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6. THE NOBLE ART OF PROBLEM SOLVING

A Critical View on a Swedish National Test

INTRODUCTION

Problem solving is the fundament of all applied disciplines in Science and Technology, where theoretical understanding is necessary for knowledge progression and systematic structuring of a task is the key to how to solve the problem. In ancient cultures, *e.g.*, the Babylonian, the Egyptian, and the Greek, a great interest in problem solving evolved (Encyclopedia Britannica 2011; Stanford Encyclopedia, 2011). Later on, the same interest was adopted by philosophers in India and China as well as in the Middle East, and they all had in common to think that problem-solving ability was a goal to strive for. Also in modern times, problem solving is the approach to scientific and technical development. As a result, problem solving is payed great attention in the Swedish school curriculum. Thus, the current Swedish elementary school is supposed to promote the students' learning to listen, discuss, argue, and use their knowledge to

– Formulate and test assumptions and solve problems,
– Reflect on experiences, and
– Critically examine and value statements and relationships

(Skolverket, 1994a, p. 12)

as well as to ensure that each student masters basic mathematical principles and can apply them in everyday life. The same applies to upper secondary school (Skolverket, 1994b, p. 9). Hence, the Swedish school system shows an outspoken ambition to improve the problem-solving ability of the students.

Nevertheless, problem solving is considered by many students as a difficulty that needs to be overcome by memorizing. Students want to compensate the lack of understanding by memorizing rules and procedures and by trying to learn models and patterns (Idris, 2009; Bergsten et al., 1997; Malmer, 1990). They try to find procedures and algorithms and substitute numerals to a formula they learn by heart in order to reach the solution without understanding the real mathematics that is lying behind (Gordon, 1997; Miller, 1992; Hiebert & Lefevre, 1986).

"Show me all the formulas and how to do it. Then I do not need to understand!" This attitude can be described as an algorithmic approach to learning and has very little to do with problem solving. It relies on the idea that repetition of recipes should

I.-B. Skogh & M.J. de Vries (Eds.), Technology Teachers as Researchers, 101–117.

give equally good results regardless the character of the problem, and that tools such as calculators, computers, and handbooks, will solve the problem through magical power. Another behavior that occurs in the context of problem solving is "piloting", where students try to realize how a task is designed in order to find an answer without understanding the real content of the problem (Löwing & Kilborn, 2002). For those students, it is more important to reach an answer by putting together many operations using symbols without a deep understanding of the mathematical task (Oaks & Rose, 1992).

Curcio (1987) gives an anecdote about this.

> One of my favorite anecdotes [...] is a "typical" quotation from a school child to his father: "You see, Daddy: I am very good in arithmetic at school. I can do addition, subtraction, multiplication, division, anything you like, very quickly and without mistakes. The trouble is, often I don't know which of them to use." (Curcio, 1987, p. 39)

Curcio (1987) wants to prevent an algorithmic approach to problem solving by implementing a strategy to improve the problem-solving ability of students. They should rather understand 'why it should be done' than 'what to do'. Several authors (*e.g.*, Curcio, 1987; Bergsten *et al.*, 1997; Hagland *et al.*, 2005) have tried to emphasize the importance of students' capability of interpreting the problem and use various forms of expression. The use of different forms of expression may be a way to overcome the difficulties that students experience in the translation of a mathematical problem from an everyday language into mathematical symbols and reasoning. Other authors (*e.g.*, Pólya, 1957; Malmer, 1990 and 1999; Emanuelsson *et al.*, 1991, 2000; Birch *et al.*, 2000; Sarrazy, 2003) present strategies to facilitate students' work in problem solving. One of the most well-known strategies is found in the classical book by Pólya (1957) "How to Solve It."

The question is whether theoretical approaches to problem solving are easily accessible for teachers or students. There should be an easy method for the noble art of problem solving – a structural method that all students can embrace. In the past, structuring a problem was considered an indispensable prerequisite for success in solving it. This was valid for any problem of mathematical character, regardless of whether it was a purely mathematical task or an application in science, technology, or engineering. It is not absolutely certain that school children or college students of the past agreed with this, but they had no other choice but to abide by their teacher's instructions.

In current Swedish school, the requirement of structuring a problem seems to be outdated. This applies not only to the compulsory school system, but also to college and university education. Many students seem to believe that it is enough to produce a reasonably correct answer to a problem, without actually describing how to get there. Their minds are shifted from the essential task of mathematics, which is reasoning and thinking, to be focused only on producing a solution. Consequently, they give more significance to the answer and not to the process itself (Feinstein, 2006).

Some students also believe that an incorrect reasoning, or reasoning with incomplete or even inaccurate justification, can be considered acceptable as long as the answer is correct.

> Many learners, either naturally or through conditioning, tend to get "obsessed" with the answers to a problem. This can happen to such a degree that the actual process involved in obtaining that answer can begin to seem irrelevant […] when the teacher does not give the student full credit, it may be confronted with a confused look and a response of, "why didn't I get full credit, I got the right answer?". (Feinstein, 2006, p.302)

The question is whether such an approach is acceptable. Teachers need to shift the attention of the students from this algorithmic attitude to mathematical thinking. As a result, students will become "less preoccupied with finding the answers and more with the thinking that leads to the answers" (Anthony & Walshaw, 2009; Fraivillig et al., 1999).

In applied subjects relying on mathematical reasoning, such as science or technical subjects, a theoretical understanding is the absolute basis for knowledge progression. One area of a subject is strictly based on another, and weak basic skills in one area will inevitably lead to failure in the following. Often the reason for an inadequate knowledge base is that the student did not understand what reasoning led to the answer to a question – the main thing seems to be that the produced answer is consistent with the answer given at the end of the text book (Feinstein, 2006).

To systematically structure the task is the key method for a student to sort out what the task is about, what theoretical reasoning can be applied, how to solve the problem based on these theoretical grounds, and, finally, solve the problem in a trustworthy and descriptive way. A good idea is to imagine somebody trying to follow the reasoning without any help other than the description/solution produced by the student. An even better idea is to assume that the reader should be able to repeat the exercise based on the methodology provided in the description by the author. Usually, this is a requirement for the content of scientific articles in science and engineering, *i.e.*, they must be possible to repeat. So why not teach students at an early stage to relate to problem solving in the same way?

TO TELL A GOOD STORY

Most people are fascinated by a good story. Nothing can capture a young child's attention as when someone is telling a story. With sparkling eyes the child falls into the story and eventually becomes a part of it in his own imagination. When the story is over, the call *Again! Please*, *again!* or *One more time!* from the child is inevitable. And if the narrator does not repeat the story word by word, sharp criticism is directed from the young listener.

All educational activities are based on this effect. It is a task for the teacher to capture interest by presenting knowledge in narrative form in an understandable, methodically structured, and trustworthy way. Moreover, if it is possible for the teacher to dress the concept in a fascinating costume, the educational foundation to open the door of learning is obtained.

Knowledge of mathematical or applied nature can also be described for students in this way. It is precisely this ability that characterizes a good teacher! Mathematics and its applications have the advantage of being structured in a logical manner. This feature provides the tools to present problem solving as a good story.

What is Characterizing a Good Story?

Every good story is logically structured in order to gradually capture the interest of the reader and engage the imagination. A good story can be divided in four essential parts: the preamble, the plot, the building of tension, and, finally, the resolution. In addition, two more parts may be added to brighten up the narrative: the prologue and epilogue. Now it is time to connect storytelling to problem solving of a mathematical task. Exactly the same reasoning can be performed in any mathematical application subject in science, technology, or engineering. Let us begin with the four necessary ingredients of a good story.

The preamble. The preamble consists of a narrative interpretation of the task using the problem description. Often it can be a rather wrapped-up task – a task where the key issue and input data are mixed with irrelevant or less important facts. Such descriptions are not unusual in an engineer's everyday life, particularly in consultative work. Generally, a customer or client cannot resolve the core issue, but gives all known facts in an unsorted way. It is thus expected of the consultant to be able to sift data and other information, in order only to use facts that are important and give up irrelevant information. In the preamble, also a discussion of what is relevant and needed to solve the problem is necessary. Irrelevant information is confusing, but one can seldom assume that a task only contains facts essential for the solution of the task.

The plot. The plot is an examination of the information extracted in the preamble. This includes the application of theories – mathematical theories if the task is a purely mathematical task, or science or technical theories and their underlying mathematics if the task is of applied nature. The plot also contains the theoretical/mathematical solution of the problem with all assumptions, statements and disposition.

The building of tension. In order to increase the tension, in the hope of soon being able to see the result of previous intellectual effort, the story now turns into a phase of utilizing the given data. Possibly, additional data have to be retrieved, however not from the problem description but from common sources such as tables and handbooks, manuals, or other publicly available sources. Numeric values are now processed according to the theoretical plan that emerged in the plot. At this stage, there is a risk for many mistakes, which is considered to be part of the tension.

Numerical processing of data may also involve digital tools for the calculations, e.g., calculators or computers. However, the result is never better than the way in which relevant data are processed. To avoid careless behavior and fribble in this process is a challenge many students appreciate. A heartfelt desire to perform correct numerical processing of the data is a great way to increase the tension in this good story.

The resolution. Now to the final touch, i.e., the very resolution of the story. The answer to the mystery is presented and clarified. It is not only to get to the end of the numerical calculations, but also to provide the reader with information about what really is the answer to the problem. The answer should be given with emphasis to the problem by repeating the main issue and giving the answer in that context. Without this punch, the story will lose value and ends up in a haze. The reader should not be forced to scroll back in order to recall the main issue.

Now the story is basically told, but a for quality improvement, two additional steps could be added without much effort: the prologue and the epilogue. The more important of these two steps is the epilogue.

The epilogue. The epilogue is a pure control function. Now the answer should be checked to be reasonable with respect to the main issue of the problem. For example, in a science task a calculation of the size of a germ or a microbe can hardly correspond to the distance between the Earth and the Sun. Without this assessment of reliability in the result of the problem, mistakes will not be discovered by the problem solver and storyteller, but will probably have the reader to ridicule the whole story. Verification of the answer as reasonable for the problem is a fairly simple task that students rarely understand the importance of, or at least use very sparingly. In the epilogue, the storyteller should also ensure that the answer is representative for the problem issued, and if everything that has to be considered is taken into account. This retrospect gives an opportunity for reflection.

The prologue. A prologue should naturally appear first in the story. Here, the task is repeated literally. By means of the prologue, the story not only becomes complete and independent of other texts, but the prologue will also be helpful in the preamble to interpret the problem and make a personal description of the task.

AN EXAMPLE FROM A SWEDISH NATIONAL TEST

In order to clarify the methodology proposed for the noble art of problem solving, an example is selected from a national test in Mathematics A, *i.e.*, the first mathematics course in Swedish upper secondary school. The selected national test is an open document as evidenced by the following statement:

> The Swedish National Agency for Education has decided in 2010–12-07 that Mathematics test A for the spring semester of 2010 will not be reused. (Skolverket, 2011a, p.1)

The test consists of two parts, Part I and Part II. Part I is a so-called calculator-free part with 14 tasks to be solved, of which 12 tasks only require answers without underlying calculations or analysis. The student has 90 minutes available to solve this part of the national test. As an example for the noble art of problem solving task number 14 was chosen, partly because it is text based and partly because it is the only MVG task in Part I. MVG is the highest grade given in the test corresponding to Excellent. Although Part I is called calculator-free, it is allowed to use a calculator when solving task number 14.

An important reason for the choice of task number 14 as an example in this article, is the mathematical character of the problem which is of great importance in most technical subjects. The problem is geometrical and concerns the volume of cylinders. Such a problem, and the methodology of solving it, is easily transferred as part of the content in general technology subject as well as specialized subjects, *e.g.*, mechanics, vehicles, energy systems, heat generation and distribution, water distribution and sewage, building construction, electricity, electronics, and many more. This transfer is even more obvious in university engineering programs.

The student is informed that task number 14 is an investigating task, which is presumed to take longer time than the other tasks of part I. In a box under the task itself in the test booklet, the student is informed about what the teacher must take into account in the assessment. Thus, the following is read by the student in connection with the mathematical problem of task number 14 (translated to English by the authors):

When assessing your work, the teacher will take into account

- what mathematical knowledge you have shown and how well you have performed the task
- how well you have explained your work and motivated your conclusions
- how well you have presented your work. (Skolverket, 2011a, p.2)

The Problem

Task number 14, entitled *The rolled paper*, is described in both text and figures. The text-based part of the task is the following (translated to English by the authors):

A rectangular paper can be rolled up into a pipe (a cylinder) as shown in the figure *(omitted here)*.

A pipe is made from a quadratic paper with the side of 10 cm.

- The pipe diameter becomes 3,2 cm, approximately. Determine the volume of the pipe (cylinder).
- Show that the pipe diameter is 3,2 cm, approximately, when the paper side is 10 cm.
- If the length and width are different in size, two different pipes (cylinders) can be produced depending on how the paper is rolled.

- Using rectangular papers of dimensions 10 cm × 20 cm, two different pipes are produced. Determine the volumes of the two pipes (cylinders).
- Compare these two volumes and determine the relationship between volumes.
- Examine the relationship between the cylinder volumes of papers with other dimensions of the sides. What affects the volume ratio between the high and low cylinder?
- Show that your discovery is valid for all rectangular papers.

(Skolverket, 2011a)

To this information some pictures are given showing different ways of rolling the paper.

Assessment Instructions

The assessment instructions for teachers (Skoverket, 2011b) state that task number 14 should be aspect assessed using a matrix. The starting point must be a positive assessment in which students receive credits for "the merits of the solution and not be penalized for errors and flaws" (p. 4). However, only answers without justification will not be given any credits.

Furthermore, the following guidelines (translated to English by the authors) for the assessment of task number 14 are given:

For full score, a correct solution with acceptable answer or conclusion is required. The presentation must be sufficiently detailed and articulated in such a way that the reasoning easily can be followed. A correct method or explanation of how the task can be solved will get credits, although followed by an error such as miscalculation. If the student also completes the task correctly, more credits are given. (*ibid.*)

Student Work G – A Pattern for Assessment

In the assessment instructions for teachers (Skolverket, 2011b) seven handwritten examples of students' solutions to task number 14 are given, indicating different levels of achieved quality levels. The solutions are named "Student Work" (followed by a subsequent letter of order) to emphasize that this is nothing else but realistic examples of how the solutions may look like. Hence, in this document Student Work A to Student Work G are found. The fact that real student solutions are chosen as examples in the assessment instructions could suggest that these examples do not necessarily represent an ideal image of the solutions for the Swedish National Agency for Education. Nevertheless, these are the only examples of guidance given to the grading teachers.

In this article we have chosen to focus on the student work that will reach the highest quality level and score the grade MVG (Excellent) according to the Swedish National Agency for Education, *viz.* Student Work G. In the following, the information in Student Work G is reproduced (and translated to English where necessary by the authors) (p. 16):

1. $\frac{10}{\pi} \approx 3{,}18 \qquad 1{,}59^2 \cdot \pi \cdot 10 = 79{,}42\,\text{cm}^3$

2. $\frac{10}{\pi} \approx 3{,}2$

3. $\frac{10}{\pi} \approx 3{,}18 \qquad 1{,}59^2 \cdot \pi \cdot 20 = 158{,}85\,\text{cm}^3$

4. $\frac{20}{\pi} \approx 6{,}39 \qquad 3{,}195^2 \cdot \pi \cdot 10 = 320{,}69\,\text{cm}^3$

5. $\frac{320{,}69}{158{,}85} = 2{,}0198\,(\approx 200\%)$

   One of the volumes is twice as large. (Small difference due to decimals)

   $$\frac{\frac{10}{\pi}}{\frac{20}{\pi}} = 0{,}5 \qquad \frac{\frac{20}{\pi}}{\frac{10}{\pi}} = 2$$

6. Paper 30×40 cm

   $$\frac{30}{\pi} \approx 9{,}55 \qquad \left(\frac{9{,}55}{2}\right)^2 \cdot \pi \cdot 40 = 2865{,}21 \text{ cm}^3$$

   $$\frac{40}{\pi} \approx 12{,}73 \qquad \left(\frac{12{,}73}{2}\right)^2 \cdot \pi \cdot 30 = 3818{,}28 \text{ cm}^3$$

   $$\boxed{\frac{2865{,}21}{3818{,}28} = 0{,}75 \qquad \frac{30}{40} = 0{,}75}$$

7. Paper $x \times y$

   $$\frac{x}{\pi} \qquad \left(\frac{x}{2\pi}\right)^2 \cdot \pi \cdot y = \frac{x^2 y}{4\pi} = V_1$$

   $$\frac{y}{\pi} \qquad \left(\frac{y}{2\pi}\right)^2 \cdot \pi \cdot x = \frac{y^2 x}{4\pi} = V_2$$

   $$\frac{V_1}{V_2} = \frac{\frac{x^2 y}{4\pi}}{\frac{y^2 x}{4\pi}} = \frac{x}{y}$$

   The relation between sides = the relation between volumes

DISCUSSION

Regardless of whether the solutions published in the assessment instructions are produced by students or not, one can conclude that these give signals to the grading teachers about how problems should be solved. The Student Work G is an almost unparalleled taciturn piece of art. This phenomenon is quite typical for students to transfer also to subjects using mathematics in application. In Student Work G, one can only with greatest effort follow the logical structure and reasoning that the student wants to communicate. However, it should be noted that some solutions in the assessment instructions for task number 14 are significantly more extensive in both running text and clarifying figures. Nevertheless, Student Work G should be discussed on the grounds that it is used as an example for the highest quality level by the Swedish National Agency for Education. There is no reason to doubt that the given solution of Student Work G is presenting the mathematical skills that are expected for the grade MVG, *i.e.*, Excellent. It is the communicative element in the student work that is criticized.

The assessment instructions for teachers (Skolverket, 2011b) are providing the following general guideline (translated to English by the authors):

> The report should be sufficiently detailed and articulated in such a way that the reasoning easily can be followed (p. 4).

Furthermore, the assessment instructions for teachers are stating as one of the MVG qualities that the student is presenting a well-structured solution using the correct mathematical language (p. 5). In the assessment matrix to task number 14, this is further emphasized through the following information on structure and mathematical language:

> How clear, precise, and complete the student's report is, and how well the student is using mathematical terms, symbols and conventions (p. 6).

Student Work G – as a Good Story

In the following, each part of the Student Work G is examined, based on the clarification above by the Swedish National Agency for Education, regarding how clear, precise, and complete the student's report is. Since the student has chosen to number the individual mathematical elements of task number 14, so will this analysis follow the chosen numbering. For the sake of clarity, also the original problem description is given element by element marked with the word Problem.

Problem #1:

A pipe is made from a quadratic paper with the side of 10 cm. **Prologue**
The pipe diameter becomes 3,2 cm, approximately.
Determine the volume of the pipe (cylinder).

Solution according to Student Work G:

$\frac{10}{\pi} \approx 3{,}18 \qquad 1{,}59^2 \cdot \pi \cdot 10 = 79{,}42\,\text{cm}^3$

Solution as a good story:

Calculate the volume of the cylinder. **Preamble**
The height of the cylinder = h
Quadratic paper => all sides are equal
The diameter of the cylinder = d, the radius = r

Theoretical expression for the volume of the cylinder: **Plot**

$$V = \pi \cdot r^2 \cdot h \qquad where \quad r = \frac{d}{2}$$

$$\Rightarrow V = \pi \cdot \left(\frac{d}{2}\right)^2 \cdot h$$

Numerically: **Building of tension**

$$d = 3{,}2\,\text{cm},\ h = 10\,\text{cm}$$
$$\Rightarrow$$
$$V = \pi \cdot \left(\frac{3{,}2}{2}\right)^2 \cdot 10 = 80\,\text{cm}^3$$

Answer: The volume of the cylinder is 80 cm^3. **Resolution**
Is the answer reasonable and **Epilogue**
representative for the problem? Yes!
Is everything that should be considered
taken into account? Yes!

Problem #2:

Show that the pipe diameter is 3,2 cm, approximately, **Prologue**
when the paper side is 10 cm.

Solution according to Student Work G:

$\frac{10}{\pi} \approx 3{,}2$

Solution as a good story:

Show that $d \approx 3{,}2\,\text{cm}$ **Preamble**
if $x = 10\,\text{cm}$ (the side of the quadratic paper)
Theoretical expression for the periphery of the cylinder: **Plot**

$$x = \pi \cdot d$$
$$\Leftrightarrow$$
$$d = \frac{x}{\pi}$$

Numerically: **Building of tension**

$$d = \frac{10}{\pi} = 3{,}18 \approx 3{,}2\,\text{cm}$$

Answer: The diameter of the cylinder is approx. 3,2 cm. **Resolution**
Is the answer reasonable and **Epilogue**
representative for the problem? Yes!
Check: $3{,}2 \cdot \pi = 10$ (*cf.* $x = 10\,\text{cm}$)
Is everything that should be considered
taken into account? Yes!
Problem #3:
Using rectangular papers of dimensions 10 cm × 20 cm **Prologue**
two different pipes are produced.
Determine the volumes of the two pipes (cylinders).

Solution according to Student Work G:

$$\frac{10}{\pi} \approx 3{,}18 \qquad 1{,}59^2 \cdot \pi \cdot 20 = 158{,}85\,\text{cm}^3$$

$$\frac{20}{\pi} \approx 6{,}39 \qquad 3{,}195^2 \cdot \pi \cdot 10 = 320{,}69\,\text{cm}^3$$

Solution as a good story:
Two papers 10 × 20 cm makes cylinders A and B. **Preamble**
x_A = 10 cm, h_A = 20 cm
x_B = 20 cm, h_B = 10 cm
where x_A and x_B is the periphery of each cylinder.
Calculate V_A and V_B.
Theoretical expression for the volume of the cylinder: **Plot**

$$V = \pi \cdot r^2 \cdot h \quad \textit{where } r = \frac{d}{2} \textit{ and } d = \frac{x}{\pi}$$

$$\Rightarrow V = \pi \cdot \left(\frac{d}{2}\right)^2 \cdot h \quad \Leftrightarrow \quad V = \pi \cdot \left(\frac{x}{2\pi}\right)^2 \cdot h$$

Numerically: **Building of tension**

Cylinder A: $V_A = \pi \cdot \left(\frac{x_A}{2\pi}\right)^2 \cdot h_A = \pi \cdot \left(\frac{10}{2\pi}\right)^2 \cdot 20 = 159{,}15 \approx 160\,\text{cm}^3$

Cylinder B: $V_B = \pi \cdot \left(\frac{x_B}{2\pi}\right)^2 \cdot h_B = \pi \cdot \left(\frac{20}{2\pi}\right)^2 \cdot 10 = 318{,}31 \approx 320\,\text{cm}^3$

Answer: The volumes of the cylinders **Resolution**
are 160 cm^3 and 320 cm^3, respectively.

Is the answer reasonable and representative for the problem? Yes! **Epilogue**

Is everything that should be considered taken into account? Yes!

Problem #4:

Compare these two volumes and determine the relationship between volumes. **Prologue**

Solution according to Student Work G:

$$\frac{320,69}{158,85} = 2,0198 (\approx 200\%)$$

One of the volumes is twice as large.

(Small difference due to decimals)

$$\frac{\frac{10}{\pi}}{\frac{20}{\pi}} = 0,5 \qquad \frac{\frac{20}{\pi}}{\frac{10}{\pi}} = 2$$

Solution as a good story:

Calculate the ratio between cylinder volumes, *e.g.*, $\frac{V_B}{V_A}$ **Preamble**

V_A = 159,15 cm^3 and V_B = 318,31 cm^3 according to Problem #3, taking the more precisely calculated values. **Plot**

$$\frac{V_B}{V_A} \approx \frac{318,31}{159,15} = 2$$

Building of tension

Answer: The ratio between volumes of the cylinders is 2, *i.e.*, one of the cylinder volumes is twice as large. **Resolution**

Is the answer reasonable and representative for the problem? Yes! However, there is a small uncertainty due to truncated values. **Epilogue**

Is everything that should be considered taken into account? Yes!

Problem #5:

Examine the relationship between the cylinder volumes of papers with other dimensions of the sides. **Prologue**

What affects the volume ratio between the high and low cylinder?

Solution according to Student Work G:

Paper 30×40 cm

$\frac{30}{\pi} \approx 9,55 \qquad \left(\frac{9,55}{2}\right)^2 \cdot \pi \cdot 40 = 2865,21 \text{ cm}^3$

$\frac{40}{\pi} \approx 12,73 \qquad \left(\frac{12,73}{2}\right)^2 \cdot \pi \cdot 30 = 3818,28 \text{ cm}^3$

$\boxed{\frac{2865,21}{3818,28} = 0,75 \qquad \frac{30}{40} = 0,75}$

Solution as a Good Story:

Choose some papers of other dimensions **Preamble**
and calculate the volume ratios of the cylinders.
What affects the volume ratio between the high and low cylinder?

Paper 1: 30×40 cm **Plot**
Paper 2: 30×50 cm

$$V = \pi \cdot \left(\frac{x}{2\pi}\right)^2 \cdot h$$

Numerically: **Building of tension**
Paper 1:

$x_{1,A}=30$ *cm*, $h_{1,A}=40$ *cm*

$$V_{1,A} = \pi \cdot \left(\frac{x_{1,A}}{2\pi}\right)^2 \cdot h_{1,A} = \pi \cdot \left(\frac{30}{2\pi}\right)^2 \cdot 40 \approx 2865 \text{cm}^3$$

$x_{1,B}=40$ *cm*, $h_{1,B}=30$ *cm*

$$V_{1,B} = \pi \cdot \left(\frac{x_{1,B}}{2\pi}\right)^2 \cdot h_{1,B} = \pi \cdot \left(\frac{40}{2\pi}\right)^2 \cdot 30 \approx 3820 \text{cm}^3$$

$$\frac{V_{1,A}}{V_{1,B}} \approx \frac{2865}{3820} \approx 0,75$$

Paper 2:
$X_{2,A} = 30$ cm, $h_{2,A} = 50$ cm

$$V_{2,A} = \pi \cdot \left(\frac{x_{2,A}}{2\pi}\right)^2 \cdot h_{2,A} = \pi \cdot \left(\frac{30}{2\pi}\right)^2 \cdot 50 \approx 3581 \text{cm}^3$$

$X_{2,B}$=50 cm, $h_{2,B}$=30 cm

$$V_{2,B} = \pi \cdot \left(\frac{x_{2,B}}{2\pi}\right)^2 \cdot h_{2,B} = \pi \cdot \left(\frac{50}{2\pi}\right)^2 \cdot 30 \approx 5968\,\text{cm}^3$$

$$\frac{V_{2,A}}{V_{2,B}} \approx \frac{3581}{5968} \approx 0,60$$

Answer: In cylinders made from different sized rectangular papers, the volume ratio between the high and the low cylinder is 0,75 for a 30×40 cm paper and 0,60 a 30×50 cm paper. It seems to be the same as the ratio between the length of the paper sides. **Resolution**

Is the answer reasonable and representative for the problem? Yes! **Epilogue**

However, the conclusion in the last sentence of the answer is somewhat vague, but is consistent with the answer of Problem #4.

Is everything that should be considered taken into account? Yes!

Problem #6:

Show that your discovery is valid for all rectangular papers. Prologue

Solution according to Student Work G:

Paper $x \times y$

$$\frac{x}{\pi} \qquad \left(\frac{x}{2\pi}\right)^2 \cdot \pi \cdot y = \frac{x^2 y}{4\pi} = V_1$$

$$\frac{y}{\pi} \qquad \left(\frac{y}{2\pi}\right)^2 \cdot \pi \cdot x = \frac{y^2 x}{4\pi} = V_2$$

$$\frac{V_1}{V_2} = \frac{\frac{x^2 y}{4\pi}}{\frac{y^2 x}{4\pi}} = \frac{x}{y}$$

The relation between sides = the relation between volumes

Solution as a good story:

Show that the presumed conclusion in the answer of Problem #5, i.e., the ratio between volumes is equal to the ratio between sides, is valid for all rectangular papers. **Preamble**

Theoretical expressions: **Plot**

Paper x × y

$$V = \pi \cdot \left(\frac{x}{2\pi}\right)^2 \cdot h$$

$x_A = x,\ h_A = y$

$$V_A = \pi \cdot \left(\frac{x_A}{2\pi}\right)^2 \cdot h_A = \pi \cdot \left(\frac{x}{2\pi}\right)^2 \cdot y = \frac{x^2 \cdot y}{4\pi}$$

$x_B = y,\ h_B = x$

$$V_B = \pi \cdot \left(\frac{x_B}{2\pi}\right)^2 \cdot h_B = \pi \cdot \left(\frac{y}{2\pi}\right)^2 \cdot x = \frac{y^2 \cdot x}{4\pi}$$

The ratio between volumes:

$$\frac{V_A}{V_B} = \frac{\frac{x^2 \cdot y}{4\pi}}{\frac{y^2 \cdot x}{4\pi}} = \frac{x^2 \cdot y}{y^2 \cdot x} = \frac{x}{y}$$

Q. E. D. **Building of tension**

Answer: The proof is showing that the presumed conclusion made in Problem #5, i.e., the ratio between volumes is equal to the ratio between sides, is valid for all rectangular papers. **Resolution**

Epilogue

Is the answer reasonable and representative for the problem? Yes! The result is consistent with previous results.

Is everything that should be considered taken into account? Yes! The proof is stringent.

CONCLUSION

The noble art of problem solving is similar to storytelling. In an educational situation where the teacher goes through a mathematical method or equivalent process of an application discipline in Science or Technology, the reader or listener should become mentally captured and gradually fascinated by the logical sequence which is such a delight in a good story.

In the best case, the storytelling in problem solving is divided into six steps:

- *The Prologue*: Recite the task literally.
- *The Preamble*: The storyteller's own description of the task. Discussion of what information given in the task is relevant for the solution process.
- *The plot*: The mathematical solution, with all assumptions, statements, and disposition, is presented.
- *The building of tension*: Numeric values are processed according to the preamble and the plot.
- *The Resolution*: The answer to the problem is presented and clarified.
- *The Epilogue*: The answer is checked to be reasonable and representative for the problem. A check if everything is taken into account that should be considered.

If this method is consistently practiced by teachers and students in mathematics education at all levels in the school system, it is easily transferred and utilized in any application subject in Science and Technology. In this way, problem solving will be more enjoyable for everybody, and the risk for mistakes in the process is minimized. Hopefully, the storytelling method in problem solving will lead to an enhanced interest among young people of higher education in Mathematics, Science, and Engineering.

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