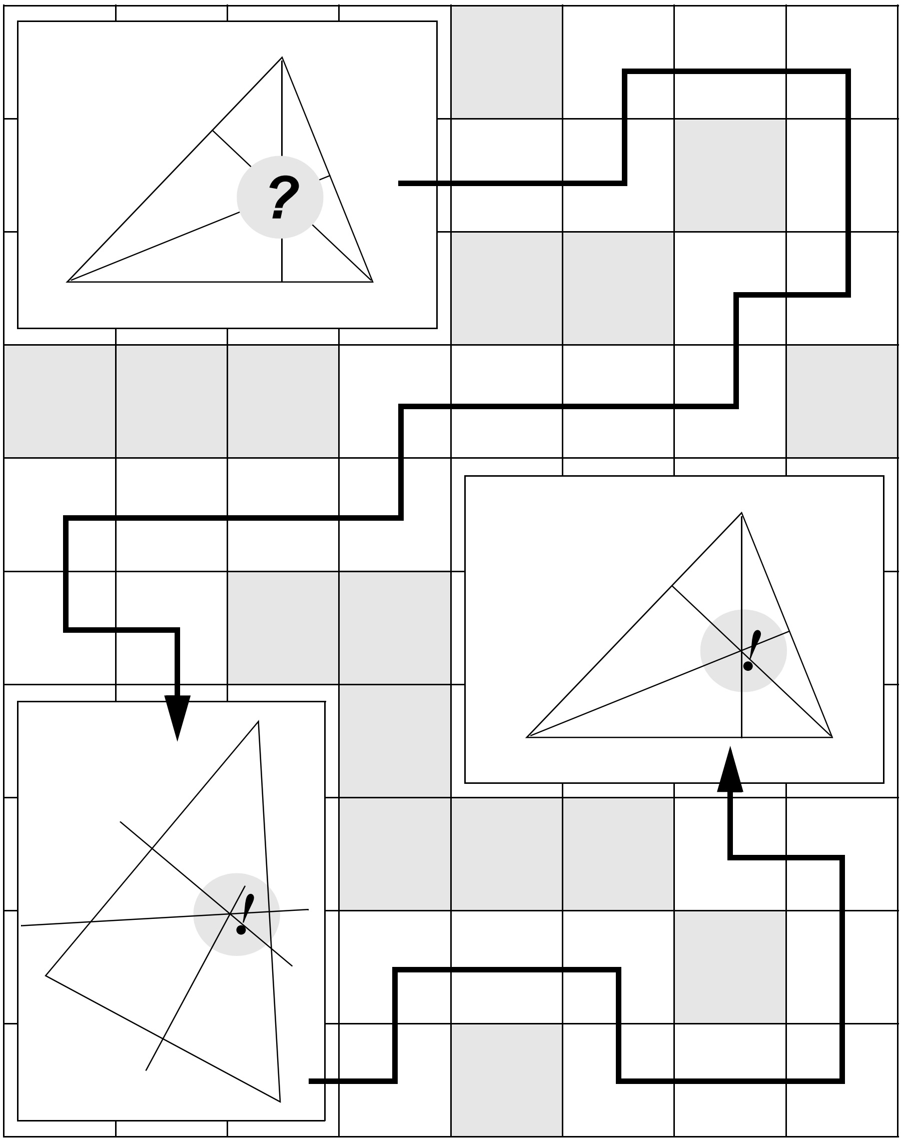
Chapter 3: Finding proofs



10. Chapter introduction

In previous chapters the proofs of the theorems were presented in little lumps. You didn't need to choose your own path. You learned:

- What proving is.
- That a proof can sometimes be written down schematically, but that you can also write down things in great detail.
- That there always is the three step form *Given - To prove - Proof*.
- That finding a proof starts with searching for an *idea*, after that you fill out the *details* and finally you write it down *properly*. You have done the last two steps elaborately.

procedure in this chapter
It is obvious what is going to happen next: practising in finding a proof. This will not be done like in the previous chapter, because then you would have not enough opportunity for your own research. We choose a different approach. Later, several search methods will be explained and sometimes be clarified with an example. After that you will be given a problem to work on. Thus the tasks you carry out are somewhat more extensive.

clues
What you need to try is: find the proofs yourself as much as possible, using the indicated search method. If you get stuck, you can glance through the *clues*, which start on page 231. But do not read these clues completely at once. Try again and again to make progress yourself – preferably with only one clue.

proof plan is the main thing
Finding a good proof plan is always the point. Your solution gives a clear proof story. It does not necessarily need to be written down in the two-columns-form of previous chapters.

methods to find proofs
In this chapter we start with using theorems about circles in small proofs. The paragraph is named: *know your theorems!*
Then there follows a short section about a pattern you have seen before. Because it is very important, it should be mentioned in our toolbox. The paragraph is named: *an important proof-pattern.*

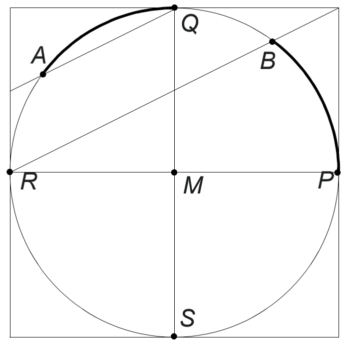
After that we will use three special methods to search for proofs. We will call them *links, split, plagiarism*

These are also the titles of the following sections. Thus the structure of this chapter is not built up mathematically, but around these search methods.

11. know your theorems!

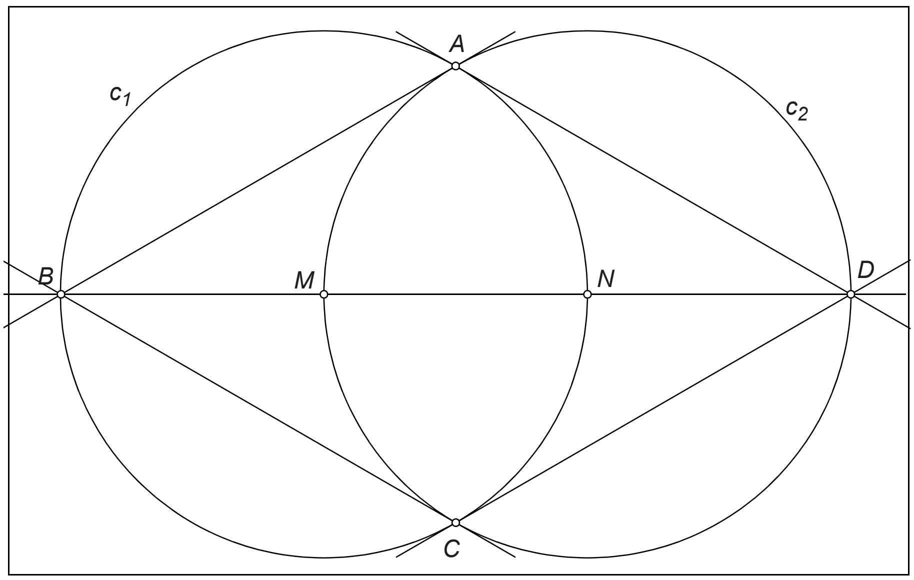
square and circle

1. A square, tangent to a circle. Given is also that *RB* and *AQ* are parallel.
a. Prove that the bold sketched arcs have the same size and that $\angle ARB$ is 45°.



Two circles: writing down proofs

2. Make a precise sketch of the figure below with compass and ruler.



M and *N* are the centers of the circles c_1 and c_2. The rest is obvious from the figure. prove the following two assertions and write your proof with full motivations.
a. *ABCD* is a rhombus.
b. The line *AB* is tangent to the circle c_2 in *A*.

Of course there are three other tangent lines: *BC* to c_2, *AD* and *CD* to c_1.

12. Reviewing an important proof-pattern

3. In the next figure three equilateral triangles have been set against the sides of triangle *ABC*. The circumcircles of the equilateral triangles seem to pass through one point. This needs to be proven!

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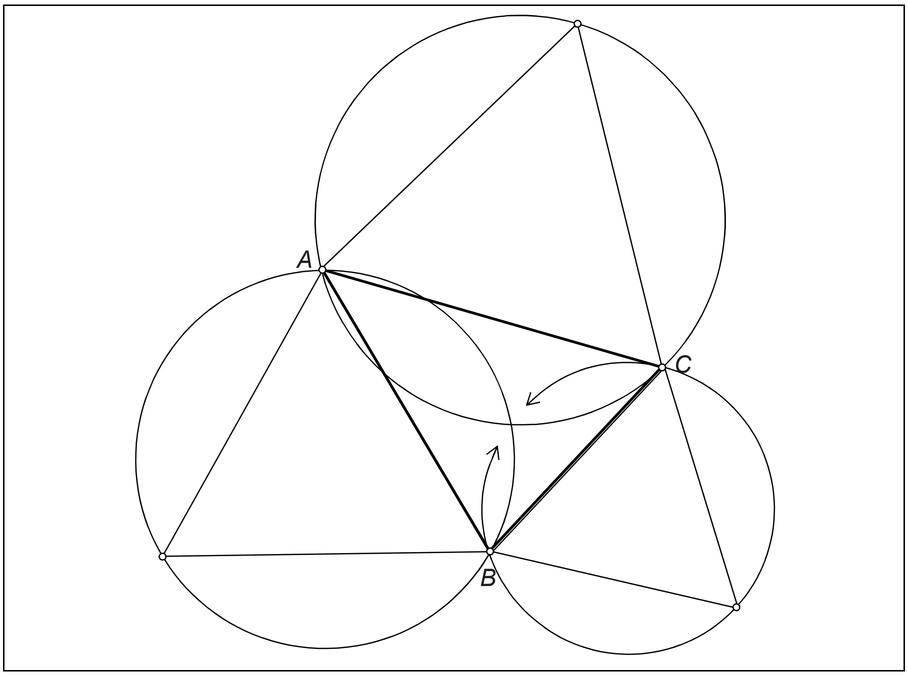


The proof pattern is:

a. Find a *characterization* for points on the small arcs.
b. Name the *intersection of two of these arcs S*
c. Show that *S* also lies on the third arc.

You used this pattern several times before!

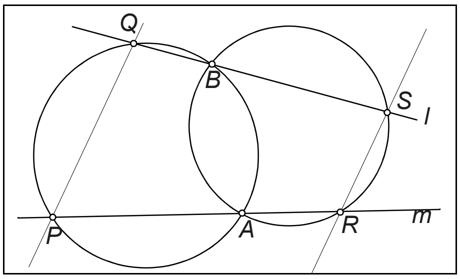
a. What is your characterization?
b. Which theorems do you use?
c. Write down the proof in the way you used earlier with the perpendicular bisector theorem in '*Distances, edges & domains*', page 76.

13. Links

finding links

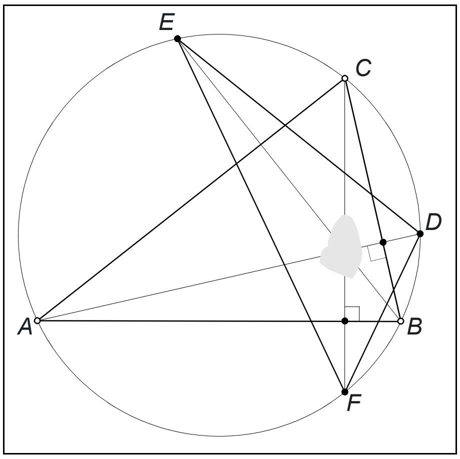
Often you encounter the situation that you need to prove an equality like $\angle X = \angle Y$ and it is not seen *directly* why these angles are equal.
If there is another angle, say $\angle Z$, of which you can prove both $\angle X = \angle Z$ and $\angle Z = \angle Y$, then you are done.
Or you may find that both $\angle X = 180° - \angle Z$ and $\angle Y = 180° - \angle Z$. Then you also hit the jackpot.
$\angle Z$ forms a *link* to get from one angle to the other. You prove the equality of $\angle X$ and $\angle Y$ *via* angle $\angle Z$. A *link* in a proof can be anything: not just one angle or one point. Sometimes you need more links, or two links at the same time.

4. Here two circles and two lines *l* and *m* through the intersections *A* and *B* of the circles are given.
To prove: $PQ \parallel RS$.
Approach: assume the idea that you need to show parallelism through indicating equal angles and search for a link. The circle and the points *A* and/or *B* of course play a role.



The next exercise leads to a familiar theorem in a new way. Thus this theorem should not be used in this proof!

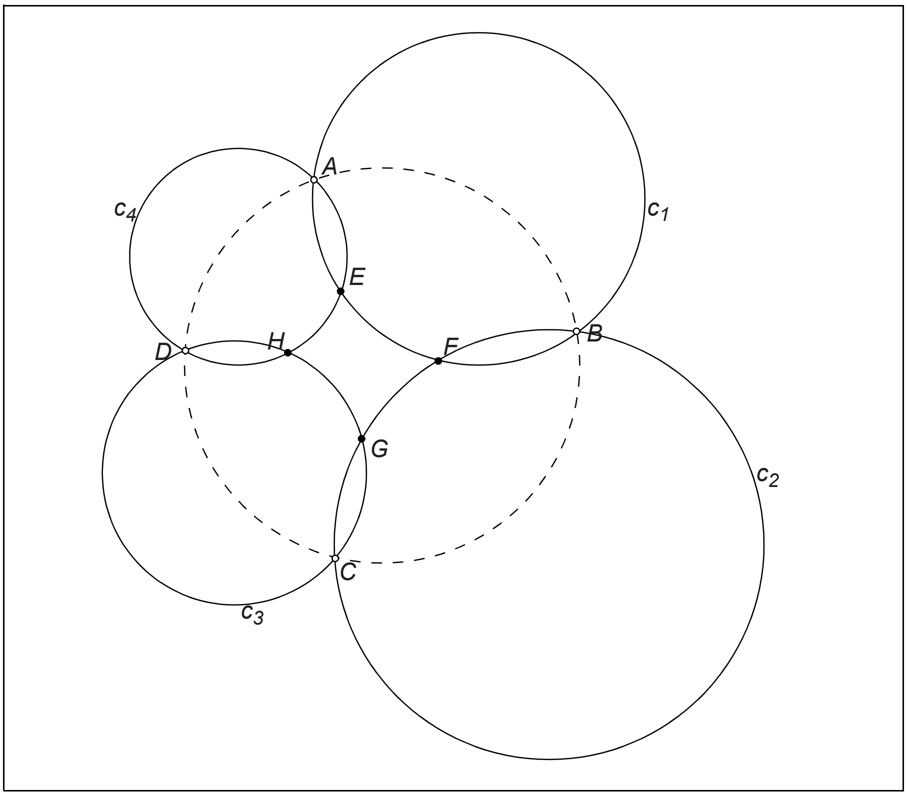
5. Given an acute triangle *ABC* with its circumcircle, the three altitudes intersect this circle in *D*, *E* and *F*.
To prove: the altitudes of triangle ABC are the bisectors of triangle DEF.
Approach:
Line *EB* should be the bisector of $\angle DEF$.
Try to find *both* angles, which should be equal to each other, somewhere else in the figure based on familiar theorems. Can these two new angles be linked via another angle?



6. How does the theorem of the altitudes follow from this? Is the proof general?

Next an example where the links will look alike, since the figure is built up from almost identical elements. You do need a substantial amount of links, but they are all of the same kind; if you operate in an orderly way, it will not be so bad.

7. The circles c_1, c_2, c_3, and c_4 intersect in the indicated manner in *A*, *B*, *C*, *D, E*, *F*, *G* and *H*. The statement you have to proof is surprising!



Prove: *If A*, *B*, *C* and *D* lie on one circle, *then E*, *F*, *G* and *H* also lie on one circle.

Clue: *EFGD* should be a cyclic quadrilateral. Connect its angles via links with angles of quadrilateral *ABCD*. Use a known theorem in the circles you know.

links in short

Often it has to be proved that two things, say *A* and *B*, are equal, while there is no theorem immediately applicable to the situation. Sometimes an extra step can be

found, a C of which you know that A and C are equal and that C and B are equal. Then C is a link. A link can be another angle, a line segment, several angles together.
When searching for links from one angle to another you have seen
- that the one angle sometimes is an angle of a cyclic quadrilateral; the opposite angle of the cyclic quadrilateral can also be a link.
- that a link angle can be on the same arc as the first one
- that link angle can be a neighbor angle of an isosceles triangle

You can also say in general:
- you often find links if you indicate equal things with symbols in the figure. At some point you will find a link or a series of links from A to B.

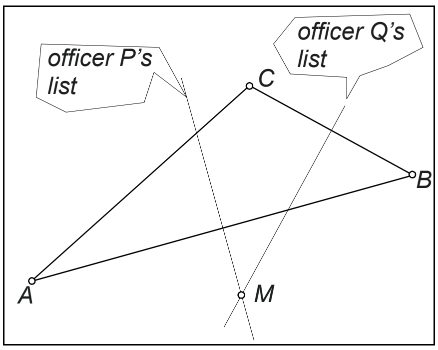
14. Splitting up conditions

bad plan
Imagine: the police know that a pickpocket, who is taller than 1.87 m and has red hair, is in a certain street. The investigation is split up over two officers as follows: On one street corner officer P collects all people who are taller than 1.87 m and officer Q collects all redheads on the other corner. The pickpocket will be the one who is standing on both corners.

... does work great in mathematics
A bad plan, clearly, in this case. Still: in geometry it often is the right plan and we have applied it many times.

a familiar example
Think back to how to find the center M of the circumcircle of triangle ABC. What was wanted was a point M to which applied:

$$d(M, A) = d(M, B) = d(M, C).$$



We sent, so to speak, two officers: Officer P searched all points with:

$$d(M, A) = d(M, B)$$

and officer Q searched all points with:

$$d(M, B) = d(M, C).$$

Or said in another way:

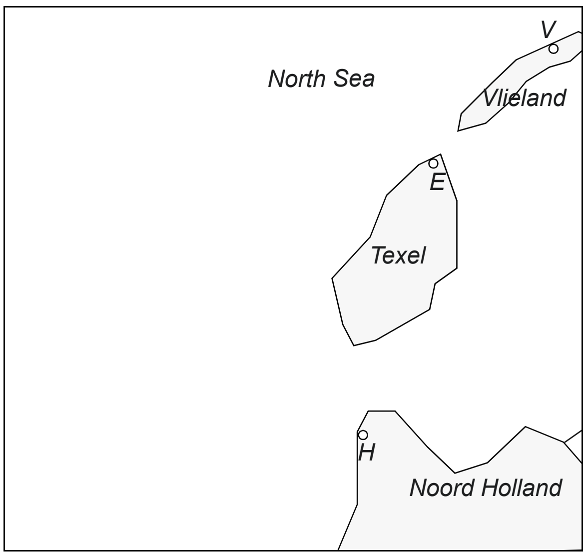
> *the condition $d(M, A) = d(M, B) = d(M, C)$ is split up in two conditions, namely $d(M, A) = d(M, B)$ and $d(M, B)= d(M, C)$.*

For each of the two, all possibilities were investigated. That resulted in two lines. The intersection of the lines met both conditions and thus is the wanted point. Next, several examples. The first one is not particularly difficult, but does illustrate the core of the method very well.

8. Sketch very accurately a triangle with sides of 12, 9 and 17 cm. You can use ruler and compass.
 a. First sketch the side AB of length 12 cm. Point C now meets two conditions. Which?
 b. Now sketch – in the terminology of a minute ago – the lists of both officers.
 c. You now find two possibilities. Are these triangles different?

9. Wanted a triangle ABC with a triangle of which you know:
$|AB| = 10$, $\angle CAB = 45°$ and $|BC| = 8$.
 a. Sketch AB and again find C using the split method.
 b. Also here there are more possibilities for C. Still the situation is different than in the previous exercise. Explain this.

10. Apply the splitting method to the following *navigation* problem. A damaged ship is (still) afloat on the North Sea. The compass no longer works. To determine their location to ask for help, the crew looks at the lighthouses of Huisduinen (H, at Den Helder), het Eierlandse Gat (E on the north point of Texel), and of Vlieland (V on the northeast side of Vlieland). The position of the ship is called S. The angles $\angle HSV$ and $\angle HSE$ can be measured easily from the ship. They are: 90° and 45°. Now show how position S of the ship on the map can be determined.



'splitting conditions' in short

Sometimes *one* point with *two* properties needs to be found. Then it is often easy
– to find the figure of all points with one property;
– and also to find the figure of all points with the other property.
A point at the cross-section of the two figures then has both properties.
In a manner of speaking: the point is the one on the lists of both officers.

Example: finding a point that has two given distances to other points.

Another example: The familiar method of finding the center of the circumcircle (the so-called 1-1-bis manner) also works with separating the two conditions: lie on the *pbs(A, B)* and lie on *pbs(B, C)*.

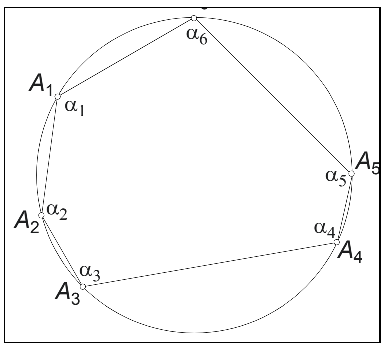
You see the split method is not only suitable for finding a certain point, but also for certain proofs. In a lot of cases it is the 1-1-bis manner, if the conditions, which are separated, are of the same type.

15. Plagiarism

If you are writing a book and then copy pieces from another writer (or from the internet), maybe slightly altered, but without acknowledgment, then you are plagiarizing. Even if you take something slightly modified so it won't show, it remains plagiarism. Also if you copy someone's ideas without acknowledgement, you plagiarize.

In this section you practice mathematical plagiarism in proving. Do not worry, the way we do it here, is a honorable cause. It works like this: You have a problem in front of you and it reminds you of something you did before. You look whether the proof can be used in the new situation, of course adapted to it.
Here are several problems where you can apply the method of plagiarism.

11. The theorem of the *cyclic hexagon*:
If the vertices of hexagon $A_1A_2A_3A_4A_5A_6$ lie on a circle and α_1, α_2, α_3, α_4, α_5 and α_6 are the angles, the following applies:
$\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6$
Prove this by plagiarizing the proof of theorem 9 of the cyclic quadrilateral as mentioned in *Distances, Edges and Domains*, page 105.



a. Prove this by dividing the hexagon in two cyclic quadrilaterals and using the theorem of the cyclic quadrilateral.

b. One of these two proofs is complete, for the other a case distinction should be made. What is up with that?

c. Formulate a *theorem of the cyclic thousand-angle*. You do not need to be very original, plagiarize.

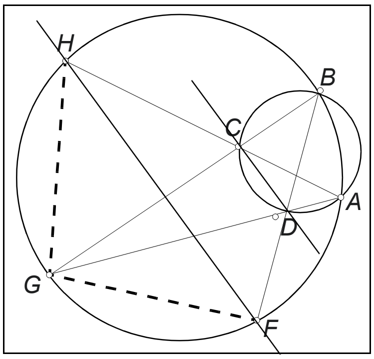
d. Does such a theorem also exist for a *cyclic pentangle* or a *thousand and one* sided figure?

The next two problems are variations on exercises from this chapter.

12. Given: two intersecting circles with points as shown in the figure.

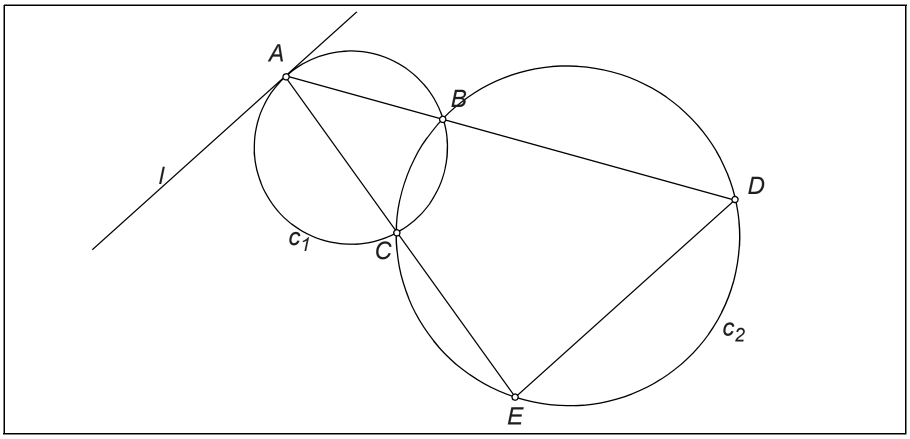
a. Show: $HF \parallel CD$.
(The position of the points was slightly different from the original exercise, but the differences are not very big.)

b. Extra for this figure, independent from question a: show that $|FG| = |GH|$.



In the next exercise the figure is a little simpler than the original on page 212. The exercises have the parallelism and the two circles in common.

13. In the figure below l is the tangent line in A to circle c_1.
B and C are the intersections of circles c_1 and c_2.
D and E are the other intersections of AB and AC with circle c_2.



Prove that l is parallel to DE.

16. Review on this chapter

finding proofs

In this chapter you have seen three different methods to find a proof.

three search methods
Sometimes you need to look for appropriate ***links***. You use these if you need to prove that two things are equal to each other, and there is no theorem that shows this relation immediately. Sometimes you need more than one link.
The method ***split of conditions*** can be used if a point has multiple properties, or if you need to prove that three figures (lines or circles) concur.
The method ***plagiarism*** was nothing new. The most important is that you get used to reusing elements from other proofs.

success never guaranteed
It is not the case that you will immediately find a proof by applying one of the methods. It also is not true that a proof cannot be found in several ways. The methods do not exclude each other, they supplement each other.

preview

In the next chapter you will explore several figures using a computer program. This leads to all kinds of conjectures, which are unproven assertions of which you have a strong impression that they are true, without having proved them. In the chapter after that you apply the techniques of this chapter to prove these conjectures.