence [1, 2, 3 : 2, 4, 6 : 3, 6, 9].

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