PROFICIENCY AND BELIEFS IN LEARNING AND TEACHING MATHEMATICS

MATHEMATICS TEACHING AND LEARNING
Volume 1

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Proficiency and Beliefs in Learning and Teaching Mathematics

Learning from Alan Schoenfeld and Günter Törner

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The photo on the front cover shows Günter Törner and Alan Schoenfeld at Marksburg Castle in the Rhine Valley, Germany, in March 2013

Photo credit: Jane Schoenfeld, 2013

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The book was principally conceived to celebrate the work of two outstanding scholars: Alan Schoenfeld and Günter Törner. First and foremost, we would like to acknowledge and thank them for supporting the idea of publishing such a book and contributing their respective chapters.

We also want to take this opportunity to thank and acknowledge all others who have been involved in the process of preparing this book. This has been a wonderful experience. Here are a few highlights of the process:

1. This book resulted from the typical process of requesting, contributing and editing chapters. However, the current volume is not typical in its content because all the chapters refer to and build upon these two scholars' work. Although mathematics education as a field does not have a long history, there have been many books published on mathematics education research and practice. However, there are few books that focus on what we can learn from and how we can build on others' work as this volume does. We would especially like to acknowledge the scholars who contributed chapters to this volume. This volume would not have been possible without their contributions detailing ways to build upon and expand the work of others.
2. Although it may seem that the contributors were all close friends, this book has actually brought together people who had never collaborated before and has provided the opportunity to develop new professional relationships. For example, although Judit is a former student of Alan's, she had not met Günter. Yeping first formally met Günter in August 2011 in College Station, Texas and started the conversation about the possibility and value of editing and publishing such a book. Thus, the work on this book has not only brought together long time friends and colleagues, but also created new professional connections and resulted in new friends. We want to thank all those who were so ready and willing to work with new colleagues and create new connections.

It is easy to see that the volume is the product of an international collaboration of scholars from different countries and disciplines. This book would not have been possible without the dedicated group of 31 contributors from six countries (Austria, Canada, Germany, Israel, UK, and the US) and we thank them for their contributions. This group of contributors also worked together as a team to review the chapters of this publication. Their collective efforts helped ensure this book's quality.

Thanks also go to a group of external reviewers who took the time to help review many chapters of the book. They are Ann Ryu Edwards, Rongjin Huang, Jennifer Lewis, Katie Lewis, Hélia Oliveira, and Constanta Olteanu. Their reviews and comments helped improve the quality of many chapters.

Finally, we want to thank Nikki Butchers for her assistance in proofreading many chapters of this book and Michel Lokhorst (publisher at Sense Publishers) for his patience and support. Michel's professional assistance has made the publication a smooth and pleasant experience, with this book's timely publication as the first volume of the new book series on *Mathematics Teaching and Learning*.

PART I

INTRODUCTION

YEPING LI AND JUDIT N. MOSCHKOVICH

1. PROFICIENCY AND BELIEFS IN LEARNING AND TEACHING MATHEMATICS

An Introduction

INTRODUCTION

This volume was sparked by the fact that Alan Schoenfeld and Günter Törner were both celebrating their 65th birthdays in July 2012. The book started out as part of a Festschrift to celebrate that event. Although the volume was not ready in time for their birthdays, the result is a belated celebration in print to recognize their contributions to the field of mathematics education. Alan and Günter share much more than simply their birth month and year. In addition to being colleagues in mathematics education, Alan and Günter are long time friends and mathematicians by training. Their background in mathematics led them to pay close attention to the details of mathematical work. Although Alan and Günter followed different professional trajectories in the field of mathematics education, their work exemplifies how mathematicians can make important contributions to mathematics education research. Alan and Günter are well respected in the international mathematics education community for their work, and this volume is meant to be a tribute to their scholarly achievement.

This volume, however, is not a collection of papers written by Alan and Günter. Instead, this is a collection of papers that show how other researchers have connected to, learned from, and built upon Alan and Günter's work. In Alan's own words this volume is, "... a chance for some good scholars to advance the field, with their own work as outgrowths of things they may have done with Günter and me. And contributing to the field is what books should be for!" (personal communication). It is in this spirit of moving the field of mathematics education forward that we offer this volume to show our deep appreciation of the work done by Alan and Günter.

Identifying a theme for the volume was at once easy and challenging. On one hand, it was not difficult to find a theme because Alan and Günter's works have covered such a broad scope of topics. However, attempts to cover these varied topic areas would not have generated a coherent volume. On the other hand, because Alan and Günter have devoted their efforts to studying key questions in learning and teaching mathematics, we found that the themes of understanding and improving mathematics teaching and learning should be at the core of such a volume. This has been a central theme in mathematics education, as researchers aim to not only understand the nature of proficiency, beliefs, and practices in mathematics learning and teaching, but also identify and assess possible influences on students' and teachers' proficiency,

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beliefs, and practices in learning and teaching mathematics. These topics have fascinated researchers from various backgrounds, including psychologists, learning scientists, mathematicians, and mathematics educators. Among those researchers, Alan Schoenfeld in the United States and Günter Törner in Germany, are internationally recognized for their contributions. Thus, the theme of "proficiency and beliefs in learning and teaching mathematics" emerged as a focus from the broader scope of Alan and Günter's work. The second theme for the book is a focus on what we can learn from and how we can build on other researchers' work. In that spirit, it offers both a look back and a look forward at where research in mathematics education has been and where it can go. It is especially important to have such volumes to reflect on the trajectory and direction of the field, so that we can build a coherent body of research.

Alan and Günter's professional work drew a group of international scholars to contribute to this book. The spirit of international collaboration is evident throughout this volume. Indeed, this book would not have been possible without the contributions of multiple scholars from six different countries. Contributors to this volume were invited because they had worked closely with Alan or Günter or used Alan or Günter's work in their current research. There are certainly many more researchers in mathematics education that fit that category than the number of chapter contributors in this book. We want to note that several scholars who have worked closely with Alan or Günter were invited but were not able to contribute a chapter due to previous commitments. Their absence from this collection should not be interpreted as an omission, but as a reflection of how busy researchers are in this particular community.

This book is also the inaugural volume of the new international book series on "Mathematics Teaching and Learning," the first book series on mathematics education published by Sense Publishers. This series aims to provide an outlet for sharing the research, policy, and practice of mathematics education and promote teaching and learning of school mathematics at all school levels as well as through teacher education around the world. This book series is designed to have a broad international readership and serve the needs of information exchange and educational improvement in school mathematics. The spirit of international collaboration evident in this volume provides a starting point for this book series to promote research in mathematics teaching and learning around the globe.

STRUCTURE OF THE BOOK

This book focuses on Alan and Günter's scholarly contributions to the study of proficiency and beliefs in learning and teaching mathematics. To provide readers with an overview of Alan and Günter's work, Part I of the book provides two chapters that serve as an introduction to and summary of Alan and Günter's work, respectively. However, this book is not simply a collection of Alan and Günter's scholarly work. Instead, the volume is designed to offer scholars an opportunity to present their own work and reflect on how their work connects with or builds upon

Alan or Günter's work. These chapters thus make up the main body of the book in subsequent sections.

Chapter authors were asked to propose topics related to proficient performance, beliefs, and practices in mathematics teaching and learning. The resulting 12 chapters reflect how different researchers have used and expanded Alan or Günter's work. These chapters are then organized into three sections according to their focuses.

Part II focuses on "Proficient Performance, Beliefs, and Metacognition in Mathematical Thinking, Problem Solving, and Learning." Three chapters are included in this part: one on problem-solving skill development, one on beliefs, and one on social metacognitive control. Although these three chapters are diverse in terms of their focus, each one makes important connections with Alan and Günter's work. In particular, Kristina Reiss, Anke M. Lindmeier, Petra Barchfeld, and Beate Sodian followed Alan and Günter's work in mathematical problem solving to study problem solving as an integrated part of students' thinking and learning. In their chapter, they extended problem solving to elementary school children's understanding and problem solving in the case of data analysis, statistics and probability. Likewise, Christine Schmeisser, Stefan Krauss, Georg Bruckmaier, Stefan Ufer, and Werner Blum built upon Alan and Günter's work when studying the beliefs of mathematics teachers on the nature of mathematics and on the teaching of mathematics. Ming Ming Chiu, Karrie A. Jones and Jennifer L. Jones built upon Alan's work on metacognitive control that focused on an individual's regulation of his/her thinking to study social metacognitive control, which focuses on groups' cognitive monitoring and control activities.

Part III on "Proficient Performance, Beliefs, and Practices in Mathematics Teaching, and Ways to Facilitate Them," includes six chapters. This is a set of contributions that focus on teaching, teachers' beliefs, and their professional development and efforts to improve teaching. The larger number of contributions received and the wide scope of issues addressed in this part suggest that issues related to mathematics teaching and teacher professional development have received more attention over the past decade. It is commonly acknowledged that the quality of teachers and their teaching is key to the success of students' mathematics learning (CBMS, 2001, 2012; NMAP, 2008; NRC, 2010). However, how to measure and improve the quality of teachers and their teaching has been a great challenge to educational researchers and mathematics educators. In various ways, these six chapters built upon Alan and Günter's work on problem solving, teaching, beliefs, and teachers' professional development (e.g., Pehkonen & Törner, 1999; Schoenfeld, 1985, 1998, 2010; Schoenfeld & Kilpatrick, 2008; Törner, 2002) to further the research on these topics.

Part IV on "Issues and Perspectives on Research and Practice" includes three chapters, each one from a different perspective – that of an educational engineer, a mathematician, and a mathematics education researcher – to connect to and reflect on research and practice in mathematics education. The chapter "Methodological Issues in Research and Development" by Hugh Burkhardt, uses the perspective of an educational engineer to build on Schoenfeld's seminal contributions to methodological issues, reviews some of the choices researchers in education face, and proposes how to improve the impact and influence of educational research on practice and policy. The chapter "A Mathematical Perspective on Educational Research," by Cathy Kessel, describes how the experience of being a mathematician might shape one's perspective on mathematics education research, discusses what it might mean to have a mathematical orientation, and illustrates how that orientation shaped Schoenfeld's research. The chapter "Issues Regarding the Concept of Mathematical Practices," by Judit Moschkovich, explores this component of Schoenfeld's framework for the study of mathematical problem solving and considers how we define mathematical practices, theoretically frame the concept of practices, connect practices to other aspects of mathematical activity, and describe how practices are acquired.

The book concludes with Part V, with contributions from Alan and Günter where they reflect on the chapters and then look ahead. These two chapters provide a look back, as these two researchers had the unusual opportunity to see collected in one place, the ways that others have made connections to their work. The chapters also provide a look at the present, as the authors describe their current research. And they furthermore provide a look forward, as Alan and Günter help us think about where the field might need to go next.

REFERENCES

Conference Board of the Mathematical Sciences (CBMS). (2001). *The mathematical education of teachers*. Available: http://www.cbmsweb.org/MET_Document/index.htm (accessed October 2012).

Conference Board of the Mathematical Sciences (CBMS). (2012). *The mathematical education of teachers II*. Available: http://www.cbmsweb.org/MET2/met2.pdf (accessed November 2012).

National Mathematics Advisory Panel (NMAP). (2008). *Foundations for success: The final report of the national mathematics advisory panel*. Washington, DC: US Department of Education.

National Research Council (NRC). (2010). *Preparing teachers: Building evidence for sound policy*. Committee on the Study of Teacher Preparation Programs in the United States. Washington, DC: National Academy Press.

Pehkonen, E., & Törner, G. (1999). Teachers' professional development: What are the key change factors for mathematics teachers? *European Journal of Teacher Education*, *22*(2–3), 259–275.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education*, *4*(1), 1–94.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-directed decision making and its educational applications*. New York: Routledge.

Schoenfeld, A. H., & Kilpatrick, J. (2008). Toward a theory of proficiency in teaching mathematics. In D. Tirosh & T. Wood (Eds.), *The international handbook of mathematics teacher education: Tools and processes in mathematics teacher education* (Vol. 2, pp. 321–354). Rotterdam: Sense Publishers.

Törner, G. (2002). Mathematical beliefs. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 73–94). Dordrecht, the Netherlands: Kluwer Academic Publishers.

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AN INTRODUCTION

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2. ABOUT ALAN H. SCHOENFELD AND HIS WORK

INTRODUCTION

Born in 1947, Alan Schoenfeld began his career as a research mathematician. After obtaining his bachelor's and master's degrees in mathematics in the late sixties, he continued his doctoral study in mathematics at Stanford University, earning a PhD in 1973. He became a lecturer at the University of California at Davis, and later a lecturer and research mathematician in the Graduate Group in Science and Mathematics Education at the University of California at Berkeley. During that time at Berkeley, he became interested in mathematics education research – an interest that has kept him in the field of mathematics education since. After academic appointments at Hamilton College and the University of Rochester, Alan was invited back to U.C. Berkeley in 1985 to strengthen the mathematics education group. He has been a full professor since 1987, and is now the Elizabeth and Edward Conner Professor of Education and an Affiliated Professor of Mathematics in the mathematics department. He has also been a Special Professor of the University of Nottingham since 1994.

Over the past 35 years, Alan Schoenfeld has exemplified what a fine scholar can accomplish through the tireless pursuit of excellence in research on topics that have had broad and long-lasting impact. Through his research, he has played a leading role in transforming mathematics education from a field focused on specific concepts and skills to one where the ability to use them effectively to tackle complex non-routine problems is now a central performance goal. His research has brought together different disciplines and perspectives to tackle complex and important topics in mathematics education.

His work is internationally acclaimed with more than 20 books and numerous articles published in top journals in mathematics education, mathematics, educational research, and educational psychology. The scope and depth of his scholarly impact is evidenced not only in terms of over 12,000 citations[1] of more than 200 scholarly publications, but also the vision and cutting-edge knowledge that he provides through his research.

Alan Schoenfeld's achievements have been recognized in the awards he has received, culminating in the 2011 Felix Klein Medal of the International Commission on Mathematics Instruction "in recognition of his more than thirty years of sustained, consistent, and outstanding lifetime achievements in mathematics education research and development" (ICMI, 2012).

The following sections outline three aspects of Alan Schoenfeld's achievements: his scholarly work as a researcher, his contributions and achievements as

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a leader, and his accomplishments as an educator. We describe each of them in a bit more detail and point to references that will be helpful to those who want to learn more.

RESEARCH

Mathematical problem solving, learning, and teaching

Alan Schoenfeld's work shows a life-long pursuit of deeper understanding of the nature and development of proficiency and beliefs in mathematical learning and teaching. Starting with work on mathematical problem solving in the late 1970s, he broadened his research interests in the mid-1980s to embrace mathematical teaching and teachers' proficiency. In moving beyond separate concepts and skills to their integrated role in problem solving he has combined the profound with the practical to an unusual extent. He has developed his theoretical contributions through deep analysis of his own experiments, usually carried out in down-to-earth classroom situations rather than laboratory settings. In this way, he has advanced the understanding of the processes of problem solving in school mathematics, teacher decision-making, and much else. His work has helped to shape research and theory development in these areas, which have been focal topics in mathematics education over the past three decades.

Studies of the nature and development of mathematical proficiency

The work of Pólya and others on the processes of problem solving was based on introspection – they reflected on the way mathematicians solve problems and suggested a set of heuristic strategies that, if adopted by students and others, might improve their ability to solve non-routine problems. While these heuristics are highly plausible, many nice theoretical ideas in education do not work well in practice. Well aware of this, and inspired by Pólya and the rather stylized work on problem solving being developed in the new field of artificial intelligence by Newell, Simon, and others, Alan Schoenfeld set out to study students' problem solving and how it might be improved through instruction. First in the SESAME group at Berkeley, then in the courses in undergraduate mathematics that he taught at Hamilton College and the University of Rochester, he did a series of careful empirical studies on the behaviour of students when they tackle problems that are new to them.

What are the key results that he found on the learning and teaching of problem solving? First, perhaps, that the Pólya heuristics are sound – but inadequate for students in classrooms. For example, it is not enough to tell students to "look at some specific simple cases"; the kind of "simple case" that is likely to be productive depends on the type of problem. For example, in pattern generalizations small n is often helpful, while in many game problems the end game is often the best place to start. Through his observations and subsequent analyses, Alan recognized that, for students to be able to solve problems effectively, strategies need to be instantiated in a set of tactics (or sub-strategies) that are specific to the type of problem being solved. These tactics need to become part of the solver's knowledge base, which must be developed from this perspective, taking Pólya's "Have you seen a problem like this before?" to a deeper level and a broader horizon.

The results of this research were first published in a series of papers, then brought together in his seminal book *Mathematical Problem Solving* (Schoenfeld, 1985a); it is a measure of the scope and quality of the experiments and his analysis that none of the many books on this intensely fashionable subject have matched, let alone superseded, the reputation and influence of this one. Alan remains "Mr. Problem Solving" within the international mathematical education community.

With this early focus, Alan was a pioneer in combining perspectives and theories from cognitive science with those from mathematics and mathematics education. He built upon his research on mathematical problem solving to examine and understand the mathematical cognition and metacognition that are in play in the problem solving process. His work in this broadened topic area again led the field to look beyond the surface of mathematical problem-solving behaviour (e.g., Schoenfeld, 1987, *What's all the fuss about metacognition*; Schoenfeld, 1992a, *Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics*).

While many other researchers focused solely on cognition in the 1970s and 1980s, Alan became keenly aware of its limitations. In particular, he drew attention to the importance of beliefs and social cognition in mathematics education and the cognitive sciences. His vision, and the knowledge generated from his own research, encouraged the field to attend to multiple dimensions in the process of mathematical problem solving and learning (Schoenfeld, 1983, *Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving focuses in intellectual performance*; Schoenfeld, 1989, *Explorations of students' mathematical beliefs and behavior*).

Research on teaching and teachers' decision-making and proficiency

Alan Schoenfeld's work on students' problem solving can be seen as the beginning of a sequence of studies of human decision-making that is still ongoing. From 1985 on, his attention began to move from students to teachers, and what determines a teacher's real-time decision-making in the classroom. One key work was published in 1988 after he observed the classroom instruction of a well-reputed mathematics teacher. In this article, Alan put forward contrasting points of view on what can be considered as "good" teaching, and what a teacher should know and be able to do to improve its process and outcomes (Schoenfeld, 1988, *When good teaching leads to bad results: The disasters of 'well-taught' mathematics courses*). Since then, he has devoted more and more effort to studying and understanding the teaching process. In a series of detailed studies of individual lessons, he developed a model of teaching that describes a teacher's minute-by-minute decisions in terms of three dimensions: his or her knowledge, goals, and beliefs (later he preferred the term orientations) about mathematics and pedagogy. Further, he gave evidence that one can infer these for an individual teacher from an analysis of their lessons. Representative publications from this line of work include *Toward* *a theory of teaching-in-context* (Schoenfeld, 1998) and *Models of the teaching process* (Schoenfeld, 2000).

More recently, he has extended this model to other kinds of "well practiced" goal-oriented behaviour – for example, a physician's diagnostic interview or other skilled work such as electronic trouble-shooting.

Like the work on problem solving a generation ago, this body of work has been published in a series of papers over the years, and brought together in his recent book *How we think: A theory of goal-oriented decision making and its educational applications* (Schoenfeld, 2010). It is, of course, too soon to say if this book will have the impact and longevity of *Mathematical problem solving*, but early reactions to it suggest that it may. For example: in a review that appeared in the Journal for Research in Mathematics Education entitled "Boole, Dewey, Schoenfeld – Monikers bridging 150 years of thought," the authors (Sriraman & Lee, 2012) argue that Schoenfeld continues and expands on the tradition of rigorous mathematical and philosophical explorations of those two historical giants. The final paragraph of the review says:

> *How We Think* is an important resource for mathematics education as well as the decision-making sciences, because like George Boole's seminal work, Schoenfeld axiomatizes and represents teacher decision making symbolically and representationally, and boldly applies to other situations the "laws" that govern action. . . . The book is highly recommended to anyone interested in self-analyzing teaching practice, researching teacher practices, or building a program of research, or who is simply interested in how we think. (p. 354)

In another review, in ZDM, Abraham Arcavi (2011) writes that the book is: "a must read for researchers, graduate students, mathematics educators and teachers." He claims this is so:

For theoretical reasons:

> . . . reading this book is a must for members of the mathematics education community, not only because of the standing of its author and his writing style (from which one can learn a lot about how to write) but also because of the issue it addresses which is at the core of today's agenda: mathematics teaching and the need for theoretical frameworks to study it (which are scant compared to the abundance of theories of mathematics learning). (p. 1019)

For meta-theoretical reasons:

> . . . this book addresses the very nature of research in mathematics education. To what extent is our discipline "scientific" (à la hard sciences)? Which methods and tools should we use in order to be consistent, general and somehow rigorous in our analyses and in presenting findings (p. 1019)

and for practical reasons, in which the theory is shown to have positive practical impact, he concludes:

> . . . the tools of the theory are offered to teachers not as mere academic constructs but as practical ways of unfolding and reflecting upon teaching decisions and actions and how knowledge, goals and orientations shaped them. (p. 1019)

Improving educational practice through research

While Alan Schoenfeld's primary research interests have focused on getting deeper insights into the processes of learning and teaching for proficiency development, he has never regarded this as enough. He has moved outside conventional academic research in education, putting huge effort into turning research insights into significant impact on educational practice.

Most researchers hope that their work will influence and improve educational practice in some way, but any causal link is usually long and tenuous. Alan has taken active steps to look for more-or-less direct ways to establish such links and make them effective. For many reasons, this is never easy or straightforward. Education is a highly political field in which the influence of research on policy makers is limited, and usually confined to diagnosis of problems rather than the development of robust ways to overcome them. "Common sense," which policy makers believe they possess in abundance, is usually preferred to research in the choice of new initiatives. Alan has devoted substantial effort to changing this, in specific ways and more generally. He has done this while fully recognizing that work of this kind carries little academic credit within the conventional educational research community.

Standards and curricula

During the 1980s, the leadership that the U.S. National Council of Teachers of Mathematics (NCTM) had traditionally provided to the profession in the U.S. was increasingly influenced by research in mathematical education, particularly that on problem solving. In this, Alan Schoenfeld was playing a leading part both in the research itself and in spreading its influence. The 1989 NCTM Standards gave equal prominence to mathematical processes and content areas, for the first time in such documents. The Standards inspired the National Science Foundation to launch an unprecedented effort in the development of curricula that would make possible the realization of the standards in U.S. school classrooms. While Alan's contribution to the 1989 Standards was indirect, when they came to be revised as *Principles and standards for school mathematics* (NCTM, 2000), he led the writing team for the High School standards. Now, about 30 years after the insight research on problem solving, their impact in classrooms is significant.

Assessment

Soon after funding a series of Standards-based curricula, the U.S. National Science Foundation began to support the development of standards-based assessments. Alan Schoenfeld recognized the influence of the high-stakes tests on what happens in most classrooms. Though his early research on problem solving was design research (before the term was coined), he recognized that meeting this new challenge required a substantial team, built around outstanding designers. He agreed to lead an international collaboration, built around the Nottingham Shell Centre team, to pioneer the development of a "balanced assessment"[2] in the United States. That strand of work has continued. Currently, he has led the drafting of the content specification of one of the national assessment consortia of US states whose stated goal, based on the new Common Core State Standards, is essentially to develop a form of balanced assessment to be used for testing across much of the United States.

"What works?"

Choices between competitive curricula have long been made on the basis of "professional judgment" rather than reliable evaluative information of how each works across the great variety of classrooms. In the U.S., the Bush administration had the admirable goal of improving on this, establishing the "What Works Clearinghouse" to review research evidence and make recommendations. Unfortunately, the combination of a flawed methodology and the lack of enough good evaluative research to review made the enterprise unlikely to do much good – any evaluation of a curriculum is no better than the tests used in that evaluation, and many of the tests that had been used were seriously flawed. Alan Schoenfeld was invited to serve as the "subject expert" for the Clearinghouse's comparative study of mathematics curricula. Well aware of the problems, he accepted and worked both to improve the methodology and to ensure that the power of the evaluations was not overstated. When the Clearinghouse refused to make appropriate changes, Schoenfeld resigned and published a full explanation of his reasons (Schoenfeld, 2006). The clearinghouse lost credibility and with it, the potential to impose a dangerously narrow view of evaluative research and mathematical understanding.

Diversity in mathematics education

International studies suggest that, while children of prosperous middle-class parents do comparably well in diverse countries and cultures, the correlation with socio-economic status and ethnicity is significant; it is particularly large in the U.S. Alan Schoenfeld has long seen this as a core challenge to the field and devoted considerable energy to studying and improving the situation through projects in disadvantaged school districts. The DiME (*Diversity in Mathematics Education*) project aimed to build an ongoing community of researchers who would dedicate their careers to working on issues of equity, diversity, and mathematics education. An indication of its success, including two dozen PhDs and the research they continue to produce, is that the DiME Center was awarded AERA's Henry T. Trueba Award for Research Leading to the Transformation of the Social Contexts of Education at the 2012 AERA Annual meeting. The DiME also resulted in changes in district policies as well as a series of influential papers (Schoenfeld, 2002a, 2009) that make a good case for working with school districts to achieve equity. His work with the U.S. National Research Council's Strategic Educational Research Partnerships project, in collaboration with the San Francisco Unified School District, is another example of his attempts to "make a difference." In this project, San

Francisco school officers identified the major challenges that needed to be worked on (e.g., the failures of minority students on high stakes assessments in middle school) and Alan's team worked to address the issues.

Research methodologies to improve the field and to have systematic impact

Beyond these specifics, Alan Schoenfeld has worked to develop research methodology in ways that strengthen the influence of research on educational practice. He has, in doing so, aimed to put research in mathematics education on a firmer methodological foundation. Drawing from his background as a mathematician, Alan has always sought to bring rigor to research in mathematics education – to move it toward being an "evidence-based" field with high methodological standards. Early on, he argued (to some effect) that researchers should make their data available, along with rich enough descriptions of their research methods such that readers could themselves examine the data and follow the chains of inference. He has done so over his career, producing studies (e.g., his problem solving work and the work on teaching and teachers' decision-making) that make both substantive and methodological contributions. By being "inspectable," his work is open to challenge – and it has withstood the test of time. A series of methodological papers spanning more than 30 years (e.g., Schoenfeld, 1980, 1985b, 1992b, 1994, 2002b, 2006) has contributed to building a sounder foundation for the field.

For example, his handbook article *Research methods in (mathematics) education* (Schoenfeld, 2002b) examines the limitations and strengths of standard methodologies in new ways. In this paper he identifies three dimensions of any research study: Trustworthiness, Generalizability and Importance. He points to the tensions between these, particularly for the typical single-author study or PhD study, with their limited time and personnel resources. Trustworthiness is, of course, essential, but well-controlled detailed studies on a small scale lack the empirical warrants for the generality of the insights that are so often suggested in the final sections of research papers – and are rightly seen as essential for use in design and development. Generalizability requires studying a much wider range of parallel situations to see how general and robust the insights prove to be – yet such replication, an essential part of the scientific method, currently carries little academic credit in education.

Alan moved on to a broader systemic agenda in the paper *Improving educational research: Toward a more useful, more influential, and better-funded enterprise* (Burkhardt & Schoenfeld 2003), which discusses what the field of education can learn from the methods of research and development used in other research-led practical fields such as medicine and engineering. He listed the changes that are needed to make education such a field, covering its academic value system and the impact-focus that have led other pure research fields to have substantial societal impact – and, following from this, to receive substantial support for the coherent long-term programs of linked research and development that are needed.

ADVANCING EDUCATIONAL RESEARCH AND PRACTICE AS A LEADER

The quality and central relevance of Alan Schoenfeld's work have earned him leadership positions in important professional associations in education, mathematics, and mathematics education. Among many other leadership roles, he has been an elected member of the U.S. National Academy of Education since 1994, a member of its Executive Board in 1995, and Vice President in 2001. He served as the President of the American Educational Research Association (AERA). Through his leadership roles, he has moved forward changes in the research community that promise to improve educational research and program development – and to have substantial societal impact. In his AERA's presidential address (Schoenfeld, 1999, *Looking toward the 21st century: Challenges of educational theory and practice*), Alan laid out an agenda filled with high priority studies in the coming decades.

A mathematician by training, Alan has, throughout his career, also sought to bring together the communities of mathematicians and mathematics educators in the common cause of educating young people in ways that will be mathematically productive. Much of this work is "invisible," by way of committee service. In one year Alan was simultaneously a member of the American Mathematical Society's Committee on Education, chair of the Mathematical Association of America's Committee on the Teaching of Undergraduate Mathematics, and a member of the National Council of Teachers of Mathematics' Research Advisory Committee. He has, both through frequent presentations at meetings and through publications, served as an emissary between the communities of researchers in mathematics and mathematics education. His 2000 article *Purposes and methods of research in mathematics education*, published in the *Notices of the American Mathematical Society* is an example. He has also, by virtue of his editorial responsibilities (including serving as a co-founding editor of the book series *Research in Collegiate Mathematics Education*), worked to build the community of mathematicians conducting mathematics education research at the tertiary level.

Throughout his career Alan Schoenfeld has fulfilled all that has been asked of him as a leader of the educational research community, nationally and internationally. He has made significant contributions in bringing research to provide practical benefit to students and teachers locally, regionally, nationally, and internationally.

NURTURING A NEW GENERATION OF SCHOLARS AS AN EDUCATOR

Alan Schoenfeld sees the mentoring of graduate students and scholars as an important part of his professional work. He has devoted time and energy to nurturing young scholars in mathematics education. He is known as one of the pioneers of a form of "Apprenticeship learning" in graduate instruction, in which students learn by doing as well as by reading. He has nurtured students' development of research understandings and created a community of learners that engages seriously and productively with research issues. Going beyond graduate instruction, his contributions to fostering a new generation of scholars in mathematics education is illustrated by the following examples:

Alan seeks to use funded projects to integrate his research work with preparing a new generation of scholars through funded projects. To do so, he has worked to obtain grants for research, development and supporting graduate students. Over his career, he has obtained close to US$40 million in project funding, with the majority of these projects funded by the U.S. National Science Foundation. With this support, Alan has established and sustained research groups through which he has educated graduated students. For example, across its three sites the Diversity in Mathematics Education project produced two dozen researchers whose focus is on issues of diversity.

Through his work as mentor, Alan is making a qualitative difference. The vast majority of his students have gone on to solid academic positions, and are regularly achieving tenure and advancement to full professor at research-extensive institutions in the United States and other countries. A substantial number of the PhDs who studied with him have, themselves, come to wield significant influence in mathematics education both nationally and internationally. A notable instance is Liping Ma's (1999) seminal book *Knowing and teaching elementary mathematics*, which directly benefited from the postdoctoral support and mentoring that Alan provided to her.

The Senior Scholar Award that Alan was given in 2009 by the Special Interest Group for Research in Mathematics Education (SIG/RME) of the American Educational Research Association (AERA) mentions his contributions both to program building at Berkeley and to the specific contributions of a "generation of doctoral and post-doctoral students who, by adopting and adapting your research focus on mathematical cognition, have developed additional research that is breaking new theoretical ground in the study of mathematical thinking." In 2013 Alan was given AERA's Distinguished Contributions to Research in Education award. AERA describes the award as its "premier acknowledgment of outstanding achievement and success in education research. It is intended to publicize, motivate, encourage, and suggest models for education research at its best."

This record of achievement should satisfy anyone – even Alan Schoenfeld.

NOTES

[1] Data from Google Scholar.

[2] A working definition: "Teaching to such a test will lead teachers to deliver a rich curriculum, balanced across learning and performance goals."

REFERENCES

Arcavi, A. (2011). Alan H. Schoenfeld: How we think – A theory of goal-oriented decision making and its educational implications. *ZDM – The International Journal on Mathematics Education*, *43*, 1017–1019.

Burkhardt, H., & Schoenfeld, A. H. (2003). Improving educational research: Toward a more useful, more influential, and better-funded enterprise. *Educational Researcher*, *32*(9), 3–14.

ICMI, (2012). Felix Klein Medal Citation, see http://www.mathunion.org/icmi/.
Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.
Schoenfeld, A. H. (1980). On useful research reports. *Journal for Research in Mathematics Education*, *11*, 389–391.
Schoenfeld, A. H. (1983). Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving focuses in intellectual performance. *Cognitive Science*, *7*, 329–363.
Schoenfeld, A. H. (1985a). *Mathematical problem solving*. Orlando, FL: Academic Press.
Schoenfeld, A. H. (1985b). Making sense of two-person problem solving protocols. *Journal of Mathematical Behavior*, *4*, 1985.
Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. Schoenfeld (Ed.), *Cognitive science and mathematics education*. Hillsdale, NJ: Erlbaum.
Schoenfeld, A. H. (1988, Spring). When good teaching leads to bad results: The disasters of well taught mathematics classes. *Educational Psychologist*, *23*, 145–166.
Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education*, *20*, 338–355.
Schoenfeld, A. H. (1992a). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.
Schoenfeld, A. H. (1992b). On paradigms and methods: What do you do when the ones you know don't do what you want them to? Issues in the analysis of data in the form of videotapes. *Journal of the Learning Sciences*, *2*, 179–214.
Schoenfeld, A. H. (1994, December). A discourse on methods. *Journal for Research in Mathematics Education*, *25*, 697–710.
Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education*, *4*(1), 1–94.
Schoenfeld, A. H. (1999, October). Looking toward the 21st century: Challenges of educational theory and practice. *Educational Researcher*, *28*(7), 4–14.
Schoenfeld, A. H. (2000). Models of the teaching process. *Journal of Mathematical Behavior*, *18*, 243–261.
Schoenfeld, A. H. (2002a, January/February). Making mathematics work for all children: Issues of standards, testing, and equity. *Educational Researcher*, *31*(1), 13–25.
Schoenfeld, A. H. (2002b). Research methods in (Mathematics) Education. In L. English (Ed.), Handbook of International Research in Mathematics Education, pp. 435-488. Mahwah, NJ: Erlbaum.
Schoenfeld, A. H. (2006). What doesn't work: The challenge and failure of the what works clearinghouse to conduct meaningful reviews of studies of mathematics curricula. *Educational Researcher*, *35*(2), 13–21
Schoenfeld, A. H. (2007). Method. In F. Lester (Ed.), *Handbook of research on mathematics teaching and learning* (second edition, pp. 69–107). Charlotte, NC: Information Age Publishing.
Schoenfeld, A. H. (2009). Working with schools: The story of mathematics education collaboration. *American Mathematical Monthly*, *116*(3), 197–217.
Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. New York: Routledge.
Sriraman, B., & Lee, K.-H. (2012). Boole, Dewey, Schoenfeld – Monikers bridging 150 years of thought: A review of *How We Think*. *Journal for Research in Mathematics Education*, *43*, 351–354.

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HANS HEINRICH BRUNGS AND YEPING LI

3. ABOUT GÜNTER TÖRNER AND HIS WORK

INTRODUCTION

Günter Törner is a Professor of Mathematics at the University of Duisburg-Essen in North Rhine-Westfalia, Germany. He was born in Germany in July 1947 and received his Master Diploma (geometry, algebra) in 1972, then two years later his PhD from the University of Gießen, Germany. The supervisors of Günter Törner's graduate studies were Dr. Benno Artmann and the famous geometer Dr. Günter Pickert. His dissertation study was honored with a price for its excellence by the University in 1975. Günter Törner then taught mathematics at the Technical University of Darmstadt (Germany), graduated (Habilitation) in 1977 with a work on Algebra, and went to the University of Paderborn in 1977, finally joining the Duisburg University half a year later in 1978 at the age of 31 as a Full Professor of Mathematics and Didactics. Today he is still engaged in the same university after rejecting honorable invitations from the University of Darmstadt and University of Bayreuth.

As a research mathematician, Günter Törner's research revolves around noncommutative valuation structures, right cones, and associated rings. He started his mathematical career with his doctoral dissertation on *Hjelmslev planes* (see Törner, 1974). These planes were named after the Danish mathematician Johannes Hjelmslev (1873–1950) who defined them as a sort of "natural geometries" – meaning that distinct lines may meet in more than one point, like what may happen in a real drawing. Associated with each (projective) Hjelmslev plane $\mathcal{H}$, there is a natural homomorphism $\mathcal{H} \rightarrow \mathcal{P}$ from a Hjelmslev plane $\mathcal{H}$ onto an ordinary projective plane $\mathcal{P}$. In a "desarguesian case" the geometric axioms for $\mathcal{H}$ are able to construct coordinate rings which are local rings. The coordinate structures are no longer fields, but (noncommutative) local subrings possessing unique chains of left/right ideals. Günter Törner went on to study these rings in detail in his PhD thesis and extensively since then.

In 1973, it happened that Günter met Hans-Heinrich Brungs (University of Alberta, Canada) who is an internationally recognized researcher in noncommutative valuation theory. It turned out that these Hjelmslev rings are strongly related to *noncommutative valuation rings*. After a few years laying geometry aside, Günter Törner made the so-called right chain rings and cones the main topic of his research in pure mathematics and published more than 40 papers in peer-reviewed journals. To further this line of thought, it should be noted that today only a few mathematicians continue doing research in Hjelmslev planes, however, these Hjelmslev rings – in the finite case – have obtained growing attention in coding theory, e.g.,

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in the theory of module codes. At the moment Günter Törner is using his insights into these rings to develop a new personal field of research. He is cooperating with companies in the field of scheduling theory and optimization.

Besides his research in mathematics, Günter Törner entered into the field of mathematics education early in his Darmstadt time while also teaching courses at upper secondary schools in the neighborhood. Those activities in school mathematics education were initially just a requirement in Darmstadt and the early years in Duisburg, but led to a passionate involvement with education that Günter maintains till today. Indeed, Günter is one of the few well-respected scholars all over the world who have been working and being engaged in these two close yet separate research fields: mathematical research and mathematics education.

In the following sections, we shall focus on three aspects of Günter Törner's work and achievements (i.e., his scholarly work as a mathematician, contributions and achievements as a mathematics educator, and accomplishments as an educator and leader), describing them in a bit more detail and pointing to those references that will be most helpful to those who may want to learn more.

SCHOLARLY WORK AS A MATHEMATICIAN

In the following sub-sections, Günter Törner's several major scholarly achievements and contributions in mathematics will be described and summarized.

Prime segments and the classification of rank one chain rings and cones

Günter Törner was the first to prove that there are *three* types of rank one chain rings. This classification also holds for cones, even right cones in groups, and Dubrovin valuation rings. Constructing examples in the exceptional case proves to be a great challenge.

A subsemigroup H of a group G is called a *cone* of G if $H \cup H^{-1} = G$, and a subring R of a skew field F is called a *chain ring* of F if $R \setminus \{0\}$ is a cone in F^*, the multiplicative group of F, or equivalently, $a \in F \setminus R$ implies $a^{-1} \in R$. A subsemigroup H of a group G is a *right cone* of G if $aH \subseteq bH$ or $bH \subset aH$ for elements $a, b \in H$ and $G = \{ab^{-1} \mid a, b \in H\}$. A subring R of a skew field F is a *right chain ring* of F if $R \setminus \{0\}$ is a right cone in F^*.

The group G is *right ordered* if it contains a cone H with $H \cap H^{-1} = \{e\}$, e the identity of G, and $a \leq_r b$ for a, b in G if and only if $ba^{-1} \in H$. This right order on G agrees with the similarly defined *left order* if and only if $ba^{-1} \in H$ implies $a^{-1}b \in H$, that is $a^{-1}Ha = H$, the cone H is invariant and (G, H) is an *ordered group*.

If the chain ring R of F is invariant, $a^{-1}Ra = R$ for $0 \neq a \in F$, then the nonzero principal ideals $\{bR \mid 0 \neq b \in R\}$ form an invariant cone in the orderd group $G = \{aR \mid 0 \neq a \in F\}$ with $aRa'R = aa'R$ defining the multiplication. In that case R is a, possibly non-commutative, valuation ring as considered in Schilling, (1950); or if F is commutative, a classical *valuation ring*.

A non-empty subset I of a cone H is a *right ideal* if $IH \subseteq I$, left ideals and *ideals* are defined similarly. An ideal I of H with $I \neq H$ is called *prime* if $A \supset I, B \supset I$ for ideals $A, B \subseteq H$ implies $AB \supset I$, and *completely prime* if $ab \in I, a \notin I$ implies $b \in I$ for $a, b \in H$.

A pair $P' \supset P''$ of completely prime ideals of H is called a *prime segment* $P' \cdot\!\!\supset P''$ of H if there is no further completely prime ideal of H between P' and P'', for a minimal completely prime ideal P' of H the pair $P' \supset \phi$ is also considered as a prime segment, i.e., we allow $P'' = \phi$ for $P' \cdot\!\!\supset P''$.

Let Q be the union of ideals L of H with $P' \supset L$ for the prime segment $P' \cdot\!\!\supset P''$. If $Q = P''$, then there are no further ideals between P' and P'', the prime segment $P' \cdot\!\!\supset P''$ is called *simple*.

If $P'^2 = P'$ and $P' \supset Q \supset P''$, then Q is a prime ideal but not completely prime; the prime segment is exceptional. In the remaining cases $P' \supset P'^2$ or $Q = P'$ there exists for $a \in P' \setminus P''$ an ideal $I \subseteq P'$ of H with $\cap I^n = P''$ and $a \in I$; this is equivalent with $P'a = aP'$, we say the prime segment is exceptional in this case.

The prime segments of invariant chain rings or cones are invariant, and even though a chain ring R of a finite dimensional division algebra D may not be invariant, its prime segments (see Gräter, 1984) are invariant. Cones with simple prime segments were constructed in Smirnov (1966) and chain rings with simple prime segments were obtained, among others, in Mathiak (1981), Brungs and Törner (1984a), and Brungs and Schröder (1995).

Dubrovin (1994) gave the first example of a chain ring with an exceptional segment and a detailed classification of rank one cones and chain domains was given in Brungs and Dubrovin (2003). These examples were obtained as chain rings in skew fields generated by a group ring $K[G]$ over a right ordered group G with a cone H with exceptional prime segment.

Associated prime ideals and prime segments are used to describe the right ideals of cones and right cones in Brungs and Törner (2009), if all prime segments are invariant. This ideal theory for right cones is applied in Brungs and Törner (2012) to study completions of chain rings. Pseudo convergent sequences are defined for chain rings and used to investigate and construct chain rings R that are I-compact for certain classes of right ideals of R. Some of the results of Krull (1932) and Ribenboim (1968) are generalized, and some results are obtained that appear to be new even in the commutative case.

In an earlier paper on completions (see Brungs-Törner, 1990) we showed that contrary to the commutative case maximal immediate extensions of right chain rings are not necessarily complete.

In the paper by Brungs, Marubayashi, and Osmanagic (2000) the classification of rank one cones is extended to rank one Dubrovin valuation rings R, see the section on extension. Prime segments for R are defined by neighbouring Goldie primes $P' \cdot\!\!\supset P''$, primes for which R/P' and R/P'' are Goldie rings, and the exceptional case is characterized by the existence of a prime ideal Q so that R/Q is not Goldie.

Extensions

Commutative valuation rings play an important role in number theory and algebraic geometry as well as in ring theory. Let V be a valuation ring of a field K contained in a field F, then by a result of Chevalley there exists an extension B of V in F, that is a valuation ring B of F with $B \cap K = V$.

MacLane (1936) shows that a rank one valuation ring of a field K has infinitely many extensions in a simple transcendental extension $F = K(x)$ of K. Many other authors have considered this and related extension problems.

In Brungs and Törner (1984a) the authors construct extensions $\hat{R}$ of a chain ring R of a skew field D in the Ore extension $D(x, \sigma, \delta)$ of D for σ a momomorphim of D and δ a σ-derivation of D. In particular it is shown that $\hat{R}$ can be a chain ring with $\hat{R}$, $J(\hat{R})$ and (0) as its only ideals even though R may be commutative or of infinite rank.

Other authors have considered related extension problems (see Brungs & Schröder, 2001). For the more general case of the skew field of quotients of the group ring $K[G]$ in the case were G is a right ordered group, K is a skew field with chain ring R and $K[G]$ is an Ore domain (see Brungs, Marubayashi, & Osmanagic, 2007). The question whether the group ring $K[G]$ over a right ordered group G with cone H, $U(H) = \{e\}$ is embeddable into a skewfield F is known as Malcev's problem and remains open. Even if F exists it is dificult to construct in F a chain ring R associated with H. If G is ordered, or equivalently if H is invariant, generalized power series can be used to construct an embedding of $K[G]$ into a skew filed F and to obtain in F a chain ring R associated with H.

In Brungs and Gräter (1989) it is shown that there are at most n chain rings R with $R \cap K = B$ in a finite dimensional division algebra D with center K and $D : K = n^2$. These extensions of B in D are conjugate, but none may exist.

By replacing skew fields by simple artinian algebras Dubrovin (1984) defines a *Q-valuation ring* as a subring R of a simple artinian algebra Q with an ideal M so that R/M is simple artinian and for every q in $Q \setminus R$ exist r_1, r_2 in R with $r_1 q, q r_2$ in $R \setminus M$. Matrix rings over chain domains form one class of examples for Q-valuation rings. He proves that for every valuation ring V in the center K of a simple algebra Q, finite dimensional over K, there exists a Q-valuation R with $R \cap K = V$. That any two such extensions R of V in Q are conjugate in Q was proved in Brungs and Gräter (1990).

If in the definition of Q-valuation rings R, now called *Dubrovin valuation rings*, the algebra Q as well as R/M are assumed to be skew fields, then R is a chain domain. There now exists a rich theory of Dubrovin valuation rings, and one of the applications of these various valuation theories is the better understanding and construction of certain division rings and algebras (see Marubayashi, Miyamoto, & Ueda, 1997).

Structure of chain rings

It is tempting to ask whether the structure theorems of I.S. Cohen (1946) for noetherian commutative complete valuation rings can be extended to noetherian right chain domains R with $J(R) = zR$, $\bigcap z^n R = (0)$, which are complete with respect to the topology defined by using the $z^n R$ as neighborhoods of 0.

If R contains a skew field F of representatives of R/zR, then R will be isomorphic to the power series ring $F\big[[z, \delta_0, \delta_1, \ldots]\big]$ with elements $\alpha = \sum_{i=0}^{\infty} z^i a_i$ for $a_i \in F$ and

$$(1) \qquad az = za^{\delta_0} + z^2 a^{\delta_1} + \ldots + z^{n+1} a^{\delta_n} + \ldots$$

defining the multiplication where $a \in F$ and the δ_i are certain mappings from F to F.

Conversely, given a sequence $(\delta_0, \delta_1, \delta_2, \ldots, \delta_n, \ldots)$ of maps δ_i from F to F, we will say that this sequence is admissable if the multiplication as given by (1) does define a power series ring $F\big[[z, \delta_0, \delta_1, \ldots]\big]$. P.M. Cohn had shown that $(id, \delta, \delta^2, \ldots, \delta^n, \ldots)$ is an admissable sequence for δ a derivation of F.

In Brungs and Törner (1984b) a familiy of admissable sequences with $\delta_0 = id$ and $\delta_n = g_n(\delta)$, is given, with F a commutative field of characteristic zero, δ a derivation of F and the $g_n(x)$ polynomials in $K[x]$ with K the subfield of constants of F. The generating function

$$H(x, y) = \sum_{0}^{\infty} g_n(x) Y^{n+1}$$

is used in the proofs. Many additional results about admissable sequences were obtained by, among others, Vidal and Roux; Martin Schröder, who was primarily interested in skew fields with a rank one valuation, also investigated admissable sequences very carefully without publishing his results. Contrary to the commutative case there may not exist a field F of representatives for R/zR whenever R and R/zR have the same characteristic (see Vidal, 1977). The structure of finite chain rings is well understood and these rings are used in coding theory.

Right chain domains

Günter Törner has always been intrigued by the fact that many results for chain rings also hold for right chain rings (see Bessenrodt, Brungs, & Törner, 1990), but that there are instances where the results will be strikingly different.

(i) The over rings of a chain ring R in a skew field F are given as localizations of R at completely prime ideals, but right chain rings S exist with just two completely prime ideals and infinitely many over rings in their skew field of fractions (see Brungs & Törner, 2012).

(ii) A noetherian chain domain R has only the non-zero right ideals $z^n R$, $n \in \mathbb{N}$, if $J(R) = zR$, whereas the semigroup of right ideals of a right noetherian right chain domain can be isomorphic to the semigroup $H_I = \{\alpha \mid \alpha < \omega^I\}$ of ordinal numbers less than ω^I for any power of ω, the order type of $\mathbb{N}$, under addition (see Ferrero & Törner, 1993).

(iii) Frege, around 1900 is concerned with, among other things, the construction of real numbers. He essentially asks whether a right cone H of a group G with $U(H) = \{e\}$ must also be a left cone. A negative answer is given in a paper by Adeleke, Dummett, and Neumann (1987). Further examples of right cones that are not left cones are given in Brungs and Toerner (2002).
(iv) Even though the classification of segments and rank one cones carries over to right cones, we have not been able to prove that the infinite list of possibilities in the exceptional case for cones is also complete for right cones.

Discrete mathematics, applied mathematics

The above sections deal only with a selection of certain areas in pure mathematics that were influenced by Günter Törner's research. Besides collaborating with, among others, M. Ferrero on distributive rings and Chr. Bessenrodt on overrings and locally invariant valuation rings, he wrote (with Bessenrodt and Brungs) a set of lecture notes on right chain rings that were a valuable source of information for researchers.

We want just to remark that Günter was and is active in collaborating with firms, modelling mathematical problems which belong to *Discrete Mathematics*. E.g. Günter is showing expertise in the optimization of work flows and machine schedules as well in pricing structures within the energy markets. It is a characteristic of Günter that he likes to be a translator between various problem fields in industry and economies and the relevant mathematical fields.

SCHOLARLY WORK AS A MATHEMATICS EDUCATOR

Over the past 30 years Günter has been passionate about mathematics education, and occupied with didactical research. His research and interest in mathematics education have changed gradually, which presents three topic areas in mathematics education (mathematical problem solving, beliefs, and mathematical teachers' professional development) that have different focuses but internal connections. We will highlight Günter's work in these three topic areas as follows.

Mathematical problem solving

Although Günter's main research agenda was in pure mathematician at the early stage of his professional career, he paid close attention to finding ways of helping students learn mathematics better in specific content areas. Going beyond his own work in mathematical problem solving as a mathematician, Günter took initial efforts of reflecting and studying mathematics instruction and problem solving in geometry and linear algebra. He already had 10 journal articles published on these topics before 1990. Mathematics content areas as calculus, linear algebra and stochastics seemed to be the most important topics Günter thought prospective teachers should learn and be good at in the 1970s and 1980s. In addition to journal article publication, Günter also contributed to mathematics education through publishing textbooks. In 1980 he published a school textbook on linear algebra with his doctoral supervisor Benno Artmann (Artmann & Törner, 1980), which transformed the ideas and concepts of the famous Gilbert Strang's textbook. This was the first textbook in Germany to unsheathe the arithmetical concepts for vectors. In 1983 Günter published a monograph on didactics of calculus together with his colleague Werner Blum (Blum & Törner, 1983), which is still useful today. In 1985 Günter also proudly coauthored a new edition of Ineichen's book on stochastics (Ineichen & Törner, 1985). The same book was originally used as a textbook for Günter in his school times 20 years before.

In the 1980s problem solving was the focus in mathematics education in the United States. Because Günter was internationally orientated, he followed this initiative to start problem solving research in Germany. He did some empirical research with Walter Szetela (at the University of British Columbia), organized in-service training courses and coached some PhD thesis around problem solving. Of course, some small elements of problem solving were integrated in the German curriculum, however, it became evident that problem solving as an independent topic would not receive sufficient credits in the German system in comparison with some other societies. Problem solving can be taken as a pedagogical philosophy within the classroom but not as an extra subject. So nearly nothing was changed. Although Günter soon shifted away from problem solving research in mathematics education, he has not really given up his work in problem solving in school mathematics for students and teachers. In fact, he later co-edited a special issue of ZDM on problem solving with Alan Schoenfeld and Kristina Reiss in 2007 (Törner, Schoenfeld, & Reiss, 2007).

Mathematical beliefs

During the course of mathematics teaching and problem solving research, Günter stumbled over *misleading, nonflexible, and inadequate beliefs* that needed to be changed. The observation was in general similar to the experiences in the 1980s in the States, although the beliefs in question might be of different type. Nevertheless, successful problem solving can only be established as long as beliefs are in favor of that type of mathematical work. These observations explain why beliefs research became highly important to Günter.

In fact, he was the first researcher in mathematics education in Germany in the 1990s involved in research about beliefs. Realizing the importance of beliefs in mathematics teaching and learning but the lack of relevant research, Günter called beliefs at that time a hidden variable. Günter was granted the privilege to have a series of workshops on mathematics education with a focus on mathematical beliefs in the Mekka of mathematics – in Oberwolfach (Black Forest). Nearly all mathematicians know the name of this small village high up in the Black Forest of Germany. These workshops led to the publication of seven proceedings about research on mathematical beliefs from 1995 to 2000. Building upon these works, Günter co-edited and published a special book on beliefs with Leder and Pehkonen in 2002 (Leder, Pehkonen, & Törner, 2002). This book has been well circulated in the mathematics education community and has promoted further research on beliefs.

Beliefs have been a main topic in Günter's mathematics education research. Over 50% of his 75 article publications since 1990 are related to the topic of beliefs. With his training as a mathematician, Günter tended to bring mathematical rigor to educational research, including the topic of beliefs. For example, he noticed that the existing literature on beliefs failed to provide a consistent and coherent definition of and understanding about beliefs. He thus introduced a mathematical way to structure and formalize key aspects of the concept of beliefs (Törner, 2002).

> Because of the rigor and clarity of his approach, Törner has provided a theoretical model that deepens our insights into the complexities entailed in processes of human believing. In doing so, he opens new possibilities and starts a new conversation in this field of considerable significance for mathematics education. (Presmeg, 2008, p. 97)

Mathematical teachers' professional development

Günter has worked with mathematics teachers since the early stage of his professional career. However, mathematics teachers' professional development did not take a center stage until the 21st century. His work on mathematics teachers' professional development builds upon his research on mathematical problem solving and beliefs, and aims to find ways to help improve mathematics teachers' proficiency in mathematics together with beliefs.

The significance of his work on mathematics teachers' professional development relates closely to the scope of such work and its contribution. With a grant support obtained in 2005 for a large project on in-service teacher training (Mathematics done differently), Günter has devoted much of his efforts to the development and research of a new program: continuous professional development (CPD).

The project "Mathematics done differently" aimed to bring the best trainers and experts together to provide on-demand training and so selected an arbitrary list of topics. Eighty-eight trainers were involved to provide 406 courses to 8657 participating mathematics teachers. In this project, CPD of mathematics teachers needed to be an integral part of school development and, therefore, is far beyond trivial. Building upon this initiative, Günter is now working with others to establish the German Center of Mathematics Teacher Education (DZLM) that has recently been funded by the Deutsche Telekom Foundation.

ACCOMPLISHMENTS AS A LEADER AND EDUCATOR

Being a research mathematician and mathematics educator, Günter has had many opportunities to learn about different perspectives about mathematics and mathematics education. He sometimes described his experiences and feelings as follows: Frequently he realizes that his fellow mathematicians claim to know everything about mathematics learning and ask ironically why researching such an obvious phenomenon but just go ahead to learn mathematics. Yet, intensive learning of mathematics is necessary but not sufficient in many cases when insights from the mathematics education community are much needed. Such an ignorance leads Günter to flee to the mathematics education community, but soon he becomes aware that many of these colleagues in mathematics education think they know mathematics, but really only have a limited and narrow world view of mathematics that is developed through their careers. The epistemological nature of mathematics can only be recognized by continuously living research in mathematics. So Günter is devoted to seeking bridges to connect these two communities.

As one of a few scholars in Germany working internationally, simultaneously, and continuously, in the two "distant" areas of mathematical research and mathematics education, he was elected in 1997 to the Executive Board of the German Mathematical Society (DMV) and has been serving on this board since, as Secretary in its directorate since 2005. He also serves as the chairman of an international committee on mathematics education within the European Mathematical Society (EMS). Günter is also a founding member of the National Center for Mathematics Teacher Education in Germany (DZLM) that was recently initiated by the Deutsche Telekom Foundation. With his leadership roles in mathematics and mathematics education in Germany and internationally, Günter has promoted communication, understanding, and collaboration between mathematicians and mathematics educators.

With his scholarly work in mathematics and mathematics education, Günter has been a great educator and advisor of 12 PhD students in mathematics since 1987, and 13 PhD students in mathematics education since 1992. Many of his doctoral students have gone on to academic positions and make important contributions in mathematics and mathematics education both nationally and internationally. For example, Bettina Roesken-Winter is Günter's formal student and now a professor of mathematics education at Ruhr-Universität Bochum. Her research clearly connects Günter's with a focus on the role of the affective domain for the teaching and learning of mathematics, the professional growth of mathematics teachers, and the interplay of teacher cognition, beliefs, and practice. She is also leading the DZLM's department that offers professional development for mathematics teachers in upper secondary education.

SUMMARY

Günter Törner is one of a few scholars internationally who are well respected in both mathematics and mathematics education communities. He has made sustained and outstanding contributions over a long career to many aspects of research in mathematics and mathematics education. He is persistent in pursuing new ideas and collaborations with passion about mathematics and mathematics education. Such a spirit also led him to pursue a few very interesting projects on the basis of cooperation with firms and institutions, in traffic, in scheduling theory, and in information processing. In these projects Günter again applies his expertise in discrete mathematics and provides research opportunities and training for his PhD students.

We should also note that Günter is married to a teacher of mathematics and German language who teaches at a comprehensive high school in Germany. As indicated by Günter, their marriage is happy with two sons of which they are extremely proud. Now they are also very happy grandparents of a granddaughter at age 4. Besides this – which is more than a footnote for Günter – he has been, and still is, highly engaged voluntarily in a German Christian church for more than four decades.

ACKNOWLEDGEMENTS

The authors are grateful and feel very fortunate that we met Günter Törner. We especially want to thank Günter for his help when developing this chapter. We are certain that our friendship with Günter will continue. Hans also hopes that his mathematical collaborations with Günter, which gave him enjoyment and satisfaction, will produce a few more results.

REFERENCES

Adeleke, S. A., Dummett, M. A. E., & Neumann, P. M. (1987). On a question of Frege's about right-ordered groups. *Bull. London Math. Soc.*, *19*(6), 513–521.

Artmann, B., & Törner, G. (1980). *Lineare Algebra – Ein Grund- und Leistungskurs*. Göttingen: Vandenhoeck & Ruprecht.

Bessenrodt, Chr., Brungs, H. H., & Törner, G. (1990). Right chain rings. Part 1 (Completely revised version). *Schriftenreihe des Fachbereiches Mathematik, Universität Duisburg*, *181*.

Blum, W., & Törner, G. (1983). *Didaktik der Analysis*. Göttingen: Vandenhoeck & Ruprecht.

Brungs, H. H., & Dubrovin, N. I. (2003). A classification and examples of rank one chain domains. *Trans. Amer. Math. Soc.*, *355*, 2733–2753.

Brungs, H. H., & Gräter, J. (1989). Valuation rings in finite-dimensional division algebras. *J. Algebra*, *120*(1), 90–99.

Brungs, H. H., & Gräter, J. (1990). Extensions of valuation rings in central simple algebras. *Trans. Amer. Math. Soc.*, *317*(1), 287–302.

Brungs, H. H., & Schröder, M. (1995). Prime segments of skew fields. *Can. J. Math.*, *47*, 1148–1176.

Brungs, H. H., & Schröder, M. (2001). Valuation rings in Ore extensions. *J. Algebra*, *235*, 665–680.

Brungs, H. H., & Törner, G. (1984a). Extensions of chain rings. *Math. Z.*, *185*, 93–104.

Brungs, H. H., & Törner, G. (1984b). Skew power series rings and derivations. *J. Algebra*, *87*(2), 368–379.

Brungs, H. H., & Törner, G. (1990). Maximal immediate extensions are not necessarily linearly compact. *J. Australian Math. Soc.*, *49*, 196–211.

Brungs, H. H., & Törner, G. (2002). Left valuation rings, left cones, and a question of Frege. In F.-V. Kuhlmann, S. Kuhlmann, & M. Marshall (Eds.), *Valuation theory and its applications* (Volume I, pp. 81–87). Volume 32. Toronto: Fields Institute.

Brungs, H. H., & Törner, G. (2009). Locally invariant and semiinvariant right cones. *Communications in Algebra*, *37*(8), 2616–2626.

Brungs, H. H., & Törner, G. (2012). I-compact right chain domains. *Journal of Algebra and its Applications*, to appear.

Brungs, H. H., Marubayashi, H., & Osmanagic, E. (2000). A classification of prime segments in simple artinian rings. *Proc. Amer. Math. Soc.*, *128*(11), 3167–3175.

Brungs, H. H., Marubayashi, H., & Osmanagic, E. (2007). Gauss extensions and total graded subrings for crossed product algebras. *J. Algebra*, *316*(1), 189–205.
Cohen, I. S. (1946). On the structure and ideal theory of complete local rings. *Trans Amer. Math. Soc.*, *59*, 54–106.
Dubrovin, N. I. (1984). Noncommutative valuation rings. *Trans. Moscow. Math. Soc.*, *1*, 273–287.
Dubrovin, N. I. (1993). The rational closure of group rings of left orderable groups. *Mat. Sbornik*, *184*(7), 3–48.
Ferrero, M., & Törner, G. (1993). On the ideal structure of right distributive rings. *Comm. Algebra*, *21*(8), 2697–2713.
Gräter, J. (1984). Über Bewertungen endlich dimensionaler Divisionsalgebren. *Results in Math.*, *7*, 54–57.
Ineichen, R., & Törner, G. (1984). *Stochastik – Ein Grund- und Leistungskurs*. Göttingen: Vandenhoeck & Ruprecht.
Leder, G. C., Pehkonen, E., & Törner, G. (Eds.). (2002). *Beliefs: A hidden variable in mathematics education?* Dordrecht, the Netherlands: Kluwer Academic Publishers.
MacLane, S. (1936). A construction for absolute values in polynomial rings. *Trans. Amer. Math. Soc.*, *40*(3), 363–395.
Marubayashi, H., Miyamoto, H., & Ueda, A. (1997). *Noncommutative valuation rings and semi-hereditary orders*. Dordrecht: Kluwer Academic Publishers.
Mathiak, K. (1981). Zur Bewertungstheorie nicht kommutativer Körper. *J. Algebra*, *73*(2), 586–600.
Presmeg, N. (2008). Mathematizing definitions of beliefs. In B. Sriraman (Ed.), *Beliefs and mathematics*, The Montana Mathematics Enthusiast Monograph 3 (pp. 93–97). Charlotte, NC: Information Age Publishing.
Schilling, O. F. G. (1989). *The theory of valuations*. Providence: American Mathematical Society.
Smirnov, D. M. (1966). Right-ordered groups. *Algebra i Logika*, *5*(6), 41–59 [in Russian].
Törner, G. (1974). Eine Klassifizierung von Hjelmslev-Ringen und Hjelmslev-Ebenen. *Mitt. Math. Sem. Gießen*, *107*, 77 pp. [MathRev 50:5618].
Törner, G. (1976). Unzerlegbare, injektive Moduln über Kettenringen. *J. Reine Angew. Mathematik*, *285*, 172–180.
Törner, G. (2002). Mathematical beliefs – A search for a common ground: Some theoretical considerations on structuring beliefs, some research questions, and some phenomenological observations. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 73–94). Dordrecht, the Netherlands: Kluwer Academic Publishers.
Törner, G., & Hoechsmann, K. (2004). Schisms, breaks, and islands – Seeking bridges over troubled waters: a subjective view from Germany. In D. McDougall, (Ed.), *Proceedings of the 26th Conference for the International Group for the Psychology of Mathematics Education*, North America Chapter (PME-NA), Vol. 3 (pp. 993–1001), Toronto, Canada.
Törner, G., Schoenfeld, A., & Reiss, K. (2007). Problem solving around the world: Summing up the state of the art. *ZDM – The International Journal on Mathematics Education*, *39*(5–6), 353–563.
Vidal, R. (1977). Contre-exemple non commutatif, dans la théorie des anneaux de Cohen. *C.R. Acad. Sci. Paris, Sér. A-B*, *284*(14), 791–794.

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PART II

PROFICIENT PERFORMANCE, BELIEFS, AND METACOGNITION IN MATHEMATICAL THINKING, PROBLEM SOLVING, AND LEARNING

KRISTINA REISS, ANKE M. LINDMEIER, PETRA BARCHFELD AND BEATE SODIAN

4. DEVELOPING PROBLEM SOLVING SKILLS IN ELEMENTARY SCHOOL

The Case of Data Analysis, Statistics, and Probability

INTRODUCTION

If somebody asked us to address the merits of Alan Schoenfeld and Günter Törner for mathematics education in a single sentence we would probably argue that they are mathematicians and mathematics educators who are able to think mathematically in both contexts and to share their way of thinking with both communities. Both are extensively involved in the topic of mathematical problem solving. They not only contributed to that topic through their own work, but also emphasized it as an important topic of educational research (Schoenfeld, 1992; Törner & Zielinski, 1992; also Törner, Schoenfeld, & Reiss, 2007).

Thinking mathematically and solving mathematical problems successfully are skills, which evolve in the course of learning mathematics in a school context as well as in an everyday context. How these different contexts may contribute to an understanding of mathematics will be discussed in this paper. We will present a study on the statistics competencies of elementary school children from grades 2 through 6.

The topic "data and probability" was included in the mathematics curriculum of recent publications of school standards in many countries (Common Core State Standards Initiative, 2010; Kultusministerkonferenz, 2004; National Council of Teachers of Mathematics, 2000). According to these standards, children should learn to

> formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them; select and use appropriate statistical methods to analyze data; develop and evaluate inferences and predictions that are based on data; understand and apply basic concepts of probability. (National Council of Teachers of Mathematics, 2000)

Some publications and in particular the German standards for school mathematics even suggest including this topic in the elementary school curriculum of grades 1 through 6. However, understanding data and probability has long been considered as possible development at a relatively late point in childhood development (Inhelder & Piaget, 1958). Research studies provide evidence that young students are hardly able to deal with more than basic concepts of data and probability and prefer the intuitive use of concepts and methods (Fischbein, 1987; Reiss & Winkelmann, 2008). The demands of standards for school mathematics on the one hand and

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children's abilities to understand the topic on the other hand might therefore be seen as conflicting issues.

There are some studies suggesting that children might favor deductive rather than stochastic thinking. This is probably a result of how mathematics is introduced in their classrooms. Moreover, as topics like fractions, ratio, and proportion are hardly taught in lower grades, children lack basic mathematics concepts as prerequisites. However, research from developmental psychology suggests that even young children may have an initial understanding of evidence-based scientific reasoning. A study by Koerber, Sodian, Thoermer, and Nett (2005) revealed e.g. that children aged 4, 5, and 6 were able to correctly interpret simple examples of covariation in data, which were presented in an everyday context. In our view, this evidence-based scientific reasoning and mathematical activities involving data and probability have an important aspect in common as they are grounded on a similar philosophy of science, which may be characterized by its experimental research paradigm. Accordingly, we regard evidence-based scientific reasoning and stochastical argumentation as closely related with respect to learning and understanding.

In the following, we will concentrate on specific aspects, namely the understanding of basic probability concepts, the characteristics of sampling, and the need for base rate information. Our analysis takes data from children in grades 2, 4, and 6 ($n = 160$) into account. In particular, our study gives information on how data analysis strategies evolve in the case of contingency tables. Descriptions of individual students' strategic choices help us to understand decision-making in probabilistic contexts.

The tasks presented to the children can be regarded as either mathematical or everyday problems. The use of the word "problem" is due to the idea that elementary school students may successfully deal with tasks that involve the use of data and probability but do mostly not know straightforward means for their solution. This definition fulfills the criteria for a problem-solving activity: A task is regarded a problem if a person wants to accomplish it but does not have the means for an immediate transformation of the initial state into this goal state (Duncker, 1935; Schoenfeld, 1985, 1992; Törner, 2009). Problem solving includes overcoming a barrier and finding a solution. Obviously, this definition suggests that problem solving is an individual activity that depends on the specific knowledge and skills of a person, which have to be combined with adequate heuristics and control strategies (Schoenfeld, 1985) in order to solve the problem. In the following, we will focus on problem-solving activities related to evidence-based scientific reasoning.

SCIENTIFIC THINKING AND REASONING

In the following, we will use the term "scientific" in the sense that it does not include mathematics but refers to natural science and to an experimental and evidence-based research paradigm in particular. Scientific thinking in this interpretation is characterized as a set of intentional, strategic, and meta-strategic processes aiming at the acquisition of new knowledge (Sodian & Bullock, 2008). It is based on information derived from experiments and addresses the evaluation of hypotheses. Koerber et al. (2011) identify three components. In their view, scientific thinking includes the understanding of experimental methods, the ability to interpret data, and an adequate epistemological understanding. Moreover, scientific thinking is closely connected to scientific reasoning, which is regarded as a specific kind of argumentation. In particular, reasoning scientifically is generally not a part of everyday experiences (Bullock, Sodian, & Koerber, 2009).

In a correspondent conception, scientific reasoning can be regarded as the ability to coordinate theories or beliefs, hypotheses, and corresponding evidence (Sodian, Zaitchik, & Carey, 1991). In this context, the notions of theory and beliefs refer to the cognitive representation of a situation. This understanding of theory incorporates formal as well as informal theories and can also refer to implicit or intuitive theories. If beliefs are examined in respect to their correctness, they become hypotheses. Evidence refers to information about a situation and stems from empirical observation (Sodian, Zaitchik, & Carey, 1991; Kuhn & Pearsall, 2000). Hypotheses can be tested if evidence is available that either supports or falsifies that hypothesis.

If evidence stands in contrast to an individual's theories and/or beliefs, scientific thinking and reasoning becomes particularly important. Successful processes lead to an adaptation of the individual's belief system, so that scientific understanding – as the result of successful scientific thinking and reasoning – improves (Kuhn, 2011). Scientific thinking fails, for example, if theory and evidence are not sufficiently coordinated or if evidence is "remodeled" in order to match an individual theory.

The processes of scientific reasoning can be structured according to the four phases of inquiry, analysis, inference, and argument. These phases can be understood as forming a cycle of scientific investigation (Kuhn, 2011). Scientific reasoning presupposes an initial problem and the idea to investigate this problem ("inquiry"). An "analysis" of its components leads to adequate implications and conclusions ("inference") and their formulation as (scientific) "arguments."

PROBLEM SOLVING AND SCIENTIFIC REASONING

Scientific reasoning is – at least in Germany – not explicitly taught in elementary school. However, scientific phenomena occur in the German elementary school curriculum (e.g., "floating and sinking" in early science education). An approach to such topics may typically be guided by the question "why" things happen the way one can observe them. Classroom discussions will probably initiate problem-solving processes.

The problem-solving framework suggested by Schoenfeld (1985) is an adequate tool for analyzing these processes. It is self-evident that a basic domain knowledge concerning the mathematical content is important for understanding a problem and therefore for successful problem solving. In the case of data and probability, this knowledge will encompass knowledge of concepts (e.g., important notions like sure, possible, impossible) as well as knowledge of procedures (e.g., throwing dice and interpreting the result). With respect to the heuristics of evidence-based scientific reasoning, one may think of searching supportive evidence or counter-examples as well as hypotheses and alternate hypotheses. Moreover, mathematical heuristics like estimation strategies can facilitate the evaluation of numerical data. Control processes include a monitoring of the solution process, coordination between theory and evidence, and keeping track of calculations or estimations. Beliefs about science, the role of evidence, the nature of chance or predictive arguments frame the reasoning process.

PROBABILITY AND CHANCE IN EVIDENCE-BASED SCIENTIFIC REASONING

The ability to analyze empirical evidence requires an evaluation of data resulting from scientific experiments. From a mathematical point of view, the corresponding processes can be assigned to uncertainty and chance, so that empirical data can be examined with statistical means. However, data analysis is not restricted to mathematical contexts. There are numerous everyday situations, which ask for the correct interpretation of empirical data (e.g., identifying the effectiveness of a medical treatment).

Research in developmental psychology concentrates mostly on these everyday experiences and their evaluation but also provides evidence that even young children can recognize systematical aspects. Elementary school students can e.g. differentiate between a controlled and a confounded experiment (Bullock & Ziegler, 1999). However, several studies show that these results cannot be generalized to a more mathematically oriented context. In particular, work by Green (1982) as well as work by Fischbein and Schnarch (1997) reveals that children and adults tend to use inadequate models of probability concepts.

The conflicting results in these different research areas seem to be quite consistent. Accordingly, it should be interesting to combine the underlying ideas and methods to clarify the mutual interactions of everyday and mathematical contexts. In particular, the intuitive approaches of young children in these different contexts have not been addressed in research but should be better understood in order to model suitable learning trajectories and to define and choose adequate teaching procedures.

RESEARCH QUESTIONS

Our study aimed to describe elementary school students' competencies (grades 2, 4, and 6) in basic statistics/scientific reasoning and the development of those competencies during elementary school. We addressed components of their declarative, as well as procedural knowledge by presenting appropriate problem-solving situations. Research questions involve the understanding of basic concepts and procedures as well as of problem-solving strategies in an everyday as well as in a mathematics context.

(1) Do elementary school children understand basic statistical concepts and principles?
(2) Are elementary school children able to solve basic problems concerning evidence-based scientific reasoning?
(3) Which problem-solving strategies do elementary school children use when dealing with evidence-based scientific reasoning?

SAMPLE AND METHOD

The sample consisted of 158 elementary school children (90 male, 68 female; 52 children from grade 2; 53 children from grade 4; 53 children from grade 6). All children took part in two individual standardized interviews lasting for 20 minutes each. These interviews encompassed items asking about their basic understanding of probability concepts, base rates and sampling procedures, and contingency tables. In one interview, items were presented in a content-oriented everyday context. In the other interview, items were presented in a more abstract-formal mathematical context. The sequencing of the interviews (content-oriented interview first or abstract-formal interview first) was randomly assigned to each child. Children were interviewed in a school setting but outside their classrooms by trained interviewers. All interviews were videotaped in order to allow a differentiated coding and an in-depth analysis of the children's arguments.

Children's basic understanding of probability concepts was assessed by asking them to judge events as certain, possible, and impossible. In the everyday context, items were chosen according to Shtulman and Carey (2007). Children were asked to evaluate statements such as "it is possible to find a crocodile under one's bed." In the mathematical context we used a simple game of chance based on drawing an object from an urn. Children were presented bags with cubes of different colors and the composition of the bags was shown to them. These bags contained for example two red and eight blue cubes or seven red and three yellow cubes. The children had to decide if it was possible to draw a blue cube from a bag. They had to choose from answers corresponding to a sure, an impossible, or a possible event. Ten items were presented in the everyday context, and nine items were presented in a formal context. As stated above, the two sets of items were presented in different interviews at different points of time.

Further interview tasks intended to assess children's understanding of base rates and sampling procedures. First, students were confronted with data from an experiment without base rates information. They had to judge whether the data was suitable to verify the hypothesis that should be tested in the experiment. The context referred to testing a medical treatment. Children were supposed to realize that testing a medication in an experiment with a random sample requires information about the base rate of sick persons. The tasks assessed whether students spontaneously showed an understanding for this need of base rates. If the students did not show a spontaneous understanding, their attention was drawn on base rates ("prompting") and it was regarded whether they were then able to see the need for base rates.

This task was implemented only in the everyday context. Second, children were given examples of data drawn from samples of different quality (a singular case, a small sample, a large sample, a non-representative sample, etc.). Depending on the specific problem, they were asked whether conclusions from these data were useful or not. The interviewers asked the children to present their ideas but interfered with specific questions when the children did not solve a problem correctly ("prompting"). Thus, we assessed whether the children showed a full and spontaneous understanding of different sampling characteristics, and whether they were able to understand these characteristics with prompts or did not understand the sampling characteristics.

Finally, a set of items used contingency tables. In such tables, data from scientific experiments (comparison of two conditions with respect to a positive or negative outcome) could be displayed (see Table 1). The children were presented contingency tables and were asked to evaluate the data and to draw a conclusion. Evidence-based scientific reasoning was needed here in order to decide if a specific condition leads to better results than another condition. With respect to this experiment, several strategies had been described in different research studies (for an overview see Zimmerman, 2007; McKenzie, 1994). Universally valid strategies take into consideration the information of all four cells. Deficient strategies – which are nonetheless often efficient strategies – are e.g. the comparison of cell a versus cell b (rule of thumb: if cell a is bigger than cell b, condition 1 is efficient), or the simple evaluation of the maximum of all four cells (rule of thumb: if cell a is the maximum, then condition 1 is efficient).

Table 1. Exemplary contingency table.

	Outcome positive	*Outcome negative*
Condition 1	Cell a	Cell b
Condition 2	Cell c	Cell d

Contingency tables were embedded in the everyday condition (EL) with suitable cover stories. A story introduced a person as a researcher who tried different fertilizers on plants in order to identify the best one. His results were presented in the four cells in the form of healthy and withered plants. A blue fertilizer and a yellow fertilizer had to be judged with respect to their effectiveness (see Figure 1 for an example "tree growing well" vs. "growth discontinued" and Figure 2 for another example "flower growing well" vs. "growth discontinued"). The more abstract-formal mathematical context (AF) was based on games of chance involving different urns. The children were told that a specific urn would include manipulative materials, namely red and blue objects, which might be cubes or beads. After drawing from an urn forty times and placing back the objects, a specific contingency table showed the results.

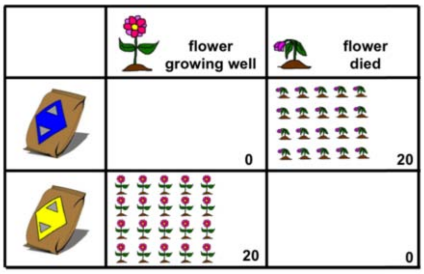

Figure 1. Example table used in the daily-life context, item 1.

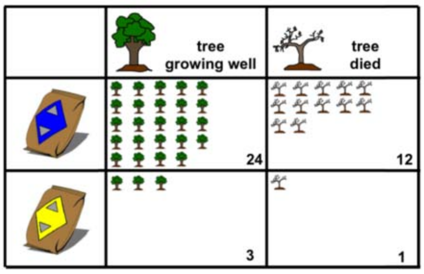

Figure 2. Example table used in the daily-life context, item 4.

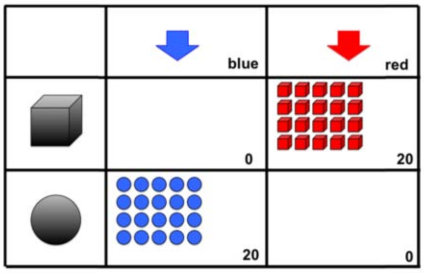

Figure 3. Example table used in the abstract-formal context, item 1.

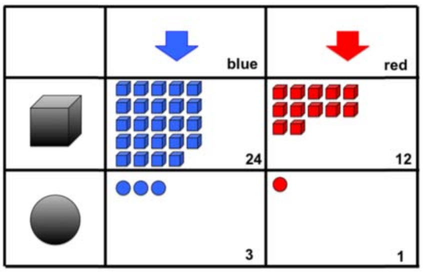

Figure 4. Example table used in the abstract-formal context, item 4.

Children were asked to look at this result and then to argue which object they would draw in order to get a blue one. As cubes and beads are distinguishable by touch, one can purposefully draw a cube or a bead from a bag with an unknown mixture of red and blue cubes and beads. Figures 3 and 4 show examples with different outcomes of cubes and beads in different colors. The tasks were introduced based on bags (the "urn") and the manipulative materials.

Experiments in both contexts included outcomes with different probabilities. In all cases, children were encouraged to give reasons for their choices. Table 2 provides an overview of the types of interview tasks.

The students' responses to tasks, which were used to assess their understanding of basic concepts of probability and basic evidence-based problem-solving, were coded as correct or incorrect. Accumulated solution rates (given as percentages) were used for further analyses. Solution rates were calculated separately for each context condition. Children's understanding of the need for base rates was coded as "no understanding," "prompted understanding," or "full understanding." The understanding of sampling procedures was coded according to one of four levels of understanding (see Table 2). Children who were assigned to "no understanding of sampling procedures" accepted vague explanations and did not see the need for empirical evidence in a scientific reasoning context. Children who were assigned to "basic understanding" favored empirical evidence over an explanation. Some of the children, who did not spontaneously show basic understanding, however, were sensitive to other sample characteristics like representativeness when they were

Table 2. Overview of interview tasks.

Type of task	*Items*	*Coding*	*Context AF / EL*	*Number of items*
Assessment of resources	Need for base-rates	- no understanding - prompted understanding - full understanding	EL	*Not a scale*
	Characteristics of samples	- no understanding - no basic understanding - basic understanding - full understanding	EL	*Not a scale*
	Basic probability concepts	- correct - incorrect	EL AF	10 9
Problem-solving	Basic problems of evidence-based reasoning presented via contingency tables	- correct - incorrect *(plus analyses of strategies)*	EL AF	4 8

asked for this specific attribute (code "no basic understanding"). Children who were assigned to "full understanding" spontaneously addressed sample characteristics like representativeness and sample size. A detailed qualitative analysis of individual students solutions of tasks complemented the quantitative analyses.

RESULTS

Understanding of probability concepts, base rates, and sampling procedures

In the following, we will summarize the results of students' basic understanding (for more information see Reiss, Barchfeld, Lindmeier, Sodian, & Ufer, 2011; Lindmeier, Reiss, Ufer, Barchfeld, & Sodian, 2011). We differentiate between grades 2, 4, and 6.

Many children showed an *understanding for the need of base rates*. In particular, in grade 2, nearly 30% of the students focused on this concept without prompting. In grade 6, almost 55% of the students spontaneously focused on base rate information. Moreover, with specific prompting, 70% of the students in grade 2 gave attention to base rates information. In grades 4 and 6, more than 90% of the children showed an understanding for the need of base rates after they had been prompted. The differences between grade 2 and grade 6 students turned out to be significant, all other differences were not significant ($\chi^2(2; N = 105) = 7.06$, $p = 0.03$ between grades 2 and 4, $\chi^2(2; N = 106) = 2.30$, $p = 0.32$ between grades 4 and 6; tests were conducted using Bonferroni adjusted alpha levels of 0.025 per test).

Most elementary school students understood *characteristics of sampling procedures*. Between 10 and 15% of the students through all grades showed a good comprehension and between 50 and 80% had at least a basic knowledge regarding characteristics of sampling procedures. The difference between 2nd graders and 4th graders was significant ($\chi^2(3; N = 105) = 11.10$, $p = 0.01$), however, the difference between 4th graders and 6th graders was not significant ($\chi^2(3; N = 106) = 1.11$, $p = 0.78$; tests were conducted using Bonferroni adjusted alpha levels of 0.025 per test).

We accumulated the number of correct solutions on tasks concerning *basic understanding of probability* concepts in both context conditions. For further analyses, we chose to use the relative number of correct solutions in order to adjust the different numbers of tasks provided in the everyday and the abstract-formal context condition. These accumulated solution rates are displayed according to grade level in Table 3. Obviously, students had an understanding of probability concepts in both conditions. Differences between grades 2 and 6 proved to be statistically significant as tested by univariate analyses of variance (AF condition: $F(2,155) = 13.66$, $p < 0.01$, $\eta^2 = 0.15$, post-test according to Tukey: $p < 0.01$ between grade 4 and 6 as well as between 2 and 6; EL condition: $F(2,155) = 11.51$, $p < 0.01$, $\eta^2 = 0.13$, post-test according to Tukey: $p = 0.03$ between grade 4 and 6, $p < 0.01$ between grade 2 and 6). Small context differences could be observed ($t(157) = -3.71$, $p < 0.01$, Cohen's $d = 0.36$). The abstract-formal tasks seemed to be somewhat easier for students at these grades compared to everyday tasks. It should be noted, that the everyday tasks were presented in textual form only, whereas the abstract-formal tasks included numerical information.

However, with further analyses of the subset of tasks involving improbable events, we found that students in grade 6 still had specific difficulty in understanding (Table 3). The accumulated rate of solutions for improbable events was approximately 65% in the abstract-formal condition and 50% in the everyday condition. Both solution rates are comparably low. These findings are consistent with results of Shtulman and Carey (2007) who also revealed that it is difficult for children between the ages of 4 and 8 to differentiate between events with low probability and impossible events.

Table 3. Understanding basic probability concepts.

Solution rates (SD)		*Basic understanding of probability*		*Subset of items with improbable events*	
	N	*AF*	*EL*	*AF*	*EL*
Grade 2	52	0.63 (0.20)	0.60 (0.12)	0.32 (0.34)	0.31 (0.25)
Grade 4	53	0.70 (0.22)	0.66 (0.14)	0.49(0.40)	0.41 (0.24)
Grade 6	53	0.83 (0.15)	0.72 (0.11)	0.67 (0.34)	0.50 (0.22)

Our results suggest that elementary school children have a good understanding of important concepts related to evidence-based scientific reasoning. In particular, second graders were able to understand the need for base rates, at least after prompting, and basic principles of sampling procedures. Moreover, they could differentiate between events according to their probability. Students in grade 6 showed a deeper understanding of most of the concepts presented in this study. Nearly all of them were able to understand the need for base rates and the characteristics of sampling procedures after they had been correspondingly prompted.

Problem-solving

Basic problems of evidence-based reasoning

Analyzing contingency tables proved to be a difficult task for elementary school students. This can be documented by their solution rates. In Table 4, exemplary solution rates are presented for some specific items.

Table 4. Evidence-based scientific reasoning: Exemplary solution rates.

Item	*Solution Rates AF (SD)*			*Solution Rates EL (SD)*		
(blue/red cubes vs. blue/red beads)	Grade 2	Grade 4	Grade 6	Grade 2	Grade 4	Grade 6
Item 2	0.79	0.75	0.83			
(10/10 vs. 0/20)	(0.41)	(0.43)	(0.38)		*not used*	
Item 3	0.63	0.75	0.87			
(12/6 vs. 8/14)	(0.49)	(0.43)	(0.34)		*not used*	
Item 4	0.06	0.17	0.21	0.04	0.13	0.21
(24/12 vs. 3/1)	(0.24)	(0.38)	(0.41)	(0.19)	(0.34)	(0.41)
Item 6	0.15	0.28	0.43	0.15	0.40	0.60
(6/7 vs. 11/16)	(0.36)	(0.45)	(0.50)	(0.36)	(0.49)	(0.49)

Data suggests that, at all grade levels, some students were able to correctly interpret contingency tables provided these tables had an easy structure. However, almost all children had difficulties with more complex contingency tables. For example, item 2 introduced an arrangement with a conditional probability of zero for blue beads. In the 2nd grade, 79% of the students were able to decide that based on the displayed results cubes were the better choice in order to get a blue object. Solution rates were also high for item 3, the only item with one of the conditional probabilities higher than .5 and the other conditional probability lower than .5. Students in grade 2 solved this problem correctly with a solution rate of 63%. In grade 6, 87% of the students suggested to draw a cube in order to get a blue object.

Items 4 and 6 were part of a subset of items that were offered in parallelized versions in the abstract-formal and in the everyday contexts. Both were particularly difficult for young students. Item 4 had the lowest rates of correct solutions in both conditions and proved to be difficult for children at all grade levels. Although this item was constructed with an easy multiplicative structure, students were obviously not able to use this structure and to choose a cube as the better choice. Item 6, in contrast, represented a more difficult numerical structure. The exact numerical interpretation of the data should be difficult for young students without prior instruction on fractions and proportional reasoning. Both topics are part of the grade

6 curriculum in German schools and not presented in earlier grades. The results reflect difficulties of 2nd graders and 4th graders in evaluating this table correctly and deciding for cubes. Grade 6 students have a solution rate of 43% (AF) and 60% (EL) but their decisions show high variation in both conditions.

Students' difficulties in dealing with contingency tables were analyzed further with respect to the specific grades. In this analysis, scores for the items in both contexts were accumulated into a scale and the relative rate of success was calculated for each child. We then concentrated on the development of these individual relative solution rates. Progression in solution rates with age was found only for the everyday condition and only between grades 4 and 6 (analysis of variance for the 4-item scale score in each condition, AF: $F(2{,}155) = 2.89$, $p = 0.06$, $\eta^2 = 0.04$; EL: $F(2{,}155) = 9.35$, $p < 0.01$, $\eta^2 = 0.11$). There was no general context effect. However, for tables with simple distributions (with probabilities of only 0 or 1), the everyday context was a facilitating factor. In addition, the level of students' justifications was higher for items in the everyday than in the abstract-formal condition, since students took more cells into account. However, students did not succeed in integrating the information correctly (see Lindmeier, Reiss, Ufer, Barchfeld, & Sodian, 2011, for details).

Strategical decision-making

The quantitative results presented above were complemented by qualitative analyses directed at the identification of students' understanding of contingency tables. In the following, we will present some first results, which illustrate the quantitative findings. We present excerpts from the interviews with three 4th graders who performed differently on the contingency table tasks. For this analysis, we will use three items from the abstract-formal context of varying difficulty (according to the empirical results).

We will provide data from three children, Joe, Mary, and Ann, as their performance on basic problems of evidence-based reasoning ranged in the upper, middle, and lower 20%-percentile. See Table 5 for further characteristics of Joe (aged 9 years, 9 months), Mary (10 years, 6 months), and Ann (10 years, 4 months). Joe showed understanding of base rates after prompting and a good understanding of sampling procedures. Moreover, he showed positive results for the tasks on basic probability concept. Mary understood the need for base rates after prompting, and had a spontaneous basic understanding of sampling procedures. However, she scored low in the tasks on basic understanding of probability concepts. Ann showed an understanding of base rates but had problems with basic sampling procedures. She did, however, well on tasks on basic concepts of probability.

The following transcript excerpts refer to item 3 (12 blue cubes/6 red cubes vs. 8 blue beads/14 red beads, $\chi^2 = 13.3$). All excerpts start with the student's answer to the question: "What would you choose in order to get a blue object?"

Joe

Joe: A cube.
I: Why would you choose a cube?

Table 5. Performance of Joe, Mary, and Ann.

	Joe (ID 123)	*Mary (ID 197)*	*Ann (ID 162)*
Age	9;9	10;6	10;4
*Basic problems of evidence-based reasoning	1	3	5
*Basic concepts of probabilities in different contexts	2EL 3AF	3EL 5AF	2EL 1AF
Need for base-rates	Prompted understanding	Prompted understanding	Full understanding
Sampling procedures	Full understanding	Basic understanding	No basic understanding
*Understanding of science	2	3	1
*Cognitive abilities	4	4	2
*Mathematical abilities	1	3	1
*Working memory abilities	1	1	2
*Verbal abilities	1	3	3

** given as the 20%-percentile of sample the student is located in (1:top 20%, ..., 5:low 20%)*

Joe: Because there are less red cubes, /ehm/ than red beads, and there are more blue cubes than blue beads.

Mary

Mary: A cube.

I: A cube, ok. Why would you choose a cube?

Mary: Because, /ehmehm/ of the beads, if I took only beads then /ehm/ I would have got only 8. Well, then I would have won anyhow, but, /ehm/ I would nevertheless collect more of the cubes as you have, you would surely win then.

Ann

Ann: Cube.

I: Why do you think that a cube is better than a bead?

Ann: . . . mm, or, I think, it does not matter.

I: Does not matter. Why does it not matter?

Ann: Because it is the same color. Blue is blue and blue wins.

The following transcript excerpts refer to item 4 (24 blue cubes/12 red cubes vs. 3 blue beads/1 red bead, $\chi^2 = 0.11$). Moreover, they start with the student's answer.

Joe

Joe: A bead.

I: Why would you choose a bead?

Joe: /Ehm/ because there are more "odds," because there are only four beads and three there of them are blue, there are more odds /ehm/ to draw a blue one thereof than from the blue cubes, because there are also more red cubes there.

Mary

Mary: A cube, because there are /ehm/, somebody had luck, apparently, 24 cubes and only 3 beads. Yes, and the others had red, these are 13 and

/ehm/ yes, I would /ehm/ take the cube.

Ann
Ann: Again, it does not matter.
I: Why do you think, it does not matter?
Ann: Because, again, blue is blue.

The last transcript excerpts refer to item 6 (6 blue cubes/7 red cubes vs. 11 blue beads/16 red beads, $\chi^2 = 0.11$).

Joe
Joe: /Ehm/ a cube
I: And why would you choose a cube?
Joe: Because there are, /ehm/ well ... If you take the 16 minus 11 then there are 5 left. Well, those are 5 red ones. And 7 minus 6, there is one red left. So you have with the 7 cubes more odds /ehm/ to draw something blue, because – well, because ... because there is only one red cube more than there are blue cubes. Well, together.

Mary
Mary: I would take a bead, because 11 were drawn thereof and of the /ehm/ cubes, only 6 were drawn.

Ann
Ann: It does not matter.
I: Why doesn't it matter, whether you take a cube or a bead?
Ann: Because blue is blue.

The interview transcripts give evidence that 4th graders vary significantly in their ways of accessing probability concepts. Some children are able to explain complex stochastic situations, while others lack the ability to comprehend simple concepts.

Joe can be regarded as a successful problem solver with respect to the tasks on statistics and probability. Quite often, he considered all information provided in the specific situation and built arguments to support his choice. Obviously, he used varying but mostly appropriate strategies. For example, he chose a comparison of cell frequencies for the solution of item 3, a comparison of cell differences for item 6, and a proportional strategy for item 4. Therefore, he was able to deal correctly with most of the problems.

Mary was not very flexible in her problem-solving strategies. She used the same strategy for all items and compared the number of objects in the target color (blue cubes vs. blue beads). Moreover, she did not fully understand the setting and interpreted the table as resulting from different people (e.g., in item 4).

Ann concentrated her attention on the idea of equal chances. In all tasks she argued that a specific choice would not matter. This behavior was in line with her difficulty of understanding sampling procedures. She did not show an understanding for the need or the use of empirical evidence to draw inferences. However, her basic understanding of probability concepts was high. Thus, she recognized the influence of chance of the game context, but was not able to use the information given about outcomes in order to evaluate her chances.

DISCUSSION

Our study reflects children's difficulties in understanding problems of data and probability but also reveals aspects of successful handling of the topic. It thus contributes to a more elaborated comprehension of children's understanding of data and probability.

Understanding base rates is an important issue in learning statistics and stochastic thinking. It is therefore important to learn that even young children feel the need to consider base rates. It might be difficult for them to see this need but they are clearly able to address it after being prompted for this aspect. Moreover, young children are also able to understand sampling procedures. The comparison between students from grades 2, 4, and 6 showed a significant development with respect to both aspects between 2nd grade and 4th grade as well as 6th grade. As there is hardly any regular mathematics instruction on this topic, this knowledge might increase as part of the children's general cognitive development.

In all grades, there were some students who showed a basic understanding of probability concepts. These percentages grew significantly with the age of the children. In particular 6th graders were able to successfully apply these elementary concepts. However, there were important differences between using a more formal-mathematical context and an everyday context. The formal context turned out to be easier for the children to judge. This result should not be overly interpreted but provides evidence that fundamental mathematical principles might be successfully presented before applying them to everyday contexts and that understanding is not necessarily in line with an immediate application.

Contingency tables are not part of the elementary school curriculum, but children are able to grasp simple ideas connected to this type of representation. However, most children are obviously unable to generalize their knowledge to more complex tables. The data shows difficulties, which are independent of the specific context. In our view, the qualitative data hints that these difficulties might have their origin in a more unidimensional way of thinking. Children concentrate on specific features of a table and are not able to integrate all the information included in the table to a consistent statement. It should be noted that there are important individual differences. Whereas a majority of young children are not able to adequately deal with this problem, some show an elaborated understanding, which allows generalizations and the correct interpretation of complex contingency tables.

Learning and understanding mathematics can be regarded as a kind of problem solving. This is particularly true for data and probability as a mathematical context. Accordingly, it is useful to recognize children's dealing with probabilistic situations from a problem-solving perspective. We take the theoretical approach of Schoenfeld (1985, 2011) and consider our results with respect to resources, heuristics, control strategies, and beliefs.

Resources include declarative as well as procedural (mathematical) knowledge. In our study, we embedded problems in contexts that were easy to understand. Everyday situations, as well as drawing objects from an urn, obviously did not presuppose a specific knowledge. The results suggest that full conceptual understanding of data and probability is more than correctly applying specific concepts. However, declarative as well as procedural knowledge probably develops in the course of elementary education as a part of general cognitive development. Heuristics is seen here as a broadly used concept which may include specific as well as general strategies such as working backwards or drawing a diagram, which are important for problem-solving. In our study, children did not usually use such techniques. Moreover, we could not identify any use of control strategies. Children did not tend to question their own thinking but stuck mostly to strategies they had used before, independent of their adaptability to different problems. Young children thus confirmed results from research at the secondary level (Reiss, Hellmich, & Thomas, 2002). Children's beliefs were not explicitly addressed in this study. However, the interview data underlined their prominent role for learning processes. Children's everyday ideas on data and probability might have an important impact on the way they judge situations inside and outside of mathematics.

Data, statistics, and probability are prominent fields of the mathematical sciences with important applications in many scientific domains. It is therefore essential to have their precursors integrated in school mathematics. However, our study reveals severe difficulties of young children's thinking in this context. In our view, whether this topic should be integrated early in the mathematics class like suggested by the standards of Kultusministerkonferenz (2003) and National Council of Teachers of Mathematics (2004) should be discussed in more detail. In particular, the prominent individual differences in intuitive probabilistic thinking should be better addressed. It is necessary to investigate these differences in more detail and to provide more empirical evidence. At least, if data and probability are a topic in the elementary classroom, we must provide more elaborated instruction on these concepts in order to support mathematics teachers in their classroom work. Integrating this topic in the mathematics classroom presupposes in our view a better connection between children's thinking and conceptualization of probability principles and the mathematical point of view. The special topic of data, statistics, and probability thus reflects a general goal for mathematics instruction, which has been formulated by Schoenfeld (1992). Moreover, the individual differences suggest a better consideration not only of the mathematical content but also of children's views of this content (Törner, 2004).

At the beginning of the 21st century, challenges of educational theory and practice were addressed in an article by Schoenfeld (1999). He pointed out "arenas," in which research progress would be necessary in the coming years. Two aspects have particularly influenced our research, namely the need of enhancing our knowledge of transfer processes ("How do we make sense of the ways in which people use knowledge in circumstances different from the circumstances in which that knowledge was developed") and the "relationship between curriculum development and research on thinking and learning." We hope we were able to contribute to these challenges of research in mathematics education.

REFERENCES

Bullock, M., & Ziegler, A. (1999). Scientific thinking: Development and individual differences. In F. E. Weinert & W. Schneider (Eds.), *Individual development from 3 to 12* (pp. 38–54). New York, NY: Cambridge University Press.

Bullock, M., Sodian, B., & Koerber, S. (2009). Doing experiments and understanding science: Development of scientific reasoning from childhood to adulthood. In W. Schneider & M. Bullock (Eds.), *Human development from early childhood to early adulthood*. Mahwah, NJ: Lawrence Erlbaum.

Common Core State Standards Initiative (CCSSI) (2010). Common Core State Standards for Mathematics. Retrieved 2011/11/15 from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

Duncker, K. (1935). *Zur Psychologie des produktiven Denkens*. Berlin: Springer.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht: Reidel.

Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic intuitively based misconceptions. *Journal for Research in Mathematics Education*, *28*, 96–105.

Green, D. R. (1982). *Probability concepts in 11–16 year old pupils*. University of Technology, Loughborough, England Report of Research, CAMET.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence*. New York, NY: Basic Books.

Koerber, S., Sodian, B., Kropf, N., Mayer, D., & Schwippert, K. (2011). Die Entwicklung des wissenschaftlichen Denkens im Grundschulalter. *Zeitschrift für Entwicklungspsychologie und Pädagogische Psychologie*, *43*(1), 16–21.

Koerber, S., Sodian, B., Thoermer, C., & Nett, U. (2005). Scientific reasoning in young children. Preschoolers' ability to evaluate covariation evidence. *Swiss Journal of Psychology*, *64*, 141–152.

Kuhn, D. (2011). What is scientific thinking and how does it develop? In U. Goswami (Ed.), *The Wiley-Blackwell handbook of childhood cognitive development*. West Sussex: BlackWell Publishers.

Kuhn, D., & Pearsall, S. (2000). Developmental origins of scientific thinking. *Journal of Cognition and Development*, *1*, 113–129.

Kultusministerkonferenz (KMK). (2004). *Bildungsstandards im Fach Mathematik für den Primarbereich*. Beschluss der Kultusministerkonferenz vom 15.10.2004. München: Luchterhand.

Lindmeier, A., Reiss, K., Ufer, S., Barchfeld, P., & Sodian, B. (2011). Umgang mit wissenschaftlicher Evidenz in den Jahrgangsstufen 2, 4 und 6: Stochastische Basiskonzepte und Kontingenztafelanalyse. In R. Haug & L. Holzäpfel (Eds.), *Beiträge zum Mathematikunterricht 2011*. Münster: WTM.

McKenzie, C. R. M. (1994). The accuracy of intuitive judgment strategies: Covariation assessment and bayesian inference. *Cognitive Psychology*, *26*(3), 209–239.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Reiss, K., & Winkelmann, H. (2008). Kompetenzstufenmodelle für das Fach Mathematik im Primarbereich. In D. Granzer, O. Köller, A. Bremerich-Vos, K. Reiss, & G. Walther (Eds.), *Bildungsstandards Deutsch und Mathematik. Leistungsmessung in der Grundschule* (pp. 120–141). Weinheim: Beltz.

Reiss, K., Barchfeld, P., Lindmeier, A.M., Sodian, B., & Ufer, S. (2011). Interpreting scientific evidence: Primary student's understanding of base rates and contingency tables. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (pp. 33-40). Ankara (Turkey): PME.

Reiss, K., Hellmich, F., & Thomas, J. (2002). Individuelle und schulische Bedingungsfaktoren für Argumentationen und Beweise im Mathematikunterricht. In M. Prenzel & J. Doll. (Eds.), *Bildungsqualität von Schule: Schulische und außerschulische Bedingungen mathematischer, naturwissenschaftlicher und überfachlicher Kompetenzen*. *45. Beiheft der Zeitschrift für Pädagogik* (pp. 51–64). Weinheim: Beltz.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334-370). New York, NY: MacMillan.

Schoenfeld, A. H. (1999). Looking toward the 21st century: Challenges of educational theory and practice. *Educational Researcher*, *28*(7), 4–14.

Schoenfeld, A. H. (2011). *How we think. A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Shtulman, A., & Carey, S. (2007). Improbable or impossible? How children reason about the possibility of extraordinary events. *Child Development*, *78*, 1015–1032.

Sodian, B., & Bullock, M. (2008). Scientific reasoning – Where are we now? *Cognitive Development*, *23*(4), 431–434.

Sodian, B., Zaitchik, D., & Carey, S. (1991). Young cildren's differentiation of hypothetical beliefs from evidence. *Child Development*, *62*, 753–766.

Törner, G. (2004). Mathematische Weltbilder und ihr Einfluss auf Bildung. *Wissenschaftler und Verantwortung – Mitteilungen der Gesellschaft für Verantwortung in der Wissenschaft (GVW)* , *12*(3), 15–29.

Törner, G. (2009). *Mathematical problem solving around the world: Patterns and trends*. Paper presented at the EARLI conference, August 2009, Amsterdam (the Netherlands).

Törner, G., & Zielinski, U. (1992). Problemlösen als integraler Bestandteil des Mathematikunterrichts – Einblicke und Konsequenzen. *Journal für Mathematikdidaktik*, *13*, 253–270.

Törner, G., Schoenfeld, A. H., & Reiss, K. (2007). Problem solving around the world: Summing up the state of the art. *ZDM – The International Journal on Mathematics Education*, *39*, 353.

Zimmerman, C. (2007). The development of scientific thinking skills in elementary and middle school. *Developmental Review*, *27*, 172–223.

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5. TRANSMISSIVE AND CONSTRUCTIVIST BELIEFS OF IN-SERVICE MATHEMATICS TEACHERS AND OF BEGINNING UNIVERSITY STUDENTS

INTRODUCTION

In recent years, the professional competence of teachers has become more and more important as a field of educational research. Teachers create learning opportunities and have a crucial influence on subject related and interdisciplinary achievement of students' educational goals (cf. Baumert & Kunter, 2013a; Reusser, Pauli, & Elmer, 2011). In the context of the COACTIV study a competence model was designed in which competence aspects such as *professional knowledge*, *motivational orientations*, *self-regulative skills*, but also *beliefs* (Figure 2) play an important role. *Professional knowledge* and *beliefs* are often subsumed under the concept of expertise, as they are assumed to be built up by learning processes and can become more differentiated with advanced experience (Baumert & Kunter, 2013a; Bromme, 2008; Woolfolk Hoy, Davis, & Pape, 2006).

In the COACTIV study, the latter two expertise aspects have been proven empirically to be substantial predictors regarding the quality of mathematics lessons as well as the learning achievement of secondary school students (Baumert & Kunter, 2013b; Voss, Kleickmann, Kunter & Hachfeld, 2013a).

Other empirical studies dealing with teachers' individual beliefs and concerning the subject of mathematics and the teaching of mathematics show that those beliefs affect the students' ways of approaching mathematical tasks and how they learn mathematics (Grigutsch, Raatz, & Törner, 1998). Furthermore it is assumed that the *teachers' beliefs* concerning communication and interactions in the classroom and in specific classroom organization have an essential effect on the *students' beliefs*. Therefore, beliefs can be seen as a major starting point for teachers' education and training (Voss et al., 2013a; Kaiser & Maaß, 2006).

The development of mathematics teachers' *content knowledge* and *pedagogical content knowledge* at various points of time in their training was also investigated in the context of the COACTIV study (cf. Krauss, Baumert, & Blum, 2008). Besides the sample of secondary mathematics teachers in the COACTIV main survey (Krauss et al., 2013), students of advanced courses of grade 13 and mathematics teacher trainees at the end of their studies were tested in the context of a validation study. A major result of this study was that the content knowledge and pedagogical content knowledge of teachers improve substantially during university education (Krauss et al., 2008).

Y. Li and J.N. Moschkovich (eds.), Proficiency and Beliefs in Learning and Teaching Mathematics, 51–67.

Up to now, concerning both expertise aspects that in the COACTIV study have proven to contribute to student learning, the question of development has been investigated only for the aspect of professional knowledge. Concerning individual beliefs on the subject of mathematics and the teaching of mathematics in the framework of COACTIV a test instrument was designed – referring to Schoenfeld (1989, 1992) and Törner and Grigutsch (1994; see also Grigutsch, 1996) – and applied (only) with in-service teachers (Voss et al., 2013a; but see Voss, Kunter, & Baumert, 2013b). Therefore, the question of whether and how those beliefs change in the course of the teacher training still remains unanswered.

For that reason we have conducted an additional study on subjective beliefs for the present article, this time with a sample of mathematics teacher trainees at the beginning of their studies. We will present these results in the following and through the comparison of these results with the beliefs of the in-service COACTIV teachers we can speculate on possible fundamental changes in these beliefs during teacher professionalization.

We will begin with a short overview on the COACTIV study itself, and we will explain the theoretical grounds and the conceptualization of the beliefs in the framework of the COACTIV study (a more detailed description can be found in the COACTIV book publication, Voss et al., 2013a). Then we will focus on the research instrument which has been formulated referring to Schoenfeld, Törner and Grigutsch, and finally results will be presented and possible consequences for the development of beliefs will be discussed.

THE COACTIV STUDY

The COACTIV project on Professional Competence of Teachers, Cognitively Activating Instruction, and the Development of Students' Mathematical Literacy aimed at conceptualizing and assessing a broad spectrum of teacher competencies, personality variables, and work-related variables in the context of secondary mathematics instruction. The project was funded by the German research foundation (DFG) from 2002 to 2006 (directors: Jürgen Baumert, Berlin; Werner Blum, Kassel; Michael Neubrand, Oldenburg) and surveyed the mathematics teachers whose classes participated in the PISA 2003/2004 longitudinal assessment in Germany (see Prenzel et al., 2004, for details of PISA 2003 and its German extension to 2004, and Prenzel et al., 2006, for details of the longitudinal German component).

The close relationship between COACTIV and PISA allows, for the first time in Germany, a combined analysis of large-scale data on teachers, their lessons, and their students within a common technical and conceptual framework (Figure 1). Whereas the achievements of students and their personality variables were assessed in PISA (right column), their teachers were surveyed in COACTIV (left column). Parallel questionnaires on lessons (middle column) were administered to both the students (in PISA) and the teachers (in COACTIV) ("multi-perspectivity"). Note that Figure 1 depicts only a fraction of the constructs assessed.

On average, the COACTIV 2003/04 teacher assessment took a total of about 12 h, distributed over the course of a school year. Besides knowledge tests, a broad

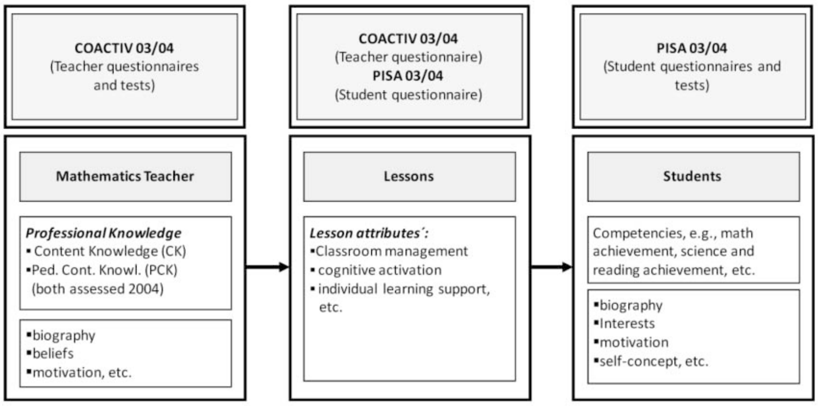

Figure 1. Conceptual connection of the COACTIV 2003/04 study and the PISA 2003/04 study with example constructs.

battery of newly developed (or adapted) instruments tapped teachers' biographical variables, motivational orientations, professional beliefs (see next chapter), and self-regulation (for an overview of the COACTIV instruments, see Kunter et al., 2013). The students (PISA classes) were administered tests and questionnaires on two school mornings (approx. 4 hrs each). The structure of the data allows us to use structural equation modeling to test various causal hypotheses, based on the assumption that the teacher influences the lessons, which in turn influence student learning (as indicated by the arrows in Figure 1). For a general overview on the COACTIV findings, see the COACTIV book publication by Kunter et al. (2013).

THE BELIEFS OF TEACHERS IN THE COACTIV COMPETENCE MODEL

In order to define which qualifications a teacher must fulfill to transform mathematical subject matter successfully into learning opportunities, the concept of "competence," as outlined by Weinert (2001), was taken as a starting point in the context of the COACTIV study. The concept of competence which describes personal and basic learnable prerequisites to cope with specific requirements was applied by Weinert regarding *work-related* requirements and was therefore called "action competence." Besides knowledge and skills, the term "action competence" includes motivational, meta-cognitive and self-regulatory characteristics (Weinert, 2001). It should be noted that competence cannot be understood as a one-dimensional skill; rather it should be treated as a complex set of skills which can be analytically differentiated according to their competence facets (see Figure 2).

For structuring *professional knowledge*, besides beliefs one of the central competence aspects, we go back to the differentiation between content knowledge, pedagogical content knowledge and general pedagogical knowledge which was introduced by Shulman (1986), and this taxonomy in COACTIV is complemented by the categories of specific knowledge on organization and interaction (Hiebert, Gallimore, & Stigler, 2002; Sternberg & Horvath, 1995) and with knowledge on

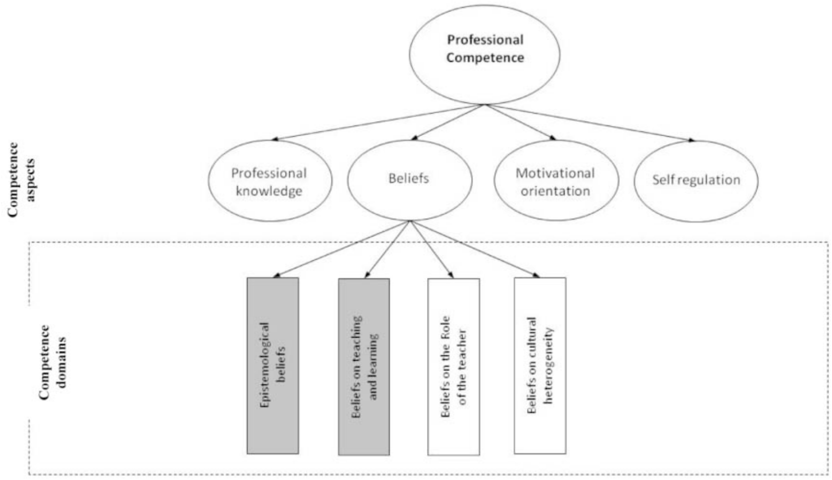

Figure 2. The COACTIV competence model with sub-structuring of the beliefs. The competence domains which are examined in the present paper are highlighted grey.

counseling which is necessary for the conversation with non-professional people (laymen) (Bromme, Jucks, & Rambow, 2000). In the theoretical framework of COACTIV, teachers' professional knowledge includes these five competence domains, which can be further subdivided into various specific competence facets (for an explication see Baumert & Kunter, 2013a).

Like professional knowledge, epistemological beliefs and subjective theories which teachers have concerning the subject and the learning of a subject, have regulative functions for the presentation of the subject matter and for the organization of learning opportunities (Leinhardt & Greeno, 1986; Peterson, Fennema, Carpenter, & Loef, 1989). In addition it must be assumed that motivational characteristics such as perceived self-efficacy or teachers' interests can be seen as a function which controls teachers' behavior in the classroom. Therefore, the professional action competence includes professional knowledge, subjective beliefs, motivational orientations and aspects of self-regulation (see Figure 2).

Definition and conceptualization of beliefs

Although teachers' beliefs play an important role in educational research, no common definition existed until today. That is why terms like "subjective theories," "mathematical beliefs," "conceptions," "philosophy," "ideology perception," "world views," "dispositions," "perceptions," "ideas" and "attitudes" can be encountered when browsing through the literature (cf. Pehkonen & Törner, 1996; Törner, 2002; Törner & Grigutsch, 1994). Alongside this heterogeneity of terms, no mutually accepted definition of beliefs can be found. In general, all definitions have one basic assumption in common, which says that beliefs are structuring the way we face the world and therefore influencing our perceptions, goals and connected action plans (cf. Köller, Baumert & Neubrand, 2000; Törner, 2002; Voss et al., 2013a). In the context of the COACTIV study, beliefs are defined as

> psychologically held understandings and assumptions about phenomena or objects of the world that are felt to be true, have both implicit and explicit aspects, and influence people's interactions with the world. (Voss et al., 2013a)

Besides various definitions, beliefs are quite often summarized into clusters in empirical studies. A manageable categorization has been suggested by Calderhead (1996). Calderhead distinguishes beliefs concerning (1) the *learners and their learning*, (2) the *teachers*, (3) the *subject*, (4) the *knowledge on teaching*, and (5) *oneself*. A further systematization is made by Woolfolk Hoy, Davis, and Pape (2006) who separate beliefs with respect to different system levels to which they refer (cf. Voss et al., 2013a, see as well Op't Eynde, De Corte, & Verschaffel, 2002):

Level 1: Beliefs on oneself, e.g. on own skills as a teacher or on the role of the teacher.

Level 2: Beliefs on the actual teaching and learning context which, considering the subject of mathematics, can be subdivided into beliefs on mathematical knowledge (epistemological beliefs) and on learning and teaching of mathematics.

Level 3: Beliefs on the educational system and the social context which, e.g., include beliefs on the cultural heterogeneity at school.

In the present chapter we will focus on beliefs on the actual teaching and learning context (Level 2), i.e. epistemological beliefs and beliefs on learning and teaching mathematics.

Epistemological beliefs

In psychological research epistemological beliefs are conceptualized by Hofer and Pintrich (1997; cf. Köller et al., 2000; Voss et al., 2013a). They distinguish the learners' epistemological beliefs on the nature of knowledge and beliefs on the nature of knowing. An overview on the distinction of both dimensions can be found in Table 1 (Duell & Schommer-Aikins, 2001; Hofer & Pintrich, 1997; Voss et al., 2013a).

Further categorizations for the epistemological beliefs of teachers and learners within the psychological tradition have, for example, been suggested by Perry (1970) and by Schommer (1990; see also Schommer, Crouse, & Rhodes, 1992). A more detailed characterization of these categorizations can be found in Voss et al. (2013a).

Since the beginning of the 1980s, mathematics education research has been concerned with beliefs. Of great significance is Schoenfeld's work concerning students' beliefs on the nature of mathematics (1989, 1992) which he based on his epistemological conception of mathematics and labeled *mathematical conception of the world*. According to Schoenfeld, students' typical mathematical conceptions of the world are the following (cf. Schoenfeld, 1992, p. 359):

– Mathematical problems have one and only one right answer.
– There is only one correct way to solve any mathematical problem – usually by the rule the teacher has most recently demonstrated to the class.
– Ordinary students cannot expect to understand mathematics. They can simply expect to memorize it, and to apply what they have learned mechanically and without understanding.
– Mathematics is a solitary activity, done by individuals in isolation.
– Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
– The mathematics learned in school has little or nothing to do with the real world.
– Formal proof is irrelevant to processes of discovery or invention.

Table 1. Two dimensions of epistemological beliefs on the nature of knowledge and the nature of knowing.

Beliefs on the nature of knowledge	**Beliefs on the nature of knowing**
Simplicity of knowledge (Knowledge as an accumulation of isolated facts or knowledge as highly interrelated concepts)	*Source of knowledge* (Knowledge acquisition as the accumulation of established truths or as a process of social construction)
Certainty of knowledge (Knowledge as outliving truths or relativistic concepts of knowledge as modifiable and dependent on context)	*Justification and validation of knowledge* (Justification of knowledge through objective procedures or a coexistence of multiple theories)

Schoenfeld's mathematical conception of the world was examined intensively and developed further in the German-speaking world by Törner and Grigutsch (1994; see also Grigutsch, 1996). On the basis of their research they developed a questionnaire to capture students' mathematical conceptions of the world, which assessed five analytically separable epistemological dimensions (Törner, 2002; cf. Kaiser & Maaß, 2006; Köller et al., 2000): (1) *The aspect of formalism* (Mathematics is constructed in a strictly logical and deductive way), (2) *the aspect of schema* (Mathematics is an additive accumulation of concepts and rules), (3) *the aspect of process* (Doing mathematics means to reflect on problems), (4) *the aspect of application* (Mathematics is relevant in many applied domains) and (5) *rigid orientation on schemas* (there is only one single solution for each mathematical task, which has to be learnt by heart).

These five dimensions can be subsumed under two general principles, according to which mathematics can be seen *statically* or *dynamically* (Grigutsch, 1996; Törner & Grigutsch, 1994). In the static view, mathematics is understood as an abstract system which consists of axioms, concepts and the relations between concepts (as well as propositions) and can be regarded as "completely interpreted" mathematical theory. From that point of view, teaching mathematics means the learning of definitions, facts, rules and routines. The dynamic view assumes that mathematics is an action which starts with questions and problems and leads to a collection of experiences and the discovery of principles, which can be arranged to systematic statements on various levels. In dynamic mathematics lessons, ideas and thinking processes take priority. The main point is, that mathematics is (re-)developed here. Nonetheless, these two points of view are not mutually exclusive, and are often referred to as the "Janus-faced" character of mathematics (see Törner, 2002).

In the context of the TIMS study, Köller et al. (2000) developed an instrument to assess students' mathematical conceptions of the world based on the research of Perry (1970) and Schommer (1990) as well as of Schoenfeld (1992) and Törner and Grigutsch (1994). This instrument contains the following scales (cf. Köller et al., 2000, p. 240f.):

1. *Mathematics as a creative language* (e.g. "A mathematical theory and a piece of art are similar as they both are the result of creativity").
2. *Mathematics as a process of discovery* (e.g. "One day, the mathematicians will have revealed the whole of mathematics").
3. *The schematic conception of mathematics* (e.g. "The derivation or the proof of a formula is not important to me; the crucial thing is that I am able to use it").
4. *Rigid schema orientation* (e.g. "There is always only one solution in mathematics").
5. *Instrumental importance of mathematics* (I. Mathematics as socially useful instrument, e.g. "Mathematics help to describe economical processes," and II. Mathematics as useful instrument in school or daily life, e.g. "Everything I learn in mathematics, I can use in other subjects").

Beliefs on learning and teaching of mathematics

Besides epistemological beliefs on the subject itself, teachers are equipped with beliefs on the learning and teaching of a subject, or, how students learn and how students should be taught respectively (see Voss et al., 2013a).

Concerning this aspect, Kuhs and Ball (1986) distinguish between three mutually exclusive positions (cf. Voss et al., 2013a): (1) *learner-focused* (mathematical learning is an active construction process in learning communities), (2) *content-focused with an emphasis on conceptual understanding* (the focus of mathematical learning is put on conceptual understanding) and (3) *content-focused with an emphasis on performance* (the focus of mathematical learning lies in the successful application of mathematical rules and procedures).

Beliefs on the teaching and learning of mathematics are often categorized alternatively according to an *orientation towards knowledge acquired at school* and an *orientation towards child development*. Teachers pursuing the *orientation towards knowledge acquired at school* are quite often convinced that teaching consists of passing on knowledge to the learners, who should be able to reproduce that knowledge afterwards. In contrast, teachers who follow the *orientation towards child development* refer to their students' individual needs in their lessons and take a conceptual understanding of mathematical contents as the goal of their lessons (cf. Voss et al., 2013a).

In the context of the COACTIV study, the various facets and components of epistemological beliefs as well as beliefs on the learning and teaching of mathematics have been summarized to a spare model on the basis of the following integrative view.

Integrative view on beliefs (according to Voss et al., 2013a)

Both epistemological beliefs and beliefs on the learning and teaching of mathematics can be subdivided into two fundamental perspectives: the *constructivist* and the *transmissive* (Voss et al., 2013a; Schmotz, Felbrich, & Kaiser, 2010; Staub & Stern, 2002). Within the constructivist view, teaching and learning processes should be student-oriented, meaning that learners already have some preconceptions and should be able to act in an independent and active way. In such lessons it is the task of the teacher to create an appropriate learning environment which supports the students' construction of knowledge. A mathematics teacher, following this view, is sure that mathematics can be seen as a process. On the other hand, the transmissive view understands lessons as a process in which learning means the passing of knowledge on to the learner who acts in large parts as a passive recipient. The focus of such mathematics lessons lies in the demonstration, repetition and incorporation of typical examples. For teachers pursuing this view, mathematical knowledge is an objective fixed collection of facts and procedures.

In the context of the COACTIV study it is assumed that transmissive epistemological beliefs and transmissive beliefs on the teaching and learning of teachers usually coincide in a transmissive orientation and analogously, that constructivist epistemological beliefs and constructivist beliefs on the teaching and learning usually coincide in a constructivist orientation. These orientations are to be seen as ideal-typical extremes, both called "theoretical learning beliefs." Similar to the static and dynamic view of Schoenfeld, Törner and Grigutsch, these orientations are not mutually exclusive. In other words: A teacher can be equipped with elements of both constructivist and transmissive beliefs at the same time.

In the COACTIV study, the structure of theoretical learning beliefs of secondary in-service teachers has been investigated according to whether constructivist and transmissive orientations constitute the final poles of one dimension or two distinct dimensions. Analyses found that both latent constructs reveal a negative correlation of $r = -0.67$. This indicates that constructivist and transmissive orientations are not independent from one another, as teachers with high values concerning the transmissive orientation had lower values concerning the constructivist orientation and vice versa. Therefore, these orientations cannot be seen as two extreme poles of one dimension but should be taken as two distinct dimensions which correlate negatively. A detailed explanation of these structures can be found in Voss et al. (2013a).

RESEARCH QUESTION

Following the assumption that we are talking about a facet of expertise, beliefs should change in the course of the teacher training "in the desired direction." With respect to the professional knowledge, a "positive" change obviously means an increase in the respective knowledge category. But what does a positive change mean with respect to constructivist and transmissive beliefs? Former results show that constructivist orientations can be seen as advantageous (Staub & Stern, 2002; Stipek et al., 2001; Op't Eynde, De Corte, & Verschaffel, 2002). The COACTIV study was even able to show empirically that constructivist beliefs have a positive influence and transmissive beliefs have a negative influence on lesson quality and students' learning (Voss et al., 2013a). Therefore, a positive change of theoretical learning beliefs might mean a growth of constructivist beliefs and/or a reduction of transmissive beliefs. However, it should be noted that – in contrast to our hypothesis – Pajares (1992) summarizes research results that show that beliefs are relatively consistent over time.

In the following, we would like to investigate the hypothesis that students, just before beginning their teacher training at university, are equipped with more distinctive transmissive orientations than the COACTIV teachers, but are equipped with a less distinctive constructivist orientation. Thus, we will report results of a cross-sectional comparison between student teachers and the teachers which have been examined in the COACTIV sample. Note that our hypothesis implies that the preferable development of professional beliefs (towards an increase in constructivist beliefs with a decrease of transmissive beliefs at the same time) happens within traditional teacher training and with accumulated work experience (and without especially conceptualized "belief-training programs," which are claimed by some authors, but which are not implemented in typical German teacher training, cf. Voss et al., 2013a; Kaiser & Maaß, 2006).

In order to answer our query, the questionnaire regarding theoretical learning beliefs which had already been used in the COACTIV study with 325 secondary mathematics teachers (cf. Voss et al., 2013a) was additionally applied to 94 teacher trainees. Although reliable assertions considering the change of beliefs can only be made to a limited extent because of the comparison of cross-sectional data (longitudinal data would have been more appropriate for that), we herewith take a first step towards an answer of our research question.

METHOD

Sample

In the context of a preparatory course (a course for school leavers before starting their university years), which was realized at the University of Regensburg during the winter 2010/2011 semester in cooperation with the Ludwig-Maximilian-University of Munich, 94 beginning mathematics students filled in a questionnaire on beliefs.

Table 2. Overview on the subscales of the instrument measuring beliefs.

Content area	Theoretical Learning Foundation	
	Transmissive	**Constructivist**
Nature of knowledge	Mathematics as a toolbox	Mathematics as a process
Learning and Teaching of Mathematics	Clarity of solution procedure Receptive learning from examples and demonstrations Automatization of technical procedures	Independent and insightful discursive learning Confidence in the mathematical independence of students

This two week course was offered especially for teacher trainees and took place three weeks before the actual start of their studies. This course focused particularly on the development of competences which are important for further teacher training. The main goals of this course were: Being able to make connections between mathematics at school and mathematics at university, becoming acquainted with typical practices in studying mathematics, socializing with other students, and exploring the university.

The investigated sample of the COACTIV study consisted of 325 secondary mathematics teachers (for a detailed description of this representative teacher sample see Voss et al., 2013a).

Instrument

The subscales, conceptualized in COACTIV, can be arranged in a four-square-table (see Table 2).

The beliefs on the *nature of knowledge* (epistemological beliefs) have been conceptualized in terms of *mathematical conceptions* of the world with respect to Schoenfeld. For operationalization, a revised version of the questionnaire of Grigutsch, Raatz and Törner (1998) was used, which includes the two subscales "*mathematics as process*" (constructivist orientation) and "*mathematics as toolbox*" (transmissive orientation).

For measuring the beliefs on *learning and teaching of mathematics*, five subscales were used, which have been designed within the framework of COACTIV based on Fennema, Carpenter and Loef (1990). Two of those subscales served to collect data on constructivist beliefs and three served to gather data on transmissive beliefs.

The questionnaire included 44 items in total for the seven subscales of Table 2. The students had to give their opinion considering each of those 44 statements with respect to a four-level scale: "false," "rather false," "rather true," "true." The phrasing of each item can be taken from Table 3.

Table 3. Forty-four items of seven subscales. The participants of the preparatory course had to tick each of those statements with regard to a four-level-scale (1 = false, 2 = rather false, 3 = rather true, 4 = true).

Content area	Transmissive view	Constructivist view
Nature of Knowledge	**Mathematics as Toolbox (5 Items)** Mathematics consists of learning, remembering and application. Mathematics is a collection of processes and rules, which tell exactly how to solve tasks. If you want to solve a mathematical task, you have to know the correct procedure. Otherwise you are lost. Mathematics is the memorization and application of definitions and formulas, of mathematical facts and procedures. Almost all mathematical problems can be solved through direct application of familiar rules, formulas and procedures.	**Mathematics as Process (4 Items)** In Mathematics you can find and try many things on your own. Mathematics subsists on new ideas. Mathematical tasks and problems can be solved in many different ways correctly. If you deal with mathematical problems, you can find something new (coherence, rules, concepts) very often.
Learning and Teaching of Mathematics	**Clarity of solution process (2 Items)** When dealing with tasks with multiple solutions it is safer to restrict oneself to one single solution. When dealing with tasks with multiple solutions, it is normally better to restrict oneself to the demonstration and illustration of one single solution.	**Independent and insightful discursive learning (12 Items)** Teachers should encourage students to look for their own solution of mathematical tasks, even if those are inefficient. Students should have the chance to explain their solution detailed, even if it is wrong. It is important to change the structure of the task in the same content again and again, in order to lead the students towards mathematical thinking. Students learn mathematics best when discovering solution processes on their own in relatively simple tasks. It is important for students to discover independently, how to solve text and application tasks. Students should be allowed, to think about their own solutions of simple tasks, before the teacher demonstrates how to do it. In mathematics learning goals can best be reached if students find their own methods to solve the task. It is helpful for understanding mathematics, if students are allowed to discuss their own ideas of solving tasks. Teachers should allow students having difficulties to solve text problems to proceed with their own approach to the problem. When dealing with application tasks, students should have the chance to justify their own process. Teachers should allow students to discover their own procedure for simple mathematical tasks. Teachers should encourage student to think about solutions to simple questions on their own.
	Receptive Learning from examples and demonstrations (12 Items) Teachers should convey detailed procedures for the solution process of tasks. Students learn best when solution processes are demonstrated. Most students have to be showed plenty of examples of how to solve tasks. Students learn best through the demonstration of sample tasks. Students learn best when following the teachers' instructions. Computing processes should be practiced before expecting a student to understand those procedures. Weak students are overcharged with tasks, which demand mathematical thinking. They best learn through demonstration. Students can become good problem solvers, when strictly following the teachers' instructions. To be successful in mathematics, students have to be careful listeners. Students need detailed instructions on how to solve text problems. Students cannot be expected to understand the meaning of calculation procedures before mastering their application. Students best learn mathematics through the teachers' illustrations and explanations	**Confidence in the mathematical independence of students (5 Items)** Most students can find solutions for simple application tasks on their own. Usually students can find out themselves how simple tasks are to be solved. Students can find the solution for many tasks without instructions as well. Students are able to develop calculating procedure on their own with the aid of appropriate materials. Most students are able to find solutions for many mathematical tasks without any help of adults as well.
	Automatization of technical procedures (4 Items) The most efficient solution process of one category of tasks should be implemented through practicing. The acquisition of numerical factual knowledge should precede the deeper understanding of operations. Students do not understand a mathematical operation before they have crucial parts of the respective numerical factual knowledge available. Frequent practicing of algorithmic tasks is essential for the acquisition of numerical factual knowledge.	

RESULTS

Table 4 gives an overview on the results of the present study (mean scores of the seven scales for the beliefs of the participants of the preparatory course) and compares them with the corresponding sample results of the in-service mathematics teachers in the COACTIV study (Voss et al., 2013a).

As can be seen in Table 4, all results are in the directions of the research hypothesis stated. While the teachers display higher means in all three sub-facets of the constructivist beliefs than the students, the opposite is true for the four sub-facets of the transmissive beliefs. However, the size of the effects varies with respect to the single subscales. The largest difference between teachers and students appears in the sub-facet "receptive learning from examples and demonstrations" with an effect size of $d = -0.45$ ($p < 0.001$). The belief that learning can best be achieved by simply demonstrating the mathematical content seems to be reduced during professionalization. In contrast, in-service teachers display a higher belief that the independency of the learners is important in order to achieve understanding when compared to the beginner teachers. Altogether, the results support the hypothesis that beliefs change during professionalization towards higher expertise. Although we did not gather longitudinal data, a change of beliefs seems to be the probable cause for the observed differences.

It is important to note that none of the COACTIV teachers received special training in the field of beliefs (this is assumed because the idea of changing beliefs is a rather new idea in the field of teacher-training, which has not been implemented in regular training of pre- and in-service teachers in Germany yet). Therefore, the observed differences seem to occur mostly because of "regular" teacher-training and "natural" learning in the job.

SUMMARY AND DISCUSSION

In the present paper we investigated the beliefs of students that participated in a preparatory course and compared these beliefs with the corresponding beliefs of mathematics teachers investigated in the COACTIV study. Conceptualizing the beliefs of mathematics teachers, we followed Voss et al. (2013a) and combined the belief of the nature of knowledge called "mathematics as a toolbox" with the three beliefs on teaching and learning mathematics called "clarity of solution process," "receptive learning from examples and demonstrations" and "automatization of technical procedures" into the aggregated construct of "*transmissive orientation*." This aggregated belief was conceptually compared to the belief of "*constructivist orientation*," which was a combination of the belief of the nature of knowledge called "mathematics as a process" and the two beliefs on teaching and learning mathematics called "independent and insightful discursive learning" and "confidence in the mathematical independence of students."

The main results were: Although the reliabilities of the seven scales were somehow lower for the students of the preparatory course than for the COACTIV teachers, it can be concluded that teachers tend to follow constructivist learning

Table 4. Means M and standard deviations SD of the scale scores of the seven scales of beliefs and reliabilities α of the students of the preparatory course (on the right of the slash) and the COACTIV teachers (on the left of the slash).

N = 325 COACTIV teachers N = 94 students	Number of items	M (SD) $_{teachers}$ / M (SD) $_{students}$ *effect size d* / *p-value (significance)*	Reliability (Cronbach's alpha) $\alpha_{teachers}/\alpha_{students}$
Constructivist beliefs			
Mathematics as a process	4	3.36 (0.47) / 3.32 (0.43) *d = 0.09* *p = 0.46*	0.67/0.54
Independent and insightful discursive learning	12	3.35 (0.39) / 3.23 (0.41) *d = 0.31* *p = 0.01***	0.88/0.82
Confidence in the mathematical independence of students	5	2.94 (0.54) / 2.81 (0.47) *d = 0.25* *p = 0.04**	0.81/0.62
Transmissive beliefs			
Mathematics as a toolbox	5	2.53 (0.58) / 2.71 (0.48) *d = –0.32* *p = 0.006***	0.73/0.55
Clarity of solution procedure	2	1.95 (0.66) / 2.04 (0.76) *d = –0.13* *p = 0.26*	0.76/0.76
Receptive learning from examples and demonstrations	12	2.45 (0.47) / 2.66 (0.45) *d = –0.45* *p < 0.001***	0.86/0.78
Automatization of technical procedures	4	2.75 (0.49) / 2.91 (0.48) *d = –0.33* *p = 0.005***	0.68/0.41

M = mean value, SD = standard deviation
According to Cohen (1992), d = 0.20 is a small effect, d = 0.50 a medium effect, and d = 0.80 a large effect.
* significant at 0.05-level; ** significant at 0.01-level

views more than beginning students, and they also have much less transmissive learning views than beginning students. Although, of course, longitudinal data were required for more valid conclusions, these results can be seen as an indicator that the beliefs of pre-service teachers change during professionalization in the desired direction (for a comparable study see Voss et al., 2013b).

The results are also in accordance with the results of the P-TEDS study (Schmotz et al., 2010). In this study the authors were able to show, with samples of university students and future teachers in the second phase of teacher training, that the belief in receptive learning declines during education while process orientation becomes more pronounced (Felbrich & Müller, 2007). In the same way Köller, Baumert, and Neubrand (2000) found in the framework of the TIMS study that students (about 18 years old) strictly follow the belief of "mathematics as a toolbox" and that they are not yet familiar with the idea of the process-related character of mathematics.

The empirical results concerning the impact of beliefs on the quality of lessons and on student learning (Voss et al., 2013a; see also Kaiser & Maaß, 2006) underline the relevance of both research on beliefs and of picking beliefs out as a central theme for the education of pre-service and in-service teachers. Despite the reservations of some researchers stating that beliefs seem to be difficult to change (e.g., Pajares, 1992), it is certainly worth making an effort in this direction (see, e.g., Rolka, Rösken, & Liljedahl, 2006, for encouraging results). Voss et al. (2013) note that it would not be sufficient to simply support constructivist beliefs, but it is also important to reduce transmissive beliefs simultaneously. The present chapter confirms that efforts in this direction would maintain and enlarge a tendency that appears to take place during professionalization anyway. The question of why and how beliefs change cannot be answered by the present data. Voss et al. (2013b) suggest that the cognitive mechanism of conceptual change provides a useful framework for explaining changes in teachers' beliefs. In short, it means that teachers may be discontent with their present beliefs and thus are motivated to change their beliefs in the direction of more fruitful alternatives (for a detailed discussion of possible mechanisms underlying the change of beliefs see Voss et al., 2013b).

REFERENCES

Baumert, J., & Kunter, M. (2013a). The COACTIV model of teachers' professional competence. In M. Kunter, J. Baumert, W. Blum, U. Klusmann, S. Krauss, & M. Neubrand (Eds.). *Teachers' professional competence. Findings of the COACTIV Research Program* (pp. 25–48). New York: Springer.

Baumert, J., & Kunter, M. (2013b). The effect of content knowledge and pedagogical content knowledge on instructional quality and student achievement. In M. Kunter, J. Baumert, W. Blum, U. Klusmann, S. Krauss, & M. Neubrand (Eds.). *Teachers' professional competence. Findings of the COACTIV Research Program* (pp. 175–205). New York: Springer.

Bromme, R. (2008). Lehrerexpertise. In W. Schneider & M. Hasselhorn (Eds.), *Handbuch der pädagogischen Psychologie* (pp. 159–167). Göttingen: Hogrefe.

Bromme, R., Jucks, R., & Rambow, R. (2000). Experten-Laien-Kommunikation im Wissensmanagement. In G. Reinmann & H. Mandl (Eds.), *Der Mensch im Wissensmanagement: Psychologische Konzepte zum besseren Verständnis und Umgang mit Wissen*. Göttingen: Hogrefe.

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 709–725). New York: Macmillan.

Cohen, J. (1992). A power primer. *Psychological Bulletin*, *112*(1), 155–159.

Duell, O. K., & Schommer-Aikins, M. (2001). Measures of people's beliefs about knowledge and learning. *Educational Psychology Review*, *13*(4), 419–449.

Felbrich, A., & Müller, C. (2007). Erste Ergebnisse aus P-TEDS: Mathematische Weltbilder und Vorstellungen zum Lehren und Lernen von Mathematik. In *Beiträge zum Mathematikunterricht 2007* (pp. 573–576).

Fennema, E., Carpenter, T. P., & Loef, M. (1990). *Teacher belief scale: Cognitively guided instruction project*. Madison, WI: University of Wisconsin.

Grigutsch, S. (1996). *Mathematische Weltbilder von Schülern: Struktur, Entwicklung, Einflußfaktoren*. Unpublished dissertation, Gerhard-Mercator-Universität Duisburg.

Grigutsch, S., Raatz, U., & Törner, G. (1998). Einstellungen gegenüber Mathematik bei Mathematiklehrern. *Journal für Mathematik-Didaktik*, *19*(1), 3–45.

Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession. *Educational Researcher*, *31*(5), 3–15.

Hofer, B. K., & Pintrich, P. R. (1997). The development of epistemological theories: Beliefs about knowledge and knowing and their relation to learning. *Review of Educational Research*, *67*(1), 88–140.

Kaiser, G., & Maaß, K. (2006). Vorstellungen über Mathematik und Ihre Bedeutung für die Behandlung von Realitätsbezügen. In A. Büchter, H. Humenberger, S. Hußmann, & S. Prediger (Eds.), *Realitätsnaher Mathematikunterricht vom Fach aus und für die Praxis* (pp. 83–94). Hildesheim: Franzbecker.

Köller, O., Baumert, J., & Neubrand, J. (2000). Epistemologische Überzeugungen und Fachverständnis im Mathematik- und Physikunterricht. In J. Baumert, W. Bos, & R. Lehmann (Eds.), *TIMSS/III. Dritte Internationale Mathematik- und Naturwissenschaftsstudie. Mathematische und naturwissenschaftliche Bildung am Ende der Schullaufbahn*, Vol. 2 (pp. 229–269). Opladen: Leske und Budrich.

Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *The International Journal on Mathematics Education (ZDM)*, *40*(5), 873–892.

Krauss, S., Blum, W., Brunner, M., Neubrand, M., Baumert, J., Kunter, M., Besser, M., & Elsner, J. (2013). Mathematics teachers' domain-specific professional knowledge: Conceptualization and test construction in COACTIV. In M. Kunter, J. Baumert, W. Blum, U. Klusmann, S. Krauss, & M. Neubrand (Eds.), *Teachers' professional competence. Findings of the COACTIV Research Program* (pp. 147–174). New York: Springer.

Kuhs, T. M., & Ball, D. L. (1986). *Approaches to teaching mathematics: Mapping the domains of knowledge, skills and dispositions*. East Lansing, MI: Michigan State University, Center on Teacher Education.

Kunter, M., Baumert, J., Blum, W., Klusmann, U., Krauss, S., & Neubrand, M. (2013). *Teachers' professional competence. Findings of the COACTIV Research Program*. Springer: New York.

Leinhardt, G., & Greeno, J. G. (1986). The cognitive skill of teaching. *Journal of Educational Psychology*, *78*, 75–95.

Op't Eynde, P., De Corte, E., & Verschaffel, L. (2002). Framing students' mathematics-related beliefs: A quest for conceptual clarity and a comprehensive categorization. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education* (pp. 13–36). Dordrecht: Kluwer.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research*, *62*(3), 307–332.

Pehkonen, E., & Törner, S. (1996). Mathematical beliefs and their meaning for the teaching and learning of mathematics. *Zentralblatt für Didaktik der Mathematik*, *28*(4), 101–108.

Perry, W. G. (1970). *Forms of intellectual and ethical development*. New York: Holt, Rinehart and Winston.

Peterson, P., Fennema, E., Carpenter, T. P., & Loef, M. (1989). Teachers' pedagogical content beliefs in mathematics. *Cognition and Instruction*, *6*, 1–40.

Prenzel, M., Baumert, J., Blum, W., Lehmann, R., Leutner, D., Neubrand, M., et al. (2004). *PISA 2003. Der Bildungsstand der Jugendlichen in Deutschland: Ergebnisse des zweiten internationalen Vergleichs*. Münster: Waxmann.

Prenzel, M., Baumert, J., Blum, W., Lehmann, R., Leutner, D., Neubrand, M. et al. (2006). *PISA 2003: Untersuchungen zur Kompetenzentwicklung im Verlauf eines Schuljahres*. Münster: Waxmann.

Reusser, K., Pauli, C., & Elmer, A. (2011). Berufsbezogene Überzeugungen von Lehrerinnen und Lehrern. In E. Terhart, H. Bennewitz, & M. Rothland (Eds.), *Handbuch der Forschung zum Lehrerberuf* (pp. 478–495). Münster: Waxmann.

Rolka, K., Rösken, B., & Liljedahl, P. (2006). Challenging the mathematical beliefs of preservice elementary school teachers. In J. Novotná et al. (Eds.), *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 441–448). PME, Prague.

Schmotz, C., Felbrich, A., & Kaiser, G. (2010). Überzeugungen angehender Mathematiklehrkräfte für die Sekundarstufe I im internationalen Vergleich. In S. Blömeke, G. Kaiser, & R. Lehmann (Eds.), *TEDS-M 2008. Professionelle Kompetenz und Lerngelegenheiten angehender Mathematiklehrkräfte für die Sekundarstufe I im internationalen Vergleich*. Münster: Waxmann.

Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal of Research in Mathematics Education*, *20*(4), 338–355.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers in Mathematics (NCTM)* (pp. 334–370). New York: Macmillan.

Schommer, M. (1990). Effects of beliefs about the nature of knowledge on comprehension. *Journal of Educational Psychology*, *82*(3), 498–504.

Schommer, M., Crouse, A., & Rhodes, N. (1992). Epistemological beliefs and mathematical text comprehension: Believing it is simple does not make it so. *Journal of Educational Psychology*, *84*(4), 435–443.

Shulman, L. S. (1986). Paradigms and research programs in the study of teaching: A contemporary perspective. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 3–36). New York: Macmillan.

Staub, F., & Stern, E. (2002). The nature of teacher's pedagogical content beliefs matters for students' achievement gains: Quasi-experimental evidence from elementary mathematics. *Journal of Educational Psychology*, *94*, 344–355.

Sternberg, R., & Horvarth, J. (1995). A prototype view of expert teaching. *Educational Researcher*, Aug–Sep, 9–17.

Stipek, D. J., Givvin, K. B., Salmon, J. M., & MacGyvers, V. L. (2001). Teachers' beliefs and practices related to mathematics instruction. *Teaching and Teacher Education*, *17*(2), 213–226.

Törner, G. (2002). Epistemologische Grundüberzeugungen – Verborgene Variablen beim Lehren und Lernen von Mathematik. *Der Mathematikunterricht*, *48*(4/5), 106–130.

Törner, G., & Grigutsch, S. (1994). Mathematische Weltbilder bei Studienanfängern: Eine Erhebung. *Journal für Mathematik-Didaktik*, *15*(3/4), 211–252.

Voss, T., Kleickmann, T., Kunter, M. & Hachfeld, A. (2013a). Mathematics teachers' beliefs. In M. Kunter, J. Baumert, W. Blum, U. Klusmann, S. Krauss, & M. Neubrand (Eds.), *Teachers' professional competence. Findings of the COACTIV Research Program* (pp. 249–271). New York: Springer.

Voss, T., Kunter, M., & Baumert, J. (2013b). Transmissive and constructivist beliefs of pre-service and in-service teachers (submitted).

Weinert, F. E. (2001). Concept of competence: A conceptual clarification. In D. S. Rychen & L. H. Saganik (Eds.), *Defining and selecting key competencies* (pp. 45–65). Seattle, WA: Hogrefe & Huber Publishers.

Woolfolk Hoy, A., Davis, H. & Pape, S. (2006). Teachers' knowledge, beliefs, and thinking. In P. A. Alexander & P. H. Winne (Eds.), *Handbook of educational psychology* (pp. 715–737). Mahwah, NJ: Erlbaum.

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6. BUILDING ON SCHOENFELD'S STUDIES OF METACOGNITIVE CONTROL TOWARDS SOCIAL METACOGNITIVE CONTROL

INTRODUCTION

When students work on mathematics problems, they select strategies, adapt them in response to feedback, allot time and make many other decisions to optimize their performance (Schoenfeld, 1985). These metacognitive control decisions involve strategic planning, self-monitoring and intentionally adapting problem solving paths to achieve a specific goal (Schoenfeld, 1985).

While most studies have primarily focused on metacognitive control by individuals (Miller & Geraci, 2011; Kaplan, 2008), recent studies (Chen, Chiu, & Wang, 2012; Larkin, 2009) suggest that examining its social corollary, social metacognitive control, could help students work together more effectively to solve problems. Social metacognitive control is group members' monitoring and control of others' knowledge, emotions, and actions as well as one's own. In addition to the traditional components of individual metacognitive control, social metacognitive control includes group interaction and social influence. When mathematics problems are solved by groups rather than individuals, team members who can monitor and control other's behaviors effectively can increase their likelihood of solving difficult problems.

This chapter shows how studies on social metacognitive control extend past research on individual metacognitive control. After briefly summarizing metacognitive control, we consider its advantages and the challenges to using it effectively. Then, we examine social metacognitive control, and its advantages and challenges. Lastly, we examine the research on teaching students metacognitive control or social metacognitive control.

METACOGNITIVE CONTROL

Originally defined as "cognition about cognitive phenomena" or more simply "thinking about thinking" (Flavell, 1979, p. 906), metacognition has two main components: metacognitive knowledge and metacognitive control (Flavell, 1979; Nelson & Narens, 1990; Otani & Widner, 2005). Metacognitive knowledge of one's state of mind, concepts and strategies allows one to understand oneself, retrieve the appropriate strategies from memory and to apply them at the appropriate time. For example, metacognitive knowledge includes knowing when one is tired. Metacognitive knowledge also includes knowing each strategy's applicability to specific

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tasks, conditions of use and effectiveness. In the case of a mathematics student, he or she learns not only mathematics concepts and skills but also when, where, and how to apply them.

Unlike metacognitive knowledge, which is simply having the knowledge of self, task, and strategy variables needed to accomplish a goal, metacognitive control involves one's ability to use metacognitive knowledge to monitor and regulate one's thinking and actions. After a person recognizes that he or she is tired (metacognitive knowledge), he or she could decide to take a break (metacognitive control). Metacognitive control during problem solving includes monitoring problem solving progress, deciding on the next step, and allocating resources (Schoenfeld, 1987).

Schoenfeld: Metacognitive control during mathematics problem solving

Schoenfeld's studies (1985, 1987, 1988, 1992) have shown the crucial role of metacognitive control during mathematical problem solving of novel, ill-structured problems. When students have sufficient content knowledge to solve a problem, they may still fail to do so because they lack suitable metacognitive control to select, continue, or abandon a specific strategy. For novices, a metacognitive control failure often involves quickly selecting a strategy without evaluating their prior knowledge, making a plan, or understanding the complexities of the problem (Schoenfeld, 1985). Schoenfeld (1992) describes a novice's "wild goose chase" problem solving approach as: "read, make a decision quickly, and pursue that direction come hell or high water." While novice undergraduate mathematics majors focused on specific formulas and equations, expert mathematicians sought to understand the goal structure of a problem and to identify which mathematical tools might help.

Consider a hypothetical example of two 10th grade students, Rita and Danny, using their metacognitive control skills to solve a trigonometry problem: what is the height of a tree given its distance to a point on the ground and an angle of elevation? After drawing a right triangle connecting the top and bottom of the tree to the point, they labeled its height with the letter "x," the distance from the bottom of the tree to the point (20) and the angle of elevation (39°).

Rita: First, let's figure out if we should use sine, cosine or tangent.
Danny: Okay, so label the sides with opposite, hypotenuse and adjacent.
Rita: Shouldn't we figure out which angle we are dealing with first?
Danny: What angles do we know? What do you think?
Rita: This one and this one. (Rita points to the angle of elevation and the right angle.)
Danny: If we use our right angle, this is the hypotenuse. (Danny correctly labels the hypotenuse as the side across from the right angle.)
Rita: But if we use that angle, how do we know which one is opposite and which is adjacent? They are both touching that angle, so are they both adjacent? Is that right?
Danny: Well, if they were both adjacent, we would have adjacent over

hypotenuse, and we would use cosine.

Rita: No, that can't be right – we can't have two adjacent sides.

Danny: So let's use the other angle we know. (Danny points to the 39° angle.) That makes the tree opposite side and the 20 the adjacent side. (Danny correctly labels the sides of the right triangle.)

Rita: Alright, we've got it now – we use tangent for opposite over adjacent, so the tangent of 39 is x over 20 . . .

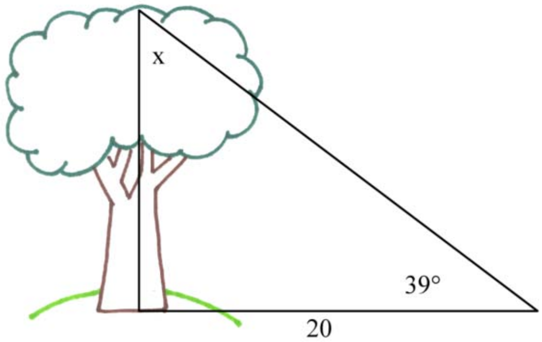

Figure 1. Right triangle trigonometry problem.

Rita and Danny make several important metacognitive decisions that help them solve the problem. First, they proceed systematically through each step of their problem solving process. They select strategies such as "labeling the side of the triangle" for a specific, articulated purpose – in this case, whether to "use sine, cosine, or tangent." Unlike the "wild goose chase" Schoenfeld (1987) described, Rita and Danny monitor and politely criticize one another's work through questions about each other's understandings of the problem (e.g., "Shouldn't we figure out which angle we are dealing with first?"). Furthermore, they use conditionals to justify their solution procedures ("if they were both adjacent, we would have adjacent over hypotenuse, and we would use cosine"). Likewise, they use warrants for their criticisms ("we can't have two adjacent sides"). Through regularly monitoring, evaluating and justifying one another's ideas, they progress toward a correct solution.

Benefits of metacognitive control

Metacognitive control can affect both mathematics problem solving processes and their outcomes. By acquiring further knowledge and relevant resources to aid problem solving, metacognitive control often yields enhanced problem solving skills and problem solving success. Combined with metacognitive control's greater autonomy, these problem solving successes tend to increase long-term academic achievement (Calskin & Sunbul, 2011; Perels, Dignath, & Schmitz, 2009).

Improved problem solving processes
Metacognitive control can increase both awareness of cognitive and emotional experiences and autonomy over learning. While greater metacognitive knowledge enables more effective metacognitive control, greater metacognitive control likewise motivates students to seek out relevant metacognitive knowledge (Hacker & Bol, 2004). In the above example, after Danny suggests labeling the sides, Rita's critique, 'Shouldn't we ..."; a move of metacognitive control, motivates the search for metacognitive knowledge ("which angle we are dealing with *first*?"). Hence, metacognitive knowledge and metacognitive control mutually influence one another.

After identifying the relevant information, Rita and Danny can try to access or acquire it ("What angles do we know?") from their prior knowledge, from the problem situation, or from each other, thereby yielding additional resources for solving the problem, what Holton and Thomas (2001) have called *self-scaffolding*. Unlike scaffolding initiated and directed by expert others (such as teachers), self-scaffolding depends on a student's independent use of his or her own metacognitive control to identify and obtain the desired resources. While self-scaffolding is limited by a learner's conceptual knowledge, its personal nature can align closely with his or her individual needs and provide customized self-help during mathematics problem solving.

Improved outcomes
The enhanced metacognitive awareness and self-scaffolding provided by metacognitive control can improve problem solving, satisfaction, and mathematics achievement. Students exerting greater metacognitive control typically identify more metacognitive knowledge that is relevant, acquire more relevant resources through self-scaffolding and spend more sustained, meaningful time on a problem (Chiu & Kuo, 2009). As a result, these stronger mathematics problem solving skills typically help students solve more problems (Desoete, 2009). For example, Rita and Danny can transfer these problem solving practices (e.g., monitoring their problem solving moves against the problem situation, evaluating the utility of specific strategies and revising their procedure as needed; see Desoete, 2009) to other trigonometry problems and other mathematics problems outside of trigonometry.

Students with greater metacognitive control also typically have more autonomy, which heightens their satisfaction after successfully solving a problem. By taking control of their learning, actively deciding on, adapting, and modifying one's cognitive efforts, learners take greater responsibility for the problem and have greater ownership of it (*autonomy*). Hence, students who exert greater metacognitive control and successfully solve a problem often feel greater satisfaction (Hacker & Bol, 2004). In the above example, Rita and Danny's frequent uses of first person plural pronouns ("we," "let's" [let us]) and possessives ("our") show their shared ownership of the problem, which helps account for their autonomy and enthusiasm when they eventually solve it ("alright, we got it now"). In contrast, students who follow their teacher's algorithm to compute a solution are less likely to enjoy their solution (Hacker & Bol, 2004).

As students with greater metacognitive control often have better mathematics problem solving skills, greater autonomy, more problem solving success, and greater satisfaction, they are more likely to attempt difficult problems, persevere, flexibly adapt their strategies and ultimately learn more. As a result, they are more likely to have higher mathematics achievement (Desoete, 2009).

Challenges of metacognitive control

Despite its benefits, many students experience difficulties while exercising metacognitive control including extra metacognitive demands, inaccurate evaluations and poor self-scaffolding (Lerch, 2004; Efklides, 2011) Metacognitive control requires dividing resources among cognitive and metacognitive processes, which can hinder accurate evaluations, effective scaffolding and ultimately, mathematics problem solving and learning.

Metacognitive demands

Metacognitive processes divide attention and can take mental resources away from other cognitive processes (Salonen, Vauras, & Efklides, 2005). Divided attention may cause distractions or confusion of information across cognitive or metacognitive tasks. In the above example, simultaneously considering whether to "use sine, cosine or tangent" could result in confusions and errors. Furthermore, as human brains have limited mental resources, allocating some resources to metacognitive control necessarily entails reducing cognitive resources in other areas, especially when there are tight time constraints (Holton & Clarke, 2006). As operating with fewer cognitive resources might yield inadequate or flawed thinking, excess allocation of mental resources to metacognitive processes might reduce learning or performance. For example, when Rita raises three questions in the above example, monitoring her thinking on each question could be extremely taxing and hinder her judgment. Thus, allocating mental resources to metacognitive demands can result in divided attention or fewer cognitive resources, either of which can result in confusions or errors that hamper mathematics problem solving or learning.

Inaccurate evaluations

Strategic planning and monitoring requires the ability to accurately assess and evaluate progress. Thus, one of the difficulties of metacognitive control is that students are often ill prepared or cognitively unable to accurately assess their progress (Cavanaugh & Perlmutter, 1982). This inability can interfere with mathematical accuracy and achievement, as the decision to continue, adapt or abandon a specific strategy can be made on incorrect information. For example, if Rita had incorrectly evaluated Danny's proposal that the two sides "were both adjacent," she would have computed an incorrect answer and failed to solve the geometry problem. While accurate evaluations allow detection of errors and continuation on a successful problem solving path, inaccurate evaluations can overlook errors and derail suitable problem solving paths.

Poor self-scaffolding

As traditional metacognitive control assumes independent problem solving, a student acquires additional information and problem solving resources through self-scaffolding. However, this self-scaffolding is limited to that student's personal expertise and knowledge. As students are often inaccurate in their self-evaluations (Chiu & Klassen, 2010), their self-scaffolding might be flawed. As a result, they may improperly allocate mental resources, choose inappropriate strategies or mis-schedule their time. For example, Schoenfeld (1992) found that novices often continued to use the same (ineffective) strategies when problem solving simply because it was the most routine or successful in the past. This reluctance to change strategies might stem from familiarity with a specific approach, past success with this approach, or difficulty in adapting other strategies to this situation (Chiu, Kessel, Moschkovich, & Munoz, 2001). Thus flawed or ineffective self-scaffolding can limit problem solving efforts and ultimately mathematics achievement.

SOCIAL METACOGNITIVE CONTROL

While metacognitive control is an individual and self-centered activity, social metacognitive control extends control and monitoring activities to groups. While metacognitive control is control over regulation and evaluation of one's own knowledge, actions and emotions, social metacognitive control is monitoring and control of the regulation and evaluation of other's knowledge (Chiu, 2008). Social metacognitive control links metacognitive judgments with communication and applies one's subjective metacognitive experiences to a group context (Salonen, Vauras, & Efklides, 2005).

In the previous vignette of Rita and Danny, the students attend to both their own and the other's knowledge, actions and emotions through invitational questions, evaluations and responses. For example, when Rita and Danny ask each other questions such as "What do you think?" and "Is that right?," they are trying to monitor and control their individual problem solving efforts, assess their communal understanding of the mathematics task, invite each other to participate and attend to each other's actions (and emotions). By monitoring and evaluating Danny's problem solving actions, Rita notices, identifies, articulates and explains a flaw in his work ("no, that can't be right – we can't have two adjacent sides"). Rather than blaming him, she depersonalizes the source of the flaw ("that can't be right") and uses shared positioning ("we") to critique politely, thereby helping him save face. Danny accepts her criticism, considers an alternative and uses their shared knowledge to suggest that they use a new tactic, "So, let's use the other angle we know." By using social metacognitive control strategies, they understand, evaluate and build on each other's thinking respectfully to create new, useful information that furthers their mathematics problem solving.

Benefits of social metacognitive control

While metacognitive control in its traditional sense is largely an individual pursuit, social metacognitive control involves group members in monitoring and controlling one another's knowledge, emotions, and actions (Chiu & Kuo, 2009). As such, social metacognitive control can yield several benefits: distributed cognitive and metacognitive demands, reciprocal scaffolding and enhanced motivation. With adequate social metacognitive control, groups can divide a complicated task into different sub-tasks and allocate them appropriately to distribute cognitive and metacognitive demands. Furthermore, when working collaboratively, a group member can articulate his or her cognitive and metacognitive processes to increase their visibility, thereby modeling them and allowing social metacognitive feedback. By focusing on their shared responsibilities, communal goals, risks and rewards, group members can develop stronger group identities and motivations than individuals working on the same tasks. As a result, social metacognitive control can help a group solve mathematics problems that are too difficult for a single group member.

Distributing cognitive and metacognitive demands

When group members understand one another's skills, talents and strengths (transactive metamemory; Wegner, 1995), they can apply their social metacognitive control to decompose a complex problem into sub-problems and allocate them to group members according to their individual skills and strengths (Chiu & Pawlikowski, 2013). Rather than trying to solve the entire mathematics problem, each group member faces fewer cognitive and metacognitive demands and can focus on suitable, smaller and simpler individual responsibilities. For example, while Danny labels the hypotenuse, Rita monitors and evaluates his actions. With fewer demands and distractions, each group member is less likely to make mistakes and more likely to solve the sub-problems (Chiu & Kuo, 2009).

As decomposing a mathematics problem into sub-problems suitable for group members is often difficult, young students require assistance and instruction from a teacher before learning to do so on their own (Chiu & Kuo, 2009). As students learn more about both the domains of mathematics problems and their group members' strengths and weaknesses, their social metacognitive control improves and they are able to decompose and distribute sub-problems to group members more effectively.

Modeling and social metacognitive feedback

Working in cooperative groups enhances the visibility of group members' cognition and metacognition, thereby enabling modeling and social metacognitive feedback. Group members model for one another as they use their understanding to try to solve a sub-problem. Thus, group members can see and hear one another's cognition and metacognition (e.g., Rita says, "First, let's figure out if we should use sine, cosine or tangent"), which can help them interpret the mathematics problem and make effective metacognitive control decisions (Nelson, Kruglanski, & Jost, 1998; Jost, Kruglanski, & Nelson, 1998).

Group members also give one another social metacognitive feedback tailored to the specific situation in the form of questions, relevant information, evaluations, and proposals for further action (Hmelo-Silver, 2006; Wittenbaum, et al., 2004). Questions can indicate inadequate understanding ("what angles do we know?"), to which a group member tries to respond with relevant information ("this one and this one" [points to angle of elevation and right angle], *transactive memory*, Wegner, 1995). Questions can also serve as evaluations, often polite critiques that seek to change the problem solving path ("shouldn't we figure out which angle we are dealing with first?" (Chiu, 2008)). On the other hand, agreements attempt to continue the current problem solving trajectory ("Okay, so . . . " Chiu, 2000). Lastly, group members can respond with proposals for further action ("Okay, so label the sides with opposite, hypotenuse and adjacent" (Chiu, 2001)). All of these social metacognitive feedbacks seek to monitor and control the group's problem solving, and their success depends on a person's understanding of both the mathematics problem and the other group members.

Positioning and motivation

Apart from the cognitive and metacognitive benefits suggested earlier, social metacognitive control can also influence student motivation through positioning (Davies & Harre, 1990). When group members position themselves together (e.g., Rita and Danny's "we," "us" and "our" rather than "I" or "you"), they highlight their shared situation and are more likely to share responsibilities, risks and rewards, which strengthens their sense of working together as a group. With less personal risk and a lower cost of failure, group members may feel less anxious and more motivated to work on the problem together. Furthermore, group members with a strong sense of teamwork are more motivated to work for the group's benefit. In contrast, positioning oneself in opposition to others (I vs. you) tends to separate oneself from the group, reduce group cohesion, weaken group cohesiveness and reduce motivation to work for the benefit of the group. Hence, social metacognitive control can enhance shared positioning, group identity and motivation or highlight separate positioning, reinforce individual identities and weaken motivation.

Challenges of social metacognitive control

Just as there were parallel benefits between metacognitive control and social metacognitive control, so too are there parallel difficulties. As with metacognitive control, social metacognitive control is difficult because it requires additional cognitive and metacognitive demands, is influenced by inaccurate self-evaluations and may result in unhelpful or ineffective feedback. Since social metacognitive control involves an added dimension of student interaction, there are additional difficulties associated with the social content. Students must be sensitive to contextual and situational factors (Efklides, 2009) that can influence performance, such as status effects, poor communication skills and cultural or personal differences.

Extra demands on mental resources

Effective social metacognitive control enables group members to allocate sub-problems and responsibilities efficiently among themselves to reduce cognitive and metacognitive demands for enhanced group problem solving. As personal metacognitive control requires dividing their attention between personal executive and evaluation tasks, social metacognitive control might further require attention to the cognitive and metacognitive processes of others, thereby placing further demands on brain resources (Chiu & Kuo, 2009). Hence, poor social metacognitive control allocation of sub-problems across group members can increase demands on brain resources rather than reduce them.

Inappropriate feedback

Limited individual content knowledge or poor social metacognitive control can result in unsuitable feedback than hinders group members (Holton & Thomas, 2001). Inappropriate feedback can be incorrect, misleading, or poorly timed, which can result in misallocation of resources, pursuit of inappropriate strategies or mis-scheduling of time (Chiu & Kuo, 2009). These poor judgments can frustrate group-mates and reduce their motivation.

Status effect

When group members perceive some members to have higher status than others, they value those people and their contributions more highly, which can hinder effective social metacognitive control. For example, group members might allocate sub-problems based on status rather than on their skills and strengths, which can hinder subsequent group problem solving (Cohen, 1994). Furthermore, group members might selectively invite and defer to high status members' opinions while discouraging, undervaluing, or outright ignoring lower status members' ideas (Chiu & Khoo, 2003). Thus, excessive attention to status can distort social metacognitive control toward excessive agreement with higher status members, which can reduce group productivity and hinder their mathematics problem solving.

Poor communication skills

Interpersonal communication skills are an essential component of social metacognitive control but do not develop automatically (Goleman, 1998). To implement social metacognitive control effectively, students must develop sufficient communication skills (Kaiser, Cai, Hancock, & Foster 2001; Gresham, Sugai, & Horner, 2001). For example, if Rita had weaker language skills and said "No, cannot have two adjacent sides," Danny might have interpreted the criticism as a rude, face attack (Tracy, 2008), retaliated and damaged their social relationship and subsequent problem solving. Even if Danny did not retaliate, he might have stopped giving suggestions or otherwise contributing, which would have hurt the entire group.

Understanding individual and cultural differences

While group member diversity can help generate more diverse ideas (Pelled, Eisenhardt, & Xin, 1999), insensitivity to cultural, racial or gender differences can lead to miscommunication and greater status effects that hinder effective social metacognitive control (Chiu & Kuo, 2009). As social metacognitive understanding of other group members is essential to effective social metacognitive control, a poor understanding of group members' individual and cultural differences can undermine the establishment of group norms, facilitate misinterpretation of group member actions, increase and result in misallocation of sub-problems across group members. Hence, learning how to value and capitalize on individual differences can aid group members' social metacognitive control and mathematics problem solving.

TEACHING METACOGNITIVE AND SOCIAL METACOGNITIVE CONTROL

Students do not naturally develop metacognitive and social metacognitive skills – indeed many adults have poor metacognitive skills (Glenberg, Wilkinson, & Epstein, 1982; Hartman, 2001). Thus, students need opportunities to learn these skills, practice them and receive feedback to improve them (Dirkes, 1985).

Difficulties of teaching metacognitive control

Despite its importance, metacognitive control instruction can be challenging for both students and teachers. Unlike other skills that are visible and easy to describe, metacognitive control is covert and requires students to consider aspects of their thinking they may have never considered. Using metacognitive control requires students to step beyond rote mathematics procedures and intentionally reflect on their thinking (Conner & Gunstone, 2004). Students who were successful with rote procedures can be uncomfortable or unwilling to stray from their historically useful methods. Moreover, social metacognitive control also requires social, interactional, and communicative skills.

Metacognitive control can entail increased cognitive load and mental demands on students. For students accustomed to rote procedures, the sudden infusion of metacognitive thinking may be met with apprehension or unease (Conner & Gunstone, 2004). Since metacognitive control requires extensive mental activity (Bruer, 1993), developing the knowledge, awareness, and control of the lower level cognitive skills needed for metacognitive control can be a tedious process for learners.

Meanwhile, teachers might not value metacognitive or social metacognitive control, or lack sufficient knowledge or sufficient preparation to teach students metacognitive skills. Teachers with poor metacognitive or social metacognitive control skills may not value them and hence not teach them to their students. Even teachers willing to teach metacognitive or social metacognitive control often lack the skills themselves (Garner & Alexander, 1989; Glenberg, Wilkinson, & Epstein, 1982) and hence cannot teach them to their students. Even if the teacher was able to effectively plan, monitor and evaluate their thinking, they may or may not know how to model and teach these skills to their students. Thus, without appropriate teacher preparation, metacognitive control instruction can present significant challenges.

Teaching metacognitive control is also difficult due to the fact that effective metacognitive control instruction must be grounded in domain-specific problem solving tasks (Brown et al., 1983; Pressley, Etten, Yokoi, Freebern, & Meter, 1998; Pressley et al., 1992). Chiu and Kuo (2009) recommend a gradual development of metacognitive and social metacognitive skills within a specific domain.

Schoenfeld on teaching metacognitive control

Schoenfeld (1985) advocates teaching metacognitive control as students work in small groups on novel problems. He suggests that teachers use small group work time not only to monitor progress about mathematical thinking but also to ask probing questions about students' metacognitive control strategies:

1. What (exactly) are you doing? (Can you describe it precisely?)
2. Why are you doing it? (How does it fit into the solution?)
3. How does it help you? (What will you do with the outcome when you get it?)

These questions encourage students to articulate and justify their rationale, evaluate their reasoning and detect errors in their thinking. The question "Why are you doing ________?" for example, encourages students to analyze their work, consider their strategies and ask subsequent questions. Though students often cannot answer these questions initially, Schoenfeld (1985) argued that continued, deliberate focus on them will develop students' metacognitive control to answer them.

While these questioning techniques did not make these students mathematics experts, their metacognitive control began to resemble more closely that of mathematicians. Schoenfeld (1985) found that the number of students who used metacognitive monitoring techniques (organizing a mathematical plan, intentionally selecting a problem solving method and monitoring the problem solving process during all stages of the process) rose 40% after metacognitive questioning techniques were used during small group interaction. Even when students faced novel problems, they continued to use the metacognitive questions as scaffolds for their thinking. He concluded that not only can metacognitive approaches be taught but students can transfer them to different contexts.

Teaching models of metacognitive control and social metacognitive control

Building opportunities to develop metacognitive and social metacognitive control habits in everyday classroom instruction is central to learning and applying metacognitive control (Lambert, 2000). Based on others' successful instruction (e.g., Gillies & Ashman, 1996; Perels, Dignath, & Schmitz, 2009; Webb, Nemer, & Ing, 2006), Chiu and Kuo (2009) present a three-part model for designing activities to improve students' metacognitive control. At the most basic level (level 1), when students have only minimal domain knowledge, they recommend relatively easy tasks that individual students can complete. For example, a teacher might ask students to identify a subgoal that would help them solve a mathematics problem or detect and correct an error. These metacognitive tasks are embedded in the mathematics content and are largely teacher directed.

At the next level, responsibility for metacognitive tasks gradually shifts from the teacher to the student (Schoenfeld, 1992). These tasks are more complex and are completed first as a whole class, and then in pairs. The teacher guides students to consider more complex metacognitive tasks, such as helping students evaluate their work, suggesting strategies or asking questions about the metacognitive thinking of others.

Finally at the most advanced level, social metacognitive control skills can be embedded into explicit whole class instruction and then applied to pairs and small groups. Students face authentic, challenging problems that require both communication and metacognitive skills. By developing metacognitive skills in this way, the teacher can promote social discourse (Tanner & Jones, 2000) as students try to clarify, justify and modify their beliefs based on the feedback from their peers. While a teacher may need to initially assign collaborative roles or provide students with cues, students gradually receive more opportunities for greater social metacognitive control and to initiate it on their own. Prompts to promote social metacognitive control can include those that activate students' prior knowledge of one another, those which ask students reflect on and evaluate others' ideas for accomplishing an activity, or those which encourage students to build on one another's ideas (Chiu & Kuo, 2009).

Metacognitive training can have a significant impact on both cognitive and metacognitive performance. In their meta-analysis, Hattie, Biggs and Purdie (1996) found that effective metacognitive training programs share several characteristics. First, metacognitive skills are embedded within lessons. Programs with direct and isolated instruction of metacognitive strategies were inefficient and did not improve students' mathematics achievement. They found that when students were taught how, when and why a specific strategy was used they were more likely to use it in others contexts. Furthermore, metacognitive training that was embedded into daily classroom routines was more effective than metacognitive training that was not embedded. Classroom teachers typically understand their classroom dynamics and students' behaviors, so they can adapt their teaching of metacognitive skills to fit student needs.

Notably, Perels, Dignath, and Schmitz (2009) tested these conditions in a general education mathematics classroom of 53 sixth grade students. For the control group, the teacher taught one class only the mathematics content (multipliers and divisors). For the experimental group, the same teacher taught the same mathematics content but also included metacognitive instruction. After nine lessons, pre-test and post-test evaluations showed that the combined metacognitive and cognitive instructional approach was superior to cognitive instruction alone and promoted greater transfer of both metacognitive control and mathematical understanding of multipliers and divisors. Similar results have been shown in a number of mathematics-based classroom settings with diverse student populations at both the middle and high school levels and different assessment measures (Cardelle-

Elawar, 1992; Desoete, Roeyes, & Buysse, 2001; Bannert & Mengelkamp, 2008; McDonald & Boud, 2003) Metacognitive research using developmentally appropriate strategies such as concept mapping and think aloud protocols have been effectively used on students as young as pre-school age (Figueiredo et al., 2004; Gallenstein, 2005).

In addition to short-term improvements in academic and metacognitive skills, metacognitive instruction also has transferrable, long-term consequences on student learning. In her 2-year longitudinal study of 66 third-grade students for example, Desoete (2009) showed that the students in the experimental group receiving metacognition training had both higher metacognitive skills and problem solving skills than the control group both immediately after the intervention as well as a year later in fourth grade. These results suggest that metacognitive training can have long-term effects on students' mathematical problem solving skills.

CONCLUSIONS AND DIRECTIONS FOR FURTHER RESEARCH

Schoenfeld's studies of mathematics students' metacognitive control spurred further research on this topic and also on social metacognitive control. Whereas metacognitive control uses metacognitive knowledge to regulate one's thinking, actions and emotions, social metacognitive control uses knowledge of group members to regulate their thinking, actions and emotions.

Metacognitive control motivates acquisition of greater metacognitive knowledge and relevant resources to aid problem solving, thereby enhancing mathematics learning and problem solving skills. However, metacognitive control requires dividing resources among cognitive and metacognitive processes, which can increase memory demands, hinder accurate evaluations, and ultimately harm mathematics problem solving. Likewise, teaching metacognitive control faces several challenges: its covert nature, increased memory and mental demands, lack of appreciation for the value of this control, inadequate metacognitive skills among teachers and insufficient training to teach metacognitive skills. However, several metacognitive training programs have successfully taught students metacognitive control, which helped them learn more mathematics and develop problem solving skills. These successful programs share the following two characteristics: embedded metacognitive control instruction within mathematics lessons and instruction by the classroom teacher rather than an outsider.

Meanwhile, social metacognitive control can serve to divide a problem into sub-problems, allocate the sub-problems to group members according to their strengths, increase the visibility of cognitive and metacognitive processes to allow for modeling and suitable feedback, and foster group identity to enhance motivation. Challenges of effective social metacognitive control include additional cognitive and metacognitive demands, inaccurate evaluations, unsuitable feedback and contextual factors (e.g., status effects, poor communication skills and cultural or personal differences). Lastly, how to teach students social metacognitive control to improve their mathematics problem solving remains an open question.

As social metacognitive control is still a developing area of research, further studies are needed to extend and apply it to heterogeneous classroom situations, diverse student populations, special education and "at risk" populations. Longitudinal studies on social metacognitive interventions can test whether it affects long-term mathematics problem solving and achievement. Furthermore, there is little existing research on programs designed to promote students' social metacognitive control while learning mathematics. Future studies can evaluate programs that teach social metacognitive control. Other variables such as timing of social metacognitive interventions, the appropriate amount of scaffolding required and the level of teacher participation in the process can also be examined.

Finally, as discussed earlier, teachers play a critically important role in promoting social metacognitive skills. However, there has been little research on the types of professional development that can improve teacher awareness and effectiveness. Thus, future research can examine how to help teachers develop students' metacognitive and social metacognitive skills.

REFERENCES

Bannert, M., & Mengelkamp, C. (2008). Assessment of metacognitive skills by means of instruction to think aloud and reflect when prompted. Does the verbalisation method affect learning? *Metacognitive Learning*, *3*, 39–58.

Brown, A. L., Bransford, J. D., Ferrara, R. A., & Campione, J. C. (1983). Learning, remembering, and understanding. In J.H. Flavell & E.M. Markman (Eds.), *Handbook of child psychology: Vol. 3. Cognitive development* (4th ed., pp. 77–166). New York: John Wiley and Sons.

Bruer, J. T. (1993). *Schools for thought: A science of learning in the classroom*. Cambridge, MA: MIT Press.

Calskin, M., & Sunbul, A. M. (2011). The effects of learning strategies instruction on metacognitive knowledge, using metacognitive skills and academic achievement. *Educational Sciences: Theory and Practice*, *11*(1), 148–153.

Catdelle-Elawar, M. (1992). Effects of teaching metacognitive skills to students with low mathematics ability. *Teaching and Teacher Education*, *8*(2), 109–121.

Cavanaugh, J. C., & Perlmutter, M. (1982). Metamemory: A critical examination. *Child Development*, *53*, 11–28.

Chen, C., Chiu, M. M., & Wang, Z. (2012). Social metacognition and the creation of correct, new ideas: A statistical discourse analysis of online mathematics discussions. *Computers in Human Behavior*, *28*(3), 868–880.

Chiu, M. M. (2000). Group problem solving processes: Social interactions and individual actions. *Journal for the Theory of Social Behavior*, *30*(1), 27–50.

Chiu, M. M. (2001). Analyzing group work processes: Towards a conceptual framework and systematic statistical analyses. In F. Columbus (Ed.), *Advances in psychology research* (Vol. 4, pp. 193–222). Huntington, NY: Nova Science Publishers.

Chiu, M. M. (2008a). Flowing toward correct contributions during groups' mathematics problem solving: A statistical discourse analysis. *Journal of the Learning Sciences*, *17*(3), 415–463.

Chiu, M. M. (2008b). Effects of argumentation on group micro-creativity: Statistical discourse analyses of algebra students' collaborative problem solving. *Contemporary Educational Psychology*, *33*, 383–402.

Chiu, M. M., & Khoo, L. (2003). Rudeness and status effects during group problem solving. *Journal of Educational Psychology*, *95*, 506–523.

Chiu, M. M., & Kuo, S. W. (2009). Social metacognition in groups: Benefits, difficulties, learning, and teaching. In C. B. Larson (Ed.), *Metacognition: New research developments* (pp. 117–136). Hauppauge, NY: Nova Science Publishers.

Chiu, M. M., Kessel, C., Moschkovich, J., & Munoz, A. (2001). Learning to graph linear functions: A case study of conceptual change. *Cognition and Instruction*, *19*(2), 215–252.

Chiu, M. M., & Pawlikowski, M. J. (2013). Social metacognition and micro-creativity. In E. G. Carayannis (Ed.), *Encyclopedia of creativity, invention, innovation, and entrepreneurship*. New York: Springer.

Cohen, E. G. (1994). Restructuring the classroom: Conditions for productive small groups. *Review of Educational Research*, *64*(1), 1–35.

Conner, L., & Gunstone, R. (2004). Conscious knowledge of learning: Accessing learning strategies in a final year high school biology class. *International Journal of Science Education*, *26*, 1427–1443.

Davies, B., & Harre, R. (1990). Positioning: The Discursive Production of Selves. *Journal for the Theory of Social Behavior*, *20*(1), 43–63.

Davis, E. A., & Linn, M. C. (2000). Scaffolding students' knowledge integration: Prompts for reflection in KIE. *International Journal of Science Education*, *22*(8), 819–837.

Desoete, A. (2009). Metacognitive prediction and evaluation skills and mathematical learning in third-grade students. *Educational Research and Evaluation*, *15*, 435–446.

Desoete, A., Roeyers, H., & Buysse, A. (2001). Metacognition and mathematical problem solving in grade 3. *Journal of Learning Disabilities*, *34*, 435–449.

Dirkes, M. A. (1988). Self-directed thinking in the curriculum. *Roeper Review*, *11*(2), 92–94.

Efklides, A. (2009). The new look in metacognition: From individual to social, from cognitive to affective. In B. C. Larson (Ed.), *Metacognition: New research developments* (pp. 137–151). New York: Nova Science

Efklides, A. (2011). Interactions of metacognition with motivation and affect in self regulated learning: The MASRL model. *Educational Psychologist*, *46*(1), 6–25.

Figueiredo, M., Lopez, A. S., Firmino, R., & de Sousa, S. (2004) "Things we know about the cow:" Concept mapping in a preschool setting. In A. J. Cañas, J. D. Novak, & F. M. González (Eds.), *Concept maps: Theory, methodology, technology. Proceedings of the First International Conference on Concept Mapping* (Vol. I, pp. 125–133). Pamplona, Spain: Universidad Pública de Navarra.

Flavell, J. H. (1979). Metacognition and cognitive monitoring: A new area of cognitive-developmental inquiry. *American Psychologist*, *34*, 906–911.

Gallenstein, N. L. (2005). Never too young for a concept map. *Science and Children*, *43*(1), 44–47.

Garner, R., & Alexander, P. A. (1989). Metacognition: Answered and unanswered questions. *Educational Psychologist*, *24*(2), 143–158.

Gillies, R., & Ashman, A. (1996). Teaching collaborative skills to primary school children in classroom-based work groups. *Learning and Instruction*, *6*, 187–200.

Glenberg, A. M., Wilkinson, A. C., & Epstein, W. (1982). The illusion of knowing: Failure in the self-assessment of comprehension. *Memory, & Cognition*, *10*, 597–602.

Goleman, D. (1998). *Working with emotional intelligence*. New York: Bantam Books.

Gresham, F. M., Sugai, G., & Horner, R. H. (2001). Interpreting outcomes of social skills training for students with high-incidence disabilities. *Exceptional Children*, *67*(3), 331–344.

Hacker, D. J., & Bol, L. (2004). Metacognitive theory: Considering the social-cognitive influences. In D. M. McInerney & S. Van Etten (Eds.), *Big theories revisited* (Vol. 4, pp. 275–297). Greenwich, CT: Information Age Publishing.

Hartman, H. J. (2001). *Metacognition in learning and instruction: Theory, research and practice*. Dordrecht: Kluwer Academic Publishers.

Hattie, J., Biggs, J., & Purdie, N. (1996). Effects of learning skills interventions on student learning: A meta-analysis. *Review of Educational Research*, *66*(2), 99–130.

Hmelo-Silver, C. E. (2006). Design principles for scaffolding technology-based inquiry. In A. M. O'Donnell, C. E. Hmelo-Silver, & G. Erkens (Eds.), *Collaborative learning, reasoning, and technology* (pp. 147–170). Mahwah, NJ: Erlbaum.

Holton, D., & Clarke, D. (2006). Scaffolding and metacognition. *International Journal of Mathematical Education in Science and Technology*, *37*(2), 127–143.

Holton, D., & Thomas, G. (2001). Mathematical interactions and their influence on learning. In D. J. Clarke (Ed.), *Perspectives on practice and meaning in mathematics and science classrooms* (pp. 75–104). Dordrecht: Kluwer Academic Publishers.

Jost, J. T., Kruglanski, A. W., & Nelson, T. O. (1998). Social metacognition: An expansionist review. *Personality and Social Psychology Review*, *2*, 137–154.

Kaiser, A. P., Cai, X., Hancock, T. B., & Foster, M. E. (2002). Teacher reported behavior problems and language delays in boys and girls enrolled in Head Start. *Behavioral Disorders*, *28*, 23–29.
Kaplan, A. (2008). Clarifying metacognition, self-regulation, and self-regulated learning: What's the purpose? *Educational Psychology Review*, *20*(4), 477–484.
King, A. (1991). Effects of training in strategic questioning on children's problem-solving performance. *Journal of Educational Psychology*, *83*(3), 307–317.
Lambert, M. A. (2000). Using cognitive and metacognitive learning strategies in the classroom. *Preventing School Failure*, *44*, 81–82.
Larkin, S. (2009). Socially mediated metacognition and learning to write. *Thinking Skills and Creativity*, *4*(3), 149–159.
Lerch, C. M. (2004). Control decisions and personal beliefs: Their effect on solving mathematical problems. *The Journal of Mathematical Behavior*, *23*(1), 21–36.
McDonald, B., & Boud, D. (2003). The impact of self-assessment on achievement: The effects of self-assessment training on performance in external examinations. *Assessment in Education*, *10*, 209–220.
Miller, T. M., & Geraci, L. (2011). Training metacognition in the classroom: The influence of incentives and feedback on exam predictions. *Metacognition Learning*, *6*, 303–314.
Nelson, T. O., & Narens, L. (1990). Metamemory: A theoretical framework and some new findings. In G. H. Bower (Ed.), *The psychology of learning and motivation* (Vol. 26, pp. 125–140). New York: Academic Press.
Nelson, T. O., Kruglanski, A. W., & Jost, J. T. (1998). Knowing thyself and others: Progress in metacognitive social psychology. In V. Y. Yzerbyt, G. Lories, & B. Dardenne (Eds.), *Metacognition: Cognitive and social dimensions* (pp. 69–89). London: Sage.
Otani, H., & Widner, R. L. (2005). Metacognition: New issues and approaches. *The Journal of General Psychology*, *132*(4), 329–334.
Pelled, L., Eisenhardt, K., & Xin, K. (1999). Exploring the black box: An analysis of work group diversity, conflict, and performance. *Administrative Science Quarterly*, *44*, 1–28.
Perels, F., Dignath C., & Schmitz, B. (2009). Is it possible to improve mathematical achievement by means of self-regulation strategies? Evaluation of an intervention in regular math classes. *European Journal of Psychology of Education*, *14*(1), 17–31.
Pressley, M., Etten, S. V., Yokoi, L., Freebern, G., & Meter, P. V. (1998). The metacognition of college studentship: A grounded theory approach. In H. Dunlosky & A. Graesser (Eds.), *Metacognition in educational theory and practice* (pp. 347–367). Hillsdale, NJ: Lawrence Erlbaum Associates.
Pressley, M., Wood, E., Woloshyn, V. E., Martin, V., King, A., & Menke, D. (1992). Encouraging mindful use of prior knowledge: Attempting to construct explanatory answers facilitates learning. *Educational Psychologist*, *27*, 91–110.
Salonen, P., Vauras, M., & Efklides, A. (2005). Social interaction – What can it tell us about metacognition and coregulation in learning? *European Psychologist*, *10*, 199–208.
Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Hillsdale, NJ: Erlbaum.
Schoenfeld, A. H. (1988, Spring). When good teaching leads to bad results: The disasters of "well taught" mathematics classes. *Educational Psychologist*, *23*(2), 145–166.
Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.
Tanner, H., & Jones, S. (2000). Scaffolding for success: Reflective discourse and the effective teaching of mathematical thinking skills. *Research in Mathematics Education*, *2*(1), 19–32.
Tracy, K. (2008). Reasonable hostility: Situation-appropriate face attack. *Journal of Politeness Research*, *4*(2), 169–191.
Webb, N. M., Nemer, K. M., & Ing, M. (2006). Small-group reflections: Parallels between teacher discourse and student behavior in peer-directed groups. *Journal of the Learning Sciences*, *15*(1), 63–119.
Wegner, D. M. (1995). A computer network model of human transactive memory. *Social Cognition*, *13*, 319–339.

Wittenbaum, G. M., Hollingshead, A. B., Paulus, P. B., Hirokawa, R. Y., Ancona, D. G., Peterson, R. S., Jehn, K. A., & Yoon, K. (2004). The functional perspective as a lens for understanding groups. *Small Group Research*, *35*, 17–43.

Zellermayer, M., Salomon, G., Globerson, T., & Givon, H. (1991). Enhancing writing-related metacognitions through a computerized writing partner. *American Educational Research Journal*, *28*(2), 373–391.

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PART III

PROFICIENT PERFORMANCE, BELIEFS, AND PRACTICES IN MATHEMATICS TEACHING, AND WAYS TO FACILITATE THEM

PESSIA TSAMIR, DINA TIROSH, ESTHER LEVENSON, RUTHI BARKAI AND MICHAL TABACH

7. THE CAMTE FRAMEWORK

A Tool for Developing Proficient Mathematics Teaching in Preschool[1]

INTRODUCTION

This chapter is concerned with developing teachers' knowledge for teaching mathematics in preschool. Like Alan Schoenfeld, we are concerned with teachers, in this case preschool teachers, knowing school mathematics in depth and in breadth. Like Günter Törner, one of the founders of the MAVI (Mathematical Views) conference, we are concerned with the affective side of teacher education. The framework we present in this chapter combines both cognitive and affective aspects related to facilitating proficient mathematics teaching in preschool.

Recently, the issue of mathematics education for preschool children has come to the fore. A joint position paper published in the United States by the National Association for the Education of Young Children (NAEYC) and the National Council for Teachers of Mathematics (NCTM) stated that "high quality, challenging, and accessible mathematics education for 3- to 6-year-old children is a vital foundation for future mathematics learning" (NAEYC & NCTM, 2002, p. 1). In various curricula, there are now specific and sometimes mandatory recommendations for including mathematics as part of the preschool program. For example, in England the non-statutory Practice Guidance for the Early Years Foundation Stage (2008) suggests ways of fostering children's mathematical knowledge from 0 to 5 years. In Israel, the National Mathematics Preschool Curriculum (INMPC, 2008) is mandatory and contains specific guidelines and objectives for children from 3 to 6 years.

The preschool teacher plays an integral role in fostering children's mathematical abilities. "It is up to her to devote attention both to planned mathematical activities as well as mathematical activities which may spontaneously arise in the class and to pay attention to the mathematical development of the children" (INMPC, 2008, p. 8). Yet in Israel, as in many countries, attention to mathematics teacher education is mostly given at the elementary and secondary levels (Arcavi, 2004; Kaiser, 2002). All too often, preschool teachers receive little or no preparation for teaching mathematics to young children (Ginsburg, Lee, & Boyd, 2008). Studies have shown that few preschool teachers are familiar with the mathematical content taught in local schools (Starkey et al., 1999) and that some preschool teachers spend less than one percent of class time on mathematics activities (Farran, Lipsey, Watson, & Hurley, 2007).

With this in mind, it is not surprising to find an increased call for strengthening the preparation of preschool teachers for teaching mathematics. The National Association for the Education of Young Children (NAEYC) and the National Council

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for Teachers of Mathematics (NCTM) recommend that "teachers of young children should learn the mathematics content that is directly relevant to their professional role" (p. 14). Similarly, the Australian Association of Mathematics Teachers (AAMT) and Early Childhood Australia (ECA) published a joint position paper recommending that early childhood staff be provided with "ongoing professional learning that develops their knowledge, skills and confidence in early childhood mathematics" (2006, p. 3).

This chapter describes a professional development program for preschool teachers, which aims to promote the knowledge necessary for teaching mathematics in preschool. The chapter is divided into three sections. The first section introduces the *Cognitive Affective Mathematics Teacher Education* (CAMTE) framework, used in planning and implementing the program. Recognizing that knowledge and affective issues are interrelated and influence teachers' proficiency (Pehkonen &Törnerr, 1999; Schoenfeld, 1992; Schoenfeld & Kilpatrick, 2008; Törner, 2002), the framework and program take into consideration teachers' knowledge as well as self-efficacy beliefs. The second part of the chapter illustrates how the above framework was used to plan and implement the professional development program. Segments of the program are presented. We also illustrate how, in line with Schoenfeld's (1999) call to treat teachers as professionals, the program provides preschool teachers with opportunities to collaborate with their colleagues in the planning of lessons. The third section offers some initial results of our investigation of teachers' knowledge and self-efficacy beliefs before and after intervention.

THE COGNITIVE AFFECTIVE MATHEMATICS TEACHER EDUCATION (CAMTE) FRAMEWORK

In this section we present the theoretical framework which guides our program and our investigation of teachers' knowledge and self-efficacy beliefs. It begins with a discussion of teachers' knowledge for teaching and a brief review of self-efficacy along with its relationship to knowledge. We then present the framework and how it relates to preschool teachers' knowledge for teaching mathematics.

Teachers' knowledge for teaching

In framing the mathematical knowledge preschool teachers need for teaching, we draw on Shulman (1986) who identified subject-matter knowledge (SMK) and pedagogical content knowledge (PCK) as two major components of teachers' knowledge that are necessary for teaching. In our previous work (Tabach et al., 2010), we found it useful to differentiate between two components of teachers' SMK: being able to produce solutions, strategies, and explanations and being able to evaluate given solutions, strategies, and explanations. These aspects of teachers' knowledge are connected. In order to evaluate different solutions or strategies that might arise in the classroom, the teacher must know different solutions and strategies. These aspects of SMK require the teacher to know school mathematics in depth and breadth (Schoenfeld & Kilpatrick, 2008). Our framework employs these aspects of SMK. Regarding PCK, we draw on the works of Ball and her colleagues (Ball, Thames, & Phelps, 2008) who differentiated between two aspects of PCK: knowledge of content and students (KCS) and knowledge of content and teaching (KCT). KCS is "knowledge that combines knowing about students and knowing about mathematics" whereas KCT "combines knowing about teaching and knowing about mathematics" (Ball, Thames, & Phelps, 2008, p. 401). Schoenfeld and Kilpatrick's (2008) provisional framework for proficiency in teaching mathematics also called for "knowing students as thinkers, knowing students as learners, [and] crafting and managing learning environments" (p. 322).

Within the domain of numbers, preschool teachers' SMK includes, for example, knowledge about counting, operations, relations and a variety of possible ways and methods of rationally examining and explaining one's solutions. Teachers' KCS includes, for example, knowledge of young children's non-conservation of numbers (Piaget & Inhelder, 1958). Within geometry, preschool teachers' SMK includes, for example, defining geometrical concepts and identifying various examples and non-examples of two and three-dimensional figures (solids) and ways of justifying this identification. Teachers' KCS includes, for instance, knowledge of which examples and non-examples children intuitively recognize as such (Tsamir, Tirosh, & Levenson, 2008). In both domains, KCT includes knowledge of designing and assessing different tasks, affording students multiple paths to understanding.

Self-efficacy

The affective domain of learning and teaching includes sub-domains such as emotions, attitudes, values, and beliefs. Beliefs can further be divided into subject-specific beliefs (e.g., beliefs about the nature of mathematics), beliefs about teaching and learning (e.g., beliefs about how learning takes place), beliefs about teaching and learning a specific subject, and self-beliefs (i.e., beliefs about oneself). Among the various self-beliefs, such as self-concept, self-esteem, and self-efficacy, efficacy beliefs have been found to have a strong association with academic performance and may be considered a sub-construct of self-concept (Pietsch, Walker, & Chapmen, 2003). The CAMTE framework draws on Bandura's (1986) social cognitive theory which takes into consideration the relationship between psychodynamic and behaviouristic influences, as well as personal beliefs and self-perception, when explaining human behaviour. Bandura defined self-efficacy as "people's judgments of their capabilities to organize and execute a course of action required to attain designated types of performances" (1986, p. 391). Hackett and Betz (1989) defined mathematics self-efficacy as "a situational or problem-specific assessment of an individual's confidence in her or his ability to successfully perform or accomplish a particular [mathematics] task or problem" (p. 262). The CAMTE framework addresses teachers' mathematics self-efficacy as well as their pedagogical-mathematics self-efficacy, i.e., their self-efficacy related to the pedagogy of teaching mathematics. Teacher self-efficacy has been related to a variety of teacher classroom behaviours that affect the teacher's effort in teaching, and his or her persistence and resilience in the face of difficulties with students (Ashton &

Table 1. The Cognitive Affective Mathematics Teacher Education Framework.

	Subject-matter		*Pedagogical-content*	
	Solving	*Evaluating*	*Students*	*Tasks*
Knowledge	Cell 1: Producing solutions	Cell 2: Evaluating solutions	Cell 3: Knowledge of students' conceptions	Cell 4: Designing and evaluating tasks
Self-efficacy	Cell 5: Mathematics self-efficacy related to producing solutions	Cell 6: Mathematics self-efficacy related to evaluating solutions	Cell 7: Pedagogical-mathematics self-efficacy related to children's conceptions	Cell 8: Pedagogical-mathematics self-efficacy related to designing and evaluating tasks

Webb, 1986). Studies report that teachers with a high sense of self-efficacy are more enthusiastic about teaching (Allinder, 1994) and are more committed to teaching (Coladarci, 1992).

Regarding the relationship between knowledge and self-efficacy, Swars, Daane, and Giesen (2006) found that pre-service teachers' self-efficacy for teaching mathematics was associated with their content knowledge. Teachers who expressed their understanding of mathematics perceived themselves as effective teachers. Similarly, Tschannen-Moran and Johnson (2011) claimed that effective instruction is partly dependent on the teacher's belief in her or his ability to use knowledge appropriately when performing tasks. There also seems to be a relationship between self-efficacy and knowledge which is related to professional development. Wheatley (2002) claimed that teachers' efficacy doubts may cause a feeling of disequilibrium which in turn may foster teacher learning. Similarly, another study found that the higher a teacher's sense of self-efficacy, the higher the motivation to actively participate in professional development (Brady et al., 2009). In addition, that study found that teachers who had learned more were likely to increase their self-efficacy.

The design of our program and the accompanying study was based on the CAMTE framework (see Table 1). In cells 1–4, and in cells 5–8, we address teachers' knowledge and self-efficacy respectively.

Below, we illustrate the different cells of the framework within the domain of number concepts, focusing on teachers' knowledge for teaching counting and enumeration.

Counting refers to saying the number words in the proper order and knowing the principles and patterns in the number system as coded in one's natural language (Baroody, 1987). For the purpose of this article we define "enumerating" as "counting objects for the purpose of saying how many." This is in line with the Hebrew terminology used in the Israel curriculum which differentiates between counting

Table 2. Examples for the Cognitive Affective Mathematics Teacher Education Framework.

	Subject-matter		*Pedagogical-content*	
	Solving	*Evaluating*	*Students*	*Tasks*
Knowledge	Cell 1: Compare the number of elements in two sets using a variety of strategies; enumerate the following large collection of items using a variety of strategies	Cell 2: Evaluate the following strategies for comparing the number of elements in two sets; evaluate the following justification for why one set has more elements than another set.	Cell 3: Which number symbols are more difficult for children to learn? What are children's common mistakes related to the counting sequence?	Cell 4: Which tasks have the potential to foster children's acceptance of the one-to-one principle necessary for enumerating? Which tasks will assess children's counting and enumerating skills?
Self-efficacy	Cell 5: Teachers' beliefs related to their ability to enumerate a large collection of items in multiple ways.	Cell 6: Teachers' beliefs in their ability to evaluate various strategies for enumerating.	Cell 7: Teachers' beliefs in their ability to identify children's common mistakes related to counting and enumerating.	Cell 8: Teachers' beliefs in their ability to design tasks that will promote children's correct and efficient enumerating strategies.

(ספירה) and enumerating (מניה). (Due to the differences between the usage of the terms in English and in Hebrew, at times we use counting interchangeably with enumerating and trust that the reader will understand the specific meaning according to the context.) Gelman and Gallistel (1978) outlined five principles of enumeration. The three "how-to-count" principles include the one-to-one principle, the stable-order principle, and the cardinal principle. The two "what-to-count" principles include the abstraction principle, and the order-irrelevance principle. Teachers are not always aware of all of the above principles or that the skill of enumerating actually includes several skills. For each cell we offer specific examples (see Table 2).

The framework was used to plan and implement our professional development program and to study preschool teachers' knowledge (SMK related to producing and evaluating solutions and strategies and PCK related to knowledge about students, tasks, and mathematics) and self-efficacy to teach mathematics in preschool. The framework served as an organizing tool and as a set of checks and balances. We used it to ask ourselves: What do preschool teachers need to know to teach mathematics in preschool? Are we paying attention to different types of knowledge? Are we devoting time to each of the different elements signified by the different cells? Although each cell focuses specifically on a different piece of the knowledge and self-efficacy puzzle, the different elements are often intertwined. This is illustrated in the next section as we describe segments of our program.

PLANNING AND IMPLEMENTING PROFESSIONAL DEVELOPMENT FOR PRESCHOOL TEACHERS

For the past several years, we have been providing professional development for preschool teachers with the intention of promoting their mathematics knowledge for teaching as well as their mathematics and pedagogical-mathematics self-efficacy. Some of our programs extended for only a few weeks while others continued for as long as three years. In this section, we describe how the CAMTE framework was used to plan segments of our program which aimed to enhance teachers' knowledge and self-efficacy for teaching number concepts. We begin by focusing on specific elements of knowledge promotion (Cells 1–4) and continue with a discussion of self-efficacy (Cells 5–8).

Promoting teachers' knowledge of producing and evaluating solutions: Cells 1 and 2

Cells 1 and 2 of the CAMTE framework focus on teachers' knowledge of solving mathematical tasks as well as evaluating solutions to mathematical tasks. In order to plan mathematical activities within the domain of number concepts to be implemented with teachers, we reviewed the Israel National Mathematical Preschool Curriculum (INMPC, 2008) and curricula from other countries. These curricula often provide examples of the types of activities which might be implemented in the classroom along with the mathematical objectives of the activities. The curriculum in Israel lists several topics under the heading of number concepts including counting, enumeration, arithmetic operations, number relationships, and set relationships. One of our goals was to strengthen and deepen teachers' knowledge of set relationships including comparing the number of elements in sets and creating sets with an equal number of elements. Knowledge of mathematics also includes knowledge of mathematical processes such as reasoning, communication, and problem solving. Enumeration may include comparing the number of elements in two sets using different strategies and evaluating which of those strategies are appropriate for a given situation.

In one of our programs (described in more detail in Tirosh, Tsamir, Levenson, & Tabach, 2011) teachers were given the following task to solve:

Here are two sets A and B:
$A = \{1, t, \alpha\}$ $B = \{7, w\}$
Is the number of elements in sets A and B equal? Yes / No
How did you reach this conclusion?

Figure 1. Comparing the number of elements in two sets – triggering enumeration.

It may seem that the question above is artificial and that perhaps an everyday context would be more appropriate for preschool teachers. However, a more concrete task might not have afforded the teachers the same opportunity to discuss the abstraction of number concepts. In addition, familiarizing teachers with set notation and the use of brackets allowed us to discuss sets in more general terms later on, rather than always refer to specific examples. We also note that in Hebrew, the word for a set of concrete objects is the same word used for the mathematical notion of sets. Thus, although it may seem to the reader that the language of sets may seem too formal for preschool teachers, it is actually quite a familiar term.

The sets in the above example have an unequal number of elements. The explanation given by all teachers referred to enumerating the elements in both sets. As one teacher wrote:

> Set A has three elements and set B has two elements. In order to determine if two sets have the same number of elements, you need to count the number of elements in each set.

The first question (Figure 1) was designed to trigger the use of enumeration as a comparison strategy and indeed, the teachers used only this method for set comparison. In Figure 2 we present questions that were designed to surprise the teachers.

These questions challenged the teachers to consider methods other than enumeration for comparing the number of elements in two sets. In fact, it is not always possible to count the elements of a set. This was highlighted by the paired dancers of Question 2 (see Figure 2) where a one-to-one correspondence between men and women indicated equivalence between the two sets. In addition, there are situations when although counting may be applicable, it is not always preferable. Thus it is important to be able to evaluate strategies as well as final answers. This point was highlighted when discussing the two sets in question 5.

> Instructors: Did anyone count the number of green and red pills taken each week?
>
> Collective answer: No.
>
> Instructors: Even though you can count them, is it the most efficient method?

As the instructors pointed out, although it is possible to calculate how many green tablets and how many red capsules were prescribed for each week it is clear that one-to-one correspondence may also be used. For each green tablet taken during the first week a red capsule was taken during the second week.

2. At a dance party all the students danced in couples, a boy and a girl in each couple. No pupils were left without a partner.
   Z = {The boys}   W = {The girls}
   Is the number of elements in set Z equal to the number of elements in set W? Yes / No
   How did you reach this conclusion?
3. Given the sets:
   X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}   Y = {a, b, c, d, e}
   Is the number of elements in set X equal to the number of elements in set Y? Yes / No
   How did you reach this conclusion?
4. Given the sets:
   P = {a, b, c, d, e, f}   V = {a, b, c}
   Is the number of elements in set P equal to the number of elements in set V? Yes / No
   How did you reach this conclusion?
5. Dan was ill. The doctor prescribed one green tablet every 3 hours for the first week. Then, in the second week he was ordered to take a red capsule every 3 hours.
   G = {The green tablets} R = {The red capsules}
   Is the number of elements in set G equal to the number of elements in set R? Yes / No
   How did you reach this conclusion?

Figure 2. Comparing the number of elements in two sets – triggering various methods.

In relation to questions 3 and 4 in Figure 2, the teachers raised another method – visual comparison: "I can see which set has more elements." This method was accepted but with constraints. It was agreed that when two sets have a dramatically different number of different elements, then one may "see" that one has more than the other.

The instructor's comment brings the discussion back to valid methods of comparison such as counting, one-to-one correspondence, and the subset method, which is introduced below. The teachers were then invited to take a closer look at the sets in question 4.

Lilly: The bigger one (set P) includes in it the smaller one (set V)

Instructors: So, we have another method. When one set is a proper subset of another, then which has more?

Valerie: The subset has less. (Note that Valerie's reply refers to the case of finite sets only. The case of infinite sets is not discussed at this stage.)

Joy: So, (the method of) seeing is called the subset method.

Eve: The proper subset method is not appropriate for the sets in question 3.

The last comment by Eve suggests that she is beginning to differentiate between the methods for comparing the number of elements in a set and that she is aware that not every method is appropriate for every situation.

To summarize, the above episodes illustrated how teachers' knowledge for solving number tasks as well as their knowledge of evaluating methods for solving number tasks, was enhanced. (Additional examples of how teachers' knowledge of evaluating solutions was promoted may be found in Tirosh, Tsamir, Levenson, and Tabach (2011).) It also shows how teachers' knowledge of evaluating solutions is connected to their knowledge of producing solutions. After the teachers experienced solving problems in different ways, most of them were better equipped to evaluate the benefits and disadvantages of the different strategies. The teachers, with guidance from the instructors, came up with five different methods for comparing the number of elements in two sets: counting, one-to-one correspondence, differences "seeing," and the subset method. Furthermore, teachers began to relate various activities already implemented with children to the counting principles and to the comparison methods. For example, Joy noted how she often paired up the children in her class for some activity and how this connects to the one-to-one correspondence concept. Regarding the evaluation of solutions, teachers became aware that certain situations and tasks encourage the problem solver to use different methods and that in certain cases some solutions may not be as acceptable as other solution methods. Thus, we also see a connection between teachers' knowledge of producing and evaluating solutions, and their knowledge of tasks.

Promoting teachers' knowledge of children's conceptions and of tasks: Cells 3 and 4

Tasks play a major role in mathematics education at any age and preschool is no exception. While children are "playing" they are also learning and forming concepts, as well as acquiring the habits of mind that will accompany them as they develop. Observing children implementing such tasks also provides us the opportunity to learn about children's conceptions (Ginsburg, 2006). In our program, tasks play a central role in developing teachers' knowledge. Above, we illustrated how engaging teachers with tasks facilitated the development of their knowledge related to cells 1 and 2 of the framework. Also important is developing teachers' knowledge of how students may interact with tasks (Cell 3 of the framework) and developing teachers' knowledge of designing tasks to be implemented with children (Cell 4). Often, these last two elements are inter- related. This is illustrated in the section below.

When considering the types of tasks that teachers implement with their young students, we differentiate between tasks intended to teach or enhance students' knowledge as opposed to tasks which seek to assess students' knowledge. While the difference may be subtle, and perhaps any task could be viewed in both lights, we make this differentiation and discuss it with teachers in order to sharpen their knowledge of the different aspects of tasks that need to be considered before, during, and after implementing the task.

Because standardized curriculum materials are less available for preschool teachers in Israel, a major part of our program is spent on developing, along with the teachers, different tasks. Developing a task includes discussing the point of the task, the raw materials which will be used for the task, and the directions which will be given to the children. Care is taken to include the teachers' ideas, reaped from their own experience. The teachers know what raw materials are available to them and which materials may best suit different tasks. They are also aware of their students' backgrounds and home environments.

In the following excerpts we describe a session where preschool teachers discuss tasks that may be used to assess children's knowledge of enumeration.

I: According to the curriculum guidelines, by the end of kindergarten a child should be able to enumerate 30 objects. In order to assess if a child can enumerate we first need to ask him to count without giving him objects. If he doesn't know the number sequence, he will not be able to enumerate objects. Now, how many objects should I place before the child to enumerate?

(Teachers offer different amounts.)

I: I probably shouldn't start with 30 because he may know how to enumerate but the large amount can make it difficult. How about 10 items? Why isn't a good idea to start with 10?

T1: It's a large number.

T2: Because we have 10 fingers.

T3: Automatically, they say 10.

I: Right. How about 8? (The instructor places 8 identical bottle caps on the table.) Many times, a child expects there to be 10 so he won't necessarily take care to point to each one at a time. Instead, he might run his finger quickly over the items saying the numbers from 1 to 10. So, if we place 10 items in front of the child, we may not be able to discern if he understands the one-to-one principle. So, 10 is not a good number for an assessment task. A good assessment task tests one principle or one piece of knowledge at a time.

In the above segment, we see how developing teachers' knowledge of tasks is intertwined with their knowledge of children's conceptions. In designing an assessment task, one needs to take into account possible children's strategies and how the specific task may encourage or discourage specific strategies. Knowing that ten is a benchmark number for children, and knowing that children may automatically count until ten regardless of the number of actual items to be counted, guides the instructor and teachers in choosing a different amount of items. It is also important to consider the types of items to be counted. The instructor points out that if the items are of two colours, the child may count each colour group separately. She recommends starting by having the children count a set of homogenous objects such as bottle caps, easily accessible to the teachers and then afterwards checking what happens with heterogeneous items.

Finally, the teachers need to consider the wording of the question or of the instructions they will give children.

I: What should we ask the child?

T6: (We should ask the child . . .) “How many are there?”

I: So, if the child points to each item and counts 1, 2, 3 till 8 then we know that he has the one-to-one correspondence principle. But do we know if he understands the principle of cardinality? So, after he counts, we will ask again, “How many items are there?”

T7: The first time we ask him how many there are, we are essentially requesting him to enumerate the items. It’s instead of saying, enumerate the items. The second time we ask, “How many are there?” we are assessing the cardinality principle.

I: Correct. Some children will begin to enumerateagain from the beginning. And if we ask them again how many items there are, they will probably start enumerating again from the beginning. But, some children do not have the one-to-one correspondence principle; they say the counting sequence correctly while running their fingers over the items (the instructor demonstrates this action) but if they end at 7 or 9, when asked again how many there are, they will say whatever number they end up with. They understand the principle of cardinality. So, we can use this task to discern both the one-to-one correspondence principle and the cardinality principle.

In the above segment, we see again how promoting teachers’ knowledge of tasks is intertwined with promoting their knowledge of children’s conceptions. How might children react to the question, “How many items are here?” Will they all react the same way? What does this tell us regarding the child’s conception of number and counting? Finally, we point out that in the specific program described above, teachers chose several assessment tasks to implement with individual children in their kindergarten classes, video-taped the task implementations, and then brought the video tape back to the course to be viewed and discussed with the other teachers and with the instructor. Many of the teachers chose the same tasks, allowing for a fruitful discussion focusing on the students’ strategies for solving the tasks and what can be learned about students’ conceptions from implementing such tasks.

To summarize, it may be said that the CAMTE framework served to guide us as we developed different aspects of teachers’ knowledge for teaching preschool mathematics. In general we focused on knowledge aspects represented in Cells 1 and 2 prior to focusing on Cells 3 and 4 but of course, as we developed teachers’ knowledge of children’s conceptions and their knowledge of tasks, we referred back to previously discussed issues. Thus, it may be said that the learning was cyclic and in fact the knowledge cells are all related. Teachers needed to learn the enumeration principles (Gelman & Gallistel, 1978) before they could appreciate how different tasks and different implementations of tasks can assess children’s knowledge of each of the principles. For example, without knowledge of the cardinality principle, teachers in the episode cited above may not have accepted the need to ask the child a second time: How many are there? Likewise, in order to appreciate students’ difficulties, for example, with employing one-to-one correspondence, teachers first needed to experience for themselves how this principle is connected to enumeration. In order to build tasks that can promote the different skills necessary for enumerating and the different strategies which may be used to enumerate items in a set, teachers needed to evaluate different strategies. When teachers brought back their video tapes for collective reviewing, the opportunity was taken to evaluate the children's strategies evident in the tape, refining knowledge aspects relevant to Cells 1, 2, and 3. These discussions ignited further discussion of how the implemented tasks may be revised, bringing us to Cell 4. In the next section, we show how the issue of self-efficacy was woven throughout the program.

Taking into consideration self-efficacy: Cells 5, 6, 7, and 8

In planning our program, care was taken at each step to consider self-efficacy. When addressing the issue of self-efficacy we considered both teachers' mathematics self-efficacy (Cells 5 and 6), as well as their pedagogical-mathematics self-efficacy (Cells 7 and 8). Self-efficacy beliefs are not only domain specific (e.g. mathematics, history, science) and content specific (e.g. within mathematics there is arithmetic, geometry, etc.), but may also be task specific (e.g. what is the child asked to do) and situation specific (e.g. is the task implemented in class, outside, individually, in a group) (Pajares, 1996; Zimmerman, 2000). Taking this into consideration, we began each new topic with a series of self-efficacy questions related to specific tasks where teachers were asked to denote on a scale of 1-4 their ability to perform certain tasks. For example, within the domain of geometry, questions associated with Cell 5 related to ability in defining a triangle and identifying a given figure as a triangle or a non-triangle. Questions associated with Cell 6 related to ability in evaluating a definition of a triangle and evaluating an explanation for why a given figure is or is not a triangle. Within the context of number concepts, questions associated with Cell 7 related to identifying number symbols which children find difficult to learn, pointing to arrangements of items which children find difficult to count, pointing to numbers which children find difficult to say which number comes next. Questions associated with Cell 8 related to ability in designing tasks which can assess children's knowledge of counting till 30, designing tasks which can promote children's knowledge of the number symbols from 0 to 9, promoting children's knowledge of the number combinations for seven.

What does it mean to work with teachers taking into consideration their self-efficacy? To begin with, we acknowledge that our goal is not only for teachers to have a high self-efficacy, but that this self-efficacy should correspond to actual performance. If teachers have a high self-efficacy in performing some teaching task, but cannot perform the teaching task in reality, we have missed our goal. If teachers have a low self-efficacy, despite their being very capable, then we have again missed our goal.

How do we achieve our goal? A crucial step towards success is having teachers recognize when they do not know something or do not have the ability to perform some task. Without this step, teachers may not feel the need to actively participate in the program. In a previous study (Tirosh & Tsamir, 2009) we reported on a preschool teacher who began our program with a high mathematics and a high pedagogical-mathematics self-efficacy related to teaching triangles. We showed that as the intervention progressed, the teacher came to realize how much she did not know, which in turn caused her self-efficacy to fall. By the end of the program, however, her self-efficacy as well as her SMK and PCK for teaching triangles, rose. That study indicated that in some cases a temporary, initial decrease in a teacher's self-efficacy is instrumental for achieving our ultimate goal.

Another important aspect of our program related to the issue of self-efficacy is making the goal of a lesson explicit to teachers. In order for teachers to correctly assess their ability to perform a task, we feel it is imperative that during instruction, they are aware of the new abilities they are forming. Thus, for example, at the start of a lesson related to counting tasks, the instructor announced:

> Today we will build assessment tasks. The idea is to build a task that can test and analyse the child's way of thinking and to discuss together possible children's answers in order to know how to continue working with the child ... In order to focus on children's ways of thinking, each one of you will implement the same task in their class and bring the results here so we can discuss together the results. We are talking about assessment tasks (the instructor writes this on the board). We do not always have the time to do this in class but it is very important so that we can know where the individual child stands.

During the lesson, the instructor reminded the teachers of the curriculum guidelines and what the child should know by the end of kindergarten. She also discussed with the teachers the difference between having students enumerate items which are exactly the same and count items which are different. For example, when discussing different number combinations that make up the number five, the instructor suggests a task that involves using five items which are exactly the same. One of the teachers suggested that it would be better if the items were not exactly the same in order to help children see the different ways of building five. At that point the instructor reminds her, "Assessment tasks are different from teaching tasks. We do not want to intervene. We want to see what strategies the children will employ." The instructor also discussed the difference between counting tasks which focus solely on the number sequence and enumerating tasks which focus on counting objects:

> Enumeration is not the same as knowing the counting sequence ... Our goal is to be able to sit with a child and find out if he or she is able to enumerate objects in a set, in what circumstances he is able to and in what circumstances he is unable to. We want to be able to assess what piece of the puzzle is missing and what might be holding the child back from being able to enumerate the objects.

By making the purpose of a lesson and what the students will be able to do at the end of a lesson explicit to the teachers, we aimed to raise the teachers' awareness of the knowledge and skills they were building.

Discussing the issue of self-efficacy up front was also part of the program. The difference between being able to perceive their students' abilities and being able to perceive their own abilities was discussed with the teachers. As the instructor said:

> We discussed together how we see ourselves, how the children see themselves, and how we see the children. It is not the same thing ... Self-perception, in general, is how we think of ourselves. Self-efficacy is how we perceive our ability to fulfill tasks. I judge my own ability. Of course, it is even more important to make connections between how I see myself and how I really am.

The instructors went on to discuss the children, and then the teachers themselves. Teachers were also reminded of how, in the beginning of the program, they themselves admitted that they did not have the necessary knowledge to teach geometry and that towards the middle of the year they began to feel more attuned to their knowledge, discussing the differences in teaching children about triangles and teaching them about quadrilaterals. In other words, teachers were aware of their abilities to teach specific topics and not others. So, too, with children; it is important to give children positive feedback and have them build a positive sense of self-efficacy. However, we should also provide opportunities for them to succeed so that their positive self-efficacy will be real and based on their own experiences. According to Bandura (1986) performance attainments are an important source of self-efficacy; successes raise self-efficacy while repeated failures lower them. Thus, when children are given opportunities for success they are apt to believe more positively in their abilities than those without these experiences. On the other hand, self-efficacy beliefs also have an impact on performance. Thus, we may say that self-efficacy beliefs and performance have a reciprocal relationship. Finally, appropriate feedback may be especially important for young children who have little experience of their own to reference. To summarize, by discussing children's self-efficacy as well as their own self-efficacy, teachers became aware of the necessity to have and promote a positive self-efficacy that is correlated with actual task performance.

An inherent part of our program is formative as well as summative assessment. In building this part of the program, we also used the CAMTE framework as a guideline. We are currently using the framework to investigate preschool teachers' knowledge and self-efficacy in several areas. In the following section we report on initial results of one of our programs, focusing on knowledge and self-efficacy related to counting and enumeration tasks.

INVESTIGATING TEACHERS' KNOWLEDGE AND SELF-EFFICACY: CELLS 4 AND 8

In this section we present results of investigating 17 preschool teachers' knowledge of assessment tasks (Cell 4) related to counting and enumeration and their corresponding self-efficacy beliefs (Cell 8). Their experience ranged between one and 25 years. These teachers voluntarily participated in one of our programs which included 10 three-hour lessons spread over a period of eight months. Approximately a third of the lessons were centred around number concepts, including counting and enumeration. All teachers were teaching children ages 4–6 years old in municipal preschools, were licensed to teach preschool, and had a bachelor degree. Before beginning these lessons, teachers were asked to fill out questionnaires, investigating their knowledge and self-efficacy beliefs. At the end of the course, teachers filled out the same questionnaires.

Two self-efficacy questions related to counting and enumeration assessment tasks appeared on the questionnaire. Teachers were asked to rate, on a scale of 1-4 their agreement with the following statements:

1. I am able to build tasks which can assess children's knowledge of counting till thirty.
2. I am able to build tasks which can assess children's knowledge of enumerating 8 objects.

Following these questions, teachers were asked:

3. Which tasks would you give children to assess their knowledge of counting till 30?
4. Which tasks would you give children to assess their knowledge of enumerating eight objects?

Ample room and time was given for the teacher to write many tasks. Questionnaires were filled out with the instructor present. Teachers' knowledge was coded along two issues. The first was related to teachers' knowledge of the difference between tasks that assess counting and tasks that assess enumerating. In other words, were the tasks offered in questions 3 and 4 appropriate for each question? The second issue was related to the variation of tasks. That is, did the teacher present a variety of different tasks, or were they essentially the same? The notion of variety is described in more detail in the results section.

We begin by offering some general results for the group of preschool teachers who participated in this study, first their self-efficacy beliefs and then the types of tasks they presented on the questionnaires. We then focus on two individuals and describe their results in more detail.

The Wilcoxin Signed-Ranks test was used to compare self-efficacy scores before and after participating in the program. Results indicated that teachers' self-efficacy for building tasks with which to assess children's knowledge of the counting sequence, were significantly higher after participating in the program ($z = -2.3111$, $p = 0.021$). Likewise, teachers' self-efficacy for building tasks with which to assess children's knowledge of enumeration after participating in the course was significantly higher than before participating in the course ($z = 2.486$, $p = 0.013$). In other words, teachers' self-efficacy with regard to building counting and enumeration assessment tasks significantly increased.

When we considered teachers' knowledge of counting and enumeration tasks, we first analysed the presented tasks to see if teachers differentiated between a task that could assess a child's knowledge of the counting sequence and a task that could assess children's enumeration skills. As was discussed previously, enumerating objects includes several skills beyond knowing the counting sequence. As Gelman and Gallistel (1986) noted, the child must employ one-to-one correspondence and know that the last number reached signifies the amount of objects in the set (the cardinality principle). A teacher who would have a child enumerate several objects in a set in order to assess their knowledge of the counting sequence, may not take into account the complexity of enumerating objects in a set and is in fact, assessing the child's enumeration skills. Before the program, 14 teachers presented enumeration tasks for tasks that could assess a child's knowledge of the counting sequence. In fact, two teachers specifically wrote that the same tasks used for assessing children's knowledge of the counting sequence could be used to assess children's knowledge of enumeration. After the program, five teachers offered enumeration tasks, such as counting the number of children who came to class, when asked for tasks that could assess a child's knowledge of the counting sequence.

In addition to differentiating between counting and enumeration tasks, we also analysed the richness and variety of tasks teachers presented. For example, knowledge of the counting sequence does not only include being able to count forward from 1 to 30 or 50. It includes being able to count forward from a number other than one, being able to count backwards, being able to count by 2s, and more. Thus, if we want to assess a child's knowledge of the counting sequence, we need to ask more of the child than just counting forward from one. Before the program, all but one of the teachers related solely to tasks which had the children counting from one forward. None of the teachers considered asking the children to count backwards and none of the teachers considered asking the child what number comes before or after some other number. After completing the program, eight of the teachers presented a rich variety of tasks which took into consideration more than the child being able to count forward from one.

Regarding enumeration tasks, both before and after the program teachers presented tasks which included enumerating different types of objects. For example, one teacher said that she would set a table with eight settings and would have the children count the number of plates on the table, the number of forks, and the number of spoons. Another teacher said that she would place eight blocks on a table and have the children count them and then she would place eight crayons on the table, and so forth. Before beginning the program, only two teachers referred to different arrangements of objects. That is, will the eight objects to be counted be placed in a pile without order, or a line, or a circle? After the program, seven teachers presented tasks which included specific mention of the arrangement of the items to be counted and how the arrangements should be varied. In addition, after the program, four teachers included tasks which assessed the child's ability to count out eight objects from a set containing a greater amount of items.

Increases in self-efficacy are most noteworthy when they are accompanied by increases in knowledge. Such was the case with Gloria, a novice teacher with only one year of experience. On the pre-test self-efficacy questions, Gloria partly agreed (a 2 on the scale from 1–4) with each statement. When asked to present tasks that could be used to assess children's knowledge of the counting sequence till 30 she wrote: "counting the children who came to class, counting chairs, counting things." When asked to present tasks that could be used to assess children's knowledge of enumeration she wrote: "everything that I said above, worksheets – to colour and circle (amounts of items)." It is clear that in the beginning, Gloria does not differentiate between enumeration and knowing the counting sequence. Nor does she offer much of a variety of tasks. From her low self-efficacy evaluation, it seems that she was aware of her insufficient knowledge in this area. On the post-test, she offered the following tasks for evaluating knowledge of the number sequence: "for example, to count till 30, to count from some number forward (not necessarily from one), to count from some number backwards, like from seven." For assessing enumeration skills she offered the following: "to enumerate objects in a row, in a circle, to enumerate objects spread around in no order, to enumerate a pile of objects, to change the amount by adding an object." In accordance with her increased knowledge, her self-efficacy increased. On the post-test questionnaire she fully agreed with both self-efficacy statements.

Not all teachers were aware of their lack of knowledge in the beginning. Such was the case of Melanie, a teacher with 24 years of experience. On the pre-test she fully agreed with both self-efficacy statements yet when asked to present counting tasks, she merely wrote, "jumping games" and for enumeration tasks she wrote "games with balls." It is possible that by "jumping games" she was referring to the rhythmic counting of numbers which goes along with jumping. It is unclear what she meant by "games with balls." Interestingly, on the post-test, her self-efficacy decreased by a degree while her knowledge improved slightly. While her suggested counting tasks remained within the realm of counting according to a beat, her enumeration tasks now included counting objects and then changing their arrangement and having the children count them again. Although we would have liked to see more richness in her tasks, we feel it is noteworthy that Melanie's self-efficacy is now more closely related to her actual knowledge. As noted previously, this may be a necessary initial step towards learning. We would like to hope that increased awareness of one's actual ability could lead to continued attendance in professional development. We also note that approximately one month after filling in the second questionnaire, Melanie handed in her final project. As part of her project she chose to assess a child's knowledge of the counting sequence. She wrote, "I wanted to check if he knows how to count without objects. Does he know and recognize the accepted sequence, forward and backward, and what is called counting from the middle." Here we see that Melanie has indeed improved her knowledge of assessment tasks for the counting sequence and is able to point to a variety of appropriate tasks.

CONCLUSION

Schoenfeld and Kilpatrick (2008) provided a provisional framework for proficiency in teaching mathematics which included: "knowing school mathematics in depth and breadth, knowing students as thinkers, knowing students as learners, crafting and managing learning environments, developing classroom norms . . . , building relationships that support learning, reflecting on one's practice" (p. 322). Through our program, guided by the CAMTE framework, we attempted to enhance many of these elements among preschool teachers. For example, working on set comparison allowed teachers to see how the concept of equivalence learned at an older age can grow out of simple number concepts such as counting and one-to-one correspondence. It also allowed the teachers to gain an appreciation for solving problems in multiple ways. Knowing students as thinkers and as learners and crafting learning environments were also touched upon as teachers planned tasks to implement in their class, discussed possible student reactions, and tried out different tasks with their young students.

An important factor affecting teachers' decision making are the resources available at the moment. "An individual's resources include his or her knowledge, but also include the social and material resources that are available to him or her" (Schoenfeld, 2011, p. 459). One of our goals is providing preschool teachers with the necessary resources that will facilitate their teaching of mathematics. Looking back, promoting knowledge was, of course, a large part of our program. However, we also helped teachers to build material resources such as tasks. In addition, by allowing time for practicing task implementation and role-playing, and by having teachers reflect together on instances that occurred in their classes, we hoped to build social resources that they could draw on in the future.

"Reflection is the ultimate key to one's professional growth as a teacher" (Schoenfeld & Kilpatrick, 2008, p. 348). Likewise, Pehkonen and Törner (1999) found that one of the conditions for affecting teacher change is "to organize such situations in which teachers could reflect on their thinking and actions" (p. 261). While this is true for all teachers, it may be especially true for preschool teachers. In Israel, children ages 4–6 learn in preschools which are often physically set apart from elementary schools as well as other preschools. Thus, these teachers have few opportunities to exchange ideas and are often on their own when it comes to making decisions and reflecting on situations. An important element of our program is providing time and structure for teachers to reflect on their mathematics teaching. The importance of this aspect was expressed by one teacher who reported about the experience she had recording herself while she engaged a child with enumeration tasks:

> It was interesting to watch myself. During class time I never see myself. It (the video) is a good tool. You can stop [the video-tape], think, watch it again, and then reflect. It really helped me to learn about myself and about the children . . . I saw that I was more confident in myself, more skillful with regard to conducting the assessment task. I see how I improved each time.

For this teacher, reflecting on her practice also served to boost her belief in her ability, illustrating the premise of the CAMTE framework that cognitive and affective issues are intertwined when it comes to facilitating proficient mathematics teaching.

In this chapter we introduced the CAMTE framework, and investigated its use in planning, implementing, and assessing professional development. Our research showed the viability of planning professional development that takes into consideration both the promotion of knowledge and self-efficacy. As mentioned earlier, our framework builds on other works, such as Ball and her colleagues but takes a different look at knowledge, focusing on knowledge for producing solutions and knowledge for evaluating solutions, two types of knowledge which are separate but related. Our framework also takes into account the interaction between promoting knowledge along with self-efficacy beliefs. Initial results of our study with preschool teachers found that they do not always differentiate between counting and enumerating tasks and they are not always aware of the variety of counting and enumerating tasks which may be implemented with young children. Our study provides evidence that professional development can make an impact on these aspects of preschool teachers' knowledge for teaching counting and enumerating. The CAMTE framework is a tool that could be used during formative as well as during summative assessment. As such, it is a useful tool for teacher educators. However, the framework could also be used by the teachers themselves. During professional development they could use it in order to track their progress. Afterwards, as they worked with children, they could use it to identify aspects of their own knowledge that might need improvement, where they might feel that they lack the necessary knowledge to achieve a certain teaching goal. They may also use the framework to prepare lessons as well as to reflect on their work with children. That being said, we recognize that the framework takes into consideration some variables and leaves out others. Both Schoenfeld (2011) and Törner (2002) recognized that teachers' beliefs about pedagogy, about mathematics in general, and about mathematical topics specifically may affect their teaching. Teacher educators may also want to consider these aspects of teachers' beliefs when aiming to facilitate proficient mathematics teaching in preschool.

NOTES

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REFERENCES

Allinder, R. M. (1994). The relationship between efficacy and the instructional practices of special education teachers and consultants. *Teacher Education and Special Education*, *17*, 86–95.

Arcavi, A. (2004). Education of mathematics teachers. In *Educating for the future. Proceedings of an International Symposium on Mathematics Teacher Education* (pp. 227–238). Malmö, Sweden: The Royal Swedish Academy of Sciences.

Ashton, P. T., & Webb, R. B. (1986). *Making a difference: Teachers' sense of efficacy and student achievement*. New York: Longman.

Australian Association of Mathematics Teachers and Early Childhood Australia (AAMT/ECA). (2006). *Position paper on early childhood mathematics*. Retrieved January 20, 2009, from http://www.aamt.edu.au/documentation/statements.

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching. *Journal of Teacher Education*, *59*(5), 389–407.

Bandura, A. (1986). *Social foundations of thought and action: A social cognitive*. Englewood Cliffs, NJ: Prentice Hall.

Baroody, A. J. (1987). *Children's mathematical thinking: A developmental framework for preschool, primary, and special education teachers*. New York: Teacher's College Press.

Brady, S., Gillis, M., Smith, T., Lavalette, M., Liss-Bronstein, L., Lowe, E., North, W., Russo, E., & Wilder, T. D. (2009). First grade teachers' knowledge of phonological awareness and code concepts: Examining gains from an intensive form of professional development and corresponding teacher attitudes. *Reading and Writing*, *22*(4), 425–455.

Coladarci, T. (1992). Teachers' sense of efficacy and commitment to teaching. *Journal of Experimental Education*, *60*, 323–337.

Farran, D. C., Lipsey, M., Watson, B., & Hurley, S. (2007). *Balance of content emphasis and child content engagement in Early Reading First program*. Paper presented at the American Educational Research Association Annual Meeting, Chicago, IL.

Gelman, R., & Gallistel, C. (1978). *The child's understanding of number*. Cambridge: Harvard University Press.

Ginsberg, H. P. (2006). Mathematical Play and Playful Mathematics: A guide for early education. In D. G. Singer, R. Michnick Golinkoff, & K. Hirsh-Paslek (Eds.), *Play = Learning. How play motivates and enhances children's social-emotional growth* (pp. 145–168). New York: Oxford University Press.

Ginsburg, H. P., Lee, J. S., & Boyd, J. S. (2008). Mathematics education for young children: What it is and how to promote it. *Social Policy Report*, *XXII*(I), 1–22.

Hackett, G., & Betz, N. (1989). An exploration of the mathematics self-efficacy/mathematics performance correspondence. *Journal for Research in Mathematics Education*, *20*(3), 261–273.

Israel National Mathematics Preschool Curriculum (INMPC). (2008). Retrieved April 7, 2009, from http://meyda.education.gov.il/files/Tochniyot_Limudim/KdamYesodi/Math1.pdf.

Kaiser, G. (2002). Educational philosophies and their influence on mathematics education? An ethnographic study in English and German mathematics classrooms. *ZDM – Zentralblatt füĄr Didaktik der Mathematik/International Reviews on Mathematical Education*, *34*(6), 241–257.

National Association for the Education of Young Children & National Council of Teachers of Mathematics (NAEYC & NCTM). (2002). *Position statement. Early childhood mathematics: Promoting good beginnings*. Available: www.naeyc.org/resources/position_statements/psmath.htm.

Pajares, F. (1996). Self efficacy believes in academic settings. *Review of Educational Research*, *66*(4), 543–578.

Pehkonen, E. & Törner, G. (1999). Teachers' professional development: What are the key change factors for mathematics teachers? *European Journal of Teacher Education*, *22*(2–3), 259–275.

Piaget, J., & Inhelder, B. (1958). *The growth of logical thinking from childhood to adolescence*. New York: Basic Books.

Pietsch, J., Walker, R., & Chapman, E. (2003). The relationship among self-concept, self-efficacy, and performance in mathematics during secondary school. *Journal of Educational Psychology*, *95*(3), 589–603.

Practice Guidance for the Early Years Foundation Stage. (2008). Retrieved April 9, 2009 from www.standards.dfes.gov.uk/eyfs/resources/downloads/practice-guidance.pdf.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.

Schoenfeld, A. H. (1999). Looking toward the 21st century: Challenges of educational theory and practice. *Educational Researcher*, *28*(7), 4–14.

Schoenfeld, A. H. (2011). Toward professional development for teachers grounded in a theory of decision making. *ZDM – Mathematics Education*, *43*, 457–469.

Schoenfeld, A. H., & Kilpatrick, J. (2008). Toward a theory of proficiency in teaching mathematics. In D. Tirosh & T. Wood (Eds.), *The international handbook of mathematics teacher education: Tools and processes in mathematics teacher education* (Vol. 2, pp. 321–354). Rotterdam: Sense Publishers.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.

Starkey, P., Klein, A., Chang, I., Qi, D., Lijuan, P., & Yang, Z. (1999). *Environmental supports for young children's mathematical development in China and the United States*. Albuquerque, NM: Society for Research in Child Development.

Swars, S. L., Daane, C. J., & Giesen, J. (2006). Mathematics anxiety and mathematics teacher efficacy: What is the relationship in elementary preservice teachers? *School Science and Mathematics*, *106*(7), 306–313.

Tabach, M., Levenson, E., Barkai, R., Tirosh, D. Tsamir, P., & Dreyfus, T. (2010). Secondary school teachers' awareness of numerical examples as proof. *Research in Mathematics Education*, *12*(2), 117–131.

Tirosh, D., & Tsamir, P., Levenson, E., & Tabach, M. (2011). From preschool teachers' professional development to children's knowledge: Comparing sets. *Journal of Mathematics Teacher Education*, *14*, 113–131.

Törner, G. (2002). Mathematical beliefs. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 73–94). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Tsamir, P., & Tirosh. (2009). Affect, subject matter knowledge and pedagogical content knowledge: The case of a kindergarten teacher. In J. Maaß & W. Schläglman (Eds.), *Beliefs and attitudes in mathematics education: New research results* (pp. 19–32) Rotterdam, the Netherlands: Sense Publishers.

Tsamir, P., Tirosh, D., & Levenson, E. (2008). Intuitive nonexamples: The case of triangles. *Educational Studies in Mathematics*, *69*(2), 81–95.

Tschannen-Moran, M. & Johnson, D. (2011). Exploring literacy teachers' self-efficacy beliefs: Potential sources at play. *Teaching and Teacher Education*, *27*(4), 751–761.

Wheatley, K. (2002). The potential benefits of teacher efficacy doubts for educational reform. *Teaching and Teacher Education*, *18*(1), 5–22.

Zimmerman, B. J. (2000). Attainment of self-regulation: A social cognitive perspective. In M. Boekaerts, P. R. Pintrich, & M. Zeidner (Eds.), *Handbook of self-regulation* (pp. 13–39). San Diego, CA: Academic Press.

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8. INTEGRATING NOTICING INTO THE MODELING EQUATION

INTRODUCTION

Understanding teacher cognition in the moments of instruction has become increasingly important to the mathematics education community over the past two decades. Various reforms and policies make it clear that to support student learning, instruction must be based, at least in part, on the ideas that students raise in class, the reasoning that ensues, and the representations that are used (CCSSI, 2010; NCTM, 2000; NRC, 2001). Teaching, as a result, must be responsive; teachers must adapt their lessons as they unfold, often making decisions about how to proceed in the midst of instruction. But how does such in-the-moment decision- making occur? What are the cognitive processes involved as teachers carry out instruction?

One productive program of research that examines these questions is the teacher-modeling work conducted by Alan Schoenfeld (e.g., 2010). Schoenfeld asks "What is the teacher trying to achieve at the moment" (2010, p. 9), and "How did that become the teacher's goal for that moment?" Schoenfeld's research emphasizes that teachers' practices are comprised of established routines that are based in a teacher's goals, resources, and orientations. In addition, Schoenfeld explores how goals are activated and prioritized in situations that deviate from a teacher's expectations for a given lesson.

Our own work takes a different approach to studying teachers' in-the-moment actions. Specifically, we investigate the nature of teacher noticing. We ask "To what do teachers typically attend during instruction?" and "How do teachers decide where to pay attention during instruction?" (i.e., Sherin, Russ, Sherin, & Colestock, 2008). We focus on the dynamic relationship between a teacher's efforts to identify significant moments of instruction and the teacher's interpretation of those moments.

Though different, both programs contribute valuable information to the study of teachers' in-the-moment cognition. The goal of this chapter is to examine the relationship between these research programs. In particular, we consider what might be gained from integrating an explicit focus on teacher noticing into Schoenfeld's "modeling equation," that is, his procedure for unpacking the moment-to-moment actions of teachers. Similarly, we ask how Schoenfeld's advances in teacher modeling can enhance our own study of teacher noticing and our understanding of how teachers' attention guides their decision making.

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In what follows, we begin by reviewing Schoenfeld's approach to teacher modeling. We draw attention to the key components of the model and demonstrate his analytic methods with an episode from an eighth-grade classroom. Next we describe our research on teacher noticing and present two examples from our data set. We then apply Schoenfeld's constructs of goals, resources and orientations to these examples. In doing so, we illustrate how noticing serves as both a catalyst for and a product of teacher decision-making. To conclude, we reflect on how integrating noticing into models of teaching has altered our understanding of how noticing – and thus teaching – works.

MODELING THE TEACHING PROCESS

In the mid 1990s Schoenfeld turned his attention from modeling mathematical problem solving and tutoring to modeling the domain of teaching (e.g., Schoenfeld, 1998). In particular, he took up the task of making sense of teaching by modeling the moment-by-moment decision making of teachers. One of the central ideas of Schoenfeld's work is that if we understand a teacher's goals, resources, and orientations, then we can construct a coherent explanation of a teacher's actions during instruction. Doing so involves identifying established routines that a teacher relies on, routines that are based in those goals, resources, and orientations. Furthermore, when unforeseen events occur, Schoenfeld maintains that an analytic focus on goals, resources, and orientations allows us to make sense of a teacher's responses by considering how that teacher reprioritizes his or her goals based on existing resources and orientations. But what exactly does Schoenfeld mean by *goals*, *resources*, and *orientations*?

Key components of Schoenfeld's model

Goals

Broadly speaking, goals are the conscious or unconscious objectives that a teacher hopes to attain. According to Schoenfeld, teachers hold multiple goals at multiple grain sizes. At any given time, for instance, a teacher might hold an overarching goal, a content and/or social goal, as well as several local sub-goals. Furthermore, different goals may become activated (and deactivated) at different points throughout a lesson. For example, a teacher's overarching goal may remain in play throughout a lesson, while the local goals shift as the teacher moves the class through particular segments of the lesson. Goal prioritization is based on whatever the teacher considers to be most important at a given moment. In Schoenfeld's model, teachers make decisions that are consistent with their goals, and teachers draw on their resources to achieve them.

Resources

In using the term resources, Schoenfeld refers primarily to the cognitive resources, or knowledge, that an individual brings to a situation. Schoenfeld emphasizes that there are a range of types of knowledge that individuals possess. These include procedural and conceptual knowledge, as well as knowledge of isolated facts and problem-solving strategies, all of which have the potential to influence the decision-making process. Furthermore, Schoenfeld considers knowledge to be associative. We come to recognize familiar situations and draw on established "knowledge packages" (2010. p. 27) to respond to such events. In addition, new connections among knowledge elements are established as we engage in new experiences. Along with cognitive resources, Schoenfeld notes that teachers may also draw on material and social resources during instruction. It is this collection of resources to which Schoenfeld refers, and which teachers draw on to achieve their goals.

Orientations

Schoenfeld uses the term orientations to incorporate the notions of disposition, belief, and value. He explains that, in particular, a teacher's attitudes towards teaching and learning shape how the teacher interacts and responds to students. Thus, for example, the beliefs a teacher holds concerning what it means to learn mathematics, how a classroom should be organized, and who (or what) should hold the place of authority in the classroom can play a key role in how resources are applied and which goals are activated at the moment. Moreover, a description of a teacher's orientation should specify the conditions under which a particular orientation is likely to be activated.

In his recent book, *How we think* (Schoenfeld, 2010), Schoenfeld details the modeling process through three examples of teaching. In doing so, he demonstrates that, through the lens of goals, resources and orientations, what at first glance might seem like random behavior on the part of the teacher, is instead behaviour that is quite coherent. The implications of this work are particularly noteworthy because he effectively models not just a single type of teaching, but a variety of types of teaching practices. Furthermore, based on his model, Schoenfeld makes suggestions concerning effective levers for productively influencing teacher practice.

A mini-example of Schoenfeld's modeling process: Crowd Estimation problem

Modeling a lesson involves first partitioning the lesson into segments that correspond to the main activities that took place in class. Next, each of these segments is decomposed into sub-segments that reflect a finer-grained parsing of the lesson activities. This iterative process continues until the entire lesson is decomposed into small segments of activity called "action sequences." As a result of this process of decomposition, a skeletal form of the model for a given lesson is produced. Each segment is delineated by an initial triggering event and a final terminating event. Furthermore, for each segment corresponding goals, orientations, and resources are identified to justify the teacher's actions in that segment. This process often leads to the discovery of patterns of goal activation and corresponding actions on the part of the teacher.

To illustrate this modeling process, we consider a 12-minute whole-class discussion from an eighth grade mathematics lesson taught by David Louis. While

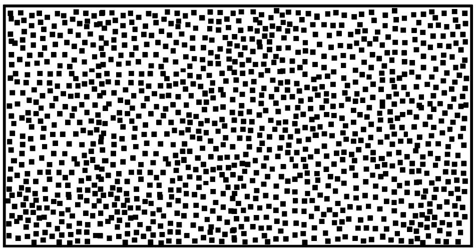

Figure 1. Estimate the population of the crowd shown in the picture.

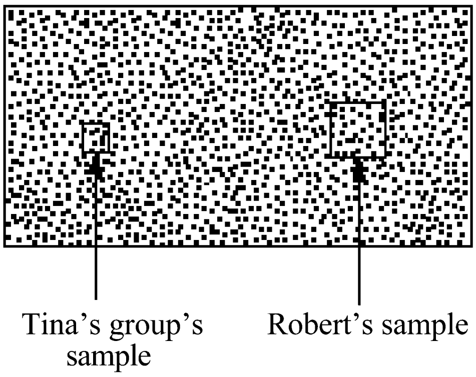

Figure 2. Two proposed solutions to the Crowd Estimation problem.

Schoenfeld typically models entire lessons, we use this mini-example as a way to give the reader a taste of the kind of modelling that Schoenfeld undertakes. (Our description of the following lesson draws on Russ, Sherin, and Sherin, 2011.)

The lesson comes from a unit on comparing and scaling (Lappan et al., 1997). Students were given a picture of a rectangle densely filled with dots (Figure 1) and told to imagine that the picture was an aerial photograph of a crowd with each dot representing a person. Working in small groups, students estimated how many people are in the photo. Tina's group shared their solution, explaining that they divided the original rectangle into 126 small squares, counted 17 dots in one of the small squares and estimated the total population by multiplying 17 by 126.

Mr. Louis then asked the class, "What do people think about this group's method?" Several students responded, including Robert who suggested using bigger squares to establish a more accurate estimate of the population. Robert explained that "with smaller squares there may be a bunch of dots packed into a small area. In just that particular area or something. Or there might have been not a lot of dots" (see Figure 2).

Mr. Louis turned to the class for comments: "What do you think about what Robert just said?" Some students voiced their agreement with Robert but Jeff suggested they find the average number of dots in 10 small squares. "It would have been better if instead of … one small square … they took ten squares from all random spots that were small size and divided the total of all the groups by 10."

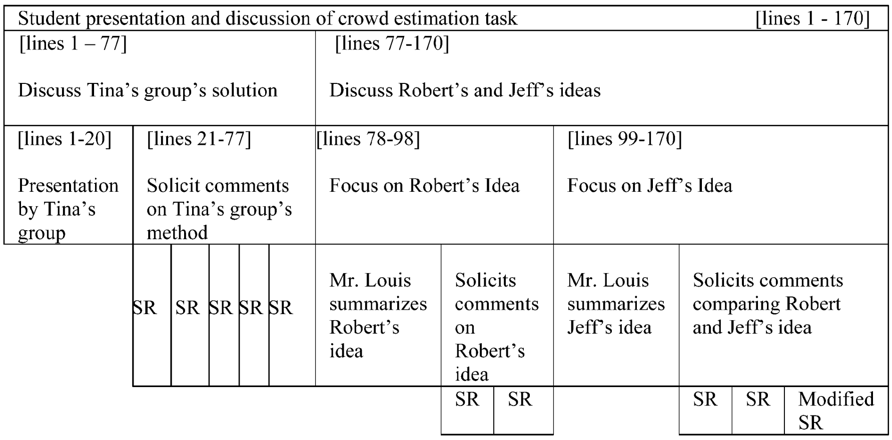

Figure 3. The iterative partitioning of a lesson segment.

After a few minutes, Mr. Louis drew the class' attention specifically to Robert's and Jeff's ideas. "We have two competing ideas here." He drew a diagram to illustrate the different approaches and encouraged the students to compare and contrast the two methods. "Which way do you think would produce the most accurate estimate of the population?"

As the class discussed Robert's and Jeff's methods, students raised a number of issues, including the role of averaging ("[For] a better estimate you have to have an average."), the context in which the sample was drawn ("Robert's methods would be better if ... the big squares had the same number of dots each time") and the relationship between the samples ("Is Jeff's method just ... making the square ten times larger?").

Parsing the lesson segment

This portion of the lesson can be partitioned into two main episodes: one in which the class explores Tina's group's solution and a second in which the class discusses two additional strategies, that of Robert and Jeff (Figure 3). The first episode can be further divided into two smaller episodes, Tina's group's initial presentation of their solution strategy, and then a whole-class discussion of the strategy. During this discussion, Mr. Louis uses a particular discourse routine in which he first solicits a student's idea, then asks another student to rephrase the idea, and finally asks for comments on the idea. This discourse routine is used five times during the episode as noted in Figure 3. (SR is used to refer to this "solicitation routine.")

In the second part of the episode, Mr. Louis explicitly focuses the class on strategies offered by Robert and by Jeff. For each strategy, Mr. Louis first summarizes the student's approach, and then asks members of the class to elaborate. In doing so, he again uses his familiar solicitation routine. In the final cycle of the solicitation routine, Mr. Louis modifies the routine somewhat, as he pursues a student's comment about sampling.

Resources, goals, and orientations

Mr. Louis' teaching during these episodes is guided by two overarching goals: to use students' ideas to structure lessons, where possible, and second, that students should comment substantively on each other's ideas. These goals are guided by Mr. Louis' orientation that learning mathematics should be a sense-making activity for students, and that talking about one's thinking and the thinking of others is a key component of an effective learning environment. Mr. Louis has strong pedagogical and subject matter knowledge. He often structures his lessons similarly – with a student presentation and discussion, followed by Mr. Louis choosing select methods for the class to discuss further (see Sherin, 2002 for more information on this approach). While Mr. Louis had not precisely anticipated the methods raised by Robert and Jeff, he was in familiar territory and recognized these two methods as central to the mathematical goals he wanted students to examine. Late in the discussion, when one student asked about a situation in which the two sampling methods might reveal different results, Mr. Louis modified his familiar discourse approach. Rather than asking students to respond the question, he provided an explanation of the issue that had been raised to the entire class. The question that Schoenfeld's modeling process answers is: What drives Mr. Louis' decision-making in this episode of instruction? For example, what goals and orientations does Mr. Louis' have that led to his decision to have students comment on Tina's solution? Or, what resources does Mr. Louis draw on when deciding to compare Robert and Jeff's idea?

In our work we are interested in a different, but related, set of questions. When we look at Mr. Louis' instruction in this episode we wonder not just about what drives the decisions he makes at any moment, but also what led him to interpret those moments of instruction as requiring a decision. Given a particular orientation and set of goals, the field of what a teacher might attend to is still fairly large. Our question then is why and how any particular moment stands out to the teacher.

The classroom is a complex environment, with many things happening simultaneously. A teacher cannot notice everything with equal weight; instead the teacher must choose where and to what to attend in the midst of this complexity. For this episode of Mr. Louis' teaching, we wonder how, amongst all those things that were happening, did Mr. Louis come to understand (perhaps tacitly) Tina's presentation as a "decision point" – a time to decide among various pedagogical moves? What did he "see" in that solution that led him to decide to have students comment extensively on it?

To answer these types of questions we investigate teacher noticing, that is, where and how teachers decide to focus their attention during instruction. A teacher might attend, for example, to the level of noise in the classroom, to students' solutions to a particular problem, or to how students respond to each other's questions. In the episode with Mr. Louis, we saw him attending to the particular solutions of his students. We can go further to say that we saw him noticing how students were making sense of the affordances of the different solution methods. We can imagine another teacher who might have noticed something else – perhaps the clarity of the presentations or even the correctness of the student solutions. Had he noticed something different, Mr. Louis may not have made the pedagogical decisions that he did when he did.

TEACHER NOTICING

For the past 15 years, we have been engaged in a program of research designed to examine the nature of teacher noticing. We argue that teacher noticing is a key component of teaching expertise, particularly in the context of current mathematics education reforms. The idea that noticing is a component of expertise is not a new claim. Experts in diverse domains have been found to be able to recognize meaningful patterns in their areas of expertise. For example, chess experts are better able to identify layouts on a chess board than novice players (Chase & Simon, 1973). Of course, chess layouts consist of static pieces while the classroom represents a much more dynamic situation. Thus, it seems likely that the act of noticing during instruction is more complex than what has been studied previously. In addition, current reforms call for teaching that is responsive and flexible, in which teachers respond to student ideas as they arise during instruction. This approach towards teaching seems to rely strongly on teachers' in-the-moment noticing abilities.

While noticing is used in everyday language to indicate the general observations that a person makes, here we use the phrase *teacher noticing* to refer to the processes through which teachers manage the "blooming, buzzing confusion of sensory data" with which they are faced (Sherin & Star, 2011, p. 69). In particular, we understand noticing to involve two main processes: *attending to particular events in a classroom* and *making sense of those events* (Sherin, Jacobs, & Philipp, 2011). As stated above, teachers must decide what to pay attention to in the classroom, as well as what not to pay attention to. Furthermore, teachers are not just passive observers of those events to which they do attend. Instead, they interpret what they notice and therefore make sense of the situation in light of the ongoing lesson.

Examining the nature of teacher noticing has highlighted the consequential nature of noticing on teaching. A teacher can only respond to what he or she notices. Returning to the mini-example from Mr. Louis' class, a teacher who did not notice that Tina's method would provide a reasonable estimate presumably would not have decided at that moment to open the class to discussion about the affordances of her method. Thus, one aspect of our work considers how shifting a teacher's notice might serve as a catalyst for changing that teacher's instruction. We would therefore like a model of teaching that accounts for our intuitive sense that noticing impacts teachers' in the moment decision-making.

Studying teacher noticing

We have recently taken a novel approach to exploring teacher noticing. With the use of new digital technologies, we have asked *teachers* to identify moments of instruction that stand out to them as interesting, thereby capturing what teachers

Figure 4. Temporal distribution of teacher-captured clips over one class period.

notice in the moment of instruction. Our methodology involves the use of a small wearable camera attached to a hat. The camera features selective archiving, which allows teachers to record 30 seconds of video immediately after the event has taken place. Thus teachers can capture an event to record immediately after it occurs. We have thus far given the camera to a range of high school math and science teachers and asked them over a period of several days to "capture what's interesting." We follow up with an interview of the teachers so that they can describe to us why they chose to capture each of the selected moments.

We have found this approach to be fairly effective. Teachers can successfully use the camera to capture moments and they seem to do so discerningly. We do not, for example, see teachers capture moments only at the beginning or end of lesson, or in regular intervals. This leads us to believe that teachers are tagging in a selective manner, much as we suspect their noticing operates. We think this indicates that they are somewhat aware of their noticing during teaching; this is not a wholly unconscious process. Furthermore, while some teachers seem to be on the lookout for certain kinds of information that they expect to tag, for other teachers, noticing is unplanned; they simply wait to see "what stands out" as a lesson proceeds (Sherin, Russ, & Colestock, 2011). Through this work, we have begun to characterize moments that stand out to teachers as interesting. We find that teachers notice a range of different kinds of issues in the classroom, some that relate to students, others to their teaching, to subject matter, to organization, and to school context.

Two examples of teacher noticing

To demonstrate the types of analyses we undertake in our study of teacher noticing, we elaborate with two examples from a high school algebra class. The teacher, Ray Bryant, was in his fifth year of teaching at an urban public high school in a large Midwestern city. Mr. Bryant used an integrated curriculum, covering topics from algebra, geometry, and statistics in his class. The school day was organized in a block schedule, with class periods of 90 minutes, meeting three times a week. In the class Mr. Bryant selected for this study, students were arranged in six groups of five. He typically organized instruction with students first working in their groups to prepare presentations on the previous nights' homework or in-class problems. Next a student from each group would present the group's solutions to the class. This was followed by a whole-class discussion of the problem, as well as the introduction of concepts and methods by Mr. Bryant.

Our standard analysis starts with looking at what the teacher notices overall during the course of the class. Figure 4 presents a timeline of the noticed moments for the day. As one can see, Mr. Bryant selected moments to capture throughout the lesson. These moments reflected many different topics of interest: of the 10 clips, 5 were related to issues of student thinking, 2 to student characteristics, 2 to climate, and 1 to pedagogy. In addition, we examined the types of classroom participant structures in which the clips were selected; and again a variety was found, with half taking place during whole-class activities and half during group work.

Example 1. Early in the lesson, Mr. Bryant was planning to move rather quickly through an example when a student asked a question about the absolute value problem the class was working on. Mr. Bryant chose to capture this moment with the camera. In his after-class interview about this moment, Mr. Bryant explained "[This] was one of those critical moments where ... I had just planned on brushing right through that ... but ... I made a decision to stop and see where this was going to go because this one student obviously had something he wanted to share with the class." In terms of the focus of the teacher's attention, we would code this instance as taking place during whole-class work and being related to student mathematical thinking.

Example 2. Later in the lesson, students were working in groups while Mr. Bryant circulated around the room. When he approached Clarissa's group, Mr. Bryant's attention was drawn to the fact that the students were talking and doing calculations out loud, but they were not recording their joint work. He captured this moment with the camera and explained in the interview, "I [captured] that because I walked up to that group and they were clearly all discussing what was going on ... [they] were talking about the [problem] ... all five of them ... but nobody's writing anything down." In terms of the focus of the teacher's attention, we would code this instance as taking place during group work and being related to classroom management.

APPLYING SCHOENFELD'S MODEL TO EPISODES OF NOTICING

Our approach to studying teacher noticing, while useful for providing an overall sense of what teachers pay attention to, provides somewhat limited information about what drives teacher noticing in a particular moment. We find that applying Schoenfeld's modeling process can add to our understanding of teachers' in-the-moment noticing. To illustrate, let us once again consider the moments Mr. Bryant captured, but now with the features of Schoenfeld's model in mind. We skip over the partitioning work, however, and treat each 30-second tagged moment as a single episode in his model.

Revisiting example 1

Schoenfeld's modeling approach provides us with tools to answer the question: What drives Mr. Bryant's instructional decision to stop and explore the student's question? When coupled with our attention to noticing, that question becomes: What drives Mr. Bryant's attention to moments in which stopping and exploring student questions is an appropriate instructional decision? An overarching goal of

Mr. Bryant's instruction is that students' ideas will drive the mathematical learning of the class and he structures his classroom with this in mind. Students regularly present their ideas in class and multiple solutions are typically welcomed. In this instance, Mr. Bryant did not anticipate the student's question but he had the resources (pedagogical content knowledge) that allowed him to understand that the students' question was a significant one. Thus, because of his knowledge of mathematics and his goal of student sense making, this is the kind of moment that will stand out to him as significant.

Revisiting example 2

As with example 1, Schoenfeld's work allows us to think about what goals, resources, and orientations Mr. Bryant might have that drive or constrain Mr. Bryant's noticing of students' failure to write down their ideas. Presumably, Mr. Bryant has several goals in mind for his students during class. One overarching key goal is for students to work together to learn mathematics. Mr. Bryant believes that students learn best when they are talking and working with peers, explaining and justifying their ideas to one another. This orientation and goal is evidenced in the way that Mr. Bryant has arranged his classroom and the extended class time he devotes to group work. Further, Mr. Bryant applies his knowledge of mathematics teaching in support of this goal. For example, he generally circulates during group work in order to advance students' thinking through questioning (Smith & Stein, 2011). When he approaches Clarissa's group however, a new goal is activated. He notices that students are engaged productively with the mathematics but he realizes that is not enough – they are not recording their ideas, and given the discussion he wants to have in class tomorrow, students will need a record of the work they have done so far. His overarching goal is therefore still in play, but a new local goal is prioritized by what he notices; having student record their thinking.

These examples highlight the way that Schoenfeld's model adds depth to our understanding of specific moments teachers captured as interesting. In particular, they tell us something about why the moment, or more generally this kind of moment, is likely to stand out to a specific teacher.

INTEGRATING NOTICING INTO SCHOENFELD'S MODEL

Thus far we have illustrated that we can learn more about what teachers notice, and particularly why they notice what they do, by drawing on Schoenfeld's modeling approach. At the same time, it seems reasonable to us that, given our assertion that noticing is a key component of teaching expertise, we should expect a model of teaching to account for teacher noticing. So where is the construct for noticing in Schoenfeld's model? How does it fit in with the existing model components?

We suggest that noticing is an important part of the teacher decision-making process that is currently implicit in Schoenfeld's model. For example, Schoenfeld writes about teachers behaving along the lines of implicit flow charts where if/then statements are asked (e.g., Does a student response require clarification?

Do circumstances require further discussion?). These questions require teachers to notice in order to make decisions about an appropriate response. Similarly, he describes the case of a teacher having to decide whether a student's statement is in line with the teacher's agenda for the lesson. We propose that this kind of reflection necessarily involves the teacher noticing what the student said.

Furthermore, we maintain that the relationship between noticing and the model's existing components are bi-directional. On one hand, teacher noticing can be a product of the teacher's existing goals, orientations, and resources. Thus, the teacher's overall goals for student learning will influence what the teacher notices in the classroom. This is precisely what Schoenfeld's model illustrates if we think about "noticing" as a "decision" that teachers make. Going back to Mr. Bryant, the first example illustrates this relationship. It is because of Mr. Bryant's belief in the importance of students' ideas that the student's question captured his attention.

In addition, noticing can serve as a catalyst for cuing particular goals, orientations, or knowledge. We believe this was the case in the second example from Mr. Bryant. Noticing that his students were not writing down their work activated particular goals and knowledge for the teacher. It was likely in noticing that students were engaging in class in a particular way (not writing down their work), that Mr. Bryant came to realize that it was a goal he had for the students in class in that moment. To be clear, we are not equating noticing with a trigger or a triggering event. Instead, noticing is, to us, an awareness that allows events and ideas to "trigger" in the first place.

Revisiting the mini-example: Crowd Estimation problem

We have now used examples from our data to demonstrate how the construct of noticing can be integrated into Schoenfeld's modeling approach. However, that data was collected using a procedure specifically designed to tease out moments of teacher noticing. As such, it might be said, "Of course the idea of integrating noticing into the modeling approach makes sense for data about teacher noticing. But does it also make sense for the more traditional data of classroom instruction that Schoenfeld typically analyzes?" Obviously, we would like the answer to be yes.

To explore that question, we return to the small episode of Mr. Louis' instruction that we analyzed at the start of the chapter in order to show that supplementing Schoenfeld's modeling analysis with an explicit focus on noticing can help make sense of teacher-decision-making.

For Mr. Louis, his goal of using student ideas as the central mathematical content of the lesson kept him "on the look out" for potentially rich student thinking. Still, with this goal in mind there was quite a bit of student thinking during the discussion that could have been "noticed." To Mr. Louis, Robert and Jeff's ideas appeared as particularly consequential. Thus, while his decision-making is driven by his resources, goals, and orientations, noticing plays a central role as well. In addition, the fact that Mr. Louis noticed the affordances of the various solution methods may have caused a shift in his goals away from facilitating discussion among student ideas to a more teacher-led discussion of the different solutions. We do not suspect that Schoenfeld would disagree with our analysis in terms of noticing (in fact he acknowledges the role of teacher noticing in Schoenfeld, 2011). However, his analysis does not highlight what we consider to be an essential, dynamic part of the teacher decision-making process.

CONCLUSION

Modeling the complex phenomena of teaching and learning has long been a goal of education research. Scholars have attempted to develop models using constructs that give a balance of explanatory power and parsimony as well as intuitive appeal and novel insight. In this chapter, we have described and illustrated one of our field's predominant models of teaching – Schoenfeld's model of teacher decision-making during instruction. While we (and he!) believe this model highlights several important aspects of teaching expertise, we also raise the subject of what and how teachers notice during instruction influences – or interacts with – their decision-making process. In particular, we use several examples to suggest that teacher noticing can be understood as both a catalyst for and product of mathematics teachers' decision-making. In doing so we suggest how noticing might be productively and explicitly integrated into Schoenfeld's model of teaching.

Stepping back from the specifics of our examples, we might ask what this modeling exercise has bought us. Part of the value of developing models is that it not only allows us to better understand the model as a whole, but that it also gives us insight into the individual component constructs that make up the model. That is, knowing how the constructs interact with one another – how they fit together – gives us some information about the character and nature of the constructs themselves. In our case, articulating how teacher noticing could be integrated into Schoenfeld's model of teaching has highlighted for us what type of "thing" teacher noticing is.

Initially, we may have thought of the things teachers notice merely as "triggering events." Thinking of teacher noticing in that way leads us to ask questions such as: What events do teachers notice? What does the activity of noticing entail and how can someone get better at that activity? In this conceptualization of noticing, noticing is an activity that can be isolated, performed, and possibly even practiced.

However, as we began to integrate noticing into Schoenfeld's model, we realized that other conceptualizations of noticing were possible. Rather than understanding noticing as a localized activity, we began to see teacher noticing as a kind of heightened awareness that constantly underlies teacher practice. In Schoenfeld's model, noticing might then be one of the pre-existing conditions that gives rise to various decisions (one of the "ifs"), or it might be part of the background that dictates how likely particular rules within the model are to be cued. Such conceptualizations lead to questions such as: How conscious is this noticing awareness? If a teacher notices some aspect of classroom activity that is (in)consistent with his goals, how likely is he to be aware of it and decide to pursue it? When mismatches between knowledge and noticing happen, what takes priority?

We are just beginning to understand the implications of this shift in how we understand the nature of noticing. However, we are confident that the exercise of placing noticing within Schoenfeld's model of teaching will be a key step in moving forward with that understanding.

REFERENCES

Chase, W. G., & Simon, H. A. (1973). Perception in chess. *Cognitive Psychology*, *4*, 55–81.

Common Core State Standards Initiative (CCSSI). (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/the-standards/mathematics.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (1997). *Comparing and scaling: Ratio, proportion, and percent – The Connected Mathematics Projects*. Palo Alto: Dale Seymour.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Research Council (Eds.). (2000). *How people learn: Brain, mind, experience, and school*. Washington, DC: National Academy Press.

Russ, R., Sherin, B. L., & Sherin, M. G. (2011). Images of expertise in mathematics teaching. In Y. Li & G. Kaiser (Eds.), *Expertise in mathematics instruction* (pp. 41–58). New York: Springer.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education*, *4*(1), 1–94.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Schoenfeld, A. H. (2011). Noticing matters. A lot. Now what? In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 223–238). New York: Routledge.

Sherin, B., & Star, J. R. (2011). Reflections on the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 66–78). New York: Routledge.

Sherin, M. G. (2002). A balancing act: Developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*, *5*, 205–233.

Sherin, M. G., Jacobs, V. R. & Philipp, R. A. (2011). Situating the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 3–13). New York: Routledge.

Sherin, M. G., Russ, R. S., & Colestock, A. A. (2011). Accessing mathematics teachers' in-the-moment noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 79–94). New York: Routledge.

Sherin, M. G., Russ, R., Sherin, B. L., & Colestock, A. (2008). Professional vision in action: An exploratory study. *Issues in Teacher Education*, *1*(2), 27–46.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematics discussions*. Reston, VA: National Council of Teachers of Mathematics, Reston, VA.

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ILANA SEIDEL HORN

9. TEACHING AS PROBLEM SOLVING

Collaborative Conversations as Found Talk-Aloud Protocols

INTRODUCTION

Alan Schoenfeld uncovered critical aspects of problem solving, identifying the way that learners use resources, heuristics, control, and beliefs to guide their activities around non-standard mathematical problems. In his groundbreaking research, he used talk-aloud protocols during problem solving sessions with undergraduates and audio recorded them to analyse their thinking. His investigation of students' talk and choices led him to develop his now well-known problem-solving framework (1985). As Schoenfeld's student, I share his deep curiosity in how people make sense of the world – only for me, the people were mathematics teachers and the problems were instructional.

Using ideas from ethnomethodology (Hymes, 1974; Garfinkle, 1967), I have spent the last 10 years analysing teachers' collaborative conversations, viewing them as naturally occurring talk-aloud protocols. From this perspective, I examine teachers' problem solving by looking at how they identify and articulate challenges in their work, as well as how they make progress on understanding these problems of teaching.

While Schoenfeld posed problems to his study participants, the teaching problems I examine emerge during interactions. In this way, I look at how teachers formulate as well as solve problems during collective work. This broader view necessitates an analysis of how teachers' knowledge and understandings of their work contribute to problem formulation and modelling as they represent, diagnose, and pursue problems of practice through their conversations.

In this chapter, I illustrate some key findings of my research on teachers' collaborative talk, demonstrating the places where "found" problem solving episodes corroborate and extend Schoenfeld's framework for mathematical problem solving. Like Schoenfeld, I find differences in how participants' beliefs, resources, and strategies influence their progress. Because I begin my analyses at the level of problem formulation, my work highlights the socially negotiated nature of problem solving. By articulating to and extending Schoenfeld's framework, this chapter contributes to a more general framework of human problem solving.

WHY STUDY TEACHERS' COLLABORATIVE CONVERSATIONS?

In the United States, teaching is an isolated profession. Teachers tend to work in their classrooms with little collegial interaction. Typically, other adults in the school only visit to evaluate performance. Even then, such visits are infrequent.

Y. Li and J.N. Moschkovich (eds.), Proficiency and Beliefs in Learning and Teaching Mathematics, 125–138.

These conditions complicate teachers' learning of "ambitious practice": forms of teaching that involve a wide range of students in rigorous forms of content (Lampert, Boerst, & Graziani, 2011). Mathematics teaching in this mode engages students in sense making and requires challenging forms of pedagogy. For example, building instruction on student thinking shifts teachers' attention away from the clear presentation of topics to a more complex practice of working from students' ideas. This latter form of teaching often leads to unanticipated classroom interactions. While this shift has the potential to improve students' mathematical learning and engagement, it also increases the uncertainty of teachers' work, requiring adaptation, responsiveness, and improvisation even for the most sophisticated of practitioners.

My research takes as a point of departure these observations about the isolated conditions of teaching and the challenging ambiguity of ambitious practice. I argue that the former exacerbates the latter. That is, professional isolation works against teachers developing more demanding teaching practices by leaving teachers on their own to diagnose and work through the inevitable teaching dilemmas that arise in ambitiously-oriented classroom environments.As a consequence, these desired "ambitious" mathematics teaching practices may not take a firm root, even among the most well-intentioned practitioners.

Numerous studies bear out this supposition. Many teachers implement superficial changes that take the form of ambitious pedagogies, without fulfilling their intended function (Spillane, 2000). For instance, they may have students work on a cognitively demanding task, but in implementation, make the task into a procedural one (Stein et al., 2008). Given the isolated conditions of their learning, it is understandable that teachers might, for instance, arrange students in groups but otherwise carry on with teaching as usual. Alternatively, teachers may tinker with new classroom structures and then abandon them, finding them unworkable in the existing structures of schooling, with its 45-minute class periods or 150 daily student contacts (Horn, 2012; Kennedy, 2010).

Despite the overall trend towards conservatism in teaching (Cuban, 1993; Lortie, 1975), there are a few documented examples of places where teachers sustain innovation and, consequentially, yield higher-than-expected student achievement (Boaler, 2002; Boaler & Staples, 2008; Bryk et al. 2010). These settings share two characteristics. First, teachers work collectively. Second, their collective work aims to increase student learning (Bryk et al., 2010; Lee & Smith, 1996; McLaughlin & Talbert, 2006). This may seem like an obvious arrangement, but it is definitely not normative, particularly in secondary mathematics (Stodolsky & Grossman, 2000).

Because of the relationship between collective work and increased student achievement, some reform efforts have embraced professional learning communities as a panacea, going so far as to mandate collaboration (Hargreaves, 2007). However collective work toward the goal of increased student learning is necessary but not sufficient for these kinds of outcomes. The feeling is that this correlative relationship keeps showing up because, in some instances, teacher collectives organized in this way support ambitious practice. The mechanisms that might explain the relationship have, to date, been underspecified. With this in mind, I examine mathematics teachers' collective problem formulation and solving to understand how these contribute to teacher learning in the service of ambitious practice in mathematics classrooms.

TEACHERS' PEDAGOGICAL PROBLEM FORMING AND PROBLEM SOLVING

Found talk-aloud protocols

Like Schoenfeld, I use records of talk to understand sense making. However, instead of posing a problem in a laboratory setting, I audio- and video-record teachers' naturally occurring workplace conversations with their colleagues.To find the moments of sense making in the messy stream of talk, I reduce the data by focusing on what I call *episodes of pedagogical reasoning* (EPRs). EPRs are the moments in teacher-to-teacher talk where issues or questions about teaching practice are brought out and accompanied by some elaboration of reasons, explanations, or justifications. In this sense, EPRs are "found" talk-aloud protocols of teachers' pedagogical problem solving. As such, they vary tremendously across participants, settings, and events. EPRs can be single turn utterances like, "I'm not using this worksheet because it bores the kids," or they can be long, multi-party and multi-turn conversations, taking dozens of minutes to unfold as teachers explore and elaborate different facets of an issue.

Longer EPRs are initiated by *problem framing* talk. Framing refers to how issues are defined through activities and interactions (Goffman, 1974). For instance, a teacher may raise the issue of student heterogeneity by framing the problem as an issue of *ability*:

> There's kids that know a lot, and then there's kids that, you know, feel like they're slow learners. (Horn, 2007, p. 50)

The contrast between knowing a lot and feeling like a slow learner invokes a problem frame of student ability; an individual trait that gives teachers limited options. Alternatively, teachers can frame student heterogeneity with an emphasis on the social sources of differential student performance:

> [K]ids who feel like they have low status will just continue to play that role because that's what they feel like they are supposed to do. (Horn, op cit., p. 54)

This second frame explains student heterogeneity through a lens of *status* – their social and academic desirability in the view of others. Frames are important resources in pedagogical problem posing and solving. In this example, the teachers' different frames for student heterogeneity constrain their sense making and responding. Students' ability may be less tractable than their social status, creating a different pedagogical problem space and different options for teacher response.

In this way, EPRs reveal underlying conceptions of mathematics teaching that shape not only problem definitions but locally specified understandings of reasonable teacher action. In other words, a teacher who interprets heterogeneity through an *ability* frame may see giving some students more challenging tasks than others as a reasonable response. On the other hand, a teacher who sees heterogeneity as partially rooted in social status might seek ways to address status dynamics in the classroom, perhaps by giving rich mathematical tasks to draw out the mathematical strengths of low status students (Horn, 2012).

Depending on the interactional organization of a particular group, the nature of the sense making varies. Because I am not designing the problem-solving task (as Schoenfeld did in the laboratory sessions), problem-solving activity depends heavily on existing relationships, norms, and routines. At one end of the spectrum, where norms of questioning others' thinking are not in place, teachers may briefly air their understanding of some issue in teaching. At the other end of the spectrum, teachers may have ample opportunities to critically engage an issue, considering alternative explanations in ways that support deeper understandings or even conceptual change (Hall & Horn, 2012). The *ability* and *status* framings of student heterogeneity, for instance, arose in the same conversation. The group had sophisticated norms and routines for extending and revising each other's problem frames. These frames, in turn, were grounded in a well-developed, taken-as-shared vision of ambitious teaching (Horn, 2010).

This spectrum of social arrangements and their influence on the pedagogical problem space has some resonances in Schoenfeld's results. In his studies of undergraduates' problem solving, most participants assumed that mathematics problems should be mere exercises, solvable within two minutes. In contrast, expert mathematicians did not assume that all problems were readily solvable and were prepared to approach a novel mathematics task with different strategies until they found a productive way in. Like the teachers I study, Schoenfeld's participants' notions of what constitutes a reasonable problem and solution colored their engagement. Some teachers in conversation may simply pose teaching issues as routine exercises to dispatch with. Teachers' views about fundamental elements of their work – teaching, students, and content – become both the means of working through problems and for learning about them. What Schoenfeld called "beliefs" govern important aspects of teachers' problem solving activity. This creates a conundrum in teacher workgroups, since the learning opportunities are heavily constrained by existing (and/or socially acceptable) conceptions, potentially making collaboration most beneficial when teachers with sophisticated pedagogical thinking are involved (Horn & Kane, under review).

Framings are the foundational difference in teachers' collaborative problem solving. They are rooted in the social and interactional resources I have identified in my studies. As shown in Figure 1, the social resources documented include collegial relationships, which shape a group's capacity for conflict, shared goals, and taken-as-shared epistemic stance on the work of teaching. This latter resource is deeply related to the group's moral commitments in their work as teachers. The social organization, in turn, shapes the interactional resources within a collaborative group. These include conversational routines, teaching principles, conversational category systems, and representations of practice. I will elaborate each of these resources in the following two sections, starting with the social resources and then getting into the interactional details. I conclude this chapter with a discus-

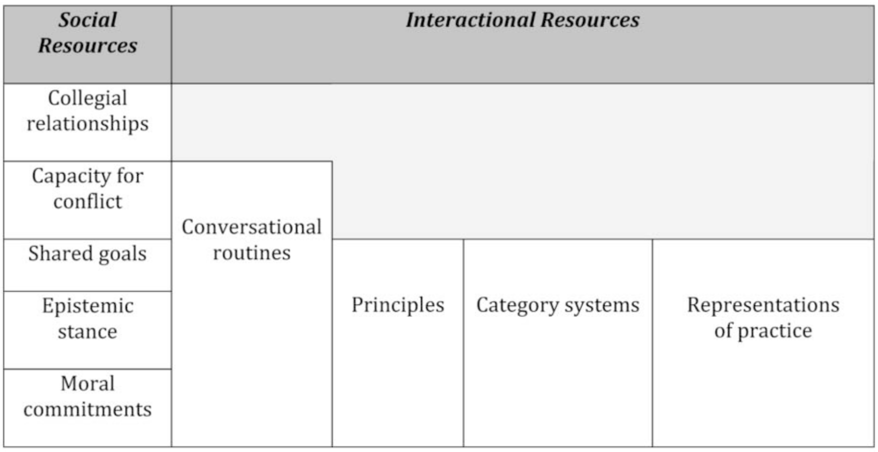

Figure 1. A hierarchy of social and interactional resources in teachers' collaborative problem solving.

sion of how my account of teachers' problem formulation supports and extends Schoenfeld's early work on problem solving.

Organization of collaborative problem solving

The social resources for teachers' conversations shape how problem formulation and problem solving unfold. For this reason, context becomes critical in my analyses of teachers' collective sense making. Social dimensions I account for when describing teachers' collegial conversations and their relationship to learning include: teachers' relationships, capacity for conflict, shared goals, epistemic stances, and moral commitments. All of these are social accomplishments of a group and go beyond an analysis of individual beliefs, stretching to conceptions that are viable and enacted within any particular workgroup (Horn, 2007).

Collegial relationships

Successful teacher collaborations are organized around positive collegial relationships. This may seem self-evident, but trust and engagement support teachers in sustaining ambitious practice. In supportive settings, teachers can try new things and flounder, knowing that sympathetic colleagues will back them and help them recover. With positive relationships, teachers reported a sense of mutual accountability emerging through regular meetings focusing on teaching problems. In interviews, these teachers often compare their workgroup membership to having exercise partners: the social arrangement creates greater accountability to challenging self-improvement goals. Ambitious forms of mathematics teaching are difficult to sustain, particularly when students and institutions typically press for other kinds of teaching. In this way, positive relationships motivate participants to persist, even in the face of trouble or uncertainty (Horn, 2012).

Even when conversations do not support in depth problem solving, teachers with good (or even decent) collegial relationships are generally glad for the opportunity to talk to each other on a regular basis. At a minimum, they garner emotional support by getting feedback from people other than their students (Metz, 1993). Simple story swapping – whether griping or joking – may provide emotional relief and an adult audience for work that is almost exclusively viewed and judged by children (Little, 1990).

Capacity for conflict

Positive relationships also provide a greater capacity for conflict. Of course, individual teachers may or may not be aligned with broader improvement-oriented purposes of talking to colleagues, even in the most productively organized teacher group. Conflict is an inevitable feature of teacher collaboration (Achinstein, 2002; Grossman, Wineburg, & Woolworth, 2001). From a learning perspective, conflict is vital. Differing viewpoints need an opportunity to come to light for people to change their mind or deepen their understanding of their own positions. In the earlier example of framing, the initial conception of student heterogeneity lay in an *ability* explanation. Through interaction, this frame was challenged and reworked into a *social status* explanation. In this case, a capacity for conflict – for different opinions to be expressed and explored – contributed to teachers' collaborative problem solving.

Shared goals

The most dynamic teacher groups organize around a clear goal premised on teaching as a complex endeavour. This alignment (Wenger, 1998) becomes a resource for collective learning. For instance, I studied a workgroup organized around the goal of de-tracking. Their conversations often emphasized finding activities that supported multiple forms of student mathematical competence (Horn, 2005, 2006, 2007, 2012). Another workgroup aimed to increase students' success in their first year college preparatory mathematics class (Horn, 2012; Horn & Kane, under review). Individual teachers not aligned with the groups' respective purpose did not participate as successfully, whether because they wanted to preserve traditional forms of teaching or because they found the work of examining students' thinking too demanding.

Epistemic stance

The social dimensions of collaborative problem solving shown in Figure 1 are deeply related. For instance, goals support problem framings. These, in turn, communicate an epistemic stance on teaching (Hall & Horn, 2012). Epistemic stances are the enacted perspectives on what can be known, how to know it, and why it is of value.

Teacher groups enact stances on good teaching that vary in complexity. Sometimes, visions of good teaching focus on what the teacher *does*, such as motivating students and presenting ideas clearly. Other times, visions of good teaching focus on the *interactions* among tasks, relationships, classroom discourse, student activity, and content that constitute a learning environment. This latter, more complex view requires simultaneous consideration of students' social, emotional, and cognitive states, individually and as a group, as well as on the organization of learning activities on multiple time scales to the end of supporting greater student mathematical understanding (Horn & Kane, under review; also see Lampert, 2001).

This range of epistemic stances becomes a critical aspect of teachers' collaborative talk, as each stance prioritizes some conversations over others. For instance, Horn and Little (2010) examining the conversations of two groups of teachers committed to improving practice and valuing student learning. Despite having similar goals, one group consistently provided substantial airtime for sharing pedagogical problems, turning toward the details of practice, while the other steered away from these toward the more concrete (and institutionally valued) tasks such as co-planning lessons, turning away from these details (Little & Horn, 2007). These differences in emphasis manifest different epistemic stances. The former enacts a view that teaching problems are worthy of attention and potentially soluble, while the latter focuses on the primacy of lesson planning – regardless of the kinds of trouble teachers face in their classrooms.

Moral commitments

In teaching, epistemic stances reflect moral commitments. This is another place where learning about teaching differs from learning mathematics. Moral commitments involve questions about the role and obligation of teachers – how much should teachers strive to teach all students, how much should they present engaging lessons and hope for the best? – which are matters of interpretation (Bartlett, 2004). People may have different epistemic stances on mathematics that influence their problem solving behaviour (e.g., *mathematics is hierarchically organized and must be learned sequentially* vs. *mathematics is a set of connected ideas whose relationships should be understood*), but stances on the work of teaching have a more explicitly moral dimension.

Misalignments can occur between a teacher's conception of their role and the group's taken-as-shared conception. To illustrate role conception with a simple issue, consider this: how much do teachers need to make themselves available for extra tutoring? Answers to such questions are part and parcel of a teacher's epistemic stance and are often locally negotiated. When moral commitments vary strongly, these differences tax a group's capacity for conflict. In turn, the group may not be a productive place for teachers' learning. Individuals with outlier role conceptions can, for instance, disrupt conversations by persistently airing alternative framings of problems and redirecting conversational focus (Horn & Little, 2010).

This section highlighted the social organization of teachers' collaborative problem solving. In the best case, teachers share a goal in their joint work, supported by a common epistemic stance and related moral commitments. These provide the basis for problem formulation and exposition, as the relatively shared view of what matters in teaching allows for the exploration of valued topics. Once teachers' share their thinking, having positive relationships allows for conflicts, while simultaneously providing mutual accountability toward improvement goals. In the laboratory and classroom scenarios that Schoenfeld studied, these social dimensions did not require as much negotiation. For instance, if students are invited to a problem solving session, they have already agreed on the activity with the observer. Differences of epistemic stance and capacity for conflict may need to be negotiated by student groups. However, the epistemic issues of mathematical problem solving typically do not have the same moral implications as those of teaching. Schoenfeld's identification of beliefs and resources as crucial to the problem solving process come the closest to mapping on to these social resources; but the social nature of teacher problem formulation and solving furthers our understanding of other kinds of problem solving behaviour. I will discuss these implications in the final section of this chapter.

Interactional resources in teacher talk

As I described earlier, framings emerge as problems are defined through activities and interactions (Goffman, 1974). Problem definition, whether explicit through talk or implicit through the organization of activity, communicates an epistemic stance. Defining an activity as, for instance, serving the goal of de-tracking, communicates that such work is both knowable and doable.

Of course, problem framings as manifest through social organization cannot be entirely separated from the specific interactional resources in teachers' talk. In my work, I have identified four interrelated interactional resources rooted in teacher groups' social organization while being analytically distinct. They are: *conversational routines*, *teaching principles*, *conversational category systems*, and *representations of practice*. Like the social resources described in the prior section, interactional resources contribute to the socially negotiated problem framings, which in turn constrain solutions and actions. In a sense they form the building blocks for teachers' understandings and progress on to pedagogical problems. In the remainder of this section, I will elaborate these four interactional resources.

Conversational routines

Conversational routines are the patterned ways that groups structure work-related talk and function in teacher professional communities. These routines differentially position teacher workgroups to forge, sustain, and support learning and improvement (Horn & Little, 2010; Little & Horn, 2007). As a feature of social organization, conversational routines convey goals, epistemic stances, and moral commitments, while also providing the means for engaging in or avoiding conflict. In this way, they are shaped (and shape) the organizational resources described in the previous section.

Two workgroups studied by Horn and Little (2010) both focused on student learning. However, the difference in epistemic stance – the enacted view of what is knowable and how to know it – gave rise to conversational routines that provided substantially different airtime to problems of practice. To support the collective exploration of teaching problems, one group developed a re-visioning routine that supported the reconceptualisation of teaching. The re-visioning routine entailed a pattern in teachers' talk in which they elaborated, reconsidered, or revised their understanding of complex teaching situations throughparticular and emotionally involving accounts of classroom events (Horn, 2010).

In an extensive EPR within this group, a new teacher, Alice, ran into trouble in her geometry classroom. She took some time during the group's meeting to describe the "mayhem" that erupted as a Geoboard lesson on triangle area veered away from her vision of the lesson. Her colleagues asked her a number of clarifying questions, prompting her to elaborate the nature of the trouble. For instance, after she described the general "squirreliness" of her class, a senior colleague asked:

> Alice, can you identify the source of the squirreliness? Like (fear is) that they, they wanted to play with the Geoboards but didn't have time to do it?

This question encouraged Alice to further specify her students' response to her lesson, requiring her to consider aspects of the classroom that had not been portrayed in her initial account. Such probing questions were a regular feature of this group's conversations. In this way, their conversational routines supported a revision of Alice's trouble, providing a different framing of the problem she encountered and, consequentially, changing the possibilities for addressing it.

Principles

Taking a closer look at the content of teachers' talk, we found all teacher groups used *teaching principles*, or propositions that serve as the foundation for pedagogical reasoning (Horn & Little, 2010; Horn & Kane, under review). Principles occur in teachers' talk and focus the analysis of a teaching problem on any combination of teaching, students, or content. For instance, I coded statements like, "Being consistent with routines help students understand expectations," and smaller claims like, "Starting a new unit is a good time to start fresh" as teaching principles. While both principles express a stance on teaching and students, neither engages issues of mathematics. Additionally, teaching principles may be more or less explicitly stated and operate on different time scales.

In a comparative analysis of workgroups made up of teachers at three different levels of instructional accomplishment, *principles* turned out to be the primary window into distinctions across teachers' collective thinking about students, teaching, and mathematics (Horn & Kane, under review). In comparing the collaborative talk of teachers who were beginning, emergent, and sophisticated in their ambitious mathematical instruction, we explored the differences in how they represented and thought through critical issues in their work. The sophisticated teachers tended to use more *multidimensional principles*, focusing simultaneously on teaching, students, and mathematics more often than either the beginning or emergent teacher groups. This reflected the most sophisticated group's more complex view of teaching expressed and exhibited in their own classrooms. Aside from the differences in the content of the principles, teachers' deployment of principles revealed important differences in conversational processes. The most sophisticated teacher group tended to use principles with greater frequency, grounding their pedagogical reasoning in well-articulated stances on teaching. In this way, principles reflected epistemic stance as expressions of what one knows as well as how one knows something in the work of teaching. At the same time, by honing in on certain problems as worth attention, the deployment of principles also asserted a moral stance.

Category systems

Teachers' conversational category systems classify things in the world in everyday talk. For instance, a teacher might refer to "slow" or "fast" students, a "hard" or "easy" class. A component of frames, these systems model problems of practice and communicate assumptions about students, subjects, and teaching.

In an analysis of conversational category systems (Horn, 2007), I examined how teachers' conversational categories for students played out in two different mathematics teacher groups facing a "mismatch" problem – the sense that their students' achievement levels were not well alignedwith intended school curricula. As the teachers talked through the problem, one group maintained static categories of student ability and motivation: students were essentialized as fast, slow, or lazy. In this framing, the only viable solution to the Mismatch Problem was to lessen the demand of the curriculum to accommodate the problem as they understood it. In contrast, another workgroup saw student abilities as malleable: students were fast and slow at certain mathematical things, but these descriptors were not fixed student characteristics. (The earlier discussion about problem frames for heterogeneity is from this same discussion.) In this light, teachers could shift the nature of classroom activities to allow different students' strengths to come into play, keeping the curriculum's rigor intact. As a building block of problem framings, conversational category systems do much to model problems and shape teachers' problem and solution space. That is, taxonomies like *fast* or *slow kids* help teachers sort out what happens in the classroom so that they can specify problems and propose reasonable courses of action in response.

Representations of practice

A persistent dilemma in teachers' collaborative problem solving is that consultations are almost necessarily asynchronous with active instruction. While co-planning is a common activity for teacher workgroups, it often obscures classroom discourse that can transform lessons in qualitatively different ways. Critical aspects of teaching are only accessible, then, as teachers render them in conversation.

This is another important distinction between mathematics problems and teaching problems. Mathematics problems can be fixed to a page or a computer screen with their fundamental features intact. Teaching problems often occur in real time, during interactions between students or between students and a teacher. Stabilizing these moments to create a common object of inquiry (Bransford et al., 2000) requires means for representing it.

To this end, teachers employ representations of classroom practice (Little, 2003). Representations of practice included curriculum artifacts, student work, and stories, or classroom talk. This last category provides a means for constructing the interactive part of teaching via *replays* and *rehearsals* (Horn, 2005, 2010). Teaching replays are blow-by-blow accounts of classroom events, while rehearsals are generalized or anticipatory versions of the same.

To illustrate how representations support problem framing and solving, consider the following EPR excerpt. A group of teachers was discussing a common phenomenon in mathematics teaching: students freezing up when faced with fractions. This group of teachers was in the midst of a unit involving the slopes of lines, which are typically represented as a ratio of "rise over run." Darla, an accomplished and experienced teacher, explained how she avoided students' fraction freeze, beginning with a replay:

> DARLA: [W]e had to tell them all the way through lines: "These are not fractions, they just look like fractions, they're rates of change," because the minute they look like fractions, the kids are like, "I'm out." (*Puts hands up*). I'm like, "No, no, no. They only look like fractions, they're really just rates of change."
>
> WENDY: Wow. Yeah.
>
> DARLA: Did you (*to Hoa*) yeah – you probably weren't here for this.
>
> HOA: I probably need to call them rates of changes now so I don't get freaked out kids anymore.
>
> DARLA: Oh, if you call them – rates of change. And you never say it as three-fourths, you always say "it goes up 3 for every 4." It's a WHOLE different experience (*laughs*).

Darla elaborated the problem of students' emotional response to fractions by replaying how she presented slopes to students ("These are not fractions they just look like fractions, they're rates of change"). A teaching principle buttressed the replay, helping her colleagues understand the purpose of the reported teaching move ("the minute they look like fractions, the kids are like, "I'm out"). In the fourth turn of talk, Hoa reconsidered her approach to introducing slopes ("I probably need to call them rates of change so I don't get freaked out kids anymore"). In the final turn, Darla presented a rehearsal, demonstrating how teachers can strategically avoid the term "fractions" while maintaining the mathematical concept.

The problem of "fraction freeze" was specified through this conversation. Darla represented students' alternative response when she renamed fractions as rates of change. This pedagogical problem lies in the real time interaction of teachers and students. This exploration can shape the teachers' planning at a level of students' affective experiences, which, in turn, influences their learning.

COMPARING TEACHERS' PEDAGOGICAL PROBLEM SOLVING TO STUDENTS' MATHEMATICAL PROBLEM SOLVING

Problem solving activity in teacher workgroups and that of undergraduates working on math problems differed in two fundamental ways. First, the teachers' work involved defining problems as well as solving them. In this way, their own pedagogical understanding became a critical resource not only for their responses but also for the very definition of the teaching problems themselves. In contrast, Schoenfeld's undergraduates were given problems that carried with them a certain view of mathematics – they instantiated critical dimensions of what is captured in the social resource analysis, such as goals, epistemic stance, and moral stance. While individual mathematical problem solvers bring their personal goals to the activity, the direction "*solve this problem*" narrows the field of possible goals more clearly than happens in teacher workgroups. The work of problem definition leaves many aspects of goals open for interpretation. Likewise, merely posing a complex mathematics problem asserts a certain epistemic stance: *this form of mathematics is knowable and do-able by you*. In contrast, the epistemic (and related moral) stances on teaching are highly contested and negotiated in the very work of problem definition.

These phenomenological level differences relate to analytical ones: the social resources for the teachers' problem solving activity become critical for the analysis, as they play heavily into problem frames. Problem frames not only shape problem definition but problem solution as well. Again, because the domain of teaching knowledge is not as well defined as the domain of mathematics, and this ambiguity gets heightened and leaves more opportunity for the social context to shape activity. While Schoenfeld's work takes him into the social world of the classroom, his framework better supports the analysis of individual activity than the socially negotiated problem solving of a teacher workgroup.

This analysis of similarities and differences between students' mathematical problem solving and teachers' collaborative pedagogical problem solving contributes to a more general understanding of problem solving activity. This articulates to Schoenfeld's (2011) recent work on goal-directed activity. In his framework, he identifies "resources" as critical to decision-making in classroom teaching. This category might be further specified by the social resources introduced here; that is, relationships, capacity for conflict, shared goals, epistemic stance, and moral commitments guide teachers' learning in a classroom community as well as in a teacher collective. In addition, by highlighting the deeply social nature of problem solving, I propose that an analysis of teaching problems requires a consideration of not only their cognitive demands, but also their social demands. In this way, teachers' successes and difficulties in carrying out teaching practices and addressing related problems could be viewed not merely as a consequence of their individual competence but also as fundamentally shaped by their social environments. For this reason, their successes and difficulties require an analysis of the social resources for their practice.

REFERENCES

Achinstein, B. (2002). Conflict amid community: The micropolitics of teacher collaboration. *Teachers College Record*, *104*(3), 421–455.

Bartlett, L. (2004). Expanding teacher work roles: A resource for retention or a recipe for overwork? *Journal of Education Policy*, *19*(5), 565–582.

Boaler, J. (2002). *Experiencing school mathematics: Traditional and reform approaches to teaching and their impact on student thinking*. Mahwah, NJ: Erlbaum.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of railside school. *Teachers College Record*, *110*(3), 608–645.

Bransford, J. D., Brown, A. L., & Cocking, R. (2000). *How people learn: Brain, mind, experience, and school*. Washington, D.C.: National Research Council.

Bryk, A. S., Sebring, P. B., Allensworth, E., Luppescu, S., & Easton, J. Q. (2010). *Organizing schools for improvement: Lessons from Chicago*. Chicago: The University of Chicago Press.

Cuban, L. (1993). *How teachers taught: Constancy and change in American classrooms, 1890–1990*. New York: Teachers College Press.

Garfinkel, H. (1967). *Studies in ethnomethodology*. Englewood Cliffs, NJ: Prentice-Hall.

Goffman, E. (1974). *Frame analysis*. Boston: Northeastern University Press.

Grossman, P. L., Wineburg, S. S., & Woolworth, S. (2001). Toward a theory of teacher community. *Teachers College Record*, *103*(6), 942–1012.

Hall, R. P., & Horn, I. S. (2012). Talk and conceptual change at work: Analogy and epistemic stance in a comparative analysis of statistical consulting and teacher workgroups. *Mind, Culture, Activity*, *19*(3), 240–258.

Hargreaves, A. (2007). Sustainable professional learning communities. In S. Stoll & K. S. Louis (Eds.), *Professional learning communities: Divergence, depth, and dilemmas*. Berkshire, UK: Open University Press.

Horn, I. S. (2005). Learning on the job: A situated account of teacher learning in high school mathematics departments. *Cognition and Instruction*, *23*(2), 207–236.

Horn, I. S. (2006, Winter). Lessons learned from detracked mathematics departments. *Theory into Practice*, *45*(1), 72–81.

Horn, I. S. (2007). Fast kids, slow kids, lazy kids: Framing the mismatch problem in mathematics teachers' conversations. *Journal of the Learning Sciences*, *16*(1), 37–79.

Horn, I. S. (2010). Teaching replays, teaching rehearsals, and re-visions of practice: Learning from colleagues in a mathematics teacher community. *Teachers College Record*, *112*(1), 225–259.

Horn, I. S. (2012). *Strength in numbers: Collaborative learning in secondary mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Horn, I. S. & Kane, B. D. (Under review). Tracing the development of pedagogical reasoning in mathematics teachers' collaborative conversations. Cognition and Instruction.

Horn, I. S., & Little, J. W. (2010). Attending to problems of practice: Routines and resources for professional learning in teachers' workplace interactions. *American Educational Research Journal*, *47*(1), 181–217.

Hymes, D. (1974). *Foundations in sociolinguistics: An ethnographic approach*. Philadelphia: University of Pennsylvania Press.

Kennedy, M. M. (2010). Attribution error and thequest for *teacher quality*. *Educational Researcher*, *39*(8), 591–598.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Lampert, M., Boerst, T., & Graziani, F. (2011). Using organizational resources in the service of ambitious teaching practice. *Teachers College Record*, *113*(7), 1361–1400.

Lee, V., & Smith, J. B. (1996). Collective responsibility for learning and its effects on gains in achievement for early secondary school students. *American Journal of Education*, *104*, 103–147.

Little, J. W. (1990). The persistence of privacy: Autonomy and initiative in teachers' professional relations. *Teachers College Record*, *91*(4), 509–536.

Little, J. W. (2003). Inside teacher community: Representations of classroom practice. *Teachers College Record*, *105*(6), 913–945.

Little, J. W., & Horn, I. S. (2007). "Normalizing" problems of practice: Converting routine conversation into a resource for learning in professional communities. In L. Stoll & K. S. Louis (Eds.), *Professional learning communities: Divergence, detail and difficulties* (pp. 79–92). Maidenhead, England: Open University Press.

Lortie, D. (1975). *School teacher: A sociological study*. Chicago: University of Chicago Press.

McLaughlin, M. W., & Talbert, J. E. (2006). *Building school-based teacher learning communities: Professional strategies to improve student achievement*. New York: Teachers College Press.

Metz, M. H. (1993). Teachers' ultimate dependence on their students. In J. W. Little & M. W. McLaughlin (Eds.), *Teachers work: Individuals, colleagues, and contexts* (pp. 104–137). New York: Teachers College Press.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (2011). *How we think: A theory of goal-directed decision making and its educational applications*. New York: Routledge.
Spillane, J. P. (2000). Cognition and policy implementation: District leaders and the reform of mathematics education. *Cognition and Instruction*, *18*(2), 141–179.
Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340.
Stodolsky, S. & Grossman, P.L. (2000). Changing students, changing teaching. *Teachers College Record*, *102*(1), 125–172.
Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge: Cambridge University Press.

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10. RESEARCHING THE SUSTAINABLE IMPACT OF PROFESSIONAL DEVELOPMENT PROGRAMMES ON PARTICIPATING TEACHERS' BELIEFS[1]

RATIONALE

The question of how to effectively promote mathematics teachers' professional development is of great interest and is being discussed in various papers (e.g., Krainer & Zehetmeier, 2008; Loucks-Horsley, Stiles, & Hewson, 1996; Maldonado, 2002; Sowder, 2007; Zehetmeier, 2010; Zehetmeier & Krainer, 2011). In this context, the question of sustainability is of particular relevance. Despite its central importance for both teachers and teacher educators, research on sustainable impact is generally lacking within teacher education disciplines (Datnow, 2005; Rogers, 2003).

This chapter addresses this issue and provides findings from two case studies focusing on the sustainable impact of a nation-wide project to promote mathematics and science teaching in Austria (IMST[2]; see Krainer, 2003, 2008). Two former project participants were revisited to gather data concerning the project's impact some years after its termination. The chapter puts a particular emphasis on the sustainable impact to the teachers' beliefs. Moreover, the factors that fostered or hindered this sustainability are carefully examined.

Although the issue of teachers' beliefs is not a really new topic (see e.g., Ernest, 1989a; Leder, Pehkonen, & Törner, 2002; Maaß & Schlöglmann, 2008; Schoenfeld, 1987; Thompson, 1984), the discussions and empirical studies are far from closed. Leder (2008) analysed the Research Reports and Short Oral Communications at the MERGA and PME conferences in 2007 to find out which of them are related to beliefs and mathematics education. This analysis showed that in both conferences approximately half of the contributions referred to beliefs. Leder (2008) highlights "the mathematics education research community's continuing interest in the ways students' and teachers' beliefs affect mathematics learning and instruction" (p. 51).

THEORETICAL FRAMEWORK

In this section, we give a short overview regarding the theoretical background of the chapter's central notions and concepts.

Professional development

Teachers are considered to play a central role when addressing professional development programmes: "Teachers are necessarily at the center of reform, for they

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must carry out the demands of high standards in the classroom" (Garet, Porter, Desimone, Birman, & Yoon, 2001, p. 916). Ingvarson, Meiers, and Beavis (2005) sum up:

> Professional development for teachers is now recognised as a vital component of policies to enhance the quality of teaching and learning in our schools. Consequently, there is increased interest in research that identifies features of effective professional learning. (p. 2)

Impact

Goals and outcomes of professional development programmes are of great interest, for the participating teachers and the facilitators in particular. In this context, the question of possible levels of goals and outcomes is important: Which levels of goals and outcomes are possible?

In most papers that put an emphasis on the question of goals (and thus the potential outcomes) of teachers' professional development, teachers' learning is the main focus (see e.g., Guskey, 2000; Lipowsky, 2004, 2010; Sowder, 2007; Zehetmeier, 2008). From a holistic perspective (according to Pestalozzi's idea of learning by head, heart, and hand; e.g., Brühlmeier, 2010), the major indicators for describing teachers' learning are their knowledge, beliefs, and practice: There is an ample body of literature discussing the mutual relationship between any two (e.g., Da Ponte & Chapman, 2006; Liljedahl, 2008; Song & Koh, 2010) or three (Carrington, Deppeler, & Moss, 2010; Ernest, 1989b; Haslauer, 2010; Zehetmeier, 2008) of these. However, the situation is rather complex since each of these notions can be defined in different ways.

Teachers' knowledge, for example, can be differentiated into content knowledge, pedagogical knowledge, and pedagogical content knowledge (Shulman, 1987); but it can also be regarded as knowledge about learning and teaching processes, assessment, evaluation methods, and classroom management (Ingvarson et al., 2005); yet other foci are expressed by the notions of attention-based knowledge (Ainley & Luntley, 2005) or the knowledge quartet (Rowland, Huckstep, & Thwaites, 2005).

At the teachers' practice level, the focus may be on classroom activities and structures, teaching and learning strategies, methods, or contents (Ingvarson et al., 2005).

Similarly, teachers' beliefs can include different aspects of beliefs about mathematics as a subject, and its teaching and learning (Leder, Pehkonen, & Törner, 2002). It also includes participating teachers' perceived professional growth and their satisfaction (Lipowsky, 2004), their perceived efficacy (Ingvarson et al., 2005), and teachers' opinions and values (Bromme, 1997). In addition, teachers' attitudes (e.g., to which extent teachers like or dislike mathematics and how this probably changes) and interests (e.g., in specific topics, questions to investigate in their own teaching) would be worth considering. Mason (2004) provides an alphabetical list of associated and interrelated terms to work out what beliefs actually are:

> A is for attitudes, affect, aptitude and aims; B is for beliefs; C is for constructs, conceptions and concerns; ... X is for xenophilia (perhaps); Y is for yearnings and yens; Z is for zeitgeist and zeal. (p. 347; the entire alphabetical list is provided there)

In this sense, Leder (2008) claims that the notion of "belief" still serves "as a convenient synonym for a host of other words" (p. 51), and that there is a "frequent failure to distinguish carefully and consistently between beliefs and other affective factors" (ibid.).

Törner (2002) analyses various definitions of beliefs in the literature and proposes a mathematical model to catch the key aspects of all these definitions and to "achieve a precise definition" (p. 73). This model "focuses on belief object, range and content of mental associations, activation level or strength of each association, and some associated evaluation maps" (ibid.). For each of these four foci, Törner (2002) introduces mathematical symbols. He defines:

> In short, a belief constitutes itself by a quadruple B=(O, C_O, μ_i, ε_j), where O is the debatable object, C_O is the content set of mental associations (what is traditionally called a belief), μ_i is the membership degree function(s) of the belief, and ε_j is the evaluation map(s). (p. 82)

Fostering factors

What are the factors promoting and fostering the impact of professional development programmes? Literature and research findings concerning this question point to a variety of factors. In particular, the factors fostering the effectiveness of professional development programmes are of central importance. However, these factors are rather manifold and complex. This is also true for the underlying theoretical concepts. This complexity can be reduced to three dimensions (see Krainer, 2006; see also Krainer & Wood, 2008; Lachance & Confrey, 2003; Llinares & Krainer, 2006; Sowder, 2007; Stein, Smith, & Silver, 1999):

– Content factors (high level and balance of subject-related action and reflection)
– Community factors (high level and balance of individual and social activities, in particular fostering community-building within and outside the professional development programme)
– Context factors (high level and balance of internal and external resources and support)

A detailed review of literature concerning these factors is provided in Zehetmeier (2008) or Zehetmeier and Krainer (2011).

Sustainability

The expected outcomes of professional development programmes are not only focused on short-term effects that occur at the end of the programme, but also on long-term effects that emerge (even sometimes years) after the programme's termination. Effects that are both short-term and long-term can be considered to be sustainable. So sustainability can be defined as the lasting continuation of achieved benefits and effects of a programme or initiative beyond its termination (DEZA, 2002). As Fullan (2006) points out, short-term effects are “necessary to build trust with the public or shareholders for longer-term investments” (p. 120). Besides these short-term effects, long-term effects need to be considered as well; otherwise the result could be to “win the battle, [but] lose the war” (ibid.). Hargreaves and Fink (2003) state,

> sustainable improvement requires investment in building long term capacity for improvement, such as the development of teachers’ skills, which will stay with them forever, long after the project money has gone. (p. 3)

Moreover, analysis of sustainable impact should not be limited to effects that were planned at the beginning of the project; it is also important to examine unintended effects and unanticipated consequences that were not known at the beginning of the project (Rogers, 2003; Stockmann, 1992).

In the literature, there are even more definitions of the notion of sustainability (e.g., Anderson & Stiegelbauer, 1994; Fullan, 2006; Hargreaves & Fink, 2003; Stockmann, 1992). The common ground of all these definitions of sustainability is the focus on durable continuation. At the same time, in most cases the extent of this duration remains open. It is unclear, whether sustainability means, for example three months or ten years of continuation. “If the time limit of sustainability is not set exactly (in some cases unlimited), the verification of sustainability is not possible” (Stockmann, 1992, p. 27). So each analysis of sustainable impact has to define the time frame for sustainability.

THE AUSTRIAN IMST^2 PROJECT

The initial impulse for the IMST^2 project in Austria came from the 1995 TIMSS achievement study (Third International Mathematics and Science Study). In particular, the results of the Austrian high school pupils (grades 9 to 12 or 13) in the TIMSS advanced mathematics and science achievement test, shocked the public. The responsible federal ministry launched the IMST research project (1998–1999) in order to analyse the situation (see Krainer, 2003).

This research identified a complex picture of diverse problematic influences on the status and quality of mathematics and science teaching: For example, mathematics education and related research was seen as poorly anchored at Austrian teacher education institutions. Subject experts dominated university teacher education, while other teacher education institutions showed a lack of research in mathematics education. Also, the overall structure showed a fragmented educational system consisting of lone fighters with a high level of (individual) autonomy and action, but little evidence of reflection and networking (Krainer, 2003; see summarized in Pegg & Krainer, 2008).

The analyses mentioned above led to the four year project IMST^2 (2000–2004). The project (Krainer, 2003) focused on the upper secondary school level and involved the subjects of biology, chemistry, mathematics, and science. The two major tasks of IMST^2 were

- The initiation, promotion, dissemination, networking and analysis of innovations in schools (and to some extent also in teacher education at university); and
- Recommendations for a support system for the quality development of mathematics, science and technology teaching.

In order to take systemic steps to overcome the "fragmented educational system," a "learning system" (Krainer, 2005) approach was taken. It adopted enhanced reflection and networking as the basic intervention strategy to initiate and promote innovations at schools.

Besides stressing the dimensions of reflection and networking, "innovation" and "working with teams" were two additional features. Teachers and schools defined their own starting point for innovations and were individually supported by researchers and project facilitators. The IMST^2 intervention built on teachers' strengths and aimed to make their work visible (e.g., by publishing teachers' reports on a website). Thus, teachers and schools retained ownership of their innovations. Another important feature of IMST^2 was the emphasis on supporting teams of teachers from a school.

Teachers' participation in IMST^2 was voluntary. They could choose among several priority programmes (e.g., "basic education" or "teaching and learning processes") according to major challenges concerning mathematics and science teaching. In general, teachers in these priority programmes were supported by mathematics and science educators and experienced teachers. The priority programmes can be regarded as small professional communities that not only encouraged each participant to proceed with his or her own project, but also generated a deeper understanding of the critical reflection of one's own teaching, by means of actions research methods.

METHOD

Research design

This research follows a case study design (Stake, 1995; Yin, 2003), because this approach is particularly suited for analysing the impact of innovations:

> The usual survey research methods are less appropriate for the investigation of innovation consequences. [...] Case study approaches are more appropriate. (Rogers, 1995, p. 409)

Similarly, Hancock and Algozzine (2006) state:

> Through case studies, researchers hope to gain in-depth understanding of situations and meaning for those involved. (p. 11)

The case studies presented here are historic (Merriam, 2001), intrinsic (Stake, 1995) and explanatory (Yin, 2003), since they analyse the teachers' development over time, focus on the particular teachers' cases, and look for the respective professional developments' fostering conditions.

Former research

This analysis is based on the results of a former research project: In 2005, 11 case studies were generated to describe and explain specific aspects regarding the impact of the IMST^2 project (Benke, Erlacher, & Zehetmeier, 2006; see also Krainer, 2005). These case studies' results could highlight various levels of impact; for example, teachers' mathematical knowledge, beliefs, or teaching practice.

Recent research

In 2010, the idea was born to revisit these case studies and analyse the project's impact after five more years, from an ex-post perspective (Zehetmeier, in preparation). For this purpose, semi-structured interviews were again conducted with the teachers who had taken part in the IMST^2 project; interviews were also conducted with the teachers' respective colleagues, schools' principals, and former project facilitators. The data gathered in 2010 was analysed and contrasted with the 2005 case studies' results.

This combination of former and current research projects resulted in a set of quasi-longitudinal case studies. This chapter provides the findings of two of these case studies, focusing on teachers' beliefs.

Research questions

This chapter's focus is the impact of a professional development programme eight years after the programme's termination: The teachers participated in the programme from 2000 to 2002. A comparison of the 2005 results with recent data from 2010 allows a thorough discussion of the following questions: Which of the 2002 and 2005 impact concerning teachers' beliefs is still effective in 2010? Which impact did disappear? Which are the respective factors that fostered or hindered the sustainability of impact?

Data

The case studies include data from various sources and time periods to gain validity by "convergence of evidence" (Yin, 2003, p. 100): data collection was done during 2001 and 2010 and contains documents (teachers' written project reports; 2002) and archival records (first author's artefacts; 2001–2004). Moreover, interviews were conducted with the teachers, their colleagues, and their principals in 2005 and 2010.

The 2005 interviews were semi-structured. This means that the interview structure was based on document analysis of existing data, which identified various levels of impact. The 2005 interviews were designed to both investigate the sustainability of identified impacts, and reveal other types of impacts which were not already coded by the researchers. Therefore, the questions were both closed (e.g., is the impact you described in the 2002 project report still effective?) and open (e.g., which other impact of the IMST[2] project is still effective?).

The 2010 interviews were semi-structured, too. The interviews should both work out which of the 2005 impacts were still effective (or not) in 2010, and identify other types of impacts which were not realised by research until 2010. Therefore, the interviews were based on document analysis of the existing data (including the findings of the 2005 case studies) and were designed accordingly.

Analysis

Data analysis included both, inductive and deductive elements (Altrichter & Posch, 2007). In a first step, all data from before 2005 was analysed and – according to the research questions – coded inductively. The second step included deductive analysis of the 2005 data: interviews and case studies were coded according the theoretical framework to analyse both the impact (on the levels of the respective teachers' knowledge, beliefs, and practice) and the respective fostering (or hindering) factors. The 2005 case studies' results were validated by means of member checking. Then, the 2010 interviews were planned, conducted, and analysed. Here, the data was again coded both inductively and deductively in order to be able to combine and contrast these recent results with the former ones.

Data was analysed by qualitative content analysis (Mayring, 2003) in order to identify common topics, elaborate emerging categories, and gain deeper insight into teachers' professional growth over time.

Validity

Creswell (2007) identified eight verification procedures for qualitative studies and recommends that at least two of them are given to ensure validity. In this study, four of these verification procedures were present: prolonged engagement, triangulation, negative case analysis, and rich description: The contact with the teacher has spanned more than one year in the contexts of the project, and the time span under research lasted for more than eight years (prolonged engagement). Data came from a variety of sources (triangulation by convergence of evidence, see above). Research results were refined with regards to disconfirming evidence until any disagreements among the findings were eliminated (negative case analysis). Finally, the case study provides detailed information about all persons and activities relevant for this research (rich description).

RESULTS

This chapter presents findings from two case studies[2] related to secondary mathematics teachers' professional development. Within these case studies, two teachers are in the focus: Andy and Barbara.[3] The following sections provide the teachers' background, their respective case studies with a particular focus on the professional development's impact on their beliefs, and the factors which fostered or hindered the sustainability of this impact.

Background

Andy and Barbara are teaching in a secondary school in a medium-sized town in Austria. In 2010, about 560 pupils were taught in 22 classes at this school. The school started a school development process in order to foster quality management and development in 1997: A mission statement was formulated, a school programme was created, and participative steering structures were established. Based on that, the school had available an organisational framework which allowed innovations to be broadly discussed and implemented. In 2000, Andy and Barbara started – individually and independently from each other – their participation in $IMST^2$.

In the following, the impact of Andy and Barbara's participation in the $IMST^2$ project regarding their beliefs is provided. In particular, the 2005 results are contrasted with the recent 2010 data. This allows discussion of the question "what impacts from 2005 were still there in 2010?"

The case of Andy

Andy is a secondary mathematics teacher with 32-years of teaching experience in 2010. From 2001 to 2002, he participated in the $IMST^2$ project for one school year. His starting point was of particular interest: He wanted to "provide and perform mathematics teaching which is efficient and appropriate for pupils" (Andy, 2002, project report, p. 1). In order to meet this goal, he intended to get feedback from the pupils regarding his teaching practice. Therefore, he conducted an action research project to find out more about his "pupils' preferences and aversions" (Andy, 2002, project report, p. 6). The results of Andy's action research project pointed to various positive aspects of his teaching. However, one particular issue was evaluated rather critically by the pupils: In many cases, Andy urged pupils to calculate on the blackboard. While his intention was to support and encourage the pupils, they perceived these situations as taking an examination and being exposed to observations by their class-mates. Inspired and surprised by this finding, Andy tried to analyse this issue more deeply. So he conducted another questionnaire with a particular focus on calculating on the blackboard. Additionally, he kept a research diary to record his and his pupils' behaviours and moods during the phases of blackboard calculations.

Another consequence of Andy's action research project was his desire to know more about his teaching. Thus, he gathered feedback not only from his pupils and project facilitators, but also from his colleagues regarding his teaching practices.

> It was very important for him to receive external perspectives regarding his explication of intentions. So he initiated a system of mutual classroom observations with two colleagues. (Andy's project facilitator, 2005, interview)

Andy's beliefs

The 2005 data indicated some changes in Andy's beliefs: In particular, the teacher's self-esteem was enhanced. He did not need to guess, rather he could know, for example, that his teaching was regarded favourably by his pupils. "This allowed him to plan and implement innovative teaching practices in a very self-confident manner" (Benke et al., 2005, p. 42). This impact was sustained: In the 2010 interview, the open question was posed: "Which impact is still effective, almost ten years after the programme's termination?" Andy's first answer was: "Definitely, the courage to go my way and to advance the things I do" (Andy, 2010, interview). Andy stated that his participation in the programme "laid the basis for my self-esteem. I dare, I try, and I still have a good feeling" (ibid.).

Another impact on Andy's beliefs concerns his awareness of the importance of clearly and explicitly explaining his intentions to the pupils – particularly whenever pupils had to perform in front of the class. This belief was still present in 2005: Henceforward, Andy explained his objectives before urging pupils to calculate on the blackboard, "to eliminate the threatening aspects of this situation" (Benke et al., 2005, p. 43). This impact was sustained: In 2010, for example, Andy stated:

> One of my particular concerns is still the calculation on the blackboard. These situations should be burdened as little as possible. This is sustainable knowledge which will be important until my retirement. (Andy, 2010, interview)

The case of Barbara

Barbara is a secondary mathematics teacher with 38-years of teaching experience in 2010. She took part in the IMST[2] project for two school years, from 2000 to 2002. Her main objective was to integrate open learning environments into her mathematics classes. The central ideas of these settings are to enable independent and autonomous learning processes as well as to allow individual goals and working schedules. In her first year of participation, Barbara used open classroom settings for pupils' practicing the content. Within the framework of an action research project, she used questionnaires and interviews to discover more about her pupils' perspectives on this kind of teaching. The results showed that they really liked this kind of learning environment. Also, their mathematical competencies increased.

In the second year of participation, Barbara decided to integrate open learning settings not only during practice phases, but also when introducing and developing new content: "I wanted to know more about possible obstacles when pupils have to acquire new knowledge for themselves" (Barbara, 2002, project report, p. 4). In a ninth grade class, she chose the topics *trigonometric functions* and *rectangular triangles* to implement open learning environments. Again, she used action research methods to get information regarding possible obstacles. She used tape recordings of pupils' conversations when working on new content. The analysis revealed that the pupils appreciated this setting and they had no problem learning the new content. In particular, some pupils learned much more when they worked individually as they did before in teacher-centred settings.

As a consequence of Barbara's action research project, she planned to put even more effort into the implementation of open learning settings. Moreover, she wanted to get feedback from her colleagues regarding these teaching practices.

Barbara's beliefs

In the context of her participation in IMST[2], Barbara's beliefs regarding open learning environments changed: She could see that these settings had positive effects on pupils' content knowledge, as well as on their self confidence. In particular, there were positive changes regarding low-performing pupils' self-esteem, as well as the further development of high-performing pupils' competencies. Moreover, Barbara's pupils had more fun and were less anxious in her mathematics lessons. This impact was sustained: Barbara still held these beliefs in 2005 and 2010. This enabled her to create and implement innovative teaching methods. For example, since Barbara is convinced of the importance of time resources for these open settings, she is very conscious of providing enough resources in each implementation phase. The school's principal stated: "This had very positive effects on the didactics of our mathematics lessons. In particular, the open learning settings represent sustained impact" (Principal, 2005, interview).

The participation in the IMST[2] project also caused another effect concerning Barbara's beliefs: She developed a reflective stance towards the content and the method of her teaching. This stance was mirrored by her belief about the value of feedback: Topics like classroom atmosphere and teaching quality were discussed with her pupils on a regular basis: "Now I see the value of discussing questions of good mathematics teaching together with the pupils" (Barbara, 2005, interview). This impact was sustained: In 2005, as well in 2010, Barbara was convinced of the importance of critically evaluating one's own teaching. "It is important to reflect on good and problematic aspects of my work" (Barbara, 2005, interview).

The project had yet another impact on Barbara's beliefs: She actively facilitated her pupils' cooperation and communication, because:

> I am convinced that the pupils learn much more when they work in groups autonomously and when they experience that they can solve the tasks for themselves. (Barbara, 2005, interview)

This impact was sustained: Even after the programmes' termination, Barbara provided time for her pupils' open learning:

> This remained: I facilitate their individual work and provide time for this. [...] I have the courage to do so. (Barbara, 2010, interview)

Andy's and Barbara's colleagues' beliefs

Andy and Barbara's participation in IMST[2] initiated a culture of mutual feedback and evaluation at their school. While at first only a few colleagues were convinced, more and more colleagues joined this feedback group as time went on. In 2003, the whole teaching staff decided to establish a school-wide evaluation system: Each teacher conducted a questionnaire or took part in a quality circle of two or three teachers, visiting each other, giving mutual feedback, and regularly discussing teaching and instructional quality.

> All this began with a small questionnaire in my mathematics class, and now we have this feedback system with 120 teachers participating enthusiastically. (Andy, 2005, interview).

In 2010, this system of mutual feedback is persisting. However, the number of participating teachers peaked off, because "now, this immediate need is no longer given. The most important and interesting things are already said" (Andy, 2010, interview). Similar to Andy, Barbara also stated:

> The colleagues can do it, if they want. But this opportunity is no longer used as often as in the first years after the programme's termination. (Barbara, 2010, interview)

There are still about ten active quality circles. In particular, the school's novice teachers gladly make use of this opportunity to learn from their experienced colleagues.

Influencing factors

This section provides findings regarding the factors which emerged as fostering or hindering the sustainability of impact. In particular, the following factors turned out to be effective in the case studies:

Fostering factors

One of the central factors that fostered the sustainability of impact was the engagement of the school's principal. Andy and Barbara stated that she had great interest in their activities: she asked them on a regular basis about their experiences, or about their professional development activities; when returning from IMST[2] seminars or workshops, they felt "like coming home where you are welcome with all your positive and negative feelings" (Andy, 2010, interview); the principal enabled both teachers to present their ideas in several school boards and committees:

> We reported in conferences and staff meetings, so our colleagues could become acquainted with our activities and ideas. And so all this could be developed and sustained. (Andy, 2005, interview)

Barbara sums up this necessary condition:

> The principal must not only tolerate the teachers' activities; a fostering principal has to promote and emphasize professional development – again and again. (Barbara, 2005, interview)

The school had an efficiently organized management structure, which represented another fostering factor: According to the principal, these structures allowed innovations to be disseminated among the teachers, provided access to information and examples of good practice, and facilitated particular working groups by providing time and space for their respective members. "We had a vivid working group of mathematics teachers who actively strove towards high quality teaching" (Principal, 2005, interview).

Another major fostering factor was the teachers' professional environment: Andy and Barbara stated that their colleagues, as well as the principal and the parents gave repeatedly positive feedback. "This continued even in the aftermath of the IMST^2 project" (Andy, 2010, interview). In particular, the pupils' reactions to teachers' activities fostered the sustainability: "They are working highly concentrated; they have fun and are motivated; they make positive experiences in mathematics lessons. All this is very important for me" (Barbara, interview, 2010).

Another factor fostering the sustainability of impact represented the direct usability of innovative practices. For example, Andy collected information for feedback purposes and could react immediately on current classroom conditions. He stated: "I simply like this feedback, which is anonymous, authentic and honest. I don't want to miss this" (Andy, 2010, interview). Additionally, the teachers experienced personal benefit, which also helped the impact persist after the programme's termination. Andy stated:

> Even after two years, this system of mutual classroom visitations is still in progress – without being imposed by the principal or school administration, just because we all know its value. (Andy, 2005, interview)

He concluded: "This is still effective" (Andy, 2010, interview).

Both the teachers and principal highlight the role of the IMST^2 project facilitator as a fostering factor. This expert not only supported the teachers' activities; she acted as a "critical friend": she introduced an external perspective, gave professional oral and written feedback, and provided alternative interpretations regarding the teachers' classroom practices. In particular, she supported each of the teachers individually according to his or her needs (see Jungwirth, 2005). At the same time, both teachers could act independently and autonomously: they were empowered to teach their own way and to frame their individual professional development.

Yet another fostering factor was represented by the IMST^2 workshops and seminars. The teachers stated that these events enabled them to communicate and network with colleagues from other schools, which was very important for their professional development: "Each of these meetings was both a source of good ideas and a clear confirmation of my own work" (Andy, 2010, interview).

Hindering factors

A hindering factor was the decreasing collegial engagement. During the teachers' participation in IMST^2, Andy and Barbara's colleagues were highly interested and keen to cooperate with them. Barbara stated that, as time went by, this engagement decreased for several reasons: On the one hand, the novelty was gone: "By and by, an innovation is no longer new; it is no longer something special; rather one thing among others" (Barbara, 2010, interview). On the other hand, teachers who were not interested from the very beginning continued articulating their opposing perspectives: "There are always some colleagues, who don't think much of things like that; not everybody likes mutual classroom observations and feedback groups" (Barbara, 2010, interview). So the impact on the colleagues' practice level decreased over time (see above).

DISCUSSION

The programme impacted the teachers' beliefs. Most, but not all, of this impact was sustainable: Andy's beliefs changed in a sustainable way regarding his self-esteem and his awareness concerning the importance of explanations. Barbara's beliefs were changed sustainably in terms of a more reflective stance towards her teaching, a higher appreciation of feedback, and a heightened awareness of her pupils' cooperation. On the level of the teachers' colleagues' beliefs, the impact regarding a culture of mutual feedback could not be sustained in full extent.

The factors that fostered the sustainability of these impacts are quite similar to those provided by Tatto and Coupland (2003). They conducted a literature review concerning the change of teachers' beliefs. Besides others, they found that "beliefs are expected to change if educational interventions provide more and better . . . opportunities for reflection either individual or with peers" and "opportunities for understanding one-self vis-à-vis challenging and novel situations in a secure environment" (p. 177).

The findings provided in this chapter mainly refer to impact on the respective teachers' beliefs.[4] However, when analysing a professional development programmes' impact, it is not always fully clear whether some impact can be categorised on the knowledge, beliefs, or practice level (see section "Impact" above); there might be some ambiguities or intersections. For example, Barbara changed her belief concerning the value of feedback, which led to a change in her evaluation practice; at the same time, this changed practice again modified and reinforced her beliefs regarding this issue. Thus, when analysing case studies' data, it was highly important to be sensitive to these complex connections. The case studies provided in this chapter used methods of member checking to limit possible ambiguities.

If professional development programmes are designed to have sustainable impact, it is reasonable to carefully examine the factors which could foster (if they are present) or hinder (if they are lacking) this sustainability. If some of these factors are highly interconnected with and dependent on the existence of the programme, then these factors could be substituted for alternative ones that are less connected or not connected at all to the professional development programme's existence. This highlights the significance of fostering factors which are (as much as possible) temporally independent from the professional development programme. Moreover, this also points to the importance of the factor follow-up support opportunities (Ingvarson et al., 2005; Maldonado, 2002; Mundry, 2005), which should be considered in the conception of professional development programmes.

This chapter points to less collegial engagement as a hindering factor for sustainable impact (see above). This finding may seem trivial at first, since it is sometimes natural to have less engagement over time. However, trivial factors particularly tend to be taken for granted and unchangeable (since they seem to be present anyway). This may lead to overlooking chances and opportunities that come along with them. Thus, as a consequence of the cases' results, we can conjecture: Designing and applying some measures for maintaining and preserving collegial engagement can enhance the sustainability of impact. This observation underlines the importance of putting more emphasis on the social factor in mathematics teacher education research (see e.g., Llinares, & Krainer, 2006).

This leads to another similar and highly important question which should be considered in the conception of professional development programmes: Which of the influencing factors can actually be controlled or affected by the professional development programme? (And which cannot?) Maybe the most important factors lie beyond the programme's realm. In this case, it could also be reasonable to look out for alternative and supplementary factors that can be provided and influenced by the programme itself.

In other words: Professional development programmes which strive to cause sustainable impact should be designed by carefully considering the following questions: Which factors are dependent from the programme itself? Which factors are located beyond the programme's realms?

SUMMARY

The findings provided in this chapter point to content, community, and context as central issues; in particular, when analysing professional developments programmes' sustainable impact, these three Cs seem to be of crucial importance:

- Content: Both case study teachers' goals were to learn about their own teaching and to change their instructional practices. The professional development programmes' contents met their respective interests, which again enhanced the sustainability of impact.
- Community: Both teachers exchanged and discussed their experiences with colleagues from their own and other schools, with their professional development programme facilitators, and with university staff. This social networking made them part of a community and enhanced the sustainability of impact. In turn, less collegial engagement turned out to be a hindering factor concerning the professional development programme's long-term effects.
- Context: Both teachers were embedded in a supportive environment. The principal ensured in-school support, while the professional development programme provided various levels of support beyond the particular school. Both in-school and beyond-school support turned out to foster the sustainability of impact.

The next step should be to conduct further research and evaluation to get new results regarding the relevance of these factors. These findings should be again integrated into the conception of future programmes.

NOTES

[1] This chapter is based on Zehetmeier and Krainer (2011, see references). However, it is modified and extended to a second case study and is putting a particular emphasis on the issue of teachers' beliefs.

[2] These two case studies were chosen due to convenience reasons: They are the ones whose data are already analysed. Upcoming analysis of the remaining case studies' data will allow thorough discussion

of communalities and differences concerning the respective cases' impacts and fostering (or hindering) factors.

[3] Theachers' names are pseudonyms. However, all data were cross-checked by them.

[4] The major indicators for describing teachers' learning are their knowledge, beliefs, and practice (see section "Impact" above). Further impacts on the knowledge and practice levels are provided in Zehetmeier and Krainer (2011).

REFERENCES

Ainley, J., & Luntley, M. (2005). What teachers know: The knowledge base of classroom practice. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (pp. 1410–1419). Sant Feliu de Guíxols, Spain: European Research in Mathematics Education.

Altrichter, H., & Posch, P. (2007). *Lehrerinnen und Lehrer erforschen ihren Unterricht* [Teachers researching their practice]. Bad Heilbrunn, Germany: Klinkhardt.

Anderson, S., & Stiegelbauer, S. (1994). Institutionalization and renewal in a restructured secondary school. *School Organization*, *14*(3), 279–293.

Benke, G., Erlacher, W., & Zehetmeier, S. (2006). Miniaturen zu IMST^2 [Miniatures of the IMST^2 project]. Unpublished paper. Alpen-Adria-Universität Klagenfurt, Austria: Institute of Instructional and School Development.

Bromme, R. (1997). Kompetenzen, Funktionen und unterrichtliches Handeln des Lehrers [Expertise, tasks and instructional practice of teachers]. In F. Weinert (Ed.), *Enzyklopädie der Psychologie. Band 3. Psychologie des Unterrichts und der Schule* (pp. 177–212). Göttingen, Germany: Hogrefe.

Brühlmeier, A. (2010). *Head, heart and hand. Education in the spirit of Pestalozzi*. Cambridge, MA: Open Book Publishers.

Carrington, S., Deppeler, J., & Moss, J. (2010). Cultivating teachers' beliefs, knowledge and skills for leading change in schools. *Australian Journal of Teacher Education*, *35*(1), 1–13.

Creswell, J. (2007). *Qualitative inquiry & research design. Choosing among five traditions* (2nd edition). Thousand Oaks, CA: Sage Publications.

Da Ponte, J., & Chapman, O. (2006). Mathematics teachers' knowledge and practices. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 461–494). Rotterdam, the Netherlands: Sense Publishers.

Datnow, A. (2005). The sustainability of comprehensive school reform models in changing district and state contexts. *Educational Administration Quarterly*, *41*(1), 121–153.

DEZA – Direktion für Entwicklungshilfe und Zusammenarbeit. (2002). *Glossar deutsch* [German glossary]. Bern, Switzerland: DEZA.

Ernest, P. (1989a). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249–254). London: Falmer Press.

Ernest, P. (1989b). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching*, *15*(1), 13–33.

Fullan, M. (2006). The future of educational change: System thinkers in action. *Journal of Educational Change*, *7*, 113–122.

Garet, M., Porter, A., Desimone, L., Birman, B., & Yoon, K. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal*, *38*(4), 915–945.

Guskey, T.R. (2000). *Evaluating professional development*. Thousand Oaks, CA: Corwin Press.

Hancock, D., & Algozzine, B. (2006). *Doing case study research*. New York, NY: Teachers College Press.

Hargreaves, A., & Fink, D. (2003). Sustaining leadership. *Phi Delta Kappan*, *84*(9), 693–700.

Haslauer, E. (2010). *Teachers' knowledge, beliefs, and classroom practices in relation to English language learners in mainstream classrooms in Midwestern schools*. Dissertation. Madison, WI: University of Wisconson.

Ingvarson, L., Meiers, M., & Beavis, A. (2005). Factors affecting the impact of professional development programs on teachers' knowledge, practice, student outcomes and efficacy. *Education Policy Analysis Archives*, *13*(10), 1–28.

Jungwirth, H. (2005). Die Arbeit mit den Schulen. Beispiele aus Mathematik. In IUS (Ed.), *Ergebnisbericht zum Projekt IMST² 2000–2004* (pp. 146–153). Alpen-Adria-Universität Klagenfurt, Austria: Institute of Instructional and School Development.

Krainer, K. (2003). Innovations in mathematics, science and technology teaching (IMST2): Initial outcome of a nation-wide initiative for upper secondary schools in Austria. *Mathematics Education Review*, *16*, 49–60.

Krainer, K. (2005). Pupils, teachers and schools as mathematics learners. In C. Kynigos (Ed.), *Mathematics education as a field of research in the knowledge society. Proceedings of the First GARME Conference* (pp. 34–51). Athens, Greece: Hellenic Letters (Pubs).

Krainer, K. (2006). How can schools put mathematics in their centre? Improvement = content + community + context. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlková (Eds.), *Proceedings of 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 84–89). Prague, Czech Republic: Charles University.

Krainer, K. (2008). Innovations in mathematics, science and technology teaching. In J. Vincent, J. Dowsey, & R. Pierce (Eds.), *Connected Maths. MAV Annual Conference 2008* (pp. 199–212). Brunswick, Vic: The Mathematical Association of Victoria (MAV).

Krainer, K. (2011). Teachers as stakeholders in mathematics education research. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 47–62). Ankara, Turkey: Middle East Technical University.

Krainer, K., & Wood, T. (Eds.). (2008). Participants in mathematics teacher education: Individuals, teams, communities and networks. *International handbook of mathematics teacher education*, Vol. 3. Rotterdam, the Netherlands: Sense Publishers.

Krainer, K., & Zehetmeier, S. (2008). *What do we know about the sustainability of professional development programmes?* Paper presented in Discussion Group 7 at ICME 11 in Monterrey, Mexico (July 7, 2008).

Lachance, A., & Confrey, J. (2003). Interconnecting content and community: A qualitative study of secondary mathematics teachers. *Journal of Mathematics Teacher Education*, *6*, 107–137.

Leder, G. (2008). Beliefs: What lies behind the mirror? In B. Sriraman (Ed.), *Beliefs and Mathematics: Festschrift in honor of Guenter Toerner's 60th birthday* (pp. 39–54). Missoula, MT: Information Age Publishing.

Leder, G., Pehkonen, E., & Törner, G. (2002). *Beliefs: A hidden variable in mathematics education?* Dordrecht, the Netherlands: Kluwer Academic Publishers.

Liljedahl, P. (2008). *Teachers' beliefs as teachers' knowledge*. Paper presented at the Symposium on the Occasion of the 100th Anniversary of ICMI (Rome, 5–8 March 2008).

Lipowsky, F. (2004). Was macht Fortbildungen für Lehrkräfte erfolgreich? [What makes teacher professional development successful?]. *Die Deutsche Schule*, *96*, 462–479.

Llinares, S., & Krainer, K. (2006). Mathematics (student) teachers and teacher educators as learners. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education. Past, present and future* (pp. 429–459). Rotterdam, the Netherlands: Sense Publishers.

Loucks-Horsley, S., Stiles, K., & Hewson, P. (1996). Principles of effective professional development for mathematics and science education: A synthesis of standards. *NISE Brief*, *1*(1), 1–6.

Maaß, J., & Schlöglmann, W. (2008). *Beliefs and attitudes in mathematics education: New research results*. Rotterdam, the Netherlands: Sense Publishers.

Maldonado, L. (2002). Effective professional development. Findings from research. Retrieved from http://www.collegeboard.com, accessed August 16, 2012.

Mason, J. (2004). Are beliefs believable? *Mathematical Thinking and Learning*, *6*(3), 343–351.

Mayring, P. (2003). *Qualitative Inhaltsanalyse* [Qualitative content analysis]. Weinheim, Germany: Beltz.

Merriam, S. (2001). *Qualitative research and case study applications in education*. San Francisco, CA: Jossey-Bass Publishers.

Mundry, S. (2005). *What experience has taught us about professional development*. National Network of Eisenhower Regional Consortia and Clearinghouse.

Pegg, J., & Krainer, K. (2008). Studies on regional and national reform initiatives as a means to improve mathematics teaching and learning at scale. In K. Krainer & T. Wood (Eds.), *International handbook of mathematics teacher education, Vol. 3: Participants in mathematics teacher education: Individuals, teams, communities and networks* (pp. 255–280). Rotterdam, the Netherlands: Sense Publishers.

Rogers, E. (1995). *Diffusion of innovations*. New York, NY: Free Press.
Rogers, E. (2003). *Diffusion of innovations* (5th edition). New York, NY: Free Press.
Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255–281.
Schoenfeld, A. (1987). *Cognitive science and mathematics education*. Hillsdale, NJ: Lawrence Erlbaum Associates.
Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–22.
Song, E., & Koh, K. (2010). *Assessment for learning: Understanding teachers' beliefs and practices*. Paper presented at the 36th International Association for Educational Assessment (IAEA) Annual Conference in Bangkok, Thailand, 22–27 August, 2010.
Sowder, J. (2007). The mathematical education and development of teachers. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 157–223). Greenwich, CT: NCTM.
Stake, R. (1995). *The art of case study research*. Thousand Oaks, CA: Sage Publications.
Stein, M. K., Smith, M. S., & Silver, E. A. (1999). The development of professional developers: Learning to assist teachers in new settings in new ways. *Harvard Educational Review*, *69*(3), 237–269.
Stockmann, R. (1992). *Die Nachhaltigkeit von Entwicklungsprojekten* [The sustainability of development projects]. Opladen, Germany: Westdeutscher Verlag.
Tatto, M. T., & Coupland, D. B. (2003). Teacher education and teachers' beliefs: Theoretical and measurement concerns. In J. Rath & A. C. McAninch (Eds.), *Teacher beliefs and classroom performance: The impact of teacher education* (pp. 123–181). Greenwich, CT: Information Age.
Thompson, A. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, *15*, 105–127.
Törner, G. (2002). Mathematical beliefs – A search for a common ground: Some theoretical considerations on structuring beliefs, some research questions, and some phenomenological observations. In G. C. Leder, E. Pehkonen, & G. Törner, (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 73–94). Dordrecht, the Netherlands: Kluwer Academic Publishers.
Yin, R. (2003). *Case study research: Design and methods* (3rd edition). Thousand Oaks, CA: Sage Publications.
Zehetmeier, S. (2008). *Zur Nachhaltigkeit von Lehrer/Innenfortbildung* [The sustainability of teacher professional development]. Doctoral Thesis. Klagenfurt, Austria: Alpen-Adria-Universität Klagenfurt.
Zehetmeier, S. (2010). Aktionsforschung in der Lehrerfortbildung: Was bleibt? [Action research in teacher education. What remains?] In F. H. Müller, A. Eichenberger, M. Lüders, & J. Mayr (Eds.), *Lehrerinnen und Lehrer lernen. Konzepte und Befunde der Lehrerfortbildung* (pp. 197–211). Münster, Germany: Waxmann.
Zehetmeier, S. (in preparation). *Tracking the sustainable impact of professional development*.
Zehetmeier, S. & Krainer, K. (2011). Ways of promoting the sustainability of mathematics teachers' professional development. *ZDM – The International Journal on Mathematics Education*, *43*(6/7), 875–887.

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11. CAPTURING MATHEMATICS TEACHERS' PROFESSIONAL DEVELOPMENT IN TERMS OF BELIEFS

INTRODUCTION

This article deals with two prominent topics in mathematics education: the role of beliefs about mathematics and its teaching and learning, and the continuous professional development of mathematics teachers. Besides providing some research findings to connect these areas, the purpose of this paper is to honor two researchers who specifically influenced my work and research interests.

First, I would like to start by sharing a little anecdote about a car ride with Alan Schoenfeld and Günter Törner. It was 2004, the year that began my research career, and I was eager to discuss aspects of beliefs while travelling with Alan and Günter. At some point, I asked Alan what he thought of the relationship between knowledge and beliefs, and he gave a nice illustration of how those variables interconnect from his point of view:

> Alan took a piece of paper and made a propeller out of it, which he then let fall down. The propeller spun around and sank to the floor. Then he asked me for explanations why the propeller took such a long time to get to the ground in comparison to a normal sheet of paper. My answers were concentrated on finding some physical explanations. Next, he said, *ok, you provided some knowledge on physics that might explain what you observed, but why did you not say it is magic? Although you have knowledge of magic you did not give this as an explanation because you do not believe in magic. That is how knowledge and beliefs go together*.

Later on, I learned more about Alan's theory of *Teaching-in-Context* and how the relationship between knowledge, goals, and beliefs can be seen from a researcher's prospective. However, I have always kept in mind this nice story that contributes such exploratory character.

Second, I would like to share a discussion that I had with Günter Törner while designing the professional development initiative *Mathematics Done Differently*, which will be elaborated on later. The main idea of the project has been to provide in-service training courses that really address what teachers need:

> At the very beginning of the initiative, Günter seriously contemplated the question, *What do teachers want to have in their in-service training courses?* Günter used the following metaphor to explain his point of view: *Assuming that we opened a shop in Berlin, on the famous street "Unter den Linden," to offer professional development products for teachers, we should ask ourselves what*

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> *we ought to put on the shelves*. Then he said that he did not know. Of course he had some ideas in mind but these were mostly based on *his* experiences and *his* theoretical background. So, he argued that we needed to ask the teachers themselves.

Thus, a questionnaire was distributed among teachers from all school types and approximately 1800 data sets were gathered.The analysis yielded huge amounts of information about teachers' *retrospective* experiences concerning their professional development as well as their *prospective* views, which they expressed in terms of needs and expectations (Roesken, 2011).

In this chapter, theoretical aspects will be discussed that provide a framework covering mathematics teacher professional development from a teacher's perspective. I will elaborate on why beliefs are a decisive parameter for discovering teachers' decision-making processes even when aspects of their professional development are under investigation. In particular, Alan Schoenfeld's framework will provide theoretical lenses through which to explore issues not only related to teachers actions in the classroom, but also related to their professional development.

The previously mentioned professional development initiative *Mathematics Done Differently* will be outlined. The philosophy of the project echoes Günter Törner's ideas and theoretical approaches to mathematics teachers' professional development. The data presented was gathered from teachers participating sessions provided by the project. Specific attention will be paid to the role of teachers' previous experiences, beliefs, and variables that affect any change processes. In this article, ideas that were initially presented in my dissertation work are further developed to capture aspects of teacher professional development in terms of beliefs (cf. Roesken, 2011).

MATHEMATICS TEACHER PROFESSIONAL DEVELOPMENT: ATTENDING TO THE ROLE OF BELIEFS

Educational reforms constitute demands that teachers are supposed to meet given that they are assigned a decisive role for gaining improvements in the classroom (Garet, Porter, Desimone, Birman, & Yoon, 2001). In most countries, changes in education have taken place that brought different issues to the agenda, for instance, learning standards for students and professional standards for teachers. Such new trends and developments have resulted in output orientations, derived from reforms in education, politics and research (Cooney, 2001; Day & Sachs, 2004, Cochran-Smith & Zeichner, 2005). In this context of change, balancing the efforts to meet *the needs of the system* and the *needs of teachers* within it (Day, 1997; Krainer, 2001) is one of the biggest challenges. When viewing professional development from a teacher's perspective, the starting point is daily practice and instruction (Cochran-Smith & Lytle, 2001; Roesken, Hoechsmann, Törner, 2008). Regarding teaching quality, Krainer (2005) concludes that the teachers themselves have to work incessantly for what constitutes good mathematics teaching. Taking this perspective seriously, teachers' needs define what constitutes appropriate professional development (Roesken, 2011).

Manifold variables are discussed in the literature in the field of professional development (Goodson & Hargreaves, 2003). However, teachers' knowledge, beliefs, and instructional strategies are the focus when the impact of initiatives is questioned (Day, 1999; Sowder, 2007; Zehetmeier & Krainer, 2012). Another area of research addresses the affective domain acknowledging that human learning in general can be described by the three components: cognition, motivation, and emotion (Meyer & Turner, 2002). In his 1992 Handbook chapter McLeod defines the affective domain as encompassing emotions, attitudes, and beliefs. An overview on how to distinguish the concepts is given by Philipp (2007) who, based on a literature review, defines affect as "a disposition or tendency or an emotion or feeling attached to an idea or object" (p. 259). Philipp (2007) further takes up the idea by McLeod that "affect is comprised of emotions, attitudes, and beliefs" (p. 259). Nevertheless, most research addressing these psychological categories of the mind has been carried out by elaborating on one of those (Hannula, 2004).

Beliefs, for instance, play a major role in Schoenfeld's (1998) theory of *Teaching-in-Context*, which models teaching primarily as function of a teacher's knowledge, goals, and beliefs. Taking Schoenfeld's early work on problem solving as a starting point, the following definition of beliefs can be considered:

> Belief systems are one's mathematical world view, the perspective with which one approaches mathematics and mathematical tasks. One's beliefs about mathematics can determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. Beliefs establish the context within which resources, heuristics, and control operate. (1985, p. 45)

In his later work, Schoenfeld concretizes the role of beliefs as follows:

> Beliefs can be interpreted as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior. (1992, p. 358)

> Beliefs are mental constructs that represent the codifications of people's experiences and understandings. (1998, p. 19)

One outcome of such a theoretical approach lies in "identifying practices and knowledge that support desired kinds of teaching, as well as tools for examining various forms of professional development and their impact" (Schoenfeld, 1999, p. 6). As implication for professional development, these parameters serve as tools to identify practice, provide information about how several issues interact, and how the dynamic can be influenced. In his recently published book, Schoenfeld (2010) replaces the concepts of knowledge and beliefs by resources and orientations, and he points out that the latter category consists of beliefs, values, biases, and dispositions. More specifically, he underlines the significance of the following aspects as theoretical basis for capturing mathematics teachers' professional development:

> The notion of orientation/resource/goal clusters is a lens through which teacher activity can be examined – and studies of coherence and change along these

> dimensions could be very interesting and useful. Specifically, research on professional development can seek to document the evolution of teachers' orientations, resources and goals as the teachers work at changing their practice. (p. 194)

Törner, Rolka, Roesken, and Sriraman (2010) give an example of how a teacher's actions in the classroom can be understood by looking through Schoenfeld's theoretical lenses, and which decisive role beliefs can play in the context of professional development.

We all know Cohen's (1990) impressive example for the constraints of professional development by his well-known case study on a teacher named Mrs. Oublier. The portrayal, as Sowder (2007) puts it, serves as "a generic description of a class of teachers who have misinterpreted the principles underlying the professional development they received" (p. 160). Mrs. Oublier was very open to implementing new curriculum material and activities, that is, "she eagerly embraced change, rather than resisting it. She found new ideas and materials that worked in her classroom, rather than resisting innovation" (Cohen, 1990, p. 311). However, the change initiated by the obtained professional development remained at the surface (cf. Pehkonen & Törner, 1999). Accordingly, Cohen (1990) concludes that

> [Mrs. Oublier's] teaching does reflect the new framework in many ways. For instance, she had adopted innovative instructional materials and activities, all designed to help students make sense of mathematics. But Mrs. O. seemed to treat new mathematical topics as though they were part of traditional school mathematics. (p. 311)

What is striking is that although the teacher was open to new approaches, well-established beliefs, knowledge, routines and scripts were not simply replaced. Instead, the new experiences were added or assimilated. Sowder (2007) claims the following point was crucial:

> Mrs. Oublier had little opportunity for sustained guidance and support. She had much to unlearn, but no one to help her do this unlearning. The lessons here for the need for sustained professional development and mentoring are significant. (p. 160)

Pehkonen and Törner (1999) report on a similar observation and stress the influence of the established teaching style as follows:

> Teachers can adapt a new curriculum, for example, by interpreting their teaching in a new way, and absorbing some of the ideas of the new teaching material into their old style of teaching. (p. 260)

Again, it is the old style of teaching based on established knowledge and beliefs that impedes any change in teaching practice. A comparable case study is reported by Törner, Rolka, Roesken and Schoenfeld (2006), who analyze the teaching practices of an experienced teacher after having participated in an in-service training course on using open-ended tasks in mathematics teaching. Since it was not the focus of the study to examine the effectiveness of the professional development event, it turned out that the teacher's beliefs were a hindrance to a successful implementation of new ideas. Törner et. al (2006) showed how old beliefs established over a long period conflict with new beliefs adopted recently. Those examples show how beliefs can play an important role as hidden variables in the field of professional development and that it is crucial to keep their significance in mind.

Regarding the processes that are decisive for successfully implementing issues of professional development, Clarke and Hollingsworth (2002) remind us that from a historical viewpoint "teacher change has been directly linked with planned professional development" (p. 948). However, teacher change has become a critical issue in recent years. For good reasons, the question *Who has the agency?* has been addressed in the discussion about change processes (Hannula, Liljedahl, Kaasila, & Roesken, 2007; Sullivan, 2007). The perspective taken in this chapter is that we cannot change another person. Likewise, Day (1999) puts it, "teachers cannot be developed (passively). They develop (actively)" (p. 2). All we can do is to provide opportunities for teachers to change; the teachers themselves hold the "ownership of change" (Sullivan, 2007, p. 152).

Clarke and Hollingworth (2002) identify perspectives on change processes, among them "change as growth or learning" (p. 948) as primarily aligning with current professional development efforts. Likewise, Sowder (2007) emphasizes that change is a "process rather than an event, it must be considered in terms of continuous growth over time" (p. 97). Again it is Schoenfeld (2000) who reminds us that "teacher knowledge leads naturally to the issue of growth and change of teacher-knowledge – and hence to issues of teacher learning and professional development" (p. 20). Pehkonen and Törner (1999) add that these processes are dependent on personal factors: that is, any development may vary in pace according to a teacher's personality:

> Everybody who has worked in teacher in-service training has surely recognized the following odd situation: there are some teachers who have reached the pedagogical goal of the in-service course already in the very beginning. And on the other hand, there might be some teachers who have difficulties adapting to the first ideas. (p. 261)

In this chapter, the focus will be on premises for teacher change and the influential parameter of beliefs, which impacts on teachers' choices regarding professional development. In particular, data will be presented that was collected during the course of *Mathematics Done Differently*, a professional development initiative that was run in Germany (Roesken & Törner, 2008; Roesken, 2011), and was designed specifically with respect to teachers' needs.[1]

MATHEMATICS DONE DIFFERENTLY

This chapter is concerned with presenting the initiative *Mathematics Done Differently* for fostering mathematics teachers' professional development for which Juerg Kramer, Humboldt-Unversität zu Berlin, and Günter Törner, Universität Duisburg-Essen, were responsible. The project began in September 2006, lasted until fall 2011 and was then successfully transferred into a larger project concerned with the establishment of a new *German Center for Mathematics Teacher Education* (DZLM[2]) in Germany.

As the foundation of *Mathematics Done Differently*, data was collected in October 2006 using an online questionnaire that was distributed among teachers in Germany so as to acquire important information about their needs and expectations. Initial courses addressed those needs in different thematic fields. The courses were offered on the homepage and made available for interested teachers. *Deutsche Telekom Stiftung*, a foundation related to the company that is internationally known as t-mobile, funded the initiative. Many of the partners involved in the project are experts from psychology and pedagogy departments who support all relevant decisions so that mathematicians, educators, and psychologists may work and bring together knowledge from different but related disciplines.

Aims and scope

The initiative gathers in-service training courses that have already been conducted successfully in Germany and makes them available nationwide. One main concern has been not to "reinvent the wheel" but to consider and involve expertise from colleagues in the form of already established professional development offers. Of course, there are many acknowledgeable initiatives in Germany. Unfortunately, most of them are only locally known. Thus, one intention of the project was to provide a new and comprehensive platform announcing successful approaches nationwide. While the educational system in Germany is decentralized due to federalism, *Mathematics Done Differently* has aimed to expand opportunities for professional development by spreading and broadening regional programs. Additionally, the project seeks to meet the unique needs of teachers and consequentially, courses are designed regarding teachers' specific needs.

Tandem approach

There has been a clear shift in European mathematics teacher education to elaborate on both teacher education as a field of practice and as research (Adler, Ball, Krainer, Lin, & Jowotna, 2005). This trend is reflected in one of the main parameters of the initiative since a *tandem* offers the in-service training course, consisting of a university teacher for mathematics or mathematics education and a practicing teacher. From their very design, courses are sure to combine research and practice, i.e., research knowledge and teacher knowledge. Evaluation data shows that this aspect is particularly highly valued by teachers (Roesken, 2011). They are interested in new developments in research but appreciate that a colleague also takes care that practical issues are not neglected (cf. Roesken & Törner, 2008). However, the role of theory is to provide an interpretative framework that encompasses experiences in practice and allows for initiating and guiding reflection. The school teacher is assigned a supporting role while principally pursuing issues relevant to practice. Since the views of teacher educators and teachers typically differ, the various viewpoints and accentuations contribute to a comprehensive picture of the single topic.

Course system: Courses à la carte and courses on demand

The courses offered by *Mathematics Done Differently* comprises courses both à la carte and *courses on demand*, i.e. we address supply and demand-oriented professional development. Courses *à la carte* focus on subject matter knowledge, pedagogical content knowledge or methodology. Contrary to traditional settings, where teachers are primarily expected to change their teaching to more or less explicit goals formulated by the implementer, these courses are not rigidly designed, but flexible enough to take into consideration specific teacher concerns. All courses endeavor to offer possibilities for development and enhancement of teachers' knowledge, beliefs and instructional strategies.

At present, the project homepage[3] offers more than 40 courses that address currently interesting topics like *Geometry unplugged* or *Don't be afraid of stochastics* refer. However, topics like discrete mathematics, modeling, technology, open-ended tasks, educational standards, and interdisciplinary courses can be booked. Courses that explicitly provide a different view on mathematics and its teaching and learning complete the range of options. One particular objective is not only to recommend cognitive challenging courses, but to also address teachers' beliefs, which are crucial for any developmental processes as shown in the theory (cf. Schoenfeld, 2010).

Courses on demand are explicitly designed for teachers' needs. Teachers have even more input in planning their professional development when asking for a specific course that is not comprised in the online suggestions. These courses are rather demanding to organize and to design since they initiate a time consuming procedure encompassing the following steps:

- supporting and encouraging teachers to specify their needs
- classifying the request with regard to research
- reviewing literature
- searching for experts that may serve as trainers
- designing the course
- offering the course on the project homepage

Collaborative work

Courses are not offered for single teachers but for groups of no less than 15 teachers from either one school or neighboring schools. Ideally, the majority of mathematics teachers from a single school participate in the course. That is, the project clearly intends to initiate staff development in school. Issues imparted at an in-service training are not likely to be transferred into the classroom when the teacher obtains no support from colleagues or is even criticized for the innovative approach (Roesken & Törner, 2008). Even though teachers apply for a course as a group, a particular teacher is responsible for the organization of the in-service training on-site and for getting into contact with the trainers. This includes scheduling the course as well as negotiating the specific needs of the group. The prearrangement is

Table 1. Amount of courses offered by Mathematics Done Differently.

Project Year	*Courses*	*Participants*
1st Year	8	150
2nd Year	77	1,634
3rd Year	143	3,024
4th Year	103	1,689
5th Year	75	2,160
Total	406	8,657
Waiting List	62	
Total	468	

rather important in order to make the course precisely match the needs of the group of teachers.

Flexibility of the project

One essential conclusion of the project was that we, as teacher educators, are part of a learning system. To design and run a project is not a static endeavor, but part of a developing process that must be refined and adjusted (Roesken & Törner, 2008). It is crucial that we document our own process of revising the initial approach and acknowledge that we are also developing professionally (cf. Loucks-Horsley et al., 2003). The learning system results from the fact that the involved parties enter an educational discourse that might be rather controversial and in which both the teacher educators and the teachers involved take the position of experts and learners (Roesken, 2011). Our learning processes as implementers of the project can be linked to the flexibility of the program which we were able to provide.

The so-called philosophy of our project has also evolved. Though our intention was surely not guided by compensating for deficits, we started by rather naively acknowledging the role of teachers as learners (cf. Roesken & Törner, 2008). In the initial project phase, the courses were mostly supply-oriented, even though the proposed themes were left rather open with regard to individual implementation. Over the time, due to project presentations and conversations with teachers, the demand-oriented approach has become increasingly important.

Overview of the course situation

Meanwhile, the project has been run for five years and, as mentioned earlier, has been successfully implemented into the new *German Center for Mathematics Teacher Education*. In Table 1 we give an overview which gives insight to how many courses have been provided during the five years of the project and how many are on the waiting list.

Table 2. Overview on how many courses have been performed in the different Federal States of Germany.

Federal State	*Courses performed*
Baden-Wuerttemberg	19
Bavaria	17
Berlin	36
Brandenburg	13
Bremen	4
Hamburg	5
Hesse	25
Mecklenburg-Vorpommern	21
Lower Saxony	39
North Rhine-Westfalia	198
Rhineland-Palatinate	44
Saarland	1
Saxony	21
Saxony-Anhalt	10
Schleswig-Holstein	8
Thuringia	0

More than 8,000 teachers have participated in the project so far. However, the courses are not of the type "one size fits all" but teachers can negotiate their specific requirements in their individual schools with trainers. The educational system in Germany is rather fragmented compared to other countries. Since lower secondary education comprises three different school types with respect to students' abilities, in-service teacher training needs to be adequately aligned. The training courses do not only address school types from primary to upper secondary school separately, but also the interfaces between them. For instance, the course *Children Invent Mathematics* brings together kindergarten educators and primary school teachers and contributes to an exchange among them.

Table 2 provides information about how the courses are distributed in the different Federal States.

The professional development courses have reached 15 of the 16 Federal States, an indication that the initiative is appreciated nationwide. We also presented short-term workshops at specific events in order to gain a wider audience for our professional development initiative.[4]

SYNTHESIS AND RESEARCH QUESTIONS

The empirical data that will be presented in the following is part of a larger study that captures dimensions relevant for mathematics teachers' professional development (cf. Roesken, 2011). In this chapter, the focus is on teachers' previous experiences and beliefs related to professional development, and how those might support or hinder any change processes. In particular, the following research questions were pursued:

– What previous experiences influence the reception of current professional development events?
– What beliefs do teachers hold towards professional development?
– What affective issues are relevant and how do teachers perceive any change processes?

EMPIRICAL APPROACH AND METHODOLOGY

In connection with the project *Mathematics Done Differently*, first *quantitative* data was gathered concerning teachers' experiences, expectations and needs regarding professional development. Second, *qualitative* data was collected by conducting interviews with teachers throughout the course of the project. Third, observations made while monitoring many of the in-service training courses completed the overall picture. The focus of the empirical study presented in this chapter is on the second aspect; for a comprehensive overview see Roesken (2011).

Basic principles and methodological justification

It has been of particular concern to use the words of the teachers themselves to show what professional development looks like from their perspective. Thus, interviews were used which allow for giving broad insight into the teachers' perspectives. For their quality, interviews rely on the nature of the interactions between the persons involved; this also includes the interviewers themselves (Partington, 2001). Kvale (1996) refers to the literal meaning of the term interview as actually being an *inter-view*, for example people exchanging their views on a specific topic create a socially situated interaction in which knowledge evolves by dialogue.

All interviews were semi-standardized. An interview guide, indicating the topics to be covered and their sequence in the interview, was employed (cf. Kvale, 1996). The guide contained a general outline of the relevant topics as indicated by the research questions.

Collection of data

The interviews were scheduled to last about 40 minutes. In reality, the interviews varied from 20 to 60 minutes; the setting was dependent on the teachers' preference. Nine randomly chosen teachers who applied for an in-service training course provided by *Mathematics Done Differently* participated in the interview study; three of them were male. Most of the teachers were interviewed in a room of their own school and during their working hours. Teachers were from different school types encompassing both primary and secondary education. The age range was from 32 to 61, yielding an average of 50 years. Most teachers were experienced, five of them looked back on more than 20 years of teaching, three of them on more than ten years while one teacher was rather young and possessed only four years of work.

All teachers were assigned some special role within the school community. Some of them were quite familiar with being actively engaged in professional development, for instance filling the role of a teacher leader.

Interviews were undertaken in the German language by the author of the chapter. Responses of participants were recorded on tape and later carefully transcribed verbatim. In any case, a student assistant provided a first transcript that afterwards was conscientiously reviewed by the author and partly retyped. Those parts selected to be subjected to intensive analysis were then translated into English. The aim was to translate literally as far as possible, but also in an accessible way. However, the data analysis also implicated listening to the recorded interviews several times.

RESULTS OF THE EMPIRICAL STUDY

The data were explored by content analysis to generate categories for the various descriptions and explanations provided by the teachers interviewed. Variants in content analysis are huge and have been discussed intensively in the research literature. The content analysis applied to the present interview data encompassed both categories that were initially derived deductively during a theory-driven approach to the data and categories that were inductively derived while supplementarily emerging from the data. That is, the formulation of categories was guided by the research questions and the pre-existing dimensions as provided by the quantitative analysis (Roesken, 2011), which cannot be presented in this article. The presented data is restricted to encompass the categories *previous experiences*, *beliefs*, and *affective issues*, and their possible effect on the process of change.

The analysis of the qualitative interviews encompasses the following steps as introduced by Lamnek (2005, p. 402):

(a) Transcription
(b) Single analysis
(c) General analysis
(d) Control phase

All interviews were analyzed individually. This process included marking the significant text passages to make them accessible for the content analysis.

Teachers' previous experiences, beliefs, and affective issues

In the following, all data will be presented as anonymized comments, i.e. the names that occur in the analysis are not the original ones. Teachers' statements are not simply presented, but also equipped with background information that helps the reader understand the relevance of the single quotation.

In the light of teachers' experiences, crucial aspects of professional development that depend on their beliefs are discussed. Formed over a substantial period of time, it is mostly these strongly held beliefs that impede any change processes and developments, and even when unconsciously held these beliefs give rise to a considerably reserved attitude, as the following example will show.

Previous experiences

Deborah requested a demand-oriented course for her mathematics department that was designed for the specific needs of her school. For her, it was important to agree in advance on what the course could offer with respect to the needs of her mathematics team. When she looks back on different experiences with in-service training, she states the following:

> *Deborah*: Once again, I surely expect a new attempt from the in-service training course.

She then explains some previous experiences of the staff with in-service training and she mentions the following:

> *Deborah*: In this respect, we had an in-service training course before, whereof the younger colleagues thought rather positively: hm, this is something for me and the elderly ones thought: I won't go there any longer. Therefore, they didn't even participate in the next course because they were partly not really met where they were. Unfortunately, no opening-up, in the sense that I had actually hoped for, did occur.

Deborah mentions a crucial point, i.e. to pick up teachers where they are. Undoubtedly, in-service education follows the rules of learning. According to this, teachers' experiences and their hitherto existing knowledge in particular are decisive for any process of accommodating new information.

Deborah's school is very experienced in being a self-organized school. Issues of autonomously organizing professional development of the entire teaching staff are, therefore, endeavors that have already been practiced successfully. In this respect, the following quotation gives some illuminating hints to related experiences:

> *Deborah*: Well, as a self-organized school we are used, well for five years now, to develop certain rights and duties. We've done that with enthusiasm related to different areas. In this context, we had compulsive in-service training. These courses provided many experiences for the colleagues, some of which they wanted and some of which they didn't want, some that pleased them and some that did not. And now, in retrospect, the head of the school takes the view that if we need in-service training courses then they need to be tailored to the colleagues or the department or the group. Now we've got that, this one is the third one we are organizing for ourselves. Not all went well, and with the one we will get from you, we don't know yet. One went very well, and with one, we fell flat on our faces, although the agreements were very concrete. Again, that led to considerable resentment.

Based on previous experiences, Deborah lends weight to the needs of the teaching staff and emphasizes the preparation of the in-service training course with particular regard to an adequate prearrangement. Deborah also raises an interesting point, namely that experiences in the field are also important for teachers to clearly define what they want and what they do not want. To reach a point where it is possible to announce individual needs is a process that takes time. She further explains the feelings of resentment when the training course does not appropriately meet the needs of the teachers.

> *Interviewer*: You were disappointed then.

> *Deborah*: Yes, I was disappointed because I spent two hours to get the agenda and wishes of the teaching staff across very concretely, and for me it was an enormous disappointment that these wishes were not fulfilled. The colleagues were also disappointed because again, they invested time and actually, they still stayed where they were. In-service training is also time-consuming for teachers, and they feel like wasting their precious time.

Deborah takes up this topic again, and explains further, why teachers are frustrated when in-service training courses do not meet their expectations and needs:

> *Deborah*: Well, we all have a lot to do. That is exactly the point, I think, why teachers are very sensitive in case someone steals their time. It means that they have to stay here, that they have expectations, and that they want to take something with them. When there is nothing, they could have prepared their lessons in that time, or engaged in other forms of developing lessons.

Peter is part of the teaching staff for which Deborah was the main organizer of the in-service training course. He also refers to previous experiences with an in-service training course that went beyond the issues that were relevant for him and the group of teachers. He, thus, explains in few words what he is expecting from the upcoming course:

> *Peter*: And, concerning this subject, we have already attended other in-service training courses, and now we've got this course, and we expect from it that it is better related to our situation, and that we can take something with us.

Peter feels frustrated because of the past events. For him, it is important that the next course will meet his expectations and needs. In particular, he is interested in getting support for dealing with heterogeneity in grade 11 when students come together from different school types, and do not even possess proficiency in basics like calculation of percentages or the rule of three.

Beliefs

Expectations of teachers are high and, as the above-mentioned statements showed, they are mainly based on previous experiences. In this context, the decisive role of beliefs towards professional development must not be neglected. This aspect will be explored by the comments given below. However, it should be noted that these experiences are mostly not acquired in relation to the initiative *Mathematics Done Differently*. Jack for instance, a very experienced teacher who has been teaching for 34 years now, was asked for the most important issue in the context of in-service education and he states the following:

> *Jack*: Once again, to get this idea, to get new incitements.

At first, he generally refers to new incitements as being most central. But in the following, what can be understood from his words is how the previous experiences became decisive for his overall attitude towards in-service education. This attitude is obviously not only acquired with respect to hitherto attended in-service training courses, but reflects the whole conception of his learning:

> *Jack*: What I consider important for an in-service training course is that someone tells me what one can do and not like this, "try it out and try it out again." [. . .] This is not effective.

By these previous experiences, beliefs are clearly accentuated. Jack is very disappointed by a specific type of in-service training. He was then asked what has proved to be effective for him and he announces the following:

> *Jack*: No, in former times I also learnt by listening to somebody who said something to me. [. . .] I listened to it and then I tried it on my own. That's how we learnt at the university. We went to attend the lectures, then we got the exercises, we did the exercises together with colleagues, with students. Why should I change that?

He possesses strong beliefs about his learning processes and needs that arise from those beliefs – which have been built over a long period, even going back to his own learning at university. Not surprisingly, he comments on any process of change and development as follows:

> *Jack*: I only have to work here for four more years, why should I change my methods?

This comment sounds quite disillusioning, but of course, there are developments that have led to such an unconciliable position. In the course of the interview, Jack reports on the many changes that he has encountered in his life as a teacher and which were primarily set from the outside. His resistance to change has been accompanied by trusting his own approach, which, as he indicates, has also proved to be very effective in terms of his students' performances. In the next remark, he tackles a very interesting point:

> *Jack*: My elderly colleagues, who are just a bit older than I am, who just left, they always said: set theory came, set theory went.

By the comment he points out how the teachers of his school reacted when they felt they were not taken seriously by hastily placed educational changes. That is, one consequence that might occur when the needs of teachers are disregarded is that they withstand change. In this context, Jack provides some interesting thoughts concerning the many current developments in education, which also contribute to a better understanding of his position.

> *Jack*: Part of education are calmness and composure, one needs to have time, one has to deal with the children, the juveniles. One needs to have time, so that one helps them to advance, not only subject-specific, and that doesn't work when constantly, always something is adapted. And then there comes something new here and there, which is not properly thought through at all.

Professional development can have many facets and wear many hats; even a position like the one just mentioned is surely not just an individual opinion. Issues that do not reach the realm of the teachers entail learning processes that might be contra-productive for those looking from an outside perspective, and effective from an inside perspective.

Beliefs are highly subjective and therefore vary according to the different bearers. In any discussion, beliefs can be differentiated with respect to the different objects they are attached to. However, the beliefs section so far has been concerned with rather negative influences of belief. In the following example, the inspiring effects and the creative power of beliefs are highlighted.

Edith organized two different in-service training courses for her school, which were part of the *à la carte* program of *Mathematics Done Differently*. The interview was held after she had attended both courses. She points out that she got some insights while attending the in-service training course that were not relevant for her students, but led to new awareness for herself:

> *Edith*: But for me, it was a mathematical highlight that once more pleased me. Well, that it is simply enlightened from a different view, so that one not only preserves the overview from above, but that one sees, aha, there is something more than just the things we are doing, that is really important.
>
> *Interviewer*: That one gets another view on mathematics?
>
> *Edith*: Yes, that one has this meta-level.

Edith raises an interesting point that is clearly related to the issue of beliefs when she mentions how she came to see mathematics in a different way. For her, looking at the subject from a meta-level provides essential awareness and information for her daily work, even though an immediate benefit in terms of concrete teaching advice is not provided. In the following, she explains how her students profit from such an elaborated experience:

> *Edith*: What teacher would I be if I said, “Math ohh,” but instead to make it clear for the students, I say, look at how beautiful it actually is and what things have to do with math, and this is nice, the inspiration. [...] But what is inside the students’ heads, is that in school mathematics, there are so many abstract issues, like formulas and calculating, and they don’t see where math is included in real life. They don’t open their eyes. So the course is good and those are impulses that I even got for myself through the in-service training course.

Likewise, in many of the teachers’ statements getting new insights and ideas is mentioned as a decisive aspect while attending an in-service training course. This aspect goes beyond simply obtaining new information towards yielding a new viewpoint, or even a higher standpoint, as described in Edith’s statements above.

Affective issues

Much research in the field is concerned with the cognitive domain, whether in terms of knowledge or partially in terms of beliefs. What is considerably neglected is the affective domain. In the following we will look at affective aspects that underline how teachers' positive attitudes are primarily influencing any process that takes place in the aftermath of a professional development event. Excerpts point out some interesting coherences. Edith, for instance, describes the these experiences while attending the in-service training course:

> *Edith*: I can really see that they [her colleagues] have had fun and that they were looking forward to, and they even said that they would look forward to the next in-service training course.

Edith notices that her colleagues took much pleasure in engaging during the course and, on top of this, that they were delighted to obtain an additional session. In the next remark, she provides some more information about what processes took place after the course:

> *Edith*: It is amazing, it is really amazing, this "flashlight," it is such a pity that you [the interviewer] did not hear what the colleagues said at the end of the course, colleagues who initially were tired of attending in-service training courses.
>
> *Interviewer*: Those who were actually tired of attending in-service training, what did they say?
>
> *Edith*: Oh, it was terrific, and as I said, I am looking forward to the next in-service training course.
>
> *Interviewer*: Fine.
>
> *Edith*: Or that questions occur like, how can we do that, we could do some team teaching together. Yes, really new ways open up, that colleagues then say, oh couldn't we teach such a lesson together.

In this short interview excerpt, Edith aptly describes how even colleagues that had not actually been very interested in attending an in-service training course, shifted their opinions and were open to getting involved in issues provided by the specific course. Deborah also refers to affective variables in the area of professional development, but with a quite different focus:

> *Deborah*: My department is extremely team-oriented. People support each other, nobody holds back anything. What is even more positive is that everybody is allowed to complain. [. . .] Nobody has to be afraid, that one is looked at disapprovingly and someone thinks: no wonder with him or her, or what else it could be. Because the doors are closed, when you as a teacher disappear into the classrooms, and that I must say is outstanding. We are a group of very young colleagues, very open.

Deborah raises an important issue, i.e. that teaching is a lonely endeavor since the classroom doors are closed. Nevertheless, in her school, the barriers disappear due to exceptional collegial support that contributes to an atmosphere of trust. The in-service training course that is offered according to the specific needs of this group of teachers will be integrated into an already existing supportive atmosphere. That is, teachers have been working together to support each other to enhance their teaching, sharpen the previously existing skills, or try out new teaching approaches. Next, Deborah reports about two teachers who do not possess real openness with respect to their professional development:

> *Deborah*: Although, especially in mathematics the composition of the teaching staff is so that I have two very experienced colleagues that are teachers of the old school, who are not very courageous regarding new ideas, who rather in the first place see the problem: that will go awry, or I would waste too much time on that, or they [the students] will not be able to do so, who are very critical towards others ways, and at the same time often complain that everything fails. To make this discrepancy apparent and to break it open is a difficulty that exists at the moment.

The teachers within the department are, in their own way, open to encountering new ideas, and Deborah's attitude is very sensitive to the different needs of the teaching staff. Obviously, teachers possess an elaborated value system, which makes them too soon resistant to any change processes. Deborah describes this phenomenon aptly, when she states that those teachers see the problem rather than the good idea. Because of their *a priori* critical attitude, these teachers miss out on the chance to gain new understandings.

Changes that are initiated by an in-service training course sometimes might not lead to direct improvement, but entail developments of a more global character. Edith, for instance, reports the following incident that, among others, has arisen from the single in-service training course:

> *Edith*: Well, that is really much, and what, for instance, is a good example is what has risen from this in-service training provided by *Mathematics Done Differently*, what has risen from that for us, [...], is that next year, for instance, we'll get the exhibition "*Mathematik zum Anfassen*," to our school.

With this statement, the teacher illuminates what general movement was generated by the input of the course. As she further explained in the interview, the teaching staff did not only decide to apply for more in-service training courses, but agreed upon several specific events for the school. In particular, they arranged an appointment with the minister of education of North Rhine-Westphalia for visiting the school.[5]

CONCLUSIONS

A crucial aspect is put on the agenda by Malara and Zan (2002), who point out that "most studies are about teachers but not with or for teachers" (p. 554). Thus, the approach of this work is to stress the *with* and *for*. In this respect, the empirical findings shed light on aspects relevant for teachers regarding professional development in general and in-service education specifically. Teachers discussed their previous experiences, revealed their beliefs and provided some insight in affective issues that influence their perception of change processes.

In general, beliefs serve as affordances in mathematics teaching and learning; this observation can also be expanded to professional development. Beliefs are linked to the self-concept of the bearer, and function as a kind of self-assertion, which protects him or her against uncomfortable ideas (cf. Goldin, Roesken, & Törner, 2008). The interview data showed how beliefs and affect could impede an open attitude towards new ideas. Teachers possess experiences related to in-service education that establish various expectations. As one teacher revealed in the interview, her expectations did not only address the topic of the in-service training course but also the possible effects for the staff development.

Drawing back on Schoenfeld's work and the definition he provided in 1992, the role of beliefs in the context of professional development can be captured as follows:

> Beliefs can be interpreted as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in *professional development*.

In particular, the data revealed how beliefs support the individual in conceptualizing and engaging in professional development. The diversity of viewpoints contributes to a bigger picture framing teachers' reality of professional development. Most offers in the field of professional development concentrate on topics, on reflection and on collaboration, but the domain of beliefs is mostly neglected. In East Asian countries, a more established tradition can be seen in exploring teachers' beliefs and values on teaching and students learning, before, for instance, the implementation of new curricula takes place (Qian, 2010).

Some teachers possess an unconciliatory position concerning developmental processes and are therefore resistant to any progress. In general, change can either be exciting or frightening depending on how it is viewed, based on hitherto experiences. If one takes a rather negative attitude toward professional development, offers are easily experienced as a "*me against*" situation. Obviously, teachers do possess an elaborated beliefs system, which can impede being really open to new ideas and suggestions. Unfortunately, such a critical attitude might lead teachers to miss a chance to gain new incitements and awareness.

On the other hand, teachers talked about how the in-service training contributed to developing new insights, to looking on the subject from a meta-level, and finally to yielding a new view or a *higher* standpoint. Obviously, one important role of professional development is to provide challenging experiences so that new ways for teachers can open up, as an example in the interview data showed. There are few if any beliefs with which the bearer associates no affective loading (Goldin, Roesken & Törner, 2008), like the above-mentioned remarks indicate. Beliefs are interwoven with affective variables, like *math is fun* or *math is fascinating*, as one teacher stressed. Teachers' positive attitudes primarily influence any process that occurs in the aftermath of a specific professional development event. Obviously, an atmosphere of trust as well as emerging enthusiasm are good indicators for pursuing new issues. Teachers from the same department who support each other, who are open and frank, are more likely to benefit from professional development.

In his talk at the ICME 11 Conference in Monterrey, Mexico, Jeremy Kilpatrick discussed Felix Klein's work on the double discontinuity that teachers encounter on their way from school to university and back to school. Particularly, he stressed that Klein's concern was to provide opportunities for teachers to obtain a higher standpoint, a notion that is sometimes labeled as an advanced standpoint. However, the English translation does not adequately meet the German expression since Klein's original aim, as Kilpatrick pointed out, was that he wanted the future teacher to stand above his or her subject, and to arrive at a more panoramic view. Since the results presented in this work profoundly emphasize the relevance of beliefs and affect in the context of professional development, any such offer should provide teachers with rich opportunities to obtain such an elaborated view.

Professional development is often dominated by black and white thinking. Issues are either considered good or bad, or statements like *teachers should*, *teachers must*, or *teachers need* are issued. What is easily forgotten is that such statements do not consider the teachers' voices. Törner (2008), while referring to a statement of Felix Klein, reminds us of the following:

> Teacher pre-service and in-service education needs to be thought from a teacher's perspective, since the efficacy of the personality matters much more than methods or curricula. Not until we succeed in such professionalism, and we are able to create a new approval culture, a real incentive for lifelong learning will be given.

Obviously, what he his referring to is Felix Klein's vision: articulated 100 years ago, and still relevant today:

> In particular, I would like to let the individuality of a teacher's confer freedom. I believe more in the effectiveness of personalities than that of the sophisticated methods and curricula. (as cited in Schubring, 2000, p. 70)

Profoundly respecting and cherishing the teachers and their needs allows for a vision of professional development that is *for* and *with* teachers, instead being simply about them. In the true tradition of Felix Klein, the work of two researchers is honored in this article: Günter Törner's work contributed to the overall philosophy, design and re-design cycles of the project *Mathematics Done Differently* making sure that the project was a teacher project. Alan Schoenfeld's work assigned beliefs a major role as "codification of teachers' experiences," and significant parameters in the context of change.[6]

NOTES

[1] Parts of the theoretical framework follow the ideas explored in Roesken (2011, pp. 1–28).

[2] http://www.dzlm.de

[3] http://www.dzlm.de/dzlm.html?seite-232

[4] The description of the project is partly taken from Roesken (2011, pp. 71–86).

[5] The interview data presented in this chapter is part of a larger investigation which can be found in Roesken (2011, pp. 111–138).

[6] More detailed ideas are reported in Roesken (2011, pp. 139–150).

REFERENCES

Adler, J., Ball, D. L., Krainer, K., Lin, F., & Jowotna, J. (2005). Reflections on an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics*, *60*(3), 359–381.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, *18*, 947–967.

Cochran-Smith, M., & Lytle, S. L. (2001). Beyond certainty: Taking an inquiry stance on practice. In A. Lieberman & L. Miller (Eds.), *Teachers caught in the action. Professional development that matters* (pp. 45–58). New York: Teachers College Press.

Cochran-Smith, M. & Zeichner, K. M. (Eds.). (2005). *Studying teacher education. The report of the AERA panel on research and teacher education*. Mahwah, NJ: Lawrence Erlbaum Associates.

Cohen, D. K. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis*, *12*, 311–329.

Cooney, T. (2001) Examining the notion of teacher change and its moral implications. In F. L. Lin & T. J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 9–32). Dordrecht: Kluwer.

Day, C. (1997). In-service teacher education in Europe: Conditions and themes for development in the 21st century. *Journal of In-service Education*, *23*(1), 39–54.

Day, C. (1999). *Developing teachers: The challenges of lifelong learning*. London: Routledge Falmer.

Day, C., & Sachs, J. (2004). Professionalism, performativity and empowerment: Discourses in the politics, policies and purposes of continuing professional development. In C. Day & J. Sachs (Eds.), *International handbook on the continuing professional development of teachers* (pp. 3–63). Berkshire: McGraw-Hill.

Garet, M. S., Porter, A. C., Desimore, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results form a national sample of teachers. *American Educational Research*, *38*(4), 915–945.

Goldin, G., Roesken, B., & Törner, G. (2008). Beliefs – No longer a hidden variable in mathematical teaching and learning processes. In J. Maass & W. Schloeglmann (Eds.), *Beliefs and attitudes in mathematics education: New research results* (pp. 9–28). Rotterdam: Sense Publishers.

Goodson, I., & Hargreaves, A. (2003). Series editors' preface. In J. Sachs (Ed.), *The activist teaching profession* (pp. ix–xi). Philadelphia: Open University Press.

Hannula, M. S. (2004). *Affect in mathematics education*. Annales Universitatis Turkuensis B 273. Finland: University of Turku.

Hannula, M. S., Liljedahl, P., Kaasila, R., & Roesken, B. (2007). Researching relief of mathematics anxiety among pre-service elementary school teachers. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 153–156). Seoul: PME.

Krainer, K. (2001). Teachers' growth is more than the growth of individual teachers: The case of Gisela. In F. Lin & T. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 271–293). Dordrecht: Kluwer.

Krainer, K. (2005). Was guter Mathematikunterricht ist, muessen Lehrende staendig selbst entscheiden! Spannungsfelder als Orientierung zur Gestaltung von Unterricht. In C. Kaune, I. Schwank, & L. Sjuts (Eds.), *Mathematikdidaktik im Wissenschaftsgefüge: Zum Verstehen und Unterrichten mathematischen Denkens. Festschrift für Elmar Cohors-Fresenborg* (Vol. 1, pp. 165–178). Osnabrueck: Forschungsinstitut fuer Mathematikdidaktik.

Kvale, S. (1996). *Interviews*. London: Sage.

Lamnek, S. (2005). *Qualitative Sozialforschung*. Weinheim, Basel: Beltz.

Loucks-Horsley, S., Love, N., Stiles, K. E., Mundry, S., & Hewson, P. W. (2003). *Designing professional development for teachers of science and mathematics*. Thousand Oaks, CA: Corwin Press.

Malara, N., & Zan, R. (2002). The problematic relationship between theory and practice. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 553–580). Mahwah, NJ: Lawrence Erlbaum Associates.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York: Macmillan.

Meyer, D. K., & Turner, J. C. (2002). Discovering emotion in classroom motivation research. *Educational Psychologist*, *37*(2), 107–114.

Partington, G. (2001). Qualitative research interviews: Identifying problems in technique. *Issues in Educational Research*, *11*(2), 32–44.

Pehkonen, E. & Törner, G. (1999). Teachers' professional development: What are the key change factors for mathematics teachers? *European Journal of Teacher Education*, *22*(2/3), 259–275.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics* (pp. 157–224). Charlotte, NC: Information Age Publishing.

Roesken, B., Hoechsmann, K., & Törner, G. (2008). *Pedagogies in action: The role of mathematics teachers' professional routines*. Paper presented at the Symposium on the Occasion of the 100th Anniversary of ICMI (Rome, 5–8 March, 2008).

Roesken, B., & Törner, G. (2008). *Mathematics done differently – An innovative approach to furthering the professional development of German teachers*. Paper presented at the 2008 annual meeting of the American Educational Research Association, New York, USA, March 24–28, 2008.

Roesken, B. (2011). *Hidden dimensions in the professional development of mathematics teachers – In-service education for and with teachers*. Rotterdam: Sense Publishers.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education*, *4*(1), 1–94.

Schoenfeld, A. H. (1999). Looking toward the 21st century: Challenges of educational theory and practice. *Educational Researcher*, *28*(7), 4–14.

Schoenfeld, A. H. (2010). *How we think. A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Schubring, G. (2000). Felix Kleins Gutachten zur Schulkonferenz 1900: Initiativen fuer den Systemzusammenhang von Schule und Hochschule, von Curriculum und Studium [Felix Klein and mathematics education: Connecting school and university, curricula at schools and universities]. *MU Der Mathematikunterricht*, *46*(3), 62–76.

Sowder, J. (2007). The mathematical education and development of teachers. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 157–224). Charlotte, NC: Information Age Publishing.

Sullivan, P. (2007). Researching change in early career teachers. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 153–156). Seoul: PME.

Törner, G. (2008, June, 21). Kongress Statements. Schule + Bildung. Innovationskongress, Neuss. Retrieved January 5, 2009, from http://www.uni-due.de/mathematik/agtoerner/roe_prodev.shtml

Törner, G., Rolka, K., Roesken, B., & Schoenfeld, A. (2006). *Teacher monologue as a safety net: Examining a German mathematics classroom situation through the lens of Schoenfeld's theory of teaching-in-context*. Paper presented at the 2006 Annual Meeting of the American Educational Research Association, San Francisco, USA, April 7–11, 2006.

Törner, G., Rolka, K., Roesken, B., & Sriraman, B. (2010). Understanding a teacher's actions in the classroom by applying Schoenfeld's theory Teaching-in-Context: Reflecting on goals and beliefs. In G. Kaiser & B. Sriraman (Eds.), *Theories of mathematics education. Seeking new frontiers* (pp. 401–417). Berlin Heidelberg: Springer.

Qian, C. (2008). *Teachers' beliefs and mathematics curriculum reform: A story of Chongqing*. Paper presented at the 11th International Congress on Mathematical Education, Monterrey Mexico, TSG 30.

Zehetmeier, S. & Krainer, K. (2011). Ways of promoting the sustainability of mathematics teachers' professional development. *ZDM, The International Journal on Mathematics Education*, *43*(6/7), 875–887.

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12. MATHEMATICIANS AND ELEMENTARY SCHOOL MATHEMATICS TEACHERS – MEETINGS AND BRIDGES

INTRODUCTION

> There is no question that mathematicians are going to be called on to teach teachers. The tide is already turning in this direction. (Sultan & Artzt, 2005, p. 53)

> As practitioners of the discipline, research mathematicians can bring valuable mathematical knowledge, perspectives, and resources to the work of mathematics education. (Bass, 2005, p. 430)

The involvement of mathematicians in mathematics education is as old as mathematics education itself. Very prominent mathematicians, such as Felix Klein and Hans Freudenthal, are considered precursors or even founding fathers of mathematics education as an academic field of study. Many well-known researchers in mathematics education started their career as research mathematicians, like Alan Schoenfeld and Günter Törner, to whom this volume is dedicated. Indeed, what could be more natural than mathematicians being intensively involved in mathematics education? However, it seems that after mathematics education established itself as a discipline, the role of mathematicians has been less prominent than expected. Paradoxically, mathematicians are often critical of this new discipline.

In the last decade, and possibly as a positive reaction to the unfortunate effects of what was called the "math wars," many avenues for dialogue have been initiated between research mathematicians and mathematics educators about the goals, content and pedagogy of the mathematics curriculum. However, the direct involvement of mathematicians in the practice of mathematics education, in teacher education for example, rarely receives careful scrutiny.

This chapter is an attempt to contribute to understanding the possible roles that might be played by mathematicians in mathematics teacher education for elementary school teachers. It describes and analyzes a professional development program (PD). The program is run by mathematicians (a research mathematician and graduate doctoral students from an internationally renowned Mathematics department) who teach in-service courses for elementary school teachers in Israel. These mathematicians work as a group, coordinating their lesson plans and collectively reflecting on them before and after implementation. The team works mostly on the basis of their mathematical insights with occasional consultations with mathematics education experts. The overarching goal of the course is to deepen and

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broaden teachers' understanding of central concepts in elementary mathematics. This course is a unique experience in Israel on a number of counts:

- Usually, instructors in professional development courses for elementary school mathematics teachers are experienced fellow teachers, or teacher leaders appointed by the Ministry of Education, or curriculum developers and occasionally mathematics educators.
- The knowledge and perspectives that the mathematician-instructors bring to the course, their modest experience with elementary school teaching, and their beliefs and attitudes regarding the nature of mathematics and its teaching are not at all typical of elementary school PD.
- The content of this PD is also unusual. Elementary math content knowledge is generally conceived as straightforward, in spite of research contradicting this view, e.g. (Ma, 1999). A common first reaction may be: "what else is there to learn about the four basic arithmetical operations?" PD for elementary school teachers tends to focus on pedagogical content knowledge (PCK) – how best to teach particular topics, how to address student errors and misconceptions, etc. In contrast, this PD aims to focus on subject matter content knowledge, as conceived by mathematicians who have little or no expertise in pedagogy.
- The instructors develop their own lesson plans. Many times they design their own exercises and problems and refine them in collective team discussions, in either face to face or virtual meetings. Also, after most of the lessons they produce reflective reports, analyzing what worked and what did not work.

This chapter is based on data collected during several lessons in these courses,[1] focusing on the mathematician-instructors – the professional knowledge they bring to bear, their attitudes and beliefs, and how all these impinge on the way they envision elementary mathematics, and how they influence their didactical choices and their teaching decisions. We document the emergence of insights, both mathematical and pedagogical in nature, which developed either during the instructors' preparation/reflection sessions or during interactions in class with the teacher-participants. The teachers' side of the story is no less interesting, and will be reported elsewhere.

In our analyses, we target both theoretical and practical contributions. Theoretically, we discuss the blending and interaction between types of knowledge towards a growing understanding of the construct of mathematical knowledge for teaching (MKT), e.g. Ball et al (2008). Practically, the PD described here may serve as the basis for future opportunities for mathematicians teaching elementary mathematics teachers.

BACKGROUND

This chapter is based on observations collected from two academic years, 2010–2011 and 2011–2012. The professional development course includes ten 3-hour sessions. There are separate tracks for grades 1–2 and for grades 3–6.

The first few sessions of 2010–2011 did not bode well. The mathematics instructors intended to focus on deepening and broadening the teachers' mathematical knowledge. One of the questions they asked themselves was "what do we know that the teachers do not?" Their answers to this question did not include "how to teach elementary mathematics" – a topic they were careful to avoid. Some instructors were very explicit about this point and stated openly to the participants: "I know nothing about teaching elementary math, I can't tell you how to teach it, and I won't." Once it was clear that the mathematicians' contribution would be in the realm of mathematical content, their next question was "what could we possibly contribute to the understanding of such apparently straightforward topics?" Their assumption was that part of the intricacies of teaching elementary mathematics, especially in the lower grades, lies in recognizing and addressing subtleties in the subject matter and its conceptual underpinnings. This was at the core of the expertise they brought to the course. Once they were explicit about it, the challenge was to expose these subtleties to the teachers in the PD. One of the first approaches the mathematicians adopted for selecting appropriate activities for the teachers was estrangement – a technique designed to gain new insights on the familiar by reflecting on the unfamiliar. Estrangement (*dépaysement* in French) is an anthropological term which literally means going out of one's country. It was used frequently by the anthropologist Lévi-Strauss, as described, for example, by Hénaff (1998), and has also been used in some studies of mathematics education (Barbin, 2011). Two typical examples of this approach, implemented in the PD, consisted of working on base-5 arithmetic in order to gain an appreciation of the structural subtleties of base 10, and reviewing cardinality and counting through learning to compare infinite sets. This approach – taking a step back to gain a broader perspective – may be typical of the way mathematicians think and work. However, it was not a great success with the teachers, who tended to judge such topics (base 5, infinite sets) as irrelevant to their teaching, and thus not at all what they were hoping to gain from the PD. They felt that what they most needed in order to improve their practice was practical tools, for example, activities they could take to class, tips for teaching particular topics, how to deal with student difficulties, etc. Grade 1-2 teachers, in particular, did not feel a need to deepen their understanding of the subject matter, which they considered quite straightforward. The teachers did not hesitate to voice their dissatisfaction with the mathematicians' approach, and the instructors felt they needed to adapt their approach to meet the teachers' expectations of *relevance*.

This mismatch of expectations could be seen as a sure promise of failure, with the subsequent feelings of frustration to be felt by instructors and teachers alike. However, the story evolved in a different and rather fruitful path.

In the 2011–2012 PD the instructors attempted to address the teachers' feedback from the previous year, but they did not completely adopt the teachers' views on what would make the PD relevant. Their interpretation of the demand for relevance was shaped by two factors: what they thought the teachers needed (better understanding of the content), and what they felt they could provide as mathematicians. They eventually came up with a number of activities which blended subject matter content knowledge (the mathematicians' expectation) and pedagogical content knowledge (the teachers' expectation).

The instructors' need to address the teachers' discontent with their unfulfilled expectations, coupled with the conviction that mathematical content should remain at the core of the course, made them re-think their courses of action and figure out how to address the former without renouncing the latter. We will analyze the instructors' moves to conciliate these seemingly opposing goals, and describe how this shaped what is now considered a successful PD program not only by the instructors and participants, but also by officials from the Ministry of Education.

We focus on how the mathematicians brought their mathematical knowledge and their beliefs about mathematics to bear on various aspects of teaching, as follows:

- Mathematical content
  - Unpacking elementary topics into their components
  - Unpacking tools for doing mathematics
  - Preparing for how elementary topics will eventually interact with future advanced topics on the horizon of the students' knowledge
- Pedagogical issues
  - Anticipating and addressing student difficulties, errors and misconceptions
  - Designing activities for the PD, bearing in mind how these activities might play out in the teachers' classrooms

MATHEMATICAL CONTENT – UNPACKING ELEMENTARY TOPICS

> I have observed, not only with other people but also with myself . . . that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question. (Freudenthal, 1983, p. 469)

This quote reflects one of the central challenges in teaching mathematics, especially elementary mathematics, which includes "unpacking" the mathematical content – reviewing what it is made up of and what it really involves for a learner, appreciating the conceptual nuances, and disentangling the multiplicity of seemingly similar meanings for the same or connected concepts.

The following are examples of how the mathematicians unpacked some concepts in elementary math, armed with their knowledge of advanced mathematics and their experience practicing it. We show examples of two kinds of mathematical unpacking, one referring to specific subject matter concepts (e.g. counting), and another referring to the practices for doing mathematics (e.g. proofs and justifications, definitions). It may well be that mathematicians' knowledge of advanced mathematics is not strictly *necessary* for unpacking elementary concepts, nor is it sufficient, but it does appear to be highly instrumental. In some cases, we see how the mathematicians' knowledge not only pointed the way to unpacking concepts but interestingly, it also yielded insightful pedagogical implications.

Unpacking counting and the concept of number

There are two different definitions of natural numbers – the set theoretic definition (attributed to Frege and Russell) and the Peano axioms. One can clearly know, operate flawlessly with, and teach natural numbers without being aware of either of these definitions, but awareness of them proved to be productive in unpacking the concepts of number and counting. The set theoretic definition has more affinity with the act of counting objects in a set, whereas the Peano axioms, based on the successor operator, tend to be aligned with the act of counting by saying the numbers out loud one after the other without a specific reference to objects and without a specific goal of establishing the "cardinality of a given set" ("rote" counting). These are two different aspects of counting that children need to learn. Awareness of the two definitions, and of the equivalence between them, helped the mathematicians see the differences and connections between these two aspects, and it also contributed to their re-visiting of the basic operations of addition and subtraction and their properties. Rote counting is related to Peano's concept of *successor* (what comes next), whereas counting elements of a set is closer to Frege and Russell's construction, where the number 3 is equated with the equivalence class of all sets having 3 elements. Proving that Frege and Russell's construction satisfies the Peano axioms (something the mathematicians considered doing in the PD, but realized would not be relevant for the teachers) helps illuminate the connection between the two counting competencies. To begin with, the number 3 is an operator that acts on objects – "3 flowers," "3 birds," etc. Children eventually need to abstract the concept, and see the equivalence of all sets of a particular cardinality. This is very similar to the equivalence inherent in the set theoretic definition of numbers. This parallel between what the mathematicians need to prove and what the children need to understand helped the mathematicians see what there is to learn in this seemingly trivial topic, and the mathematical basis helped to make this explicit.

Comparing the cardinality of two sets (which set has more elements) can be based on counting, but it is in fact possible to make such comparisons without knowing how to count. It is possible to set up a 1-1 correspondence between the elements of the two sets, and see which – if either – has elements left over. Mathematics students typically encounter this principle in an under-graduate course in set theory, where 1-1 correspondences are used to compare infinite cardinalities. In fact, the existence of such a correspondence is taken to be the definition of equal cardinalities. Furthermore, 1-1 correspondence is a more fundamental concept than counting, since counting the objects in a set relies on setting up a 1-1 correspondence between the set's objects and the first natural numbers (1, 2, 3, ...). Teachers often overlook this comparison strategy, perhaps due to their preoccupation with mastering the skill of counting, and may be completely unaware of the concept of 1-1 correspondence and its role in counting objects. This is another example of how their knowledge of advanced mathematics helped the mathematicians regain insight into the foundations of elementary mathematics. The question of how this mathematical content can be presented to the teachers is a separate issue, which will be addressed in the section on designing activities.

Addition is defined differently in the two constructions of numbers. In the set theoretic definition, addition is based on set unions – putting together two collections of objects. In the Peano approach, addition is based on the repeated application of the successor operator, namely starting from one number and counting-on as many times as the second addend indicates. The mathematicians found it is easier to prove the commutative principle in the set theoretic construction of numbers than by relying on the successor. For them, this implied something about how the property may be understood by children. Adding 11 to 2 (counting-on 11 starting from 2) does not feel at all the same as adding 2 to 11 (counting-on 2 starting from 11). In fact, it is not at all obvious why the results should work out to be the same! This corresponds to the difficulty in proving the commutative principle based on Peano's axioms. On the other hand, the union of two sets (one having 2 objects, one having 11) is symmetrical. Thus, on the basis of the set theoretic definition of numbers, the commutative principle is obvious and its proof is straightforward. Through this connection between mathematicians' definitions and children's models of addition, the mathematicians gained some pedagogical insight: the commutative property is more obvious in some contexts than in others, and should be introduced to children accordingly.

Unpacking the associative property of multiplication

The distinction between multiplication's commutative property, $a \times b = b \times a$, and associative property, $(a \times b) \times c = a \times (b \times c)$ may be confusing for students and teachers alike. The confusion may be related to the following: the combination of the two properties boils down to *when you need to multiply a list of numbers, you can do it in any sequence you like*. So why separate this simple statement into two distinct properties, if procedurally they seem indistinguishable? The answer is provided in university algebra – some mathematical domains (e.g. non-commutative groups) have one property and not the other, so they must be considered as distinct. It is questionable whether this argument would convince a student, or even a teacher. One of the instructors came up with a convincing argument without resorting to advanced mathematics. The example he worked with was: There are 5 buses, 40 children on each bus, and each child has 2 parcels, how many parcels are there in total? The teachers suggested a number of ways to calculate the result, including: $(5 \times 40) \times 2$, $5 \times (40 \times 2)$, and $(5 \times 2) \times 40$. The last of these calculations is the easiest, starting with the obvious (5×2). The instructor asked how we know that all of the above yield the same answer. There was no consensus – both the commutative and the associative properties were suggested. In fact, the third way of calculating follows from the first or the second by applying both the commutative and the associative properties, $5 \times (40 \times 2) = 5 \times (2 \times 40) = (5 \times 2) \times 40$, but the instructor did not take this formal route. Instead he returned to the problem and its contextual meaning. What does 5×40 represent? The total number of children. What does 40×2 represent? The number of parcels per bus. Either of these quantities may be the first calculation in our solution (an observation that in fact demonstrates the associative property). But what does 5×2 represent? It does not represent anything meaningful in the problem! So, we can explain the commutative property (perhaps by means of the array model), and we can explain the associative property (as demonstrated above), but the combination of the two – *multiply in any order you like* – is a conclusion that does not follow naturally from the meaning of problem. This helped illustrate to the teachers that there are indeed two distinct properties, having distinct explanations.

Unpacking equality

In students' early encounters with the equality symbol it is usually taken as a call for action, for example, "$7 \times 2 =$" is read as an invitation to carry out a calculation (see, for example, Saenz-Ludlow & Walgamuth, 1998). Mathematicians are aware of the sophisticated multiplicity of other meanings (e.g. Freudenthal, 1983), where equality is first and foremost an equivalence relationship. This became salient in the topic of division with remainder, where the equivalence breaks down. Adopting the American notation, $7 : 2 = 3R1$, but $3R1$ is also the result of $10 : 3$. May we conclude that $7 : 2 = 10 : 3$?! Conversely, $7 : 2$ and $14 : 4$ should be equal, but one is $3R1$ and the other is $3R2$. The implication is that in this context, the equality may only be read from left to right (implying a call for action), contrary to the most basic requirements of equivalence. This clash was so critical for the mathematicians that they actually engaged (themselves) in the task of inventing alternative notations to circumvent the problem, for example, $7 : 2 = 3R(1 : 2)$, which reminds us that the remainder (1) is a result of division by 2. Note how this notation may also be seen as a step towards fraction notation, since after becoming knowledgeable with fractions, we will eventually write $7 : 2 = 3\frac{1}{2}$.

Unpacking the concept of average

The common definition of average (arithmetic mean) learned at school is usually procedural – add all the numbers in a list, and divide by the number of numbers you added. The instructors, who tended to take a more conceptual approach to knowing and learning of mathematics, aimed at unpacking the concept and unfolding its multiple facets. For example, they decided to focus on alternative definitions of the concept. One instructor suggested: *given a list of numbers, the average is the number such that when you add up all the (signed) differences from it, you get 0*. This can be considered a definition nearer the meaning of average than the traditional definition, and in some cases it can be practical for finding the average, or for checking if a given number is indeed the average. This alternate definition mirrors a sequence that is typical of university mathematics – define a construct, investigate it to find its properties, and then define a new construct based on one of these properties. The new construct may be identical to the original one (if the property is necessary and sufficient, as is the case with the alternate definition of average), or a generalization of the original one (if the property is necessary but not sufficient).

Another alternative view of the average can specifically rely on a useful representation: *when two numbers are represented on the number line, their average is represented by their midpoint*. This fact is often overlooked, even when teaching in Hebrew where there is a strong semantic connection between the words for "average" and "middle" which share the same root (*memutza* and *emtza*, respectively).

The theoretical background that the mathematicians brought to this topic had many implications. It provided the flexibility to invent and justify ad hoc calculation strategies: for example, to find the average of 81, 87, 88, and 89 one can find the average of 1, 7, 8, 9, then add 80 to the result. It also contributed to their awareness of likely pitfalls. One example was the question raised by an instructor of whether in order to find the average of many numbers it is acceptable to partition them, find the average of each part, and calculate the average of the averages. Will it work? Always? Sometimes? How does this connect to the topic of weighted averages? In this case, unpacking the concept resulted in the identification of the teachers' fragmented knowledge of it. Based on a single instance where the average of averages gave the correct result, some of the teachers conjectured that this would always work. Inspired by the fact that this procedure does sometimes give the correct result, the mathematicians proceeded to further unpack the concept, to clarify under exactly what conditions this strategy yields the correct result. When partitioning the list of numbers that need to be averaged, one needs to give each partition its relative weight. Weighted averages is a topic all mathematicians are familiar with: for example, in the context of basic probability, where the expectation of a random variable is the average of all possible values weighted by their probability. The concept of average exemplifies the extent to which some topics of elementary school curriculum are just the tip of a very rich set of connected concepts. The mathematicians' ability to unpack that richness contributed to the identification and analysis of potential knowledge flaws and the understanding of what this concept entails, including the ideas presented above for promoting computational fluency and flexibility.

MATHEMATICAL CONTENT – UNPACKING TOOLS FOR DOING MATHEMATICS

> … what people do is a function of their resources (their knowledge, in the context of available material and other resources), goals (the conscious or unconscious aims they are trying to achieve), and orientations (their beliefs, values, biases, dispositions, etc.) (Schoenfeld, 2010, p. xiv)

Schoenfeld's theory of goal-oriented decision making is primarily concerned with in-the-moment teaching decisions, but can also be related to a much broader scope and applied to our analysis of the PD instructors' teaching decisions. So far we have described and analyzed examples of how the mathematicians' knowledge contributed to the unpacking of elementary mathematical content, and consequently to supporting some of their instructional decisions. However, meta-mathematical issues and mathematical practices, such as mathematical conventions, definitions, proofs, justifications, and explanations were just as important a teaching goal as the content. In this, the mathematicians were guided by their orientations, including their beliefs about the nature of mathematics and its established practices. In this section we describe how these meta-mathematical topics and mathematical practices were unpacked, and how such unpacking shaped decisions and actions.

Number naming conventions

What, if anything, is wrong with the number "thirty eleven?" Is it a correct result for the problem $27+14$? The teachers tended to see this as an unfinished procedure: the tens were added (30), and so were the ones (11), but regrouping of the 11 ones was neglected. The mathematicians' focus was different. They did not automatically assume that numbers must have unique names. Indeed, in some contexts one thousand nine-hundred eighty four may legitimately be named in English nineteen hundred eighty four. Non-unique names are not an intrinsic problem, as long as you feel comfortable with equivalence classes. The question of uniqueness led the mathematicians to a search for alternative naming conventions in a variety of languages. Consider, for example, the Welsh word for 78, which translates to – *two nines and three twenties* – or the Alamblak word for 87 – *twenty two-and-two, and five, and two*. These are indeed unusual naming conventions, but is there anything wrong with them? Do they have any advantages over our naming convention? The *mathematical* point is that, given the intrinsic arbitrariness of naming, we should not ask ourselves if a naming convention is correct, but rather how practical and how unambiguous it may be. The criteria for practicality that the mathematicians focused on were mathematical in nature – does the convention give each number a *unique* name, how well does the naming convention support estimation (*one-hundred less two* may give a better sense of the order of magnitude than *ninety-eight*), does the convention support simple lexicographical comparison (all numbers that begin "seventy..." are greater than all numbers that begin "sixty..."), and perhaps most importantly: how well is the naming convention aligned with the standard place value notation. This last point has cognitive and didactic implications as well – a naming system aligned with the place value notation may be easier to master, and may even support conceptual understanding of place value principles. Browning and Beauford (2011) found this to be the case with the Chinese naming convention.

Unpacking proof

The concept of mathematical proof is not at all trivial (see, for example, Lakatos, 1976). Nonetheless, mathematicians' ideas about what constitutes a mathematical proof are bound to feed into their unpacking of elementary mathematics. We will show some examples of such unpacking of *proof*, referring also to *definition*, which seems to be strongly linked to proof. For example, the way we show that a number is even depends on the way we define evenness in the first place. We note that in this context we clump together the concepts of proof, justification, and explanation.

There are many distinctions one can make between these concepts (e.g. Levenson & Barkai, 2011), but we are more interested in what they have in common.

Areas and perimeters of rectangles are topics in the elementary math curriculum. One issue is the relationships between these two concepts. Children should learn that in some conditions area grows with the perimeter, but not in all cases. Given a rectangle, it is generally possible to find one with greater perimeter and smaller area, or with smaller perimeter and greater area. Of all rectangles having a given perimeter, the square has the greatest area. This is a weak version of the well know isoperimetric inequality and it can easily be proven using the algebraic equivalence: $(x - a)(x + a) = x^2 - a^2 \leq x^2$, where $4x$ is the perimeter and x the side of the square, but this is not feasible using elementary school techniques. The mathematicians' commitment to proving mathematical claims (and not just stating them) was the motivation to search for a convincing geometric proof/justification of this claim, accessible by means of elementary school math. On the basis of some examples, they showed that whenever we extend one side (p) of the rectangle by 1 unit, and shorten the other side (q) by 1 unit, we add a narrow rectangle which increases the area by $(q - 1)$, and remove a narrow rectangle which decreases the area by p. As long as p is not shorter than q, the net result is a decrease in the total area. The teachers felt this proof was something they could take to their own classrooms. It also served to show why the square has the greatest area, and that the area decreases the more "squished" the rectangle is. In searching for a proof, and in coming up with this one, the mathematicians acted in a manner consistent with these beliefs:

- There should be no magic in mathematics. Every fact should have a proof.
- The proof must be comprehensible, based on what is already known.
- The proof should say something about *why* the statement is true.

A common enrichment activity is the famous problem of adding all integers from 1 to 100. Solving this problem by pairing numbers with equal sums ($1 + 100$, $2 + 99$, $3 + 98$ ...) is often attributed to the young Gauss. This process yields a general answer, $\frac{n(a_1+a_n)}{2}$, but there is a snag – the pairing process assumes an even number of addends. It is possible to patch up the proof for the odd case, but this is inelegant. One of the instructors presented a version of this problem in the PD. The task was well known to the teachers – finding the total number of Hanukkah candles required for the eight-night celebration (2 on the first night, 3 on the second, ... 9 on the eighth). This particular problem has an even number of addends, so the pairing solution works, but the instructor was aware of the incompleteness of the argument for the general case, and felt that a more general argument was called for, even though the teachers felt no need to generalize the problem. This commitment yielded an elegant proof inspired by a non-mathematical aspect of the problem story: there is an alternative Hanukkah tradition where the number of candles decreases, namely 9 on the first night, 8 on the second, and all the way down to 2 on the last. The proof was based on the following observation: If you light candles according to *both* traditions, you will light 11 candles on every night, for a total of 88 candles, 44 according to each tradition. This version of the proof works equally well for an even or an odd number of addends. Introducing this proof was consistent with the belief that:

– Claims and proofs should be as general as possible.

Unpacking definition

Even in elementary mathematics, many terms need to be defined accurately. Choosing a definition, or perhaps more than one definition, has pedagogical implications. What constitutes a definition in elementary mathematics? Should it specify what constitutes a non-example as well as what constitutes an example? Should it be parsimonious, or should it be redundant, namely rich in superfluous details? In what ways does it support us when we attempt to prove (or explain) that some object does or does not satisfy the definition? Should we have a multitude of definitions? If so, they should be equivalent, but how do we know they really are? What are the advantages and disadvantages of particular choices of definitions? These are some of the questions that the mathematician considered when choosing, offering, using or creating definitions. We will show several instances of the mathematicians grappling with these questions.

In a previous section we described how the concept of average was unpacked, aided by the instructors' knowledge of mathematics. We mentioned that the usual working definition for average was operational – add all the numbers and divide by the number of numbers you added. The instructors felt that this definition was deficient – it lacked a good feel for what the average really is. One instructor decided to provide a second definition: the number such that when you add up all the (signed) differences from it, you get 0. This definition draws attention to the fact that average is "between" the numbers – if some are greater than the average (positive differences), then for the sum of differences to be 0, others must be smaller than the average (negative differences). The instructor did not prove the equivalence of these definitions – this would have been difficult without algebra – but did show that in examples where the average had been calculated, it had this property. This approach to mathematical definitions is consistent with the beliefs:

– Definitions should say something meaningful about the concept being defined.
– Multiple definitions for a concept are desirable.
– It may be difficult to rigorously prove the equivalence of definitions, but this issue should not be ignored. Some motivation or justification should be provided.

There are many different ways one may define an even number, some based on the properties of numbers (e.g. divisible by 2 without remainder, multiple of 2), some based on properties of sets (e.g. a set has an even number of elements if its elements can be arranged in pairs). Clearly, the working definition that we have in mind will influence the way in which we prove (or explain) why a number is or is not even. One of the instructors designed the following activity in order to support the making of explicit connections between definitions and proofs by the teachers.

> Students were asked if the number of legs in the classroom is even. Five answers follow. Which answers are correct? How would you respond to each of the students' answers? In your opinion, is there a best answer? Which? What is the implicit definition for even behind each answer? Which definition would you choose to use in your classroom? The five responses were: 1) Yes, because the number of legs is twice the number of people. 2) Yes, because each person adds 2 legs to the total, so when we add them all up we get $2 + 2 + \cdots$. 3) Yes, because there is an even number of people in the classroom. 4) Yes, because we can divide the legs into two groups – left legs and right legs. 5) Yes, because we can divide the legs into two groups – boys' legs and girls' legs.

In this activity we see how the desirability of multiple definitions and a (non-rigorous) focus on their equivalence were implemented in the task design. Moreover, we see how the design of the task reflects a shift in the underlying reasons for such desirability, from mathematical (or meta-mathematical) to pedagogical. Namely, an integral part of mathematical activity is to produce alternative definitions and check for and prove equivalence. This reason may not apply to one's teaching needs, yet knowing and inspecting alternative definitions may still be central for teaching practices (i.e. how to address students' productions).

In including options with flawed arguments (3 and 5 above), this task also provides an opportunity to address the meta-mathematical topic of logical reasoning, and is consistent with the belief that:

- Conclusions should follow *logically* from definitions.

MATHEMATICAL CONTENT – PREPARING FOR TOPICS ON THE HORIZON

The teachers participating in the PD tended to have specialized knowledge, based on their experience of teaching no more than one or two different grade levels. In an expectations questionnaire administered at the beginning of the course, teachers showed little interest in topics "on the horizon," namely topics that their students will learn in later grades, which they themselves do not teach. Moreover, they tended not to recognize which of the topics they teach will be crucial foundations for more advanced knowledge. The mathematicians built on their background in order to make explicit connections between current and future topics, and included recommendations on how to teach some elementary topics in a way that will support more advanced topics later on. This is consistent with what Ball (1993) describes as "mathematical horizon" for teaching.

Equations

In the section on unpacking the concept of equality we saw how the instructors' awareness of equality "on the horizon" – as it is used in middle-school algebra – guided their approach to it in the context of elementary school arithmetic.

Subtraction

There are many situations that can serve as the basis for understanding the operation of subtraction – removal of objects from a set, comparison of the cardinality of two sets, distance on the number line, and more. It was not always clear to the teachers why they need more than one. The mathematicians brought an important consideration to this question – some situations extend to fractions or to negative numbers better than others, for example, distance on the number line can be very instrumental for understanding why $4-(-2)$ is the same as $4+2$. Similar considerations apply to different approaches to multiplication, where the area model provides meaning and visual support when the factors are fractions. These considerations are not purely mathematical; they lie at the confluence of mathematics and didactics. Nonetheless, they seemed to be quite foreign to the teachers, especially for those who teach one or two grades, and for whom just one view of these topics seems to suffice for their work.

PEDAGOGICAL ISSUES – DIFFICULTIES, ERRORS AND MISCONCEPTIONS

Anticipating and recognizing student errors and misconceptions is at the heart of teachers' expertise. Hill et al. (2008) have developed test items regarding this aspect of teaching expertise, and have shown that skilled teachers outperform research mathematicians in anticipating and identifying student difficulties. Our mathematician-instructors were no exception – they were not very knowledgeable on these matters either. One instructor stated that a certain type of problem *must be considered difficult, since it appears so rarely in textbooks*, implicitly admitting that he is not an expert on what is difficult for students. Often the instructors would appeal to the teachers for their pedagogical insight – "*Is this difficult for your students? Is it something they can do?*" The teachers welcomed this kind of question, and were glad to be able to bring their expertise to bear.

In spite of their lack of expertise, the mathematicians coped with the issue of student difficulties on the basis of their own proficiency, supplying a complementary perspective to that of the teachers. In this section we illustrate how the mathematicians' knowledge served as a springboard for their understanding of and their suggestions for coping with common student errors and misconceptions.

Counting errors

As described above, counting was unpacked into rote counting, 1-1 correspondence with the natural numbers, and invariance under permutations of the set elements. Omitting any of these ingredients may lead to error. For example, skipping elements in counting amounts to a correspondence not defined on the whole set of elements. A correspondence not *well defined* is a way to describe and explain rote counting not synchronized with the ticking off of the elements. A correspondence that is not 1-1 may result in counting an object twice, or in skipping others. Not accepting invariance under permutations may cause children to repeat their counting in a different sequence, not quite expecting the same result. It was their understanding of the mathematics that helped the mathematicians anticipate these potential errors and difficulties.

Problems with unknowns and the equality symbol

As mentioned above, the equality symbol is often seen by children as a call for action – "Solve!" Realizing this, and realizing that problems with an unknown ($3+? = 5$) require a different interpretation of the symbol – as equivalence – one of the instructors suggested that the unknown be covered by a curtain (3 +  = 5). He actually cut one out from a curtain catalog and stuck it on the whiteboard. Placing the curtain over the unknown implied that someone had solved the problem in the past (in-line with the "call for action"), and now we are detectives trying to recreate what the problem must have been in the first place. The instructor's suggestion may be seen as a bridge between equality as a call for action and as an equivalence relationship. The inspiration for the idea came from a pedagogical "trick" the instructor had been shown by a teacher in a different context.[2] This example shows how the instructor appropriated a design idea underlying a didactical tool developed to attain an educational goal (weaning students from the need to count from 1) in order to enrich a narrow interpretation of the equality sign. In this case, the task and the didactical tool were firmly based on a worthwhile mathematical idea, and the mathematician-instructor was not only capable of making that idea explicit, but he also appropriated the design principles and applied them to the design of an artifact, illustrating a new idea. The issue of designing activities is discussed further below.

Misconceptions in vertical subtraction

In one activity, based on Ernest (2011), in the spirit of Brown and Burton (1978), the teachers attempted to uncover and explain student errors in vertical subtraction, and to predict how these students would solve some new problems. The instructors stressed the following: *What is the* **conceptual** *misconception behind the procedural error, what in the student's previous learning might be responsible for this misconception, how would you help this student master the procedure, can you suggest a correct procedure based on this student's erroneous one?* We see in these questions a blend of mathematical and didactical points of view.

PEDAGOGICAL ISSUES – DESIGNING ACTIVITIES

What makes a "good" problem for elementary mathematics PD? In previous years, the main criticism of the PD was that not all activities[3] were *relevant* for the teachers. For teachers, relevance meant having a direct impact on what they bring to and do in classrooms. The teachers made it very clear from the start that what would serve them best would be "prêt-à-porter" problems, namely those that they could "use in our classrooms tomorrow morning," as is. The instructors generally accepted the premise that they should provide some activities that the teachers could use in their own classrooms, but also remained faithful to their goal of teaching mathematics. One of the ways these two distinct goals were conciliated was by capitalizing on the teachers' engagement with problems they found interesting and "relevant," and using them as springboards to discuss the underlying mathematics. Some of the resulting problems used in the course were very productive in this respect – they were rich enough to provide a context for both views of relevance: on the one hand, deepening the teachers' understanding but on the other hand, appropriate for the teachers to use (possibly with some modifications) in their classrooms. One of the instructors went to great lengths to make sure his activities would reach the teachers' classrooms. One of the homework assignments consisted of choosing an activity from the PD, adapting it for classroom use, implementing it in one of their lessons, and reporting on it in a future PD session. The adaptations the teachers made enabled us to infer some of their beliefs about mathematics and teaching. Discussing this in the PD provided the instructor with opportunities to bring to bear mathematical and meta-mathematical issues such as: elegant solutions, alternative explanations (provided either by teachers or their students), math is not only about solving exercises, etc. The tasks described above (evenness of the number of legs in the classroom, alternative naming conventions for numbers), played out as productive activities in the PD. The following are some additional examples.

Place the digits

In this activity, teachers needed to use 3 given digits (1, 4, 9) to construct a multiplication problem with the greatest possible result. Many of the teachers resorted to trial and error, but this turned out to be a good context for re-visiting the associative and distributive properties – why is 4×90 the same as 40×9 (associative property), and why is 9×41 greater than 4×91 (distributive property). The 4-digit version of this problem presented further opportunities to deepen the math in the context of problems which can be brought as is to the classroom.

The Gelosia method for multiplication

Many of the instructors presented a procedure for multi-digit multiplication which was unfamiliar to the teachers. The procedure dates from the 15th century and is illustrated by the example given in Figure 1 ($934 \times 314 = 293276$) taken from the Treviso Arithmetic (1478), as it appears in (Smith, 1958, pp. 114–117).

In this method there are 9 partial products, each one the result of multiplying two 1-digit numbers. This eliminates the need for most of the carrying in the standard procedure. Furthermore, the partial products do not need to be staggered; the correct place value is achieved by adding partial products along the diagonal lines. As mathematicians, the instructors were intrigued by the mathematics behind the procedure, and thus decided on its appropriateness for the PD, as it provides opportunities for deepening the understanding of multiplication and place value. Moreover, they felt that the activity was within the range of what the teachers

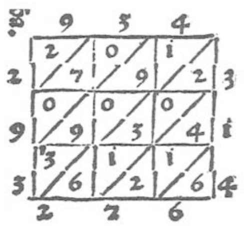

Figure 1. First printed example of the Gelosia multiplication.

could take to their own classrooms. The instructors were somewhat disappointed to learn that most teachers perceived the activity quite differently. Instead of using it as an enrichment activity to deepen their students' conceptual understanding, most teachers saw it as a potential remedial tool – an alternative procedure they could offer to students who had not mastered the standard procedure. As such, they did not dedicate much effort to the question of how and why the procedure works. We are not sure why so many teachers chose not to use this activity for enrichment. Perhaps they felt they had more pressing topics to teach. Or perhaps they were more impressed by the procedural aspects of the method (a reliable and easy-to-remember way to multiply) than the conceptual issues it raises. This may mirror differences between the mathematicians' and the teachers' attitudes towards mathematics in general. In spite of the instructors' disappointment, we consider this a productive activity. The teachers were highly engaged, deepened their understanding – which should eventually have beneficial effects on their teaching, and were willing to bring the alternative method to their classrooms – though with a different purpose in mind than that of the instructors.

Focusing on one-to-one correspondence

In the section on unpacking mathematical content we described how the mathematicians came to focus on the concept of 1-1 correspondence as foundational, even more basic than counting. Their insight was based on infinite sets, where 1-1 correspondence is the only way to compare cardinalities. How can this topic be introduced to the teachers in a way that is meaningful for them? One of the mathematicians with some programming capabilities found a creative approach. He asked himself what is special about infinite sets. In this context, the main point is that they cannot be counted. Thus, what he needed was a finite set that cannot be counted. He prepared a game applet in which blue and red balls move around the screen in random motion. The goal is to determine whether there are more red or blue balls. Counting is not a feasible strategy due to the balls' motion, but the applet does allow the player to pair up a blue and a red ball, at which point they are both removed from the screen. Players proceed to pair up balls until they are all exhausted, or until balls of only one color remain. In this game players make implicit use of the principle of 1-1 correspondence in order to solve a comparison problem without counting. The designing of this game is an example of how the mathematicians used their advanced knowledge of mathematics to uncover some of the less obvious foundations of elementary mathematics, and yet found ways to share their insights with the teachers, in a context that is relevant, playful, and is grounded in the most elementary mathematics.

DISCUSSION

This PD seemed doomed from the start. The teachers enrolled hoping to enrich their practice with new tools of the trade – activities for their classrooms and teaching tips-and-tricks. The mathematician-instructors had something else in mind – using their mathematical knowledge to deepen the teachers' mathematical understanding. Yet in spite of the chasm between these expectations, the PD was considered a success by all involved. The teachers' feedback indicated their satisfaction, the instructors felt they were teaching effectively and indicated that the teachers participated actively, and the ministry representatives – who occasionally sat in on sessions – were pleased with what they saw and heard. Furthermore, some of the teachers are utilizing their newfound knowledge. One teacher testified that her principal recently sat in on her math class. When he asked her what she was teaching, her proud reply was "what I learned in the PD last week." We now take a step back and try to explain what worked and why.

The concept of unpacking – unpacked

We have shown numerous examples of the mathematicians unpacking mathematical content. We will now attempt to unpack the concept of unpacking – reveal its elements and describe its mechanisms.

Two-way didactic transposition

Chevallard (1985) coined the term *didactic transposition* to describe the change that mathematics content must undergo from a body of knowledge *used* ("savoir savant") to a body of knowledge *taught* at school ("savoir enseigné"). Borrowing and extending this idea, we may say that the mathematicians applied a reverse-transposition: they took elementary concepts and lifted them up to the context of university mathematics. In other words, they transposed knowledge taught at school to knowledge of the professional mathematician. In this context, they employed the full power of their mathematics to deeply re-inspect the topics. Then they transposed them back to the domain of school mathematics. The first transposition may be seen as an *embedding*[4] of elementary math in the more sophisticated university math. The second transposition may be thought of as a homomorphism from university mathematics to school mathematics, aiming to maintain the structure of the discipline while scaling it down to something more palatable for students and teachers. Paraphrasing the courtroom oath, this second transposition was committed to *nothing but the truth*, but could generally not be fully faithful to *the whole* mathematical *truth*. This process of double transposition helped highlight rich mathematical connections between the elementary concepts, as was demonstrated in some of the examples above. What happens to mathematical concepts which undergo didactic transposition? Trivially, *proofs* in advanced mathematics tend to suggest how ideas may be *explained* in elementary math. Less trivially, mathematical connections between concepts tend to be mirrored in cognitive connections made by teachers and students. This was seen, for example, in the various definitions of natural numbers, mirrored in the skills of rote counting and cardinality counting, or in the way details of a mathematical proof may suggest possible student errors or misconceptions. This mirroring provided surprisingly productive insights into cognitive processes and student difficulties. For example, unpacking evenness and the meta-mathematical goal of teaching alternative definitions of this concept (and the equivalences thereof) gave birth to the activity where teachers evaluated students' correct and incorrect explanations of why the number of legs in the classroom is even.

Setting

The setting appears to be crucial as the environment needed for the unpacking to occur. The unpacking was highly situated. It took place in the context of a specific PD program in which the teachers' backgrounds and expectations were a determinant and constraining factor. Left to their own devices, the mathematicians may have remained much closer to university math, in which case their unpacking of the elementary math topics would have looked quite different. Consideration of the PD teachers' needs and explicit expectations provided a sense for the type and extent of the didactic transposition. Furthermore, the mathematicians appeared to be able and willing to learn from the teachers (e.g. the case of appropriating pedagogical insight in the context of problems with unknowns).

Knowledge and beliefs

The examples showed how knowledge of university mathematics was instrumental in unpacking elementary mathematics. Of similar importance was how the mathematicians brought their *beliefs* and their *mathematical points of view* to the task of unpacking. Their commitments to some underlying fundamental principles, even when not always articulated, were fundamental to the unpacking. In the following we list some of these principles quoting from Wu (2011), a mathematician involved in pre-college math education:

1. *Every concept is precisely defined, and definitions furnish the basis for logical deductions.*
2. *Mathematical statements are precise. At any moment, it is clear what is known and what is not known.*
3. *Every assertion can be backed by logical reasoning.*
4. *Mathematics is coherent; it is a tapestry in which all the concepts and skills are logically interwoven to form a single piece.*
5. *Mathematics is goal-oriented, and every concept or skill in the standard curriculum is there for a purpose.*

What made the program a success – Bridging the cultural gap

We believe that, in essence, this is a story of bridging a cultural gap. The gap is multi-dimensional. There is a knowledge gap, a gap between attitudes towards mathematics and its learning and teaching, and more practically, there is a gap between expectations regarding the PD. The PD was successful due to the various ways in which this gap was bridged.

Activities

We have seen how instrumental a "good" problem can be. Some of the most successful activities occurred around problems that the teachers could use in their classrooms, and at the same time provided a springboard for discussing non-trivial mathematics in the PD. Such activities, in supporting both the teachers' and the mathematicians' perspectives on mathematics, served as a bridge between their expectations. This was the case even when the mathematicians and the teachers did not ultimately agree on the role of the activity. For example, the Gelosia Method was perceived by the teachers primarily as an alternate procedure for multi-digit multiplication, suitable for their struggling students, whereas the instructors' main intention was to use it as a context for deepening the understanding of multiplication and place value, both in the PD and ultimately in the teachers' classrooms. Although the instructors were disappointed by the ways in which the teachers perceived the goal of this activity, they nonetheless provided what they aimed to provide – meaningful mathematics in the PD – and teachers received what they hoped to receive – a usable activity for their classroom.

Roles

Many activities evolved in such a way that the teachers provided valuable pedagogical input, and the mathematicians provided mathematical critique within a context of mutual interest, for example, evaluating web-based educational video clips. The teachers provided didactic criticism (e.g. use of the board, student participation) while the mathematicians provided mathematical criticism (e.g. accurate use of language and symbols, validity of logical arguments).

Mutual appropriation

The data indicate that the teachers may have started to appropriate (in the sense of Moschkovich, 2004) some of the mathematicians' attitudes towards and beliefs about mathematics, but as we said, this will be discussed elsewhere. Less trivial is the fact that the mathematicians appropriated some of the teachers' attitudes toward mathematics teaching and learning. This was evident in the blend of mathematical and didactical points of view expressed in many of the activities they designed. And finally, there is some evidence indicating that the mathematicians appropriated more of the teachers' culture than one might have expected. The instructors were often annoyed by the teachers' repeated demand for activities they could "use in class tomorrow morning." As the PD progressed, and the activities came to be designed around classroom problems, it was not uncommon to see emails from the instructors along the lines: “I’m stuck! Does anyone have a good problem I can use in the PD tomorrow?”

CONCLUSION

The literature distinguishes between two main types of content knowledge for teaching – subject matter content knowledge (SMCK) and pedagogical content knowledge (PCK). The initial chasm between the PD instructors and the teachers may be characterized in terms of this distinction. Roughly speaking, the instructors intended to build the PD around SMCK, whereas the teachers were expecting a program that would contribute to their PCK. This distinction between types of knowledge is also the key to understanding what ultimately made the PD a success. Many of the episodes we have described took place at the intersection of these types of knowledge. We have seen how a university conception of SMCK may serve as a strong springboard for developing PCK, both in the PD sessions and in the mathematicians’ preparation, and conversely, the teachers’ existing PCK was used as a springboard for conducting mathematical discussions and developing mathematical insights. These rich interconnections and mutual exchange of mathematics and mathematical pedagogy were at the core of this unique PD and they are worth exploring further. We feel that in exploring these interactions, attention should be given to the mathematicians’ special SMCK, as influenced by their university conception of elementary mathematics. The mathematicians’ knowledge of mathematics helped them unpack many elementary concepts – both mathematical and meta-mathematical – exposing their nuances and revealing what they involve for learners. Furthermore, their mathematical practices and beliefs about mathematics guided them in conveying meta-mathematical messages in the activities that they designed and conducted.

Finally, it is commonly agreed that research mathematicians can and should be involved in the education of elementary school mathematics teachers, but there have not been many models to learn from. Here we have given an account of a productive involvement, and have highlighted some ways in which the mathematicians’ contribution was special. The interactions were considered productive by both communities, by a number of different standards. Both the teachers and the mathematicians learned some mathematics and some didactics from their encounters, and both parties felt that the teachers ended up better equipped to do their job.

NOTES

[1] Most of the lessons are being recorded by the first author of this chapter and are undergoing a first round of analysis. The data are particuraly rich because both the instructors and the in-service teachers who participate in the course are candid and outspoken about their feelings and evaluations.

[2] A game designed to encourage “counting-on” – finding $3 + 2$ by counting “3 … 4, 5,” and not “1, 2, 3, … , 4, 5,” by allowing students to see only one of the addends at a time.

[3] We use the term activity for a segment of a PD session, typically a problem posed by the instructor, solutions suggested by the teachers, and discussions that followed.

[4] Injective structure-preserving mapping.

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REFERENCES

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, *93*(4), 373–397.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education*, *59*(5), 389–407.

Barbin, E. (2011). Dialogism in mathematical writing: Historical, philosophical and pedagogical issues. In V. Katz & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education* (pp. 9–16). Mathematical Association of America.

Bass, H. (2005). Mathematics, mathematicians, and mathematics education. *Bulletin (New Series) of the American Mathematical Society*, *42*(4), 417–430.

Brown, J. S., & Burton, R. R. (1978). Diagnostic models for procedural bugs in basic mathematical skills. *Cognitive Science*, *2*(2), 155–192.

Browning, S., & Beauford, J. E. (2011). Language and number values: The influence of the explicitness of number names on children's understanding of place value. *Investigations in Mathematical Learning*, *4*(2), 1–24.

Chevallard, Y. (1985). La transposition didactique: Du savoir savant au savoir enseigné. *La Pensée sauvage*, p. 126.

Ernest, P. (2011). *The psychology of learning mathematics – The cognitive, affective and contextual domains of mathematics education*. Saarbrucken: Lambert Academic Publishing.

Freudenthal, H. (1983). *The didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel.

Hénaff, M. (1998). *Claude Lévi-Strauss and the making of structural anthropology* (M. Baker, Trans.) Minneapolis: University of Minnesota Press.

Hill, H. C., Loewenberg Ball, D., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372–400.

Lakatos, I. (1976). *Proofs and refutations – The logic of mathematical discovery*. Cambridge: Cambridge University Press.

Levenson, E., & Barkai, R. (2011). Explanations, justification, and proofs in elementary school. In *SEMT'11 – International Symposium Elementary Maths Teaching* (pp. 371–372). Prague: Charles University, Faculty of Education.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. New Jersey: Routledge.

Moschkovich, J. N. (2004). Appropriating mathematical practices: A case study of learning to use and explore functions through interaction with a tutor. *Educational Studies in Mathematics*, *55*(1–3), 49–80.

Saenz-Ludlow, A., & Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol. *Educational Studies in Mathematics*, *35*(2), 153–187.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. New Jersey: Routledge.
Smith, D. E. (1958). *History of mathematics*, Vol. II. New York: Dover Publication.
Sultan, A., & Artzt, A. F. (2005). Mathematicians are from Mars, math educators are from Venus, the story of a successful collaboration. *Notices of the AMS*, *52*(1), 48–53.
Wu, H. (2011). The mis-education of mathematics teachers. *Notices of the AMS*, *58*(3), 372–384.

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PART IV

ISSUES AND PERSPECTIVES ON RESEARCH AND PRACTICE

HUGH BURKHARDT

13. METHODOLOGICAL ISSUES IN RESEARCH AND DEVELOPMENT

INTRODUCTION

This chapter builds on Alan Schoenfeld's seminal contributions on methodological issues (Schoenfeld, 1980, 1985, 1992, 1994, 2002, 2006, 2007, 2010) and on our discussions over many years of collaboration and complementary thinking: Alan with the priorities of a cognitive and social scientist with a concern for practice; I with those of an educational engineer who recognizes the importance of insight-focused research for guiding good design. Alan has primarily aimed to bring rigor to research in mathematics education – to move it toward being an "evidence-based" field with high methodological standards. The Shell Centre team has an approach to research that gives high priority to impact on practice in classrooms. The analysis here reflects the challenges that we have faced, individually and together, and their wider implications for research methods in education.

The next section tackles issues of strategy, going on in the third section to look at qualities that enable studies to make contributions to the body of research-based knowledge that is reliable enough, for example, to guide design. But strategy is not enough; in research, as in design, the details matter, so the fourth section focuses down on the essential core of education: classrooms, and what can make teaching and learning more effective. It looks at the challenges of designing such research through three case studies, each based on custom-designed research tools that fit their complementary but very different purposes, and draws some general conclusions about the design of tools for research. The examples reflect one of Alan's major methodological themes: that research should be "inspectable" so that readers can follow the chain of inference from data to claims. The fifth section draws these elements together, setting out a vision of education research that would likely be more purposeful and effective – a vision that, I believe, Alan broadly shares.

STRATEGIC ISSUES FOR RESEARCH IN EDUCATION

Across the various departments of a university there are very different styles of research. This breadth is summarised in the definition used for the UK Research Assessment Exercise (RAE, 2001) in which all UK university departments are rated every five years or so:

> 'Research' for the purposes of the RAE is to be understood as original investigation undertaken in order to gain knowledge and understanding. It includes

Y. Li and J.N. Moschkovich (eds.), Proficiency and Beliefs in Learning and Teaching Mathematics, 203–235.

> work of direct relevance to the needs of commerce and industry, as well as to the public and voluntary sectors; scholarship; the invention and generation of ideas and, images, performances and artifacts including design, where these lead to new or substantially improved insights; and the use of existing knowledge in experimental development to produce new or substantially improved materials, devices, products and processes, including design and construction.

Unusually, in research in education all of these elements can be found. But, I will argue, the whole is less than the sum of the parts – and that it doesn't need to be so.

Styles of research in education

The breadth of the above definition may surprise people. It arises from taking seriously four different traditions, characteristic respectively of the humanities, the sciences, engineering and the fine arts. The focus of both the humanities and the science approaches is the search for improved *insights*; in education these cover learning, teaching, professional development, and the behaviour of education systems and their key constituencies. The engineering research approach has a rather different priority: *impact* on systems. In education this focus is on developing products and processes that will help teachers and other professionals move to more effective practices. Fine arts are similarly concerned with products as well as analysis. Let us look at each tradition in a bit more detail.

The "humanities" approach

This is the oldest research tradition, based on scholarly acquisition of knowledge and critical analysis of it, and of other people's work. From the RAE definition it is

> original investigation undertaken in order to gain knowledge and understanding; scholarship; the invention and generation of ideas ... where these lead to new or substantially improved insights.

In the humanities there is no tradition of empirical testing of the assertions made. The key product is critical commentary – as, for example, on works of art or literature.

There is a lot of this in education. The ideas and analysis, based on the authors' reflections on their experience, are often valuable. Without the requirement of further empirical testing, a great deal of ground can be covered. This is still the most influential approach, partly because it supports the general belief that anyone can play, "expert" or not. This allows politicians to choose their own "common sense" policies.

However, since so many plausible ideas in education have not in practice led to improved outcomes across the system, the lack of empirical support is a key weakness. How can you distinguish reliable comment from plausible speculation? This has led to a search for "evidence-based education" and the emerging dominance in the research community of the "science" tradition.

The "science" approach
This style of research is also focused on better *insight*, of improved understanding of "how the world works," through the analysis of phenomena, and the building of models that help to explain them. In the RAE definition, it is again

> original investigation undertaken in order to gain knowledge and understanding; scholarship; the invention and generation of ideas ... where these lead to new or substantially improved insights.

This is the same wording as for the humanities approach, but with an additional implied requirement for empirical testing of the assertions made, which are now called hypotheses or models. Such testing takes time and effort, and narrows the range of what can be covered in a single study.

The key products are, again, assertions but now supported by evidence-based arguments and evidence-based responses to key questions. The evidence is expected to be empirical. The products are research journal papers, books and conference talks.

This approach is now predominant in the research in science and mathematics education. Such research provides insights, identifies problems, and suggests possibilities. However, it does not itself generate practical solutions, even on a small scale; for that, it needs to be linked to the "engineering" approach.

The "engineering" approach
This is directly concerned with practical *impact* – not just understanding how the world works but helping it "to work better." It does this by developing solutions to recognised practical problems in the form of tools and processes that help professionals become more effective. It not only builds on science research insights, insofar as they are available, but goes beyond them. In the RAE definition it is

> the invention and generation of ideas ... and the use of existing knowledge in experimental development to produce new or substantially improved materials, devices, products and processes, including design and construction.

Again there is an essential requirement for empirical testing of the products and processes, both formatively in their development and in evaluation. The importance of science-based insights varies, depending how far the "theory" is an adequate basis for design.

The key products are not only new tools and/or processes that work well for their intended uses and users but also new insights that come from the development process. (Below we give examples of this.) With these elements, development *is* research. However, in the academic community it is often undervalued – in some places only "insight" research in the science tradition is regarded as true research currency. I come back to these issues in the fifth section.

The "fine arts" approach
This is related to the "humanities" approach rather as "engineering" is to "science." In the RAE definition is it is "the invention and generation of ideas and, images, performances and artifacts including design, where these lead to new or substantially improved insights."

The key products are paintings, sculpture, musical compositions etc. I will say little about this approach because, though it enriches education and could do more, it is not central here.

I believe that all these research traditions have contributions to make in achieving reliable research insights in education, and in translating them into practical impact in classrooms and school systems, but that currently the balance among the four approaches is far from optimal. What balance, of effort and of "academic credit," would be most effective, and how does it differ from the current pattern? I will argue that that there should be more "engineering" research and that this needs reliable research insights to build on. The implications for "science" research in education are the focus of the third section.

Scales of research and development

My next strategic point looks at different foci of research, and the scale of research effort that is needed for each to contribute significantly to the overall challenge: *establishing a sound research-based path from insights to large scale implementation*.

Table 1. Four scales of R&D.

	Focal variables	*Typical Research and Development Foci*
Learning (L)	Student Task	R: Concepts, skills, strategies, metacognition, beliefs D: Learning situations, probes, data capture
Teaching (T)	Instruction Student Task	R: Teaching strategies and tactics, nature of student learning D: Classroom materials that are OK for some teachers
Representative Teachers (RT)	Teacher Instruction Student Task	R: Performance of representative teachers with realistic support. Basic studies of teacher knowledge and competency. D: Classroom materials that "work" for most teachers
System Change (SC)	System School Teacher Instruction Student Task	R: System change D: Tools for Change – i.e., materials for: classrooms, assessment, professional development, community relations

I find it useful to distinguish four different foci: learning, teaching, teachers, and school systems. The distinctions are summarised in Table 1, with the different research and development foci in the third column. The very different scales needed for the four kinds of study may be summarized as: a laboratory; a classroom; many classrooms; and whole school systems.

There is a crucial difference between T, which is about teaching possibilities, usually explored by a member of the research team, and RT, which is about what can be achieved in practice by typical teachers with available levels of support. Design research is often confined to T, whereas impact on practice requires going further, at least to RT. In "engineering" research in education (Burkhardt, 2006), the process of design research at T is continued through further rounds of trialing in more typical classrooms, so the products work well for a well-defined target group of real users.

Currently, the great majority of research is confined to L and T. A better balance across these different kinds of work is needed, if research and practice are to benefit from each other as they could. This has big implications for research strategy, since it is evident that RT and SC research needs larger research enterprises and longer time-scales. We return to this, too, in the fifth section.

RESEARCH INSIGHTS FOR IMPROVING PRACTICE

In this section, I look at features of insight-focused research that make it useful for guiding practice and, in particular, the design of educational materials and processes. The analysis builds mainly on Alan's first Handbook paper on methods (Schoenfeld, 2002) which has further references. In section VI he remarks:

> A very large percentage of educational studies are of the type, 'here is a perspective, phenomenon, or interpretation worth attending to,' and that their ultimate value is both heuristic ('one should pay attention to this aspect of reality') and as catalysts for further investigation.

This shows a remarkably modest level of confidence in the products of the research enterprise – a level of confidence that I share. It is illuminating to review his reasoning.

Schoenfeld's dimensions

In the paper Alan suggests three dimensions that help us to think about research claims. Briefly, they may be summarized as:

– *Generalizability*: How wide a range of phenomena does a claim cover?
– *Trustworthiness*: How well substantiated is the claim?
– *Importance*: How much should we care?

Typically, any given research report contains assertions in different parts of this 3-dimensional space, illustrated in Figure 1. Let us focus on the first two variables, G and T. A typical research study looks carefully at a particular situation – for

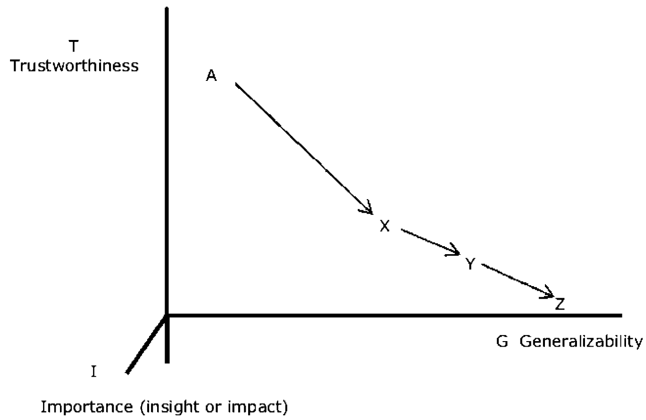

Figure 1. The trajectory of a typical research report.

example, a specific intervention based on clearly stated principles tried out in a few classrooms, collecting and analysing the teacher and student responses to the intervention. If carefully done, the results are high on T but, because of the limited range of the variables explored, low on G, shown as the zone A on the graph.

However, the conclusions section of a typical paper goes on to discuss the "implications" of the study. These are usually much more wide ranging but *with little or no empirical evidence to support the generalisations involved*. These hopes, each a greater extrapolation with fewer warrants, are illustrated as X, Y and Z in the diagram. In this example, X might represents the suggestion that most students would respond similarly, Y that it would work for teachers at all stages of professional development, Z that the design principles would work across different topics in the subject. These are essentially speculations or, a little more kindly, plausible commentary in the humanities tradition.

Only large scale studies or metanalysis can move beyond this problem and establish "zones of validity" for research insights.[1] An example from the work of Alan Bell, Malcolm Swan and the Shell Centre team on "Diagnostic Teaching" illustrates this well (Bell, Swan, Onslow, Pratt, & Purdy, 1985; Bell, 1993). This approach, now often called "formative assessment" or "assessment for learning," is based on leading students whose conceptual understanding is not yet robust into making errors, then getting them to understand and debug these misconceptions through structured discussion. The early work showed learning gains through the teaching period (pre- to post-test) similar to those of the comparison group which had standard direct instruction teaching – but without the fall-away over the following 6-months that is so familiar to teachers ("They knew it when we did it"). This is illustrated in Figure 2.

The first study was for one *mathematics topic*, with the detailed treatment designed by *one designer*, taught by *one teacher* to *one class*. It was, in Alan's words, "worth attending to." Only several studies later, when the effect was shown to be stable across many topics, designers, teachers and classes could one begin

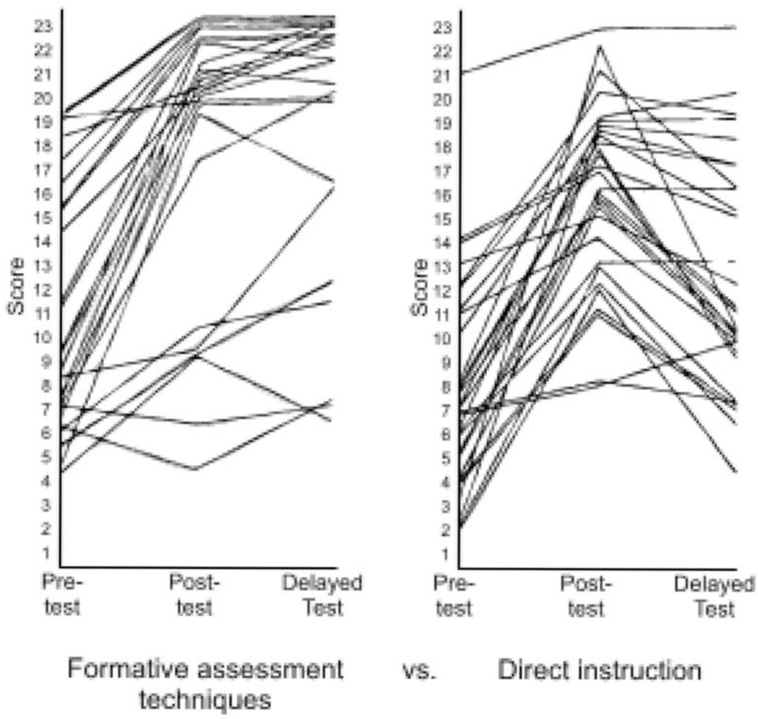

Figure 2. Typical results from work at the Shell Centre suggesting that teaching based on "cognitive conflict" techniques used in formative assessment improves long-term retention of learning.

to make reasonably trustworthy statements about "diagnostic teaching" as an approach. Even then, there remained further questions about its accessibility to typical teachers in realistic circumstances of support – an issue we return to later in this chapter.

The general point here is that much research is really about treatments, not about the principles the authors claim to study; to probe the latter one must check stability across a range of variables (student, teacher, designer and topic in this case). This typically *needs time and teams* beyond the scale of an individual Ph.D. or research grant. Other subjects arrange this; if it were more common in education, the research could have high G *and* T and, if the importance I were enough, be a reliable base to build further work on, in both the science and engineering traditions.

On importance, it is enough for the moment to say that a result can hardly be important unless it is generalizable beyond a specific study. My own perspective is that importance can come either from substantial impact on improving educational practice or from theoretical ideas of broad application with evidence for their generality.[2] Because of the scale of effort required to establish such evidence, the latter are rare.

Returning to Alan's comment with which this Section opened, very little insight-focused research has enough evidence of the generality and boundaries of its insights for them to provide a sound basis for design. Their conclusions may well be "worth attending to" but finding a range of validity is usually left to the engineers.

If you need statistics, forget it

What does this eye-catching heading mean? How can it be defended in a research field in which statistical analysis of data is so central? First, it is not a rejection of the importance of data; far from it. The key point is that:

> The large variability in implementation of educational innovations will wash out any small effects, however "significant" the gains may be from a purely statistical point of view. So only substantial clearly visible gains are likely to prove robust.

In medicine, by contrast, certain kinds of intervention, such as taking a prescription drug, can be implemented with little variation.[3] Even if the drug is only marginally effective, if used widely it can save (or, better, extend) thousands of lives; many drugs are of this kind. So randomized controlled clinical trials that show small improvements are valuable; it is these that need large samples and powerful statistical analyses. If the gains are substantial, as in the early research on antibiotics where people dying of septicemia were dramatically cured, you don't need statistics. Indeed, if it becomes clear during clinical trials that the control group is disadvantaged, the trial is immediately discontinued on ethical grounds and both groups given the treatment. My assertion is that the variability in implementation of educational initiatives is such that only where research shows clear and substantial gains are these likely to be robust and worth taking forward.

From a wider perspective, this is about the relationship between "systematic error" and "statistical error."[4] In most educational research, the systematic uncertainties are substantial. How far is the innovation actually happening, as designed? What range of strategies does the teacher use? How do teacher background, professional development, systemic support from principals/school district vary? How do all these affect outcomes? Large samples give data that is statistically more "reliable" – but uncertainties in the control of variables like those just listed, crucial to effective design and development, often make these error estimates delusory.

In education research, systematic errors dominate

This is not as despondent a message as it may seem. For example, in classroom research people say "every classroom is different"; true, but observations across mathematics classrooms, at least, show huge similarities in important ways. We have found that sample sizes of 3 to 7 are often optimal. This allows one to use always-limited research resources to collect and analyse richer data on each case, while distinguishing features that are probably generic from the idiosyncratic.

… but what about survey research?

There is one caveat to the theme of this chapter that I must mention. There *are* categories of research that yield results with well-established generality, discussed in detail in Schoenfeld (2002, 2007). For example, survey research, with all its sophistication and limitations, can be valuable in identifying widespread problems and suggesting provisional diagnoses. It is the epidemiology of education. In contrast, this chapter is focused on research that will lead to better "treatments": intervention studies, and design and development of new or improved products and processes.

However, the many variables and the problems of their control that characterize education limit the diagnostic value of the data, which is of limited depth even in sophisticated surveys, making inference far more challenging than, for example, in the Doll studies that produced such a persuasive case for the harmful effects of smoking. Even there, establishing the causal connection was decisive to wide acceptance.

RESEARCH TOOLS FOR THE ZONE OF INSTRUCTION

Alan's analysis of mathematical problem solving (Schoenfeld, 1985) identified four levels of activity in the problem solving process: overall control, strategic plans, tactical decisions, and the technical skills in carrying them out. It offers a useful way to think about all problem solving, including our goal here: devising more effective methods for educational research. The argument so far has been about strategy; this needs to be complemented with something on tactical and technical aspects. Handling these well is crucial to the research enterprise. Details matter. This section seeks to exemplify that.

I have chosen to focus on research on the activities of teachers and their students in the classroom for several reasons. Elmore (2011) calls this, and those things that impinge directly on it such as teaching materials and professional development, "the zone of instruction." This is where educational improvement happens; the rest is, at best, merely supportive. Further, within classroom research, classroom observation is the most challenging single aspect. Of course, other kinds of information are important: student work, student and teacher responses to questionnaires probing their activities and attitudes, teacher logs and teaching materials, are all important sources of complementary information.

I shall also concentrate on observation because of the richness of the information that is in principle available and the challenges of selecting and collecting what is significant in a form that can be analyzed to yield useful insights, both specific and general. I believe that there is no adequate substitute for structured observation, expensive though it is in time, and therefore resources. Equally, it is an area of research design with opportunities for improvement.

Classroom observation: three case studies in data selection

The flood of available information in a mathematics lesson is overwhelming. A television picture transmits millions of bytes per second, yet it can capture only a small part of what is visible in a classroom – missing, for example, most student discussion and written work. At a less information-theoretical level, the information flow is still unmanageable. Selection is inevitable. The research challenge is to understand what is going on, so as to select, to capture, and to analyse, what is most cost-effective for the purposes of the research. There are inevitable tensions, and the necessary trade-offs, in optimizing the selection and collection of data in research. The theme here is "horses for courses" – that the optimal choices depend on the phenomena on which the research chooses to focus, and theoretical ideas it seeks to test. Cost-effectiveness is at the core of the design challenge.

Any research design also needs to look at how best to communicate the analysis to the "positive-thinking skeptics' that form any good research community. To make the research process explicit, Alan has long argued (see e.g. Schoenfeld, 1980) that researchers should make their data available, along with rich enough descriptions of their research methods such that readers could themselves examine the data and evaluate the inferences. He has done so over his career, producing "inspectable" studies that make both substantive and methodological contributions.

From the myriad of published "observation schedules" (see e.g. Good & Brophy, 2002), I have chosen these three because they all seek to capture aspects of the richness that is present in mathematics classrooms, each combining breadth with attention to detail. These cases illustrate three very different approaches to capturing what happens, each with a different balance of priorities. The first emphasizes depth of understanding of teachers' decision making, down to the level of their individual "moves' in a lesson; the goal was to construct a theoretical model of a specific area of human problem solving: teaching. The second was designed to find how far the pattern of dialogue in classrooms changed when teachers used specific new materials; the complementary goals were to elicit some design principles, and to provide feedback for refining the materials, so the study needed to cover many lessons. Both achieved their very different goals. The last (still in development) has a balance of these priorities, covering many lessons with a focus on the mathematical nature of the discussion and teacher professional development over a year.

Teacher decisions focus

The first case comes from Alan's long running "teacher modeling" program, published in a series of papers and brought together as the core of his book: *How we think: A theory of goal-oriented decision making and its educational applications* (Schoenfeld, 2010). This study is based on an extremely detailed analysis of video of three lessons, taught by very different teachers: two highly experienced and innovative, the third a recent graduate. The goal of the research was ambitious: to understand every move the teacher made in the lesson in terms of three dimensions: *their knowledge*, *goals* and *orientations* (earlier called beliefs). Knowledge is defined broadly, including mathematical knowledge and skills, pedagogical content knowledge, and knowledge of pedagogical strategies, tactics and skills. The meaning of goals and orientations will become clearer through the example below.

The data is presented in three parallel streams, the latter two subdivided, with time increasing down the page. The streams are increasingly analytic, namely:

– *a full transcript* of the dialogue

– *a parsing of the dialogue*, with levels of increasing detail, from the major activities of the lesson down to the smallest self-contained episodes.
– *a graphical representation of goals and orientations* in the form of continuous vertical bars, shaded to show the level of activity of each at that point in the lesson.

This graphical representation can be seen both as a fine-grained description of the lesson as it unfolded and as the basis for a model of the teacher's decision making: a model equipped with the knowledge, goals, and orientations found in the graphical representation would produce decisions consistent with those of the teacher.

These elements are illustrated for two short sections of the lesson (from Schoenfeld, 2010, chapter 5) in Figures 3 and 4. The teacher is a distinguished science teacher and the lesson is about criteria for choosing "the best number" from a set of measurements. The teacher motivates the discussion in terms of tests of blood alcohol concentration; the students then make multiple measurements on something more accessible – the length of a table.

The way the analysis is structured and communicated exemplifies Alan's belief, noted above, that readers must be able to follow the data and its analysis in enough detail to allow them to critically review the author's thinking – the opposite of "trust me" styles of commentary in some research. Here I can give only a flavor of the way this is done and the tools he developed to do it.

Figure 3 shows how the transcript of the opening of the lesson, which is largely organizational, is parsed at increasing levels of detail. The focus here and throughout is the detailed attention given to understanding the raw data, epitomised by the transcript but enriched by other aspects of the video.

Figure 4 shows how the parsing of the more complex dialog in the core of the lesson, on choosing an appropriate summative measure, is analysed into the bar notation which shows the active goals and orientations at each point in the dialogue. The main goals and orientations, noted at the bottom of the figure, show that, while goals g and l are concerned with the key content to be learned, the other goals are focused on the classroom dynamics that will support the learning processes, reflecting the teachers orientations rather as tactics support strategy.

From analyses of this kind, Alan and his students and collaborators, have built up a theoretical model of teachers' decision making. I hope this brief sketch will encourage the reader to enjoy the rigor of the analysis by reading *How we think*, which goes on to apply a similar methodology to other areas of human real-time decision making, including medical diagnosis.

Classroom discussion focus

In this case the goals and the context were quite different. The study complemented and supported the ITMA program of design and development of educational software. ITMA (Investigations on Teaching with a Microcomputer as an Aid) focused on a single computer with a large (TV) display in each classroom – an approach that realistically reflected the level of hardware provision at the time.[5] The project leader, Rosemary Fraser, had found in her own classroom that simple non-routine

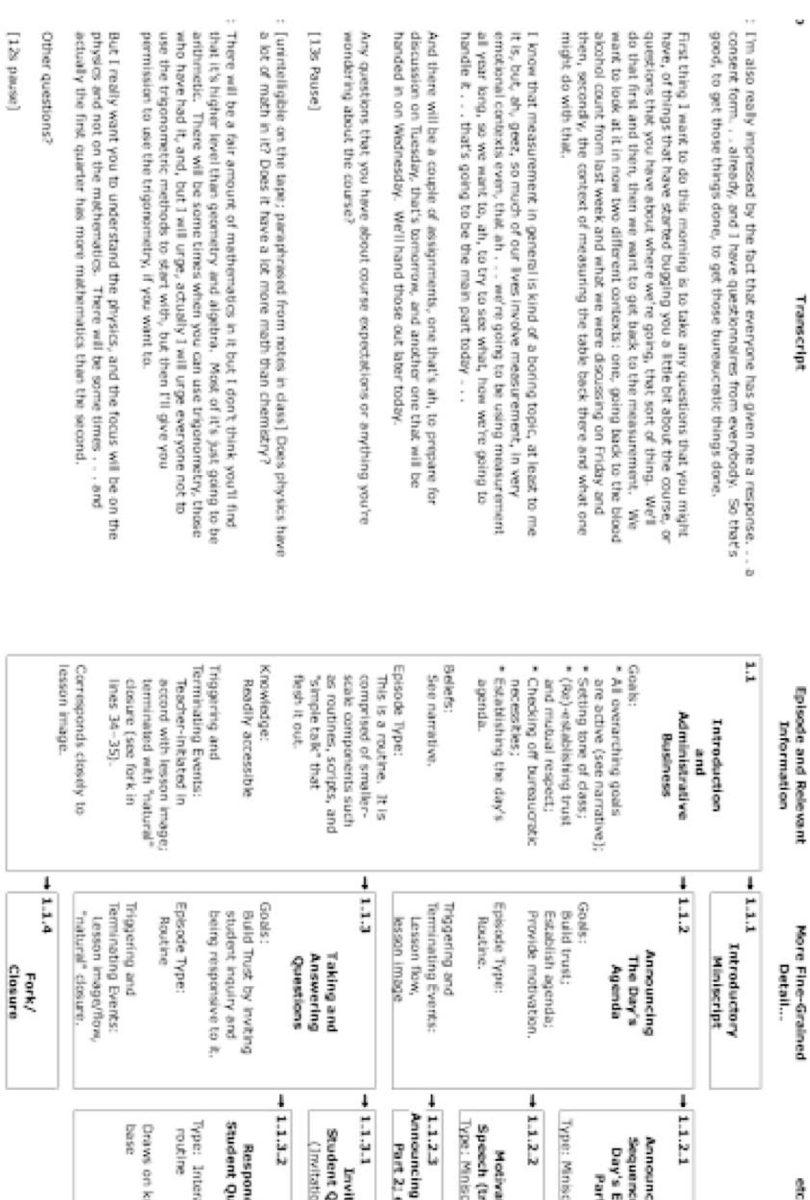

Figure 3. A multi-level parsing of the introductory episode of the lesson.

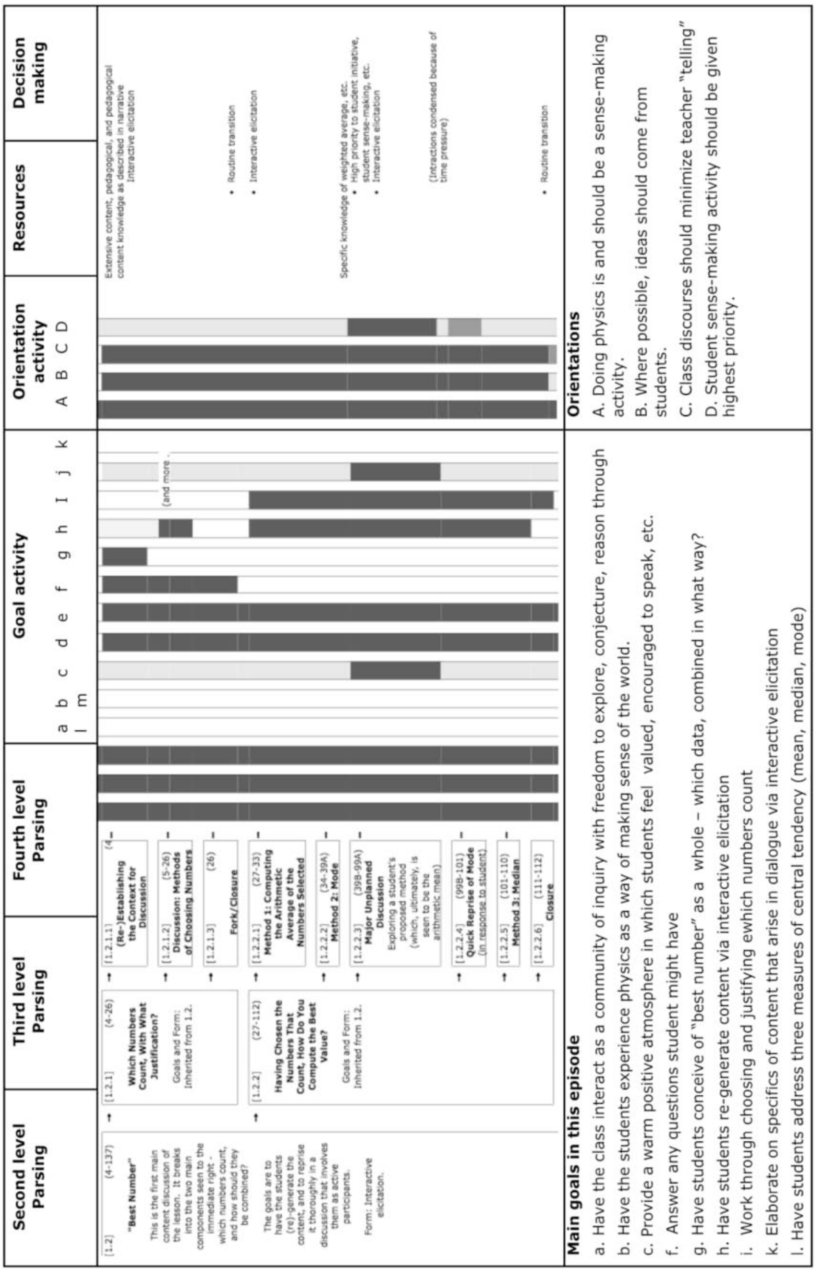

Figure 4. Deciding on the best number to summarise a set of measurements.

problem solving software of this kind promoted student engagement and mathematical discussion. The ITMA team of teacher-programmers designed and developed many examples of such software, along with lesson notes for the teacher.

The research goal was to understand better what happens when this material is used by a variety of teachers in their classrooms, and to exploit that understanding in the design and development of such software and accompanying curriculum support. Seventeen teachers agreed to choose and use 10 lessons from the draft collection, and to be observed in their normal teaching and in using these new lessons.

We found that structured classroom observation was essential to capture the changes in the pattern of interpersonal dynamics that the team had found in their own classrooms. How far would the materials lead other, more typical teachers to work in similar ways? We needed a lesson observation protocol that would enable observers to capture key information within the time and effort we could afford. With about 200 lessons to study, we decided on one hour for "live" observation and a brief post-lesson discussion with the teacher, with about one further hour for the analysis of that lesson.

We decided to design our own observation system, based on an intense open-minded study of 10 examples of lessons on video. Three of us viewed these lessons many times, discussing what we could see that seemed to us significant for our purposes. Terry Beeby, the graduate student, got to know the lessons so well that, whenever in our discussions a type of event was suggested as significant, he could quickly find similar examples for discussion.

We were particularly interested in those things that differed from teacher to teacher, and from conventional mathematics lessons to those using ITMA software. Of the things we saw in the video lessons, the variation in the patterns of discourse were particularly striking, with profound changes from the teacher-directed nature of most British mathematics lessons. The outcome of this tool design process was *SCAN – a systematic classroom analysis notation for mathematics lessons* (Beeby, Burkhardt, & Fraser, 1980). Key features include:

– Use of a shorthand notation (rather than box-ticking or diagrams)
– Three timescales: *activities* within the lesson, self-contained *episodes*, linguistic *events*
– *Events* include: **q**uestion, **e**xplanation, **i**nstruction, **h**ypothesis, **m**anagement, social **g**ambit with qualifiers for:

  Initiator: assumed to be the teacher; pupils **p** or numbered
  Depth: α recall of a single fact, β familiar exercise, γ extension
  Guidance: **1** detailed, **2** specific, **3** open
  Correctness: $\surd$ correct, **x** wrong, **?** unclear

This method of developing observation tools, through the intensive study of a sample of videos to identify and classify events that are significant from the point of view of the study, is of general value. In this respect, it is rather like the previous example, though covering many lessons made cost-effectiveness more important.

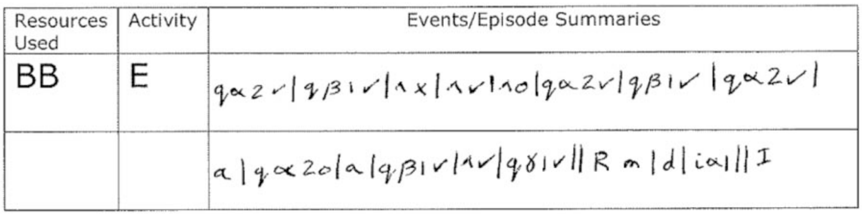

Figure 5. A SCAN record of a simple lesson opening.

As with any shorthand, it takes time to become fluent in the notation. But, for example, 20 teachers after three hours training on video produced very consistent "live" SCANs of a simple conventional lesson on polygons. Figure 5 shows a SCAN record of the first five-minute exposition (E) activity at the blackboard (BB).

The teacher launches the lesson with an exposition activity **E** by checking that the students know some basic definitions (a revision episode, **R**). Note the linguistic style, dominated by short questions **q**, mostly of single facts α with fairly close guidance **1** that elicited correct responses $\surd$. (^ signals a repeat of the question.) The teacher then initiates a second activity (I) of individual student work (W1 on the following line, not shown); in this he gives the formal definition of polygons **d**, then gives detailed instructions i for a simple activity: working some similar cases. This is a teacher who uses the Q&A mode of exposition, which is common in the UK. He gives a lot of support to his students, while keeping them on a short leash intellectually. The SCAN provides detailed semi-quantitative evidence of this.

The three lesson extracts in Figure 6 are more interesting, in themselves and for the purpose of the study. They show different teachers working with the same piece of software: a simple "function machine" program. In using it you give JANE a number; when you press the answer key, she gives you one back. The question is "What does JANE do to numbers?" (There are six girls, who multiply, and six boys, who add different numbers. You can go on to a "function of a function" investigation, involving a boy then a girl or vice versa.) The mathematical purpose is to develop students' hypothesis generation and an awareness of the implications and limitations of evidence – that a counter example kills a conjecture but many examples are not a proof. You also practice mental arithmetic.

What did we learn from the full SCANs, along with the lesson materials, some student work and the less-structured notes of the observer? The lesson worked well for all three teachers, with nearly all students focused throughout. The three teachers worked in very different ways. These show in the simple statistics in Table 2. Note, for example, the differences in the distributions of α, β and γ questions and the number of pupil explanations across the three lessons

Looking at the rhythm of each lesson, even these short extracts show that Teacher A established a rhythm of very short "search successful" (**SS**) episodes; these continued through the lesson, exhaustively repeating the same pattern before going onto the two children challenges, which then became exercises in combining operations the class had already worked out. In contrast, Teacher C had much longer episodes, collecting multiple alternative hypotheses and delaying closure. Later,

Table 2. Comparative statistics on 3 teachers working with JANE.

Lesson		*A*	*B*	*C*
questions asked				
(resolved)	α	17(15)	7(5)	3(0)
	β	15(15)	17(14)	10(10)
	γ	7(6)	9(7)	15(9)
explanations		8	12	6
assertions/instructions		1	2	6
student questions		1	0	0
student explanations		0	4	11

after collecting much confirmatory evidence on one hypothesis, he asks "Can we be sure?"; after a long pause with no response, he squares his mathematical and pedagogical consciences with "Well, we can be pretty sure" – which seems fair, in the universe of 11 year-old students who are too young to have the concept of rigorous proof (Bell, 1976).

One of these teachers also taught the polygon lesson of Figure 6; the reader is invited to guess which one from the evidence in the SCANs on their styles.

The outcomes of this work included both improved lesson units and their associated software, and some insights with wider implications for design. I will mention one: the *roles analysis* (Burkhardt, Fraser, Coupland, Phillips, Pimm, & Ridgway, 1988). In analysing the SCAN data, the researchers were struck by the various roles played in the classroom dynamics by the teacher, the students and the computer. Far from the computer being an inanimate tool, it was clear from the reactions that each piece of software gave it a personality,[6] as with "What does Jane do to numbers?" Detailed study identified about 30 roles, which we boiled down to 6 main groups, shown in Table 3. Most of the names are self-explanatory. Counsellors advise, they do not direct or explain. A *Resource* supplies information, but only when asked.

Table 3. Classroom roles distributed among teacher, students and microcomputer.

Directive roles	*Facilitative roles*
Manager	Counsellor
Explainer	Fellow Student
Task-Setter	Resource

In regular mathematics lessons, most teachers take the directive roles, the students *(Fellow) student*, and the resources are inanimate – typically textbook and worksheets. In lessons with the ITMA software, the software on screen took over

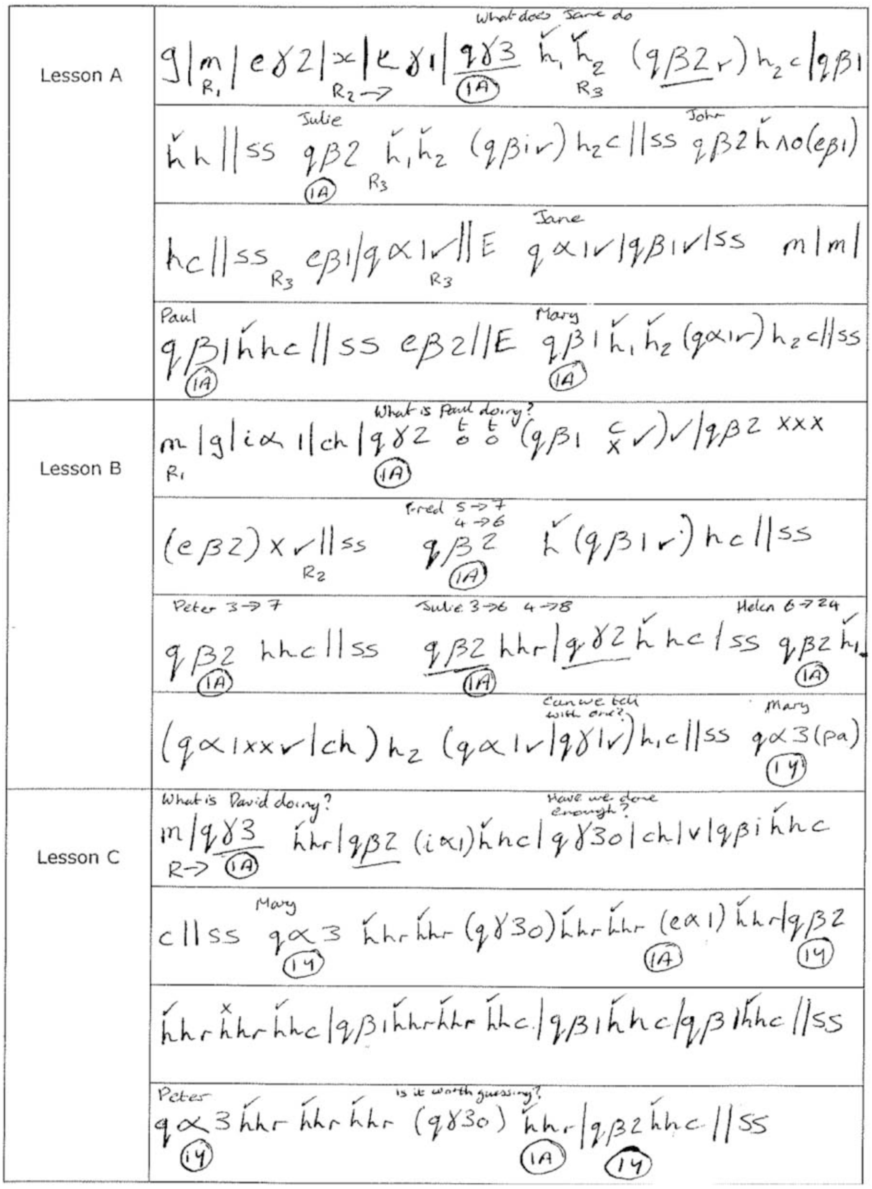

Figure 6. SCAN records of three teachers working with JANE.

much of the *manager*, *explainer* and *task-setter* roles, and the teachers moved to play *counsellor* and *fellow student*. Somewhat to our surprise, without any prompting teachers moved from the front of the class, talking with students about "What's it doing?" Sixteen of the seventeen teachers made these role shifts naturally. (The exception stood proudly next to the screen throughout the lessons, sharing its role!) Alan has never liked SCAN, primarily because it does not record the specific mathematical content of the discussion in each episode. (The teaching materials and student work samples do, of course, partly fill this need but they are not linked to specific events in the lesson by the SCAN record.) The next case describes our current efforts to meet this concern in a protocol that combines something of the economy of SCAN with a deeper look at the mathematical structure of the discourse.

Mathematical discussion focus

The Mathematics Assessment Project (MAP) is developing lesson materials that support formative assessment for learning in US classrooms. The power of formative assessment for learning, when it is done well, was summarized in the metanalysis of Black and Wiliam (1998). Their and others' subsequent work has approached the challenge of making formative assessment happen through professional development; they find long-term and intensive work with teachers is needed, making the challenge of "going to scale" something between very expensive and unrealistic. The MAP lessons are a product of the first engineering research on supporting formative assessment for learning primarily through teaching materials (MAP, 2012).

The Shell Centre design team is led by Malcolm Swan, with Alan as PI of a Shell Centre-Berkeley collaboration. The previous emphasis on professional development reflects the fact that these lessons take most teachers of mathematics well outside their pedagogical and mathematical comfort zone. The lessons provide support for teachers in this broadening of their professional capacity. They are being used in school systems across the US to support the implementation of the Common Core State Standards for Mathematics. The initial reception has been enthusiastic.

Structured classroom observation of trial lessons has guided two iterations of revision of each lesson. Now we need to learn in more detail and more depth about what happens as teachers gain experience in using these materials. Alan is leading the team in a program of research in which the design of an appropriate protocol for observation and analysis will play a central role.

We plan to observe 20 teachers, each using 10 of the formative assessment lessons in the course of a school year, along with some of their normal teaching. Each lesson will be videoed. Nonetheless, as in the design of SCAN, for an analysis of around 200 lessons, cost-effectiveness is a prime consideration. The development of the protocol is ongoing but the current version has the following features.

In the large, the goal of the research is to produce an analytic scheme that captures the things that research indicates are the essential aspects of a lesson –

the goal being to document the relationship between the presence and frequency of those classroom behaviors and the depth of student learning. The former will be captured by the analytic scheme, the latter by robust tests of student understanding such as the Balanced Assessment/MARS tests.

The characteristics of such a research-in-practice analytic scheme must be radically different than those of the scheme in the section on "Teacher decision focus." The analyses in Schoenfeld's book took years to produce; in contrast, a SCAN coding can be done in real time. The goal of the current analyses is to produce a coding of a lesson in no more than twice real time (a real-time observation plus the same amount of time to convert one's observational notes into a formal coding record), while at the same time being directly sensitive both to important classroom activities and the quality of the mathematics being discussed.

After much experimentation, the MAP team converged on a scheme that has five "process- or practice-related" dimensions and one focused content-related dimension. Ultimately, these are coded in five different types of classroom activity.

First, the dimensions for analysis. The research team believes that each of the following dimensions are central in examining classrooms:

1. Mathematical focus, coherence, and accuracy. Is the mathematics discussed rote and mechanical, or are procedures connected to underlying concepts? Do the students have the opportunity to do sense-making? If the students do not have the opportunity to engage with meaningful mathematics, they are not going to learn it.

2. Cognitive Demand. Classroom observation shows that, when students encounter difficulty, many teachers provide "help" that actually removes the main challenges from the task, lowering the level and depriving the students of the opportunity for productive struggle. Are classroom interactions structured so that students have the opportunity to grapple meaningfully with the mathematics?

3. Access. Which students get to participate. Are most of the students involved, or only a select few?

4. Agency: Accountability and authority. Do students have the opportunity to speak and write mathematics, to become expert and share that expertise?

5. Uses of assessment. Does the teacher obtain information about student understandings, formally or informally, and use that information in ways that allow the lesson to build on student understandings and address misunderstandings?

6. Domain specifics. If a lesson focuses on a particular topic, what is the most important mathematics in that topic? Does the lesson grapple with that content? The sixth dimension is handled separately. For each of the first five we have a general rubric on a 3-point scale, outlined in Table 4.

This is a broad summary. In fact, we employ context-specific versions of this rubric for each of the following classroom activities:[7]

- teacher giving directions (setting up or modifying tasks for student work)
- teacher exposition of mathematical ideas (this may be in the form or lecture or classroom summary)
- classroom discussion of mathematical ideas, in which there are student contributions;

Table 4. Dimensions and levels for the MAP observation protocol.

Level	Focus, Coherence & Accuracy	Cognitive Demand	Access	Agency: Authority and Accountability	Uses of Assessment
1	Skills-oriented focus; little or no attention to concepts and connections.	Content is proceduralized to where it becomes rote.	No apparent effort to improve access; uneven pattern of participation.	Teacher presents information and judges student work	No evidence of *collecting or using* student reasoning.
2	Some attention to concepts and connections, but little explanation	Students are supported in making connections between procedures and concepts	Some efforts to invite student participation	Students have some time to engage/explain, but their role is often reactive; the bottom line is teacher authority.	Student reasoning is elicited or referred to and corrected when in error.
3	Significant attention to explanations of procedures, concepts, & connections between them	The teacher's hints or scaffolds support students in "productive struggle" in working complex problems and building understandings	Clear efforts to invite and support broad student participation	Students are expected and encouraged to explain and respond to mathematical ideas.	Student reasoning is referred to and discussed, sometimes affecting directions of classroom discussion.
	Focus, Coherence & Accuracy	Cognitive Demand	Access	Agency: Authority and Accountability	Uses of Assessment

– students seek to clarify mathematical ideas and/or reveal confusion
– connecting to prior knowledge (can be during set-up, or when discussing work on problems)

This scheme is still under development, but preliminary testing indicates that it has some face validity with teachers, and meets the constraints discussed above –

lessons can be coded in no more than twice real time, and with some degree of consistency. Time will tell with regard to the scheme's utility. "Watch this space."

In summary

The purpose of this section has been to make and illustrate three points in the challenging context of designing tools for classroom observation:

- Designing reasonably efficient methods of data selection, capture and analysis is at the heart of good research design.
- The design will always involve trade-offs, with the balance determined by the project's research priorities – this implies the design should normally be custom-tailored and, of course, "mixed methods."
- The earlier in the process that redundant data can be discarded, the lower the cost – provided, of course, that you don't throw away essential data.

The three cases outlined here reflect different priorities. Each was a choice that suited the purpose in hand. All three could be improved and extended with additional resources.

I have featured this detailed technical aspect of research for several reasons:

- the central importance of capturing rich data from the classroom;
- the interesting challenges of doing observation well;
- the potential that technology offers in this area.

There are already devices that link written notes to an audio recording, so that the touch of the special pen on a note replays the audio from the moment it was made, allowing easy reconsideration and expansion of interesting events. Apps for both tablets and smartphones will allow us to show on screen a rich analytical framework for observation, so that observers' input can more easily be made in real time, and captured automatically for analysis. As ever, we will have to be vigilant that the technology does not impose standard solutions that undermine the research quality.

TOWARDS MORE PRODUCTIVE RESEARCH: A "SYSTEMS" PERSPECTIVE

This section brings together the strategic and tactical issues discussed so far into a set of suggestions on changes in the grand strategy for research in education that would enable it to make a greater contribution.

The argument builds on previous sections and the synthesis in the paper "Improving educational research: towards a more useful, more influential and better-funded enterprise" (Burkhardt & Schoenfeld, 2003). Looking at education in comparison with other fields, this paper identifies six elements that are needed for a research program to have impact on practice. These are shown in Table 5.

The paper goes on to look in more detail at the various barriers to such change, and ways in which they might be overcome. Here we discuss the implications for various key communities – researchers of various kinds, teachers, schools, school systems and policy makers.

Table 5. Elements needed for research to improve practice.

1. Robust mechanisms for taking ideas from laboratory scale to widely used practice. Such mechanisms typically involve multiple inputs from established research, the imaginative design of prototypes, refinement on the basis of feedback from systematic development, and marketing mechanisms that rely in part on respected third-party in-depth evaluations. These lab-to-engineering-to-marketing linkages typically involve a strong research-active industry (for example, the drug companies, Bell Labs, Xerox PARC, and IBM).
2. Norms for research methods and reporting that are rigorous and consistent, resulting in a set of insights and/or prototype tools on which designers can rely. The goal, achieved in other fields, is cumulativity – a growing core of results, developed through studies that build on previous work, which are accepted by both the research community and the public as reliable and non-controversial within a well-defined range of circumstances. (Work on the cutting edge is something else, of course, with some uncertainties and controversy in every field of research.)
3. A reasonably stable theoretical base, with a minimum of faddishness and a clear view of the reliable range of each aspect of the theory. Such a theory base allows for a clear focus on important issues and provides sound (though still limited) guidance for the design of improved solutions to important problems.
4. Teams of adequate size to grapple with large tasks, over the relatively long time scales required for sound work of major importance in both research and development.
5. Sustained funding to support the Research-to-Practice process on realistic time scales.
6. Individual and group accountability for ideas and products; do they work as claimed, in the range of circumstances claimed?

"Importance" – For whom?

In most societies, the long-term goal for education is to improve the outcomes for children in terms of performance and attitude – the range of things they can do well, know about, use effectively, and enjoy. How to achieve this is a high-profile issue of policy and politics.

The educational research community surely shares these goals. How well is it structured to focus on them? Like any community, it has its own agendas and inward-looking concerns. The great majority of researchers are in academic institutions, so the community needs systems for evaluating work and selecting people for appointment, tenure and promotion. Research in education has a value system that guides these judgments, outlined in Table 6.

Table 6. Current academic priorities tend to favor.

new results	*over*	replication and extension
trustworthiness	*over*	generalizability
small studies	*over*	major programs
personal research	*over*	team research
first author	*over*	team member
new ideas	*over*	results that can be relied on
disputation	*over*	consensus building
journal papers	*over*	products and processes

It will be clear from the argument so far that these are not the priorities that are likely to lead to building a body of reliable detailed research that can underpin design, and thus build a direct link from research to improved practice. Indeed the second and third sections and Table 5 suggest that they are likely to have the opposite effect. How has it come to be this way?

First, how do these priorities serve the internal needs of the research community? If you look for a fundamental measure of quality in research in any field, it is self-referential.

Impressing key people in your field

is the prime criterion. Each field turns this into a set of quasi-objective criteria. How did education come to the pattern in Table 6?

There is a pattern of pressures on researchers that helps explain. Researchers, being human, tend to like research similar in style to their own. Academics are usually only part-time researchers, with substantial loads of teaching, and administration of courses. Yet, to be seen as successful, they are expected to produce several journal publications a year. Acceptance by journal reviewers depends on the studies being seen as "trustworthy." Ph.D. students need to be trained in research and to produce publishable work within three or four years. Assigning credit is more difficult with multiple authors, let alone large teams. As explained in the third section, all these factors encourage neat small-scale "science" studies. Partly because of the limited empirical warrants that such studies provide, there is a continuing acceptance of commentary in the humanities tradition – interesting and plausible new ideas get published, noticed and cited, despite the paucity of evidence on their validity and generalizability. Replication, a key element in scientific research, is simply not sexy.

All this does much to explain why education lacks a body of generally accepted research results; in other research-based fields there is often intense disputation, but only at the cutting-edge of new research. There *is* a modest body of research in education that is beyond dispute within fairly well-defined boundaries. To take one example, there is a "common sense" policy in some US states of making students who fail repeat a grade; yet many studies have shown that this produces little or no improvement in performance for most students and a large drop-out rate. Many design principles, like those mentioned in the section on SCAN, are supported by a solid body of evidence from design research (though much of it is unpublished). There are other examples. But building a growing body of reliable evidence requires careful work, with replication across a variety of circumstances to establish boundaries of validity of the insights involved. Because such work does not fit the current academic value system, little of it is done.[8]

How is this avoided in other fields?

What can education learn from science, engineering and medicine that would mitigate these pressures and improve the value system for research? There are various elements. In every field of research, significant new ideas and discoveries always have the highest prestige – but they have to earn it. Because there is an established body of research results, and theoretical models that reflect it, any new suggestion will have implications – so new results must be tested. Other researchers in the same area will seek to replicate the ground-breaking study, to probe its research design and analysis for weaknesses and alternative explanations of the result. There is prestige in being active in these sub-communities.

In many fields, the key experiments can *only* be done well by substantial teams over periods of years.[9] (The core of my argument is that education is such a field.) Mechanisms have been developed for giving appropriate credit to individuals, according to their contributions to the work of the team. Ph.D. students are given specific jobs of experimental design, construction or analysis to carry through, and to write up in the wider context of the whole experiment as their dissertation.[10]

Underlying all this is, of course, money. In science, engineering and medicine it is accepted that serious research needs explicit funding, for the salaries of research team, including the time of leading academic researchers, and for the equipment and running costs of the enterprise. This has led to billion dollar budgets in science, engineering and medicine with government-funded initiatives that, if successful, are taken over and developed further by research-based industries. Antibiotics, nuclear energy, electronics, the internet and the world wide web are only some of a broad spectrum of examples where this has happened.

What is the situation in education? Tens of thousands of people in universities around the world do research as part of their academic work. While there is little or no marginal funding for most people, the total cost of their research time is substantial.[11] Could the impact be increased by a more coherent system?

There are agencies that fund research in education, but they have budgets that are orders of magnitude smaller than for science, engineering and medicine. History may help us to understand why. Research budgets in science and medicine were small a century ago; they boomed only during and after the second world war, when these fields produced results with a practical payoff that society recognized and wanted, including the notable examples just mentioned. Though the need in education is well-recognized, educational research has yet to make that breakthrough. *To do so, it will need to have a direct beneficial impact that society recognizes*.

Which brings us back to "importance," the third dimension in Alan's classification of research studies. The discussion so far implies that criteria for assessing importance should take impact on practice very seriously. For this the engineering research approach provides the cutting edge of the research enterprise, turning reliable insights from other research into design principles, tools and processes of direct use in practice. Equally, this needs reliable insights from science research to build on. It is encouraging that funding agencies in education tend to put most of their money into studies that they believe will have direct impact on practice. They are still far from achieving the kind of coherent support that is summarized in Table 5.

Why doesn't it happen? A key reason is the absence of serious evaluation. There are few substantial studies of widely available materials. Those there are tend to be profoundly inadequate, often looking only at student learning outcomes – usually scores on tests that assess only a subset of the learning goals. The ambitious *What Works Clearinghouse* review of mathematics curricula illustrates many of the problems, both in methodology and in lack of adequate research input. Schoenfeld (2006) vividly tells the unhappy story.

While if they were well done, such comparative reviews might help client school systems make better informed choices, they give no guidance on how to improve the products. For that one needs to know, in detail like that discussed above:

- what actually happens in classrooms
- with teachers at various levels of professional development
- using specific materials of various kinds
- with students of various abilities and backgrounds, as well as
- outcomes across the whole range of goals.

The skills needed for such work are in the mainstream of insight-focused research in education but the scale means that it needs large teams, and is therefore expensive. I estimate that to get enough high-quality information to guide the next round of improvement to the NSF mathematics curricula would cost around $100 million, comparable to what was spent in their development. Such knowledge in depth would move the field forward. It looks expensive but we will show that such costs are trivial in the context of the education system.

As it is, published curricula are evaluated the same way that movies, plays and restaurants are reviewed. The differences between well-presented draft materials and a well-engineered product that works well are not obvious on inspection. So it is not surprising that publishers see no need to pay the higher costs of research-based development. As a result, education has no research-based industry of the kind that, in other research-led fields, takes much of the engineering load of turning prototypes into robust products.

The new balance – A vision for an effective research community

Let us look in a bit more detail as to the sort of pattern of research that would make educational research the "go to" community for policy makers seeking to improve education, as medical research is when health issues arise. Table 5 makes it clear that major changes are needs, leading to coherent ongoing programs of research and development. There are many ways this might be achieved. Here I outline a model that draws together the diagnoses of system problems so far into an explicit "solution" that might provide, at least, a basis for useful discussion.

The changes I envisage include three strands, listed here with their aims in terms of the knowledge, goals and beliefs behind government decision making:

- *Evaluation*, so that both current problems and the impact of initiatives can be recognized and understood.
  This will enhance government knowledge of the current situation and, more importantly, provide evidence to encourage their currently-intermittent belief that these problems need well-engineered solutions.
  This will include both survey research and the collecting of much more detailed information on the implementation and outcomes from specific initiatives, independently carried out but on a basis, and using research tools, agreed

with the developers and their funders. These studies will have a formative focus, as well as providing summative information to guide policy.

- *Development*, so that, in recognised problem areas, well-engineered products and processes are developed to help professionals realize their and the system's goals more effectively.

  This will reinforce government knowledge of effective change processes, and gradually undermine their belief that "the profession" will be able to find good solutions to any problem (a belief that all professions encourage). This will be based around ongoing programs in specific areas by established research teams, with two or three working in parallel on major challenges (again as in frontier science and engineering).
- *Cumulative research*, so that the community builds a body of research, with established reliability and bounds of validity, that goes beyond "worth paying attention to," providing a solid foundation: for design, better than authors' experience; for policy, better than politicians "common sense."

This will encourage governments to ask for advice from the research community, and to take it, recognizing that there is a zone of reliable knowledge they do not own. (Advice to government in other fields is always based on the accepted body of research results, plus warnings of uncertainties.)

This approach will require building research collaborations in each important area, with groups doing parallel studies on important issues in varied but related circumstances using common treatments and instruments. The challenges of tool development (discussed in the context of classroom observation, above) and the collecting and analysis of adequate data sets will be shared in a co-ordinated way. The goal of each group is a set of results that can only be challenged at the boundaries.

Note that there is no mention here of changing the educational goals of government. There are, of course, disagreements – for example, about the appropriate balance between general education and specialised study and training. However, much the largest mismatch is between current shared intentions and actual outcomes in practice. Finally, one must never forget the prime goal of democratic governments: to get re-elected, which militates against controversial change and spending money. However, many governments have made a commitment to evidence-based policy, at least at the rhetorical level. There are some intermittent signs that they will move forward with this on some fronts.

Funding will be needed for most of these things to happen. The next section estimates the costs of doing research-for-practice reasonably well. However, it is worth looking at what might be achieved within the enormous existing resources represented by the research time of the academic community. There are opportunities.

Evaluation of the kind sketched above lies within the skill set, if not the current practice, of educational researchers. Given its crucial role in convincing politicians that research pays off in their terms, it is here that the best route to bootstrapping a substantial investment in research may lie.

Building an accepted research base offers a major opportunity to the research communities to undertake longer-term research with replication to explore the generality, and the boundaries, of interesting results.

Engineering and design research teams enjoy relatively good financial support from government, reflecting their perceived value in developing robust solutions to difficult challenges. However, their funding is rarely even medium-term, each project being a one-off; closer links with the research enterprise in their institutions could "bridge" the funding gaps more effectively than at present, if more of their colleagues saw design and engineering research as of value.

These things all imply that collaboration must be recognized as positive, requiring changes in the current academic value system. This remains a major challenge. Money can help: even modest amounts of funding would allow these things to happen, and give researchers some feeling of recognition that is different from acceptance of their papers by journals. At least as rewarding is for researchers to see their work having beneficial impact on children and teachers; specific mechanisms for this should be part of research designs. Most academic researchers will continue doing what they do but there are enough of them for even a modest shift in the balance of research styles to have real impact.

What would all this cost?

A research-based approach costs much more than simple authorship – the standard approach in which experienced professionals write down and publish what has worked for them, without thorough developmental testing. Research-based design and development normally needs several rounds of trials, with rich and detailed feedback from a variety of classrooms guiding the revision and refinement of the products. It becomes part of a continuing program of formative feedback, which contributes both new insights and new products to the overall program.

One can get a rough estimate of costs from some examples; in current terms, adjusted for inflation:

- NSF mathematics curricula in the US were funded in 1990 at rather more than $1 million per school year of 180 lessons; the second round of implementation funding plus inflation raises the cost to around $15,000 per lesson.[12]
- Shell Centre development of 3 week "replacement units" in the 1980s cost £100,000 for 15 lessons, around $30,000 per lesson now.
- The formative assessment lessons in our current MAP development are costing around $30,000 per lesson.

If we accept $30,000 as a typical estimate, what would the cost implications of this approach be for the whole curriculum in the US? Let us err on the high side:

- 25 hours a week for 40 weeks a year for 13 years ~ 13,000 hours;[13] double this for children with different needs
- $30,000 per hour lesson ~ $800 million

A round of total re-development would take at least 10 years ~ $80 million per year. The annual running cost of the US K-12 education system ~ $400 billion.

Cost of high-quality materials development ~ 0.02% of turnover!

If a country won't spend that, it isn't serious – or it doesn't believe a research-based approach has significant advantages. In practice, not everything will need to be redeveloped every time . Clearly, cost should not be an issue; selling the concept of research-based development, and its more effective organization, are the challenges.

Technology may possibly offer a way forward here, because the costs of programming justify the costs of systematic design and development. So far technology has had minimal impact on modes of learning in mainstream mathematics and science education, which have become seriously out of line with the way mathematics and science are done outside school. Small scale work over the last 30 years has shown enormous potential in many diverse modes of use of technology, but no curricula in which technology is fully embedded have yet been developed. This is largely because of a mismatch of timescales: a seriously innovative curriculum takes 10–25 years to develop while the technology changes every few years, so there has been no stable "platform" for which to design. That situation may be changing. There are exciting current initiatives that are developing curricula without printed materials, where every student has a tablet computer. However, the challenge of doing this well tends to be underestimated. Realizing the potential of the technology will need fine designers who have explored and absorbed its affordances, so they can again focus on students and teachers.

The status and roles of "theory"

Finally, as a coda to this chapter, some comments on theory. Theory is seen as the key mark of quality in educational research. I am in favour of theory. (Indeed, in my other life, I am a theoretical physicist.) However, in assessing its roles in any field, it is crucial to be clear as to how strong the theory is. From a system point of view (Burkhardt, 1988), the key question is:

How far is this theory an adequate basis for design?

Again, it useful to look across fields. In aeronautical engineering, for example, the theory is strong; those who know the theory can design an airplane at a computer, build it, and it will fly, and fly efficiently. (They still flight test it exhaustively.) In Medicine, theory is relatively weak, but getting stronger. Despite all that is known about physiology and pharmacology, much development is not theory-driven. The development of new drugs, for example, is still often done by testing the effects of very large numbers of naturally occurring substances; they are chosen intelligently, based on analogy with known drugs, but the effects are not predictable and the search is wide. However, as fundamental work on DNA has advanced, and with it the theoretical understanding of biological processes, designer drugs with much more theoretical input have begun to be developed. This process will continue – indeed there is now work, for example, on cancer drugs tailored to an individual's specific tumour.

In the range and reliability of its theories education is a long way behind medicine (perhaps 100 years), let alone engineering (at least 350 years). The much-quoted theories in education are ambitious. By overestimating their strength, damage has been done to children – for example, by designing curricula based on behaviorist theory. The current dominance of constructivism is similarly inadequate, though less dangerous. Its incompleteness is more obvious, since it is impossible to design a curriculum built only from constructivist principles. It is not that behaviourism or constructivism are wrong; indeed, they are both right in their core ideas, but they are incomplete and an inadequate basis for design. Physicists would call them "effects." The harm comes from overestimating their power, ignoring other effects.

Let me illustrate this distinction with an example from meteorology. "Air flows from regions of high pressure to regions of low pressure" sounds and is good physics. It implies that air will come out of a popped balloon or a pump. It also implies that winds should blow perpendicular to the isobars, the contour lines of equal pressure on a weather map, just as water flows downhill, perpendicular to the contour lines of a slope. However, a look at a weather map shows that the winds are closer to parallel to the isobars. That is because there is another effect, the Coriolis Effect. It is due to the rotation of the earth which twists the winds in a subtle way, clockwise around low pressure regions. (They go round the other way in the Southern Hemisphere.) In education there are many such effects operating. We have mentioned some of them but, as in economics, it is impossible to predict just how they will balance out in a given situation.

Some more modest theories have a better track record. "Teaching to the test" in systems with high-stakes testing is a good example; it summarises a general reality. The first two cases in the classroom observation section also exemplify this. Alan's studies of teaching, outlined earlier, provide solid evidence that knowledge, goals and beliefs are key variables to focus on – a valuable theoretical guide in the design of professional development, which has often chosen a much narrower agenda, often just knowledge of mathematics. The concept of "role shifting" and the way it deepens mathematical discourse in the classroom emerged from the study in the section on SCAN; it has since proven a robust design principle. Table 7 shows an example (Swan, 2008) of theory in the design of teaching materials in mathematics focused on conceptual understanding.

These more modest theories, sometimes called heuristics, are *phenomenological* in that they may be seen as summarizing a body of data on a group of phenomena. Every research field relies on such theories. An example from physics and engineering is Young's theory of elasticity. It says that how much a body stretches is proportional to how hard you pull it, with a constant of proportionality "Young's modulus" that is a property of the material. This phenomenological theory also covers what happens if you pull it too hard, notably when it breaks. The fundamental theory underlying this is quantum mechanics. (Young's modulus for metals is one of the few cases where you can actually calculate the coefficient from the underlying theory.) *However, such phenomenological theory is key in airplane design*.

Table 7. An example of phenomenological theory (Swan, 2008).

Teaching design for conceptual understanding is more effective when we:

- Use rich, collaborative tasks. The tasks we use should be accessible, extendable, encourage decision-making, promote discussion, encourage creativity, encourage "what if?" and "what if not?" questions. Students should not need to start or finish at the same point, enabling everyone to engage with the activity.
- Develop mathematical language through communicative activities. Mathematics is a language that enables us to describe and model situations, think logically, frame and sustain arguments and communicate ideas with precision. Students do not know mathematics until they can "speak" it. Interpretations for concepts remain mere "shadows" unless they are articulated through language. We find that many students have never had much opportunity to articulate their understanding publicly.
- Build on the knowledge learners already have. This means developing formative assessment techniques so that we may adapt our teaching to accommodate learning needs. Lessons do not follow the traditional pattern for explanation followed by exercise. Instead, the teacher asks expose and assesses existing ways of thinking and reasoning before explaining. The teacher listens to the discussions before joining in, then attempts to prompt students to articulate their thinking and reasoning. Teacher explanation follows this discussion, it does not pre-empt it.
- Confront difficulties rather than seek to avoid or pre-empt them. Effective teaching challenges learners and has high expectations of them. It does not seek to "smooth the path" but creates realistic obstacles to be overcome. Confidence, persistence and learning are not attained through repeating successes, but by struggling with difficulties. Conceptual obstacles are part of design, deliberately included to provoke discussion.
- Expose and discuss common misconceptions and other surprising phenomena. Learning activities should expose current thinking, create "tensions" by confronting learners with inconsistencies and surprises, and allow opportunities for resolution through discussion. The activities encourage misconceptions and alternative interpretations to surface so that they may be discussed. Conflicts originate both internally, within the individual, and externally, from an individual's interpretation of another person's alternative viewpoint.
- Use higher-order questions. Questioning is more effective when it promotes explanation, application and synthesis rather than mere recall. Teachers are encouraged to prompt students to reflect and explain through the use of open prompts that begin "Explain why . . . "; "Show me an example of . . . "; "How do you know that . . . ?"
- Make appropriate use of whole class interactive teaching, individual work and cooperative small group work. Collaborative group work is more effective after learners have been given an opportunity for individual reflection. Activities are more effective when they encourage critical, constructive discussion, rather than argumentation or uncritical acceptance. Shared goals and group accountability are important. Teachers are advised to gradually establish "ground rules" for discussion among students and then behave in ways that encourage dialogic and exploratory talk.
- Encourage reasoning rather than "answer getting." Often, learners are more concerned with what they have "done" than with what they have learned. Aim for depth rather than for superficial "coverage," telling students that comprehension is more important than completion. The teacher's role is to prompt deeper reasoning by asking students to explain, extend and generalize.
- Create connections between topics both within and beyond mathematics. Learners often find it difficult to generalise and transfer their learning to other topics and contexts. Related concepts remain unconnected. Effective teachers build bridges between ideas, so design in multiple connections between different representations.
- Recognise both what has been learned and also how it has been learned. What is to be learned cannot always be stated prior to the learning experience. After a learning event, however, it is important to reflect on the learning that has taken place, making this as explicit and memorable as possible. Allow students to share their findings through the public display of their work. Encourage students to extend and generalise their ideas by making small changes to the examples, and then to explicitly formulate rules for equivalence. This helps the teacher recognise and value the contributions of students, extending and institutionalising them.

What do phenomenological theories in education look like. Like the examples in the classroom observation section, they are specific and well-defined. The set of design principles in Table 7 builds on Malcolm Swan's own research (Swan, 2005) and earlier work by the Shell Centre team and by other design researchers. They are an example of phenomenological theory that has developed and proven robust over many years of application to the design of materials; nonetheless they and the field could benefit from further replicative studies.

I believe that the research enterprise should devote more effort to developing solid reliable phenomenological theories for specific areas, reflecting the balance of research in other fields. The growth of design research, which has this agenda, is encouraging. Such phenomenological theories build evidential warrants through further testing of their robustness and limitations, by their creators and by other designers. This process will, over time, build a knowledge base that others can rely on.[14]

However, it would be to repeat the common mistake to overestimate the completeness of theory. *In design, details matter* – they have important effects on outcomes that are not determined by theory. For the foreseeable future, design skill and empirical development will remain essential for turning research into tools to support practice, with theoretical input providing useful heuristic guidance.

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NOTES

[1] All theoretical models in science have limits of validity. "Universal theorems are for mathematics, certainly not for mathematics education" (Henry Pollak).

[2] In other fields, these carry comparable prestige. The physicist John Bardeen won two Nobel Prizes, one of each kind, for the invention of the heart of modern electronics, the transistor, and for the theory of superconductivity.

[3] Even here, there is variation; some patients do not take their drugs as prescribed.

[4] "Uncertainty" is a better term; "error" often implies that "somebody made a mistake."

[5] Thirty years later, electronic whiteboards are now widely available – and perfect for this mode of use.

[6] That is why we described this as a "teaching assistant" mode of computer use, hence ITMA.

[7] Although there are myriad variants of classroom activity structures, we have found that the following five types span most of the activities of interest, and that almost every classroom episode is one of these types.

[8] The Campbell Collaboration http://www.campbellcollaboration.org/ECG/Education/index.php, modelled on the Cochrane Collaboration in medicine, seeks to establish a body of accepted results

through metanalysis but, in my view, the absence of a stream of replication studies means that it lacks the "feedstock" for such an approach. The Bush administration's "What Works Clearinghouse" suffered both from that and from a deeply flawed methodoology.

[9] Particle physics is an extreme example of "big science." The experiments at the CERN Large Hadron Collider have involved thousands of Ph.D. physicists, engineers and computer scientists over two decades, costing billions of dollars, with more to come. Papers will have hundreds of authors. This may be unattractive to some but, when it has to be done, it can be.

[10] It is worth recalling that the Ph.D. was created as a research training degree, in contrast to other doctorates (D.Sc., D.Litt, etc.) that reflect substantial professional achievement.

[11] 10,000 people on salaries of $50,000 spending 40% of their time on research, probably an underestimate, totals $200 million a year.

[12] These are order-of-magnitude estimates, avoiding "spurious precision."

[13] *Fifteen thousand hours* is the title of a famous UK study of schools (Rutter et al., 1982).

[14] This is one of the strategic goals of ISDDE, the International Society for Design and Development in Education. http://www.isdde.org/isdde/index.htm.

REFERENCES

Beeby, T., Burkhardt, H., & Fraser, R. (1980). *SCAN: A systematic classroom analysis notation for mathematics lessons*. Nottingham: Shell Centre.

Bell, A. W. (1976). *On the understanding of proof*. Ph.D. Thesis, University of Nottingham

Bell, A. (1993). Principles for the design of teaching. *Educational Studies in Mathematics*, *24*(1), 5–34.

Bell, A., Swan, M., Onslow, B., Pratt, K., & Purdy, D. (1985). *Diagnostic teaching for long term learning*. Report of ESRC Project HR8491/1: Shell Centre for Mathematical Education, University of Nottingham.

Bell, A., Swan, M., Crust, R., & Shannon, A. (1993). *Awareness of learning, reflection and transfer in school mathematics*. Report of ESRC Project R000-23-2329. Shell Centre for Mathematical Education, University of Nottingham.

Black, P. J., & Wiliam, D. (1998) Assessment and classroom learning. *Assessment in Education*, *5*, 7–74; see also *Inside the black box: Raising standards through classroom assessment*. London: King's College London School of Education (2001).

Burkhardt, H. (1988). The roles of theory in a "systems" approach to mathematical education, article in honor of prof. Hans-Georg Steiner's 60th birthday. *International Reviews on Mathematical Education, ZDM*, *5*, 174–177.

Burkhardt, H. (2006). From design research to large-scale impact: Engineering research in education. In J. Van den Akker, K. Gravemeijer, S. McKenney, & N. Nieveen (Eds.), *Educational design research*. London: Routledge.

Burkhardt, H., & Schoenfeld, A. H. (2003). Improving educational research: Toward a more useful, more influential, and better-funded enterprise. *Educational Researcher*, *32*(9), 3–14.

Burkhardt, H., Fraser, R., Coupland, J., Phillips, R., Pimm, D., & Ridgway, J. (1988). Learning activities & classroom roles with and without the microcomputer. *Journal of Mathematical Behavior*, *6*, 305–338.

Elmore, R. (2011). The (only) three ways to improve performance in schools. http://www.uknow.gse.harvard.edu/leadership/leadership001a.html; see also *Instructional rounds in education: A network approach to improving teaching and learning*. Cambridge: Harvard Education Press.

Good, T., & Brophy, J. (2002). *Looking in classrooms*, 9th edition. Allyn and Bacon.

MAP. (2012). Material from the Mathematics Assessment Project may be found at http://map.mathshell.org.uk/materials/.

RAE. (2001). http://www.rae.ac.uk/2001/Pubs/1_98/1_98cd.html.

Rutter, M., Maugham, B., Mortimore, P., & Elston, J. (1982). *Fifteen thousand hours: Secondary schools and their effects on children*. Cambridge: Harvard University Press

Schoenfeld, A. H. (1980). On useful research reports. *Journal for Research in Mathematics Education*, *11*(5).

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1992). On paradigms and methods: What do you do when the ones you know don't do what you want them to? Issues in the analysis of data in the form of videotapes. *Journal of the Learning Sciences*, *2*, 179–214.
Schoenfeld, A. H. (1994). A discourse on methods. *Journal for Research in Mathematics Education*, *25*, 697–710.
Schoenfeld, A. H. (2002). Research methods in (mathematics) education. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 435–488). Mahwah, NJ: Erlbaum.
Schoenfeld, A. H. (2006). What doesn't work: The challenge and failure of the what works clearing-house to conduct meaningful reviews of studies of mathematics curricula. *Educational Researcher*, *35*(2), 13–21.
Schoenfeld, A. H. (2007). Method. In F. Lester (Ed.), *Handbook of research on mathematics teaching and learning* (second edition, pp. 69–107). Charlotte, NC: Information Age Publishing.
Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. Routledge.
Swan, M. (2005). *Improving learning in mathematics: Challenges and strategies*. Sheffield: Teaching and Learning Division, Department for Education and Skills Standards Unit.
Swan, M. (2008). Designing a multiple representation learning experience in secondary algebra. *Educational Designer*, *1*(1), http://www.educationaldesigner.org/ed/volume1/issue1/article3/index.htm.

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14. A MATHEMATICAL PERSPECTIVE ON EDUCATIONAL RESEARCH

INTRODUCTION

I share with Schoenfeld and Törner the experience of being educated as a mathematician, then learning to work in mathematics education research. I write this chapter from this perspective, drawing on my experiences of work with both mathematicians and education researchers. In this chapter, I describe how being a mathematician might shape one's perspective on mathematics education research. I describe what it might mean to have a mathematical perspective and illustrate how it can be seen as shaping Schoenfeld's research in mathematics education. In doing so, I discuss two phenomena that Schoenfeld describes in his "accidental theorist" articles (1987, 2010):

Being "trained" by the discipline to have a mathematical perspective. In general, this means being "theory-neutral" in one's mathematical work:

> In mathematics, unless one worries about foundations (logic), one just goes about one's work: The rules of the game are so well established that one simply forges ahead, working on what one hopes is the next meaningful and significant problem. After all, a proof is a proof is a proof; people schooled in mathematics know what one is and how to produce one. (Schoenfeld, 2010, p. 104)

Why a mathematician might go from a "theory-neutral" to a "theory matters" stance in education research as Schoenfeld describes:

> My work in education started near the dawn of cognitive science, and I happily adapted tools from artificial intelligence to the study of human thinking and problem solving. . . . This stance did not ignore theory . . . but it made somewhat passive use of it. As I evolved as a researcher, however, I came to realize that being explicit about theory and models helped me clarify what I was trying to understand and to test and refine my ideas. (2010, p. 104)

Like Schoenfeld, my experience with mathematics, mathematics education, and mathematics education research has occurred primarily in the United States. Unlike Schoenfeld, my interest in theory began early, beginning with the foundations of mathematics, and later extending to philosophy of science. Again unlike Schoenfeld, I make my living outside of academe, working as a consultant in mathematics education. In practice, this requires listening to and interpreting the views of mathematicians and mathematics education researchers, then formulating and justifying claims in ways that satisfy members of both groups.

Y. Li and J.N. Moschkovich (eds.), Proficiency and Beliefs in Learning and Teaching Mathematics, 237–256.

This chapter draws on my particular background: education as a logician in a mathematics department, interest in history and philosophy of mathematics, teaching experience as a mathematics instructor in a variety of colleges and universities, further education and research experience in mathematics education with Schoenfeld and others at the University of California, and work experience as a consultant. The last includes extensive experience editing reports and books that involve contributions from mathematicians and education researchers. Some examples, in chronological order, are: *Mathematical Thinking and Problem Solving* (Schoenfeld, 1994), the Mathematical Sciences Education Board *High School Mathematics at Work* (1998), the National Council of Teachers of Mathematics *Principles and Standards* (2000), the Conference Board of the Mathematical Sciences *Mathematical Education of Teachers* (MET) report (2001), *Assessing Mathematical Proficiency* (Schoenfeld, 2007), the Mathematical Sciences Research Institute's *Teaching Teachers Mathematics* (2009), the second MET report (2012), and the *Progressions for the Common Core State Standards* (in preparation).

MATHEMATICIANS' VIEWS OF SCIENCE AND EDUCATIONAL RESEARCH

To parody Jane Austen, "It is a truth universally acknowledged that one's experiences shape one's perspective." Cognitive scientists might put "perspective" in terms of scripts, schemas, frames, or cultural models. Such constructs describe how humans perceive a situation to be of a certain type and act in ways the perceiver considers appropriate. In this chapter, I will use the terms "perspective" and "viewpoint" in talking about perception of the situation rather than choice of action. I'll take it as axiomatic that one's experiences shape one's perspective.

Images of how geographical location is associated with one's perspective on the rest of the world are given in satirical maps. A famous example is Saul Steinberg's "View of the World from 9th Avenue" (Figure 1). In this drawing, "perspective" in the sense of "cognitive perspective" is shown via geometrical perspective and the presence or absence of details. The foreground shows a perspective drawing with details of cuboid-shaped buildings, right-angled streets, tiny people, and scattered vehicles in New York City. Across the Hudson River stretches a plain . . . featureless except a few names of cities and states, three lumps whose meaning is obscure, and some greenery near Las Vegas. Across the Pacific Ocean, at the horizon, are China, Japan, and Russia.

This drawing has inspired numerous others that connect geographical location with cultural perspective (e.g., "View of the World from Bedford Avenue" suggests that New York City's cultural center has shifted from Manhattan to Brooklyn, "View of the World from Pennsylvania Avenue" portrays the viewpoint of the U.S. government, and "How China Sees the World"). It can also serve as a metaphor for natural scientists' views of research on human behaviour – in particular, for mathematicians' views of mathematics education research.

In this view, the neighbourhood of Ninth Avenue is the richly detailed and organized world of mathematics, densely populated and often involving close-

Figure 1. Saul Steinberg's "A View from 9th Avenue." © The Saul Steinberg Foundation/Artists Rights Society (ARS), New York. Reprinted with permission.

knit social and intellectual relationships (of mathematicians, see Zuccala, 2006, figures 5 and 6) and mathematical relationships (of mathematical objects).

Mathematics education research is located far away in the desert across the Hudson. One obvious reason to depict mathematics education research as an almost featureless desert is that its objects of study are very different. A major focus of research in mathematics education is the behaviour of humans rather than the behaviour of mathematical objects. Consequently, a major focus is the collection and analysis of empirical data. In general, mathematicians are at the opposite end of the spectrum. Their experiences with empirical data are more likely to have occurred in situations with data that are relatively straightforward to collect, e.g., in introductory physics courses, marks that indicate the location of a falling object at a given time or, in high school or grade school, measurements of a plant's growth. Thus, in the drawing, mathematicians concerned with empirical problems might be placed near the Hudson River or in New Jersey, within easy commuting distance, if not in the same neighbourhood. Statisticians are somewhere beyond, with no specific location in this picture.[1]

Why might mathematicians have the perspective that I have sketched? In this section, I answer this question by making use of the axiom that one's experiences shape one's perspective, giving some details about mathematicians' experiences and how those experiences might shape a mathematical perspective.

I describe three types of experiences that seem to be common to many mathematicians:

> *Experiences of laypeople*. Being a layperson with respect to any type of systematic study of human actions, such as history, psychology, anthropology, sociology, economics, political science, or education.
>
> *Experiences of teaching mathematics.*
>
> *Experiences of being educated as a mathematician.*

Because this chapter is addressed mainly to education researchers (for whom the content of the first two categories is likely to be familiar), my main focus is the third category. This is not meant to suggest that the first two categories are unimportant or insubstantial, but rather to save space and avoid repeating what has been said so well elsewhere.

Experiences of laypeople

Statistics suggest that natural scientists and engineers tend not to associate with researchers who study human behaviour. For example, statistics collected in the United States between 1900 and 2008 show that a high proportion of partnered or married couples with one mathematician include another mathematician, natural scientist, or engineer (Kessel, 2009). Anecdotes suggest similar proportions for parents, siblings, and children of mathematicians.

Thus, like many people, mathematicians' views of fields such as education research seem less likely to be shaped by discussions with practitioners than by news or social media. Limitations of these media are caricatured in Figure 2. This shows some common slips in interpretation of experimental science: mistaking correlation for causality and neglecting the conditions under which an experiment occurs.

Although mathematicians are not likely to make slips such as confusing correlation and causality, their perceptions of research in education may be affected by media accounts in other ways. Media accounts of education often focus on test results and generally omit technical details of test construction, scoring, and administration. For example, constructs such as "mathematical reasoning ability" are often not distinguished from scores on the tests that purport to measure them (e.g., the SAT). In this particular case, a further complication is that "the construct 'mathematical reasoning' is only vaguely defined in most testing organizations that produce measures of this construct" (Gallagher & Kaufman, 2005, p. 317).[2] My experience with discussions (live and online) among mathematicians and other natural scientists suggests that details such as differences between an operational definition and the meaning of the construct measured rarely seem to surface, nor do concerns such as confirmation or sample bias. However, the use of social media such as blogs may change this. Already, there are examples of scientists' blogs (e.g., Language Log, Neuroskeptic) that discuss methodological considerations for claims reported in news media and in books written for general audiences.

Figure 2. "Piled Higher and Deeper" by Jorge Cham, www.phdcomics.com Copyright 2009 by Jorge Cham. Reprinted with permission.

Experiences of mathematics teachers

Teaching is a large part of professional life for many mathematicians, particularly those at academic institutions. Those who do not currently teach often have been teaching assistants as graduate students.

The experience of teaching appears to shape a teacher's beliefs about students' mathematical knowledge. Years ago, Schoenfeld noted:

> Thanks to a National Science Foundation grant I got a videotape machine, and actually looked at students' problem solving behaviour. What I saw was frightening.
>
> Even discounting possible hyperbole in the last sentence, one statement in the previous paragraph sounds pretty strange. I'd been teaching for more than a decade and doing research on problem solving for about half that time. How can I suggest that, with all of that experience, I had never really looked at students' problem solving behaviour? ... What I saw was nothing like what I expected, and nothing like what I saw as a teacher. That's because as teachers (and often as researchers) we look at a very narrow spectrum of student behaviour. (1987, p. 33)

Many mathematics education researchers are also former teachers. Thus, they are likely to have seen how students can seem to know quite a bit less mathematics outside of teaching situations. In classrooms or office hours, students are attuned to reading the teacher's behaviour and judging whether or not they are giving a response or asking a question that is considered appropriate. (Examples for precollege German classrooms are given by Bauersfeld, 1992; Jungwirth, 1991, and others. Reactions from teachers after seeing videotapes of their classrooms suggest that they are not entirely conscious of how their actions shape students' responses, e.g., Voigt, 1998, pp. 213–214.)

Moreover, if students are solving problems in the classroom or on tests, often those problems are closely related to previous instruction (Doyle, 1988, discusses this issue for precollege U.S. classrooms). As Schoenfeld put it: "the context tells the students what mathematics to use" (1987, p. 33).

Experiences of mathematicians

Researchers in one discipline often have difficulty making sense of another. In talks and articles outside of one's discipline, it can often be difficult to perceive – let alone understand – the central constructs, understand why particular findings are considered significant, what counts as supporting evidence, and how generalizable the findings are. (In more mathematical terms, it's hard to find definitions, proofs, axioms, and rules of inference.) Mathematicians' difficulties in understanding unfamiliar disciplines may be especially noticeable when the discipline's object of study involves human behaviour (Grattan-Guinness, 1993, gives examples for history).

Education research is further complicated by not being a single discipline, but drawing on a variety of disciplines. Different researchers employ concepts, methods, and findings from psychology, anthropology, cognitive science, history, and other disciplines, with varying degrees of explicitness.

My experience, as well as that of others, suggests that cultural differences between research in mathematics and in education are profound. Gerald Goldin, who has conducted research in theoretical physics as well as mathematics education for three decades, describes the "cultural divide" between researchers in mathematics education and physics researchers.

> Early in my career, the effects could reasonably have been termed "culture shock." I became aware in the different academic communities of powerful, tacitly held assumptions, beliefs, and expectations, conflicting deeply with each other. … An acquaintance who moved several years ago from a physics department to a graduate school of education in the United States described the resulting "culture shock" to me quite seriously as greater than what she had experienced in emigrating to America from Russia. (2003, pp. 174–175)

In the remainder of this section, I discuss the nature of these differences in assumptions, beliefs, and expectations. I'll begin with cultural differences among the natural sciences.

Differences among the natural sciences: Ways of thought

It is widely accepted that there are different disciplinary perspectives in the sciences. These are the topics of jokes, mainly about mathematicians, physicists, and engineers (Gilkey, 1990; Renteln & Dundes, 2005, pp. 30–32). The mathematician Ian Stewart provides an example:

> An astronomer, a physicist and a mathematician (it is said) were holidaying in Scotland. Glancing from a train window, they observed a black sheep in the middle of a field.
>
> "How interesting," observed the astronomer, "All Scottish sheep are black!"
>
> To which the physicist responded, "No, no! *Some* Scottish sheep are black!"
>
> The mathematician gazed heavenward in supplication, and then intoned, "In Scotland there exists at least one field, containing at least one sheep, *at least one side of which is black*." (1995, p. 286)

Gilkey (who is a folklorist) characterizes her collection of jokes about disciplinary differences as "professional slurs" and asks: "Why is it that mathematicians and physicists who tell the jokes want to perpetuate these stereotypes?" The answers that she got from the joke-tellers were that the jokes illustrate differences in ways of thought. I agree, but think this explanation can be elaborated in the case of jokes about disciplinary differences. These illustrate what is salient for members of different disciplines and (sometimes indirectly) the values of those disciplines.

For example, to me the joke about the black sheep illustrates the importance of precision. Exactly how the joke illustrates precision can be described in various ways (the joke itself isn't precise!). The astronomer's and physicist's remarks can be seen as imprecise because they do not specify the constraints of the observation. Or, the joke might be viewed in terms of avoiding false generalizations or unwarranted assumptions. For example, the astronomer seems to assume that all Scottish sheep are similar in colour. The physicist assumes only that if a sheep is black on one side, then it is black on another side. I hasten to add that the valuing of precision among mathematicians often seems confined to mathematics rather than to interpreting empirical observations (as in the joke).

For some years, precision has been a topic of discussion among mathematicians interested in mathematics education. It is now a part of the Common Core State Standards for Mathematical Practice (see Standard 6, p. 7). In discussions of these standards, K–12 mathematics teachers quickly connected "precision" with precision in measurement, but seemed much less frequently to connect it with the types of mathematical precision discussed above or with care in making definitions (also a topic of jokes). To my mind, lack of precision is associated with lack of awareness that terms in school mathematics may have different meanings (e.g., Clark, Berenson, & Cavey, 2003), thus may have properties that differ – or even conflict. In academic mathematics, some standard practices help to avoid such inconsistencies. Definitions of terms are given at the beginnings of articles, and in courses and textbooks (at least for graduate students).

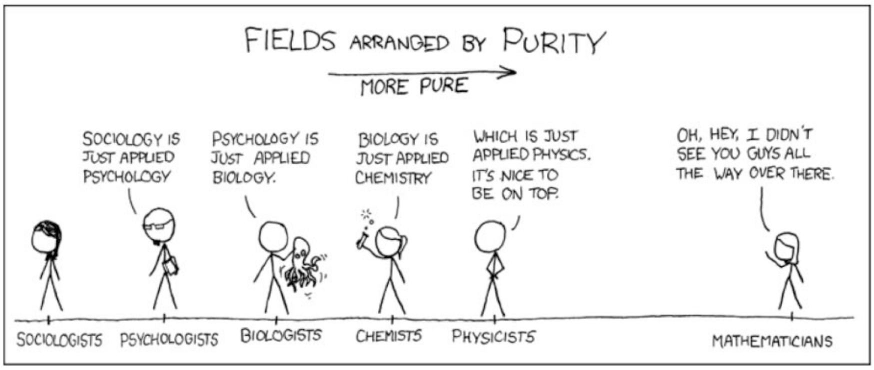

Figure 3. Randall Monroe, xkcd, 2008, June 11, http://imgs.xkcd.com/comics/purity.png.

Differences among the natural sciences: Objects and methods of study

Figure 3 illustrates the perspective of mathematicians, as caricatured by someone educated as a physicist.

To those in “purer” fields, the cartoon’s “X is just applied Y” may also be a reminder of reductionist views. These range from complex and well articulated philosophical views to actions consistent with the belief that anyone in a purer field is readily able to understand the findings of a more applied field. In physics, this is reflected by distinctions between experimentalists and theorists (the latter consider themselves superior to experimentalists, Traweek, 1988, p. 111).

Another feature of the cartoon, the distance between the mathematician and the other scientists, may suggest the view that mathematicians are far removed from reality. Perhaps reflecting my viewpoint, this was not immediately apparent to me. Instead, I think of this distance as symptomatic of a major difference between mathematics and other scientific fields with respect to empirical phenomena.

Although the cartoon is recent, the notion of such an ordering goes back to at least to Auguste Comte’s six-volume *Cours de philosophie positive* (1830–1842).[3] However, as “purer” scientists sometimes do, the cartoon neglects two aspects that were important for Comte. One is the complexity of the phenomena investigated by the different disciplines. According to Comte, these increase in complexity from right to left, with the exception of mathematics. For example, astronomy (which Comte put between physics and mathematics) is concerned with motions and positions. Along with these, physics includes forces and charges, and so on. A second aspect is the methods used by the different disciplines. In Comte’s time, astronomy had only one method, observation, and did not concern itself with such things as the chemical composition of stars. Physics included experimentation as well as observation. Biology differed (according to Comte) by employing comparison and analogy as well as observation and experiment.

Similarities among the natural sciences: Positivity

The complexity of the phenomena studied was one aspect discussed by Comte when comparing disciplines. Another was "positivity" – "precise, verifiable correlations between observable phenomena" (Laudan, 2008, pp. 375–376; see also Bordeau, 2011). According to Laudan:

> Although Comte was not the earliest writer to stress empirical verification, there is no doubt that it was largely through his influence … that the doctrine of verifiability enjoys the wide currency it has had in recent philosophy and science. (p. 376)

In the cartoon, the ordering of the disciplines follows Comte's assessment of their positivity,[4] a middle course between empiricists who claim to have no preconceptions and "armchair theorists" who do not consult empirical evidence. Instead, scientific method connects theory and empirical events. Theory predicts outcomes that can be tested by the methods of the discipline – some combination of experiment and observation.

For example, astronomical observations (orbit and mass of known planets) combined with Newton's theory of gravitation predicted the existence and location of a previously unknown planet (Neptune). Observations that confirmed the existence of the new planet provided evidence that supported Newton's theory. However, gravitational theory together with the hypothesis of another previously unknown planet did not explain another collection of observations (irregularities in Mercury's orbit). In this case, questions arose about the correctness of gravitational theory – which was eventually refined by Einstein (Hanson, 1962). Much more recently, observations of a Pluto-sized object in the vicinity of Pluto stimulated discussion of the meaning of "planet." In 2006, members of the International Astronomical Union voted to accept a definition of "planet" that reflects some of the properties (e.g., mass relative to mass of nearby objects) used in explaining earlier planetary observations. As a result, Pluto became a dwarf planet rather than a planet (Brown, 2010).

Similarities among the natural sciences: Ahistoricity

Striving toward theories that can be empirically tested is a goal shared by all in the natural sciences. However, it may often be the case that abstract ideas about relationships between theories and methods are not salient to researchers in these disciplines. As the philosopher of science Thomas Kuhn points out, during periods of "normal science," researchers within a given discipline do not feel the need to develop new methods and theories, but rather to extend and refine the existing ones. In this situation, the history of one's discipline – aside from recent developments' is not important for most natural scientists. Omission of history helps to exclude philosophical considerations about theory and methods. As Kuhn noted for physics and chemistry:

> Textbooks, however, being pedagogical vehicles for the perpetuation of normal science, have to be rewritten in whole or in part whenever the language, problem-structure, or standards of normal science change. … Once rewritten,

> they inevitably disguise not only the role, but the very existence of the revolutions that produced them. (1973, p. 137)

Given that natural scientists are educated in this way, it is not surprising that a natural scientist looks for theories as well as methods in educational research, and is disturbed by the absence of theories, but not necessarily immediately concerned about their relationship with methods. The nature of connections between theories and methods is not salient in the situation as Kuhn describes it.

For mathematics, the non-salience of these connections may be especially unsurprising because past theorems and definitions are generally expressed in modern notation and using modern concepts.[5] This makes their gist easier for a modern person to grasp, but masks differences between modern concepts and their historical predecessors, thus masking differences between ancient theorems and their modern versions. Accordingly, we see the Pythagorean Theorem (which may not have originated with Pythagoras) about relationships of geometrical magnitudes (e.g., lengths and areas) expressed in algebraic notation that developed hundreds of years later and which requires the understanding (developed in the time of Descartes) that magnitudes can be interpreted as numbers. (For a detailed and documented discussion of a similar example, see Grattan-Guinness, 2004, p. 166.)

The practice of expressing ancient theorems in modern terms conceals differences between modern methods, e.g., modern algebraic notation, as opposed to older forms of what we now, looking back, identify as "algebra." And, differences between ancient and modern conceptions of numbers seem entirely obscured. Bjarnadóttir (2007) gives an account of this phenomenon for 1 and 0 in arithmetic textbooks. In the ancient Greek view, 1 was not generally understood as a number, but as a unit. Other numbers were collections of units. This might be considered a major difference between ancient and modern conceptions. Yet, in the view of the notable mathematician G. H. Hardy, "the Greeks first spoke a language that modern mathematicians can understand" (1996/1940, p. 81). This comment occurs in his book *A Mathematician's Apology*, which many mathematicians of my generation read as students and, at least in my experience, was consistent with prevailing views.

Obscuring historical differences serves some useful functions. For example, depicting the ancient Greeks as "fellows of another college" (as Hardy put it, quoting his colleague Littlewood) helps to make salient similarities between the methods of proof used by the ancient Greeks and by mathematicians of Hardy's time (and our own). Modern mathematicians do not need to understand how concepts and notations changed in order to do mathematics. However, such simplifications of history help to support a misleading view of how mathematics developed as a discipline, reinforcing the theory-neutral view that "a proof is a proof is a proof."[6]

Similarities among the natural sciences: Oral traditions

Textbooks may also mislead about mathematical development, both historical and individual. Writing a decade after Thomas Kuhn, the ethnographer Sharon Traweek noted other aspects of physics textbooks and accompanying oral traditions. These differ from features noted by Kuhn, although they are consistent with them, and with my experiences in mathematics.

> [Particle physicists'] history of physics is a short hagiography and a list of miracles. It is this history that they teach their students: a set of oral traditions about heroes and antiheroes, detectors, and examples of "good physics judgement." (p. 78)
>
> Beyond these messages in the margin [e.g., "science is the product of individual great men," p. 78], there are instructions for the students in the body of the text, about their own status as novice physicists. ... students are urged to assume that they are not going to be an Einstein or Dirac. (p. 80)

When I was a graduate student, the corresponding message for mathematics – communicated orally to two of my contemporaries at two different universities – was: "You'll never be another Gauss." I bring up this aspect of mathematics not to note that graduate school was harsh (although it sometimes was) but to illustrate how its version of history depicts the landscape of mathematics: results proved by great men (and a few women) using – ostensibly – modern methods and notation.[7]

Another oral tradition involves jokes. As noted earlier, some jokes involve differences among the natural sciences. In my opinion, this helps to orient mathematicians to features of their own discipline by contrasting them with aspects of other sciences.

Jokes that don't rely on contrasts among disciplines may also help to orient the listener to features of a given discipline. Robert Crease (who is a physicist) connects jokes with disciplinary values: "[humor] is an especially useful tool in science, and particularly physics, precisely because it engages, fosters and celebrates the same values that the field itself depends on – namely cleverness, play and imagination." I agree – if "particularly physics" is omitted – but I think there is more that could be said. As the introduction to one collection puts it, "The selected jokes and sayings contain something essential about mathematics, the mathematical way of thinking, or mathematical pop-culture" (Cherkaev & Cherkaev, n.d.).

Here are some annotated examples of what various mathematical jokes seem to illustrate. The annotations are not meant to imply that the jokes were explicitly constructed to achieve these goals, only that they afford[8] them (as well as being funny, if one understands them).

> *Meaning of notation and some expectations about how that notation is used*: "What does ∀∃∃∀ mean?" "For every backwards A there exists an upside-down E."

This joke suggests the formal definition of limit (in calculus) in which the quantifier "for all" (∀) is followed by the quantifier "there exists" (∃). There are more notation jokes and their existence suggests the importance of notation.

> *Use and meaning of adjectives*: "What's purple and commutes?" "An Abelian grape."

This joke suggests the definition of Abelian group – a group in which the group operation (often called "multiplication") is commutative and illustrates how the adjective "Abelian" is used.

Similarities among the natural sciences: Structure-preserving diagrams

In many areas of mathematics and the natural sciences, structure-preserving correspondences between objects play important roles. Sometimes these correspondences are represented explicitly as mappings,[9] but often only the end result is represented, perhaps as a diagram with some source information. Edward Tufte (whose work focuses on visual reasoning) remarks,

> Compared to evidence presentations about *nature* (physical science), presentations about human behaviour (medicine and social science) are more descriptive, more verbal, less visual, less quantitative. (2006, p. 138)

In practice, the structure that is preserved and the correspondences among concepts, notations, and diagrams may not be described at all or not described completely, but seem generally apparent to users within the associated discipline.

These correspondences bear some similarity to grounding and linking metaphors in cognitive semantics which "preserve the inferential structure of the source domain except in those cases where target domain structure overrides" (Lakoff & Nuñez, 1997, p. 32). In this case the target domain structure is determined by the diagram, perhaps augmented by disciplinary conventions. It is my impression that well-established diagram types particular to natural science disciplines avoid the override that Lakoff and Nuñez describe, either by disciplinary conventions or caveats about their use. Some examples are cladograms in biology and Feynman diagrams in physics (Tufte, 2006, pp. 74–77). Others from mathematics (e.g., Venn diagrams) and engineering (e.g., circuit diagrams) are discussed with respect to preserving structure in *Logical Reasoning with Diagrams* (Allwein & Barwise, 1996).

Outside the natural sciences, this override is illustrated by variations in bar graph representations. Depending on their creator, bar graphs can preserve structure – or not. In reading them, a tacit assumption is often that relative sizes of numerical statistics are preserved as geometrical magnitude (which might be length or area), helping the reader to reason correctly about the relative sizes of the quantities represented. In his book *The Visual Display of Quantitative Information*, Edward Tufte discusses this assumption, gives examples of statistical graphics that violate it, and uses it to formulate a design principle:

> The representation of numbers, as physically measured on the surface of the graphic itself, should be directly proportional to the numerical quantities represented. (1983, p. 56)

Anyone who has been annoyed by defaults in graphics software (e.g., bar graphs that appear three-dimensional) will appreciate another one of Tufte's design principles:

> The number of information-carrying (variable) dimensions depicted should not exceed the number of dimensions in the data. (p. 77)

Both of these principles illustrate how target domain structure (in this case, the design of a bar graph) can override (or at least obscure) the source domain structure (in this case, a collection of statistics).

Within mathematics, annoying defaults seem to be non-existent. Within mathematical specialities, structure-preserving correspondences are often built into well-established correspondences of concepts, notations,[10] and diagrams – and sometimes noted when they are not readily afforded, e.g., "geometry is the art of right reasoning on wrong diagrams." For example, in analytic geometry correspondences between symbolic expressions and the coordinate plane preserve the following types of structure.

> For the point (1, 3), we expect its distance from the x-axis to be three times its distance from the y-axis. (That is, the lengths in the diagram are analogues of the sizes of the numbers that are the coordinates.)

> For two plotted points, say, (1, 3) and (2, 4), we can locate the result of adding their coordinates in two ways. We can use the symbolic expression, then locate the result graphically (i.e., calculate the coordinates of (1 + 2, 3 + 4) and locate the result). Or we can add the coordinates of the two points graphically on the diagram, using lengths. The result is the same because the correspondence between pairs of numbers and locations of points on the diagram preserves relationships among numbers as relationships among lengths.

More examples for the case of linear equations and their graphs are given by Schoenfeld, Smith, and Arcavi (for a summary, see their figure 24) and illustrated by software such as Function Probe (Confrey, 1991–2002).

Another example of a correspondence that preserves some structure is stereographic projection from a three-dimensional sphere minus the projection point to the Euclidean plane. (Stereographic projection can be visualized as a light shining down from the north pole of a sphere that rests on a plane. Points on the sphere are projected to their shadows on the plane.) Angle measurements are preserved by this correspondence. Area measurements are not.

With the use of structure-preserving correspondences comes the expectation that referents can be found for parts of representations. The Common Core State Standards for Mathematical Practice describe the ability to "to pause as needed … in order to probe into the referents for the symbols involved" and "explain correspondences between equations, verbal descriptions, tables, and graphs" (p. 6). Such explanations may involve identifying referents for part of one representation (e.g., a point or segment of a graph) in another (e.g., a table entry or section).

Outside the natural sciences, differences in expectations about diagrams are sometimes noticeable for diagrams with arrows. Edward Tufte gives examples under the heading "links and causal arrows: ambiguity in action" (2006, pp. 65–81), asking, in my view, an essential question about such diagrams: "But what precisely do the arrows mean?"

Similarities among the natural sciences: Structure-preserving models

Models in the natural sciences are often represented by diagrams. A model can be thought of as a collection of objects within a (more or less explicit) system of concepts (the theory) which are represented by notations and diagrams. Like a diagram, a model can be construed as a representation (although it may be a dynamic or mental representation) of a real-world situation via a structure-preserving correspondence.

For example, a model of the solar system is a collection of objects that includes planets and the sun, their masses, and the orbits traveled by the planets. Its concepts include mutual attraction as formulated by gravitational theory. Gravitational theory together with data from observations predicts the orbits, creating a correspondence between data and model. Models of the solar system went through a series of changes, each time preserving more structure in correspondence between data and models. Due to observations (stimulated by predictions of the model) Neptune was added to the collection of objects in the model. Relationships of the model (generated by gravitational theory) were changed due to refinements of the theory (stimulated by observations as compared with predictions of the model). Most recently, major objects in the model ("planets") were given an explicit definition. With each change, the resulting model corresponded more closely via the gravitational theory of that era to the accumulated astronomical data of its era.

This oversimplified history is intended to illustrate how the "precise verifiable correlations" of positivity can be construed in terms of structure-preserving correspondences between models and data. From a mathematical perspective, systems of concepts, models, and structure-preserving correspondences may be more salient than data, methods of data collection, and changes in those methods.

Summary

Being "trained by the discipline" involves oral traditions, jokes, and experiences with mathematics. These experiences shape a mathematical perspective which includes views about shared characteristics of the natural sciences and scientific methods as well as distinctions among the sciences. It includes use of heuristics (e.g., reducing a problem to a previously solved problem), precision in use of terms, care with notation, and familiarity with structure-preserving correspondences that coordinate concepts and representations. These features can reinforce each other, for example, use of structure-preserving correspondences can work to clarify referents and reduce ambiguity in definitions. Precision in definitions helps to delineate the scope of a theory. Details matter, slight differences in wording matter, and assumptions matter.

Historical changes in mathematical concepts or methods of proof are generally not salient. For many purposes, there are well-established concepts which are coordinated with notations and types of diagrams. New definitions are often made in terms of well-established concepts. This picture is consistent with Schoenfeld's remark "The rules of the game are so well established that one simply forges ahead." Only knowledge of recent changes is – or was[11] – seen as relevant for education as a mathematician.

A MATHEMATICAL PERSPECTIVE IN EDUCATIONAL RESEARCH

Years ago, when I was a newcomer to research in education, features of Schoenfeld's work began to stand out to me as being particularly easy to understand, and helping me to make sense of educational research. As I worked with Schoenfeld's research group, I began to notice other features of his work that seemed to make it easier for me to understand the field than much of the other research in education I encountered.

At least that's what I think. As Schoenfeld and others point out (e.g., 2002, p. 439), individuals' reports of their thinking processes (including memory) are notoriously unreliable as veridical accounts of events. Thus, I've used my memories as heuristics for generating features of Schoenfeld's work that seem salient from the perspective that I have described in the previous section.

Definitions

I have a vivid memory of sitting in a classroom with graduate students in mathematics education, getting the meaning of a term wrong, and eliciting annoyed reactions.

In my view, the author had not given a definition. I had decided that in education research, meanings of terms were deduced from looking at the words in the term. I had duly attempted to do that by following Humpty Dumpty's lead in decoding meanings (e.g., "'*slithy*' means 'lithe and slimy'"). For instance, "proportional reasoning" might mean "reasoning about proportions," which it does – sort of. Later, I became aware that education researchers sometimes signal the meaning of a term via citations, e.g., "proportional reasoning (Lamon, 2007)."

Schoenfeld's presentation of terms was much easier for me to understand because definitions were signaled explicitly and relationships among constructs were discussed. For example, page 74 of his 1985 book *Mathematical Problem Solving* signals that a definition is coming up. The section heading includes "What a Problem Is" and the section begins "The difficulty with defining the term *problem*." The discussion includes an example of what a problem is, as well as an example of what it is not, and how its meaning depends on the knowledge (including perspective) of the solver. His 1992 handbook article "Learning to think mathematically" goes further, identifying the different meanings of "problem" and "problem solving" used in different teaching and research traditions (e.g., pp. 338–339, 348).

This suggests how theory matters, even if theory is not explicit – or perhaps especially when theory is not explicit. The different research traditions have aims which are reflected in different assumptions about appropriate research methods. For instance, a study of problem solving in symbolic logic from the information-processing perspective "explicitly excluded any subjects who knew the meanings of the symbols" (p. 348). A study using other methods might do exactly the opposite. In the absence of an explicit connection between the two meanings of problem solving, what conclusions could be drawn from their results?

Schoenfeld's discussion of terms alerted me to the possibility that an author might emulate Humpty Dumpty, "When I use a word . . . it means just what I choose it to mean – neither more nor less – without mentioning that author X's meaning for a term may conflict with author Y's – even when author X is referring to author Y's work.[12]

Diagrams

Over the years, I've encountered diagrams in educational research articles that struck me as puzzling. These diagrams are used to illustrate objects such as models or frameworks, but aspects of their structures that were salient to me were not reflected in the descriptions of the objects and their relationships. So, for example, a diagram might consist of circles that are all tangent to a point on the left, in which each circle is called a level. The point is salient, but has no referent in the description. The descriptor "level" suggests an interpretation involving vertical organization rather than a collection of circles. Another type of diagram that puzzled me had regions of different shapes and sizes without explaining their significance. Such diagrams made me wonder whether a list or a differently shaped diagram would serve the same purpose.

This is because they seem to have structures that appear to give additional information – the override that Lakoff and Nuñez describe. Without explanation, I am left perplexed about whether or not the target structure of the diagram has overridden the source structure of the object as conceived by the author.

In his writings, Schoenfeld uses diagrams in at least two ways: to report on analyses of data (e.g., in his teacher model work) and to illustrate ideas about research methods. In both cases, from my point of view the diagrams are easy to understand. Moreover, they convey a lot of information in a very concise way.

Many of his diagram types and conventions were familiar to me. For example, I'd already had experience reading flow charts (e.g., 2010, figure 5.3). Quantitative aspects of other diagrams draw on familiar graphic conventions, e.g., the timeline diagrams for analyses of problem-solving protocols (figures 9.1B and 9.2B in *Mathematical Problem Solving*) or parsing diagrams in the teacher model work both show elapsed time as length, obeying Tufte's principle that measurements of graphical representations of quantities be directly proportional to the quantities.

Representing results of data analyses in graphical form has several advantages. As Tufte points out, "The special power of graphics comes in the display of large data sets" (1983, p. 56). A related point is Barwise and Etchemendy's contention that structure-preserving diagrams can provide a lot of information that can be read off the diagram (1996, p. 23). Barwise and Etchemendy's "inference" is inference in formal logic. Outside of formal logic, diagrams may be even more advantageous. If they preserve relevant structures in ways that readers can understand, diagrams that summarize data analysis can display relationships between constructs and data, and among the constructs themselves.

Both of these features occur in the representations Schoenfeld uses to describe analyses of teaching (e.g., 2010, figure I-1). These connect a transcript of a lesson with its analysis, representing a correspondence between a model and the data from which it's derived (in this case, the lesson video together with information such as teacher logs). In mathematics education research, it is unusual to represent data in such a comprehensive way when the data are not statistical.

Models

To Schoenfeld (and to me), "A model is more than a picture with a collection of objects and arrows" (2002, p. 475). A model is derived from data. Schoenfeld's analytic diagrams, augmented by explanations, show what the models are and how they are derived from data.

Aside from statistical models, models in educational research are not often described in ways that make their connections with data apparent.

CONCLUDING REMARKS

In my view, scientific practices reflect the goal of positivity: "precise, verifiable correlations between observable phenomena." Especially in mathematics (the most positive of sciences according to Comte), these practices include care with definitions, use of diagrams, and expectations of structure-preserving correspondences. Schoenfeld's work illustrates how these practices can be used in educational research in ways that allow public inspection of models, data, and correspondences.

In contrast with frequently expressed views about how educational research can become more scientific or positive (e.g., Lester, 2010, p. 67), this does not mean that educational research should adopt the "agricultural" methods of experimental psychology. But, it does suggest that educational research should have the goal of connecting constructs, which, like the three lumps in the desert of the Steinberg drawing, have obscure meanings and relationships.

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In more ways than one, this article would not exist without Alan Schoenfeld. Without his introduction to educational research, I would probably still be wandering in the desert. I continue to profit from his acute and helpful commentaries.

Over the years, I have had the good fortune to acquire other distal colleagues from mathematics, educational research, history, and psychology who are willing and able to explain their views. By shaping my perspective, their views have shaped this article.

In particular, Margaret Murray commented on an early incomplete version of this article, helping me to clarify murky ideas and simplify convoluted sentences. Her interest encouraged me to continue, resulting in later version that was improved by Judit Moschkovich's comments.

NOTES

[1] One way to measure distinctions among the mathematical sciences is to look at organizations and their meetings. The Conference Board of the Mathematical Sciences is an umbrella organization of professional societies that represent researchers in pure mathematics, applied mathematicians, statisticians, actuaries, logicians, and others involved with the mathematical sciences. The annual Joint Mathematics Meetings include meetings of the American Mathematical Society, Mathematical Association of America, Society for Industrial and Applied Mathematics, and (in alternate years) the Association for Symbolic Logic, but not the American Statistical Association or Society of Actuaries. Combining these measures: statisticians and actuaries are further from mathematicians than are applied mathematicians and logicians.

[2] This particular feature of tests may change, due to the assessment activity surrounding the Common Core State Standards, and more generally, due to increased attention to the nature of tests and increased participation of mathematicians in mathematics education.

[3] The activity of ordering fields according to various attributes is much older, going back at least to Aristotle.

[4] Comte did not include psychology and did include astronomy. Astronomy, which in Comte's time used only observation, shows that positivism is not confined to use of experimental methods.

[5] In the U.S., there seems to be more consciousness of history among mathematicians now than during the 1970s and 1980s. This is evidenced by textbooks such as *A Radical Approach to Real Analysis* (Bressoud, 1994, second edition 2006) which describe the historical context of their subjects or books such as *Mathematical Expeditions* (Laubenbacher & Pengelley, 1998) which include the original formulations of theorems and excerpts from primary sources.

[6] Such simplifications may result from a combination of factors. For example, until I made acquaintance with the residential college system, I interpreted "fellows of another college" as something like the difference between mathematics departments at two different universities. However, a much closer equivalent would be "resident fellows at another dormitory of my university." Similarly, it is easy for a modern reader to interpret Hardy's discussion of Greek mathematics in modern terms. Study of Greek was more widespread in Hardy's time and readers may have been more aware of Euclid's conceptions. Awareness of contemporary differences in views on laws of inference is evidenced by Hardy's footnote mentioning that some logicians would prefer to avoid reductio ad absurdum arguments such as Euclid's proof that there are infinitely many primes.

[7] It's a very convenient shorthand and I have made use of similar tactics in the previous section.

[8] Here, I follow Philip Kitcher's usage, e.g., "lettuce affords eating to rabbits" (1984, p. 12).

[9] For example, homomorphisms between groups preserve "group multiplication." This is an example of a structure-preserving mapping between two structures that satisfy the same theory (group theory). Structure-preserving mappings can also occur between two structures that satisfy different theories (e.g., Boolean rings and Boolean algebras).

[10] Much can be said about notation. Part of the "goodness" of good notation is its ability to embody relationships among the concepts it represents. Embodying such relationships can be construed as a structure-preserving correspondence between mental objects and physical representations. This feature is suggested by Kaput's notion of "action notations" in which computations can occur as opposed to "display notations" that only record results (1994, p. 101).

[11] It seems especially relevant here to stress that, as noted earlier, consciousness of history may have increased among U.S. mathematicians.

[12] A 2011 example of this lack of precision is noted by Cobb and Jackson (p. 183).

REFERENCES

Allwein, G., & Barwise, J. (Eds.). (1996). *Logical reasoning with diagrams*. New York: Oxford University Press.

Barwise, J., & Etchemendy, J. (1996). Visual information and valid reasoning. In G. Allwein & J. Barwise (Eds.), *Logical reasoning with diagrams* (pp. 3–25). New York: Oxford University Press.

Bauersfeld, H. (1992). Classroom cultures from a social constructivist's perspective. *Educational Studies in Mathematics*, *23*(5), 467–481.

Bjarnadóttir, K. (2007). The numbers one and zero in Northern European textbooks. *International Journal for the History of Mathematics Education*, *2*(2), 3–20.

Bordeau, M. (2011). Auguste Comte. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Summer 2011 edition). Retrieved from http://plato.stanford.edu/archives/sum2011/entries/comte/.

Bressoud, D. (1994). *A radical approach to real analysis*. Washington, DC: Mathematical Association of America.

Brown, M. (2010). So is Pluto a planet after all? Retrieved from http://www.mikebrownsplanets.com/2010/11/so-is-pluto-planet-after-all.html.

Cherkaev, A., & Cherkaev, E. (n.d.) Math jokes collection by Andrej and Elena Cherkaev. Retrieved from http://www.math.utah.edu/%7Echerk/mathjokes.html.

Chi, M., Feltovich, P., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science*, *5*, 121–152.

Clark, M., Berenson, S., & Cavey, L. (2003). A comparison of ratios and fractions and their roles as tools in proportional reasoning. *Journal of Mathematical Behavior*, *22*, 297–317.

Cobb, P., & Jackson, K. (2011). Assessing the quality of the Common Core State Standards for Mathematics. *Educational Researcher*, *40*(4), 183–185.

Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers*. Providence, RI: American Mathematical Society.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II.* Providence, RI: American Mathematical Society.

Confrey, J. (1991–2002). *Function probe* (v. 2.3.5, 3.0, 4.0, 4.1). Ithaca, NY: Cornell Research Foundation. (Designed with Forrest Carroll and Erick Smith, revised 2002 with Alan Maloney and Pedro Larios.)

DeFranco, T. (1996). A perspective on mathematical problem-solving expertise. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education II* (pp. 195–213). Providence, RI: American Mathematical Society.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist*, *23*(2), 167–180.

Gallagher, A., & Kaufman, J. (Eds.) (2005). *Gender differences in mathematics*. New York: Cambridge University Press.

Gilkey, C. (1990). The physicist, the mathematician and the engineer: Scientists and the professional slur. *Western Folklore*, *49*(2), 215–220.

Goldin, G. A. (2003). Developing complex understandings: On the relation of mathematics education research to mathematics. *Educational Studies in Mathematics*, *54*, 171–202.

Grattan-Guinness, I. (1993). A residual category: Some reflections on the history of mathematics and its status. *The Mathematical Intelligencer*, *15*(4), 4–6.

Grattan-Guinness, I. (2004). The mathematics of the past: Distinguishing its history from our heritage. *Historia Mathematica*, *31*, 163–185.

Hanson, R. (1962). Leverrier: The zenith and nadir of Newtonian mechanics. *Isis*, *53*(3), 359–378.

Hardy, G. H. (1996/1940). *A mathematician's apology*. New York: Cambridge University Press. (First edition published 1940.)

Institute for Mathematics and Education. (in preparation). *Progressions for the Common Core State Standards*.

Jungwirth, H. (1991). Interaction and gender – Findings of a microethnographical approach to classroom discourse. *Educational Studies in Mathematics*, *22*(3), 263–284.

Kaput, J. (1994). Democratizing access to calculus. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 77–156). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kessel, C. (Ed.). (2009a). *Teaching teachers mathematics*. Berkeley, CA: Mathematical Sciences Research Institute.

Kessel, C. (2009b). Notes on the two-body problem. *Association for Women in Mathematics Newsletter*, *39*(6) 14–16.

Kitcher, P. (1984). *The nature of mathematical knowledge*. New York: Oxford University Press.

Klein, F. (1932). *Elementary mathematics from an advanced standpoint: Arithmetic, algebra, analysis* (E. R. Hendrick & C. A. Noble, trans.). New York: Macmillan. Original work published 1908.

Kuhn, T. (1973). *The structure of scientific revolutions* (2nd ed.). Chicago: University of Chicago Press.
Lakoff, G., & Nuñez, R. (1997). The metaphorical structure of mathematics. In L. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21–89). Mahwah, NJ: Lawrence Erlbaum Associates.
Lamon, S. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Charlotte, NC: Information Age Publishing.
Laubenbacher, R., & Pengelley, D. (1998). *Mathematical expeditions: Chronicles by the explorers.* New York: Springer.
Laudan, L. (2008). Comte. In C. Gillespie, F. Holmes, & N. Koertge (Eds.), *Complete dictionary of scientific biography* (Vol. 3, pp. 375–380). Detroit, MI: Charles Scribner's Sons.
Lester, F. (2010). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. In B. Sriaman & L. English (Eds.), *Theories of mathematics education* (pp. 67–85). New York: Springer.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
National Governors Association Center for Best Practices and Council of Chief State School Officers. (2010). *Common Core State Standards for mathematics*. Washington, DC: Author.
Renteln, P., & Dundes, A. (2005). Foolproof: A sampling of mathematical folkhumor. *Notices of the American Mathematical Society*, *52*(1), 24–34.
Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
Schoenfeld, A. H. (1987). Confessions of an accidental theorist. *For the Learning of Mathematics*, *7*(1), 30–38.
Schoenfeld, A. H. (2000). Purposes and methods of research in mathematics education. *Notices of the American Mathematical Society*, *47*(6), 641–649.
Schoenfeld, A. H. (2002). Research methods in (mathematics) education. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 435–488). Mahwah, NJ: Lawrence Erlbaum Associates.
Schoenfeld, A. H. (2010). Reflections of an accidental theorist. *Journal for Research in Mathematics Education*, *41*(2), 104–202.
Schoenfeld, A. H., Smith, J., & Arcavi, A. (19). Learning. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 55-175).
Stewart, I. (1995). *Concepts of modern mathematics*. Mineola, NY: Dover.
Traweek, S. (1988). *Beamtimes and lifetimes: The world of high energy physicists*. Cambridge, MA: Harvard University Press.
Tufte, E. R. (1983). *The visual display of quantitative information*. Cheshire, CT: Graphics Press.
Tufte, E. R. (2006). *Beautiful evidence*. Cheshire, CT: Graphics Press.
Voigt, J. (1998). The meaning of empirical phenomena. In F. Seeger, J. Voigt, & U. Waschescio (Eds.), *The culture of the mathematics classroom* (pp. 191–220). New York: Cambridge University Press.
Working group. (1999). Research mathematicians and research in mathematics education. In E. A. Gavosto, S. G. Krantz, & W. McCallum (Eds.), *Contemporary issues in mathematics education* (pp. 170–171). Berkeley, CA: Mathematical Sciences Research Institute. Retrieved from http://library.msri.org/books/Book36/contents.html.
Zuccala, A. (2060). Modeling the invisible college. *Journal of the American Society for Information Science and Technology*, *57*(2), 152–168.

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15. ISSUES REGARDING THE CONCEPT OF MATHEMATICAL PRACTICES

INTRODUCTION

In the first chapter of Alan Schoenfeld's 2011 book *How we think*, he describes his original framework (1985) for the study of mathematical problem solving as having four components: knowledge base, problem solving strategies, metacognition, and beliefs. In the notes for that chapter, Schoenfeld comments that in 1992 he added a fifth category to the framework, practices, described in "Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics" (Schoenfeld, 1992). Schoenfeld also comments in that note that, although the first four categories of the framework were sufficient for examining problem solving "in the moment," the additional category is essential to examine because practices shape one's beliefs and resources. Twenty years later, although mathematics education research now includes mathematical practices, the concept still needs attention. In this chapter I address four issues regarding the concept of mathematical practices, how we 1) define mathematical practices, 2) theoretically frame the concept of practices, 3) connect practices to other aspects of mathematical activity, and 4) describe how practices are acquired.

Why should mathematics education researchers take time to carefully consider the concept of mathematical practices? Although some research since 1992 has used the phrase mathematical practices, there are many different definitions and uses of the concept. We need clear definitions and examples and more research on this topic that does not dichotomize mathematical practices and the settings in which these practices occur. A careful consideration of this concept also has implications for practice. The phrase "standards for mathematical practice" has recently been introduced to practitioners through its use in the Common Core State Standards. We are now in the process of discussing how mathematical practices appear in classrooms. Developing theoretical clarity and grounded examples will also help to develop clarity in conversations with practitioners. In the closing section I discuss issues in using mathematical practices in recommendations for teaching.

In the 1992 chapter, Schoenfeld himself identified mathematical practices as the least articulated and understood component of his framework, saying that

> Issues regarding practices and the means by which they are learned – enculturaltion – may be even more problematic. Here [. . .] we seem to know the least. (p. 365)

At the same time, Schoenfeld raised several issues: defining mathematical practices, how practices might be connected to other central aspects of mathematical

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activity, and the means by which practices are learned. The first observation regarding mathematical practices was that the term *practices* could refer to several phenomena: habits, dispositions, epistemologies, epistemological stances, epistemological goals, and epistemological sense. Another observation was that *practices* do not work alone, since the pieces are parts of a whole:

> The person who thinks mathematically has a particular way of seeing the world, of representing it, of analysing it. Only within that overarching context do the pieces-the knowledge base, strategies, control, beliefs, and practices-fit together coherently. (p. 363)

Schoenfeld described mathematical practices as being acquired through enculturation and also socialization, entry into the mathematical community, legitimate peripheral participation (Lave & Wenger, 1991), and interaction with others:

> If we are to understand how people develop their mathematical perspectives, we must look at the issue in terms of the mathematical communities in which the students live and the practices that underlie those communities. The role of interaction with others will be central in understanding learning . . . (p. 363)

Today, at least three issues remain regarding mathematical practices that need attention and further research: a) clarity in defining and theoretically framing mathematical practices, b) connecting mathematical practices to other aspects of thinking and learning; and c) describing how practices are acquired. In this chapter, I use an example from my research to describe how a Vygotskian theoretical framing of mathematical practices can address these three issues by providing a practice perspective on cognition, connecting mathematical practices to mathematical discourse, and using appropriation to describe how mathematical practices are acquired. I begin by using Schoenfeld's 1992 chapter as the foundation for a description and definition of "mathematical practices." In the first section of the chapter, I build on the initial description to distinguish among different types of mathematical practices. In the second section of the chapter I describe how a Vygotskian framing can contribute to clarifying the concept of mathematical practices. I use examples from my own work to illustrate how two central Vygotskian concepts, discourse and appropriation, address the three issues identified here. My intention here is not to provide an exhaustive review of the work on the topic of mathematical practices since 1992 or to chronicle the development of the concept of mathematical practices in mathematics education research, but instead to use an example from my own work to raise issues regarding how we frame and use the concept in research.

First, allow me to provide a little historical and personal background. When I first started working with Alan as a doctoral student in 1986, I spent time trying to understand his framework and also connect it to Vygotskian and neo-Vygotskian theories of learning (i.e. Forman, 1996; Vygotsky, 1978, 1979, 1987, 1979a, 1979b, 1984, 1985). My goal was to reconcile my theoretical commitments to Vygotskian perspectives with this framework for the study of mathematical thinking. I wondered whether the framework was compatible with Vygotskian perspectives, and if so, how. Certainly there were echoes of Vygotskian views in Schoenfeld's approach to teaching problem solving and in his study of metacognition. Metacognition, one aspect of mathematical problem solving, was a crucial aspect of Schoenfeld's framework that seemed directly connected to a Vygotskian focus on self-regulation. However, when I first looked at this framework, I initially struggled to answer several (fundamentally Vygotskian) questions, especially where one could find mediation by social-cultural artifacts in the framework. Today I see that, for me, the answer to this question lay in clarifying the category of practices, connecting practices to mathematical discourse, and using appropriation to describe how practices are acquired. Now, in retrospect, I can see that his framework informed my work by pointing me in the direction of that less articulated category, mathematical practices. Using Vygotskian theories and work in sociolinguistics, I have spent the last 20 years analysing discussions of mathematical problems among students or between a learner and an adult. My analyses have focused on identifying and describing central aspects of mathematical discourse practices. In the following sections I describe how a Vygotskian perspective, in particular the concepts of discourse and appropriation, can frame the study of mathematical practices, what they are and how they are learned.

DEFINING AND DESCRIBING MATHEMATICAL PRACTICES

In this section I discuss a central issue with the concept of mathematical practices; how we define and describe the practices of mathematicians. Much of the work in mathematics education in the last twenty years has assumed that mathematics instruction in schools needs to parallel, at least in some ways, the practices of mathematicians (for example Cobb, Wood, & Yackel, 1993; Lampert, 1986, 1990; Schoenfeld, 1992). These proposals emphasize classroom activities that parallel those of academic mathematical practice. This view of students as mathematicians suggests that student activity should parallel at least some aspects of a mathematician's practices, such as making generalizations or conjectures and subjecting these to review and refutation by a (classroom) community. Students should explore the nature of mathematical objects, make and test conjectures, and construct arguments, and instruction should emphasize abstracting and generalizing as central mathematical practices. During discussions, students are expected to engage in activities which parallel the practice of academic mathematicians such as: "constructing, symbolizing, applying, and generalizing mathematical ideas" (NCTM Curriculum Standards, 1989, p. 128); "synthesize, critique and summarize strategies, ideas, or conjectures" (NCTM Curriculum Standards, 1989, p. 67); and "explore, formulate, and test conjectures; prove generalizations; discuss and apply the results of these investigations" (NCTM Curriculum Standards, 1989, p. 128). Bringing the practices of mathematicians into the classroom is expected to create a common set of discourse practices that parallel academic mathematical practices. Students are expected to make conjectures, agree or disagree with the conjectures made by their peers or the teacher, and engage in public discussion and evaluation of claims and arguments made by others. This approach is intended to give students access to academic mathematical practices, such as the construction and presentation of mathematical proofs or arguments.

If we agree with this premise and these recommendations (which in large part, I do), then we need carefully consider what data we use to define and describe mathematicians' practices. Research needs to use descriptions of mathematicians' mathematical practices that are not idealized or imagined accounts of what mathematicians do, but, instead, are based on empirical work documenting the actual practices of mathematicians.

Accounts of mathematicians' practices

What we know about the practice of mathematicians comes largely from autobiographical reports of their own practice rather than from ethnographic studies of the practices of mathematicians. These different accounts, ethnographic and autobiographical, may contribute to the perception that mathematical practices are fundamentally different from what everyday people (Just Plain Folks in Lave's, 1988, words) do. In contrast, Leone Burton's (1999) work provides an example of a study of mathematician's practices that relies on self reports but is not autobiographical. Her study of 70 research mathematicians focused on epistemologies or how they "come to know" mathematics and uncovered several aspects of the mathematical practices of mathematicians that are worth noting because they contradict idealized versions of mathematicians' practices and perceptions of how different their practices are than the practices of other people. Burton found that mathematicians, much like other people, "often thought that they knew something but it turned out that they were not correct" (p. 132). Many mathematicians talked about how certainty feels, rather than how they achieve it, and pointed out that an expectation of certainty is unrealistic:

> Sometimes you never know. You can do it by contradiction, by finding fault. You can never know it is right. You always have to live with uncertainty. (p. 133)

Burton's interviews provide a view of mathematical practices that include intuition, insight, and aesthetics, corroborating autobiographical accounts of mathematicians work. Most importantly, Burton's study provides evidence that mathematicians' practices, like other research professions, are not one single unitary practice for all mathematicians or even for one individual, but instead involve tensions and contradictions. In particular, this study shows that the community practices of mathematicians, such as collaboration with colleagues or co-operation across areas of expertise, have changed and shifted over time, and thus are not static in time and should be described in their historical context.

Autobiographical accounts of academic mathematical practices (Davis & Hersch, 1982; Hadamard, 1945; Hardy, 1941; Schoenfeld, 1992; Tymocz, 1986) provide mathematicians' own descriptions of what mathematicians do. Mathematicians report that their practices involve aesthetic values, such as elegance, simplicity, generalizability, certainty, and efficiency. In general, when faced with a problematic situation, academic mathematicians tend to bring in as much many mathematical tools as possible to understand it. However, autobiographical descriptions of mathematicians' practices are contested within the community of mathematicians either because there are different practices across different subfields, or because there are fundamental disagreements about what it is that mathematicians actually do (Restivo, 1993).

Although autobiographical accounts of academic mathematical practices are certainly useful, they differ from ethnographic accounts in several ways. Ethnographic accounts, if these were available, would be based on more extensive data sources than autobiographical data, they would balance etic and emic perspectives on academic mathematical practices, and they would serve to illuminate these practices for non-mathematicians. Autobiographical accounts rely only on the self-reports of participants trained in mathematics, whereas ethnographic accounts triangulate self-reports by participants with other data sources. Ethnographic studies rely on the ethnographic experience of an outsider trained in ethnographic research (and perhaps some aspects of mathematics, as well), on the systematic and extensive collection of data from multiple sources, and on the interpretation of this data within a framework of cultural practices (Moschkovich & Brenner, 2000).

Other accounts of mathematical practices use historical, philosophical, or cognitive methodologies. For example, Schoenfeld's (1985) account of mathematicians' problem solving is a cognitive analysis based on think-aloud protocols. Other accounts of mathematical practices, such as Lakatos (1976) and Polya (1957), rely on a combination of introspection, historical data, and philosophical methods. I make a distinction between these autobiographical, historical, philosophical, and cognitive accounts of mathematicians' practices and the ethnographic accounts of scientific practices in laboratories, such as Latour (1987), Latour and Woolgar (1986), Traweek (1988), Ochs, Gonzales, and Jacoby (1996), or Ochs, Jacoby, and Gonzales (1994). These ethnographic studies analyze data collected at the sites where scientific activities take place, and they include observations of daily activity over extended periods of time and descriptions of how scientists define, approach, and solve new problems. They also provide an account of how scientific "facts" are produced and presented in various settings (Latour, 1987; Latour & Woolgar, 1986) or a detailed sociolinguistic analysis of scientific conversations (Ochs et al., 1994, 1996). There are many accounts of mathematicians' practices, but none of them are ethnographic in the sense described here.

Examples of ethnographic studies include research on mathematical practices in non-school settings (Carraher, Carraher, & Schliemann, 1985; Nunes, Schliemann, & Carraher, 1993; Scribner, 1984; Saxe, 1991; Lave, 1988). This body of work provides a systematic and detailed account of how "Just Plain Folks" (Lave, 1988) carry out everyday problem solving and use different mathematical tools. This research has documented how people apply mathematical concepts in everyday activity, from the basic computational and regrouping strategies that vendors use, to how builders invoke the concepts of proportion and scale in reading floor plans. One result of this work is that "just plain folks" rarely use school-taught algorithms, especially if there is a more local solution process that is efficient.

Although ethnographic accounts of the practice of academic mathematics may not yet be available, ethnographic accounts of the practices of physicists (Ochs et al., 1994; Traweek, 1988) and other laboratory scientists (Latour, 1987; Latour & Woolgar, 1986) can inform our understanding of academic mathematical practices. What can we learn about the differences between everyday practices and the practices of laboratory scientists that might apply to academic mathematical practices? One important conclusion from this work is that scientific practices are more a reflection of how scientists see the world than a reflection of scientists' use of a careful method to describe the world. Another conclusion is that scientific practice is similar to non-scientific practice in that it is contingent on daily events and subject to non-scientific constraints (Latour, 1987).

These ethnographic studies of actual laboratory practices are useful in several ways. First, they provide detailed examples that can be used to think about how conversations in science classrooms might reflect scientists' practices. Second, and perhaps most importantly, they show us how scientific practices are closer to everyday practices than we may have imagined. For example, studies show that the discursive practices of scientists vary across formal and informal situations. In more formal situations, scientists tend to omit references to their involvement in the research activities, whereas in informal discussions, they "often refer to themselves as agents in the production of knowledge" (Ochs et al., 1996, p. 330). As an example, one study of physicists at work combined ethnographic and sociolinguistic approaches to examine how physicists use talk, gestures, and graphic representations while making sense of phenomena and using a graph. Physicists at work referred to a graphical representation of a physical phenomenon as if they inhabited the graph, saying such things as "when I come down [the curve] I am in the domain state" (Ochs et al., 1994). The analysis concluded as follows:

> We hope to have demonstrated that scientists routinely blur the boundaries between themselves as subjects and physical systems as objects using a type of indeterminate construction which blends properties of both animate and inanimate, subject and object. (p. 359)

This finding seems relevant to a common description of mathematical discourse, that it is impersonal and that utterances leave out personal pronouns (Pimm, 1987). However, this study of physicists' practices makes us consider whether an impersonal style might also vary for mathematicians across formal and informal situations. Such examples of scientists' practices provide detailed descriptions that might be used in classrooms to see and hear how students are acting like scientists when they are in informal situations. Without similarly detailed examples of the practices of mathematicians in different situations based on ethnographic studies of mathematicians' work, it is difficult to know whether we are proposing that students engage in the actual practices of mathematicians or in idealized versions of mathematicians' practices.

In spite of the fact that ethnographic accounts have described the mathematical aspects of everyday practices, no research has described the everydayness of academic mathematical practices. There is a need for ethnographic accounts of how mathematicians become mathematicians and for detailed examples of how they act, talk, and see the world. Such accounts could, for example, include data on mathematicians' daily activities over extended periods of time, in different settings (conferences, academic offices, faculty meetings, discussions of a mathematical problem over lunch with a colleague, etc.). These ethnographic accounts would contribute to a more detailed view of the practices of mathematicians and, most importantly for classroom teaching, comparisons between everyday practices and academic mathematical practices that acknowledge the continuity between what mathematicians and regular folks do.

Current inquiry into the practices of mathematicians concludes there is not one mathematical practice, one way of understanding mathematics, one way of thinking about mathematics, or one way of working in mathematics (Burton, 1999):

> Out of the interviews with research mathematicians, I have a clear image of how impossible it is to speak about mathematics as if it is one thing, mathematical practices as if they are uniform and mathematicians as if they are discrete from both of these. (p. 141)

Professional mathematical discourse is also historically situated. For example, over time, the roles played in professional mathematical discourse by dialogue (Mendez, 2001) and mathematical arguments have changed:

> What was a good argument in the scientific environment of Euclid was no longer so to Hilbert; and what was nothing but heuristic to Archimedes became good and sufficient reasoning in the mathematics of infinitesimals of the seventeenth and eighteenth centuries. (Høyrup, 1994, p. 3)

Even mathematical definitions have changed over time. For example, the definition of a function has changed throughout history from the Dirichlet definition as a relation between real numbers to the Bourbaki definition as a mapping between two sets:

> At some early stage, functions were restricted to those which could be expressed by algebraic relationships. Later, the concept was extended to encompass not only correspondences which can be expressed algebraically, and later still to correspondences not involving sets of numbers at all. (Arcavi & Bruckheimer, 2000, p. 67)[1]

Mathematical practices also vary depending on purposes. Richards (1991) describes four types of mathematical discourse. Research mathematics is the mathematics of the professional mathematician and scientist. Inquiry mathematics is mathematics as used by mathematically literate adults. Journal mathematics emphasizes formal communication and is the language of mathematical publications and papers. (Richards describes this type of mathematical discourse as different from the oral discussions of the research community because written formal texts reconstruct the story of mathematical discoveries). School mathematics, being the practices typical in the traditional mathematics classroom, may share with other classrooms the initiation-reply-evaluation structures of other school lessons (Mehan, 1979). Richards points out that school mathematics has more in common with journal mathematics than with research or inquiry mathematics. Schoenfeld's work (2007) is also relevant here:

> [. . .] there is accountability to the teacher – both in terms of the traditional authority structure, but also in that the teacher is the prime orchestrator of the classroom mathematical community, and a representative of the mathematical community in the classroom. (p. 4)

In sum, research needs to make a fundamental shift away from conceiving *mathematical practices* as uniform. Mathematical practices are not singular, monolithic, or homogeneous. Mathematical practices include multiple forms ranging over a spectrum of practices such as academic, workplace, playground, street selling, home, and so on. Many more research studies are needed to better understand how mathematical practices differ depending on the setting, context, and circumstances. In particular, studies need to consider what mathematical practices learners use in each of many different settings, what knowledge and practices learners use across settings, and how to make visible the ways that learners reason mathematically across settings. Instead of asking general questions such as "Does this activity use mathematical practices?" research needs to ask how, when, and under what circumstances are multiple and varied mathematical practices connected, and consider the multiple ways that mathematical practices function in different circumstances and for different aspects of mathematical reasoning.

In documenting mathematical practices across settings, researchers should consider the spectrum of mathematical activity as a continuum rather than reifying the separation between practices in out-of-school settings and practices in school. Analyses should consider everyday and scientific practices and discourses as interdependent, dialectical, and related rather than assume they are mutually exclusive. Rather than debating whether an utterance, lesson, or discussion is or is not "a mathematical practice," studies should instead explore what practices, inscriptions, and talk mean to the participants and how they use these to accomplish their goals.

Overall, our descriptions need to shift away from dichotomies such as everyday/academic, formal/informal, or in-school/out-of-school. Research needs to stop construing everyday and academic mathematical practices as a dichotomous distinction (Gutierrez et al., 2010; Moschkovich, 2007, 2010; Scheleppegrell, 2010). A theoretical framing of everyday and academic practices (or spontaneous and scientific concepts) as dichotomous is not consistent with current interpretations of these Vygotskian constructs (e.g., O'Connor, 1999; Vygotsky, 1978, 1987). Vygotsky (and other theorists) describe everyday and academic practices as intertwined and dialectically connected.

MATHEMATICAL PRACTICES FROM A VYGOTSKIAN PERSPECTIVE

What assumptions do we make about the nature, origin, development, and acquisition of mathematical practices? Are mathematical practices individual, collective, cognitive, discursive, social, and cultural phenomena? It is impossible to answer these questions without a theoretical framing. From a Vygotskian perspective, mathematical practices are socio-cultural phenomena in the sense that they are higher order intellectual activities and originate in social interaction. They are first constructed interpersonally and then appropriated to become part of the repertoire of practices that an individual will use. A Vygotskian theoretical framing can contribute to clarifying and articulating the concept of mathematical practices. First, it provides a practice perspective on cultural activity. Second, it provides a connection to discourse as a central aspect of practices. And lastly, it describes how practices are acquired through appropriation.

I begin with a Vygotskian definition for a *practice* perspective.[2] I use the terms *practice* and *practices* in the sense used by Scribner (1984) for a practice account of literacy to

> [...] highlight the culturally organized nature of significant literacy activities and their conceptual kinship to other culturally organized activities involving different technologies and symbol systems ... (p. 13)

This definition requires that *practices* are culturally organized in nature and involve technologies or symbols systems. From this perspective, mathematical practices are social and cultural, because they arise from communities and mark membership in communities. They are also cognitive, because they involve thinking, and they are also semiotic, because they involve semiotic systems (signs, tools, and their meanings). Mathematical practices involve values, points of view, and implicit knowledge.[3]

What does a *Vygotskian* perspective of mathematical practices entail? Beyond the assumption that practices are social and cultural in origin, a Vygotskian perspective has several implications for the concept of mathematical practices:

- Social interaction that leads to learning principally involves joint activity (not just any type of interaction),
- Goals are an implicit yet fundamental aspect of practices,
- Discourse is central to participation in practices,
- The meanings for words are situated and constructed while participating in practices,
- Appropriation is a central metaphor for describing learning (but learners do not merely imitate practices, they sometimes actively transform them).

The central features of appropriation as described by Rogoff (1990) are that appropriation involves achieving a shared focus of attention, developing shared meanings, and transforming what is appropriated. Rogoff suggests that intersubjectivity may be especially important for learning to participate in practices that are implicit:

> Intersubjectivity in problem solving may (also) be important in fostering the development of "inaccessible" cognitive processes that are difficult to observe or explain – as with shifts in perspective as well as some kinds of understanding and skills. (p. 143)

In my work I have used this Vygotskian perspective to frame the study of mathematical practices. In particular, I will refer to one study that examined how interaction with a tutor (Moschkovich, 2004) supported learner appropriation of mathematical practices.[4]

Example: Appropriating mathematical practices

The example is a case study of how one student learned to use and explore functions through interaction with a tutor (Moschkovich, 2004) while using graphing software (Schoenfeld, 1990). In the article, I described how mathematical practices involve goals, meanings for utterances, and focus of attention. This account of learning mathematics focused on two mathematical practices for using functions: treating functions as objects and connecting a line to its equation (and vice versa). The tasks used in that study reflect two aspects of mathematical practices related to functions: a) seeing, talking about, and acting as if a line is an object that can be manipulated and b) seeing, talking about, and acting as if lines are connected to their equations. I argued that the focus of attention, meanings for utterances, perspectives, actions, and goals that the learner appropriated through joint problem solving with the tutor were significant not as isolated skills, but because treating functions as both objects and processes and connecting lines to equations are mathematical practices central for success in using and exploring functions.

This study complemented the cognitive analysis presented in Schoenfeld, Arcavi, and Smith (1993) describing learning in this domain. The 1993 study described that the student learned many things during these tutoring sessions: she corrected previous knowledge, added new pieces of knowledge, and made new connections between pieces of knowledge. However, the focus of my study was not *what* she learned but *how she learned through interactions with the tutor* (for a detailed analysis of what this student learned, see Schoenfeld et al., 1993). My analysis thus extended this work by describing in detail how this student's learning was mediated by interactions with the tutor.

The analysis addressed questions specific to the appropriation of mathematical practices: What *particular* aspects of mathematical practices does a learner appropriate as they gain expertise in this domain? *How* does a learner appropriate mathematical practices? This case study described how one learner, by solving problems jointly with a tutor, appropriated the focus of attention for tasks, meanings for utterances, and the actions and goals for carrying out new tasks. The focus of attention, meanings, and goals were not evident in the interactions as isolated pieces of tutor knowledge that were stated explicitly by the tutor. Rather, ways of seeing, talking, and acting were implicitly embedded in mathematical practices as the focus of attention, the meanings of utterances, the perspectives, and the actions and goals used during joint activity. I described two mathematical practices, treating lines as objects and the action of connecting a line to its corresponding equation. These two expert mathematical practices for working with functions were evident in where the tutor and student focused their attention, in how they interpreted the meanings of utterances (especially questions by the tutor), and in the goals and actions the tutor and student set and carried out to accomplish tasks.

This case study used a Vygotskian perspective and the concept of appropriation (Newman, Griffin, & Cole, 1989; Rogoff, 1990) to describe how a student learned to work with linear functions. The analysis described the impact that interaction with a tutor had on a learner, focusing on how the learner appropriated goals, actions, and meanings that are part of mathematical practices (as well as how the learner was active in transforming several of the goals that she appropriated). The paper described how a learner appropriated two mathematical practices that are crucial for working with functions (Breidenbach, Dubinsky, Nichols, & Hawks, 1992; Even, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993; Schwarz and Yerushalmy, 1992; Sfard, 1992): treating lines as objects and connecting a line to its corresponding equation in the form $y = mx + b$. The analysis of two tutoring sessions illustrated how the tutor introduced three tasks (estimating y-intercepts, evaluating slopes, and exploring parameters) that involved two mathematical practices in this domain and described how the student appropriated goals, actions, meanings, and perspectives for participating in these practices.[5] I view the practices of treating lines as objects and connecting equations to lines as both emergent phenomena during these tutoring interactions as well as already established ways of working with lines and within communities that regularly use equations and graphs. I view practices as constituted by actions, goals, perspectives, and meanings for utterances and that these goals, actions, perspectives, and meanings are embedded within the mathematical practices that experts participate in when using and exploring functions.

The tutor and student interactionally established goals, meanings, and perspectives that were embedded in tasks. These goals, meanings, actions, and perspectives were part of three tasks the tutor introduced: visually estimating the y- intercept of a line, evaluating the slope of a line, and exploring the parameters in an equation. These tasks reflect two mathematical practices: a) seeing, talking about, and acting as if a line is an object that can be manipulated and b) seeing, talking about, and acting as if lines are connected to their equations. This student's learning was an example of "insertion into an intellectual practice requiring a social use of signs and the understanding of their meanings" (Radford, 2001, p. 261).

Connecting mathematical practices to mathematical discourse

At this point, I would like to make a connection between mathematical practices and mathematical discourse. From a Vygostkian perspective, discourse is central to joint activity and the example summarized above includes meanings for utterances as a component of any mathematical practice. How we connect mathematical practices to mathematical discourse depends, in part, on how we define and use the term *discourse*. I use the phrase "mathematical discourse practices" to signal that I do not conceptualize mathematical Discourse as individual, static, or referring only to language. Instead, I assume that mathematical discourses are more than language, that meanings are multiple and situated, and that mathematical discourse practices are connected to multiple communities. Discourses certainly involve using language, but they also involve other symbolic expressions, objects, and communities. Gee's definition of Discourses (1996)[6] highlights how these are not just sequential speech or writing:

> A Discourse is a socially accepted association among ways of using language, other symbolic expressions, and "artifacts," of thinking, feeling, believing, valu-

ing and acting that can be used to identify oneself as a member of a socially meaningful group or "social network," or to signal (that one is playing) a socially meaningful role. (p. 131)

In general, discourse is not disembodied talk; it is embedded in practices. Most importantly for this discussion of mathematical practices, language, utterances, or meanings are not mathematical in themselves but as they are embedded in mathematical practices. This is the crucial connection between mathematical discourse and practices.

Moreover, mathematical discourse practices involve not only meanings for utterances but also focus of attention.[7] Mathematical discourse practices are not simply about using a particular meaning for an utterance, but rather using language in the service of particular goals while coordinating the meaning of an utterance with a focus of attention. Thus, mathematical discourse practices involve not only language but also perspectives and conceptual knowledge. Words, utterances, or texts have different meanings, functions, and goals depending on the practices in which they are embedded. Discourses occur in the context of practices and practices are tied to communities. I view mathematical discourse practices as dialectically cognitive and social. On the one hand, mathematical discourse practices are social, cultural, and discursive because they arise from communities and mark membership in discourse communities. On the other hand, they are also cognitive, because they involve thinking, signs, tools, and meanings.

We should not imagine that classroom discussions involve one single set of discourse practices that are (or are not) mathematical. In fact, we might imagine the classroom as a place where multiple mathematical discourse practices meet. As teachers and students engage in conversations, they bring in multiple meanings for the same utterances and they focus their attention on different aspects of any situation. These meanings and way of focusing attention may reflect the meanings and ways to focus attention that are common in more than any one discourse community. When discussions serve as a way to negotiate what utterances mean and where one might focus one's attention, different discourse communities are, in some sense, meeting.

There is no one mathematical discourse practice (for a discussion of multiple mathematical discourse practices see Moschkovich, 2002b). Mathematical discourse practices vary socially, culturally, and historically. Mathematical discourse practices vary across different communities for example between research mathematicians and statisticians, elementary and secondary school teachers, or traditional and reform-oriented classrooms. Mathematical arguments can be presented for different purposes such as convincing, summarizing, or explaining. Mathematical discourse practices also involve different genres such as algebraic proofs, geometric proofs, school algebra word problems, and presentations

Research needs to shift away from dichotomizing everyday and academic discourse practices. Because classroom discourse is a hybrid of academic and everyday discourses, multiple registers co-exist in mathematics classrooms. Most importantly for supporting the success of students in classrooms, academic discourse practices needs to build on and link with the language and practices that students bring from their home communities. Therefore, everyday discourse practices should not be seen as obstacles to participation in academic mathematical discourse practices, but as resources to build on in order to engage students in the formal mathematical practices taught in classrooms. For example, the ambiguity and multiplicity of meanings in everyday language should be recognized and treated not as a failure to be mathematically precise but as fundamental to making sense of mathematical meanings and to learning mathematics with understanding.

We may even need to consider that mathematical language may not be as precise as mathematicians or mathematics instructors imagine it to be. Although many of us may be deeply attached to the precision we imagine mathematics provides, ambiguity and vagueness have been reported as common in mathematical conversations and have been documented as resources in teaching and learning mathematics (e.g., Barwell, 2005; Barwell, Leung, Morgan, & Street, 2005; O'Halloran, 1999, 2000; Rowland, 1999) as well as in the work of mathematicians (Byers, 2007; Hersch, 2007). Even definitions are not a monolithic mathematical practice, since they are presented differently in lower level textbooks – as static and absolute facts to be accepted – while in journal articles they are presented as dynamic, evolving, and open to decisions by the mathematician. Neither should textbooks be seen as homogeneous. Higher-level textbooks are more like journal articles in allowing for more uncertainty and evolving meaning than lower level textbooks (Morgan, 2004), evidence that there are multiple approaches to the issue of precision, even in mathematical texts.

THE FUTURE OF MATHEMATICAL PRACTICES

Schoenfeld's framework has certainly had a great impact on how we think about, describe, and teach mathematical thinking. Twenty years later, both researchers and practitioners regularly refer to metacognition as central to solving mathematical problems. Schoenfeld's addition of practices to his framework for mathematical problem solving in 1992 not only provided a more complete framework, but also set the stage for deeper analyses of mathematical practices that followed in mathematics education research. The work summarized in this chapter is only one of many attempts to theoretically frame the concept of mathematical practices, connect it to mathematical discourse, and describe how mathematical practices are appropriated. Much more work remains to be done clarifying how we define mathematical practices, describing the connections to other aspects of mathematical activity, and studying how practices are acquired.

In closing, I would like to raise some issues regarding how we interpret and use the concept of "mathematical practices" for making recommendations for teaching mathematics. As an example, here is the list of eight "Standards for Mathematical Practice" described in the Common Core State Standards (for more details see http://www.ccsstoolbox.com/):

1. Make sense of problems and persevere in solving them.

2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

There are several issues I would like to raise about these standards. The first is that this is a list of standards, not practices.[8] The Common Core documents say that these "describe varieties of expertise that mathematic educators at all levels should develop in their students" and connect these standards to other "processes and proficiencies" such as problem solving, reasoning and proof, communication, and connections, as well as to the four strands of mathematical proficiency (adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition). The second issue is that, although it may seem obvious that we need to distinguish practicing – in the sense of repetition or rehearsal as a strategy for memorizing – from the concept of practices, we should make this distinction explicit in conversations with practitioners and laypeople who may associate learning mathematics primarily with repeating and memorizing multiplication tables.

The third issue is that as we talk about these standards for mathematical practice, we should think carefully about how we see the origin of the mathematical practices that mathematics educators value, in particular the role of social interaction in learning to participate in these practices. Interpretations and applications of these standards will depend on how we frame the concept of mathematical practices. What assumptions are we making about the nature, origin, and development of valued mathematical practices? Are practices individual, collective, cognitive, discursive, social, and/or cultural phenomena? Again, it is impossible to answer these questions without a theoretical framing. Without a Vygotskian framing, we might assume that some mathematical practices are individual in the sense that we typically accomplish the goals for these activities alone (for example when we persevere in solving problems, reason abstractly and quantitatively, model with mathematics, or look for and make use of structure). In the same vein, we might think that critiquing others reasoning is *really* social because critiquing the reasoning of others requires other people. However, from a Vygotskian perspective and a practice view as defined by Scribner, all mathematical practices are socio-cultural phenomena in the sense that they are higher order intellectual activities that *originate* through social interaction. Children and adolescents learn to participate in mathematical practices first interpersonally and then come to appropriate the practices as these become part of the repertoire of practices that an individual will later use (either alone or in the company of others).

How does this theoretical framing matter for instruction? If we leave behind the assumption that mathematical practices are socio-cultural in origin and see these practices are purely individual and mental, we will continue to see some learners as deficient because they have not yet developed proficiency in these practices and others as talented because they (supposedly) developed these practices all on their own. The question to ask is not whether a mathematical practice is always or primarily carried out through social interaction, but whether social interaction was involved in the origin of the practice, even if the practice is now accomplished alone. Another question that is relevant to instruction is what kind of social interaction is conducive to learning (appropriating) mathematical practices. Without a Vygotskian framing one might think that the social interaction necessary for student learning is telling students that these practices are important, or reminding students that they should engage in these practices often, or modelling these practices at the board. However, from a Vygotskian perspective, these examples of social interaction do not include an active learner or involve joint activity and, therefore, do not support learner appropriation of goals, focus of attention, or shared meanings for language.

The last issue is a call to clarify the meaning of "precision." It is important to consider what we mean by precision for students learning mathematics, since students are likely to need time and support as they move from expressing their reasoning and arguments in imperfect form towards more academic ways of talking. In particular, we should remember that precise claims can be expressed in imperfect language and that attending to precision only at the individual word level will get in the way of students' expressing their emerging mathematical ideas. More work is needed to clarify how best to guide students in becoming more precise in their mathematical language over time.

The belief that precision lies primarily in individual word meaning reflects a simplified view of language that is not connected to a Vygotskian theoretical perspective. From this perspective, word meaning is embedded in cultural practices. For example, we could imagine that attending to precision means using two different words for the set of symbols "$x + 3$" and the set of symbols "$x + 3 = 10$." If we are being precise at the level of individual word meaning, the first is an "expression" while the second is an "equation." However, attending to precision is not necessarily about using the precise word; a significant mathematical practice is making claims *about precise situations*. We can contrast the claim "Multiplication makes bigger," which is not precise, with the question and claim "When does multiplication make the result bigger? Multiplication makes the result bigger when you multiply by a positive whole number (or a number greater than 1, or another way to express that the number cannot be a fraction between 0 and 1 or negative)." Notice that when contrasting these two claims, precision does not lie in the individual words, nor do the words used in the more precise claim necessarily need to be formal "math" words. Rather, the precision lies in the *mathematical practice* of specifying when the claim is true. Instruction should move away from interpreting precision to mean using the precise word, and instead focus on how precision works at the claim level in mathematical practices.

NOTES

[1] Mathematical practices in historical context are also different than the situation with mathematical practices in school. See Moschkovich (2002) for a discussion of the history of academic and everyday mathematical practices.

[2] In using the terms practice and practices in the sense used by Scribner (1984), I make a distinction between the concept of practices and other common uses, for example practice as repetition or rehearsal, or practice as in "my teaching practice."

[3] Many researchers have used the concept of mathematical practices. For example, Cobb, Stephan, McClain, and Gravemeijer (2001) define mathematical practices as the "taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas" (Cobb et al., p. 126). In contrast to social norms and socio-mathematical norms, mathematical practices are specific to particular mathematical ideas. Cobb et al. (2001) make a distinction between two types of mathematical practices; those that are "normative ways of acting that have emerged during extended periods of history" and mathematical practices that are emergent in classroom activity. In my own work, I have framed mathematical practices as simultaneously emergent in current classroom activity and the result of socio-cultural historical activity. However, it is not my aim in this chapter to review the use of the concept or compare and contrast how different researchers have used the construct of mathematical practices.

[4] In another study (Moschkovich, 2008), I described how a teacher supported learner appropriation of mathematical practices during a teacher-led small group discussion.

[5] The study also described how appropriation functioned in terms of the focus of attention, the meaning for utterances, and the goals for these three tasks and examined how the learner did not merely repeat the goals the tutor introduced but actively transformed some of these goals.

[6] Gee distinguishes between "discourse" and "Discourse." For sake of clarity, I will only use the term discourse.

[7] In another study (Moschkovich, 2008), I examined how meanings for utterances reflect particular ways to focus attention. The notion of focus of attention also comes from a sociocultural framework that uses appropriation for describing learning. The 2008 analysis provided an example of how utterances have multiple meanings depending on the focus of attention during joint activity.

[8] I thank my colleague Patricio Herbst for bringing my attention to this distinction.

REFERENCES

Barwell, R. (2005). Ambiguity in the mathematics classroom. *Language and Education*, *19*(2), 117–125.

Barwell, R., Leung, C., Morgan, C., & Street, B. (2005). Applied linguistics and mathematics education: More than words and numbers. *Language and Education*, *19*(2), 141–146.

Breidenbach, D., Dubinsky, E., Nichols, D., & Hawks, J. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*, 247–285.

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics. *Educational Studies in Mathematics*, *37*(2), 121–143.

Byers, W. (2007). *How mathematicians think: Using ambiguity, contradiction, and paradox to create mathematics*. Princeton, NJ: Princeton University Press.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology*, *3*, 21–29.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *The Journal of the Learning Sciences*, *10*(1&2), 113–164.

Cobb, P., Wood, T. & Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In E. Forman, N. Minick, & C. A. Stone (Eds), *Contexts for learning: Sociocultural dynamics in children's development* (pp. 91–119). New York, NY: Oxford University Press.

Davis, P. J., & Hersch, R. (1982). *The mathematical experience*. Boston: Houghton Mifflin Co.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, *21*, 521–544.

Forman, E. (1996). Learning mathematics as participation in classroom practice: Implications of sociocultural theory for educational reform. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 115–130). Mahwah, NJ: Lawrence Erlbaum Associates.

Gee, J. (1999). *An introduction to Discourse analysis: Theory and method*. New York: Routledge.

Gutiérrez, K. D., Sengupta-Irving, T., & Dieckmann, J. (2010). Developing a mathematical vision: Mathematics as a discursive and embodied practice. In J. Moschkovich (Ed.), *Language and mathematics education: Multiple perspectives and directions for research*. Charlotte, NC: Information Age Publishing.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. New York: Princeton University Press.

Hardy, G. H. (1940). *A mathematician's apology*. Cambridge: Cambridge University Press.

Hersh, R. (2007). Book review: How mathematicians think. *Notices of the AMS*, *54*(11), 1496–1499.

Høyrup, J. (1994). *In measure, number, and weight: Studies in mathematics and culture*. Albany: SUNY Press.

Lakatos, I. (1976). *Proofs and refutations* (Rev. ed.). Cambridge: Cambridge University Press.

Lampert, M. (1986). Knowing, doing and teaching multiplication. *Cognition and Instruction*, *3*(4), 305–342.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*(1), 29–64.

Latour, B. (1987). *Science in action*. Cambridge: Harvard University Press.

Latour, B., & Woolgar, S. (1986). *Laboratory life: The construction of scientific facts*. Princeton, NJ: Princeton University Press.

Lave, J. (1988). *Cognition in practice*. Cambridge: Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.

Mehan, H. (1979). *Learning lessons: Social organization in the classroom*. Cambridge, MA: Harvard University Press.

Mendez, E. P. (2001.) A history of mathematical dialogue in textbooks and classrooms. *Mathematics Teacher*, *94*, 170–173.

Morgan, C. (2004). Word, definitions and concepts in discourses of mathematics, teaching and learning. *Language and Education*, *18*, 1–15.

Moschkovich, J. N. (2002). An Introduction to examining everyday and academic mathematical practices. In M. Brenner & J. Moschkovich (Eds.), *Everyday and academic mathematics: Implications for the classroom*. *Journal for Research in Mathematics Education*, Monograph Number 11, 1–11.

Moschkovich, J. N. (2004). Appropriating mathematical practices: A case study of learning to use and explore functions through interaction with a tutor. *Educational Studies in Mathematics*, *5*, 49–80.

Moschkovich, J. N. (2007). Examining mathematical Discourse practices. *For the Learning of Mathematics*, *27*(1), 24–30.

Moschkovich, J. N. (2008). "I went by twos, he went by one:" Multiple interpretations of inscriptions as resources for mathematical discussions. *The Journal of the Learning Sciences*, *17*(4), 551–587.

Moschkovich, J. N. (2010). *Language and mathematics education: Multiple perspectives and directions for research*. Charlotte, NC: Information Age Publishing.

Moschkovich, J. N., & Brenner, M. (2000). Integrating a naturalistic paradigm into research on mathematics and science cognition and learning. In R. Lesh & A. Kelly (Eds.), *Handbook of research design in mathematics & science education* (pp. 457-486). New Jersey: Lawrence Erlbaum Associates.

Moschkovich, J., Schoenfeld, A., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations, and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of function* (pp. 69–100). Hillsdale, NJ: Erlbaum.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

Newman, D., Griffin, P., & Cole, M. (1989). *The construction zone: Working for cognitive change in school*. Cambridge: Cambridge University Press.

Nunes, T., Schliemann, A., & Carraher, D. (1993). *Street mathematics and school mathematics*. Cambridge: Cambridge University Press.

O'Halloran, K. L. (1999). Towards a systemic functional analysis of multisemiotic mathematics texts. *Semiotica*, *124*(1/2), 1–29.
O'Halloran, K. L. (2000). Classroom discourse in mathematics: A multisemiotic analysis. *Linguistics and Education*, *10*(3), 359–388.
O'Connor, M. C. (1999). Language socialization in the mathematics classroom. Discourse practices and mathematical thinking. In M. Lampert & M. Blunk (Eds.), *Talking mathematics* (pp. 17–55). New York: Cambridge University Press.
Ochs, E., Gonzales, P., and Jacoby, S., (1996). "When I come down I'm in the domain state": Grammar and graphic representation in the interpretive activity of physicists. In E. Ochs, E. Schegloff, & S. Thompson (Eds.), *Interaction and grammar* (pp. 328–369). New York: Cambridge University Press.
Ochs, E., Jacoby, S., & Gonzales, P. (1994). Interpretive journeys: How physicists talk and travel through graphic space. *Configurations*, *2*(1), 151–171.
Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London: Routledge.
Polya, G. (1957). *How to solve it* (2nd ed.). Princeton, NJ: Princeton University Press.
Radford, L. (2001). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics*, *42*(3), 237–268.
Restivo, S. (1993). The social life of mathematics. In S. Restivo, J. P. Van Bendegem, & R. Fischer (Eds.), *Math worlds: Philosophical and social studies of mathematics and mathematics education* (pp. 247–278). Albany: SUNY Press.
Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 13–51). Dordrecht, the Netherlands: Kluwer.
Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context*. New York: Oxford University Press.
Rowland, T. (1999). *The pragmatics of mathematics education: Vagueness in mathematical discourse*. New York: Routledge, Taylor & Francis Group.
Saxe, G. (1991). *Culture and cognitive development*. New Jersey: Erlbaum.
Schoenfeld, A. (1985). *Mathematical problem solving*. New York: Academic Press.
Schoenfeld, A. (1990). GRAPHER: A case study in educational technology, research, and development. In A. DiSessa, M. Gardner, J. Greeno, F. Reif, A. Schoenfeld, & E. Stage (Eds.), *Toward a scientific practice of science education* (pp. 281–300). Hillsdale, NJ: Erlbaum.
Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.
Schoenfeld, A. H. (2007). Problem solving in the United States, 1970–2008: Research and theory, practice and politics. *ZDM – The International Journal on Mathematics Education*, *39*(5–6), 537–551.
Schoenfeld, A. S. (2011). *How we think: A theory of goal-oriented decision making and its educational applications*, Studies in Mathematical Thinking and Learning Series. New York: Routledge.
Schoenfeld, A., Arcavi, A. & Smith, J. (1993). Learning. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 55–175). Hillsdale, NJ: Erlbaum.
Schwarz, J., & Yerushalmy, M. (1992). Getting students to function in and with algebra. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 261–289). MAA Notes, Vol. 25. Washington, DC: Mathematical Association of America.
Schleppegrell, M. J. (2010). Language in mathematics teaching and learning: A research review. In J. Moschkovich (Ed.), *Language and mathematics education: Multiple perspectives and directions for fesearch*. Charlotte, NC: Information Age Publishing.
Scribner, S. (1984). Studying working intelligence. In B. Rogoff & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 9–40). Cambridge: Harvard University Press.
Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification-the case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 261–289). MAA Notes, Vol 25. Washington DC: Mathematical Association of America.
Traweek, S. (1988). *Beam times and lifetimes: The world of high energy physicists*. Cambridge: Harvard University Press.
Tymoczo, T. (Ed.). (1986). *New directions in the philosophy of mathematics*. Basel: Birkhauser.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Cambridge, MA: Harvard University Press.
Vygotsky, L. S. (1979). The genesis of higher mental functions. In J. V. Wertsch (Ed. & Trans.), *The concept of activity in Soviet psychology* (pp. 144–188). Armonk, NY: M. E. Sharpe.
Vygotsky, L. S. (1987). *The collected work of L. S. Vygotsky* (R. W. Rieber, Ed., & N. Minick, Trans.). New York: Plenum Press.
Wertsch, J. V. (1979a). The concept of activity in soviet psychology: An introduction. In J. V. Wertsch (Ed. & Trans.), *The concept of activity in soviet psychology* (pp. 3–35). Armonk, NY: M. E. Sharpe.
Wertsch, J. V. (1979b). From social interaction to higher psychological processes: A clarification and application of Vygotsky's theory. *Human Development*, *22*(1), 1–22.
Wertsch, J. V. (1984). The zone of proximal development: Some conceptual issues. In B. Rogoff & J. V. Wertsch (Eds.), *Children's learning in the zone of proximal development* (pp. 7–18). New Directions for Child Development Series, No. 22. San Francisco: Jossey-Bass Inc.
Wertsch, J. (1985). *Vygotsky and the social formation of mind*. Cambridge: Harvard University Press.

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PART V

REFLECTIONS AND FUTURE RESEARCH DEVELOPMENT

GÜNTER TÖRNER

16. LOOKING BACK AND AHEAD – SOME VERY SUBJECTIVE REMARKS ON RESEARCH IN MATHEMATICS EDUCATION

Let me first express my great thanks for the undertaking of this book, which humbled me. I am aware of the enormous amount of work accomplished by Yeping Li and Judit Moschkovich. I very much appreciate their work and efforts.

In addition, I am excited that my scientific work is part of a book in honor of Alan Schoenfeld. Our birthdays are so close that he could not be my brother: however, I regard him not only as my personal friend, but also as my teacher and I see myself as his auditor since I have learned so much from him.

This chapter is not meant to serve as a first draft of some forthcoming memoirs, of a mathematician cum mathematics educator. My academic life, fortunately, is continuing – and I am very happy to still be involved in my university as a lecturer and scientist, and recently as an ombudsman for scientific integrity.

However, I believe that the editors expect me to line out future research developments, as recommendations for young researchers. Thus I will start – according to countless reviewers' opinions – to manifest my subjectivity. Clipped and precise: Being a senior, I am allowed to speak and think freely and I do not worry about negative consequences. Since the readers of this book are expected to be mathematics educators, my comments will be restricted to mathematics education. However, some of them may also be applied to mathematics and its applications.

The list of issues, which I like to annotate, will remain exemplary and the sequence of the items is arbitrary. Some of the issues are more general, whereas others are of a more precise range of influence. I will number them in order to be able to compare topics to each other.

1. Contributing results to an international discussion. More than 30 years ago my colleague Werner Blum and I published a book on the *Didactics of calculus* (Blum & Törner, 1983). It was the first comprehensive textbook in Germany on this subject and was intended to address prospective student teachers. At that time I believed that the investigation of the main issues in this subject should be based on our views on results of German classroom analyses. Later, when the first TIMSS results were published in the late 1990s, I learned that the problems surrounding mathematics are nearly always the same no matter the country. This observation marked a personal paradigm change. I am sorry that today our intensive, in-depth discussion in Germany on the didactics of calculus is unknown to mathematics

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education communities elsewhere because of the language barrier. Thus, there was no chance to receive acknowledgement and appreciation from other researchers. By the way, I should explain that I recently accepted an invitation to work on a report comparing various European concepts on calculus. Somehow 30 years too late – but there will be a synopsis anyhow.

2. International research – The other side of the coin. There is not only the obligation to provide other researchers with your insights and publish in English, but we have to be aware of the actual international discussion. Early in the 1990s, when I was working in problem theory, I also stumbled upon beliefs and soon learned from my fellow colleague through his book that beliefs are very important variables. This was during the time when the curriculum reform by propagating problem solving in the States was at its zenith (see Alan's article and many others on that topic). I willingly accepted that the propagation of problem solving remains illusionary as long as accompanying beliefs are not also under discussion. This was the beginning of my engagement in and empirical research on the theory of beliefs and subjective theories (e.g. world views).

3. The discussion on beliefs in Germany. It was an epiphany for me personally, and it opened a new research area in Germany: the discussion on beliefs and world views, not only in the context of the problem solving, but in general as proposed by Alan Schoenfeld's and others' articles (e.g., Schoenfeld, 1985, 1998). I should also credit the empirical research of my student Stefan Grigutsch on students' world-views on mathematics, who – I am sorry to say – went after his PhD-graduation as a teacher to school and did not start a promising career as a researcher. Today, it is self-evident that one cannot understand teachers' and students' behavior without analyzing their underlying beliefs.

At the same time all PME- and PME-NA-conferences had a session on beliefs, whereas in Germany only a few colleagues worked in this area. However, times changed. While today many scientists are integrating beliefs into their models, Alan is promoting an overarching scientific construct: that of "orientation."

4. The Issue-Life-Cycle-Management in mathematics education. From business economics we know that "issues" have something like a life cycle. At the beginning only a few are aware of the relevance and the issue remains more or less invisible in general. Then there is an emergence of the topic, next there is a zenith, and soon after the issue is superseded by a new issue. So far, I do not know any research on this topic in mathematics education, but I believe in sociological laws behind issues. It will always take time for development when new terminology is entering scientific discussions. Maybe it is a law of research propagation that new terminology requires time to be implemented. Being a senior I do not believe in a continuous flow of development by new research results, for me it is more appropriate to assume erratic progress, although on a more finite scale there might be some continuous growth.

It is the task of senior scientists to bring the changes in leading issues to mind; thus I do not hesitate to mark the progress with respect to Leder, Pehkonen, and Törner's book on beliefs first published in 2002 (Leder, Pehkonen, & Törner, 2002). Ten years later there is again space for a new book, an already announced new publication to be edited by my former student, Bettina Rösken-Winter, concentrating on what we have learned since 2002 in beliefs research (Pepin & Rösken-Winter, in preparation).

Once again, what is important in motivating students (i.e. to make obvious their learning progress) is also important for researchers and their community, namely to emphasize new insights while reflecting on the previous ones.

5. Research and Implementation (R & I). The title as proposed by the editors contains the word "research." Undoubtedly research is of high importance for both mathematics and mathematics education alike, as I have learned from personal experience.

However, since there were some challenging collaborations with industry in my academic life, I have often stumbled upon two letters, R & D, which stand for "Research and Development" in industrial contexts. I called it a tandem: two fields of action, two sides of the same coin. However, in mathematics education, I have never read such a suggestion. Probably in our field we should replace the tandem R & D with R & I, where the letter I stands for implementation.

Let me give an example from industry. If a company is not able to put a promising research result into effective production, the management will not be amused, since there is no profit from the research. A company cannot afford such a drawback too often.

Do not misunderstand me: We all are aware that producing research results is not sufficient. Just as mathematicians tell their first year students: Necessary does not mean sufficient. However, the question of *implementation* is often only a marginally discussed issue and is sometimes much more difficult than research. We all know that is easy to offer a proposal, but soon realize this proposal is much more complicated. It is my observation that the question of implementation is often underexposed in our discussion on research. I freely concede that the research and implementation side of a project can rarely be executed by a single person. Maybe two persons or even two groups working in tandem are necessary. Rather seldom have I found this issue being discussed in our community. Even more, there is nearly no culture of implementation in our daily work. We should not assign the implementation process to educational policy and its administration; all of us should regard ourselves as decisive stakeholders.

To use a stylistic device, research without convincing answers regarding the implementation of its results, can be compared to the broken or unfinished pillars that can be found in old graveyards and which indicate that life is sometimes too short to finish a project.

6. The overflow of non-sustainable publications. Being a senior I am no longer dependent on the number of my publications. When I receive invitations, it is up to me to "accept" or "deny." We are drowned by the number of papers published in our community because countless research reports are accepted by PME, by ERME, or by ICME, etc. However, I think we must consider if we are really communicating in an appropriate style?

In this context, I would like to remind my readers of Gauss' philosophy *pauca sed matura* (little but also mature). It is reported that the great mathematician rejected a different proposal by a friend who tried to convince Gauss to follow the strategy, *multa nec immature* (many and not immature). Following Gauss' guideline would indeed mark progress for me.

Many research papers are too long and yet, at decisive places, are not detailed enough. I wonder who can read all these pages and think about them. Sorry, I could not offer a pragmatic solution, but my vision would be that a report is offering information on different scales or magnifications which can be set up by the reader. Usually, I start to read a paper by looking up the introduction and questioning the conclusions. Then I decide whether any of the other sections are important for me.

P. M. Cohn, an internationally highly respected algebraist, once said it is easier to *write two publications* than to read and *understand one*. Maybe he was already a professor emeritus at that time. There is definitely a need for research on the styles and layouts of communication in our community.

I dare not image the situation in ten years' time. The readers may extrapolate their current situation, count their daily emails and compare the figures to the situation five years ago – incredible. Never in the history of science has a generation had so much and so easy access to scientific results and yet been as ill-informed as today's generation of researchers.

7. The missing memory of our community. Are there still any scientists who are well-informed about the delusions and confusions within mathematics education in the 1950s? It seems to me that the memory of a generation only comprises the last thirty years. I consider this a great danger. A famous German philosopher, Wolfgang Huber, recently lamented an unacceptable consumption of the "past" and at the same time, *an illegal shortening of the "presence."*

As a mathematician I know that it is important to review all scientific results on a problem by intensively investigating the databases of Mathematics Reviews and Zentralblatt. The situation in the field of mathematics education is not as comfortable as in mathematics, although there is the MathEdu database. However, I am wondering why our community – in some parts of the world – is not supporting MathEdu, which would make this database much stronger and the management cheaper. However, to be successful representatives of mathematics and mathematics education in committees I have to communicate and to cooperate.

8. Solid findings in mathematics education. I chair the Committee of Education of the European Mathematical Society (EMS) that started a series of articles under the headline "solid findings." They are published for the audience of mathematicians, namely the readers in the EMS Newsletter.

But what is really solid in mathematics education? What marks solid insights? To cite our introductory article in the EMS Newsletter, we mean findings that:

- result from trustworthy, disciplined inquiry, thus being sound and convincing in shedding light on the question(s) they set out to answer.
- are generally recognized as important contributions that have significantly influenced and/or may significantly influence the research field.
- can be applied to circumstances and/or domains beyond those involved in this particular research.
- can be summarized in a brief and comprehensible way to an interested but critical audience of non-specialists (especially mathematicians and mathematics teachers).

I fear that we are unable to give final convincing answers. However, I believe they exist. These scientific insights should be discussed during ICME every four years. We ought to have chronicles to list the most important findings of recent years.

9. Sustainability. Continuing the discussion under 8. Sustainability is a key concept whenever you apply for a new project. Of course it is correct for everybody to strive for sustainability, but are we really honest with ourselves? Is our community honest with itself? I will stop claiming universal platitudes. I believe that sustainability needs some research on its own. At the moment, in the context of establishing a center for excellence of teaching mathematics in Germany, we are starting to learn more about the sustainability of continuous professional development for teachers.

Of course, the list of these general issues could be extended, but I would like to end here. Now is the time for me to mark some specific research fields. As always, the question how to implement our findings will remain unanswered – but at least we are aware of this deficiency. What are, from my point of view, the relevant research topics that should be stressed, maybe much more acutely? The range of potential action fields is wide, but a few are listed below.

10. Research in the reality of everyday-classroom teaching. It is the merit of David Clark to have focused on the Learners' Perspective Study.[1] Nevertheless, much more should be done. For me a challenging research question is how the *everyday teaching of mathematics* is "executed." Some of the TIMSS videos made me nervous. The problem is how to observe and investigate the normal everyday mathematics classroom. As soon as I am entering a classroom, the teacher will explicitly or implicitly react on my presence and so do the students, thus the situation is no longer a regular everyday mathematics classroom. My presence disrupts reality. Teachers often tell me: You as mathematics educators do not know what is really happening in my classroom. Yes, I would like to know, but it is nontrivial to get the relevant insights and data and then to offer paradigmatic solutions.

11. The challenge of transition from school to university. In older times when 10% of an age group was leaving school and entering university, this process was never under discussion. School "delivered" (matured) beginners and universities welcomed them. Today more than 40% of an age group leaving school begins study at a university. Thus, we are aware that there is a high drop-out rate in this process. Mathematics at university is the subject with one of the highest dropout rates. Everybody knows the mutual recrimination from university to school – the students are "not sufficiently educated and trained" – and vice-versa from school to university – university teachers are ignoring the actual school curricula and not teaching properly.

This is not a really didactical problem, but a question of how different systems are interfering and again R & I are necessary. Although I am not very happy with the current situation in Germany and I do not want to unroll details, the need to address this problem gradually becomes evident. Nevertheless, it is a question of the credibility of our community taking care and offering proposals.

12. Missing research on mathematics sociology. Of course I know that there are some articles on this topic, but I am not aware of any real research. We have to understand what mathematics education is doing to our mind and "soul" or – to use a fruitful term of Alan's – how it is influencing our *worldviews*. Mathematics education shapes a person – and in not so few cases mathematics education also misshapes or deforms; you will find some discussion in Hersh's book, *What is mathematics, really?* Look up students' reports on their mathematics teachers and you will find out how many cases of mathematics traumatization exist. There is an old proverb in Germany: A student *does not learn for the teacher*; Freudenthal once remarked that many students do not learn *because of the teacher!*

13. An ongoing challenge and a never ending task – Partnerships between mathematicians and mathematics educators. It was the privilege of my double qualification that I was accepted by mathematicians as well as mathematics educators. In a sense I lived as a commuter, as a person living in two worlds. Sometimes ignorant mathematicians frustrated me and hence, I went to visit my colleagues in mathematics education. Rather soon, however, their limited knowledge of mathematics, and their reduction of mathematics to some folk wisdom made me return to mathematicians, and I feared that nothing could be changed.

However, through my work within the Committee of Education of the European Mathematical Society (EMS) I have learned that in other countries the situation is quite different, and it is not as peaceful everywhere as in my country. There are, even in Europe, "wars" between mathematicians and mathematics educators, there is no communication (and thus no cooperation), but many accusations concerning what is done wrongly, in curricula, in textbooks etc. Why?

The reasons may be historical; I rather suspect they are due to specific cultural variables. Of course, each country has its right to define its specifications, but communities cannot afford to have "math wars" like what took place in California. Hence we need a global systematic research connecting mathematics with mathematics education, not only via research, but via R & I – and then, hopefully, students will finally be the center of attention again.

14. Networking with the outside world. Mathematics and, in particular, mathematics education are missing a sufficiently large number of lobby groups. Note that there is no real "mathematical industry," which might pay for advertisements in our journals. There is no chance to change this either. However, there are foundations, non-governmental organizations, and institutions supporting and paying for projects. Many mathematicians believe that it is dishonorable for mathematics to ask for support from third parties, but I strongly disagree! We in Germany understand that mathematics education needs more friends. Although we already have some, you can never have too many friends.

So, together with Celia Hoyles (London) and Deutsche Telekom Foundation (Bonn, Germany), in March 2013 we held the conference "Friends of mathematics education" (FoME) in Berlin for the first time, bringing together all European stakeholders interested in mathematics education. This is a first step on the way to creating powerful lobbyism for our common aim to promote mathematics education. I hope and expect that many more steps will follow.

Last, but not least: 14 is not a prime number, but I dare to end my listing here. Astonishingly,

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2.$$

As a mathematician, I would like to ask for similar patterns or a proof of the exceptional case of these five numbers. Soon, though, my colleagues would argue that this problem belongs to recreational mathematics, which is often deemed inferior. As a person engaged in discrete mathematics I would respond that recreational mathematics nevertheless leads to important research in discrete mathematics, e.g. the marriage problem gave birth to new research areas.

However, when translating this issue to mathematics education, important questions are in existence. For example, what is "recreational mathematics" from the point of mathematics education? How can mathematics education contribute to a sustainable life-long-learning of mathematics with non-trivial recreational mathematics? What are the parameters to be set? These research questions have not yet been answered.

NOTES

[1] http://www.lps.iccr.edu.au/

REFERENCES

Blum, W., & Törner, G. (1983). *Didaktik der Analysis*. Göttingen: Vandenhoeck & Ruprecht.
Leder, G. C., Pehkonen, E., & Törner, G. (Eds.) (2002). *Beliefs: A hidden variable in mathematics education?* Dordrecht, the Netherlands: Kluwer Academic Publishers.
Pepin, B., & Rösken-Winter, B. (in preparation). *From beliefs to dynamic affect systems: (Exploring) a mosaic of relationships and interactions*. Heidelberg, Germany: Springer.
Schoenfeld, A. H. (1985). *Mathematical problem solving*. New York: Academic Press.
Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education*, *4*(1), 1–94.

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17. ENCORE

INTRODUCTION

The title of this chapter has a double meaning. At the end of a lovely performance, it is a cry for more: Encore! And, there is always the question, “Et encore?” – what next? I will say a bit about where the work represented in this book, as well as my own lines of work, might be heading.

THANKS

But first, a few words of appreciation. This volume, and the collection of people who contributed to it, demonstrate the wisdom inherent in the advice, “choose your friends and colleagues wisely.” I have been blessed with the best of both. Let me start with some words of thanks to Guenter Toerner, my partner in so many things – including age. As has been noted, we share a lot: a background as mathematicians; a love of mathematics and a wish for others to experience the pleasures of mathematics the ways we have; an urge to understand mathematical thinking, problem solving, and teaching, and to use those understandings for the improvement of mathematics education. And, I should add, good wine. To Guenter, a toast: May we continue to enjoy the pleasures of each other’s company, good wine, and not coincidentally, continued work, for many years to come.[1]

All of the authors and editors of this book share many of the attributes described above, although there are interesting and productive differences. Hugh Burkhardt, for example – my partner in crime for more than 30 years – is a physicist by training, and much more of an educational “engineer” than I. He shares with Guenter a passion to engineer productive, research-based change in real-world educational settings. My partnerships with Hugh (we have had joint grants on assessment-related topics since 1991, having worked together on problem solving since 1982) have inevitably resulted in my doing far more work that I had expected, in much messier arenas that I expected, with greater impact than I could have ever hoped. Hugh, too, loves nothing better than fine food and fine wine. “Work hard, play hard” is as good a motto as I can imagine. What is life without passion?

The editors, Yeping Li and Judit Moschkovich, represent two different trajectories of collegiality. I first interacted with Yeping through a purely professional channel: He and Rongjin Huang produced a volume on Chinese mathematics education (Li & Huang, 2013) for my Taylor and Francis series *Studies in Mathematical Thinking and Learning*. Such interactions are telling: working with editors, you can see just how much they want to “get it right” and how much energy they

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are willing to put into the task. Yeping has a spare energy tank, and a drive to make amazing things happen. That is true of Judit as well (and all the authors in this collection) – but Judit's trajectory was different. Judit was in my very first cohort of PhDs. She, along with Abraham Arcavi, Ming Ming Chiu, Lani Horn, Cathy Kessel, and Miriam Sherin, spent a number of years in Tolman hall on the Berkeley campus, either as my student or as a postdoc in the "functions group" (see http://functions.berkeley.edu/). They're all different. They all have their own unique and powerful voices, and their own distinctive and powerful lines of research.

Why? Many years ago Fred Reif, who mentored me as I made the transition from mathematics into education, described the ideal student as "the one who starts out as a student and ends up as a colleague." The best colleagues, of course, are the ones who have their own points of view, and who challenge you when they think you don't have it right. I have been blessed with such students. It was simply my responsibility to help them find and harness their own passions, and co-develop the standards for thoughtful and rigorous work that would guarantee the quality of whatever they produced.

I couldn't be happier with either set of colleagues (mine or Guenter's) in this volume. They have produced a series of studies that advance both what we know and how to help good ideas get "traction" where it counts – in classrooms. Their voices, and their passions, are clear.

THE HEART OF THE VOLUME

Broadly speaking, the papers in this volume concern or are based on the characterization of what it means to understand and do mathematics in a deeply conceptual way; what it means to teach for the development of such rich mathematical thinking and doing; how to understand and support teachers in such activities; and how the larger contingencies of society and schooling shape (for better or worse) what happens in classrooms.

For the introductory chapters in Part I, I will simply express my gratitude. Part II, "Proficient performance, beliefs, and metacognition in mathematical thinking, problem solving, and learning," begins appropriately with young children.[2] As Kristina Reiss, Anke M. Lindmeier, Petra Barchfeld, and Beate Sodian note in their chapter "Developing problem solving skills in elementary school,"

> Learning and understanding mathematics can be regarded as a kind of problem solving. This is particularly true for data and probability as a mathematical context. Accordingly, it is useful to recognize children's dealing with probabilistic situations from a problem-solving perspective.

The authors' use of such a framework provides a broad way to reflect upon issues of student thinking, and of teaching: does instruction explicitly attend to resources (typically yes), heuristics (often not), control strategies (rarely), and beliefs (almost never)? This approach provides a unifying way to think not only about this topic, but also potentially across the curriculum.

From my perspective the key question is: Is doing mathematics an act of sense making, in which the formal rules of the discipline are seen as the codification of regularities in what we have come to understand about the discipline, or is it "mastering" and applying rules and procedures developed by others? The former, sense making mode (which includes the dimensions above: resources, strategies, control, and beliefs) is what mathematics learning *could* be. Properly experienced, mathematics learning is fundamentally an act of sense making (see, e.g., Schoenfeld, 1992). Unfortunately, mathematics is often experienced in the latter, "transmissive" mode. As Christine Schmeisser, Stefan Krauss, Georg Bruckmaier, and Werner Blum indicate in their chapter "Transmissive and constructivist beliefs of in-service mathematics teachers and of beginning university students," such experience is consequential. We develop our orientations toward mathematics from our experiences with it; and if we are to teach mathematics, those orientations shape not only the ways we do mathematics but also the ways in which we teach it – thus perpetuating a vicious cycle. As the COACTIV work indicates, that cycle can be broken. Behavioral change is not easy, but it can be achieved. Part of the reason that "behavioral change" is not easy is that the phrase itself is only a partial truth. We make the decisions to act in the ways we do on the basis of our knowledge and resources, goals, and orientations (Schoenfeld, 2011). Hence sustained experiences that result in changes in orientations and goals (for one's students, for example) are necessary in order for the behavioral changes to be manifested.

In "Building on Schoenfeld's studies of metacognitive control towards social metacognitive control," Ming Ming Chiu, Karrie A. Jones, and Jennifer L. Jones examine the construct of social metacognitive control, which might be thought of as a collective (rather than individual) form of monitoring and group (rather than self) regulation. Doing so – and supporting students in becoming more effective at it – makes good sense. For one thing, group work is an increasingly common classroom practice. Anyone who has watched small groups in action knows that effective collaboration doesn't simply happen; it has to be supported. Equally important, the whole idea of "internalization" suggests that the actions of social metacognitive control for the regulation of a group's thinking, actions and emotions may well pay off in terms of the individual group members' development. Hence attention to group processes may well have a double payoff.

The chapters in Part III focus on teaching and its support via professional development. As in Part II, we begin with young children: In 'The CAMTE framework: A tool for developing proficient mathematics teaching in preschool," Pessia Tsamir, Dina Tirosh, Esther Levenson, Ruthi Barkai, and Michal Tabach remind us that, just as mathematical sense making should begin early, so should its support. And there are challenges. Mathematics is not typically a favored subject of preschool teachers, and most people's (including preschool teachers') view of what math for kids is or could be is truly impoverished. Although young kids can learn a great deal about geometry by working geometric puzzles and playing geometric games, and about mathematical thinking by means of various activities (think of the math involved sharing cookies fairly, in playing Chutes and Ladders, or of the strategy involved in tic-tac-toe), most people think of pre-school math (if they think of it at all) as simple counting, rote addition and subtraction, and perhaps the rote naming of geometric figures (see, e.g., Schoenfeld & Stipek, 2011). As noted above, teachers' beliefs about the nature of mathematics are fundamental shapers of how they interact with kids over mathematics. Thus, a program that helps pre-service teachers to gain an appreciation of the connectedness of mathematics, of various ways to approach and solve problems, and of the principles underlying the arithmetic operations – and to reflect on their understandings and their teaching – is a step toward early mathematical sense making.

Sense making, whether of the mathematics or of one's teaching, is the key. And to make sense of things, you have to be aware of them. This simple statement underlies the paradigm of noticing described by Miriam Sherin, Rosemary Russ, and Bruce Sherin in their chapter, "Integrating noticing into the modeling equation." In the interests of brevity, I will put aside my urge to delve into the details of their examples – give me enough detail about an incident of teaching, and I'm happy to wade into the sea of analysis at length – and will react at the meta-level. Some years ago, Barbara Rogoff noted that any analysis foregrounds some things, while backgrounding others. Sherin, Russ, and Sherin's paper highlights the affordances of the noticing and modeling approaches, and the potential synergies in interesting ways. A key set of questions for both paradigms is, "What does a teacher notice, and why? And, having focused on something, what does the teacher do (and why)?" Consider the issue of "framing" or "stance," the broad orientation that a teacher brings to any classroom situation. There is a lovely dialectic between framing and noticing. On the one hand, one is attuned to notice particular things (or not) as a result of one's orientations.

To give a trivial example, the teacher who is oriented toward the "demonstrate and practice" mode of teaching is likely to attend to whether a student's answer is right or wrong, in order to acknowledge correctness or to correct errors. In contrast, a teacher oriented toward formative assessment is likely to attend both to potential misconceptions and partially correct ideas in what a student says, in order to build on what is solid and address the substance of what is incorrect. Thus, what one notices is a function of one's orientations (and knowledge, of course). But, this path is bi-directional: observing what a teacher attends to is likely to be a very powerful tool for inferring and documenting teachers' orientations. Moreover, attending to what a teacher notices is diagnostic: if one is interested in professional development, one needs to know what the teacher currently considers worthy of attention. There are, I think, significant riches to be gained in exploring the dialectic between the noticing and modeling perspectives.

Lani Horn's chapter, "Teaching as problem solving – Collaborative conversations as found talk-aloud protocols," provides a rich compare-and-contrast of analyses of individual problem solving with the far more complex task of analyzing the act(s) of teaching as problem solving. To reflect a bit, it is tremendously gratifying to see how far the field has come in the (gasp!) more than 35 years since I began my problem solving work. Back then it was all we could do to analyze the work of an individual, isolated in a lab, working on a problem with the only resources available being pencil, paper, and his or her brain. Teaching, as Horn rightly points out, is problem solving of a high order. The participants define the problems, and goals they set for themselves and their students can vary dramatically; they draw upon a very wide range of resources, many of which are highly social in nature. Horn concludes her chapter by proposing

> that an analysis of teaching problems requires a consideration of not only their cognitive demands, but also their social demands. In this way, teachers' successes and difficulties in carrying out teaching practices and addressing related problems could be viewed not merely as a consequence of their individual competence but also as fundamentally shaped by their social environments. For this reason, their successes and difficulties require an analysis of the social resources for their practice.

I concur. This point is fundamentally relevant when one thinks about issues of professional development, an issue I will return to in my concluding comments.

Professional development is the focus of Stefan Zehetmeier and Konrad Krainer's chapter, "Researching the sustainable impact of professional development programmes on participating teachers' beliefs." The key word is "sustainable:" the point of interventions is to make catalytic rather than ephemeral change. In the U.S., at least, professional development for teachers has often been short-term and aimed at bolstering teachers' skills. Such activities tend to have a short half-life. Part of the reason for this is, as this entire volume has made clear, that teachers' resources (including knowledge and skills), beliefs and orientations, goals, and decision making are all deeply intertwined. To have a lasting impact, professional development must also take root in teachers' belief systems. As Zehetmeier and Krainer indicate, it is possible for professional development to have a lasting impact on teachers' belief systems. They point out that beliefs are not etched in stone, and that gains can be lost if they are not supported (once again, a community issue in large measure); but, it is heartening to see that aspects of teachers' belief systems with regard to self-esteem, the importance of explanations, a more reflective stance towards teaching, and a higher appreciation of feedback lasted for many years after the intervention – and, presumably, continued to have some impact on the teachers' practice.

In the same vein, Bettina Roesken takes up the theme of "Capturing mathematics teachers' professional development in terms of beliefs" in her chapter. Roesken takes us deeply into the real world of professional development, with an important moral: the teachers we are working with are real individuals, with their own goals and motivations; and (just as in the classroom) we must "meet them where they are" if the work we do with them is to have any impact. Her statement, "Beliefs can be interpreted as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in *professional development*," is a warning flag for those who would impose their views on teachers. I take as a given, as Roesken does, that we should profoundly respect and cherish teachers and their needs. To that, I would also add that we must work to earn their trust and respect – "a vision of professional development that is *for* and *with* teachers" demands a partnership in which all of the participants feel that they are working together toward a set of important goals. And what might those goals be? Mathematical sense making, of course. But note that these must become shared goals, supported by a set of consistent, if not shared, beliefs.

In "Mathematicians and elementary school mathematics teachers – Meetings and bridges," Jason Cooper and Abraham Arcavi describe a project that took on some of those challenges – albeit on a more intimate scale than the *Mathematics Done Differently* project described by Bettina Roesken. The non-negotiable goal of the mathematicians in the collaboration they describe was mathematical sense making. As I see it the mathematicians were unwilling to simply take things for granted, but wanted to know (and make sure the teachers knew) why things are the way they are. Consider, for example, Cooper and Arcavi's description of the mathematicians' actions as they worked to find an accessible argument to show why, of all rectangles with fixed whole number perimeter, the square has largest area:

> In searching for a proof, and in coming up with this one, the mathematicians acted in a manner consistent with these beliefs:
>
> – There should be no magic in mathematics. Every fact should have a proof.
> – The proof must be comprehensible, based on what is already known.
> – The proof should say something about *why* the statement is true.

Those who know my undergraduate problem solving courses will know that I maintain precisely those "ground rules for doing mathematics" with my undergraduates! Unpacking ideas to the point that they are grounded in clear communal understandings is precisely what mathematics teaching should be about.

But there was another central shift, which also reflects my problem solving courses and my comments about partnerships when discussing Bettina Roesken's paper. Interactions between mathematicians (or anyone doing professional development) and teachers almost always start off with a power imbalance, and with both sides having a "we/they" mentality. In this case, the mathematicians knew what they wanted the teachers to learn, but assumed that the teaching and learning would be one-directional. As the process unfolded, it became clear that both groups – teachers and mathematicians – had their own expertise (pedagogical content knowledge is a real thing!). And once that becomes clear, the ground shifts. With mutual respect, and common goals that require the expertise of both groups, there is the potential for true collaboration.[3]

Part IV expands the scope yet further. As I mentioned above, Hugh Burkhardt and I have worked synergistically for more than three decades – with Hugh's passion for engineering systemic change becoming a driving force in my own life. His paper, "Methodological issues in research and development," is both a synoptic survey of major issues Hugh and I have grappled with throughout the years (many of which will be revisited in my final comments) and a serious guide to systemic change. In my opinion it should be required reading for anyone who hopes to have an impact on the system.

Cathy Kessel's chapter, "A mathematical perspective on educational research," unpacks significant commonalities in mathematical perspective not just between the two of us, but for a large part of the mathematical community. Reading the chapter brought to mind two of my favorite references. The first is the oft cited quote from Virgil, “As a twig is bent the tree inclines.” Guenter’s and my backgrounds as mathematicians have, without question, shaped the ways in which we go about educational R&D. My needs for precision, for definition, and for nailing things down in educational research (including what some people take as an almost pathological focus on modeling as a means of theory testing) are without question rooted in my experiences as a young mathematician. They have, I think, served me well. The second is different in nature. I now have a deeper understanding of why I like one of my favorite movie scenes. It’s in a 1933 release of *Alice in Wonderland*, in which W. C. Fields plays Humpty Dumpty. In a wonderfully clever exchange (see http://www.youtube.com/watch?v=KcxEukZZM0c), Humpty Dumpty plays a series of word games with Alice. When she protests, he says: “When I use a word it means what I choose it to mean – neither more nor less.” Fields, or rather Lewis Carroll, the author of *Alice in Wonderland*, tickles my mathematical funny bone. As well he should – Lewis Carroll was the pseudonym of Charles Lutwidge Dodgson, an Oxford mathematician. (He was a “don” or Fellow of Christ Church.) It is interesting how community membership brings with it a whole set of orientations, including the kinds of things one is likely to find amusing. (There are numerous web pages of mathematical jokes, like some of those in Kessel’s chapter.)

Judit Moschkovich’s chapter, “Issues regarding the concept of mathematical practices,” serves as the perfect bridge between the content of this volume and my discussion of next steps. In discussing mathematical practices, Moschkovich exemplifies the desire for clarity and precision that has been a leitmotif of this volume. Moreover, her choice of topics – mathematical practices – will be absolutely central as we take the next steps in mathematical reform. If we are to have systemic impact, we will need to understand and change the very ways in which people engage in and with mathematics, in and out of classrooms. And, we will need to understand the ways in which what takes place in classrooms is supported or inhibited by larger societal structures, in schools and beyond. The understandings we develop will have to be both at the individual and community levels; and we will need to better understand the ways in which the individual and community levels exist in dialectic. As Moschkovich notes,

> Beyond the assumption that practices are social and cultural in origin, a Vygotskian perspective has several implications for the concept of mathematical practices:
>
> – Social interaction that leads to learning principally involves joint activity (not just any type of interaction),
> – Goals are an implicit yet fundamental aspect of practices,
> – Discourse is central to participation in practices,
> – The meanings for words are situated and constructed while participating in practices,
> – Appropriation is a central metaphor for describing learning (but learners do not merely imitate practices, they sometimes actively transform them).

Our challenge is to make use of such understandings in ways that will make the mathematical practices come alive in our classrooms.

ET ENCORE

The question now is, “what next?” And, how does what comes next build on what came before? An answer, at least in proposal form, is a potential collaboration between the University of California, the SERP Institute (see http://www.serpinstitute.org/2013/) and the Oakland (California) Unified School District, OUSD, to improve mathematics instruction in the district. The proposed project is called “The Final Mile,” to represent the distance that must be travelled from the creation of the Common Core State Standards in Mathematics (CCSSM, downloadable from http://www.corestandards.org/) to their becoming a reality in Oakland classrooms. My intention in outlining the ideas in the proposal is to show how the various pieces of the puzzle highlighted in this volume might fit together.[4]

The focus of our proposal is on mathematical practices – in large part because those of us putting the proposal together believe that the real “action” in mathematics classrooms is in the ways that the classroom community interacts with the mathematics at hand. This is not to say that content will be ignored, but rather that it is a comparatively straightforward matter to work with the content descriptions in CCSSM. The challenge is to make that mathematics come alive, in ways that result in students experiencing it as a form of sense making.

Here are four “basis vectors” behind our approach.

- *Theory*

We take it as axiomatic that practical work must be theory-based, so that we can learn from the attempt. In essence, the work of systemic improvement should be approached as a design experiment – the goal being to start with a theory-based intervention, and monitor closely what happens. Evaluations of ongoing attempts should result in refinements of both the theory and of the intervention.

- *Vision*

Everybody in the district has to be on the “same page” with regard to the goals of instruction, and what counts in mathematics classrooms.[5] (There is ample evidence that when teachers get conflicting messages – whether, for example, from high stakes tests, district pacing guides and the associated assessments, administrators’ classroom visits – progress can be easily stymied.) In fact, there has to be system-wide alignment with regard to orientations, goals, and instructional practices. With such alignment, one can begin to build the resource base.

• *Resources*

Instructional resources, professional development, and teachers' professional learning communities must be developed in ways that are mutually consistent and that support the common vision.

• *Feedback*

Complex systems such as education, if they are to evolve successfully, depend on high quality feedback – which needs to be rich, detailed, and timely. The tools used to assess progress must be consistent with the common vision, and help to calibrate and support progress toward it.

In what follows I describe each of these briefly, to give a sense of how our various ideas are woven together.

On Theory

In a complex system such as a school district, multiple levels need to be addressed. One needs to theorize:

a. the nature of the desired mathematical proficiency;
b. the attributes of mathematically productive classrooms;
c. the nature of teachers' developmental trajectories; and
d. the attributes of productive professional learning experiences for teachers;
e. necessary elements of systemic coherence.

Little needs to be said regarding (a), in that the Common Core State Standards for Mathematics (CCSSM) specify the content students should learn and point to the practices we would like to see come alive in classrooms. The goals, broadly speaking, are to produce mathematically proficient students who have positive mathematical dispositions and identities. Work needs to be done to operationalize this, of course: see "On Resources" and "On Feedback."

On the subject of (b), a major focus of my work in recent years has been an attempt to delineate the dimensions of mathematically productive classrooms. Our analytic scheme, "Teaching for robust understanding of mathematics," or TRU Math,[6] was briefly described by Hugh Burkhardt in his chapter. I won't repeat the details – see Table 4 in chapter 3. The core idea is that there are five major dimensions to look at in mathematics classrooms. In brief summary, the dimensions are as follows:

1. *Focus, Coherence, and Mathematical Accuracy*. Was there honest-to-goodness mathematics in what students and teacher did? Was it focused and coherent, with concepts and procedures tied in integral ways?
2. *Cognitive Demand*. Did students engage in "productive struggle," or was the mathematics "dumbed down" to the point where they didn't?
3. *Access*. Who had the opportunity to engage? A select few, or everyone?
4. *Agency, Authority, and Accountability*. Who had a voice? Did students get to reason through things, refine arguments in discussion, and develop ownership?

5. *Uses of Assessment*. Did instruction find out what students know, and build on it?

Our full analytic scheme (see Schoenfeld, 2013, for the story of its development) provides a three-point scale for evaluating classroom practices along these dimensions.[7] The working hypothesis (which we are working to substantiate with data, now that the analytic scheme is robust) is that classrooms that score well on these five dimensions will produce students who are powerful mathematical thinkers.

A discussion of (c), the nature of teachers' developmental trajectories, can be found in chapter 8 of Schoenfeld (2011). There are two key aspects to the theory. The first, which has been discussed in this volume, is that teachers' decision making – everybody's decision making – is a function of their knowledge and resources, goals, and beliefs/orientations. Hence facilitating teachers' growth (and developing the right mindsets in administrators and other participants in the endeavor) will require attention to all three. The second is that teachers tend to act on three planes of activity, dealing with issues of

i. classroom management,
ii. implementing engaging mathematical activities, and
iii. engaging in "diagnostic teaching," i.e., attending to student understandings and shaping the lesson around those understandings.

Research indicates that beginning teachers spend a huge amount of time on (i), without necessarily being effective at it; as they become more accomplished, they spend more time on (ii), which, of course makes management less of an issue; and, truly accomplished teachers do a substantial amount of (iii), which makes (ii) more effective and means that even less time is needed for (i).

Note that the TRU Math scheme is consistent with these observations: higher scores on TRU Math will correspond, in general, to progress along a teacher's developmental trajectory. That is: developing as a teacher means becoming more proficient at the five dimensions of the TRU Math scheme.

It stands to reason, then, that professional development should operate in ways that support teachers' growth along these dimensions. As various chapters in this volume have reminded us, support structures for teaching are inherently social, must cohere with the rest of teachers' experiences, and must offer resources that fit into the fabric of their professional lives. On that score, the project will employ a form of lesson study, which (when used in the right ways) is focused on the development of materials and practices that have student thinking and learning at their core. The lesson study work will be grounded in the study and implementation of "Formative Assessment Lessons" or FALS developed by the Mathematics Assessment Project under the design leadership of Malcolm Swan (see http://map.mathshell.org/materials/index.php). The FALs are aimed at supporting teachers in employing classroom techniques that focus on rich mathematical content and practices, consistent with TRU Math. The challenge of issue (d), which we hope to address through lesson study, is to create sustainable professional learning communities within schools and across the district. District mathematics coaches will be working to foster this vision.

Regarding issue (e), systemic coherence, see “On Vision” below. The key point theoretically is that all of the major influences on what happens in classrooms need to be aligned, with the same message.

On Vision

The question is how to insure systemic coherence. The major goals and the means to achieve them need to be understood and supported at all levels of the system, from the classroom to the building to the professional learning communities within the district, to the state, given that state policies afford and constrain what happens at the district and school levels. The overall vision, as described above, involves deep student understanding (as reflected by performance on rich mathematical tasks) and rich classroom practices (as reflected by high level performance on the TRU Math scheme) orchestrated by the teacher.

In the following section, “On Resources,” I will provide a bit more detail on the tools we will be using. Here I simply note that our goal is to have the high stakes assessments, ongoing district assessments, professional development, and classroom observational tools that we will use, all be in synch. The challenge then is to undertake actions so that the varied communities at all levels of the system (including teachers and administrators) are oriented to the same goals, with enough knowledge and resources to be supportive.

Our plans are for district administrators (including site leaders) to experience the FALs and be briefed on the professional development efforts, so that they understand the main goals of PD and instruction. Administrators have already made use of a classroom observation tool (“the 5×8 card,” named for its size) that focuses on classroom discourse practices and aspects of the CCSSM practices that highlight student reasoning and engagement. The result of this work is that administrators’ classroom observations are focused on student practices (i.e., what are the students doing, saying, thinking?). The 5×8 card is consistent with the TRU Math scheme, and once the administrators (who are not necessarily “math people”) have become comfortable with observing lessons with a focus on student reasoning, they will be introduced to TRU Math. This way, we hope that stakeholders at every level of the system will be working toward the same goals: deep student understanding, as measured by the high stakes tests and district assessments, classroom practices that contribute to such understanding, and support structures such as coaching and lesson study, all of which are consistent with the values embodied in the TRU Math scheme.

On Resources

We have done our best to insure that all of the resources used in the project are philosophically and materially aligned. A consequence of CCSSM implementation (which is related to the federal “Race to the Top” program – for details on assessment see http://www2.ed.gov/programs/racetothetop-assessment/index.html) is that each of the states using CCSSM will be tracking student progress using assessments developed by one of two national assessment consortia. California will be using the assessments developed by the Smarter Balanced Assessment Consortium (SBAC: www.smarterbalanced.org/). Hugh Burkhardt and I were the lead authors of the specs for the SBAC assessments, and the items and plans released by SBAC thus far are in synch with our vision.[8] For its internal (during the year) assessments, OUSD plans to use the Balanced Assessment in Mathematics tests delivered over the past decade by the Mathematics Assessment Resource Service (MARS) (see http://www.mathshell.org/ba_mars.htm). The MARS tests, designed by Rita Crust and the Shell Center team, were based on the earlier "Balanced Assessment" materials, and were the basis for our models for SBAC development. (Hugh Burkhardt and I were the originators of the Balanced Assessment/MARS projects.)

The lessons for use in lesson study are the Formative Assessment Lessons (FALs) developed by the Mathematics Assessment Project (MAP), which features the same dramatis personae under the design leadership of Malcolm Swan. The TRU Math scheme was developed under the aegis of the MAP project and the Algebra Teaching Study (ATS) project, and we have done preliminary validations and data analyses in OUSD classrooms. The 5×8 card was co-developed by Phil Daro, one of the authors of CCSSM. It is, as noted, consistent with TRU Math. (Daro is on the advisory board for ATS and has advised us on assessment matters since the original Balanced Assessment project.) Thus, as much as we could insure, the material resources to be employed by the district are consistent with the vision – and, we will do what we can to insure that the human resources are supported in the same directions.

On Feedback

There are two levels of feedback in our planned work. The first is with regard to how the system functions. Here the MARS tests and the Smarter Balanced Assessment tests will provide information regarding student performance. The 5×8 card will serve administrators as a lens into classroom practice, and the TRU Math scheme will provide feedback for discussions between teachers and coaches, for professional learning communities, and for lesson study efforts. All of these will enable partners in the enterprise to judge progress toward project goals.

At the theoretical level, we envision the attempts to work with OUSD as a design experiment. If we are lucky enough to have the resources to carry out this project, we will have the opportunity to refine both the material and social aspects of the intervention (tools and practices, writ large) and to do theory revision. There is a good deal of room to sharpen the theoretical ideas outlined above.

And, a picture

The theory of action for "The Final Mile" is given in Figure 1.[9]

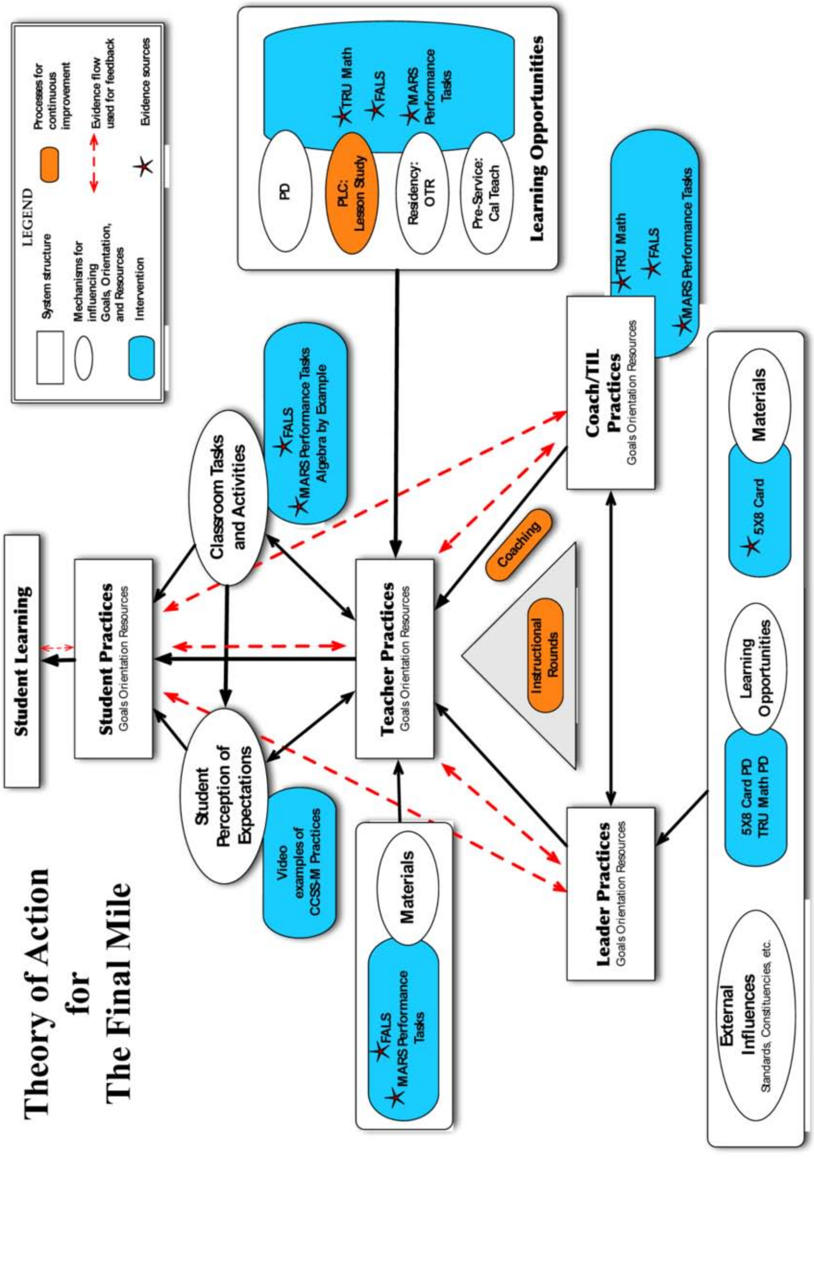

Figure 1. Theory of action for "The Final Mile."

CODA

My answer to the question "what do you want to do when you grow up?" is: "I want to keep growing up." The work is fun, and it continues to grow in lovely ways. The more we learn as a field, the more we can do, and the more we can learn. My greatest pleasure is that such work is done in the company of people like those who contributed to this volume.

Whether we get to try "The Final Mile" in the ways we hope is a decision that lies in the lap of the funding gods. But it is a beautiful prospect to contemplate, in that it could pull together – and advance – many of the currently separate strands of my work, and build on the work of the scholars in this volume. Whether we get to do that particular work remains to be seen, but I am confident that there will be plenty to do. In fact, I will be heading to Germany in a few months to work with my friend and colleague Guenter Toerner, to see if we can entice industrial and other "Friends of Mathematics Education" into creative ways of supporting the enterprise. Some good wine, and some good times, are sure to be a corollary to the process. May it always be so.

NOTES

[1] A few years ago Guenter asked me if I planned to retire. No, I said; I was having too much fun. He had to, he said: 65 is the mandatory retirement age, and he would have to find something to do. Somehow that didn't happen; he's busier than ever.

[2] I am not as extreme as Jim Kaput, who might have argued for pre-natal calculus, but I do believe that mathematical sense-making should start as soon as kids start mathematizing – and that is pretty close to birth.

[3] Addressing the power imbalances in partnerships between universities and schools is always a challenge. One very clever suggestion from Kim Seashore, in a professional development partnership my research group had with the SERP Institute (see http://www.serpinstitute.org/2013/) and the San Francisco Unified School district was to have each researcher on the team (including me and my graduate students) become a teaching assistant in the classrooms of our partner teachers. That placed us in a context where our collaborating teachers were clearly far more expert than we, with the result that our interactions (when we designed materials, etc.) were much more on an even footing. In addition, the fact that we put ourselves at risk in their classrooms helped to build a foundation for trust.

[4] The development of the proposal was a truly collaborative effort. Credit for its emergence goes to Harold Asturias, Phil Daro, Suzanne Donovan, Catherine Lewis, Catherine O'Connor, Donna Riordan, Joan Talbert, Philip Tucher, and many more, including substantial support from teams at the participating institutions (the SERP Institute, Oakland Unified School District, and the University of California at Berkeley).

[5] I'm assuming here that the vision is a good one! It could be otherwise, of course.

[6] TRU Math was developed by the Mathematics Assessment Project (with funding from the Bill and Melinda Gates Foundation) and the Algebra Teaching Study (with Co-PI Robert Floden, funded by the National Science Foundation).

[7] When the scale is used, a lesson is divided into "episodes" corresponding to activity structures (whole class, small group, and student presentations, which are subdivided if they extend past 5 minutes). Each of these episode types has a three point rubric for each dimension (see Schoenfeld, 2013).

[8] Any large-scale operation in a highly politicized context needs to make decisions on multiple grounds - cost, expediency, the willingness of state assessment directors to move into new territory, etc. Despite this, the SBAC plans and released items represent a significant step forward in large scale assessment.

[9] The SERP Team created the figure – an act far beyond my graphical skills.

REFERENCES

Common Core State Standards in Mathematics. (2010). Common Core State Standards Initiative. Accessible from http://www.corestandards.org/.

Li, Y., & Huang, R. (Eds.). (2013). *How Chinese teach mathematics and improve teaching*. New York: Routledge.

Mathematics Assessment Project (MAP) resources. Accessible from http://map.mathshell.org/materials/index.php.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.

Schoenfeld, A. H. (2011). *How we think*. New York: Routledge.

Schoenfeld, A. H. (2013). Classroom observations in theory and practice. *ZDM – The International Journal on Mathematics Education* (in press).

Schoenfeld, A. H., & Stipek, D. (2011). *Math matters: Children's mathematical journeys start early*. Report of a conference held November 7 and 8, 2011. Berkeley, CA: University of California, Berkeley. Accessible from http://earlymath.org/.

Smarter Balanced Assessment Consortium. (2012). Assessment specifications and sample items. Accessible from http://www.smarterbalanced.org/

U.S. Department of Education. Race to the top assessment program. Accessible from http://www2.ed.gov/programs/racetothetop-assessment/index.html.

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Werner BLUM got his Diploma in mathematics in 1969 and his Ph.D. in pure mathematics in 1970, both from the University of Karlsruhe. Since 1975 he has been a full professor of Mathematics Education at the University of Kassel. From 1995 to 2001 he served as the President of the GDM, the maths education society of the German speaking countries. In 2006, he received the Archimedes Award of MNU. His current research areas include empirical investigations into the teaching and learning of mathematics, (for instance on self-regulated mathematics learning or on classroom assessment) and empirical investigations into mathematics teachers' competencies. One of the main focuses of his work is quality development in mathematics teaching. Among other things, he is engaged in the development of national standards and tests in mathematics for the lower secondary level in Germany. He has done a lot of work in the area of modelling and applications of mathematics education, among other things as a continuing editor of the ICTMA Proceedings and as the editor-in-chief of ICMI Study 14. Proofs and Proving in mathematics education also belong in his areas of interest. Since 2000, he has been a member of the international PISA Mathematics Expert Group.

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Hans H. BRUNGS studied mathematics and physics at the University in Frankfurt, Germany, from 1959 onwards and obtained a Ph.D. in mathematics in 1966 with Professor R. Baer as supervisor and a thesis about unique factorization in non-commutative rings. In 1967–1968 he was a visiting Assistant Professor at Rutgers University, New Jersey, since Professors C. Faith and B. Osofsky as well as P. M. Cohn were there. In 1968 he became a Research Associate at the University of Alberta, Canada, then a Professor in 1979 and retired in 2005, and as a Professor Emeritus ever since. In 1973–1974 he visited the University in Giessen, Germany, and completed his Habilitation in 1975 with Professors G. Michler and P. M. Cohn as supervisor and examiner.

He has primarily worked on non-commutative valuation rings and was fortunate enough to be part of a very active research group in Edmonton and to find wonderful collaborators such as, N. Dubrovin, J. Fisher, J. Graeter, H. Marubayashi, E. Osmanagic, M. Schroeder, K. Strambach and G. Törner (in alphabetical order).

Hugh BURKHARDT spent the first half of his academic career working in theoretical elementary particle physics and applied mathematics. The 1960s reforms in mathematics education led him to experiment with the teaching of mathematical modelling of practical everyday life problems, working initially with undergraduates and high school teachers. His 1976 appointment as Director of the Shell Centre for Mathematical Education at Nottingham shifted the balance of his work towards K-12 mathematics, still with real world problem solving as a key learning goal and with technology as a useful tool. He became concerned with the mismatch of official goals and classroom practice, recognizing that high-stakes examinations largely determine what is taught. Consequently, based at the Shell Centre and UC Berkeley, he has led a series of assessment projects with US and UK test providers who agreed to align their tests with learning goals. He takes an "engineering" view of educational research and development – that it is about using imaginative design and systematic development to make a complex system work better, with theory as a guide and empirical evidence the ultimate arbiter. He led the foundation of ISDDE, the International Society for Design and Development in Education, and its e-journal *Educational Designer*.

Ming Ming CHIU is a Professor of Learning and Instruction at the University at Buffalo, State University of New York (UB). He deveoped a theory of social metacogniton, invented two statistics methods (multilevel diffusion analysis [MDA] and statistical discourse analysis [SDA]), and earned a National Academy of Education post-doctoral fellowship and a Young Researcher award. Using his MDA, he showed how to detect a type of corruption in the music industry (payola). Using his SDA, he tested social metacognition hypotheses by statistically modelling traditionally "qualitative data" such as conversations, individual problem solving and interviews. Specifically, he statistically identified conversation watersheds (e.g., insights, insults) that radically change classroom interactions. He also showed that social metacognitive actions (e.g., polite disagreements, wrong ideas, correct evaluations) during the three previous conversation turns (micro-time context) aided subsequent creation of correct, new ideas (micro-creativity) during cooperative learning. He also showed how teacher assessments of student progress opened doors to other teacher actions that improved students' subsequent time-on-task and problem solving. Beyond the classroom, he conducts large-scale statistical analyses of over half a million students across 65 countries, and shows how broader contexts such as economic growth, inequality and cultural values affects student learning.

Jason COOPER is working on his Ph.D. dissertation in mathematics education at the Weizmann Institute of Science under the direction of Professor Abraham Arcavi. He holds a B.Sc. in computer sciences and a M.Sc. in mathematics, both from the Hebrew University in Jerusalem. He has taught middle school and high-school mathematics, and has developed curricular materials for a variety of grade levels. He was involved in the development of a unique computer-based teaching platform for use in elementary schools. He has also worked with teachers in both pre-service and in-service education.

Ilana Seidel HORN is an Associate Professor of Mathematics Education at Vanderbilt University's Peabody College. Her work arises out of a concern with the underperformance of American secondary students in school mathematics. Her research centers on ways to make rigorous mathematics accessible to students, particularly those who have historically been disenfranchised in our educational system.

One line of her research seeks to specify the practices of ambitious and equitable teaching. What exactly do teachers need to do to teach their students effectively? Of course, identifying these practices is not enough. Teachers need support in incorporating these into their classrooms. Thus, her second line of work inquires into pre-service and in-service teacher learning, with an eye toward making teacher education at both levels more effective. Her current research examines how middle school mathematics teachers committed to instructional improvement in two high poverty school districts work to improve their instruction through collaboration. Her work has appeared in venues such as the *Journal for Research in Mathematics Education*, *Mathematical Thinking and Learning*, *Cognition and Instruction*, *Journal of the Learning Sciences*, and *American Educational Research Journal*.

Jennifer L. JONES is the Special Education coordinator at Oracle Charter School in Buffalo, NY. She has worked as a middle school mathematics teacher as well as a mathematics instructor at Niagara University where she taught calculus and statistics courses to students at both the undergraduate and graduate levels. She has also taught education courses at the University at Buffalo, State University of New York. She holds a BA in mathematics education, an MS in Special Education, and is currently working on a Ph.D. in mathematics education from the University at Buffalo, State University of New York.

Karrie A. JONES is a high school mathematics teacher at Tapestry Charter High School in Buffalo, NY. She has taught algebra, trigonometry, pre-calculus and Advanced Placement Calculus at the high school level since 2007. She has also taught mathematics and education courses at both Niagara University and the University at Buffalo, State University of New York. She holds a BA in mathematics education, an MS in Special Education, and is currently working on a Ph.D. in mathematics education from the University at Buffalo, State University of New York.

Cathy KESSEL became an assistant professor of mathematics after receiving a PhD in mathematics (specializing in logic). A few years later, dissatisfied with her students' learning, she left that position to learn about mathematics education research at the University of California at Berkeley. She now works as a consultant in mathematics education. Along the way, she has had the opportunity to pursue her interest in mathematics and gender, the history of mathematics, and related areas. Her work as a consultant often involves editing reports with contributions from mathematicians and mathematics education researchers. A recent example is *The Mathematical Education of Teachers II* (Conference Board of the Mathematical Sciences, 2012). She is a past president of the Association for Women in Mathematics.

Konrad KRAINER is a Full Professor, Member of the Senate and since 2012 Director of the School of Education at the Alpen-Adria-Universität Klagenfurt. His research interests are mathematics teacher education, school development, and educational system development. He has co-edited nine books such as the volume "Participants in Mathematics Teacher Education" of the *International Handbook of Mathematics Teacher Education* (1998). From 1994 to 2011, he led the Austrian

Educational Competence Centre for Instructional and School Development situated in Klagenfurt and Vienna. Konrad Krainer then became Associate Editor of JMTE (1998–2009), a founding and board member of the European Society for Research in Mathematics Education and a member of the Education Committee of the European Mathematical Society. Among other contributors, he gave plenary lectures at ICME-10 (co-presenter) and PME 35, and chaired the TEDS-M-Plenary Panel at ICME-12. He is the leader of the nation-wide IMST project (Innovations Make Schools Top) and co-leads the Austrian part of the EU project FIBONACCI.

Stefan KRAUSS is a Full Professor of Mathematics Education at the University of Regensburg (Germany) since 2009. Stefan Krauss passed his state examination in mathematics and physics in 1995 at the University of Erlangen-Nuremberg. After an additional study of psychology at the University of Eichstätt (1995–1998), he got his Ph.D. in psychology from the Free University of Berlin in 2001. From 1998 until 2006 he worked at the Max-Planck-Institute for Human Development in Berlin as a research scientist and afterwards at the University of Kassel, where he got his habilitation in mathematics in 2009. His research interests include research on teaching and learning, the didactics of mathematics, professional knowledge of mathematics teachers, and probabilistic reasoning.

Esther LEVENSON is a researcher and teacher at the School of Education, Tel Aviv University, and a middle school mathematics teacher. In her PhD study she investigated mathematically and practically based explanations in the elementary school: Individual preferences and socio-mathematical norms. Her research interests include: Fostering mathematical creativity among students and teachers, developing mathematical thinking among preschool students and their teachers, the connections between proving and reasoning, and the role examples play in eliciting explanations.

Yeping LI is a Professor of Mathematics Education, holder of the Claude H. Everett, Jr. Endowed Chair in Education, and Head of the Department of Teaching, Learning and Culture at Texas A&M University. His research interests focus on issues related to mathematics curriculum and teacher education in various education systems and understanding how factors related to mathematics curriculum and teachers may come together to shape effective classroom instruction. He is the editor-in-chief of the *International Journal of STEM Education* published by Springer, and also the editor of a monograph series *Mathematics Teaching and Learning* published by Sense Publishers. In addition to co-editing several books and special issues of journals, he has published over 100 articles that focus on three related topic areas of study (i.e., mathematics curriculum and textbook studies, teachers and teacher education, and classroom instruction). He has also organized and chaired many group sessions at various national and international professional conferences, such as ICME-10 in 2004, ICME-11 in 2008, and ICME-12 in 2012. He received his Ph.D. in Cognitive Studies in Education from the University of Pittsburgh, USA.

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Judit MOSCHKOVICH is a Professor of Mathematics Education in the Education Department at the University of California, Santa Cruz. Her research uses socio-cultural approaches to examine mathematical thinking and learning in three areas: algebraic thinking (in particular, student understanding of linear functions); mathematical discourse practices; and mathematics learners who are bilingual, learning English, and/or Latino/a. She received her Ph.D. from UC Berkeley, Department of Education in Mathematics Science and Technology (EMST). Before joining the faculty at UCSC, she was a lecturer at UC Berkeley, and a Researcher at IRL (Institute for Research on Learning) in Palo Alto and TERC in Cambridge MA. In addition to published articles and book chapters, she is the editor of the book *Language and mathematics education: Multiple perspectives and directions for research* (2010) and co-editor, with M. Brenner, of the JRME monograph, *Everyday and academic mathematics: Implications for the classroom* (2002). She was the recipient of a NAED/Spencer Post-Doctoral Fellowship (1995–97), the Principal Investigator of the NSF research project "Mathematical discourse in bilingual settings: Learning mathematics in two languages" (1998–2003), and Co-PI for CEMELA (Center for the Mathematics Education of Latinos/as) a Center for Learning and Teaching funded by NSF from 2004 to 2011.

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Alan H. SCHOENFELD is the Elizabeth and Edward Conner Professor of Education and Affiliated Professor of Mathematics at the University of California at Berkeley. He is a Fellow of the American Association for the Advancement of Science and of the American Educational Research Association, and a Laureate of the education honor society Kappa Delta Pi. Schoenfeld has served as President of the American Educational Research Association and Vice President of the National Academy of Education. In 2011 he was awarded the International Commission on Mathematics Instruction's Klein Medal, the highest international distinction in mathematics education.

A mathematician by training, Schoenfeld became a mathematics educator in order to understand and share the pleasures and power of mathematical thinking. His work has focused on problem solving, assessment, teachers' decision-making, and issues of equity and diversity. His current work focuses on understanding teaching practices in powerful mathematics classrooms. Schoenfeld has written, edited, or co-edited more than two hundred pieces on thinking and learning, including 22 books. His most recent book, *How we think*, provides detailed models of human decision making in complex situations such as teaching.

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Miriam G. SHERIN is a Professor of Learning Sciences in the School of Education and Social Policy at Northwestern University. Her interests include mathematics teaching and learning, teacher cognition, and the role of video in supporting teacher learning. Along with V. Jacobs and R. Philipp, she edited the volume *Mathematics teacher noticing: Seeing through teachers' eyes* which was published in 2011 by Taylor & Francis. Sherin recently served as Principal Investigator on the National Science Foundation grant Using Video to Study Teacher Learning. Recent articles appear in the *Journal of Teacher Education*, *Teaching and Teacher Education*, and the *Journal of Mathematics Teacher Education*. In April 2003, she received the Kappa Delta Pi/American Educational Research Association Division K Award for early career achievements in research on teaching and teacher education.

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Günter TÖRNER is a Full Professor within the Faculty of Mathematics at University Duisburg-Essen in Germany and the ombudsman for integrity of scientific research of his university. He studied mathematics and physics at the University of Giessen, received his Master's Diploma in 1972, his PhD in 1974 and graduated (Habilitation) with a mathematical topic. In 1978 he accepted a Full Professorship at Duisburg University. Today he is still engaged in the same university after rejecting honorable offers from other universities.

Törner's research revolves around non-commutative valuation structures and is engaged in small scheduling theory projects with companies. Till 2005 his Faculty at Duisburg campus hosted teacher education and he was (and still is) also occupied with didactical research (problem solving, beliefs theory, attitude theory and professional development) within the life-long teacher education. Working scientifically in two disjoint fields he became a member of the board and later a member of the EC of German Mathematical Society. He also serves as a chairman of the Committee of Education within the European Mathematical Society. Törner is a founding member of National Center for In-service Teachers Training that is financed by the Deutsche Telekom Stiftung.

His scientific work comprises math than 50 mathematical papers, as well as more than eighty papers in educational journals and international proceedings of conferences. Finally, he has been the advisor of more than 20 PhD students.

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Stefan ZEHETMEIER is an Assistant Professor of the School of Education at Alpen-Adria-Universität Klagenfurt. His research interests include mathematics and science teacher education, school development, action research, evaluation, and impact analyses of teacher professional development programmes. He formerly served the University of Vienna as a researcher and teacher educator in the nation-wide professional development project IMST (Innovations Make Schools Top).

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