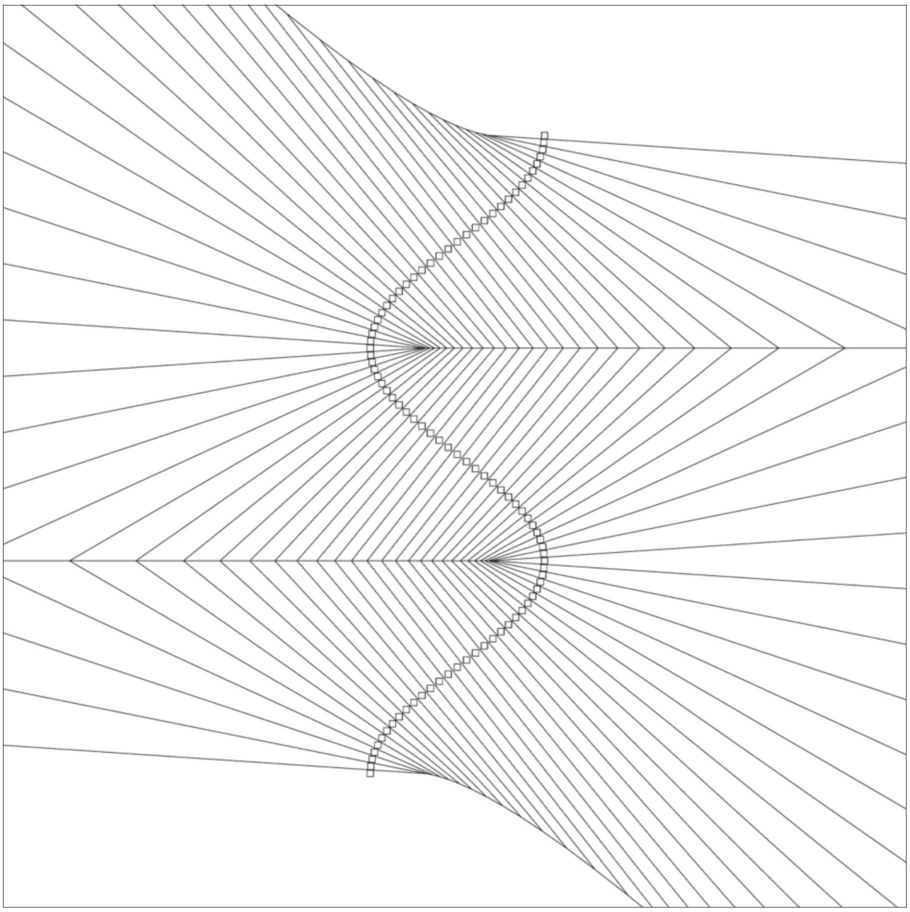
Chapter 3: Computer practical Voronoi diagrams



Introduction, software availability
In this chapter we will construct Voronoi diagrams using the computer. The chapter is of a practical nature: you will try a lot, but prove little to nothing.
Since we will be working with more than 5 or 6 centers rather fast and easily, we can also look at other properties than the ones in chapter 2.

Most of the tasks in this chapter can be done with the free applet VoroGlide (FernUniversität Hagen; made by Christian Icking, Rolf Klein, Peter Köllner, Lihong Ma)**.**

Starting Voroglide:
Go to http://wwwpi6.fernuni-hagen.de/GeomLab/VoroGlide/index.html
or search for VoroGlide with Google.

Basic use of VoroGlide
You can generate points by clicking your mouse.

- Already made points can be dragged.
- Remove points with right-mouseclicks or ***Clear*** under ***Edit***
- If you want several points on a line or circle, you can put a physical object (cup, ruler) on the screen to help you. Also, because you can only drag points as far as the border, you can use the borders as lines.
- Under ***Show*** you can choose three types of diagrams; combinations are possible. You will discover their meanings in this chapter.

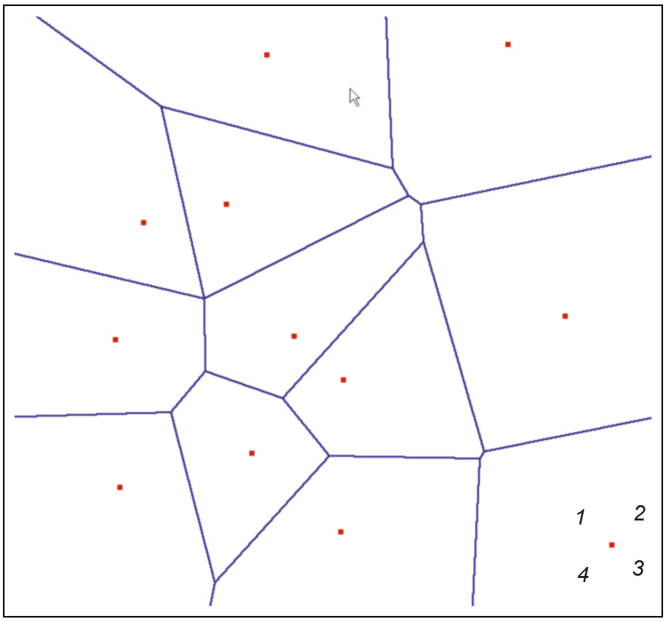
13. Introduction

You know that constructing a Voronoi diagram is based on drawing perpendicular bisectors, but finding the right line segments can be very time-consuming if there are a lot of centers. In applications of Voronoi diagrams we usually have situations with a lot of centers. We will also look for changes in the configuration when one points moves slightly. It is obvious that there is a need for computer programs that can do the time-intensive sketching. We will be working with such a program now.

start, usage
You can find information and simple directions for using the program VoroGlide on the previous page. Once the program is running, the screen will look as follows: You operate the program mainly with your mouse; you only need to click with the left button. On the left side of the screen you click points on or off. On the right side you click on the task for the program. That is all. For some assignments you will be given directions on how to use the `report`-part of the screen.

1. Start VoroGlide and explore the possibilities.

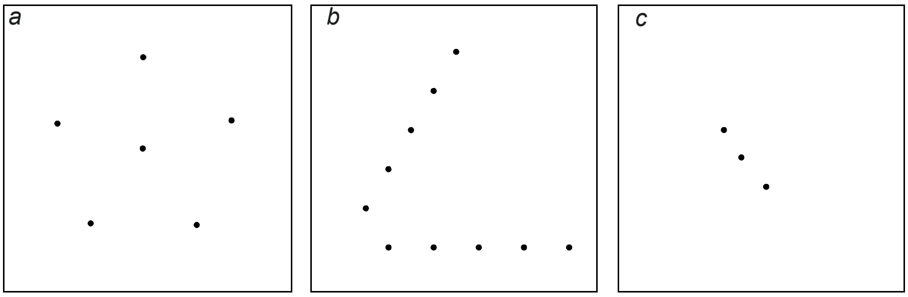
2 **a.** Make the screen look like this illustration.



b. There is a four-country point. Add a point to make it into a five-country point without drawing a circle. Indicate in the illustration above all possible places for this fifth point.

c. In the illustration there is also an 'almost' four-country point. In which of the four indicated direction should you drag the low-right center to get a real four-country-point. Think first, drag later!

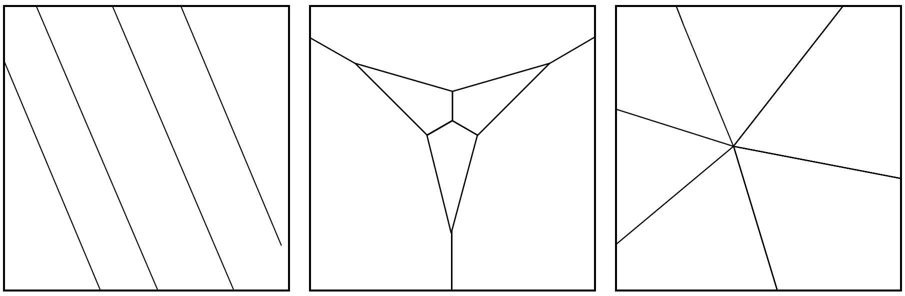
3 **a.** Create Voronoi diagrams according to the following positioning of the centers.



b. If you construct situation *b* very precisely, a number of edges will be straight behind one another. They form a line. Which line is that?

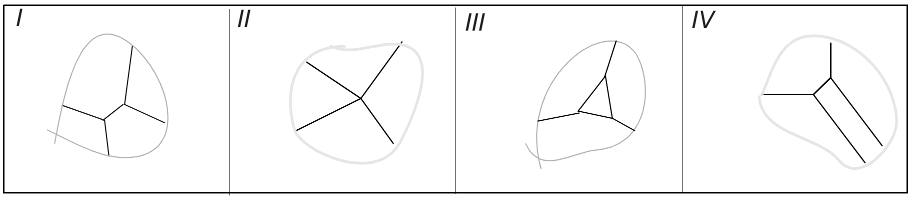
c. In situation *c* you cannot easily see a three-country-point on the screen. Now move the middle point a bit left and right. Try as best as possible to find a spot for this point which makes the diagram really not having a three-country point, also not outside the screen.

4. Construct by clever choosing the centers of the Voronoi diagrams in the figures shown below. Remember that you can also sketch circles and lines first to find out where the centers must be placed. Then indicate in the figure where the centers need to be approximately.

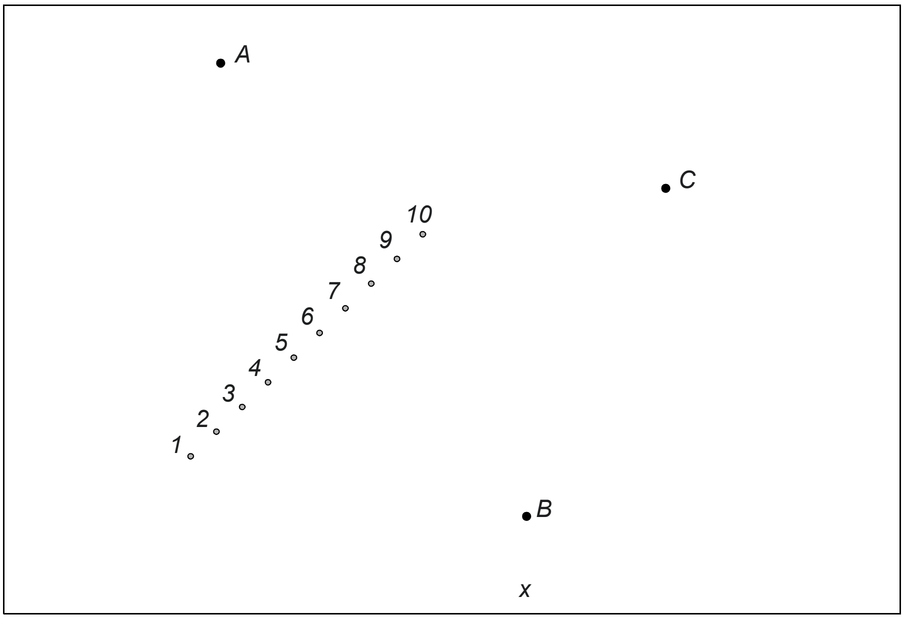


14. The influence of the fourth point

There are only four clearly distinctive diagrams possible for four points, namely these four:



5. Choose approximately three points like the black points *A*, *B* and *C* shown in the figure below. In this exercise we will add a fourth point (*D*) every time and



see what the effect of that point is. (The points 1 through 10 don't play a part until the next exercise.)

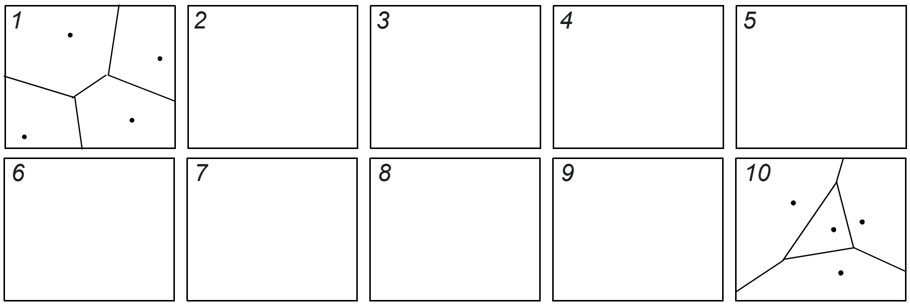
a. Indicate a fourth point in such a way that you get a type III diagram. If you move the point a little, the diagram will remain a type III.

b. Show on the screen all possible places for *D* for which the Voronoi diagram is of type II.

c. Do the same for type IV.

d. What type arises if *D* lies close to *x*?

6. If the fourth point consecutively takes the positions 1, 2, 3, ... , 10, the Voronoi diagram changes gradually. The beginning and ending of that process are given. Sketch the intermediate stages and determine the type for each state



7. From the preceding you can see that the transitions from one type to another type take place when point *D* passes through the circumscribed circle or through one of the three lines (through *A* and *B*, through *B* and *C*, through *C* and *A*).

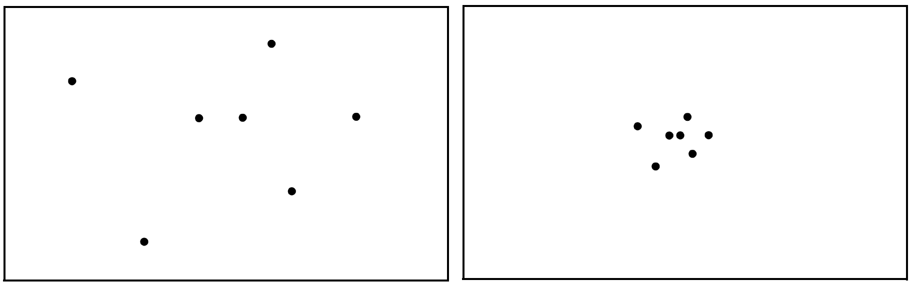
a. A type belongs to each intermediate stage. Indicate that in the figure with I, II, III or IV.

b. If you pick a point *D* 'at random', there is a greater chance for two of the four types, but the chance for the other two of the four types is very small. Why is that so?

15. Infinitely large cells, the convex hull

Meanwhile you have seen that there are cells which are enclosed on all sides by edges and that there are cells for which this is not the case. Now we will look especially at that last kind of cells.

8 **a.** For starters: make the Voronoi diagram for the situation at the left. There are only two finite cells.



b. Now make the situation at right; that is the same but four times smaller. What does the diagram start to look like when you reduce much more?

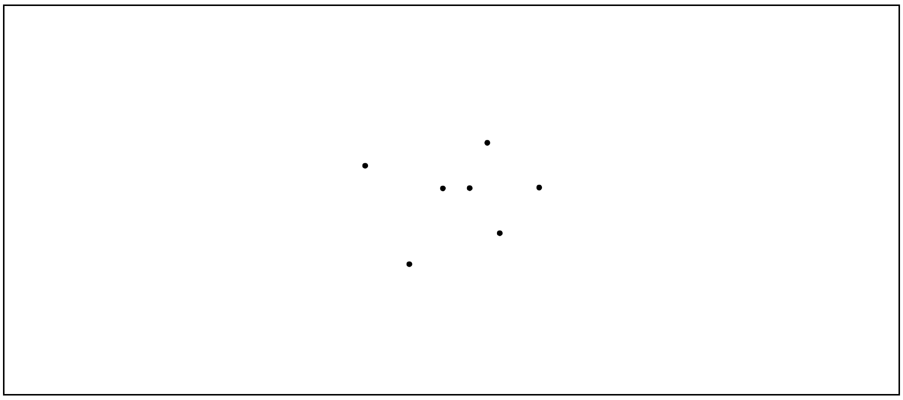
convex hull

c. Now go back to the first situation and choose *Show >> Convex Hull*.
A green closed line is added now. Imagine that the centers are nails in a board, which still stick out a bit. The green line looks like an elastic ribbon around all the points.

d. Add a couple of points *within* the hull. Are their any new infinite cells? Why is that so?

e. Add one new point, but do it in such a fashion that you add exactly one infinite large cell and that the other infinite large cells stay infinitely large.

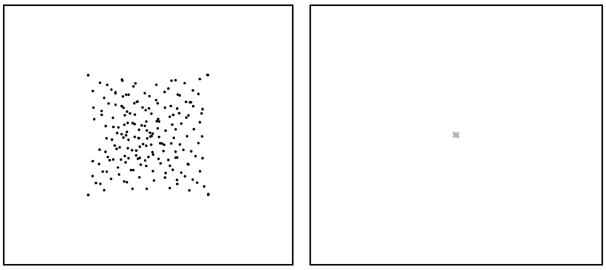
f. Here we have the situation of question **8 a** again, with some more space around. Indicate with a color the areas in which you can choose a new center in such a way that all the centers which are on the hull do not come loose from the hull.



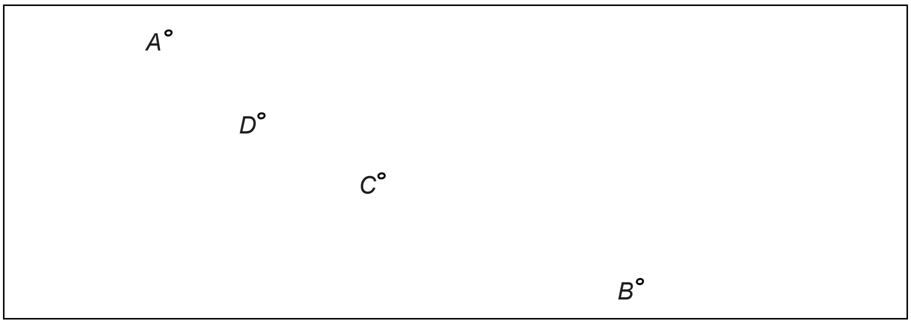
conclusion 1

The centers on the convex hull have infinitely large cells.

9. Here you see a situation where the computer cannot help you to sketch the Voronoi diagram.
This would take too long. However, you can predict how the diagram will look, if you decrease the scale a lot. Sketch that approach in the space on the right, where the little gray cloud in the middle represents the group of centers.



10 a. Construct this situation, with Voronoi diagram and hull. Verify whether the cells of *A* and *D* adjoin.



b. Sketch the circle through *A*, *D* and *B* and also the circle through *A*, *C* and *B*. Find their centers, if necessary resize the situation. Why do they need to lie on the Voronoi edge of *A* and *B* or on its extension?

c. Add within the hull of *A*, *B*, *C* and *D* a new center *E*, but in such a way that the line segment *AB* still belongs to the hull. Again, sketch the circle through *A*, *E* and *B*. Do the cells of *A* and *B* still adjoin? Why (or why not)?

We draw another temporary conclusion:

conclusion 2

Two centers which are connected with a line segment of the hull, have adjoining infinitely large cells.

11. Test with several examples whether this conclusion holds.

12. Construct an example with seven centers for which the Voronoi cells of the two furthest apart centers adjoin.

The two conclusions we drew in this paragraph look solid and reliable, but they are based only on your observations. You could prove them using the method from the previous chapter. Since it would be a lot of work and would not lead to new insights, and also since we will not build further on these conclusions, we will leave it at this.

16. Dividing the sea

The North Sea

13. In the picture on the next page you see Great Britain on the left and the Netherlands, Germany, Denmark and Norway on the right; all coastlines have been sketched using only points. Using a specialised computer program, the Voronoi diagram is drawn.

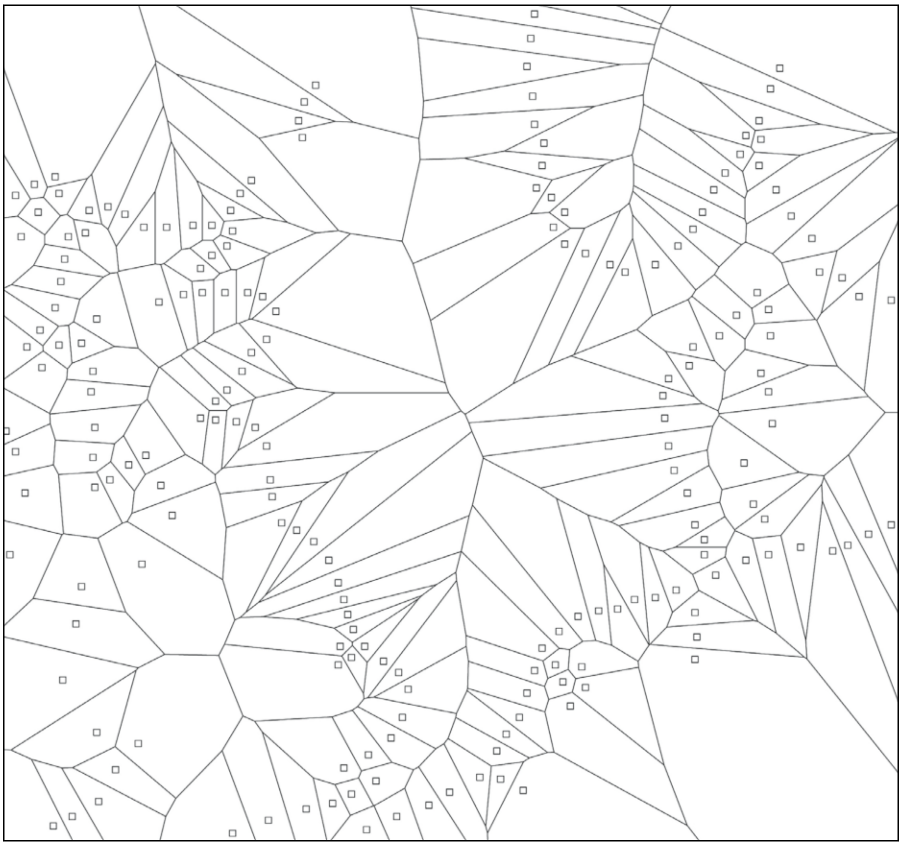
a. For oil exploitation purposes, the North Sea has been partitioned according to the nearest-neighbor-principle for whole countries. Draw this division in the picture.

b. It is possible to draw a circle which is tangent to the coast of Great Britain and Norway. Find a possible midpoint for this circle and draw it with a compass.

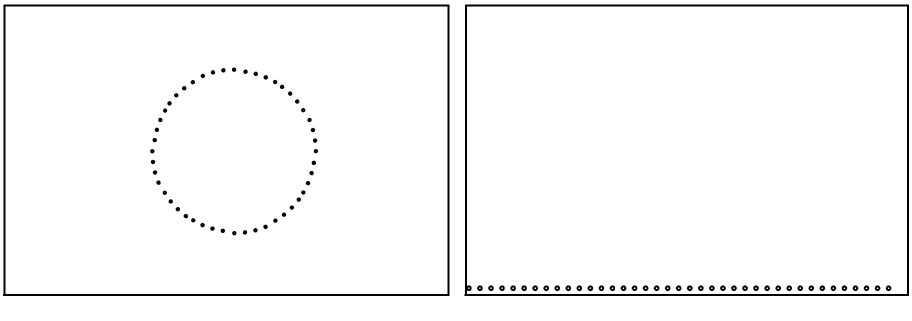
c. Find also a circle tangent to the coasts of Great Britain, the Netherlands and Denmark.

d. Because of your circle in question c there is no circle tangent to Great Britain, the Netherlands and Norway. Is this argument okay?

e. An important British oil harbor is located in the group of islands to the north-east of Great Britain (the Shetland Islands). Several years ago a mammoth tanker stranded on the rocks. Check how the North Sea would be partitioned if this group of islands belonged to a Scandinavian country. Before 1472 this was the case, but back then people did not drill for oil.



17. Exploring two more mathematical configurations



14. These two configuration can be made easily with a cardboard circle and dragging points to the bottom line.

a. Make the circle example and adjust points carefully so that all Voronoi edges meet in the center.
b. Now bring in a new center and drag that in and out of the circle. Explore what kind of cells you may get around that point. Is a new circle possible?
c. The oval shape that comes into being if the point is in the circle is an approximation of an ellipse. We will meet it again later! If the extra point is outside the circle, we find an infinite cell: it is a hyperbola, which we will also meet again later.
d. Make the points on the line. If you do well, all Voronoi edges should be parallel.
e. Now bring in a new point and explore what kind of cells you may get around that point. Does the shape looks familiar?
f. Again it is the approximation of a well known figure, the parabola. The extra point is called the focus, and the line the directrix.
g. Check this: a tangent to the parabola is the perpendicular bisector of the focus and a point on the directrix.
We will also come back to this property later!

Summary

In this computer practical you were able to try several things you have seen before, like the existence of largest empty circles. While exploring the influence of a fourth point on a diagram with three centers, it appeared that the circumscribed circle of the triangle of the three centers plays a dominant role. Furthermore, the sides of the triangle were also important. You also took a more precise look at infinite cells. Their centers turned out to lie on the convex hull. If you work with large numbers of points, you could form figures like countries on a map and explore their Voronoi borders. Using more mathematical arrangements like circles and straight lines, we discovered figures such as ellipses, hyperbolas and parabolas!