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12. MATHEMATICIANS AND ELEMENTARY SCHOOL MATHEMATICS TEACHERS – MEETINGS AND BRIDGES

INTRODUCTION

> There is no question that mathematicians are going to be called on to teach teachers. The tide is already turning in this direction. (Sultan & Artzt, 2005, p. 53)

> As practitioners of the discipline, research mathematicians can bring valuable mathematical knowledge, perspectives, and resources to the work of mathematics education. (Bass, 2005, p. 430)

The involvement of mathematicians in mathematics education is as old as mathematics education itself. Very prominent mathematicians, such as Felix Klein and Hans Freudenthal, are considered precursors or even founding fathers of mathematics education as an academic field of study. Many well-known researchers in mathematics education started their career as research mathematicians, like Alan Schoenfeld and Günter Törner, to whom this volume is dedicated. Indeed, what could be more natural than mathematicians being intensively involved in mathematics education? However, it seems that after mathematics education established itself as a discipline, the role of mathematicians has been less prominent than expected. Paradoxically, mathematicians are often critical of this new discipline.

In the last decade, and possibly as a positive reaction to the unfortunate effects of what was called the "math wars," many avenues for dialogue have been initiated between research mathematicians and mathematics educators about the goals, content and pedagogy of the mathematics curriculum. However, the direct involvement of mathematicians in the practice of mathematics education, in teacher education for example, rarely receives careful scrutiny.

This chapter is an attempt to contribute to understanding the possible roles that might be played by mathematicians in mathematics teacher education for elementary school teachers. It describes and analyzes a professional development program (PD). The program is run by mathematicians (a research mathematician and graduate doctoral students from an internationally renowned Mathematics department) who teach in-service courses for elementary school teachers in Israel. These mathematicians work as a group, coordinating their lesson plans and collectively reflecting on them before and after implementation. The team works mostly on the basis of their mathematical insights with occasional consultations with mathematics education experts. The overarching goal of the course is to deepen and

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broaden teachers' understanding of central concepts in elementary mathematics. This course is a unique experience in Israel on a number of counts:

- Usually, instructors in professional development courses for elementary school mathematics teachers are experienced fellow teachers, or teacher leaders appointed by the Ministry of Education, or curriculum developers and occasionally mathematics educators.
- The knowledge and perspectives that the mathematician-instructors bring to the course, their modest experience with elementary school teaching, and their beliefs and attitudes regarding the nature of mathematics and its teaching are not at all typical of elementary school PD.
- The content of this PD is also unusual. Elementary math content knowledge is generally conceived as straightforward, in spite of research contradicting this view, e.g. (Ma, 1999). A common first reaction may be: "what else is there to learn about the four basic arithmetical operations?" PD for elementary school teachers tends to focus on pedagogical content knowledge (PCK) – how best to teach particular topics, how to address student errors and misconceptions, etc. In contrast, this PD aims to focus on subject matter content knowledge, as conceived by mathematicians who have little or no expertise in pedagogy.
- The instructors develop their own lesson plans. Many times they design their own exercises and problems and refine them in collective team discussions, in either face to face or virtual meetings. Also, after most of the lessons they produce reflective reports, analyzing what worked and what did not work.

This chapter is based on data collected during several lessons in these courses,[1] focusing on the mathematician-instructors – the professional knowledge they bring to bear, their attitudes and beliefs, and how all these impinge on the way they envision elementary mathematics, and how they influence their didactical choices and their teaching decisions. We document the emergence of insights, both mathematical and pedagogical in nature, which developed either during the instructors' preparation/reflection sessions or during interactions in class with the teacher-participants. The teachers' side of the story is no less interesting, and will be reported elsewhere.

In our analyses, we target both theoretical and practical contributions. Theoretically, we discuss the blending and interaction between types of knowledge towards a growing understanding of the construct of mathematical knowledge for teaching (MKT), e.g. Ball et al (2008). Practically, the PD described here may serve as the basis for future opportunities for mathematicians teaching elementary mathematics teachers.

BACKGROUND

This chapter is based on observations collected from two academic years, 2010–2011 and 2011–2012. The professional development course includes ten 3-hour sessions. There are separate tracks for grades 1–2 and for grades 3–6.

The first few sessions of 2010–2011 did not bode well. The mathematics instructors intended to focus on deepening and broadening the teachers' mathematical knowledge. One of the questions they asked themselves was "what do we know that the teachers do not?" Their answers to this question did not include "how to teach elementary mathematics" – a topic they were careful to avoid. Some instructors were very explicit about this point and stated openly to the participants: "I know nothing about teaching elementary math, I can't tell you how to teach it, and I won't." Once it was clear that the mathematicians' contribution would be in the realm of mathematical content, their next question was "what could we possibly contribute to the understanding of such apparently straightforward topics?" Their assumption was that part of the intricacies of teaching elementary mathematics, especially in the lower grades, lies in recognizing and addressing subtleties in the subject matter and its conceptual underpinnings. This was at the core of the expertise they brought to the course. Once they were explicit about it, the challenge was to expose these subtleties to the teachers in the PD. One of the first approaches the mathematicians adopted for selecting appropriate activities for the teachers was estrangement – a technique designed to gain new insights on the familiar by reflecting on the unfamiliar. Estrangement (*dépaysement* in French) is an anthropological term which literally means going out of one's country. It was used frequently by the anthropologist Lévi-Strauss, as described, for example, by Hénaff (1998), and has also been used in some studies of mathematics education (Barbin, 2011). Two typical examples of this approach, implemented in the PD, consisted of working on base-5 arithmetic in order to gain an appreciation of the structural subtleties of base 10, and reviewing cardinality and counting through learning to compare infinite sets. This approach – taking a step back to gain a broader perspective – may be typical of the way mathematicians think and work. However, it was not a great success with the teachers, who tended to judge such topics (base 5, infinite sets) as irrelevant to their teaching, and thus not at all what they were hoping to gain from the PD. They felt that what they most needed in order to improve their practice was practical tools, for example, activities they could take to class, tips for teaching particular topics, how to deal with student difficulties, etc. Grade 1-2 teachers, in particular, did not feel a need to deepen their understanding of the subject matter, which they considered quite straightforward. The teachers did not hesitate to voice their dissatisfaction with the mathematicians' approach, and the instructors felt they needed to adapt their approach to meet the teachers' expectations of *relevance*.

This mismatch of expectations could be seen as a sure promise of failure, with the subsequent feelings of frustration to be felt by instructors and teachers alike. However, the story evolved in a different and rather fruitful path.

In the 2011–2012 PD the instructors attempted to address the teachers' feedback from the previous year, but they did not completely adopt the teachers' views on what would make the PD relevant. Their interpretation of the demand for relevance was shaped by two factors: what they thought the teachers needed (better understanding of the content), and what they felt they could provide as mathematicians. They eventually came up with a number of activities which blended subject matter content knowledge (the mathematicians' expectation) and pedagogical content knowledge (the teachers' expectation).

The instructors' need to address the teachers' discontent with their unfulfilled expectations, coupled with the conviction that mathematical content should remain at the core of the course, made them re-think their courses of action and figure out how to address the former without renouncing the latter. We will analyze the instructors' moves to conciliate these seemingly opposing goals, and describe how this shaped what is now considered a successful PD program not only by the instructors and participants, but also by officials from the Ministry of Education.

We focus on how the mathematicians brought their mathematical knowledge and their beliefs about mathematics to bear on various aspects of teaching, as follows:

– Mathematical content
  - Unpacking elementary topics into their components
  - Unpacking tools for doing mathematics
  - Preparing for how elementary topics will eventually interact with future advanced topics on the horizon of the students' knowledge
– Pedagogical issues
  - Anticipating and addressing student difficulties, errors and misconceptions
  - Designing activities for the PD, bearing in mind how these activities might play out in the teachers' classrooms

MATHEMATICAL CONTENT – UNPACKING ELEMENTARY TOPICS

> I have observed, not only with other people but also with myself ... that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question. (Freudenthal, 1983, p. 469)

This quote reflects one of the central challenges in teaching mathematics, especially elementary mathematics, which includes "unpacking" the mathematical content – reviewing what it is made up of and what it really involves for a learner, appreciating the conceptual nuances, and disentangling the multiplicity of seemingly similar meanings for the same or connected concepts.

The following are examples of how the mathematicians unpacked some concepts in elementary math, armed with their knowledge of advanced mathematics and their experience practicing it. We show examples of two kinds of mathematical unpacking, one referring to specific subject matter concepts (e.g. counting), and another referring to the practices for doing mathematics (e.g. proofs and justifications, definitions). It may well be that mathematicians' knowledge of advanced mathematics is not strictly *necessary* for unpacking elementary concepts, nor is it sufficient, but it does appear to be highly instrumental. In some cases, we see how the mathematicians' knowledge not only pointed the way to unpacking concepts but interestingly, it also yielded insightful pedagogical implications.

Unpacking counting and the concept of number

There are two different definitions of natural numbers – the set theoretic definition (attributed to Frege and Russell) and the Peano axioms. One can clearly know, operate flawlessly with, and teach natural numbers without being aware of either of these definitions, but awareness of them proved to be productive in unpacking the concepts of number and counting. The set theoretic definition has more affinity with the act of counting objects in a set, whereas the Peano axioms, based on the successor operator, tend to be aligned with the act of counting by saying the numbers out loud one after the other without a specific reference to objects and without a specific goal of establishing the "cardinality of a given set" ("rote" counting). These are two different aspects of counting that children need to learn. Awareness of the two definitions, and of the equivalence between them, helped the mathematicians see the differences and connections between these two aspects, and it also contributed to their re-visiting of the basic operations of addition and subtraction and their properties. Rote counting is related to Peano's concept of *successor* (what comes next), whereas counting elements of a set is closer to Frege and Russell's construction, where the number 3 is equated with the equivalence class of all sets having 3 elements. Proving that Frege and Russell's construction satisfies the Peano axioms (something the mathematicians considered doing in the PD, but realized would not be relevant for the teachers) helps illuminate the connection between the two counting competencies. To begin with, the number 3 is an operator that acts on objects – "3 flowers," "3 birds," etc. Children eventually need to abstract the concept, and see the equivalence of all sets of a particular cardinality. This is very similar to the equivalence inherent in the set theoretic definition of numbers. This parallel between what the mathematicians need to prove and what the children need to understand helped the mathematicians see what there is to learn in this seemingly trivial topic, and the mathematical basis helped to make this explicit.

Comparing the cardinality of two sets (which set has more elements) can be based on counting, but it is in fact possible to make such comparisons without knowing how to count. It is possible to set up a 1-1 correspondence between the elements of the two sets, and see which – if either – has elements left over. Mathematics students typically encounter this principle in an under-graduate course in set theory, where 1-1 correspondences are used to compare infinite cardinalities. In fact, the existence of such a correspondence is taken to be the definition of equal cardinalities. Furthermore, 1-1 correspondence is a more fundamental concept than counting, since counting the objects in a set relies on setting up a 1-1 correspondence between the set's objects and the first natural numbers (1, 2, 3, . . .). Teachers often overlook this comparison strategy, perhaps due to their preoccupation with mastering the skill of counting, and may be completely unaware of the concept of 1-1 correspondence and its role in counting objects. This is another example of how their knowledge of advanced mathematics helped the mathematicians regain insight into the foundations of elementary mathematics. The question of how this mathematical content can be presented to the teachers is a separate issue, which will be addressed in the section on designing activities.

Addition is defined differently in the two constructions of numbers. In the set theoretic definition, addition is based on set unions – putting together two collections of objects. In the Peano approach, addition is based on the repeated application of the successor operator, namely starting from one number and counting-on as many times as the second addend indicates. The mathematicians found it is easier to prove the commutative principle in the set theoretic construction of numbers than by relying on the successor. For them, this implied something about how the property may be understood by children. Adding 11 to 2 (counting-on 11 starting from 2) does not feel at all the same as adding 2 to 11 (counting-on 2 starting from 11). In fact, it is not at all obvious why the results should work out to be the same! This corresponds to the difficulty in proving the commutative principle based on Peano's axioms. On the other hand, the union of two sets (one having 2 objects, one having 11) is symmetrical. Thus, on the basis of the set theoretic definition of numbers, the commutative principle is obvious and its proof is straightforward. Through this connection between mathematicians' definitions and children's models of addition, the mathematicians gained some pedagogical insight: the commutative property is more obvious in some contexts than in others, and should be introduced to children accordingly.

Unpacking the associative property of multiplication

The distinction between multiplication's commutative property, $a \times b = b \times a$, and associative property, $(a \times b) \times c = a \times (b \times c)$ may be confusing for students and teachers alike. The confusion may be related to the following: the combination of the two properties boils down to *when you need to multiply a list of numbers, you can do it in any sequence you like*. So why separate this simple statement into two distinct properties, if procedurally they seem indistinguishable? The answer is provided in university algebra – some mathematical domains (e.g. non-commutative groups) have one property and not the other, so they must be considered as distinct. It is questionable whether this argument would convince a student, or even a teacher. One of the instructors came up with a convincing argument without resorting to advanced mathematics. The example he worked with was: There are 5 buses, 40 children on each bus, and each child has 2 parcels, how many parcels are there in total? The teachers suggested a number of ways to calculate the result, including: $(5 \times 40) \times 2$, $5 \times (40 \times 2)$, and $(5 \times 2) \times 40$. The last of these calculations is the easiest, starting with the obvious (5×2). The instructor asked how we know that all of the above yield the same answer. There was no consensus – both the commutative and the associative properties were suggested. In fact, the third way of calculating follows from the first or the second by applying both the commutative and the associative properties, $5 \times (40 \times 2) = 5 \times (2 \times 40) = (5 \times 2) \times 40$, but the instructor did not take this formal route. Instead he returned to the problem and its contextual meaning. What does 5×40 represent? The total number of children. What does 40×2 represent? The number of parcels per bus. Either of these quantities may be the first calculation in our solution (an observation that in fact demonstrates the associative property). But what does 5×2 represent? It does not represent anything meaningful in the problem! So, we can explain the commutative property (perhaps by means of the array model), and we can explain the associative property (as demonstrated above), but the combination of the two – *multiply in any order you like* – is a conclusion that does not follow naturally from the meaning of problem. This helped illustrate to the teachers that there are indeed two distinct properties, having distinct explanations.

Unpacking equality

In students' early encounters with the equality symbol it is usually taken as a call for action, for example, "$7 \times 2 =$" is read as an invitation to carry out a calculation (see, for example, Saenz-Ludlow & Walgamuth, 1998). Mathematicians are aware of the sophisticated multiplicity of other meanings (e.g. Freudenthal, 1983), where equality is first and foremost an equivalence relationship. This became salient in the topic of division with remainder, where the equivalence breaks down. Adopting the American notation, $7 : 2 = 3R1$, but $3R1$ is also the result of $10 : 3$. May we conclude that $7 : 2 = 10 : 3$?! Conversely, $7 : 2$ and $14 : 4$ should be equal, but one is $3R1$ and the other is $3R2$. The implication is that in this context, the equality may only be read from left to right (implying a call for action), contrary to the most basic requirements of equivalence. This clash was so critical for the mathematicians that they actually engaged (themselves) in the task of inventing alternative notations to circumvent the problem, for example, $7 : 2 = 3R(1 : 2)$, which reminds us that the remainder (1) is a result of division by 2. Note how this notation may also be seen as a step towards fraction notation, since after becoming knowledgeable with fractions, we will eventually write $7 : 2 = 3\frac{1}{2}$.

Unpacking the concept of average

The common definition of average (arithmetic mean) learned at school is usually procedural – add all the numbers in a list, and divide by the number of numbers you added. The instructors, who tended to take a more conceptual approach to knowing and learning of mathematics, aimed at unpacking the concept and unfolding its multiple facets. For example, they decided to focus on alternative definitions of the concept. One instructor suggested: *given a list of numbers, the average is the number such that when you add up all the (signed) differences from it, you get 0*. This can be considered a definition nearer the meaning of average than the traditional definition, and in some cases it can be practical for finding the average, or for checking if a given number is indeed the average. This alternate definition mirrors a sequence that is typical of university mathematics – define a construct, investigate it to find its properties, and then define a new construct based on one of these properties. The new construct may be identical to the original one (if the property is necessary and sufficient, as is the case with the alternate definition of average), or a generalization of the original one (if the property is necessary but not sufficient).

Another alternative view of the average can specifically rely on a useful representation: *when two numbers are represented on the number line, their average is represented by their midpoint*. This fact is often overlooked, even when teaching in Hebrew where there is a strong semantic connection between the words for "average" and "middle" which share the same root (*memutza* and *emtza*, respectively).

The theoretical background that the mathematicians brought to this topic had many implications. It provided the flexibility to invent and justify ad hoc calculation strategies: for example, to find the average of 81, 87, 88, and 89 one can find the average of 1, 7, 8, 9, then add 80 to the result. It also contributed to their awareness of likely pitfalls. One example was the question raised by an instructor of whether in order to find the average of many numbers it is acceptable to partition them, find the average of each part, and calculate the average of the averages. Will it work? Always? Sometimes? How does this connect to the topic of weighted averages? In this case, unpacking the concept resulted in the identification of the teachers' fragmented knowledge of it. Based on a single instance where the average of averages gave the correct result, some of the teachers conjectured that this would always work. Inspired by the fact that this procedure does sometimes give the correct result, the mathematicians proceeded to further unpack the concept, to clarify under exactly what conditions this strategy yields the correct result. When partitioning the list of numbers that need to be averaged, one needs to give each partition its relative weight. Weighted averages is a topic all mathematicians are familiar with: for example, in the context of basic probability, where the expectation of a random variable is the average of all possible values weighted by their probability. The concept of average exemplifies the extent to which some topics of elementary school curriculum are just the tip of a very rich set of connected concepts. The mathematicians' ability to unpack that richness contributed to the identification and analysis of potential knowledge flaws and the understanding of what this concept entails, including the ideas presented above for promoting computational fluency and flexibility.

MATHEMATICAL CONTENT – UNPACKING TOOLS FOR DOING MATHEMATICS

> … what people do is a function of their resources (their knowledge, in the context of available material and other resources), goals (the conscious or unconscious aims they are trying to achieve), and orientations (their beliefs, values, biases, dispositions, etc.) (Schoenfeld, 2010, p. xiv)

Schoenfeld's theory of goal-oriented decision making is primarily concerned with in-the-moment teaching decisions, but can also be related to a much broader scope and applied to our analysis of the PD instructors' teaching decisions. So far we have described and analyzed examples of how the mathematicians' knowledge contributed to the unpacking of elementary mathematical content, and consequently to supporting some of their instructional decisions. However, meta-mathematical issues and mathematical practices, such as mathematical conventions, definitions, proofs, justifications, and explanations were just as important a teaching goal as the content. In this, the mathematicians were guided by their orientations, including their beliefs about the nature of mathematics and its established practices. In this section we describe how these meta-mathematical topics and mathematical practices were unpacked, and how such unpacking shaped decisions and actions.

Number naming conventions

What, if anything, is wrong with the number "thirty eleven?" Is it a correct result for the problem $27+14$? The teachers tended to see this as an unfinished procedure: the tens were added (30), and so were the ones (11), but regrouping of the 11 ones was neglected. The mathematicians' focus was different. They did not automatically assume that numbers must have unique names. Indeed, in some contexts one thousand nine-hundred eighty four may legitimately be named in English nineteen hundred eighty four. Non-unique names are not an intrinsic problem, as long as you feel comfortable with equivalence classes. The question of uniqueness led the mathematicians to a search for alternative naming conventions in a variety of languages. Consider, for example, the Welsh word for 78, which translates to – *two nines and three twenties* – or the Alamblak word for 87 – *twenty two-and-two, and five, and two*. These are indeed unusual naming conventions, but is there anything wrong with them? Do they have any advantages over our naming convention? The *mathematical* point is that, given the intrinsic arbitrariness of naming, we should not ask ourselves if a naming convention is correct, but rather how practical and how unambiguous it may be. The criteria for practicality that the mathematicians focused on were mathematical in nature – does the convention give each number a *unique* name, how well does the naming convention support estimation (*one-hundred less two* may give a better sense of the order of magnitude than *ninety-eight*), does the convention support simple lexicographical comparison (all numbers that begin "seventy..." are greater than all numbers that begin "sixty..."), and perhaps most importantly: how well is the naming convention aligned with the standard place value notation. This last point has cognitive and didactic implications as well – a naming system aligned with the place value notation may be easier to master, and may even support conceptual understanding of place value principles. Browning and Beauford (2011) found this to be the case with the Chinese naming convention.

Unpacking proof

The concept of mathematical proof is not at all trivial (see, for example, Lakatos, 1976). Nonetheless, mathematicians' ideas about what constitutes a mathematical proof are bound to feed into their unpacking of elementary mathematics. We will show some examples of such unpacking of *proof*, referring also to *definition*, which seems to be strongly linked to proof. For example, the way we show that a number is even depends on the way we define evenness in the first place. We note that in this context we clump together the concepts of proof, justification, and explanation.

There are many distinctions one can make between these concepts (e.g. Levenson & Barkai, 2011), but we are more interested in what they have in common.

Areas and perimeters of rectangles are topics in the elementary math curriculum. One issue is the relationships between these two concepts. Children should learn that in some conditions area grows with the perimeter, but not in all cases. Given a rectangle, it is generally possible to find one with greater perimeter and smaller area, or with smaller perimeter and greater area. Of all rectangles having a given perimeter, the square has the greatest area. This is a weak version of the well know isoperimetric inequality and it can easily be proven using the algebraic equivalence: $(x - a)(x + a) = x^2 - a^2 \leq x^2$, where $4x$ is the perimeter and x the side of the square, but this is not feasible using elementary school techniques. The mathematicians' commitment to proving mathematical claims (and not just stating them) was the motivation to search for a convincing geometric proof/justification of this claim, accessible by means of elementary school math. On the basis of some examples, they showed that whenever we extend one side (p) of the rectangle by 1 unit, and shorten the other side (q) by 1 unit, we add a narrow rectangle which increases the area by $(q - 1)$, and remove a narrow rectangle which decreases the area by p. As long as p is not shorter than q, the net result is a decrease in the total area. The teachers felt this proof was something they could take to their own classrooms. It also served to show why the square has the greatest area, and that the area decreases the more "squished" the rectangle is. In searching for a proof, and in coming up with this one, the mathematicians acted in a manner consistent with these beliefs:

– There should be no magic in mathematics. Every fact should have a proof.
– The proof must be comprehensible, based on what is already known.
– The proof should say something about *why* the statement is true.

A common enrichment activity is the famous problem of adding all integers from 1 to 100. Solving this problem by pairing numbers with equal sums ($1 + 100$, $2 + 99$, $3 + 98$ ...) is often attributed to the young Gauss. This process yields a general answer, $\frac{n(a_1+a_n)}{2}$, but there is a snag – the pairing process assumes an even number of addends. It is possible to patch up the proof for the odd case, but this is inelegant. One of the instructors presented a version of this problem in the PD. The task was well known to the teachers – finding the total number of Hanukkah candles required for the eight-night celebration (2 on the first night, 3 on the second, ... 9 on the eighth). This particular problem has an even number of addends, so the pairing solution works, but the instructor was aware of the incompleteness of the argument for the general case, and felt that a more general argument was called for, even though the teachers felt no need to generalize the problem. This commitment yielded an elegant proof inspired by a non-mathematical aspect of the problem story: there is an alternative Hanukkah tradition where the number of candles decreases, namely 9 on the first night, 8 on the second, and all the way down to 2 on the last. The proof was based on the following observation: If you light candles according to *both* traditions, you will light 11 candles on every night, for a total of 88 candles, 44 according to each tradition. This version of the proof works equally well for an even or an odd number of addends. Introducing this proof was consistent with the belief that:

– Claims and proofs should be as general as possible.

Unpacking definition

Even in elementary mathematics, many terms need to be defined accurately. Choosing a definition, or perhaps more than one definition, has pedagogical implications. What constitutes a definition in elementary mathematics? Should it specify what constitutes a non-example as well as what constitutes an example? Should it be parsimonious, or should it be redundant, namely rich in superfluous details? In what ways does it support us when we attempt to prove (or explain) that some object does or does not satisfy the definition? Should we have a multitude of definitions? If so, they should be equivalent, but how do we know they really are? What are the advantages and disadvantages of particular choices of definitions? These are some of the questions that the mathematician considered when choosing, offering, using or creating definitions. We will show several instances of the mathematicians grappling with these questions.

In a previous section we described how the concept of average was unpacked, aided by the instructors' knowledge of mathematics. We mentioned that the usual working definition for average was operational – add all the numbers and divide by the number of numbers you added. The instructors felt that this definition was deficient – it lacked a good feel for what the average really is. One instructor decided to provide a second definition: the number such that when you add up all the (signed) differences from it, you get 0. This definition draws attention to the fact that average is "between" the numbers – if some are greater than the average (positive differences), then for the sum of differences to be 0, others must be smaller than the average (negative differences). The instructor did not prove the equivalence of these definitions – this would have been difficult without algebra – but did show that in examples where the average had been calculated, it had this property. This approach to mathematical definitions is consistent with the beliefs:

– Definitions should say something meaningful about the concept being defined.
– Multiple definitions for a concept are desirable.
– It may be difficult to rigorously prove the equivalence of definitions, but this issue should not be ignored. Some motivation or justification should be provided.

There are many different ways one may define an even number, some based on the properties of numbers (e.g. divisible by 2 without remainder, multiple of 2), some based on properties of sets (e.g. a set has an even number of elements if its elements can be arranged in pairs). Clearly, the working definition that we have in mind will influence the way in which we prove (or explain) why a number is or is not even. One of the instructors designed the following activity in order to support the making of explicit connections between definitions and proofs by the teachers.

> Students were asked if the number of legs in the classroom is even. Five answers follow. Which answers are correct? How would you respond to each of the students' answers? In your opinion, is there a best answer? Which? What is the implicit definition for even behind each answer? Which definition would you choose to use in your classroom? The five responses were: 1) Yes, because the number of legs is twice the number of people. 2) Yes, because each person adds 2 legs to the total, so when we add them all up we get $2 + 2 + \cdots$. 3) Yes, because there is an even number of people in the classroom. 4) Yes, because we can divide the legs into two groups – left legs and right legs. 5) Yes, because we can divide the legs into two groups – boys' legs and girls' legs.

In this activity we see how the desirability of multiple definitions and a (non-rigorous) focus on their equivalence were implemented in the task design. Moreover, we see how the design of the task reflects a shift in the underlying reasons for such desirability, from mathematical (or meta-mathematical) to pedagogical. Namely, an integral part of mathematical activity is to produce alternative definitions and check for and prove equivalence. This reason may not apply to one's teaching needs, yet knowing and inspecting alternative definitions may still be central for teaching practices (i.e. how to address students' productions).

In including options with flawed arguments (3 and 5 above), this task also provides an opportunity to address the meta-mathematical topic of logical reasoning, and is consistent with the belief that:

– Conclusions should follow *logically* from definitions.

MATHEMATICAL CONTENT – PREPARING FOR TOPICS ON THE HORIZON

The teachers participating in the PD tended to have specialized knowledge, based on their experience of teaching no more than one or two different grade levels. In an expectations questionnaire administered at the beginning of the course, teachers showed little interest in topics "on the horizon," namely topics that their students will learn in later grades, which they themselves do not teach. Moreover, they tended not to recognize which of the topics they teach will be crucial foundations for more advanced knowledge. The mathematicians built on their background in order to make explicit connections between current and future topics, and included recommendations on how to teach some elementary topics in a way that will support more advanced topics later on. This is consistent with what Ball (1993) describes as "mathematical horizon" for teaching.

Equations

In the section on unpacking the concept of equality we saw how the instructors' awareness of equality "on the horizon" – as it is used in middle-school algebra – guided their approach to it in the context of elementary school arithmetic.

Subtraction

There are many situations that can serve as the basis for understanding the operation of subtraction – removal of objects from a set, comparison of the cardinality of two sets, distance on the number line, and more. It was not always clear to the teachers why they need more than one. The mathematicians brought an important consideration to this question – some situations extend to fractions or to negative numbers better than others, for example, distance on the number line can be very instrumental for understanding why $4-(-2)$ is the same as $4+2$. Similar considerations apply to different approaches to multiplication, where the area model provides meaning and visual support when the factors are fractions. These considerations are not purely mathematical; they lie at the confluence of mathematics and didactics. Nonetheless, they seemed to be quite foreign to the teachers, especially for those who teach one or two grades, and for whom just one view of these topics seems to suffice for their work.

PEDAGOGICAL ISSUES – DIFFICULTIES, ERRORS AND MISCONCEPTIONS

Anticipating and recognizing student errors and misconceptions is at the heart of teachers' expertise. Hill et al. (2008) have developed test items regarding this aspect of teaching expertise, and have shown that skilled teachers outperform research mathematicians in anticipating and identifying student difficulties. Our mathematician-instructors were no exception – they were not very knowledgeable on these matters either. One instructor stated that a certain type of problem *must be considered difficult, since it appears so rarely in textbooks*, implicitly admitting that he is not an expert on what is difficult for students. Often the instructors would appeal to the teachers for their pedagogical insight – "*Is this difficult for your students? Is it something they can do?*" The teachers welcomed this kind of question, and were glad to be able to bring their expertise to bear.

In spite of their lack of expertise, the mathematicians coped with the issue of student difficulties on the basis of their own proficiency, supplying a complementary perspective to that of the teachers. In this section we illustrate how the mathematicians' knowledge served as a springboard for their understanding of and their suggestions for coping with common student errors and misconceptions.

Counting errors

As described above, counting was unpacked into rote counting, 1-1 correspondence with the natural numbers, and invariance under permutations of the set elements. Omitting any of these ingredients may lead to error. For example, skipping elements in counting amounts to a correspondence not defined on the whole set of elements. A correspondence not *well defined* is a way to describe and explain rote counting not synchronized with the ticking off of the elements. A correspondence that is not 1-1 may result in counting an object twice, or in skipping others. Not accepting invariance under permutations may cause children to repeat their counting in a different sequence, not quite expecting the same result. It was their understanding of the mathematics that helped the mathematicians anticipate these potential errors and difficulties.

Problems with unknowns and the equality symbol

As mentioned above, the equality symbol is often seen by children as a call for action – "Solve!" Realizing this, and realizing that problems with an unknown $(3+? = 5)$ require a different interpretation of the symbol – as equivalence – one of the instructors suggested that the unknown be covered by a curtain ($3 +$  $= 5$). He actually cut one out from a curtain catalog and stuck it on the whiteboard. Placing the curtain over the unknown implied that someone had solved the problem in the past (in-line with the "call for action"), and now we are detectives trying to recreate what the problem must have been in the first place. The instructor's suggestion may be seen as a bridge between equality as a call for action and as an equivalence relationship. The inspiration for the idea came from a pedagogical "trick" the instructor had been shown by a teacher in a different context.[2] This example shows how the instructor appropriated a design idea underlying a didactical tool developed to attain an educational goal (weaning students from the need to count from 1) in order to enrich a narrow interpretation of the equality sign. In this case, the task and the didactical tool were firmly based on a worthwhile mathematical idea, and the mathematician-instructor was not only capable of making that idea explicit, but he also appropriated the design principles and applied them to the design of an artifact, illustrating a new idea. The issue of designing activities is discussed further below.

Misconceptions in vertical subtraction

In one activity, based on Ernest (2011), in the spirit of Brown and Burton (1978), the teachers attempted to uncover and explain student errors in vertical subtraction, and to predict how these students would solve some new problems. The instructors stressed the following: *What is the* **conceptual** *misconception behind the procedural error, what in the student's previous learning might be responsible for this misconception, how would you help this student master the procedure, can you suggest a correct procedure based on this student's erroneous one?* We see in these questions a blend of mathematical and didactical points of view.

PEDAGOGICAL ISSUES – DESIGNING ACTIVITIES

What makes a "good" problem for elementary mathematics PD? In previous years, the main criticism of the PD was that not all activities[3] were *relevant* for the teachers. For teachers, relevance meant having a direct impact on what they bring to and do in classrooms. The teachers made it very clear from the start that what would serve them best would be "prêt-à-porter" problems, namely those that they could "use in our classrooms tomorrow morning," as is. The instructors generally accepted the premise that they should provide some activities that the teachers could use in their own classrooms, but also remained faithful to their goal of teaching mathematics. One of the ways these two distinct goals were conciliated was by capitalizing on the teachers' engagement with problems they found interesting and "relevant," and using them as springboards to discuss the underlying mathematics. Some of the resulting problems used in the course were very productive in this respect – they were rich enough to provide a context for both views of relevance: on the one hand, deepening the teachers' understanding but on the other hand, appropriate for the teachers to use (possibly with some modifications) in their classrooms. One of the instructors went to great lengths to make sure his activities would reach the teachers' classrooms. One of the homework assignments consisted of choosing an activity from the PD, adapting it for classroom use, implementing it in one of their lessons, and reporting on it in a future PD session. The adaptations the teachers made enabled us to infer some of their beliefs about mathematics and teaching. Discussing this in the PD provided the instructor with opportunities to bring to bear mathematical and meta-mathematical issues such as: elegant solutions, alternative explanations (provided either by teachers or their students), math is not only about solving exercises, etc. The tasks described above (evenness of the number of legs in the classroom, alternative naming conventions for numbers), played out as productive activities in the PD. The following are some additional examples.

Place the digits

In this activity, teachers needed to use 3 given digits (1, 4, 9) to construct a multiplication problem with the greatest possible result. Many of the teachers resorted to trial and error, but this turned out to be a good context for re-visiting the associative and distributive properties – why is 4×90 the same as 40×9 (associative property), and why is 9×41 greater than 4×91 (distributive property). The 4-digit version of this problem presented further opportunities to deepen the math in the context of problems which can be brought as is to the classroom.

The Gelosia method for multiplication

Many of the instructors presented a procedure for multi-digit multiplication which was unfamiliar to the teachers. The procedure dates from the 15th century and is illustrated by the example given in Figure 1 ($934 \times 314 = 293276$) taken from the Treviso Arithmetic (1478), as it appears in (Smith, 1958, pp. 114–117).

In this method there are 9 partial products, each one the result of multiplying two 1-digit numbers. This eliminates the need for most of the carrying in the standard procedure. Furthermore, the partial products do not need to be staggered; the correct place value is achieved by adding partial products along the diagonal lines. As mathematicians, the instructors were intrigued by the mathematics behind the procedure, and thus decided on its appropriateness for the PD, as it provides opportunities for deepening the understanding of multiplication and place value. Moreover, they felt that the activity was within the range of what the teachers

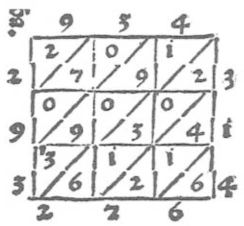

Figure 1. First printed example of the Gelosia multiplication.

could take to their own classrooms. The instructors were somewhat disappointed to learn that most teachers perceived the activity quite differently. Instead of using it as an enrichment activity to deepen their students' conceptual understanding, most teachers saw it as a potential remedial tool – an alternative procedure they could offer to students who had not mastered the standard procedure. As such, they did not dedicate much effort to the question of how and why the procedure works. We are not sure why so many teachers chose not to use this activity for enrichment. Perhaps they felt they had more pressing topics to teach. Or perhaps they were more impressed by the procedural aspects of the method (a reliable and easy-to-remember way to multiply) than the conceptual issues it raises. This may mirror differences between the mathematicians' and the teachers' attitudes towards mathematics in general. In spite of the instructors' disappointment, we consider this a productive activity. The teachers were highly engaged, deepened their understanding – which should eventually have beneficial effects on their teaching, and were willing to bring the alternative method to their classrooms – though with a different purpose in mind than that of the instructors.

Focusing on one-to-one correspondence

In the section on unpacking mathematical content we described how the mathematicians came to focus on the concept of 1-1 correspondence as foundational, even more basic than counting. Their insight was based on infinite sets, where 1-1 correspondence is the only way to compare cardinalities. How can this topic be introduced to the teachers in a way that is meaningful for them? One of the mathematicians with some programming capabilities found a creative approach. He asked himself what is special about infinite sets. In this context, the main point is that they cannot be counted. Thus, what he needed was a finite set that cannot be counted. He prepared a game applet in which blue and red balls move around the screen in random motion. The goal is to determine whether there are more red or blue balls. Counting is not a feasible strategy due to the balls' motion, but the applet does allow the player to pair up a blue and a red ball, at which point they are both removed from the screen. Players proceed to pair up balls until they are all exhausted, or until balls of only one color remain. In this game players make implicit use of the principle of 1-1 correspondence in order to solve a comparison problem without counting. The designing of this game is an example of how the mathematicians used their advanced knowledge of mathematics to uncover some of the less obvious foundations of elementary mathematics, and yet found ways to share their insights with the teachers, in a context that is relevant, playful, and is grounded in the most elementary mathematics.

DISCUSSION

This PD seemed doomed from the start. The teachers enrolled hoping to enrich their practice with new tools of the trade – activities for their classrooms and teaching tips-and-tricks. The mathematician-instructors had something else in mind – using their mathematical knowledge to deepen the teachers' mathematical understanding. Yet in spite of the chasm between these expectations, the PD was considered a success by all involved. The teachers' feedback indicated their satisfaction, the instructors felt they were teaching effectively and indicated that the teachers participated actively, and the ministry representatives – who occasionally sat in on sessions – were pleased with what they saw and heard. Furthermore, some of the teachers are utilizing their newfound knowledge. One teacher testified that her principal recently sat in on her math class. When he asked her what she was teaching, her proud reply was "what I learned in the PD last week." We now take a step back and try to explain what worked and why.

The concept of unpacking – unpacked

We have shown numerous examples of the mathematicians unpacking mathematical content. We will now attempt to unpack the concept of unpacking – reveal its elements and describe its mechanisms.

Two-way didactic transposition

Chevallard (1985) coined the term *didactic transposition* to describe the change that mathematics content must undergo from a body of knowledge *used* ("savoir savant") to a body of knowledge *taught* at school ("savoir enseigné"). Borrowing and extending this idea, we may say that the mathematicians applied a reverse-transposition: they took elementary concepts and lifted them up to the context of university mathematics. In other words, they transposed knowledge taught at school to knowledge of the professional mathematician. In this context, they employed the full power of their mathematics to deeply re-inspect the topics. Then they transposed them back to the domain of school mathematics. The first transposition may be seen as an *embedding*[4] of elementary math in the more sophisticated university math. The second transposition may be thought of as a homomorphism from university mathematics to school mathematics, aiming to maintain the structure of the discipline while scaling it down to something more palatable for students and teachers. Paraphrasing the courtroom oath, this second transposition was committed to *nothing but the truth*, but could generally not be fully faithful to *the whole* mathematical *truth*. This process of double transposition helped highlight rich mathematical connections between the elementary concepts, as was demonstrated in some of the examples above. What happens to mathematical concepts which undergo didactic transposition? Trivially, *proofs* in advanced mathematics tend to suggest how ideas may be *explained* in elementary math. Less trivially, mathematical connections between concepts tend to be mirrored in cognitive connections made by teachers and students. This was seen, for example, in the various definitions of natural numbers, mirrored in the skills of rote counting and cardinality counting, or in the way details of a mathematical proof may suggest possible student errors or misconceptions. This mirroring provided surprisingly productive insights into cognitive processes and student difficulties. For example, unpacking evenness and the meta-mathematical goal of teaching alternative definitions of this concept (and the equivalences thereof) gave birth to the activity where teachers evaluated students' correct and incorrect explanations of why the number of legs in the classroom is even.

Setting

The setting appears to be crucial as the environment needed for the unpacking to occur. The unpacking was highly situated. It took place in the context of a specific PD program in which the teachers' backgrounds and expectations were a determinant and constraining factor. Left to their own devices, the mathematicians may have remained much closer to university math, in which case their unpacking of the elementary math topics would have looked quite different. Consideration of the PD teachers' needs and explicit expectations provided a sense for the type and extent of the didactic transposition. Furthermore, the mathematicians appeared to be able and willing to learn from the teachers (e.g. the case of appropriating pedagogical insight in the context of problems with unknowns).

Knowledge and beliefs

The examples showed how knowledge of university mathematics was instrumental in unpacking elementary mathematics. Of similar importance was how the mathematicians brought their *beliefs* and their *mathematical points of view* to the task of unpacking. Their commitments to some underlying fundamental principles, even when not always articulated, were fundamental to the unpacking. In the following we list some of these principles quoting from Wu (2011), a mathematician involved in pre-college math education:

1. *Every concept is precisely defined, and definitions furnish the basis for logical deductions.*
2. *Mathematical statements are precise. At any moment, it is clear what is known and what is not known.*
3. *Every assertion can be backed by logical reasoning.*
4. *Mathematics is coherent; it is a tapestry in which all the concepts and skills are logically interwoven to form a single piece.*
5. *Mathematics is goal-oriented, and every concept or skill in the standard curriculum is there for a purpose.*

What made the program a success – Bridging the cultural gap

We believe that, in essence, this is a story of bridging a cultural gap. The gap is multi-dimensional. There is a knowledge gap, a gap between attitudes towards mathematics and its learning and teaching, and more practically, there is a gap between expectations regarding the PD. The PD was successful due to the various ways in which this gap was bridged.

Activities
We have seen how instrumental a "good" problem can be. Some of the most successful activities occurred around problems that the teachers could use in their classrooms, and at the same time provided a springboard for discussing non-trivial mathematics in the PD. Such activities, in supporting both the teachers' and the mathematicians' perspectives on mathematics, served as a bridge between their expectations. This was the case even when the mathematicians and the teachers did not ultimately agree on the role of the activity. For example, the Gelosia Method was perceived by the teachers primarily as an alternate procedure for multi-digit multiplication, suitable for their struggling students, whereas the instructors' main intention was to use it as a context for deepening the understanding of multiplication and place value, both in the PD and ultimately in the teachers' classrooms. Although the instructors were disappointed by the ways in which the teachers perceived the goal of this activity, they nonetheless provided what they aimed to provide – meaningful mathematics in the PD – and teachers received what they hoped to receive – a usable activity for their classroom.

Roles
Many activities evolved in such a way that the teachers provided valuable pedagogical input, and the mathematicians provided mathematical critique within a context of mutual interest, for example, evaluating web-based educational video clips. The teachers provided didactic criticism (e.g. use of the board, student participation) while the mathematicians provided mathematical criticism (e.g. accurate use of language and symbols, validity of logical arguments).

Mutual appropriation
The data indicate that the teachers may have started to appropriate (in the sense of Moschkovich, 2004) some of the mathematicians' attitudes towards and beliefs about mathematics, but as we said, this will be discussed elsewhere. Less trivial is the fact that the mathematicians appropriated some of the teachers' attitudes toward mathematics teaching and learning. This was evident in the blend of mathematical and didactical points of view expressed in many of the activities they designed. And finally, there is some evidence indicating that the mathematicians appropriated more of the teachers' culture than one might have expected. The instructors were often annoyed by the teachers' repeated demand for activities they could "use in class tomorrow morning." As the PD progressed, and the activities came to be designed around classroom problems, it was not uncommon to see emails from

the instructors along the lines: “I’m stuck! Does anyone have a good problem I can use in the PD tomorrow?”

CONCLUSION

The literature distinguishes between two main types of content knowledge for teaching – subject matter content knowledge (SMCK) and pedagogical content knowledge (PCK). The initial chasm between the PD instructors and the teachers may be characterized in terms of this distinction. Roughly speaking, the instructors intended to build the PD around SMCK, whereas the teachers were expecting a program that would contribute to their PCK. This distinction between types of knowledge is also the key to understanding what ultimately made the PD a success. Many of the episodes we have described took place at the intersection of these types of knowledge. We have seen how a university conception of SMCK may serve as a strong springboard for developing PCK, both in the PD sessions and in the mathematicians’ preparation, and conversely, the teachers’ existing PCK was used as a springboard for conducting mathematical discussions and developing mathematical insights. These rich interconnections and mutual exchange of mathematics and mathematical pedagogy were at the core of this unique PD and they are worth exploring further. We feel that in exploring these interactions, attention should be given to the mathematicians’ special SMCK, as influenced by their university conception of elementary mathematics. The mathematicians’ knowledge of mathematics helped them unpack many elementary concepts – both mathematical and meta-mathematical – exposing their nuances and revealing what they involve for learners. Furthermore, their mathematical practices and beliefs about mathematics guided them in conveying meta-mathematical messages in the activities that they designed and conducted.

Finally, it is commonly agreed that research mathematicians can and should be involved in the education of elementary school mathematics teachers, but there have not been many models to learn from. Here we have given an account of a productive involvement, and have highlighted some ways in which the mathematicians’ contribution was special. The interactions were considered productive by both communities, by a number of different standards. Both the teachers and the mathematicians learned some mathematics and some didactics from their encounters, and both parties felt that the teachers ended up better equipped to do their job.

NOTES

[1] Most of the lessons are being recorded by the first author of this chapter and are undergoing a first round of analysis. The data are particuraly rich because both the instructors and the in-service teachers who participate in the course are candid and outspoken about their feelings and evaluations.

[2] A game designed to encourage “counting-on” – finding $3 + 2$ by counting “3 … 4, 5,” and not “1, 2, 3, …, 4, 5,” by allowing students to see only one of the addends at a time.

[3] We use the term activity for a segment of a PD session, typically a problem posed by the instructor, solutions suggested by the teachers, and discussions that followed.
[4] Injective structure-preserving mapping.

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