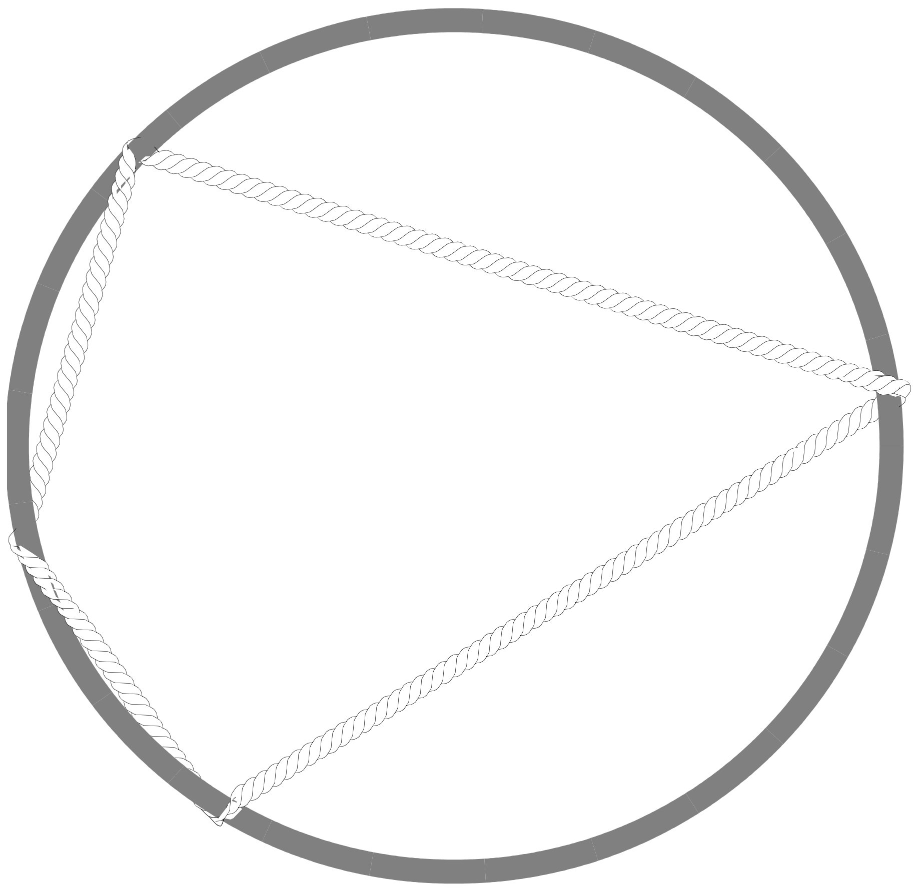
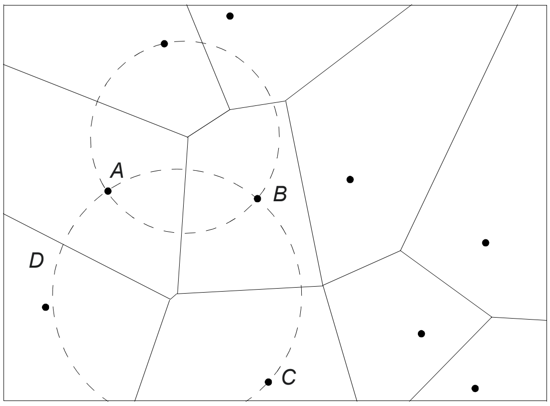
Chapter 4: A special quadrilateral



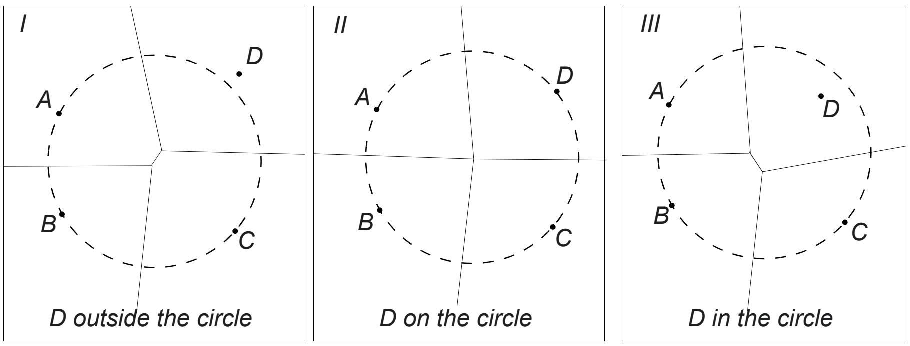
In this chapter we will continue reasoning.

In general we will deduct several things concerning distances and angles from very few given data. We will also think about the process of reasoning itself and how to write down proofs.

18. Cyclic quadrilaterals



In chapter I, when you were dealing with Voronoi diagrams, you encountered this example, in which point *D* lies just outside the circumcircle of triangle *ABC*. This circle is empty and thus the cells round *A*, *B* and *C* converge into a three-countries-point, which of course is the center of the circumcircle of triangle *ABC*. Thus *D* does not disturb the three-countries-point. For the position of the fourth point *D,* compared to the circle through the three points *A*, *B* and *C*, there are three options:



(In case III it is possible that *D* lies so close to *B* that the cell round *D* is closed.)

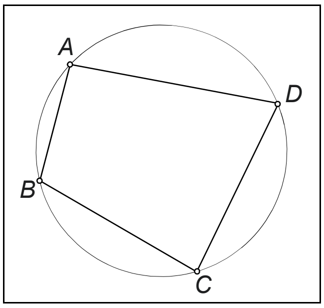
1. **a.** In case II the Voronoi diagram is very special: there is a four-countries-point.
   Why?
   b. Which vertex of the Voronoi diagram is the center of the triangle *ABC* in case I?
   c. And how about case III?

Since the sides of the quadrilateral in the special case are all *chords* of the circle, we call quadrilateral *ABCD* a cyclic quadrilateral. The **definition** is:

definition of cyclic-quadrilateral

A quadrilateral is called a **cyclic quadrilateral** if its vertices lie on one circle.

In this section we shall prove a relation between the sizes of the angles for these special quadrilaterals. Later on we shall use this relation for other purposes than Voronoi diagrams.

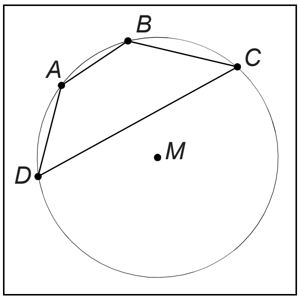


2. On the right you see a sketch of a cyclic quadrilateral *ABCD*. In the figure we didn't actually portray the main property of the circle: that there is a center *M* and that the line segments *MA*, *MB*, etcetera, have equal length.
 a. Therefore, sketch the center and the line segments *MA, MB, MC* and *MD.*
 b. The quadrilateral is now divided in four triangles. The eight angles to the vertices of the quadrilateral are equal two by two. Why?
 c. Indicate equal angles with the same symbol; use symbols like 1, 2, *, ◦, ´, ·. In each vertex you see a different combination of signs. But what do you notice when you compare the sum of the signs of *A* and *C* to the sum of *B* and *D*?

First a short footnote: The only correct answer to question **2b** is: because triangle *ABM* is isosceles. It is given that the sides are equal, namely d(*A, M*) = d(*B, M*). That you can conclude from this that the angles are equal, is based on an at the moment non-formulated theorem about isosceles triangles. We will not prove this theorem here, it is one of the things we accept for now, like earlier the triangle inequality. Later on, you will draw up a numbered list of these kinds of theorems, which have already been familiar to you for some time.
What you have just proven is the following *temporary* theorem:

Temporary theorem of the cyclic quadrilateral
In each cyclic quadrilateral ABCD holds: $\angle A + \angle C = \angle B + \angle D$.

We reached a fine result by smart reasoning. However, there are some problems left.



Problem A
The first question you need to ask yourself is: Is the theorem proven for every cyclic quadrilateral one can think of? For example, consider a cyclic quadrilateral as shown on the right. We again looked at just one special figure, which is not representative for all cases. For in this one, the proof above does not hold....

Problem B
The temporary theorem only deals with equality of $\angle A + \angle C$ and $\angle B + \angle D$. This is a bit meager. Maybe something can be said about the *size* of $\angle A + \angle C$ and $\angle B + \angle D$.

Problem C
Just like in the complete theorem about the perpendicular bisector, theorem 3, page 71, we need to know what happens if D lies inside or outside of the circle, since this is the most frequently occurring case!

19. Scrutinize proving

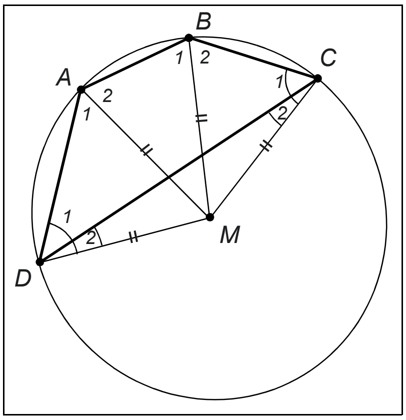
Problem A

3 **a.** What is the essential difference between this cyclic quadrilateral (last figure on previous page) and the one from exercise 2?

b. Again go over the steps of exercise 2. Where do you need to deviate from exercise **2** for this case?

It is not very difficult to find a proof for this situation. We will draw up this proof in a clearly noted form as an exercise in notation.

This is the accompanying sketch.



Now we can talk easily about all kinds of angles and parts of angles in the situation without referring to them with strange symbols.

4. In the theorem we are proving $\angle A$ plays a role. By $\angle A$ we mean the vertex angle $\angle DAB$. The sketch also shows $\angle DAB = \angle A_1 + \angle A_2$. Here we are not reasoning based on a sketch, but merely showing what we mean with all those letter notations.

a. What does $\angle D$ mean in the theorem? Write down the relation of $\angle D$ with $\angle D_1$ and $\angle D_2$.
Look carefully at how the little arches are indicated.

The complete proof could start as follows:

$\angle D_1 = \angle A_1$, $\angle A_2 = \angle B_1$, $\angle B_2 = \angle C_1$, $\angle C_2 = \angle D_2$
(since the triangles *DMA*, *AMB*, *BMC* and *CMD* are isosceles)

This is actually
a statement
with a motivation.
In the remainder of the proof we will use the data from the sketch and these four equalities to rewrite the sum of angles $\angle A + \angle C$ step by step to $\angle B + \angle D$. Between the brackets is stated why an equality holds, thus those are again

$\angle A + \angle C = \angle BAD + \angle DCB$
= (dividing angles)
$(\angle A_1 + \angle A_2) + (\angle C_1 - \angle C_2)$
= (using equal angles)
. . . .

motivations.
b. Complete this story.
The last line of this story will be
$= \angle B + \angle D$
(You could – if you get stuck – search by starting at $\angle B + \angle D$ and splitting up the angles)

For the first case (where *M* lies inside of the cyclic quadrilateral *ABCD*) you could have written down the proof in the same fashion.

5. The sketch would be different, but in the proof you would need to change some details. Which?

case distinction
We are still working on the temporary theorem of the cyclic quadrilateral. We have made a careful distinction between the cases where the center of the circumcircle lies inside or outside the quadrilateral.

6 a. Is the temporary theorem of the cyclic quadrilateral now proven for áll possible cyclic quadrilaterals? In other words: are there other situations than those where *M* lies inside respectively outside of the quadrilateral?
b. If you find another case, which of the two proofs holds?

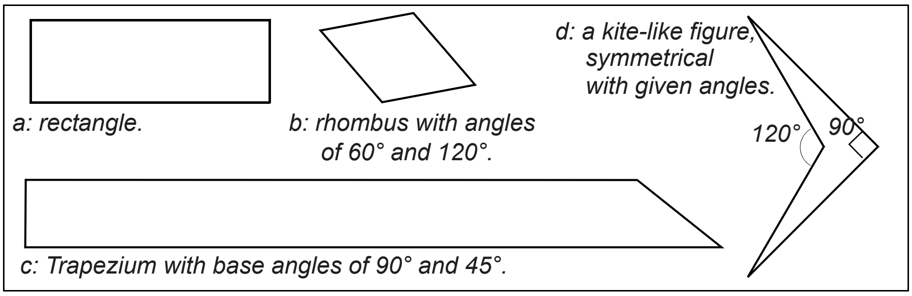
Conclusion from this part for Problem A
The temporary theorem of the cyclic quadrilateral was put under some pressure, but is eventually saved by adjusting the proof for the other case. While doing that we also practiced how to write down a proof clearly. We distinguished statements and motivations. Noting angles with indices was handy for keeping the relation between proof and sketch. On to problem B now.

Problem B
That was:
The temporary theorem only talks about the equality of $\angle A + \angle C$ and $\angle B + \angle D$. This is a bit meager. Maybe something more can be said about the size of $\angle A + \angle C$ and $\angle B + \angle D$ themselves.

7 **a.** a. Find out in exercises **2** and **4** what $\angle A + \angle C$ and $\angle B + \angle D$ are, using a protractor. The four results are not very different!
b. What is your conjecture (still to be proved!) about the sum of opposite angles in a cyclic quadrilateral?

If your statement is right, you can also say something about $\angle A + \angle C + \angle B + \angle D$ in such a quadrilateral. Before we will prove that the sum of the four vertex angles in a *cyclic quadrilateral* is 360°, we check whether we really need to restrict ourselves to cyclic quadrilaterals.
Namely, we switched from $\angle A + \angle C = \angle B + \angle D = 180°$ to $\angle A + \angle C + \angle B + \angle D = 360°$, but the last thing also holds if we have for instance $\angle A + \angle C = 140°$ and $\angle B + \angle D = 220°$. In other words: for the total sum of angles of 360° we maybe should not restrict ourselves to cyclic quadrilaterals.
We will first try to find out how general $\angle A + \angle C + \angle B + \angle D = 360°$ can be true.

8. Determine the total internal sum of angles for these examples. These are a few special cases for which it is easy to compute and determine angles.



(Note that in ***d*** the angle of 120° is not an inside-angle of the quadrilateral.)

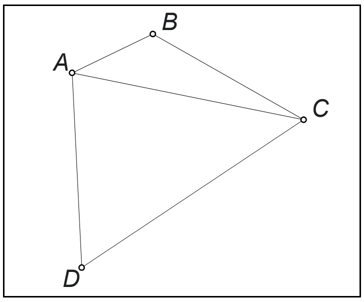
Let’s formulate the statement first as a theorem, the proof will follow.

Theorem 8

In each quadrilateral the sum of angles is equal to 360°.

You will give the proof in the form which just has been shown. For the kernel of the proof you will of course need to know where to look, but you do already know that the sum of angles in a triangle is 180°. This is something you learnt earlier which you can use now. In the numbered list at the end of this chapter we will assimilate this fact as a theorem. In short: split the quadrilateral into two triangles!

9 **a.** This is a sketch which goes with theorem **8**. However, note that it is sometimes possible that the connection *AC* does not lie inside the quadrilateral. Sketch such a case and divide that quadrilateral in two internal triangles with a connection line. As long as we do not use anything except the properties of triangles, everything will work out just fine after this case distinction.



b. Now add the necessary numbers and arches in the figure and write down the proof using the format of exercise **4**.

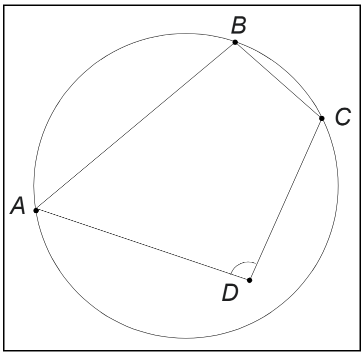
The proof of theorem 8 is now complete. It is now safe to use the theorem to improve the temporary theorem of the cyclic quadrilateral to:

Temporary theorem of the cyclic quadrilateral, improved version

For each cyclic quadrilateral ABCD holds $\angle A + \angle C = \angle B + \angle D = 180°$

Problem C

Now we will look at the situation where *D* lies within the circle through *A*, *B* and *C*.
The next assumption almost goes without saying: $\angle B + \angle D > 180°$. Imagine that you are standing somewhere on the circle, opposite *B*. If you starting walking forwards, you need to widen your view to left and right to be able to still see *A* and *C*.
The proof of $\angle B + \angle D > 180°$ shall be based on this idea: we will compare the situation in point *D* to one with a point on the circle.



10. For that point we do not choose a completely new point, but a point, which has a relation to the other points.
- **a.** Choose a point on the extension of *AD* and call it *E*. Make sure that the quadrilateral *ABCE* is fully drawn.
- **b.** Now show, using a familiar property of triangles, that $\angle ADC = \angle DCE + \angle CED$.
- **c.** Which inequality does now apply to $\angle ADC$ and $\angle CED$?
- **d.** Complete the proof of $\angle B + \angle D > 180°$ by applying the temporary theorem of the cyclic quadrilateral to *ABCE* and combining that with the result of **c**.

The proof has not been put in a strictly organized form, but this is not always necessary.
Now the following has been proven:

I. If in a quadrilateral *ABCD* point *D* lies **INSIDE** the circumcircle of triangle *ABC*, then $\angle B + \angle D > 180°$ applies.

We already knew:

II. If in a quadrilateral *ABCD* point *D* lies **ON** the circumcircle of triangle *ABC*, then $\angle B + \angle D = 180°$ applies.

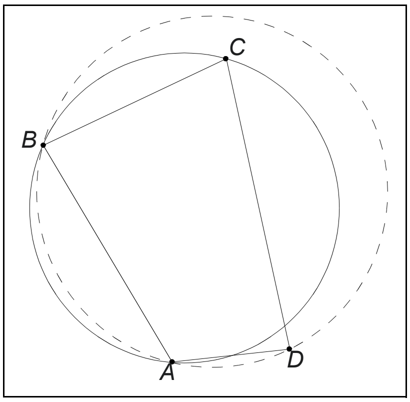
And of course we also expect that:

III. If in a quadrilateral *ABCD* point *D* lies **OUTSIDE** the circumcircle of triangle *ABC* then $\angle B + \angle D < 180°$ applies.

You could try prove III almost similar to I in exercise **10**, but you will have to check several cases. It is possible that *AD* and both intersect the circle, or one of them or none.
It is better to use the proven cases I and II in a smart way and deducting case III from there. We will do this in this extra exercise.

extra

11. First of all the sketch. *D* lies outside the circle through *A*, *B* and *C*. The dotted circle goes through *A*, *B* and *D*. It looks like the complete arch from *A* via *D* to *B* lies outside the circle through *A*, *B* and *C*.
- **a.** Argue that, using theorem 5, page 78. Actually, you need to show that both circles cannot have a third point *X* in common, because what would then be the circumcircle of *AXB*?

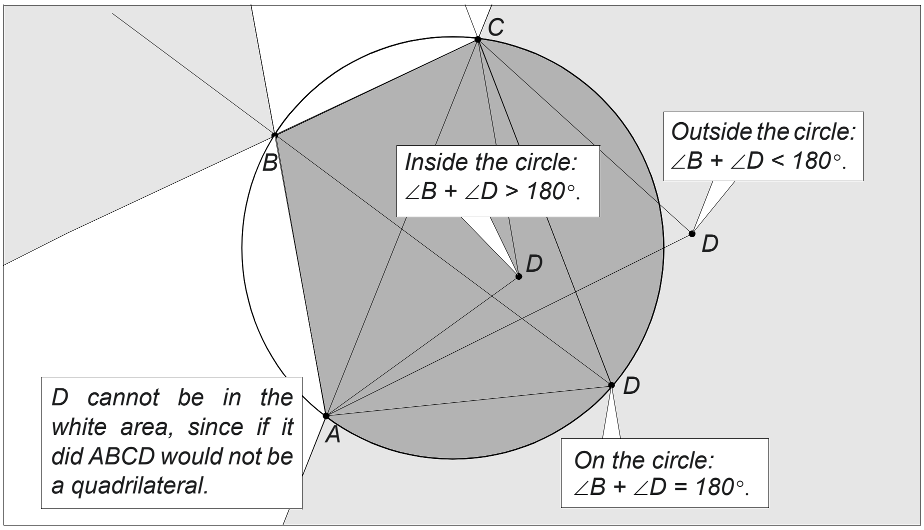


b. Thus C lies inside the circle through A, B and D. The proven part I of above now leads to an equality in where $\angle C$ occurs. Write it down.
c. Now deduct, using also theorem 8, the desired statement $\angle B + \angle D < 180°$.

Part III is proved now and we will summarize the results of this section in one theorem.

Theorem 9 (properties of the cyclic quadrilateral)
If $ABCD$ is a cyclic quadrilateral, then $\angle A + \angle C = \angle B + \angle D = 180°$.
If D lies inside the circumcircle of A, B and C, then $\angle B + \angle D > 180°$.
If D lies outside the circumcircle of A, B and C, then $\angle B + \angle D < 180$.

This deserves a survey illustration.



Remark: Since the three cases of the theorem exclude each other, you can immediately draw conclusions like:

If in a quadrilateral $\angle B + \angle D = 180°$, then that quadrilateral is a cyclic quadrilateral.

After all, if this equality holds, then the last two statements of the theorem ensure the fact that D neither lies inside nor outside the circle. Remains: on the circle.

extra: finding a good definition

12. It is said that *D* cannot lie in the white area, since then *ABCD* would not be a quadrilateral. This is rather vague as long as we not have agreed upon what a quadrilateral is. Find out what is going on and give a definition of 'quadrilateral', which exactly excludes these cases.

extra: a warning!

In mathematics it can, and will, happen that you have a proof which looks right and clear, but later on somebody finds a small error in it. It does not always mean that the proof is totally wrong; it can be fixed in most cases. You are now in such a situation.

13. Check again the position of point *E* in 10a above. It could be on arc *BC*! In that case you cannot work with *ABCE* as a quadrilateral.

a. How to fix this hole in the proof?

20. Using cyclic quadrilaterals

In this section we return to constructing Voronoi diagrams. We use what we know of cyclic quadrilaterals, so mostly theorem 9. Since that theorem talks about angles, you need to measure angles very precisely several times.

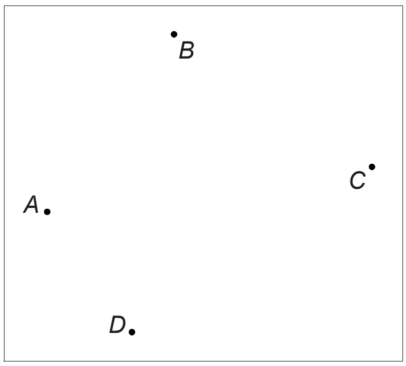
14. Given are four centers.

a. Find out whether in this Voronoi diagram of these four points the cells round *A* and *C*, or the cells round *B* and *D* adjoin.

b. Sketch all connection lines of centers which have adjoining cells with a color.

c. Finish the Voronoi diagram by sketching perpendicular bisectors.

d. Unlike exercise **4**, page 45, now you did not sketch too many perpendicular bisectors. Why?



15. Is this also true?

If a Voronoi cell is a quadrilateral, then that quadrilateral is a cyclic quadrilateral.

If necessary, give a counterexample.

16. Sketch a situation with six centers, where you have two four-countries-points in the Voronoi diagram. (Avoid the flat example that the centers which have a four-countries-point form a square or a rectangle.)

Summary of chapter 4

reasoning
In this chapter we got results through reasoning. The direction of concluding things was from former knowledge to new all the time.

Writing down proofs
You have learned that you can write down proofs in a neat way. Two aids were:

a. Indicating angles with indices: A_1, B_2, etcetera. In the sketch you can also indicate angles with symbols like $*$, $\circ$, $\times$, and $\bullet$. However, it looks kind of weird if you start your proof with: * = *. A good compromise is: indicate in the sketch equal angles with the same color or symbols and use unambiguous denominations.

b. Note the statement and motivation in this format:

a statement

with a motivation.

motivations
As motivations the following are allowed:
- references to definitions
- basic unproven known facts we agreed about (like the triangle inequality)
- statements that have been proven earlier.

theorems
Important things which we know to be true and which we will use again, are laid down in the form of a theorem. Several theorems mentioned below have not been proven in this book. Those are the first two of the following overview.

overview of theorems in this chapter

Theorem 6 (Isosceles triangle)

If in triangle *ABC*

d(*A, C*) = d(*B, C*),

then also

$\angle CAB = \angle CBA$.

The reverse is also true:

If in triangle *ABC*

$\angle CAB = \angle CBA$,

then also

d(*A, C*) = d(*B, C*).

Theorem 7 (Sum of angles in a triangle)

In each triangle the sum of the angles is equal to 180°.

Theorem 8 (Sum of angles in a quadrilateral)

In each quadrilateral the sum of the angles is equal to 360°.

Theorem 9 (properties of the cyclic quadrilateral)

If *ABCD* is a cyclic quadrilateral, then $\angle A + \angle C = \angle B + \angle D = 180°$.

If *D* lies inside the circumcircle of *A*, *B* and *C*, then $\angle B + \angle D > 180°$.

If *D* lies outside the circumcircle of *A*, *B* and *C*, then $\angle B + \angle D < 180$.