MATHEMATICS CLASSROOMS: STUDENTS' ACTIVITIES AND TEACHERS' PRACTICES

Mathematics Classrooms: Students' Activities and Teachers' Practices

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SENSE PUBLISHERS
ROTTERDAM / BOSTON / TAIPEI

A C.I.P. record for this book is available from the Library of Congress.

ISBN 978-94-6209-279-2 (paperback)
ISBN 978-94-6209-280-8 (hardback)
ISBN 978-94-6209-281-5 (e-book)

Published by: Sense Publishers,
P.O. Box 21858, 3001 AW Rotterdam, The Netherlands
https://www.sensepublishers.com/

Printed on acid-free paper

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BARBARA JAWORSKI

PREFACE

This book presents unique insights into a significant area of French research relating the learning and teaching of mathematics in school classrooms and their development. Having previously had only glimpses of this work, I have found the book fascinating in its breadth of theory, its links between epistemological, didactic and cognitive perspectives and its comprehensive treatment of student learning of mathematics, classroom activity, the work of teachers and prospective teacher development. Taking theoretical perspectives as their starting points, the authors of this volume present a rich array of theoretically embedded studies of mathematics teaching and learning in school classrooms.

The book charts the use of a theoretical/methodological perspective called *The Double Approach*, a didactic and ergonomic approach for the analyses of teaching practices (Robert & Rogalski, 2002). This approach is concerned simultaneously with the design of teaching and with its practical, ergonomic (work-based) contribution to students' learning of mathematics in classrooms. It seeks to address associated issues widely and in their full complexity recognising institutional dynamics and constraints, the impact of social and cultural perspectives and interweaving layers of activity.

The term "activity" is ubiquitous throughout, taking on two kinds of meaning, in one sense referring to the actions of students and teachers in the classroom and, in parallel, referring also to *activity* as in *Activity Theory*, a complex dynamic encompassing the wholeness of classroom learning and teaching; as Leont'ev has expressed it, "the non-additive, molar unit of life … a system with its own structure, its own internal transformations, and its own development" (Leont'ev, 1979, p. 46). The two senses are deeply entwined in the ways activity is addressed. Thus, it is not surprising that one of the foundations of the Double Approach, its "organising framework" is the sociohistorical theory of Vygotsky, and followers such as Leont'ev. In Chapter 1, Janine Rogalski writes, "the object of study consists of the activity of an individual subject with individual motivations, within a specific situation. When the subject is a teacher, it is not the "properties" or "functioning" of the teacher's position that is at issue. … Rather the issue involves questions of diversity among teachers, and the development and emergence of their individual professional competencies" (p. 3). The focus on the individual subject ("as a person-subject rather than as a didactic subject," ibid.) is perhaps somewhat more surprising, especially since it leads the authors to consider a Piagetian approach of epistemological genetics alongside Vygotsky's sociohistorical framework. The surprise is in the juxtapositioning of theories of Piaget and Vygotsky of which others scholars have been cautious, if not dismissive, due to the (supposed) incommensurability of these theoretical perspectives (see e.g., Lerman, 1996). More recently Lerman (2013) wrote “In general, drawing on more than one theoretical perspective needs some work in order to ensure that the perspectives are coherent together.” Regarding the complementarities of the two areas of theory, Rogalski writes: “In particular, the Piagetian theory looks “from the student’s side” at epistemological analyses of the mathematical objects in play, while the Vygotskian theory takes into account the didactic intervention of the teacher, mediating between knowledge and student in support of the students’ activity” (p. 23). Just one of the exciting aspects of these authors’ use of the Double Approach is to see how this theoretical juxtapositioning leads to analyses of teacher and classroom activity which make sense for the researchers and those who learn from this research.

So, what exactly is meant by the *Double Approach*? Aline Robert and Christophe Hache (Chapter 2) weave the abstract theory (above) with theoretical frameworks relating to teaching and learning in classrooms and general methodologies that follow from these frameworks. They write, “… we seek to measure the gap between the activities of students applying their knowledge (during its acquisition) analyzed *a priori*, and the activities that may actually have taken place during a regular lesson” (p. 62). At a simple level, we see an analytic progression from epistemological analysis in a mathematical topic, through a didactic analysis relating to the design of teaching, into analyses of classroom activity and inter-relations between teachers and students with, last but not least, analyses of student activity and understanding. The progression is not linear (as my list might suggest); the research lens may focus in any of these areas, or zoom out to address complex inter-relationships between them. Thus the programme is ambitious. The reader is taken through subdivisions of “the world of the study:” we read of student activity, the transition from designed task to student activity with task, levels of conceptualization (related to Vergnaud’s, 1990, ‘conceptual fields’), the nature of concepts and students’ progression with concepts in terms of generalization and formalization, the knowledge of teachers and design of tasks, wider issues in terms of systemic demands or emotional, personal and social factors. For example, relationships between didacticians and teachers are addressed, ways in which teachers adopt or adapt didactic designs, the ‘work’ of the teacher in the classroom, teacher speech patterns and representations of mathematical concepts and their relationships to student activity. Consideration of the profession of teaching and roles in teachers’ work lead to questions of teaching development and the education of new teachers.

In presenting theory and methodology in these areas, the authors move to and fro between the cognitive and the sociocultural frames so the reader is faced with challenges in making sense of the complexities involved. Unsurprisingly, this ambitious enterprise raises many questions for the reader, not least as to how theoretical complexities are translated into practice in schools and classrooms, how teachers work with researchers (or independently with these perspectives), and how researchers address inter-relationships between observations of student conceptualisation, teachers’ didactic processing in design of activity and the wider frames of educational and sociocultural impact. For example, as the authors acknowledge, "if it is difficult to analyse teaching in relation to learning, it is even more difficult to have legitimate evidence of it" (p. 58).

These questions are addressed variously in the (nine) chapters which follow in which we see the elements of the Double Approach in action with differing zooms of the research lens. Authors present a variety of methodological approaches with analyses of classroom settings, design of tasks, mathematical topics, teachers' intentions, student responses, imposed constraints and the degree of 'leeway' experienced by the teachers. For example, in Chapter 3, Eric Roditi discusses the tasks offered by four teachers in similar sixth grade classes on the topic of multiplication of decimal numbers. We read of the nature and choice of tasks, their levels of cognitive demand and their relation to curriculum guidance on the topic, suggested class time, and expectations of professional practice discerned through classroom observations and interviews with teachers. *A priori* expectations of students' activity according to designed tasks is compared with student outcomes in the tasks. Despite the commonalities of designed tasks in the four classrooms, research emphasised and categorized the variability of classroom activity depending on the ways in which individual teachers worked with students in their class.

The work of teachers and its relation with the realities of situation and context is central to methodologies employed. The relationship between teaching practices and student learning is a recurring theme. Julie Horoks writes in Chapter 6 (p. 135), "Naturally we are not questioning the teacher's work, and we will consider the different components of his/her job to explain certain choices made for his/her class." Horoks describes the use of classroom video recordings to reveal teachers' use of tasks focusing on similar triangles and to relate students' degrees of success with these tasks to the ways in which the tasks were used in the classroom. Monique Chappet-Paries, Aline Robert and Janine Rogalski, in Chapter 4, focus on classroom activity around the theorem of Pythagoras, analysing a teacher's speech patterns to gain insight into invariants in the teacher's practice and ways in which these invariants impact on students. The idea of a "teaching scenario" – a sequence of lessons and exercises around a mathematical topic such as decimal numbers, similar triangles or Pythagoras' theorem (studied a priori) – is a common theoretical construct. For example, in Chapter 7, Aurélie Chesnais discusses the implementation of the same teaching scenario (about orthogonal symmetry) by two experienced teachers in order to study regularities and variability of practices between teachers, as well as the relationship between teaching practices and student learning.

The major theme of relations between a priori analyses of tasks, teachers' implementation of tasks in the classroom and students' take-up of tasks is considered in later chapters with an added dimension, that of the use of electronic resources. In Chapters 8 and 9 these are *Electronic Exercise Bases*, consisting of mathematics exercises within an environment, which includes different types of suggestions, aids, tools (graphs, calculators, etc.), lesson reminders, as well as explanations, answer analyses or complete solutions. The scenarios here are designed around the electronic environment and its use by students with a study of, for example, how the designed situations influence the students' activity. Research reveals that the expected activity is not always the activity developed by the students and emphasizes the difficulty for students to regulate their activity while interacting with the software without teacher intervention. As well, studies address the impact of the electronic resources on the day-to-day activities of teachers and on teachers' evolving classroom practices. Chapter 10 compares activity in a dynamic geometry environment with that in a pencil and paper environment to analyze how the tasks designed for ICT environments differ (or not) from those of non ICT ones. Research explored the differences in classroom management, including ways in which the teacher assisted students, in order to understand their possible impact on students' activities.

I have given these very brief sketches of the focus of various chapters to illustrate or exemplify the pervasive themes of the book in addressing classroom complexity and the deeply inter-related nature of teaching-learning activity. Each study presents different facets of design, implementation and impact of scenarios within the real constraints of classrooms and the personal and social influences which surround classroom interactions. In the final chapter, Maha Abboud-Blanchard and Aline Robert reflect on the earlier chapters to distil elements of their findings which offer insights that are useful in considering the education of mathematics teachers, and of those who will train mathematics teachers. They ask the question, "who should be trained first – the teacher or the teacher's trainer?" (p. 235). This leads to their setting out a training programme for the trainers of mathematics teachers. They acknowledge that this is speculative and that associated research in yet to be undertaken. It nevertheless points to the ambitious scope of the book and the broad programme of research it charts.

Throughout this book the reader is made aware of many unanswered questions and challenged to consider associated theoretical and methodological issues. There is nevertheless an internal consistency and coherence to this work which revolves around the Double Approach. For English-speaking communities who have lacked opportunity to access the French literature the book opens up a wealth of new ways of thinking about and addressing unresolved issues in mathematics learning, teaching and teacher education. I recommend it wholeheartedly!

References are included in the general bibliography of the book.

FABRICE VANDEBROUCK

INTRODUCTION

This book presents several works in the field of mathematics didactics revolving around secondary school and university teaching. The specificity of the research studies in question is that they attribute as much importance to the actors (students and teachers) as to the mathematics and the school situation. These studies fit well in the very general framework of Activity Theory.

The presented researches aspire to analyze what is at play in a mathematics classroom, by varying the school situations, the environments, the contents, the teachers and the classrooms. The main objective is to study, understand, and even interpret the links between the teaching of a given mathematical content and the corresponding student learning. We seek to highlight regularities and variations of these processes in order to better understand students' acquisitions, and interpret the teachers' practices. The work as a whole leads to inferences which can contribute to the professional development of teachers by widening the range of possible activities for each teacher.

The general framework of Activity Theory, with associated development theory, is described in chapter 1, and we directly clarify how this work fits in this framework. The analyses of students' in-class activities, as they are organized by the teachers, provide us with data which allow us to tackle teachers' practices and approach students' learning: the general theory accounts for this focus and the corresponding reality splitting. Nevertheless, the way activities are assigned to mathematics and school situations is not very present in the framework of Activity and development theories. Therefore, the necessary theoretical and methodological complements are presented in chapter 2.

The main concern of this book is however not theoretical, even though its specificity borrows elements from Activity Theory and development theories which complement typically didactical tools. We seek to assign to the singular subjects (students and teachers) their place within the didactical relationship, even though the affective and social factors are not directly accounted for, despite their high importance. We develop the means to collect and analyze in a significant way, adapted to our project, data about teaching and learning allowing us to interpret the relationship between the two.

All the research studies of this book follow a common methodology presented in chapter 2, but involve, of course, indispensable adaptations which are introduced gradually. They pertain to the teaching of mathematics in middle school, high school, or the first two years of university. Some works are the fruit of individual research[1] and handle a small number of cases in an exhaustive manner, often over quite short periods of time. Others works are clusters of research studies or the fruit

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of collective research based on larger data and oriented more directly towards results which are relevant for the general research question of the book. In any event, there is always a limit to the hasty generalizations of results. Hence, there are neither definitive results nor (even less) prescriptions in our discourse.

In chapters 1 and 2, we present the theoretical frameworks and the tools used in the book, while stating the specificity of our research. Chapters 3 and 4 are concerned with the results about teachers' practices in "ordinary" classrooms. They highlight the stability of teachers' practices and also account for the diversity and variability between the teachers. Chapter 5 deals with teaching manuals and shows that exercises proposed in these manuals do not offer the teachers opportunities to diversify their student activities. Chapters 6 and 7 refer more directly to teachers' practices in relation with students' activities. Chapter 8 focuses on the activity of students in a specific teaching situation in a computerized environment. Chapters 9 and 10 deal with teachers' practices in computerized environments, in particular the comparison of teachers' activities in different environments. Chapter 11 is a large scale study about teachers' practices and the factors related to the regularity and variability of the practices. Last, chapter 12 comes as a synthesis of the book with an opening on professional development of teachers.

The different chapters can be read in a relatively independent way. In particular, it is not necessary to complete an exhaustive reading of chapters 1 and 2 in order to read the other chapters … and vice versa!

NOTES

[1] All the researchers who contributed to this book, apart from Aurélie Chesnais, Eric Roditi and Janine Rogalski, are members of the Laboratoire de Didactique André Revuz (LDAR) at Paris Diderot University

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1. THEORY OF ACTIVITY AND DEVELOPMENTAL FRAMEWORKS FOR AN ANALYSIS OF TEACHERS' PRACTICES AND STUDENTS' LEARNING

INTRODUCTION

The goal of this chapter is to propose a theoretic framework to analyze the structured activities of teachers and their students, and to provide support for some inferences regarding teachers' training in professional competencies[1] and students' acquisition of knowledge in specific disciplines.

The organizing framework is that of the theory of activity, which was established by Leontiev, enriched through a line of research originated by Vygotsky, and then exploited and developed within the field of ergonomic psychology (Leplat, 1997; Rogalski, 2004). Its fundamental components are:

– the distinction between task and activity;
– the double point of view, taking into account both the situation and the subject of the action; and
– the system of double regulation of activity, in which determining factors, and the effects of the activity, influence situational components as well as the subject. This regulation is not only retroactive, but also proactive, as a goal-oriented activity is affected when subjects adapts their actions in an attempt to produce the desired results.

Within this theoretic framework, the object of study consists of the activity of an individual subject, with individual motivations, within a specific situation. When the subject is a teacher, it is not the "properties" or "functioning" of the teacher's position that is at issue here (as would be the case for a *stricto senso* didactic perspective, which we could define as the "science of didactic processes"). Rather, the issue involves questions of diversity among teachers, and the development and emergence of their individual professional competencies. Equally relevant are considerations of the student as a person-subject, rather than a didactic subject. All this leads us to consider the Piagetian approach of epistemological genetics, together with Vygotsky's socio-historical framework, as they relate to individual development.

Taking into account the effects of the activity on the subject is an aspect of the developmental and constructivist dimension of the theory of activity (TA). Our focus is on the activity, on its determining factors and on its effects as they relate to teaching mathematics. We are particularly interested in the activity's effects on a teacher's development of professional competencies, and on a student's mathematical conceptualization. Interpreting TA within the theoretical frameworks

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of Piaget and Vygotsky enriches our approach, by defining the developmental dimension of the double regulation in terms of factors and effects, temporality, and the role of psychological tools (Vygotsky) and cognitive tools (Piaget). We include within the Vygotskian framework Bruner's findings on mediation (Wood, Bruner, & Ross, 1976), which add to our understanding of the didactic intervention of a teacher in class. We conclude with a discussion of these theoretical frameworks in order to define the tools we will use in our analysis of teaching practices and student activities in mathematics.

THE THEORY OF THE GOAL-ORIENTED ACTIVITY

The theory of activity was developed by researchers who followed Vygotsky in studying the psychology of work (later called "ergonomic psychology"). The theory was then used in professional didactics, before being "articulated" with a didactical approach to mathematics teaching, in the so-called "double approach" (Robert & Rogalski, 2002, 2005; Robert, chapter 2). The theory involves goal-oriented and motivated activities. By their actions, subjects aim to achieve task goals, and their actions are driven by motivations of the activity.

We will describe the following elements, all of which are essential to our objectives: the task-activity distinction, task structure, the various ways to analyze an activity, and the connection between the subject and the situation within the model of double regulation of activity. We will also indicate how this theoretical framework allows us to analyze the structure of teacher and student activities.

Task and activity

The task-activity distinction is central to the theory of activity. The activity relates to the subject,[2] while the task relates to the objects of the action. The definition of a *task*, as proposed by Leontiev (1975, 1984) and developed by Leplat (Leplat and Hoc, 1983; Leplat, 1997) is the "goal to be attained under certain circumstances." The *activity* is what a subject engages in during the completion of the task. This includes not only external actions, but also inferences, hypotheses, decisions, and actions the subject decide *not* to take. The activity also includes the subject's time management and personal state – workload, fatigue, stress, enjoyment of work – as well as interactions with others within the work situation. We will first consider the task, and describe its essential characteristics. We will then examine the activity developed in response to the task.

Structure of a task

The *task object* is that which is to be transformed or studied. Tasks involving material objects were originally the most studied by ergonomic psychology. Tasks for which the "objects" include human individuals (service professions, therapeutic work, teaching) or for which the goal is to learn and acquire tools for thought (being a student) require a more complex analysis. For the teacher, the goal to be attained is often described in procedural terms, with action verbs: "Teach the concept of length measurement to elementary school students," "correct a math test," "follow the curriculum." Goals can also be stated with reference to the student-knowledge relationship: "Have the student acquire the concept of length and linear units," "Have the student represent functions as mathematical objects and tools." For the student, the task is defined by the teacher's statement, and the requirements of mathematical work.

Tasks and sub-tasks

In a complex situation, the goal to attain consists of various sub-goals, whose achievement order is more or less constrained. For example, "introduce students to the concept of functions," in ninth grade, involves making documentation choices, creating lesson plans that cover one or more class sessions, defining the student tasks, conducting the in-class activities, and finally evaluating students' acquired knowledge.

The structure of the task involves transitions between the intentions of the prescribed task and the actual task as implemented

In a workplace (in the teacher's case) or learning environment (in the student's case), the subject responds to tasks assigned by a prescriber, with the framework for completion defined by the desired results and the permitted resources. This constitutes a prescribed task. But an *activity* is not a direct response to a prescribed task. The task is first redefined by the subject. To complete this task, the subject must form a representation of the task, allowing or forbidding possibilities (not always consciously), lifting or imposing restrictions, and using evaluation criteria that may differ from those of the prescription. This constitutes the *effective task*, to which the subject's activity represents a response. Misunderstandings in teaching are an expression of differences between the task anticipated by the teacher, and the task responded to by the student.

The gap between prescribed and effective tasks is inherent to the existence of two viewpoints: That of the task prescriber, and that of the task completer. The task the subject completes can differ from the assigned task for various reasons: Because the subject lacks motivation to engage in the desired actions, because the subject lacks the necessary competencies, because the subject constructed an inappropriate representation of the task, or even because of a divergence between the intended and prescribed tasks. The effective task is revealed by the subject's activity.

Analysis of the activity

In work or training situations, activity is oriented towards the completion of the task. Observable actions that permit an analysis are, first, operations on the objects of action, regardless of the aim of the research. This explains why the analysis of the activity relies on a preliminary analysis of the task, which can be understood as a psychological task analysis (Vicente, 1999) that relies on domain expertise.

However, the activity includes more than simply actions on "what to do," and includes other personal factors. For example, a teacher can assign different sets of problems from one year to another, not only because of the effect on students, but also to maintain personal motivation and avoid repetition or fatigue.

The analysis of a student task requires a didactician's mathematic expertise in order to identify what a student can do to effectively complete it. This is the aim of the *a priori* analysis presented in Chapter 2. The analysis of the teacher's task is more delicate. It is a largely discretionary task for which there is no defined procedure to follow (Leplat, 1997, p. 21). How to identify a strategy that would lead to the desired goal remains an open question, as there is no commonly accepted definition of an "expert teacher."[3] For this analysis of teachers' tasks and activities, we refer to a model of teaching as management of a dynamic human environment (Rogalski, 2003), in which the teacher mediates (Wood, Bruner, & Ross, 1976) between the student and the knowledge to be acquired (Robert & Rogalski, 2005), and in which language plays a central role (Pariès, Robert, & Rogalski, 2005).

The subject and the situation

The theory of activity depends on two other key concepts: The subject and the situation. We are interested in an *individual subject*, who has intentions and competencies (potential resources and personal constraints). Within this framework, subjects do not identify with their role, even though they may be constrained by legal and other responsibilities that act on the teacher. We can look both for commonalities between subjects and for specific aspects of their activities: What factors and organizational aspects do they share? What are the individual differences between them?

Whether students or teachers, subjects are not the sole masters of their goals or methods. They act within a *work* or *training situation*, which consists of a system of resources and constraints. Within this system, the teacher completes a set of tasks, which we can more globally consider to be a mission (the discretionary dimension of the task), tied to a prescriber (employer, supervisor) by a partially implicit contract. The teacher is acting within a context where students encounter multiple interventions (parents, teachers of other subjects, etc.) and within a process that continues during students' entire schooling. The student's situation is not limited to the tasks prescribed by the teacher under a didactic contract, but includes the social and familial environment.

We will now present the model of double regulation of activity, which can be related to issues of learning and development, as defined through the theories of Piaget and Vygotsky, and expanded by Vergnaud. Later on, we will defend the complementarity of Piaget and Vygotsky.

The model of double regulation of activity

The concept of regulation reflects the fact that the activity modifies the state of the situation as much as the state of the actor. The situation is a determining factor of the activity, and is simultaneously itself modified by the activity. This modification primarily affects the object of the activity, but can also include modification of resources and constraints. Subjects, too, both determine the activity and are modified in turn by their own activity. The situation can affect their potential for knowledge and action (competencies), their physical state (tired, sleepy, etc.), or their emotional state (happy, bored, anxious, etc.).

Figure 1 presents a schematic diagram of how this system of double regulation relates to the system of situational and subject determinants.

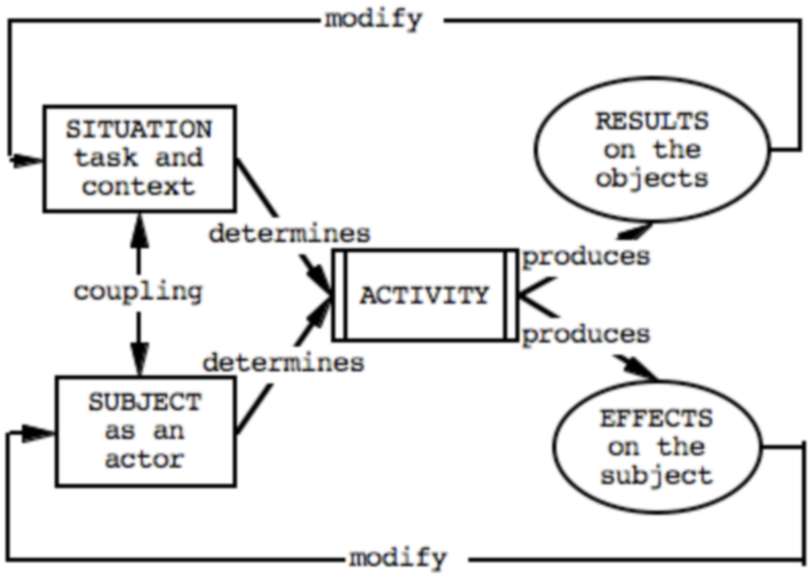

Figure 1. The double regulation includes a co-determination of the activity by situational and subject properties, as well as a double modification of the situation and the subject that is created by the results and effects produced (and by their agreement with expectations and acceptable outcomes).

This regulation can be considered in terms of short-term adjustments to action and “local” learning (such as learning how to find the inverse image of a function on a graph), or in terms of long-term development of a subject (understanding the concept of a function). The model of double regulation fits directly with the constructivist theories of Piaget and Vygotsky. It also sheds light on the issue of didactic intervention, by considering situational properties as potential producers of learning and development. Before considering this further, we must first pause to define the specific activities of the teacher and student, respectively.

A framework for defining student and teacher activities

The student, whether as an individual, set of students, or class as a whole, is a central determining factor of the teacher's activity. The choice of lesson plans (student task organization), and the unfolding of these plans in class, depends on prior knowledge of students, as well as on the possible actions the teacher believes to be possible in class. The teacher's didactic interventions in class depend on students' individual or collective activity. Completing a task produces a return effect on the teacher's activity, with an eventual adjustment both of the proposed tasks and of the teacher's own activity. Students' behavior can also contribute to the effects on the teacher, inducing fatigue, enjoyment, etc.

The teacher determines the activity of the student through the assigned mathematical tasks. During the completion of a task, the teacher mediates between the students and the mathematical concept to be acquired. This mediation can consist of assistance in getting started, procedural or constructivist assistance in completing the task, evaluation of the final product, identification of the concept in play, etc. The teacher can also participate in the construction of a student's reflexivity (for example, by demonstrating how to solve problems) and intervene in the constructivist dimension of the student's activity. Chapter 2 will explore this question.

COMPARISON OF THE THEORIES OF PIAGET AND VYGOTSKY ON DEVELOPMENT AND LEARNING

Piaget and Vygotsky each elaborated theoretical frameworks for understanding children's (and, more generally, humans') developmental processes. We will first present each researcher's scientific objectives, then the relevant elements of Piagetian constructivism, and finally Vygotsky's theoretical contributions. Putting these two frameworks in perspective highlights their commonalities, which include factors of development, a long-term perspective, and the role of tools in development (called "cognitive tools" by Piaget and "psychological tools" by Vygotsky).

Piaget's and Vygotsky's scientific objectives

For Piaget, the crucial point, distinguishing Piaget from all others in the field and rendering him irreplaceable within the scientific panorama of the 20th century, is his objective of genetic epistemology. The central question of this is how humans acquire knowledge, and how they thereby progress from children to adults capable of contributing to the development of scientific knowledge.

Piaget's aim is to "try to untangle the roots of the diverse varieties of knowledge, beginning with their most elementary forms and following their development to subsequent levels, including scientific thought" (Piaget, 2005, p.6). He notes that development of knowledge during the evolution of a species could in theory be considered part of this objective, but he chooses to begin with the development of a human child. He insists on the fact that his work has "a psychological dimension, but as a by-product … the goal is essentially epistemological" (op. cit., p. 7).

Piaget's viewpoint is therefore that of knowledge development for an *epistemological subject*, which is as much a theoretical construct as the *didactic subject* of mathematical didactics (when it defines teacher and student in terms of their role in the school system). A biologist by training, Piaget always insists on the biological roots of knowledge (Piaget, 1971, 2005, pp. 59-75).

Piaget's interest in the evolution of the structures of knowledge leads him to neglect a certain number of topics. For example, the topic of the developmental factors of a child (considered as a *psychological subject*) will not be central to Piaget's work. This is not because Piaget denies the effects of factors that are not internal to the "epistemological subject," but because his objective is to understand the internal process of development.

Piagetian constructivism claims that knowledge of objects is constructed through actions on these objects, and Piaget's goal is to demonstrate his approach's validity on the set of large domains of knowledge. These actions are not limited to physical acts on material objects, as knowledge construction can also occur through mental operations. Observation, for example, is a valid action that affects a subject's representations.

As for Vygotsky, his goal of theorization is clearly psychological, aiming to theorize the "higher functions" of thought. For him, the subject is a psychological subject, considered from the beginning to be in a social interaction with other subjects who have previously and personally developed "psychological tools" : this enables the development of knowledge. Under this model, knowledge of the world is socially preexisting in children: Their cognitive activities exist within social interactions before they are internalized into a subjective plane. This is the central, and very strong, idea of socio-constructivism: The passage from the inter-individual to intra-individual relies on the construction of psychological tools.

Vygotsky's focus is therefore profoundly different from Piaget's, with completely different objectives. Vygotsky's subject is an individual and social subject, who will construct tools for thought within social interactions. Piaget's subject is an epistemological subject, for whom the organization of knowledge (rather than mediation or tools) is the issue.

From this starting point, Vygotsky describes in a theoretical fashion the processes of learning and development, without dissociating the two. He will particularly differentiate, within a subject's "learning-development," the "everyday" concepts from the "scientific" concepts. Everyday concepts come from the everyday world, where social interactions do not have as a goal the production of an organized conceptual piece of knowledge in children. The acquisition of scientific concepts is accomplished through deliberate didactic interventions (Vygotsky, 1986, chap. 6).

Under Vygotsky's theory, scientific concepts are taught in scholastic institutions, and develop differently from everyday concepts. This deeper theoretical understanding of the evolution of concepts is directly pertinent for all didactics of a knowledge domain.

We highlight these differences between Piaget's and Vygotsky's objectives and central objects as a preface to presenting evidence that their case is not one of two psychologists with conflicting viewpoints, but rather of each making his own specific scientific contribution. Each proposes an original perspective on knowledge construction, as defended by Shayer (2003), for example. We will therefore first go deeper into the framework of Piagetian constructivism, and then describe the theoretical contributions of Vygotskian conceptualism, which are crucial for didactics of science.

Piagetian constructivism

The dominant image of Piagetian constructivism is probably that of a construction of knowledge that is internal to the subject. From this, one could see Vygotskian socio-constructivism as in opposition, taking into account the social dimension that Piaget would supposedly discard. To show that this is simply a question of perspective, we can refer to Piaget himself: "The social group plays … from a cognitive point of view the same role that the 'population' plays from a genetic point of view. … In this sense the society is the supreme unit and the individual only achieves his intellectual constructions insofar as he is the seat of collective interactions for which the level [depends] on the society as a whole" (Piaget, 1992, p. 345). The necessity of the social aspect in cognitive development is here clearly affirmed. Piaget successfully integrates the existence of two shifts during development: One associated with the individual as epistemological subject, and the other purely social. But it is the process of organization of knowledge (its structure) that will be central in the research he conducts. This "internal mechanism" of development is conceived in terms of a double regulation, retroactive and proactive,[4] for which Figure 2 presents a schematic diagram.

We can consider this double loop as a "zoomed-in" portion of the system of activity regulation (Figure 1). The object of the action is what is retained in the situation: The comparison between the intended state of this object and the observed effect releases an adjustment of the action. The feedback on the subject (which was not made explicit in Figure 1) will modify the action "upstream" through an adaptation of knowledge and schemes for action. Moreover, inasmuch as there is an intended or anticipated result, the action is also regulated proactively ("feedforward"). Piaget defines this mechanism in terms of a dialectic between *assimilation* of the new situation into the subject's strategies and conceptualizations, and *accommodation* of these concepts and of their organization. (We can think of the passage from a one-dimensional treatment of objects to a bi-dimensional treatment, for which the model is the Cartesian product.)

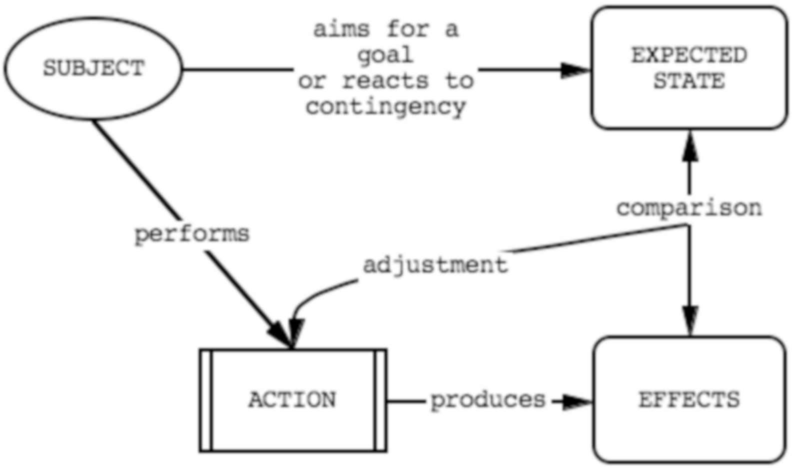

Figure 2. Schematic diagram of action regulation. The regulation takes place within a short action-adjustment loop, dependent on the comparison of the produced effect to the objective of the action. The objective can be determined through a conscious goal or can be a byproduct of the subject getting the object in his field of attention.

The development of knowledge structures results, therefore, from a double process: a *dis-equilibration* when prior knowledge structures lead the subject to expect a result that is invalidated by the action, and a *re-equilibration* after the knowledge structures are modified (Piaget, 1985). Piaget does not say that the re-equilibration is necessarily an improvement, which is to say it does not necessarily lead to a more efficient conceptualization (where "conceptualization" refers to the construction of concepts to understand and act on the world). He also describes the importance of reflective abstraction upon the subject's activity itself, and not simply on its results. This concept of reflective abstraction was operationalized for mathematics teaching by Simon et al. (Simon, Tzur, Heinz, & Kinzel, 2004).

Within the framework of Piagetian genetic epistemology, the importance of considering knowledge content was reaffirmed by Greco (a collaborator of Piaget who should be rediscovered) and included by Vergnaud in his theory of conceptual fields (see below). This consideration also leads to an analysis of the double regulation as simultaneously a "functional" adjustment of representations of the situation and of the organization of actions, and a "structural" regulation that modifies the conceptual organization and the cognitive operations of the subject. The diagrams in Figures 1 and 2 do not yet differentiate the two types of regulation, or the timeframes in play. We will return to these topics later on.

Piagetian genetic epistemology: The role of knowledge content

The Piagetian concept that has probably entered the most into psychology and science of education is that of the stages of development (sensorimotor, concrete operational, formal operational, and eventually intermediary stages as well). The other well-known concept is the idea of a "child logic," that evolves toward a scientific logic that will permit coordinated logical operations and manipulation of abstract propositions, independent of their content. In fact, the initial work of genetic epistemology on space, numbers, speed, time, and physical concepts presents successive organizations of a child's representations in each of these domains: these organizations run with a similar underlying structure.

Greco highlighted the role of content in a set of analyses brought together in a posthumous work (Greco, 1991). He provided evidence as to the importance of considering the object of the action, as well as the task. He recalls, "Within the genesis of elementary logical structures, Inhelder and Piaget … indicate steps or regular levels, but also [insist] on the fact that these steps are strongly differentiated by the nature of the material, the classifying task proposed to the child … etc." (op. cit., p. 38). He underlines the necessity of "specifying the conditions of equilibration, notably as these conditions also highlight properties of objects and tasks" (p. 39), and insists on the role played by the object of action and knowledge within the regulation that is at the heart of the development process: "The adjustment of forms to content requires a revision that reveals properties the available forms do not allow us to cover…The worrying question of restoring the role of the object in development is an integral part of Piagetian constructivism" (op. cit., p. 55).

The complementary contribution of Vergnaud's conceptual fields

As highlighted by Greco, the essential Piagetian concepts of dis-equilibration/re-equilibration, the assimilation/accommodation dialectic, and the intervention of a complex regulation process are general concepts that should be made more specific according to the knowledge content. It was Vergnaud who enlarged the Piagetian framework in theorizing the concept of the conceptual field (Vergnaud, 1982, 1990), by outlining situation classes, operational invariants, schemes for action, and representational systems, all relative to a knowledge domain.

The theory of conceptual fields "was initially elaborated in order to take into account the process of progressive conceptualization of additive structures, multiplicative structures, [and] the relationships between number and space, within algebra" (Vergnaud, 1990, p. 135). The theory articulates two epistemological approaches: That of mathematical didactics and that of developmental cognitive psychology. Broadly, from a didactic point of view, the concept of the conceptual field aims to provide a framework for analyzing the student-knowledge relationship within the didactic triangle of student, knowledge, and teacher. From the point of view of developmental psychology, the theorization in terms of conceptual field permits a joint analysis of the effects of learned concepts and of development throughout a student's mathematical education.

The notion of conceptual field

A typical example of conceptual field is that of additive structures within elementary calculus. Included with this field is a set of numerical concepts: numbers (natural numbers, initially "small" numbers), order relations (first between whole numbers, and then generalized to the number line), and addition and subtraction operations (including their properties, as well as their relationships with order relations and with classes of problems that use these operations for their solution). Trying to consider the concept of number "in itself," isolated from other concepts that render it operational, is not only ineffective for understanding what students learn, but devoid of meaning for studying teaching processes aimed at numeric concepts. The concept of conceptual field is relevant to studying student learning at a wide variety of levels, from the "everyday" conceptualization of a young child (outside of a didactic project), to the conceptualization that is the goal of a scientific lesson or of a student specializing in mathematics.[5]

Thus, at an elementary level, the construction of a complete collection of unique colored shapes, given a set of shapes and an independent set of colors, brings into play a set of concepts related to the conceptual field of the Cartesian product (Rogalski, 1985). These concepts include those of identity and difference for ordered pairs, and eventually the cardinality of sets and the distributive law (ensuring that each form is associated with each color, or vice versa). We note that the concepts in question, while "everyday concepts," are precursors of logico-mathematical concepts. Another example of such everyday concepts is the "small" cardinals, which a small child is able to manipulate after mastering its precursor, "numerosity," a quality perceived in object collections, not unique to humans but shared by a number of species. All these concepts will continue to develop and enter into the conceptual fields of additive then multiplicative structures.

In general, concepts can be considered as nodes of a network that contains all kinds of relations. A conceptual field corresponds to a part of this network that possesses characteristics relevant to a set of proposed situations. Thus, a number of concepts are contained within the conceptual field of numeric multiplicative structures, including the Cartesian product, product measure, linear transformations on **R**, multiplicative operations on various numeric sets (**N**, **D**, **Q**, **R**). Similarly, a single scientific concept can fall under several different conceptual fields. The concept of surface area, for example, is included in the conceptual fields of physical quantities, measure operations, sets of positive numbers, space and its models, it belongs to the conceptuel field of additive structures (as a "simple" measure), and of multiplicative structures (as a product of linear measures).

Schemes in the theory of conceptual fields

Vergnaud insists on the fact that, to take into account the adaptive function of knowledge, it is necessary to give a central place to the operational dimension of knowledge ("rational knowledge is operational or nothing," op. cit., p. 136). The concept of a "scheme" models this operational function. With Piaget, the scheme is defined as an invariant organization of action for a class of situations.

It should be stressed that it is not the action that is invariant, but a particular property of the action: its organization. The operational nature of the scheme reflects the possibility that the action may vary with the determinants of the subject's situation. This is what enables new situations to be met with adaptation rather than simple repetition. In the Piagetian line of research, a number of processes have been proposed for the development of schemes: a double process of assimilation and accommodation (similar to the process in play in conceptual development), processes of generalization/specification, and processes of combining pre-existing schemes.

Within the domain of learning mathematics, Vergnaud described the evolution of a counting scheme during the numeric education of children and students. This scheme involves the temporal coordination of visual focus, pointing gestures, and recitation of the list of number names, and repetition of the last word-number (an ordinal) to give the cardinality of the set ("one, two, three … seven – SEVEN!"). The child will assimilate new counting situations into a scheme initially developed for very small collections. New situations will call for an accommodation in the initial scheme. For "big" collections, operations will be added to the scheme's organization. These may include taking into account the spatial structure of the collection, using the theorem-in-action of adding the cardinals of disjoint subsets, or using an intermediate system (tally marks grouped in fives, used in manual vote counting systems, for example). The "technical" operations of addition will stem from this.

The analysis proposed in the theory of activity, together with "activity, action, operation" (Galperine, 1966; Haenen, 2000; Leontiev, 1984; Savoyant, 1979, 2005) leads to a distinction of levels within schemes.[6]

The issue of schemes of action for the student is considered in studying the development of what we could call their dexterity in executing mathematical procedures, which falls outside the scope of this book.

The importance of systems of representation

Within the theory of conceptual fields, developing "a psychological and didactic approach to the formation of mathematical concepts leads to a consideration of a concept as a set of invariants that are available for use in an action. The pragmatic definition of a concept, therefore, includes the set of situations which constitutes the reference for the various properties of the concept, as well as the set of schemes applied by subjects in these situations. However, the operative action does not constitute the entire conceptualization of reality … the use of explicit signifiers is indispensible to the conceptualization" (op. cit. p. 145). Inhelder and Piaget's declaration that "memory of a scheme is that scheme" brings us back to the issue of progression from schemes' "concepts-in-action" and "theorems-in-action" to representable concepts, following a conscious realization.

For Vergnaud, representations are two-sided: The "signifier" corresponds to their external dimension, and the "signified" to their internal dimension. External representations (signifiers) can take a variety of forms. Vergnaud, like Vygotsky, highlighted the central place of verbal language (as opposed to non-verbal; can be oral or written). The treatment of symbolic representations can be an intrinsic part of the activity. This is a crucial point for analyzing the double teaching/learning process within secondary education, particularly where algebra is concerned.

Collective history of mathematics, individual history of the student

At the moment of teaching, a process of conceptualization has already taken place within "mathematical communities." This process results in the production of "theoretical knowledge" that is at the heart of epistemological analysis in mathematical didactics. These historically constituted conceptual organizations of scientific knowledge serve as a reference to determining the relevant conceptual fields for analyzing and provoking students' conceptualizations, issued from their activity in appropriate situations.

Students' personal history and the familiar frameworks in which they act may introduce new elements in their activity, in addition to the conceptual structure of the situation. Vergnaud elaborated a typology of addition problems in which psychological and didactic relevance is denoted by important differences in student learning. His typology departs of the mathematical models that essentially limits variation to the numeric values and to the "technical" operations available for use in solving.

The "socio-constructivist" theoretical framework elaborated by Vygotsky enables taking this historical double determination into account, and outlines two concepts. The first concept is that of *social mediation*. This concept primarily intervenes through a direct intervention by an adult or a "more knowledgeable" into the activity of the student or child. This analysis was later developed by Bruner, with the goal of developing a theory of instruction (Bruner, 1996; Wood, Bruner, & Ross, 1976). The second concept is the "everyday concepts/scientific concepts" dynamic, where the latter are instruction objects, analyzable as "theoretical knowledge."

This Vygotskian framework also allows for consideration of the "learning/ development" process as one point of view on the subject, and for expanding the issue of development (which psychology and Piaget himself, traditionally limited to childhood) to cover the entire lifespan of the subject.

After recalling, below, Vygotsky's contributions, we will put into perspective the Piagetian and Vygotskian frameworks as compatible and complementary tools for analyzing teaching practices and student learning.

Vygotsky's views of didactic intervention and of development

Vygotsky's theory put social mediation, and the value of "psychological tools" in this mediation, at the heart of the development process. It thus offered a perspective that was complementary to Piaget's for studying the teaching/ learning relationship within the development of the conceptual fields of a scientific domain such as mathematics. Vygotsky also contributed to the analysis of conceptualization, showing how "everyday" concepts from the child's normal life, and "scientific" concepts that were explicitly taught, developed within a "double germination" dialectic.

Everyday concepts and scientific concepts

Chapter 6 of *Thought and Language* (Vygotsky, 1986) explores the topic of two types of concepts: everyday concepts and scientific concepts. It also describes the relationships with the learning and development of a child. Vygotsky criticizes Piaget for only being interested in the development of spontaneous or everyday concepts, and for not examining the form scientific ("taught") concepts took in a child or adolescent, and how they were integrated into development.

The distinction introduced here by Vygotsky contrasts the characteristics of concepts stemming from a child's interactions with objects in the world without didactic intervention, with characteristics of scientific concepts originating from a prior collective production, which are also objects of instruction.

Two characteristics distinguish everyday and scientific concepts: Their organization (the "structural" dimension) and their relationships with objects (the "functional" dimension. Scientific concepts are strongly tied each other by mutual relationships, including abstraction (generalization) relations, while everyday concepts can be isolated. In the activity of a very young child, the concept of *cat* is not necessarily tied by an abstraction relation to the concept of *feline* or *mammal*. Nor must it be placed in comparison to, or contrast with, the concept of *dog*. The *cat* concept is functional, and operational, without any such relationships. By contrast, the concept of *function* in mathematics (numerical, one-variable) cannot exist without that of *variables*, while the operationalization of the concept of variables assumes that of *numeric concepts*. In addition, the concept of *graph of a function* (a one-dimensional subset of the plane) and its graphical (external) representation implies a *number-space relation*, with the concepts of *number line*, *x-axis*, *y-axis*, and *coordinates*.

Everyday concepts can exist "in action" without children being conscious of them or able to verbalize them, either because the process of consciously realizing the "concept-in-action" did not take place during the child's development, or because the action does not call it to mind. By contrast, scientific concepts are explicit and exist through symbolic representations, such as language (the primary form of representation) and other symbolic mathematical systems.

Vygotsky presents the idea that an everyday concept is "glutted with empirical content." This is a strength of everyday concepts from the point of view of significance, but it is also a weakness, as the content brings with it a mass of properties, which limit conceptual constructions at a higher abstract level.

By contrast, the strength of a scientific concept comes from its generality in terms of abstraction, and the generality of the domain in which it acts (its "decontextualization"). Its strength also comes from the fact that the concept is conscious and was constructed with "words to say it," as well as from the coherence of the system of concepts to which it belongs. However, despite being more general than everyday concepts, scientific concepts display much less empirical concept, which is their weakness, in terms of significance.[7]

The dialectic of "double germination" of everyday concepts and scientific concepts
The respective characteristics of everyday and scientific concepts can lead to considering them as two conceptual categories. The historical-developmental dimension of Vygotsky's theoretical approach is in fact essential for progressing past this view, to seeing everyday and scientific concepts as developing interaction within a dialectic of "double germination."

In one direction, the germination of scientific concepts passes from the "low" to the "high," where the "high" is what is "general" and "decontextualized." This passage follows interaction with objects from the world of action (as in Piagetian constructivism). In another direction, this germination passes from high to low, supported by symbolic representations (including appropriate language) proposed in the mediation.

Within this biological metaphor of germination, everyday concepts clear the way for the germination of scientific concepts by the meaning they provide. Scientific concepts, in turn, clear the way for the germination of everyday concepts through their organization and the mediations they propose, and "pull" the everyday concepts higher.

In terms of the child, an operational piece of knowledge is acquired when the two types of concepts meet and two processes are engaged. The first process is a reorganization of everyday concepts to better organize them into a system. The second process involves extracting the meaning of scientific concepts to make them concepts for action. This process of interaction assumes a property of the development dynamic: that the dynamic takes place within a *proximal development zone*.[8]

The proximal development zone and the learning/development relationship
The proximal development zone (PDZ) is situated between the current level of development, defined by what the child is capable of doing or solving autonomously, and what the child can do or solve with the help of others (adult, teacher, more knowledgeable peer). For conceptual learning to succeed, situations should raise this zone. If they are above the PDZ, assistance can at best produce an immediate imitation (or recitation; Vygotsky speaks of "verbal mechanics"), and do not contribute to development. If the situations fall below this zone, the child/student only uses prior knowledge, and learns nothing.

The concept of PDZ is relevant for the initial development of a new conceptualization that is based on prior mathematical knowledge acquired by the student. By acting on the student in this zone, the teacher allows everyday concepts, or familiar mathematical concepts, to transform and integrate into a more detailed conceptual field. If the mediation is successful (assuming its goal is conceptualization and not the simple mastery of procedures), these concepts can go toward the mathematical concepts to which they are epistemologically tied.

The Vygotskian concept of PDZ is not only a way to discuss development/learning relationships. It transforms the understanding of these relationships. Under the Vygotskian theory, scientific conceptualization does not await, rely on, or follow spontaneous conceptual development, but instead intervenes in this development, offering new mediations through new tools for thinking. As for the learning dynamic, it too depends on the possible meeting of taught concepts and meaning brought through everyday concepts (or familiar mathematical concepts), even if the latter are weaker in terms of their generality and organization. Vygotsky thus proposes a dialectic process of conceptualization at the heart of the development processes.

Merging the Piagetian and Vygotskian frameworks

We have described the respective approaches of Piaget and Vygotsky from an epistemological point of view, which has led us to consider them non-contradictory. In addition, we have examined the parts of their respective theoretical frameworks concerning conceptual development, in terms of issues of mathematical didactics (and other didactics centered on conceptualization). These considerations lead us to highlight their commonalities, or complementarity, where Bruner "celebrated their differences," arguing that "with one thinker emphasizing the role of inner autochthonous logical[9] processes, and the other the shaping role of culture, inevitably led to sharp divergences in their approach of mental growth" (Bruner, 1996). Here we take the same path as other researchers, such as Cole and Wertsch, specialists in Vygotsky who proposed going beyond the apparent social/ individual incompatibility between the two (Cole & Wertsch, 2001). In fact, Piaget never denied the key role of the social dimension in child development, but simply did not include it in his theory. Looking at Vygotsky, his socio-constructivist theory is in no way incompatible with the concept of structuration of subject knowledge via the regulation processes of the Piagetian framework, even if he did not specifically consider these processes. The following discusses their commonalities in terms of factors of development, psychological or cognitive tools, and long-term development.

Factors of development

We highlighted above the fact that even if factors of development were not at the heart of Piaget's genetic epistemology, he nevertheless did not reduce them to interaction with the objects of the action. He stressed three general factors: biological maturation, the role of exercise and experience gained in the action, and, finally, social interactions and transmissions (Piaget & Inhelder, 1971, p. 152 ff.). The social dimension of Piaget's theory was studied more generally by DeVries (1997). As for Vygotsky, he made explicit the role of interaction with objects of the world of action within a child's development. Vygotsky's socio-constructivism is thus a materialist constructivism.

Psychological tools, cognitive tools

One central element in Vygotsky's theory is the role of tools in a child's development. He focuses particularly on psychological tools, as these have already been constructed socially. A very specific place is given to language, a psychological tool *par excellence*.

Piaget largely used language (and graphical representations) as a way to access a child's "spontaneous representations." Even so, he did not question the role of language in development, but explicitly referenced the contributions of language to cognitive tools: "… language has already been elaborated socially and contains a notation for an entire system of cognitive instruments (relationships, classifications, etc.) for use in the service of thought. The individual learns this system and then proceeds to enrich it" (Piaget & Inhelder, 1971, p. 87).

The long-term development of a subject

The study conducted by Piaget and his collaborators regarding stages of cognitive development for the large categories of thought regularly stressed the long-term nature of this development. Within the numerical domain, Vergnaud showed that the conceptual field of additive structures develops precociously, with additive operations appearing at two years, while transpositions and comparisons[10] were not mastered until the end of mandatory schooling. Data on learning spatial measurements, particularly volume, have shown the difficulty and length of the conceptualization process required to progress from a "one-dimensional" understanding of volume (a familiar concept at the end of elementary school) to the idea of a product measure (for which conceptualization is not achieved at the end of junior high school).

Vygotsky did not have the same insistence on the long-term view of cognitive development, but he did stress, in the chapter entitled "Everyday concepts, scientific concepts" (1986), that the construction of scientific concepts, like everyday concepts, only began after the child had assimilated for the first time a new meaning or term, bringing with it a scientific concept.

Didactics and extending the timeframe of development

The didactic interpretation of "the two constructivisms" has led us to extend the timeframe of development to "advanced math." The concepts already available within a conceptual field are potential *precursors* of the concepts to be learned, and have a function analogous to that of everyday concepts in the Vygotskian framework. These precursor concepts have two possible and contradictory roles, that of a precursor and that of an obstacle. The productive role of precursors is, in particular, to give meaning to new concepts (Vygotsky spoke of the "force" of everyday concepts). Their reductive role is tied to properties of concepts that are no longer valid for the new concepts to be learned (following Bachelard, French didactics calls these "epistemological obstacles"). In Piaget's theory, the duality of the productive/reductive roles can be interpreted in terms of the interplay between the processes of assimilation and accommodation.

One important contribution of the Piagetian framework, for which we find no Vygotskian analogue, is the fact that knowledge develops through a "Münchhausen effect": "One of the strength-ideas of Piagetian constructivism is that knowledge itself creates the conditions and tools of knowledge" (Greco, 1991, p. 52). There is, thus, a dynamic unique to knowledge beyond occasions of development provoked by didactic mediation.[11]

The two constructivisms: complementary theoretical tools

Each of the two main figures in constructivism recognized the strengths of the other. Vygotsky, discussing the shift toward scientific concepts of a certain number of notions of causality, remarked, "I have not looked closely at the state of children in terms of the logic they are capable of using; in this Piaget has shown overwhelming superiority" (Vygotsky, 1997). As for Piaget, he noted his agreement with Vygotsky's approach to the analysis of everyday and scientific concepts. In addition, he stressed that "The individual only achieves intellectual constructions insofar as he is the seat of collective interactions, whose level [depends] on the society of his group" (Piaget, 1992, p. 345).

Finally, a more philosophical point of agreement between the Piagetian and Vygotskian frameworks was highlighted by Bruner (1996): "The unique mystery of mind is its privacy, its inherent subjectivity. Both Piaget and Vygotsky were very explicit on this point. See Piaget (1974, pp. 28 ff.); Bruner's (1987) preface to Volume One of Vygotsky's collected works." This concordance on the subjectivity of thought reinforces the links between the two developmental frameworks and the theory of double regulation of the activity. The latter concentrates on subjects, authors of and actors in their own activity, whether they are students whose learning is the goal, or teachers who work toward student learning.

CONCLUSION: OUTLINING THE THEORY OF ACTIVITY AND THE TWO CONSTRUCTIVISMS

We presented the theory of activity and the model of double regulation of the activity, which is used in ergonomic psychology but is extendable to any completed activity, including that of the student. One component of the double regulation model is the impact of the activity on subjects themselves, which represents the developmental dimension of this model. On the topic of knowledge, we then highlighted the commonalities and complementarity of the constructivist theories of Piaget (as extended by Vergnaud's conceptual fields) and Vygotsky (with Bruner's theory of scaffolding).

The connection between the theory of activity and the "two constructivisms" thus offers a theoretical tool for a double approach from the viewpoints of mathematical didactics and the activity of the subjects in question (teacher and students). In particular, the Piagetian theory looks "from the student's side" at epistemological analyses of the mathematical objects in play, while the Vygotskian theory takes into account the didactic intervention of the teacher, mediating between knowledge and student in support of the student's activity.

The developmental dimension also calls into question the timeframe of the processes in play, particularly for students. Leplat already compared the theory of activity and Piagetian constructivism and highlighted the existence of a functional regulation as well as a structural regulation. The functional regulation leads to adjustment of the action (cf. Figure 2), within a process of micro-genesis, that can translate through procedural learning, with possible support from the teacher. This short-term regulation can also involve conceptual "primings" within the student's PDZ, between the previously acquired concepts and new mathematical concepts. The structural regulation acts over the long-term, within a process of macro-genesis, in which the conceptual structures and the student's schemes of action are transformed within an assimilation/accommodation dialectic.

From a didactic perspective, we can take the relationship between these two timeframes into account by forming the hypothesis that after concepts are "primed" (whether through a fundamental situation, an appropriate problem, or even an explicit direction from the teacher), making these concepts functional is an essential contribution to the structure of the intended conceptual field[12]. The methodological importance comes partly from analyzing the mathematical tasks proposed by the teacher in light of the potential student activity during the completion of these tasks, and partly from analyzing the didactic interventions on the activity of students in class. Two delicate elements within analyses of teaching/learning situations are the role played by the autonomous activity of the student, and the importance of the teacher's identification of the PDZ.

There are some very general avenues for research that rely on the theoretical tools presented here. There are still aspects to be clarified enabling the study of didactic interventions and student development at various levels of analysis: from the global level, consisting of relationships between the overall structures of teacher interventions and the mathematical knowledge acquired by students[13], to the "micro" level of individual interactions during class, passing through the local level of a class period.

NOTES

1 By "competencies," we refer to the sense as it is used in ergonomic psychology and professional didactics. This type of competency does not denote a set of tasks that can be completed successfully, but a set of potential resources for action of a subject. It is the same sense intended when referring to a student's competency in mathematics.

2 Note: This task-activity distinction differs from formulations encountered in various pedagogical texts. Within the theory of activity "The activities proposed to a student" would be expressed as "The tasks proposed …."

3 Berliner (2001) showed the complexity of the question of the characteristics and even definition of an "expert" teacher. A description of approaches that agrees with the one we propose here is presented by Perrenoud (2005), with regards to the question of knowledge mobilized in the analysis of teaching practices.

4 This conception should be linked to its importance for cybernetic concepts, for which regulation is a central concept, from biology to automated systems.

5 The entry point chosen for analysis here is that of concepts. This entry point is directly applicable for studying student learning in scientific disciplines (the conceptual fields are here defined with

reference to theoretical knowledge). A "dual" entry point by way of situations is used in professional didactics, with the notion of "conceptual structure of a situation," describing diverse concepts, including pragmatic ones (Pastré, 1999; Vidal-Gomel & Rogalski, 2007). Brousseau's concept of the fundamental situation could be seen as an expression of the epistemological link between concepts and situations.

[6] The analysis is complicated by the fact that these levels are relative. An action previously composed of multiple operations can become, during development, a "unitary" operation that is itself a component of a higher-level action.

[7] Within the work domain, professional didactics introduced the theoretical notion of "pragmatic concepts" as the organizers of the activity. Historically constructed by a professional community within and for a particular domain, these are neither "everyday" concepts nor "scientific" concepts (or techniques) under Vygotsky's definition. Integrated within a conceptual structure of the work situation, these concepts relate to indicators (observables) and ways of acting (Vidal-Gomel & Rogalski, 2007). In teaching, an expression such as "the class has disengaged" refers to a pragmatic concept for which teachers use various indicators and have a multitude of possible interventions (changing tasks, intervening in students' activities, etc.).

[8] A number of versions of this concept can be found in the literature: zone of near development, zone of proximal development, or even zone of potential development. I have chosen to use "proximal" to refer to this zone.

[9] Our focus, coming from didactics, is actually on the development of knowledge and particularly the process of conceptualization, rather than on general logical processes.

[10] Jean-François Richard (2004) showed that humanities students encountered serious problems in descriptive statistics related to issues of cumulative effects (requiring them to perform subtractions). My own experience has taught me that we find these types of problems in errors on credit and debit in accounting.

[11] It is the existence of the dynamic unique to knowledge that led us to consider the activity of the teacher as the management of a dynamic environment, which is the student/knowledge relationship (Rogalski, 2003).

[12] We can even make the hypothesis that making concepts functional can create meaning, under certain conditions on the density of work and the position of the student in relation to mathematics.

[13] The methodological problems of defining adequate global indicators, as much for student learning (beyond assessments of success at certain types of tasks) as for relevant properties of the teacher's intervention, remain open.

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ALINE ROBERT AND CHRISTOPHE HACHE

2. WHY AND HOW TO UNDERSTAND WHAT IS AT STAKE IN A MATHEMATICS CLASS

OVERVIEW

Chapter 1 proposed a unique general framework, organized around activities by actors. This framework specifically allows an analysis of student learning and teacher practices. The goal of this chapter is to describe, from a theoretical and methodological point of view, the use of this analysis process to examine the teaching of mathematics in school. All research presented here concerns the teaching of mathematics in middle school and high school (students age 11-18).

Our research has two goals. First, we aim to give researchers access to student learning for a given topic, in relation to the instruction they have received, within a specific school system, from a diagnostic viewpoint (analysis to understand what there is) or a prospective viewpoint (experiments to learn how to enrich the existing situation). Second, we aim in the long term to work on teacher education, particularly based on conclusions from previous analyses and on hypotheses allowed by the theoretical framework (see end of volume).

The goal of this chapter is therefore to describe the specific theoretical frameworks that we use and the general methodologies that follow from these frameworks. Later chapters will describe specific studies led under these frameworks and that use these methodologies.

We will first present the theoretical tools that we have adopted to help us understand the learning of a given topic through examination of the relationship between mathematical content and the teaching and learning of the topic. We will also look at evidence of regularity and variability among classes, teachers, and teaching practices. We will begin by examining how we aim to have access to student learning, and what kind of results we expect to see.

Our approach is, first, thoroughly didactic, in the sense that we develop all our analyses from the specific characteristics of the mathematical content to be taught. This preliminary analysis of content is connected to other in. Thus, we describe the mathematical content which keeping in mind what we will study, student learning. More precisely, in our theoretical framework, mathematical learning is associated with the concept of conceptualization (chapter 1). This leads us to connect the mathematical content studied to levels of conceptualization. These levels are defined from school curricula and from characteristics of the concepts involved in a set of tasks and in the corresponding knowledge whose use is intended (second paragraph of the first section of this chapter). This is where certain differences can arise between didacticians. For example, we ascribe significant importance, among other characteristics, to the variety of possible ways of making use of knowledge,

F. Vandebrouck (ed.), Mathematics Classrooms: Students' Activities and Teachers' Practices, 23–73.

and we have given ourselves the means to identify this variety from problem statements.

Secondly, for any given topic, we organize our analyses through the study of students' activities. Following the theory of activity, we postulate that student learning depends directly on student activities, even if these activities are partly inaccessible and differ from one student to another, and even though other elements can intervene in student learning (beginning of the first section of this chapter).

Studying student activities involves analyzing their work on assigned in-class tasks, together with anything added by the teacher during class time. We have conducted *a priori* analyses of tasks in terms of their potential for calling on mathematical knowledge (third part of the first section of this chapter). These analyses allow us to characterize how students must use their knowledge (based on our study of what takes place during class time). We complete our analysis by developing ways to analyze class periods and possible student activities, incorporating all elements added by the teacher during class that contribute to our model of possible student activities (second part of the second section of this chapter).

However, if these activities are well developed for the majority of the class, then we should also take into account, day by day, everything proposed to students on the topic to be studied. We call this sequence of lessons and exercises on a topic the *scenario*.

We seek to understand global scenarios for a given topic in terms of the intended student conceptualization. These scenarios can be understood as sequences of lessons and exercises associated with intended applications of the content knowledge. This provides an initial approach to understanding possible student activities and student learning. These global scenarios can be seen as planned "cognitive itineraries" (first part of the second section of this chapter).

All these analyses incorporate some general hypotheses about learning. These hypotheses arise from our adaptation of the combined theories of Piaget and Vygotsky for school mathematics learning, as initiated by Vergnaud (chapter 1). The analysis of proposed tasks involves elements that are assumed to have an influence on activities and therefore on learning. Thus, in terms of skill construction, the variety of what students must use in their work plays just as much of a role as the order in which students complete exercises, or the quantity of exercises completed. In other words, the possible ways to mobilize, combine, and recognize the knowledge to be used in exercises are the main factor in constructing student knowledge (along with processes of assimilation, accommodation, dis-equilibration and re-equilibration). But this depends not only on proposed tasks and the actions they may provoke, but also on the way in which these tasks are worked on by students (particularly in class in terms of the nature and quality of individual and group investment), as well as on the mediations and assistance provided by the teachers. To analyze in-class events, we also use anything that can influence student activities in terms of teacher practices, whether related to the nature of the organized work (autonomous, in groups) or to direct teacher interventions (assistance, identification of student work, use of this work, assessments, etc.).

Finally, for the last fifteen years, we have introduced the idea that teachers' choices, in and out of the classroom, are not solely determined by factors related to student learning. They also depend on numerous external constraints, which can be institutional (tied to curricula and schedules) or social (tied to classes and establishments). They can also be tied to the personality, representations, knowledge, and experience of the teacher. To better understand teachers' practices (which depend on student activities and on which student activities depend),[1] we complete our analyses based on in-class events, taking into account factors tied to the teaching profession through a didactic and ergonomic double approach (third section of this chapter). We use the word "practices" to refer to all work done by a teacher. Although our analyses are based on in-class work and teachers' activities in relation to intended student activities, we include in a teacher's "practice" all work done by that teacher, whether before, during, or after class time.

It is clear that the choice of divisions and variables to analyze is delicate, as the variables in play can be both local and global (a concept or specific exercises, for example). Variables can also be defined in terms of other variables. For example, the way we choose to describe a concept depends on what we find useful in understanding scenarios and their potential for student learning. Descriptions of scenarios, conversely, depend on the specifics of the content involved. Our descriptions of mathematical content (in terms of variety of tasks, for example) should allow for understanding of the corresponding learning process. Finally, if students' intended activities contribute to decoding teachers' in-class activities, it is these in-class activities that in turn allow us to describe possible student activities.

The remaining questions focus on this division, which we will discuss further in the fourth section of this chapter. One open question we continue to work on, for example, is the determination of significant indicators in a teacher's speech. How far should our analysis of the way a teacher addresses students go beyond examining the strict content of the message (which should also be investigated)? To what extent, and to what, are students sensitive: to repetitions, images, spoken and written information, questions, differences in presentations, etc.?

The results that these tools have allowed us to produce, as subsequent chapters will illustrate, come out of relatively recent research, in which researchers have adapted them to particular research areas by specifying, discussing, and enriching the methodology.

This work is equally applicable to the analysis of teachers' manuals, which mainly relies on *a priori* analyses of tasks and scenarios. This analysis reveals the benefits and limits of the analysis of practices from which we can infer general important characteristics, as mush for students as for trainings: intrapersonal stability, inter-teacher commonalities, variability and changes, etc. A certain number of these studies examine the integration of technology into teaching, and propose new diagnostics of teacher and student difficulties in order to design suggestions for technology use or trainings.

The reasoned descriptions we produce allow for deep understanding of what takes place in a mathematics class, over the short and medium term, in terms of consistency or diversity for a topic, for a single teacher or multiple teachers. Understanding student knowledge in the long term is difficult, as it is difficult to define and broad, and the variables contributing to this knowledge for each student are out of reach. Thus, even though some studies may relate the instruction given on a mathematical chapter, analyzed according to our criteria, to the resulting student work, they are identifying fairly local regularities that depend in part on the individuals involved, and do not claim to deduce from this prescriptive indications. We can say, by contrast, that our research can help to enrich teachers' work, by revealing variables that contribute to their choices before and during class, and by giving them ways to discover the full range of what is possible. In addition, interpreting the identified inter-teacher commonalities and variances within the chosen theoretical framework contributes to reflection on teacher trainings (conclusion of this volume).

In the first section of this chapter we will focus on student activities (intended activities, possible activities, *a minima* or *a maxima* activities, etc.). We will present a general outline of our didactic approach, and describe our mathematical *a priori* analyses of class periods, which will allow us to better understand student activities. These analyses consist of global analyses of the form and type of mathematical content to be taught, and local *a priori* analyses of assigned tasks in terms of both the intended conceptualizations to be formed of the content and student learning. In the second section, we will describe the *a posteriori* analyses of class periods, both globally (the scenarios) and locally (analyses of in-class events). These analyses allow us to discover students' activities by using teachers' activities as an intermediary. In the third section, we will detail the analyses of the teaching practices of mathematics teachers, which form the heart of the studies in this volume. Finally, in the two final sections we will discuss the general elements of the methodology, and conclude by indicating methods for comparing different "paradigms" of didactic research.

STUDENT ACTIVITIES AND A PRIORI ANALYSES OF CLASS PERIODS

Student activities and the general plan for our analyses

Student activities[2] (as well as teacher activities) consist of their actions during the completion of a task. This task can be anything from an exercise to listening to a lesson. The activity takes place within a specific situation, such as in class or at home, and consists of external mathematical actions, which may be spoken, written, or performed, as well as internal actions such as hypotheses and decisions as to what to do. These last constitute the student's "personal state." Personal state activity is not directly observable but leaves observable traces. Activities are made up of everything students do, including listening, as well as everything surrounding the actions. This allows the development of knowledge from actions. Student activities also consist of what students say, think, do not do, do not say, etc. They depend on a number of factors, including the teacher's activity, which contributes to the desired transformations in terms of knowledge. For the teacher's activity, we use lessons[3] and assigned exercises as well as work conditions in class and aspects of the teacher's speech. In Appendix 2 we present an overview of the dimensions we believe affect learning. However, there are other factors affecting learning that we do not directly consider, including emotions tied to school, socio-cultural factors that can act as a filter between the student and the school,[4] and factors connected to other circumstances outside of school. These other factors (emotional, socio-cultural, tied to circumstances, etc.) are considered as variable parameter and are taken into account, but are not independently analyzed.

We also do not consider extreme cases of students who do not participate at all in the class activities, whether because they refuse or because they do not understand the transformation expected for knowledge activities. These last students act in ways that are too different from the ways intended. By contrast, some of the studies presented introduce the concepts of action logic (success logic) or learning logic, according to the possible ways to include students in their own learning. The double regulation system from chapter 1 is only used in certain studies, most notably those that explicitly concern individual subjects.

It must be emphasized that students in the same class will not develop the same activities or follow the same course along the same "cognitive itinerary" (see above). An individual's activity also depends on the individual. Elements of differentiation may be introduced in the various studies presented.

As activities are, by definition, partly internal and inaccessible, depending on the case we will only study possible student activities. These activities are presumably close to students' effective activities, but, in light of *a priori* analyses, we cannot be sure that all students will complete them. We can even, in some cases, be sure that this is not true. When necessary, we identify ways to better approximate students' effective activities (based on computer logs, for example).

From tasks to activities: Possible, a minima, and a maxima activities

All teacher and student actions modify the possible activities, as predicted by the *a priori* analyses, and contribute to their reconstitution. Among the possible activities, we often distinguish *a maxima* and *a minima* activities. *A maxima* activities are the activities of students who begin working as soon as the teacher asks. They engage in the assigned task with some autonomy. They often have an idea on how to begin, and are able to overcome the desired adaptations. *A minima* activities are the activities of students who may be more distracted or slower. They wait until the last moment to begin, and until the teacher has given as many indications as possible. They work with less autonomy. Using a computer, we can more easily identify shifts from the predicted possible activities towards reduced and modified activities.

In each case, we try to identify the reductions, modifications, or enrichments of the activities with respect to the activities predicted by the *a priori* task analysis.

Differentiation among students

By considering real students, even if we do not always examine individual students, we come quickly to questions of differentiation. Several aspects of this differentiation can be investigated. The heterogeneity of classes depends both on differences between students, and of the composition of the class.

Our tools allow us to look at the first aspect from the viewpoints of student results and of teachers' reaction to these differences. This latter includes all the adjustments improvised by teachers, particularly while presenting the correct answer or providing constructive assistance (see further on), which can give us information regarding this differential consideration of students.

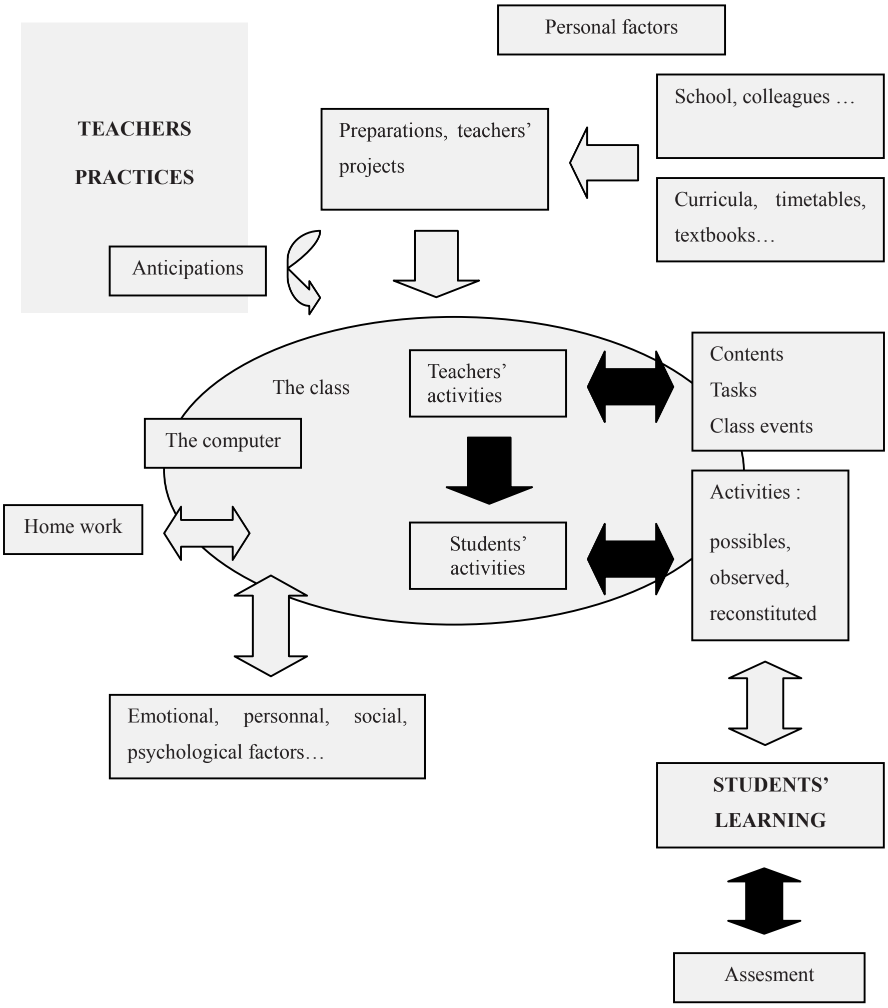
Students from the Zone of Educational Priority (ZEP)[5] have overall representations and general conceptions of school when they enter class. They may be in an environment where school is an unknown or undervalued institution. Parents in the ZEP may be disappointed by school, and young people may expect little from it. Students may be confronted by new external demands that are apparently independent of scholastic acquisitions. This can have consequences on the way they see mathematics. Effectively, if their relationship to knowledge is always an "action" (perform calculations, solve exercises, work on assigned tasks; see Charlot, Bautier, & Rochex, 1992) this does not prepare them for the necessity of seeing concepts as mathematical objects.

In class, under various conditions, students begin working on tasks and developing activities. To have positive effects on the conduct of the class as well as student learning, these tasks must be calibrated precisely to be neither too simple nor too complex. Students must move beyond the initial oral solving phase to written work and then to finding links to the knowledge goal. There are many sources of active or passive resistance for students that can be difficult to overcome.

We also observe student micro-actions that are tied to classwork in terms of overall knowledge. Not taking off a coat may indicate that the student has not fully transitioned to class time. Other such micro-actions include not listening from the start, not paying attention (or only rarely), chatting with other students, doing nothing, etc. These micro-actions can also be a permanent obstacle to engaging in the mathematical activity of the class. In particular, a short attention span can prevent students from retaining class events, an act that requires making connections and waiting until something more general emerges. At the same time, these students may be very curious and lively at times, and want to talk frequently, answer quickly, and monopolize teachers' attention. This type of attitude may be harmful in mathematics due to the cumulative nature of knowledge, and the aim of conceptualization.

A schematic for the world of the classroom

This system is a tentative method for illustrating the subdividing of the world of the study. It is very general and will be used in different ways in each study.



Possible student activities are at the center of this schematic, between teaching practices (upper left) and learning (lower right). The arrows do not have a theoretical status, but indicate links that seem to us to be important in our subdivision. If a link is not explicitly taken into account in our research, we represent it with a white arrow. If a link is taken into account, or even drives our research, it is represented by a black arrow. The objects of our research are highlighted in light gray. We highlight in dark gray elements that are observed and analyzed.

Our analyses are narrowly focused on mathematics in school situations, and attempt to take into account the relationships between the individuals involved. However, as shown in our schematic, we focus primarily on the "hands-on" teacher variables. These variables are weighted by factors beyond the classroom, which we do not have the means to completely account for.

As discussed above, our analysis of possible student activities aims primarily to estimate their learning. Our analysis of teacher practices aims to examine[6] their effects on student activities. Although we do not directly consider external factors on students, we do include some external elements in our analysis of teachers. By taking into account important institutional, social, and personal factors that affect practices, we can better weight, understand, and interpret what takes place.

However, while we can partially reconstruct elements of representation or an overall path, some elements necessarily remain inaccessibly, particularly personal elements. Specifically, for teachers we favor elements of the analyzed situation, such as the mathematics class and the students. These elements, consciously or pre-consciously, depend on actors.[7] While some indicators, particularly in some speech analyses, reveal phenomena specific to individuals, they only reinforce other analyses regarding teachers' choices. These choices primarily concern the mathematical content to be taught and class management. They may be *a priori* and/or partially improvised, but they are not unconscious.

GLOBAL ANALYSES OF THE MATHEMATICS TAUGHT AND LEARNED IN A CLASSROOM: LEVELS OF CONCEPTUALIZATION, TYPES OF CONCEPTS, RELIEF MAP

Overall, the way we choose to characterize the mathematical content to be taught should be able to serve as a reference to our analyses of teaching and learning.

We describe students' "acquisitions" in our analyses in terms of level of conceptualization. Following, Vergnaud, we define acquisitions with reference to a set of tasks whose intended resolution requires the reorganization of new knowledge into previously learned concepts, as well as the availability[8] of a certain number of aspects of the concept. These aspects include objects (definitions, theorems, properties) and tools (contextualized, within different frameworks and registers). The definitions of "tools," "objects," "frameworks," and "registers," which we adapt from those of Regina Douady, can be found in Appendix 1. We also include the definition of "viewpoint."

This starting point leads us to define a level of mathematical conceptualization to be acquired from a given curriculum and for a given concept, domain, or mathematic chapter, and contributes to defining the goals of teaching and learning (first section below). However, in order to describe the learning scenarios, we need to add other elements to this initial description of mathematics to be taught. These elements affect the choice of the cognitive itinerary to propose to students, particularly tied to the relative proximity of the new concepts to previous concepts (second section). We conclude by introducing the idea of the relief map, connected to all these aspects of concepts that we can take into account before describing a specific scenario (third section).

Relationship with the didactic transposition

We investigate the didactic transposition between theoretical knowledge and knowledge to be taught by giving ways to define what, in a given piece of knowledge, is prescribed to the study at a given moment. A historical or epistemological study is often necessary to understand what characterizes a concept, the reasons for its emergence at such a moment in history, what of it remains in a given curriculum, etc. However, as didacticians, we often use previously synthesized texts and secondary sources in our research, and do not attempt to advance epistemological research, for example, even if we re-open some questions (Dorier, 2000; Bridoux, 2011).

Level of conceptualization

For us, a level of conceptualization is a fairly large and coherent domain of mathematical work that is at least partly taught (or to be taught). It consists of:

– Fundamental axioms, either specific to the domain or borrowed from other mathematical fields. These may remain implicit at certain levels.
– A corpus of definitions (objects), theorems, and propositions. We call this corpus the level's "arsenal."
– Reasoning methods, steps, and a specific degree of rigor.
– A set of problems that can be resolved within this level.

Within a given level of conceptualization, work may take place in several different frameworks or registers. For example, the geometrical frameworks of points, vectors, numbers, analysis, or figures can all coexist, as can the different registers of Cartesian, polar or barycentric coordinates, various vector notations, complex numbers written in algebraic, trigonometric, or geometric forms, etc. The systems of representations (most notably registers) presented in chapter 1 are again in play here, with the ability to choose such a system and to pass from one to another as an important issue.

The coherence of a level of conceptualization refers to the possibility of establishing a domain's arsenal using only the fundamental axioms and initial definitions. In other words, the domain's theorems can be proved internally, using the domain's tools. This also applies to the field of problems attempted once the arsenal has been acquired. This should not be taken to mean that this should be done with students, nor that there are not other, external, methods to achieve the same results.

An example

Two levels of conceptualization underlie the geometry taught from middle school to the first years of university mathematics: "Euclidean geometry" (taught primarily in middle school), and affine and affine-Euclidean geometry (taught in the first years of university mathematics and in preparation for the teaching examination, and introduced surreptitiously in high school).[9]

In "Euclidean geometry," the fundamentals, the work following these fundamentals, and the methods of reasoning all come from Euclid. However, real numbers and area formulas are also included.

We note that these levels do not overlap, even if the body of problems that they can be used to solve may be partially shared. A certain amount of additional generality is acquired in "affine and affine-Euclidean geometry." In addition, there is no strict chronology of these levels in school, with occasional borrowings from a level of conceptualization that has not yet been presented (particularly analytic geometry beginning in middle school, juxtaposed with "Euclidean geometry").

The Erlangen program provides another level of conceptualization in geometry that we will not discuss here. The axiomatic geometry developed by Hilbert seems to us to be another candidate for our categorization, and thus a fourth example of a level of conceptualization). We will not describe these two levels of conceptualization, as instructional content is not organized on those bases.

Levels of conceptualization are not simply extensions of one another.[10] A different level of conceptualization is another way of organizing knowledge. Depending on the case, it may represent a generalization (from affine and affine-Euclidean geometry to the Erlangen program, for example), or a different focus on the fundamental axioms (from "Euclidean geometry" to Hilbert geometry), or a change of fundamental axioms (from "Euclidean geometry" to affine and affine-Euclidean geometry).

Relationships with conceptual fields

Vergnaud's (1990) conceptual fields are defined in terms of students mathematical learning. It is up to the author, as noted in chapter 1, to provide a framework that "allows for the understanding of the connections and breaks in learning in children and adolescents." The levels of conceptualization that we introduce are much more modest. They are only tied to mathematical knowledge, as developed throughout history and presented in school curricula.

The common use of the word "conceptualization," however, indicates a shared preoccupation with mathematical learning. In our case, these levels organize the mathematical knowledge to be transmitted and contribute to characterizing the expected work at each level, in relation to the variables associated with learning. The theory of conceptual fields enables the conception of a cognitive organization that students should attain for a given conceptual field and an appreciation of the corresponding itinerary.

Types of concepts

The goal, in this analysis of concepts to be taught, is to understand the relationship between new concepts and concepts students have worked with previously. In particular, we will attempt to deduce the characteristics displayed (and clearly taken into account in school curricula) of reasonable methods of introducing these concepts. One concept may have several introductions, particularly according to the progression previously adopted by the teacher.

We have identified three general types of concepts: Extensions of old concepts (with or without "crashes"), RAP concepts (responses to a problem), and FUG concepts (formalizing, unifying, and generalizing). We will discuss their respective introductions later on.

In a mathematics curriculum, "new" can include new concepts (trigonometry in 8th grade, for example) but also new frameworks (the graphical or algebraic frameworks in middle school), new objects (scalar product, introduced in 11th grade), or new theorems and properties (Thales' intercept theorem or the Pythagorean theorem in middle school). We focus on three characteristics that distinguish new concepts (or objects, or frameworks, etc.) from old concepts, and that lead to specific student work. We also analyze the function that these new concepts fulfill in the mathematical landscape where they are introduced. We define these concepts through the combination of multiple such characteristics. Our hypothesis is that each type of concept presented can be introduced in a specific and adapted way.

The *generalizing* characteristic appears when the new concept is broader than the one students currently have available. The new concept extends the old one, including it to various degrees. It may extend the domain of application, or introduce generality where there was specificity. For example, the scalar product in space generalizes the planar scalar product. Functions can also have this characteristic at the beginning of high school, as students progress from specific, affine functions, defined by their algebraic expression, to general functions, defined as a series of calculations that may not be explicit.

The *formalizing* characteristic is found in the introduction of a new formalism. This new formalism may be more or less "invasive," and is occasionally used in a limited fashion before its official introduction. New vocabulary (formulations) and symbols may occur in the formalism. For example, the formalism of the framework of elementary algebra is new, particularly due to the appearance of x. However, it also contains previously acquired symbols such as =, +, etc. and written numbers. These symbols are not always used in the same way in algebra and in elementary arithmetic. The equality sign, for example, represents in algebra not only a result but also equivalence. This is a "crash." As another example, integrals may be introduced to formalize the calculation of the area under a curve. (Integrals also have a unifying characteristic, even if it is not always displayed.)

We highlight, however, that some concepts may have multiple coexisting formalizations, which may or may not fall under different frameworks. This can be seen occasionally when the same name is given to objects. The relationships between these formalisms are not always explicit. The organization of knowledge and their representations may be hidden. Authors such as Duval (1995, 1996) have done substantial work on the non-congruent correspondences between different registers (writings). He suggests the effectiveness of explaining this aspect of the formalism, assumed to be non-transparent for students.

Examples:

– Cosine (trigonometry in right triangles, scalar product, function);

– π (formula for the area of a circle, formula for the perimeter of a circle, complex exponential);
– exponents (arithmetic, with base *e*);
– linear functions in the plane in the geometric sense (vectors)
– Various theorems

There are also concepts that, even if they are sometimes used implicitly, are not yet formalized or formalizable at a school level (Robert & Pouyanne, 2004). Arsac (1998) gives the striking example of the distinctions between what we have the right to read and say on a geometric figure (tied to concepts of convexity that are implicit in middle school), what we have the right to read without even saying (tied to concepts of area), and what must be said. Chevallard's proto-mathematical and para-mathematical concepts are of the same kind.

The *unifying* characteristic indicates that the new concept regroups, brings together, or replaces several elements that were previously treated separately. This unification is often accompanied by a simplification, but potentially also by a loss of clarity relative to the elements that were replaced. Algebraic expressions, for example, when introduced in the new framework of elementary algebra, have a unifying characteristic. The symbol x can equally designate a variable (when statements have an implicit "for any x), an unknown (when statements are only true for certain values of x), a parameter, a generalized number that may be an integer, decimal, fraction, etc. Functions also have this unifying characteristic. A function cannot be reduced to its algebraic formula or to its graphical representation, and point, global, and local examinations are necessary to characterize it.

The vector spaces introduced at the beginning of college allow polynomial, series, or vector spaces to be treated in the same way.

We characterize an initial type of concept: some concepts (objects, theorems, etc.) are extensions of older concepts. This may be because they have a generalizing characteristic, or because they are expressed with a formalism that extends a previous formalism. There are "crash-less" extensions, for which old and new work is congruent, and "crashing" extensions, which involve a change in the type of work. Multiplication of decimals, for example, is an extension of integer multiplication. There are no crashes in meaning between the two types of multiplication, but there is a difference in the solving algorithm. The scalar product in space is a crash-less extension of the planar scalar product.

A second type of concept corresponds to concepts that are viewed more as objects, and that are introduced to answer a problem. The problem may be formulated in terms that are accessible to students, and students may be able to begin a solution to it. These types of concepts have two characteristics, which may be generalizing and unifying or unifying and formalizing (for example). The Pythagorean theorem may be introduced as the solution to the problem of finding a general relationship between the lengths of the sides of a right triangle. This theorem unifies various specific situations for which students know how to perform the calculations. This is a concept of the type we call RAP (Response to A Problem). Another example of this type is the integral, when seen as the area under a curve (Robert & Rogalski, 2004). The barycenter may also be introduced as an RAP.

Finally, some concepts will have all three characteristics at once. We call these the "FUG" concepts. FUG concepts allow additional generality while unifying different pre-existing objects using a new formalism. This new formalism often offers simplifications. Two examples have been developed: series convergence and vector spaces (Dorier, 1997; Robert, 1998).

Relief map – Student difficulties and naturalization of knowledge by teachers

The "relief map" of one or more concepts to be taught is attached to the set of elements that allow us to define what is useful for the researcher (and teacher educator) to know for analyzing teaching. The map is attached to one or more curricula, and includes the mathematical characterization of these concepts. This leads to defining the intended level of conceptualization at a given moment during instruction, as well as the type of concept. The tool and object are specified, along with their integration in previous curricula and in the assumed prior knowledge of students. Other elements involving students, beyond the conceptual structure of the situation, are also defined as cognitive subjects (chapter 1).

We note here the list of previously reported student difficulties, if possible. For example, from numerous didactic studies in elementary algebra, we can introduce the idea that it is necessary to work specifically on the gap between arithmetic and algebra. The difficulties associated with this gap, such as with the new status of the equals sign or the stress on numerical proofs, are often underestimated (Grugeon, 1997).

All this allows us to specify the meaning that concepts can take at a given moment of schooling, as well as their place in the landscape of student knowledge. In addition to accessing the distance between new and old, it allows us to identify potential pressure points for teaching and foreseeable obstacles. It also facilitates our understanding of proposed activities covering the concepts, of the design of introductions to the mathematic chapters involved, and of the subsequent mathematical work to organize for students, as well as comments to develop and traps that may arise.

Finally, in terms of practice analysis, this facilitates the necessary research in the naturalization of teacher knowledge. The term "naturalized" knowledge refers to knowledge that has become transparent for professionals, but not for students. This knowledge may involve choices of frameworks or changes of viewpoint, for example. Identifying this knowledge contributes to a better appreciation of student progressions (cf. examples in elementary algebra by Lenfant, 2002). The passage from a given right triangle to the use of the equivalent property that two straight lines are perpendicular or that a certain angle is a right angle is an example of a viewpoint change with information loss. The given information is not only translated into other words within the same framework (change of viewpoint), but also we are no longer considering the triangle as a whole. We retain only the two sides forming the right angle or the angle that they form (placing us in a geometry with explicit measure).

Using a global view of the relief map on a studied concept facilitates the focus on key elements that may intervene in teaching or learning.

To go further in the analysis of taught mathematics and attempts to give a useful relief map to teaching, it is possible to introduce levels of conceptualization that cover all of formal schooling (Dorier, 1997; Robert, 1998). The concept of conceptual field (Vergnaud, 1990) is another way to access this attempt that is perhaps more adapted to the first degree. Very generally, it is useful to analyze networks of concepts that are studied together (Robert, 1992). The analyses developed by Chevallard (1992), more systematic than those indicated here, allow a very complete approach to taught mathematics, from the starting point of didactic transposition and decision theory.

A PRIORI LOCAL ANALYSES OF MATHEMATICAL TASKS

A mathematical task is, here, very generally, attached to a given statement proposed to students. It is characterized by the use of old and new knowledge to solve it. The various ways of using the knowledge are determined according to course content (theorems, definitions, properties, examples, solved exercises, etc.). What interests us here is the way (or ways) in which students can use their knowledge in the exercise. This allows us to predict possible students activities for a given problem statement, particularly in class.

These analyses are called *a priori* as they may be based only on the problem statement, without examining cases where the problem was solved by one or more students or by a class accompanied by a teacher.

The *a priori* analysis of a task leads to asking, for a curriculum, what the role is of exercises in in-class work, and what the use is that students will make of their old and new knowledge in working on the problem. This analysis, then, does not refer directly to the potential learning benefits of an exercise. We are only trying to find what activities students will be able to take part in for this exercise, with their supposed knowledge (curricula, previous lesson content, etc.). But even if they do not explicitly appear in analyses, the choices made for describing these activities are certainly not independent of hypotheses concerning learning.

For example, we determine whether or not the knowledge to be used is indicated in the problem statement, and if so, whether this indication is direct or implicit (an implicit indication may be given by the placement of the problem in the lesson progression)[11]. If the knowledge to be used is not indicated, it may be assumed to be readily available for students. This indicates the necessity of a specific and fundamental activity for students to allow them to access this knowledge or think of using it (two activities that are difficult to dissociate). We hypothesize that this activity may contribute to constructing the desired availability.

We first distinguish simple and isolated tasks (SIT), or immediate applications of a piece of knowledge without adaptation or combination. A single piece of knowledge is used, potentially with simple replacement of general inputs by the given information in the context of the exercise.

Different levels of knowledge use

When tasks are simple and isolated (SIT), we speak of student work at the technical level. When tasks[12] require adaptations of knowledge that are at least partly indicated, we speak of the level of knowledge application that can be mobilized. Students' work is not effectively analogous, depending on whether they must look for the knowledge to use (questions of why or what), or apply and adapt the indicated knowledge (questions of how). If it is up to the student to recognize the knowledge to use, we speak of the available level of knowledge application.

Rising to a certain level of knowledge application for a given task requires that the student's work on the task involve the knowledge at this level. Either this is possible for this student, and the knowledge is perhaps reinforced, or it is not initially possible, and working on the problem will perhaps contribute to transforming the student's knowledge until it is possible.

Knowledge adaptations

For other tasks (else than SIT), we determine, for each relevant piece of knowledge, the adaptations that students must do, in relation to the required recognitions, initiatives, additions, and combinations (Robert, 1998; Robert & Rogalski, 2002). This allows us to characterize individual problems, each of which may involve multiple tasks. These analyses clearly depend on the given level of schooling or the given class. We also keep track of the set of proposed tasks and their repetitions.

Recall the importance, accepted for mathematical learning, of the variety of contexts encountered, and of their interactions, particularly changes of frameworks, registers, viewpoints, and combinations of old and new.

We have developed a list of seven adaptations. We completed this list by considering activities students may have to perform using raw pieces of knowledge, and distinguishing among them recognition of properties or procedures or procedure application, or what is an introduction of intermediaries or steps, which seems to us to be another very important mathematical activity. We also distinguish combinations, links, or changes among elements such as frameworks, and work further on different types of intellectual activities that are specific to mathematics.

These adaptations (identified with a code of type Ai) may occur simultaneously. Each has a fairly large (and again, relative) spectrum:

- A1. Partial recognitions of ways of applying concepts, theorems, methods, formulas, or other types of knowledge. For geometry, this typically consists of recognizing configurations, using Thales' intercept theorem, etc. … This can range from recognizing variables and notations to recognizing formulas, conditions of applying formulas, etc.
- A2. Introduction of notations, points, or expressions as intermediaries. In geometry, this typically consists of introducing a parallel line, or naming a point to use Thales' intercept theorem.[13]
- A3. Combinations of several frameworks or concepts, point of view changes, framework or register changes, connections, or interpretations, etc. In geometry,

this typically consists of using algebraic calculations for obtaining the result (for example, solving $x^2 = 1$ within a geometry problem). Problems that involve graphical/algebraic aspects of functions automatically contain this adaptation.

– A4. The introduction of steps, or the organization of calculations or reasoning processes. This can range from the repeated use of the same theorem to reasoning *reductio ad absurdum* using this theorem. In geometry, this typically consists of using Thales' theorem and its converse four times, non-independently. The steps can be classical, somewhat forced (in examining a function), or to be determined.
– A5. Use of previous questions in solving a problem.
– A6. Using choices, which may or may not be forced (only one will lead to the correct answer).
– A7. Lack of new knowledge.

Let us examine the following exercise, as an example of a problem given at the end of middle school: "*Show that the product of two numbers, each of which can be written as the sum of two squares, can be written in the same way*."

An initial activity will be to understand and formalize the given sentence. What is "a number that can be written as the sum of two squares"? How should "can be written in the same way" be interpreted? We can note that the question is open. One step is imposed on students: to know how to form a conjecture on the result to be demonstrated. This may induce numerical experiments, but we can suppose that they will not change the search for a proof. An "elementary" response is: "If m, n, p, and q are integers, we can write:

$$(n^2 + m^2)(p^2 + q^2) = n^2p^2 + m^2q^2 + 2npmq + n^2q^2 + m^2p^2 - 2nqmp$$
$$= (np + mq)^2 + (nq + mp)^2$$

And thanks to the stability of integers under addition and multiplication, we can conclude the desired result."

In addition to elementary algebraic manipulations, it is necessary to introduce an intermediary: the algebraic expression $2npmq$ that we add and then subtract. The useful identities are, in this method, knowledge items that must be adapted.

Another possible response uses complex numbers. This represents a change of framework, as the problem statement was arithmetical. Under this method, each sum of squares is identified as the square of the absolute value of a well-chosen complex number (the intermediary). We thus write $n^2 + m^2 = |m + in|^2$, and then apply the rule that $|z|^2|z'|^2 = |zz'|^2$ (rule adaptation) and invoke the stability of integers under addition and multiplication.

In 12th grade, when students have learned complex numbers, there can therefore be a strategy choice for students.

In the case of computer-based lessons, new tasks appear in addition to the associated new activities. In all cases, task analysis should take into account the software environment, which can simplify or complicate the proposed tasks and modify the possible student activities. To a lesser extent, the analysis of textbook tasks can also take into account the environment. In that case, it consists of external indications such as titles or images that may or may not help the student's activity.

We note that there are other task analyses that less directly involve the specific knowledge to apply for a given exercise, but instead refer exclusively to the nature of the expected work: conjecture, search for a proof, applying a procedure, etc. (Stein, 1996).

Analyses in terms of competencies focus on the broad types of intellectual activities, defined independently of content (looking for information, beginning on a process, communicating, etc.). These do not refer directly to the knowledge involved, introducing a fundamental difference between competencies and knowledge. However, a competency allows for the possibility of solving tasks that are not simple, that are varied, and that are not "ready-made." This includes real-life issues, and supports the possibility of diverse available knowledge adaptations, which allows for the use of the previous analyses.

Other approaches will be presented in the last section (particularly a decision theory analysis).

A POSTERIORI LESSON ANALYSIS

Global analyses of scenarios presented to students

For a given instance of instruction, the "scenario" refers to the intended ordered set of exercises and lessons for a mathematic chapter or concept[14]. It includes evaluations and homework, with rough predictions of management (length, division of work). A scenario is understood both in terms of its "internal" qualities, on which depend the set of activities that we can predict through *a priori* analyses, and in terms of the actual events it permits, beyond its own content.

We study the overall scenario, first under the predicted method of introduction, relative to the specifics of the concept (particularly its similarity to previous concepts). This is the first introduction of the meaning of the concept. We then examine the scenario's lesson/exercise dynamic and the corresponding dynamics, which can exist between meaning and techniques (cf. what Douady in particular calls familiarization or reinvestment). Finally, the quantity and the nature of the proposed tasks are associated with the evaluation of the adaptation work proposed to students. These elements (various dynamics presented, adaptations under student initiative, etc.) are also elements that allow us to think that the implementation of a scenario would allow a larger availability of the concept, and a certain (re)organization of knowledge.

This does not completely prejudge what may happen in class. It is the analyses of class events that can provide information of this point. We call a scenario "robust" if it can play out in multiple ways in class without modifying the main element, student activities.

Scenario analyses allow, among other things, shedding light on the double role of the teacher's work:

– Design scenarios that aim at constructing meaning out of what is taught and the necessary technical acquisitions; and
– Organizing classroom events to allow convergence between what was envisioned and what takes place in reality.

We will now discuss the two dimensions mentioned above: the introduction of concepts, and the contextualization/de-contextualization dynamic.

On concept introduction

The problem is made difficult by the diversity of concepts introduced to students in curricula. In our eyes, the introductions depend on the concepts in play. From this, for us, one important variable involves the identification of the desired concepts (detailed in the previous section of this chapter) and its appropriateness for the proposed introduction.

The question of concept introduction seems to us to be addressable thanks to our classification of types of concepts.

A concept deemed an extension may be, for example, introduced through a problem, constructed with the "old" knowledge. The problem will be accessible, since the new concept is an extension, but the solving of the problem will require the new concept.

The problems arising from work on the tool/object dialectic also seem to us to correspond to extension-type concepts, particularly if the extension is without a crash (see above). These problems allow us to broaden the meaning of the new concept and the associated technique, provoking for students an initial use, as an implicit tool, of what is intended. If there is a crash, we can still introduce the new concept in the same way, but the crash can lead to an error. It is the quality of the problem, particularly in terms of the predicted internal methods of control (if possible) that will allow the obstacle to be surmounted.

The concepts that may correspond to a mathematician's responses to a problem (RAP) may be introduced by giving students a problem that they can appropriate and understand, but not solve. In our classification of adaptations under student control, this corresponds to adaptation A7 (see below). It is the teacher that will introduce the new concept, particularly the new object with, if necessary, its formalism. While working, students can partially test out to what the new concept brings a response. We understand that these introductions are more relevant to the meaning of concepts than to new techniques (particularly if there is a new formalism as well).

For FUG concepts, we hypothesize that there is no problem adapted to introduce them with meaning. The introduction that we can suggest is very partial. It is possible that the optimal strategy is to present part of the knowledge first (or very quickly), and then give students a problem to give it meaning (Dorier et al., 1998, on linear algebra; Robert, 2011, on series convergence). Unless the nature of the tool can emerge from what students already know, without them noticing that they are missing something, this process consists of leading students to work with these concepts. We may ask whether the introduction of "informational jumps" (Brousseau, 1998) does not have something to do with this type of factor. Students must be "forced" to use the elements introduced in the course, even if they are not familiar and do not initially seem efficient to students. In the case of informational jumps, there is the idea of a "forced" efficiency. For us, there is even a contract effect.

The contextualization/de-contextualization dynamics and the variety of proposed adaptations

Students' activities prepare them, to a certain extent, for the intended conceptualization. Inspired by theories of learning and research results in mathematical didactics, we have identified dimensions that may influence the quality of activities proposed to students, in relation to this intended conceptualization, and therefore the intended availability of concepts, tools, and objects. What we call the "meaning" of concepts corresponds to this characteristic of acquisitions. Adding meaning translates (and is translated by) the possibility of applying the concept wisely, in various contexts. In light of our hypotheses, is implies (and is implied by) student work that puts into play, in a dialectic manner, the tool and object characteristics of the concept, tied to the exercises and lessons, and the organization of new concepts in the entire knowledge set, tied to the variety of the proposed tasks. Ideally, this consists of introducing new knowledge within a certain continuity as much as possible. Knowledge should be introduced alongside old information, within a context that allows a particular tool-type use, relative to the intended level of conceptualization. This assures (in part) the possibility of the beginning of an autonomous construction. This implementation also assures the possibility of moving past this, thanks to a teacher that presents decontextualized object characteristics, which may be re-contextualized in other ways due to the introduced generality. We see clearly the importance of choices of tasks and lessons (knowledge presentation).

An initial issue thus relates to the way in which knowledge is introduced, in relation to the exercises. The order of what is applied during the exercises and what is presented in the lesson allows us to understand the connection between what is worked on in context and what is presented out of context.

The tasks proposed to students introduce them to diverse aspects of the concept, and diverse ways to work on it (adaptations). It is in studying the desired implementations of knowledge, and the relative variety of the adaptions, in relation to the specifics of the concept, that we can understand the span of the intended knowledge and the foreseeable reorganization of the new into the old.

We should note that each concept requires a specific analysis. An example of discussion on the scenario is given for instruction of orthogonal symmetry in sixth grade (chapter 7).

It should also be said that it is the in-class activities that lead to all these tasks. Their analysis is indispensible for understanding what is in play. These analyses are prefaced by specific *a priori* analyses of tasks, and analyses of in-class events that are discussed in the following section.

Local a posteriori analyses: in-class events

A number of factors influence students' in-class activities and, as a result, the knowledge created. These factors contribute to encouraging the transformation of presented information into individual understanding through the intermediary of student activity. There is, however, no general law that connects teaching and learning. We can only note common elements, which depend strongly on work conditions, classes, and the types of knowledge in play. For example, an action, even if repeated, does not necessarily generate a construction of knowledge. There must be a transformation of this action into an activity.

Be that as it may, it is the analyses of the relationships between the expected tasks and the events organized by a teacher during a class period that allows the researcher to understand what are the possible student activities. It consists of giving ways of answering the following question: to what extent have students performed the activity that was expected from the *a priori* task analysis? What was the nature of their activity? Recall that in general, we cannot claim to be able to access the effective activities of each student. We can only access their possible activities. However, these local analyses remain partial and should be connected to the set of what is proposed to students.

A number of examined elements contribute to the development of the answer to our question. First, the form and nature of the work are recomposed and divided into episodes (see below). These elements clarify student activity, and lead to inferring its potential non-didactic characteristics, the possible role of inter-student interactions, the importance of the activity during the class period, etc. One important source of observable elements is tied to factors added by the teacher, whether in soliciting student responses, responding to students, or developing a didactic project.

In any event, it is the supplementary analyses that allow us to recompose all the information and suggest a reading of possible student activities in a given class period.

The didactic contract,[15] along with the habits, customs, and memory of the class, also plays a role in learning. Students may, for example, engage in a task because they have understood that it is expected of them by the teacher, and not for mathematical reasons. What do they learn from this? We also take into account this type of overall interrogation.

We indicate below several dimensions that may guide our analyses.

Nature and format of student work, including autonomous work, group work, written work, oral work, etc.

In our analyses of class periods, we note, in addition to the duration of student work on the tasks proposed to them,[16] the format (as a class, in small groups, etc.) and the nature of the in-class work (re-copying, reading, calculation, investigation, written or oral, graded or not, etc.).

This allows us to bring to light, at least in part, the autonomy given to students (including not doing what the teacher expected), the role of exchanges between students, and the possibility or necessity for students to take initiative, whether tied to the intended adaptations or to others. This should naturally be completed by the manner in which this work is "recuperated" by the teacher and related to the supposed state of student knowledge.[17] To the extent that we hypothesize, following Piaget and Brousseau, the importance (and, indeed, necessity) of individual moments of knowledge construction, we understand the value of occasions where one or more students confront a problem autonomously. We therefore identify occasions where students are left to work on their own, either in the long term (a non-didactic phase) or not. During these phases, the teacher has no influence whatsoever on students, neither by assistance nor by direct or indirect indications. The nature of students' mathematical activity then depends on the object of the work, in relation to their knowledge: preliminary investigation following the introduction of a concept, or solving an exercise in a given mathematic chapter, or during a problem that cuts across several domains, etc. There are many parameters to include in analyses during the reconstruction of possible student activities.

From this point of view, class periods involving computers interest us particularly to the extent that autonomous student activities is generally more present *a priori* (see also chapter 8).

To the extent that we hypothesize that student exchanges, during interactions between students,[18] represent socio-cognitive conflicts, and that teacher interactions influence student activities, we aim to take this into account. This is particularly true in analyses of the existence and nature of the exchanges (predicted or not by the organization of the work provoked by the teacher), as well as analyses, during group phases, of the verbalizations (formulations, formalizations) requested from students. We will return to this below.

Studies in education sciences, particularly in the framework of socially underprivileged students (Bautier, 2006) has long insisted on the importance and specifics of students' written work (in all disciplines). Written work is an occasion of distancing oneself from action. We hypothesize that this is something that must be addressed in knowledge constructions. In mathematics, in particular, this written work is both a method of representation and an instance of work in formalization or symbolization. There can even be some unexpected creativity in mathematical production based on symbolic writing (drawing a figure or writing a formula on paper can put students on the path to a proof or interesting calculation that was not anticipated).

In class, the use of written work is fairly variable, as is its relationship with oral work. Very different forms of written work exist (provisional or draft work, for example). We can hypothesize that the effects of this work are not the same for different students. This dimension is still under construction in our research.

The teacher's written work (particularly at the board) is another object of study. We have previously shown the regularity of forms used by each teacher (Robert & Vandebrouck, 2003). One important question involves the role given by teachers to their own written work. Is it a simple translation of what they said aloud? Are there transformations between written and oral work? Are they indicated or implicit?

Does the writing on the board or on handouts serve as a model for students' written work? There are many such questions, whose answers can bring light to the corresponding student activities, and particularly what is left under their control (which can be a source of potential misunderstanding for students who do not decode it).

In the case of computer-based classes, this written work, and particularly its articulation with the machine work, becomes a very important variable for student activities (see chapter 8).

Teacher interventions

Multiple aspects of teacher intervention were analyzed, always in terms of their supposed influence on student activities. Some relate to the format of interactions with students, and others concern the content of the interventions (assistance, assessment, reminders, explanations, corrections and evaluations, presentation of knowledge, mathematical content, etc.). In the background, we find the attention given to the identification that the teacher makes of students' visible work, and to the possibility of profiting from it, by calibrating interventions to knowledge that is assumed to be "close" to students' level. Everything that contributes to this identification, such as questions, answers, or throwing the initiative back to students, may also be an object of study. Aspects of interactions with students have long been studied by numerous authors,[19] without considering the content in play (Postic, 1989).

One important variable that can affect the importance of interventions relates to the student knowledge to which these interventions relate, and more precisely to their degree of proximity to students' previously acquired knowledge.[20]

Focus on assistance

We define the nature of teacher assistance, identifying the moment when the assistance what given, the nature of this assistance, and the format. We present two types of assistance according to whether they modify the activities predicted *a priori*, or whether they add something to students' actions.

The first type, said to have a "procedural function," involve the assigned tasks themselves by strictly modifying activities relative to those predicted from the *a priori* analysis of the problem statement. They correspond to indications given by the teacher before or during student work, and include open-ended questions such as "What theorem can you use?" They may lead to subdividing the task into explicitly mentioned subtasks, or to having students choose a contextualized method. This changes the necessary adaptations, and can orient the activity toward more immediate processes.[21]

The other type, whose function we call "constructive," add something between the specific student activity and the desired knowledge construction that can form as a result. This may be through a simple summary of what was done, even in an immediate application (for a simple isolated task), or by reminders, partial generalizations, assessments, etc. All kinds of interventions lead students to gain perspective on what they have done, to find a slightly more general method, to discuss results, etc. This assistance can present a small de-contextualization of what students have done, by presenting the corresponding generic case, for example. It can also indicate how to do this type of task, or explain certain choices.

For a given student, procedural assistance can immediately become constructive, if the student extracts a generalization. For a different student, constructive assistance can remain procedural. If the value of the generality is not understood, the student will forget it once the exercise is over and never use it again.

Recall that adopting the framework of the theory of activity leads is to hypothesize that there is another “internal” transformation of the subject. In solving a problem, the student must live between the solving and the knowledge that will be potentially constructed through this process. This constitutes a depersonalization, a generalization, and finally a de-contextualization and/or an organization. Constructive assistance participates at least at the beginning of this process.

This process is often unsuccessful, especially for students with the most difficulties in the subject. It is one of the most fundamental issues for teachers in Zones of Educational Priority. We think it can also be assisted by developing the knowledge as a group, before assessments. This can lead students to make a place for this knowledge before they have it (Butlen & Pezard, 2003). This is another form of intervention to facilitate the previous process.

We note also that the way in which teachers consider individual students in group interactions to further their goals for the class is also a factor in the potential influence of assistance. Will the teacher look for information to regulate the interventions, particularly during the presentation of the correct answer? Will the teacher rely on the strongest students? What assessment will the teacher make of the class from individual assessments?

Focus on the quality of the speech (linguistic functions and linguistic markers)

Some finer characteristics of speech contribute to modeling student activities, particularly during their work on complex tasks (chapter 4). These finer characteristics include the nature of linguistic functions engaged during interactions, the regular use of certain linguistic markers at certain specific moments, and other characteristics tied to indicators that are yet to be determined. Through this type of speech analysis, we can better understand the way to accompany and influence student work. Whether this reinforces teachers’ other choices, or compensates for them, is an open question.

In some research, supplementary indicators are used to analyze practices. These indicators relate to teachers’ automatic actions, such as simple routine professional gestures (including oral gestures). This analysis leads to defining the speech presented to students more precisely, at finer levels, including aspects of which teachers are not aware.

A focus on correction phases and lessons (phases of knowledge presentation)
Phases during which the teacher makes use of student work represent an important variable in student activities, related to the synthesis of their actions.

We differentiate several types of correction phases (oral, written, continuous, at the end of an exercise, by the teacher, by students, etc.). This type of reflection may be compared with reflections on the role of errors in learning.

Hidden behind some errors are false or incomplete representations that may remain in place for students if nothing specific is said on the subject, or if nothing is asked of them during the correction phase to bring them to light. The presentation of the model solution can leave then in silence, unnoticed by the professor or by students.

In addition, multiple elements may come into play during the correction phase. These include the specific solution to the questions, but also what "worked" in the exercise (de-contextualization of the method), how to write it up, etc.[22] In particular, when the solution to each exercise is presented as students complete it, more general aspects of the exercise may escape students unless their attention is brought to it, overall, at a given moment.

In a way, the correction phase can be a fruitful time for the student's action. It can contribute to the transformation of the student's action into internalized knowledge. One way this can occur is if the student has succeeded at the assigned task and the teacher's confirmation and summary has allowed the student to retain some aspects. Another is if the student did not succeed, but the teacher has responded to the student's attempts and helped complete them, allowing the student to progress. We do not see all correction phases as equivalent. They should both add generality and be close to students' processes. The teacher should remain, if possible, in students' proximal development zone (PDZ).

Studies on knowledge presentation phases (lessons, institutionalization when this follows certain situational formats) are few.

However, there are several variables noted, and some overall dynamics that have already been mentioned: the order in which the different phases take place, relationships between contextualization and de-contextualization, or even the format of the course (lecture-based, interactive, dialogue-based). The nature of students' activity during the phases of the course is clearly an important variable (even if it is difficult to analyze when their only observable action is listening).

We also examine the nature of the comments on the mathematics (or "meta" comments). We determine whether these types of comments, which refer not to strictly mathematical knowledge but to a larger reflection on this knowledge, or to a reasoned presentation of possible methods, exist. These comments may also consist of an explicit external structuring of the lesson and particularly the proofs (argumentations), or recounting the emergence of knowledge in relation to the problems that have appeared, for example in the case of FUG concepts for which no introductory problem is proposed (Dorier, 2000). Meta comments may also include a presentation of subsequent occasions to use the concept.

Some studies have attempted to classify methodological comments by their distance from the intended content. This classification distinguishes general comments on work, such as methods that are independent of the specific problem where they are being used, from methods specifically adapted to the problem (Robert & Tenaud, 1988).

A few other studies have focused on teachers' examples, metaphors, or formulation progressions. These studies have examined natural language and symbolic vocabularies (cf. semiotic studies), as well as differences between spoken and written communication. Finally, class handouts and documents remain unstudied.

Work outside of class

Work outside of class probably becomes more important as students progress through school. There have so far been few studies examining out-of-class work, although much ink has been spilled on the topic (concerning "coursework inflation," for example). It seems coherent with our theoretical framework to assume that homework is not independent of classwork, and may even depend on it (Félix, 2004; Rayou et al., 2010). We do not directly include it in out analyses except when it leaves traces in the classroom: Exercises given to students to complete at home that are then corrected in class, etc.

Open questions

As we noted above, there are still some open questions in this field. These include the long-term influence and, more generally, the "size" of the topic.

Some studies take into account this important dimension. They may compare, for example, the set of activities on a mathematic chapter to students' abilities at the end of instruction. This requires adapting the methodology to handle a considerable quantity of data (chapters 6 and 7).

ANALYSES OF THE PRACTICES OF MATHEMATICS TEACHERS (THE DIDACTIC/ERGONOMIC DOUBLE APPROACH)

This section is developed a bit more than the others, to the extent that it corresponds to more recent research and to an enlargement of previous frameworks. We will present a brief history of the theoretical evolutions before describing the current method of analyzing teacher practices.

We begin by noting that the word "practices" is used to refer to everything that informs us of teachers' thoughts, which may include speech or actions. The viewpoint is long-term, and includes periods before, during, or after class. The term "activities" is reserved for specific moments within these practices, and refers to specific situations in a teacher's work: in-class activities, preparation or test-writing activities, cooperative activities, etc. The word "work" is reserved for the subset of practices within a mathematics class and in preparation for this class that constitute the heart of our analyses.

This distinction is important to the extent that we believe that it is necessary to introduce concepts specific to the study of teachers' practices and activities. For example, in some studies, we examine teachers' activities within a specific situation of integrating technology in their classrooms, and adopt an approach that is directly inspired by professional didactics. In these studies, the dialectic between "productive activity" and "constructive activity" is introduced to specify that the teachers act on and transform the situation (contributing to students' activities) but also that teachers transform themselves through a long-term development process.

A brief history: What role to give to teachers in our analyses?

Many of the researchers contributing to this project work in teacher education (in Ecoles Normales,[23] and then in Instituts Universitaires de Formation de Maîtres[24] since 1991). Whether in initial training or continuing education, the difficulty of diffusing research in mathematical didactics is constant. Why is it so difficult? A scientific explanation of this phenomenon seems necessary.

Additionally, like many didacticians in recent years, our research in the links between teaching and learning has led us to focus on effective activities in class, such as student activities provoked by the teacher's speech.

Initial attempts

Some studies on mathematics teachers were conducted in the 1990s. Starting from the observation that teachers had difficulty listening to didacticians and adopting didactic inventions, we initially wondered if this might not be due to differences between teachers and didacticians in representations of mathematics, of mathematics teaching, and of mathematics learning. It was quickly shown that this explanation was not sufficient to explain the observed differences (Marilier, 1994), and was even less useful for acting. In the first place, the expressed representations to which we had access did not sufficiently translate or explain effective practices. Secondly, teachers' difficulties in "borrowing didactic elements" and the gap between the possible and the (prescribed) didactic could not be ascribed solely to people, but also had to be related to something else, particularly what we will introduce later as the "profession."

In order to explore teaching practices, we focused on in-class speech, both in terms of the mathematical content of the tasks assigned to students, as well as additional content added by the teacher.

For several years, studies have shown that there is inter-individual variability in in-class speech, in meta commentary (Chiocca, 1995; Josse & Robert, 1993) and contextualization of the same problem. Methodologies were borrowed directly form mathematical didactics. In addition to analyzing tasks given to students in terms of the specific intended content, researchers also analyzed speech through a general categorization that distinguished structure, argumentation, non-mathematical accompaniment, etc. Only students were taken into account, and content aimed at them was the only yardstick used by the analyses.

The last study of this type was Hache's 1999 dissertation, which succeeded in regrouping certain variables tied to speech, management, and content, identifying several "universes" unique to each professor from a number of possible types. Each universe was characterized by a certain combination of the nature of the proposed tasks and the corresponding management (characterized by the nature of the elements relative to the teacher's speech). A single teacher never took from more than 3 or 4 universes.

At this stage, there were still large questions remaining beyond the observation of this diversity. These concerned not only the effects of this diversity on learning, but also on the interpretation of this variability, and the consequences we can identify for teacher education. Why do teachers, whether beginning or experienced, use such or such methods to lead a class? What variation exists for a single teacher? Between teachers? At the same time, some studies on teacher education (Masselot, 2000; Vergnes, 2001) have pinpointed beginning teachers' difficulties in borrowing elements from didactics for their practices. It may be tempting to say that we will give students a "good" problem, a problem that will prepare them for constructing the meaning of a concept, but there may nevertheless be obstacles. Instruction may consist only of introductions to concepts. All concepts may not be equally suited to this type of approach. There may be time constraints that affect the possibility of letting students work autonomously. All students may not be equal, including in how prepared they are for autonomous work. Beyond the introduction, the technique must also be practiced. Finally, teachers may succeed in reaching a certain number of students through techniques that may be very different from ours. How does this happen? And how can teachers more quickly adopt effective practices?

There was both an issue of practice comprehension (and reflection on training) and a need to go beyond the results obtained on individual variability to identify commonalities and find elements that could be modified (and how to do so), without losing sight of student learning.

Enlarging the research scope

The research discussed above led us to analyze teacher practices in terms of student learning, taking into account that these practices consisted of the exercise of the same profession of "teaching."

We thus broadened our research in two ways. First, we abandoned the exclusive link between in-class practices and intended learning to enter the universe of the profession. We chose the following option: To analyze and interpret practices (and to perhaps train them), we could not ignore the fact that these practices, while having student learning as a goal, concerned the sole exercise of the teaching profession, which could not fail to induce specific choices. This represents a significant change of viewpoint for the researcher. The second enlargement was the idea of borrowing from the theory of activity, ergonomic psychology, and professional didactics. This work is far from being finished.

We note that the beginning of this work was facilitated by the fact that we were already using elements of the theory of activity in our didactic approach to learning mathematics (inspired by work by Piaget, Vygotsky, and above all Vergnaud).

Together with J. Rogalski, we designed a theoretical procedure called the "double approach"[25] to study mathematics teachers' practices. The name "double approach" emphasizes the fact that we combine didactic analyses of students' mathematical activities with ergonomic analyses inspired by the analysis of the exercise of a profession. The didactic analyses drive part of the analyses of teachers' actions, and the ergonomic analyses complete them. Additionally, some of us have added elements directly inspired by systems of development of work activities.

From the didactic point of view, the analyses begin with the choice of global and local tasks to give to students. The *a priori* analyses of tasks thus serves to decode teachers' in-class activities, incorporating the proximity of the predicted tasks and student activities, taking into account the resulting unfolding of activities. From the ergonomic point of view, we consider the activity of the teacher as a subject, and not as an element of the knowledge-student-teacher didactic triangle. The choices for the class and in the class, relative to the diverse constraints tied to the institution and individuality thus come into play. From this perspective, we acknowledge that the activity within a particular class depends on the necessities of the profession within a didactic institution. The choices should not be completely random, and studying them from the point of view of the profession allows us to better understand the reasons behind them.

The didactic/ergonomic double approach to the analysis of mathematics teachers' practices

To analyze practices, and specifically teachers' work for the class and in class, we propose here to take into account both goals (such as student learning, but also student investment)[26] and the non-ignorable, non-temporary constraints that are imposed by the profession of mathematics teacher. These constraints decline as we take into account external factors (institutional, social, and personal) as well as different scales of work.

We develop the double approach by acknowledging the complexity and coherence of practices (De Montmollin, 1984). This framework is represented by multicomponent analyses and levels of organization that are recomposed, keeping in mind that is it these re-compositions that reveal what we seek. These qualitative analyses, based on in-class observations and completed through documents collected outside of class, aim to help us understand the student activities organized by the teacher. They also aim to identify what in a practice is fixed, variable (presenting alternatives), temporary, essential, shared, individual, or able to be enriched.

A starting point: Analyses in a mathematics class

We analyze the practices of a given teacher based on the teacher's classes and organized activities. Our observed variables are student activities as organized by the teacher, interpreted in terms of the teacher's various choices. This interpretation is deepened through various out-of-class studies that relate to the analyzed class periods and allow us to complete our analyses of the observed variables.

From one class period, analyzed in relation to possible student activities, we identify the first two components of observed practices, which we call the cognitive and mediatory components.

The cognitive component corresponds to a teacher's choices regarding content and tasks, including their organization, their quantity, their order, their inclusion within a curriculum beyond the class period, and plans for managing the class period. It can be deduced based on the cognitive itinerary chosen by the teacher for a few class periods. It also allows us to predict, for other class periods, these types of choices.

Choices corresponding to class events, and to the effective implementation in class of the chosen cognitive itinerary, make up the mediatory component. These choices may include improvisations, speech, student investment and participation, instructions, assistance to students in completing the tasks, identification of their work and the work of the teacher, validations, explanations of knowledge, etc. It also includes paths developed for different students.

These components, inferred based on one or more class periods, are then reincorporated into intervention logic, which goes beyond a single class period, allowing some long-term integration, particularly in terms of tying student activities and learning. This logic also affects the personal choices of teachers, which may be otherwise examined (see below).

Our work on the stability of experienced teachers' practices legitimizes,[27] to a certain extent, this extrapolation (see chapter 4).

The profession: Integration of professional factors that impact practices

To better define the "profession," three supplementary components of practices were introduced: The personal component, the institutional component, and the social component. They represent taking into account data that are not directly observable in class, but that must be considered for understanding certain choices. They correspond to professional factors.

First, a personal component allows us to give appropriate weight to what we see in class and to integrate it within the long term. The teacher can, actually, make choices, including those tied to the long term. As, in general, we only observe excerpts from a practice over a school year, we can only have an idea of these choices if we ask the teacher, and even this approach is insufficient. This component serves also to translate teachers' representations, which are tied to their knowledge and experience, as well as the risks they take in the exercise of their profession, and the safety they need. A profession is exercised over the long term, and we cannot consent to efforts that are too great for too long. We access these elements in general through interviews, which are best completed by watching videos of the teacher in the teacher's presence (see an example in chapter 4). There are clearly aspects in this personal component that are even more specific, tied to a teacher's psyche, which we do not explicitly take into account, although we recognize their importance. We remain in the rational, working with elements that are consciously accessible, which legitimizes some simplifications.

Teachers do not choose the transfers that may emanate from their persons. They do not choose the composition of their classes, or the circumstances. They do not choose their automated actions in advance. They do consciously choose, however, the particular content that they will present and the way they will organize and present it. The conscious rationality attached to these choices leads us to favor the corresponding analyses to the extent that we are keeping in mind teacher education, and a rational form of teacher education. This recalls the choice of the possible student activities as an intermediary to access learning.

But to exercise a profession is also to respect a certain number of constraints that may prove to be more or less contradictory with what we would have wished to do on our own. From our point of view, a teacher is not free. We have defined the institutional component by the nature of the mathematics to be taught, the curricula, the schedules, resources such as manuals, the administration, inspections, etc.

We add a social component that corresponds to the fact that the teacher is not alone in a classroom. Students affect what happens in the classroom as a group and as members of social groups. The teacher is also not alone in the institution, but is subject to pressure, expectations (from colleagues, parents, etc.) and occasionally constraints, which must not be overlooked in our interpretations of in-class event. For an extreme case that we have already encountered, consider the young teacher who is strongly discouraged from having students work in small groups because it "makes too much noise," despite the teacher's strong interest in doing so.

This, then, is our first method of analyzing practices within the double approach framework. This division into components, which are deeply intertwined within the complex system representing practices, and the reasoned re-compositions that it allows has enabled us to advance in our research, particularly in finding action logic, commonalities, and variability. In particular, a teacher's choices of mathematical content are directly implied by the very nature of the mathematics in play, as well as by imperatives of class management, by considerations tied to curricula, to the long term, and to the teacher's own representations and knowledge.

Levels of organization in teachers' work

We have identified a second type of practice analysis, still within the perspective of the double approach. This second type of analysis is more suited for examining variability and individual changes in work.[28]

One aspect of the complexity of teachers' work lies in the connections between different distinct phases of this work. Preparation, for example, is partly independent of the rest of the work, but partly influenced by the anticipation of what will happen in class. It is less constrained, particularly in terms of times, than is the unfolding of these plans in class or improvisation, which are regulated by the passage of time. Keeping the class on track does not completely ensure the goal of student success, even if the two are linked. Nor does student success ensure learning, even if they too are linked. Finally, it is not possible to exhaustively describe what teachers must do at each step of their work.[29] It is similarly impossible to completely evaluate teachers' work, as learning is difficult to measure in the medium term,[30] and difficult to directly connect to teaching, as we have already mentioned.

The three levels of organization used for this component of the analysis consider the different scales connected to the timeframe and texture of the activities to be analyzed.[31] They are directly tied to the subjects. These levels (or practice organizers) are:

– A micro level, which consists of studying actions that are automatic. This includes non-prepared speech, basic gestures, etc. (Butlen, 2007). We have also examined the method of writing on the board, which is, in part, completely automatic. Other research has analyzed shifts. We note that within these analyses, we may potentially have access to phenomena that manifest without teachers' knowledge and that may remain unconscious. The teacher may become aware of these phenomena but still have difficulty acting on them.
– A local level, concerning the daily class. This level contains preparations, class events and teacher's improvisations, and it is the level of all teacher adaptations.
– A macro level, for projects and preparations,[32] based on individual knowledge, representations, and experiences.

Some examples of practice analyses

An initial type of results involves the confirmation of the individual coherence (Vandebrouck, 2002) of practices (De Montmollin, 1984) and the fact that they become stable over time. The mediatory component, examined to a certain level of detail,[33] is the most stable (see chapter 4). Other work on teachers' use of the board confirmed the consistency of each teacher's choices, and the coherence with the chosen method of classroom management (Robert & Vandebrouck, 2003).

Another type of results helps explain the consistency of the intervention of institutional constraints in practices (close institutional components), given the diversity within the other components (see chapters 3, 6 and 7).

As we will not return to this topic in subsequent chapters, we will summarize here some previous results concerning practices of experienced teachers that we found to be common in 9th and 10th grades, as well as results from beginning teachers that demonstrate the use of levels of organization.

Examples of consistency in 9th and 10th grade class periods devoted to exercises

We will only give a summary of the results of these studies[34] (Robert, 2005a, 2005b).

Teachers favor in-class work that exclusively focuses on the new mathematical concepts being taught. This type of work does not involve much exploration of the field of problems solvable with the associated tools. Effectively, the necessity of progressing through the topic leads teacher to propose tasks that are relatively close to the lesson, that require standard applications of the concept, which must have been already seen. This leads again to favoring "decontextualized" meanings versus "contextualized" meanings. At the same time, there is little explicit maintenance of prior knowledge. There are rarely occasions of reorganization between old and new concepts. Furthermore, students are only rarely and briefly confronted with uncertainty on what they should do, which leads to minimizing student questioning of what should be used and autonomous linking of concepts.

This takes place through organized in-class events and by teacher interventions before and during student activities. We identify an unequivocal orientation of students' activities toward the desired new knowledge. This orientation is particularly enabled by a precise and rapid (indeed, immediate) consideration of these activities, with constant guidance and little time for autonomous work that is not on the final calculations. These calculations are completely outlined for the students, and form the major part of students' in-class work.

The resulting activities thus relate to tasks that, if not originally simple isolated tasks, become isolated. These tasks relate to the relevant chapter, without many adaptations of the concepts to be used. There is rarely need for structuring knowledge as an action for students, as the teachers handle this themselves. In these conditions, there is also no need for delegating control to students.

We identify therefore a certain sequencing of student activities on a concept in relatively independent moments. Students apply the tools, one after another, independently. They only need the (stacked up) tool concepts corresponding to the lesson and inspired by the teacher's subdivision of activities. The development of the dynamic between lessons and exercises can be limited on scope. It is thus the organization of student knowledge that will be one of the first victims of this time constraint.

The constraints shared by teachers of the same grade in similar establishments (schedule restrictions, curricula, effect of instructors and inspectors, substantial heterogeneity, class composition, etc.) act as if to lead to shared practices, in terms of the mediatory and cognitive components, even if there is leeway accorded to teachers that is not applied in the same way.

We cannot be sure that students' knowledge will be partitioned,[35] as students can learn things that are not explicitly taught to them (and that are therefore intended for them, more or less implicitly). But we can still ask if the common complaint of observers of the lack of "certainties" among students may come from this type of classwork. This is reinforced by a common complaint from students: "Just when we start to understand we switch chapters." One question emerges: Are there alternatives to this type of choice?

Examples of analyses of practices of beginning teachers

The study of transitory practices of beginning teachers can also illustrate this type of practice analysis. Beginning teachers (cf. Robert, Roditi, & Grugeon, 2007) (called PLC2 in France) develop practices that evolve during their first year. We call these practices "transitory," as they are not yet stable. They are usually complex, however, and we assume that their coherence is developing, due to their previous experiences and knowledge.

These beginning teachers are led to adopt a new position, that incorporates their personal component and that is tied to the exercise of a new profession, in an actual establishment. This leads them to become aware of constraints and leeway of their new profession: “Not everything is possible, either for everyone or for each person.”

Every day, in beginning teachers’ classes, we see evidence of difficulties in recognition of students and time management (the mediatory component). It is possible that the mathematical project of the class period may be central to the detriment of students, or that consideration of students may be focused on to the detriment of the mathematical project. It is as if some beginning teachers are obsessed with the reactions of the class and the concern that all students follow along, while others forget that it is to their students that they are teaching mathematics, and even display ignorance of mathematics for students.

At the level of the cognitive component, the project developed by beginning teachers is often fairly local. It covers a maximum of several class periods, and does not always fit into a coherent whole for the year, particularly in terms of the mathematical plan (Margolinas & Riviere, 2005; Bloch, 2005).

Other complementary elements were proposed in a study by Bloch (2005), who suggested that the beginning teachers did not only lack the means to organize their lesson content, particularly introductions, but also had specific difficulties regarding students’ mathematical work that could take many forms. It is, paradoxically, a lack of awareness of the necessity of having transitory constructions serve as intermediaries for students, and sometimes even a lack of awareness of the necessity of the construction of meaning. Bloch (2005) uses this to support the idea (previously used by Lenfant, 2002, in algebra) that certain beginning teachers have so internalized some mathematical concepts that they no longer see the difficulties. They are not aware of the fact that giving a rule, even with commentary, is not sufficient for all students to learn it. They may also expect formal proofs too quickly while students can proceed though more pragmatic processes. Finally, Bloch (ibid) proposes giving these teachers ways to “have students really do mathematics,” particularly by identifying the concepts in play, elaborating situations in which the topic will arise, and learning to handle them. We will return to this.

Finally, these unstable practices of beginning teachers lack sufficient operational mental images to enable nuance and adapted improvisations. Beginning teachers have incomplete or skewed images (Chesné, 2006). Even if their images are not what we call “deformed,” they may lake depth or a hierarchy in a desire to “do well” by following their training. They may temporarily erase, as much as possible, the personal component and spontaneous reactions (Chesné, 2006). These teachers thus do not involve themselves completely as such, with consequences on students who do not have a truly engaged teacher in front of their class. Other teachers may have “caricature deformations” which overly focus on the individual relationship with students or their activity, which may be more or less mathematical. Others overstress following the mathematical project they have decided to adopt, the presentation of knowledge and the course of exercises.

We hypothesize that, unable to rely on automated processes, routines, or overall depth concerning either mathematics or students prevents these beginning teachers from leaving the local level, which then becomes all there is. With a lack of connections to the micro and global levels, there is an overload on the local level.

The case of newly certified teachers, particularly those in their first positions in the Zone of Educational Priority, has led to recent studies (Coulange, 2006) that show the diversity of the potential effects of teacher training on those trained, relative to their personal component, their first position, and the details of their training. One question arises: Are there pre-existing factors that could lead a sizeable number of beginning teachers adopting certain practices over others? We will return to this in chapter 12.

RETURN TO STUDENT ACTIVITIES – A METHODOLOGICAL POINT OF VIEW – SOME QUESTIONS

Even if each study chooses elements of this general methodology to adapt, the work to be conducted in order to study the teaching of a given mathematical topic can be divided into six "acts," which include better understanding learning and teaching practices, and describing alternate strategies to try. These acts are clearly non-independent, but may be completed in various orders.

Act 1: Determining the relief map corresponding to the concept

This phase of research leads to examining epistemological or historical studies, as well as didactic studies. The goal is to define the mathematical details of the concept, to characterize its role in the curriculum (and its potential evolution), and to synthesize students' identified difficulties. This can be enriched by analyses of manuals or other resources.

Act 2: The studied or intended teaching scenario

It is clearly the reference to the relief map that allows us to appreciate the intended introduction as much as the dynamics between the lesson and the exercises and/or the richness of tasks. This is a difficult task, as it often relies on many inputs. Comparing different scenarios may help.

Act 3: A priori analysis of specific tasks, from problem statements presented to students

Chapter 5 on the analysis of manuals illustrates in detail an example of these analyses.

Act 4: Analyses of in-class events, based on observations or on video or audio recordings

Analyzed class periods are often filmed and then transcribed. The camera is placed at the back of the room, centered on the board, with the teacher as the principal actor. The students are rarely seen and cannot be heard well, but the teacher often repeats their statements.

To complete the corresponding types of analyses, we have established a rubric for studying classroom events (described in detail later on). One of the variables is the level of detail of these analyses. Depending on whether we examine speech phrase by phrase or more finely, the information we collect is different. For example, to understand the dynamics of interaction phases, it may be interesting to examine language markers (see following chapters).

We compare *a priori* analyses of problem statements, the work conditions in the classroom, and all verbal exchanges to reconstitute students' activities. We go into more or less detail depending on the specific research question.

We first take into account the established chronology. Beginning with the *a priori* analysis of the problem statement, we list student tasks and their length as they are encountered throughout the lesson, with reference to the *a priori* listing.

We thus note the task format (individual, collective, etc.), the output of the task (simple research, written work, group written work, etc.) and the types of tasks (researching, writing or speaking, listening, composing, recopying, etc.). Task types and task formats determine the nature of the work, independently of the work content. In particular, moments of silence by the teacher are taken into account as indications of an attempt to delegate the task to students. In this way, students may, to a certain extent, research, discuss, write, listen, recopy, get help, get corrected, or even be encouraged, on a single task. The nature of their work (for example, investigating as a group, writing out a response individually) gives weight to and indeed modifies the application of the knowledge induced by the tasks.

We then compare the tasks and subtasks worked on by students and the teacher's contributions: questions, rephrasings or answers, help or explanations, identifying and applying student work, presenting knowledge, generalizations… silences. Particular attention is paid to teacher assistance. Recall that procedural assistance can reduce the task to be accomplished, but can also allow students to apply themselves to work. Constructive assistance can allow students to construct new knowledge based on their own work. This corresponds to use by the teacher of what we model under the generic term "PDZ." From the students' point of view, procedural assistance may already be constructive, and constructive assistance may remain procedural (cf. connection to knowledge of students in a ZEP). Generally, all commentary added by the teacher, whether on the mathematics in play or on student work, is an important element of our analyses of in-class events (cf. meta). They can reveal what improvisations, choices, and thresholds the teacher selects, relative to what was expected. This in turn can reveal both possible student activities and teachers' logic. In terms of the phases of knowledge presentation (or "lessons"), analysis is a little more difficult, especially if students are silent. Moreover, it is imperative to be aware both of what precedes the lesson, and what immediately follows it. The lesson is often illustrated by simple examples or immediate applications, to establish the potential real or artificial link from activities before the class period to the class period.

We replace the analysis of tasks by the analysis of units, which may be present or implicit in the lesson. Possible units include definitions, theorems, properties, propositions, demonstrations, examples, commentary, illustrations, diagrams or drawings, and applications.

We then investigate these units in the same way as before. We examine particularly their existence, the length of time spent on these knowledge presentation phases, the order and placement of the presented content, and the manner in which the different comments are introduced, as well as the moment they take place. This shows us the potential way the professor introduces the object and tool aspects of a single concept (the corresponding theorems or methods).

We seek, if possible, the apparent role of improvisation and conversational exchanges during these phases, as well as students' apparent activities. Writing on the board, and differences between what is written and what is said, may also be analyzed (depending on the study). Similarly, the degree of mathematical formalization may or may not be analyzed.

It is interesting to analyze how the teacher asks students to use the lesson (and to work on it), and to note the quality of the references the teacher makes to the lesson. It is as if for some teachers the lesson serves both as a reminder for the class and as a catalog of knowledge to use.

It may be interesting to compare teaching manuals with teachers' lessons, as well as the use that is recommended for students.

Act 5: Reconstitution of students' activities, links to learning, initial questions

The preceding analyses allow us to reconstitute the traces of possible permitted or encouraged student activities. Possible activities are those that we can estimate were done, at least in part, by many students during class time. Often, we are otherwise led to distinguish *a maxima* activities and *a minima* activities. By contrast, since students work autonomously on a computer, accessing their activities is less problematic (even if in actuality no one can access the effective activities themselves). We thus speak of the observed activity (even if only the actions were observed) and we have developed several methodological ways of accessing these activities (by direct observation or thanks to trace files; see chapter 8).

That is to say, if it is difficult to analyze teaching in relation to learning, it is even more difficult to have legitimate evidence of it. We are well aware that a blank sheet is not always synonymous with a lack of learning, that an apparently trivial variation of a problem statement can affect student performance, and that there may be a large gap between what a student writes and what the student has understood and retained, or is in the process of learning. Tests, for example, provide very limited evidence of learning, polluted by social and affective factors, and tied to the teaching contract and to the necessity of having a certain level of success in a class.

Our analyses thus do not permit us to do other than pull out relatively contextualized relationships between problem statements and in-class events on one side, and student success on tests on the other. We will, however, have to content ourselves with this in certain studies (see chapters 6 and 7).

Note that, more generally, our theoretical framework does not allow us to form precise hypotheses concerning these teaching/learning relationships. It is perhaps on the edges, with students having difficulties, for example, that the compensations that may influence some students are no longer effective, or that there may be some threshold effects. For example, researchers (Castela, 1995, 2000) have shown that some students can construct knowledge without their having been explicitly taught. Are there sources of differentiation here? Are they individual or social?

At most we can give several hypotheses (see Appendix 2) on extreme cases, inferred from the general theories that we use for inspiration. First, a completely lecture-based lesson has a strong chance of preventing many students from constructing knowledge, due to insufficient student activities. A lesson with no period of knowledge exposition may also prevent students from learning, due to a lack of occasions for transforming activities into knowledge. Similar problems may be found in a lesson that contains only simple and isolated tasks, due to a lack of non-immediate activities.

Act 6: the logic of teachers' actions and analyses of their practices

This last act consists of reconstituting and recomposing, for a single teacher and then potentially several, the components that we have distinguished, keeping track of the organization into levels.

The action logic combines the cognitive and mediatory components, to better identify what can lift the constraints we take into account.

Using multiple analyses of a single teacher or of several teachers, we look for common factors in overall content choice. For beginning teachers, for example, the extra burden of the local level has led to calling on levels of organization to interpret the records. Determining professional groups (such as gender) leads to working both on components and on the levels of organization.

Diversity between practices is often expressed through logic of action (on the local choices of tasks and class progressions).

These are, for now, inferences based on initial results that lead us to suggest variabilities (what can shift, and at what cost).

A number of chapters of this book illustrate this last act.

Conclusion: Further questions — toward the long term?

We indicated above the open questions concerning appropriate indicators for studying the speech of teachers in class. These indicators can be more or less finely grained and can consist of linguistic markers or the format of exchanges. But these analyses of in-class events, based on analyses of tasks (exercises, lessons), remain essentially local. They cannot completely reveal events over the long term, although learning itself takes place over the long term. There is also no "rung" to better appreciate student activities and teacher motivations.

Teachers often do not say or do all that they intended. This feeds a double discussion of the necessity of their choices and omissions, and of the alternatives. The elements analyzed during several lessons leads to larger questions concerning other potential exercises, the lessons, the curriculum, the class, the specific teacher, the establishment, etc. Focusing on these questions allows us to go back up to the global level.

The study of lesson plans can certainly reveal part of the mathematics commonly used by students, in terms of dynamics within the project, between the lesson and the exercises, and between the meaning and the technique. They are revealed through the order in which the different parts are presented, the expected lengths of the different phases, and the quantity and variety of the tasks. But the difference between what is predicted and what actually happens, as highlighted by this chapter, is too great for this to be sufficient.

When possible, examining student work for a given chapter can certainly allow us to compare in-class events with evidence of learning. However, this is very time-consuming, as all events must be examined relative to the test. Some studies have begun to devote such resources, although they are limited in scope or in the number of parameters.

We retain, from this theory of learning and practices, the intermediaries chosen to have access to it, with the importance of coupled links (statement and in-class events) as precious indicators of factors that can vary in student and teacher activities. These activities can be understood through *a priori* analyses of expected applications of knowledge from a problem statement, compared to the applications provoked or allowed during the class. These latter applications are approximated by the work that the teacher puts into place for students. These analyses of possible activities are sometimes completed by studies of observed effective activities. More general studies of the mathematical concept in play, from the scenario in which the classes take place, and from the institutional, social, and (for the teacher) personal context, reveal recompositions respecting the complexity in play.

POSITION RELATIVE TO OTHER STUDIES

Our investigation relates to the effective class and individual subjects. We indicate in this section, in a necessarily schematic and summary way, several general characteristics of this theory relative to other foreign or French didactic theories. Note that it is impossible in a few pages to be either exhaustive or complete, and that we have selected several examples for clarifying our proposals.

We will see a large initial difference concerning whether to adopt a "*stricto senso* didactic" point of view, where subjects play a generic (or indeed epistemic) role, and are therefore all considered equivalent. This is not our viewpoint.

Concerning students and student learning

A certain amount of anglophone research is directly inspired by the theory of activity for analyzing class periods (Christiansen & Walther, 1986; Hiebert &

Wearne, 1997; Stein, Grover, & Henningsen, 1996; and others). We also find that for them, the task is the starting point for the activity, and remains external to the student, while the activity is what the student actually does, and what influences learning. They give particular importance to in-class events for analyzing student activities, and are not content to study the tasks students are proposed. However, the methodologies used are different from ours, and the activity is analyzed more in terms of overall applications of mathematical steps than applications of precise skills such as conjecturing on an open-ended task, calculating, or reasoning on the nature of a problem, internal or external to the mathematics. Here is a list of four such types of tasks given by Sullivan, Clarke, Clarke, and O'Shea (2010).

– *Type 1.* Involves a model, example, or explanation that elaborates or exemplifies the mathematics.
– *Type 2.* Situates mathematics within a contextualized practical problem to engage the students, but the motive is explicitly mathematics.
– *Type 3.* Involves open-ended tasks that allow students to investigate specific mathematical content.
– *Type 4.* Involves interdisciplinary investigations in which it is possible to assess.

Great importance is given to open tasks and to associated investigatory steps. This corresponds notably to research developed in relation to the NTCM (see below). However, characteristics tied to affective or psychological dimensions are often introduced, which partially direct the analysis of tasks and activities differently from our analyses led by knowledge *stricto senso*. Some examples are "attention" (Mason, 2003) or "challenge" (Jaworski, 1994).

We return briefly to francophone research. In the Theory of Didactic[36] Situations (TSD), when in-class events are analyzed, references and the comparison in the light of which we report observations are the model of mathematical learning defined from fundamental situations and the environment. Fundamental situations, at the heart of the theory, model a didactic procedure that in some sense forces students to use the mathematics to be acquired. The corresponding problems are developed from the deep meaning of concepts, identified from the question: What are the concepts for? Students have no other option but to use them, on the condition that they play the proposed game. Moreover, it is within the problem that they find the elements that allow them to determine if their work is correct. They do not need to wait for the teacher's input (see the "puzzle" example).

From the perspective of in-class events, Brousseau (1998) particularly highlighted the interest of the sequence of phases of action, formulation, and validation. He introduced the concept of the didactic contract that represents the potentially implicit expectations of the teacher towards the students and vice versa. We can thus gauge *a priori* the different situations (including ordinary situations) proposed to students that use the tools initially conceived for describing "ideal" didactic situations. The *a posteriori* analyses allow us to compare the gaps between what actually happens in class and the *a priori* analyses. This involves verifying if the tools the students must use to solve the problem are available to them and are well within the environment. If the situation is predicted to be a-didactic, in order to introduce a concept from a "good" problem, for example, we verify *a priori* that there is no need for the teacher, that the intended knowledge is necessarily and uniquely at work, and that the situation is within students' reach. We verify *a posterori* that the actual lesson respected the expectations. Based on the characteristics of the in-class events and the predicted tasks, the analyses explicitly confirm or reject the gaps between what took place and what could have been predicted based on the model of learning within the TSD. We can invoke the point of view of a certain possible theory of learning for analyzed situations: that analyses carry inferences on learning through the intermediary of the model (Brousseau, 1997, 1998).

For our part, we seek to measure the gap between the activities of students applying their knowledge (during its acquisition) analyzed *a priori*, and the activities that may actually have taken place during a regular lesson.

First, the description of these activities is made in reference to large dimensions that do not correspond to a constructed theoretical model, but only to large categories of variables influencing learning and depending on teachers. We should note at this point the importance of the chronology and the corresponding details of the assistance provided by the teacher in our analyses. The chronology and fine-grained analyses of speech seem to us to be often absent from analyses of the environment, where the situation is analyzed as a whole from the start.

Second, we can work equally well with short sequences as with longer ones. This is indispensable at certain moments during the research. In effect, our basic unit (the problem statement/in-class event couple) is smaller than the "situation" in the TSD sense that it replaces. We can however note that it is "of the same order" in a certain sense, as it combines content and management. But the concepts of tasks and activities do not explicitly appear in TSD. They are replaced by situations proposed to students divided into possibly a-didactic phases, referring to students' single mathematical work within each phase. The dynamic of reference situations refers uniquely to the application of the expected mathematics, made indispensable by the proposed problem and which is with internal methods of control for students. The individual variabilities therefore have no place.

Finally, we give the same importance in our approach to the introductions of concepts as to the rest of the instruction of a concept. We do not find a focus on fundamental situations or the tool/object dialectic. We also do not find the implicit increase of the introduction of concepts for the construction of knowledge by students.

The anthropological theory of didactics (TAD[37]) is another different theory. It pertains to the didactic of elements inspired by an anthropological vision of man in the world. From mathematical decision theory, we obtain systematic and systemic ways to establish an exhaustive definition of an institution's mathematical provisions in terms of one or more concepts: curriculum, manuals, or even more or less complicated mathematics. The "provisions" are everything available to students and teachers in terms of types of tasks, (legitimate) justifications, theories, etc., without taking into account conditions of instruction unrelated to the mathematics. In particular, it is not the problem statements of the exercises that are analyzed, but the types of tasks that they illustrate and that are extracted from them, independent of the specific activity that the students have to do on the task in question, which we have not given ourselves the means to determine. This leads to identifying types of elementary tasks as kinds of units that can appear in exercises, manuals, or lessons, and that serve to distinguish the instruction from the content, understood in terms of gaps or evolution. Each type of task is associated with one solving method or technique and to various justification possibilities (which may or may not be present within the analyzed elements). The whole, which can range from one concept to several mathematic chapters, and which can cover one or more curricula, is organized relative to the elements of the corresponding mathematical theories and the evidence of the transposition of this knowledge into teaching content. It is thus not in this sense that the word "task" is used in the presented work (Chevallard, 1992).

Concerning teaching practices

Many anglophone studies involve teacher beliefs, but do not introduce the occupational dimension. For example, the model developed by Schoenfeld (1998) characterizes decisions and actions as a function of one's knowledge, goals, and beliefs. These factors are clearly marked by the content that is considered. This model is used to characterize a moment within the teaching activity, included within the level of student interaction and aiming to also predict the behavior of a teacher whose factors have been previously determined, which underlies the existence of certain invariances. In addition, a certain number of studies now examine collective systems that involve communities of practice that serve more to study potential evolutions in these practices than to analyze them in relation to student learning (Wenger, 1998).

In France, in research inspired by TSD, Margolinas (1995) and others presented an analysis of teacher knowledge and its role in class, organized in levels structuring the environment. For Margolinas, working (for a teacher) is "putting in play" knowledge of different levels of mathematics and student. Studying the work consists of examining this knowledge and its interactions between these levels and then imagining ways to have this knowledge acquired.

Much of this knowledge concerns the way in which the professor understands the content within a curriculum, develops exercises to give to students, and interprets student knowledge. This type of fairly large investigation does not seem to us to take into account the chronology of the lessons, or the way they unfold in class, which become secondary. The manner in which actions, such as taking up a student's idea or offering help, are taken, are thus not described and have no place.

From the TAD viewpoint, the actual lessons that take place are systematically related to the didactic organizations that are developed independently of classes, teachers, and concepts (Chevallard, 1999). Thus, different points in the lesson (introduction, work on the technique, etc.) are taken into account. However, in these studies, the import of the effected lessons and the specifics of individual subjects, which we take into account through the intermediary of the analysis of effective activities, seem to us to be discounted.

Finally, more recently, researchers have developed a model of the action of the professor in class, which has been expanded into a model of the joint professor-student action within the framework of the compared didactic (Sensevy et al., 2000). The decomposition of the teacher's action into four dimensions, and the systematic consideration of the mesogenesis, chronogenesis, and topogenesis, also do not seem to us to be well adapted to our project of realtime introduction and individual and conceptual variabilities. The explicit theoretical references to cognitive characteristics of student knowledge and to effective student activities are not called upon.

In this step, working consists of bringing activities into play, and analyzing consists of studying the activity (what is thought, said, unsaid, done, undone, etc.). We are less interested in knowledge than in its application, and we do not reduce teachers' activities to professional gestures, even if it is interesting to introduce levels tied to the temporality and grain of these activities.

These different viewpoints clearly have consequences on teacher training and corresponding research. We will return to this in the conclusion.

Concerning professional didactics

Although our work falls within the general double approach framework, it borrows more or less specific elements from professional didactics (Pastré, 2005a, 2005b). We have allowed ourselves to be more directly influenced by research on the integration of technology into teaching practices.

For example, in some research, we study the teacher's activity in a specific situation of integrating technological tools in the classroom. Our approach is directly inspired by professional didactics. In these studies, the dialectic between "productive activity" and "constructive activity" is introduced to clarify that the teacher acts to transform what takes place by contributing to students' activities, and that the teacher is also transformed though a longterm development process (Pastré & Rabardel, 2005).

We would like to add in this section several difficulties that arise from these borrowings. They are first tied to our difficulty in defining schemes and integrating the concept of competence. They are also tied to questions of expertise. There is not always agreement in mathematics on the orientation of the teacher's action. There is far from being a universal definition of "good ways" to teach, nor common adoption of reference models for analyzing practices. Moreover, the test of a teacher's practices is not made on learning, the ultimate goal of practices (or only partially, by the intermediary of proofs for which everyone agrees to underline the insufficiencies inherent to the complexity of such an evaluation). By contrast, the teacher, and even the educational inspector or other colleagues, can verify that "the class functions" or that the students succeed on tests. It is not the goal of the action that is easily and directly tested, but only a partial subgoal of student investment, which is a no doubt necessary condition but which can not reveal learning. We wonder if a multi-expertise is not necessary in this professional domain.

The individual/invariant/generic relationship, and the differences between the action sequence and the significance for each and invariants, are unique to the teaching profession, as is working in a human open dynamic environment. Complexity, variability, and unpredictability make the subdivisions that allow us to locate the execution of the action, and no doubt require several simultaneous and interrelated approaches, problematic.

Finally, the analyses of actual work structure many ergonomic studies. In the studies on teacher practices, this type of analysis is difficult to perform, given its complexity (the presence of students, the difficulty of the evaluation). Consequently, many teacher trainings, for example, are inspired by the work desired by the trainers, without always being anchored in the real work of teachers, contrary to many formations inspired by professional didactics.

Technology in education: details on the case of work on the computer

In the case of computer-based work, a preliminary examination on the use cases of technological tools is specifically relevant, to the extent that the discussion of the longterm of computer-based lessons with classical lessons is a factor in student activities. The *a priori* task analysis should be completed by an equivalent analyses of environments, assistance, and possible feedback. Along with teacher assistance, they can modify the expected possible activities.

For in-class events, the length and nature of student work differ from one type of tool to another. Work is, however, often individual, and students have *a priori* a lot of autonomy. They can have different work progressions with different amounts of time spent on the proposed tasks. They can also benefit from individualized feedback, and can discuss with partners or with the teacher. There is thus in general more material to observe than during traditional lessons in which the rhythm is often dictated by the collective progression of the class. Students can thus be followed individually. We have access not only to their possible activities, but to their actual actions. The reconstitution of the activity is this closer to reality than it is with classical methodology. We can better determine if constructive assistance serves its purpose, and thus to go towards effects in terms of learning. However, there is a loss of generality, as only a few students can be observed.

Learning specific to the use of technological tools, combining mathematics and knowledge of the tools themselves, appear in particular in students' use of open-ended software (dynamic geometry, spreadsheets, etc.). They can justify using a theoretical approach centered on learning. It involves using an instrumental approach (Rabardel, 1995) specific to the mathematics (Artigue, 2002).

Finally, some assistance from the teacher is uniquely tied to the computer environment and the manipulation of tools. It is not found in the traditional paper-and-pencil environment (except when non-technological tools, such as a compass, straightedge, etc., are used). We introduce in some work a third type of assistance (after procedural and constructive assistance) which we call "manipulatory" (chapter 10).

Conclusion

We could diagram the main dimensions that allow us to distinguish researchers' choices.

An initial source of diversity is the connections maintained by didactic studies to the situation (the time and the problems found in the countries involved), to the terrain (the school, the students, and the teachers), and teacher trainings, in relation to the institutional conditions that are imposed on research. In concerns the position that is adopted concerning the links between the research and the instruction prescribed in a given area, the curricula, the instructions, but also the cultural or social habits. In a word, does the research have a prescriptive or prospective impact? Do they provide diagnostics or propositions to be tested (or not)? What variables are introduced? In relation to the problems to be treated, in what theoretical or conceptual framework does the research take place? What is its role?

Some studies fit directly in a given educational system, with goals of acting on the system. We note particularly anglophone studies attempting to increase the effectiveness the NCTM standards established in the 2000s to improve mathematics instruction in the US (and in other countries that have adopted the same types of standards). Analyses of tasks that are then produced, for example, seek to translate not the epistemological characteristics of the mathematics content but student skills to be accessed (looking for a solution, conjecturing, writing up a solution, etc.). Other researchers work with the same goals as, and even together with, teachers. They have analogous goals tied to the reality of what takes place in class. They declare explicit objectives of transforming instruction and improving learning (Boero, 2007). Still more look for cultural or social sources for the groups involved within the mathematics content (Radford, 2010). By contrast, some researchers focus more on characterizing universal forms that are necessary for mathematics instruction, tied to content (fundamental situations) or to understanding how knowledge evolves and diffuses, establishing for example a structure of knowledge (Brousseau, 1997; Chevallard, 1992; Sensevy et al., 2000 in France). Still other researchers attempt to clarify how teaching and learning are tied, and what are the resiliences, the invariants, and the diversities, and beyond this the local or general situation (Vergnaud et al., 1979). These latter researchers call more on theoretical frameworks, which are often models serving as reference to research, to choose before studies to adapt or develop. The first types of researchers work with local or general theoretical frameworks (or a mixture of the two) that seem appropriate to begin working on the questions to be treated. Other researchers remain working on empirical or positivist studies. In France, research on mathematical didactics was able to develop with some independence of the situation. From this, theoretical frameworks from anthropological theory to be adapted to didactics, from game theory as a model of learning, from Vygotsky and

Piaget's theories applied to the theory of activity, were developed, ahead of contextualized research.

Finally, research on training mathematics teachers in teaching or mathematics are more or less associated with didactic research. In some countries, this is even the origin of the didactic question. Here again is much diversity. We will return to this in the last part of the book.

Another distinction, which is not unrelated to the preceding and to theoretical choices, relates to what is taken into account in research in terms of students and teachers, and the corresponding variables that are introduced in studies, in connection with the theoretical or conceptual frameworks adopted and their definitions. A student learning mathematics and a teacher teaching mathematics can be "approximated" and analyzed in several (non-exclusive, non-independent ways):

– As an "epistemic" subject (studied in relation to the invariants that characterize the subject's evolution, and in terms of the subject's function or role—student or teacher).
– As an institutional subject (studied in terms of "subservience" as a function of the institutions to which the subject belongs).
– As a social subject (related to both the subject's sociocultural origins, particularly for the student, and the subject's profession, particularly for the teacher).
– As a psychological subject, with multiple points of view.
– As a cognitive subject (studied in terms of the subject's development, in terms of learning potential for the student, and practice enrichment potential for the teacher).
– As an affective subject (studied in terms of emotions, tastes, self-confidence, etc.).
– As an individual or psychic subject (with a personal history, representations, knowledge, character traits, etc.).

Even the name of the corresponding scientific field in different countries can be revealing.

One final dimension is tied to the consideration of quantitative aspects and validation in studies and, more recently, to the role given to evaluations. Do we work on individuals, on a few classes, on a large number of classes, on a category of students (possibly characterized by a "generic" representative), on a professional group (such as middle school teachers)? Is there a place to validate the conclusions, and if so, how?

This dimension is effectively tied to the mode of validation predicted in the studies, and to the theoretical frameworks. A strong coherence with a certain framework can serve as auto-validation to certain studies. This, the role given to experimentation, and more generally to all data analyses, varies widely in relation to the theoretical model and with the potential discussion of or the search to enlarge this model. If used to explore new fields of mathematics instruction, experiments serve to collect data for which the model prescribes analyses[38]. If not, if the research is led by questions that do not fall directly under a model, or that contradict it, or that borrow from general theories, experimentation can also serve to establish commonalities, hierarchies, causality, or dialectics contributing to increase or renew the understanding of the phenomena.

For now, research in France is mostly clinical. It is qualitative, and, depending on the case, often validated by comparing predictions to outcomes. There are many people who imagine changing the scale of this research. Here there is another link to the objectives. Research that is very linked to the objectives to be obtained for student learning cannot go without quantitative evaluations, even if we know the main limits: uncertainty on the causes of learning, on the moment to evaluate, on the tasks to propose (not to close or too far from the information learned), etc.

International evaluations are not always associated to didactic research. However, their results allow us to revise quantitative perspectives thanks to powerful statistical tools and extraordinary improvement of ways to improve data collection and associated processing.

Depending on the role given to the situation and the terrain in studies, and according to the types of situations considered, some variables are more or less imposed on researchers. Thus, in France, the analysis of tasks in terms of types of tasks, techniques, technologies, and theories is more adapted to an overall, more or less exhaustive analysis of the knowledge content to be taught. This analysis can critique and even break from curricula, concerning primarily institutional subjects (in various institutions. Our *a priori* analyses (presented above) are more adapted to understanding possible student and teacher activities within a given curriculum (concerning epistemic and cognitive or even social subjects). They require placing task within an overall cognitive itinerary, with reference to a conceptual field (or to a level of conceptualization). From this viewpoint, curricula can be contested through arguments that are not only epistemological but also tied to subjects (particularly students). Studies that involve teachers as psychological subjects (from a certain point of view) are also different from studies involving institutional subjects only to the extent that other variables are taken into account, such as leeway for constraints and their investment, consisting of choices, alternatives, etc., and not only constraints. Still other research very tied to the terrain has developed methodologies specific that involve diverse communities of practice and discourse created with researchers, teachers, and teacher trainers.

That being the case, researchers in mathematical didactics all have the goal of first taking into account the specifics of the mathematics to be taught—to the extent that they share the fundamental postulate of the importance (and specificity) of the nature of the content in play in the learning. But the descriptions of the mathematics necessarily depend on the nature of the students and teachers considered, and even the nature of the relationships between learning and teaching in which we are interested. Several types of relationships between teaching and learning have been studied. This, if all agree to extract, from the history and epistemology of mathematics, curricula, and their evolution, a description of the content to be taught (concepts, mathematic chapters, mathematic domains, or mathematical fields, then the very modalities of these descriptions depend largely on the didactic project and the studied "subjects," as we have already discussed above for francophone research. Some other researchers place a lot of weight on the language developed in class, and specifically study the communication that is established. Others insist on the semiotic analysis that can be done, notably from the moment instruments become involved.

If the problems are approached quantitatively, we can extract important variables that are susceptible to revealing the recorded variations. They can be used both to describe the modes of investigation and to analyze them. There is no question of looking at individual differences, tied to taking into account individual subjects. There is also no question of attributing to a specific result an interpretation that goes beyond the variables selected. This is no doubt the origin of the week impact of some international evaluations, which, wanting to include to much, do not leave much room for interpretation.

By contrast, if we wish to understand the more individual games played in class, the margins that remains at the interior of a given system, then the choice of variables selected and the types of subjects studied may be different. If, for example, we want to introduce the student as a “cognitive” subject, and the teacher as an individual subject, then we need descriptions that are not limited to the mathematics that is structured and analyzed in relation to the knowledge only. It is necessary to given, in the descriptions of content, ways to describe potentially unexpected student difficulties, and, more generally, the learning that evolves. It is also necessary to give methods to analyze what takes place in class, along with what was predicted. The concepts of “conceptual field” and level of conceptualization are a response to the first expectation, while the analyses of in-class events respond to the second. It is also necessary that these latter analyses can reveal what we are looking for. Thus, the *a priori* analyses of tasks in terms of adaptations were introduced specifically to allow for studying the outcomes of the predictions in class and to better appreciate subjects’ activities in response to tasks processed in class.

Concerning other disciplines

In France, various types of didactics have been developed with different histories. Some come from teacher trainers (didactics of the French language), and others from universities (didactics of mathematics and physics). Some interdisciplinary studies related to science (particularly mathematics and physics) begin to be conducted. This usefully increases the spectrum of questions that are posed. But this is another story.

APPENDIX 1
FRAMEWORKS, REGISTERS AND POINTS OF VIEW

We adopt the Douady’s categories (1986): a “framework” corresponds to some mathematical field (or domain), in which a given notion is introduced, not alone of course. It is characterized by some fundamental axioms, implicit or not, a corpus of definitions (objects), theorems, and propositions and a set of problems that can be resolved within this field. For instance, the middle of a segment [AB] can be studied in an analytic way (involving the analytic field), a geometric way (geometrical field), and so on.

According to Duval (1995), we call "register" a precise way of writing mathematics, using a given formalism, introducing so a semiotic view. For instance, to work on decimals numbers, some can choose such writing as 4, 567, or 4 + 5/10 + 6/100 + 7/1000: these two registers differ and inside each of them the treatments (operations) are not exactly the same.

The "points of views" refer to different ways of tackling a problem, leading often to different strategies. For instance one may consider the intersection of three lines as a point belonging to each of them, or may consider that the intersection point of two of them belongs to the third one, or may look for a transformation such that the three lines are the image of three other intersecting lines… Each point of view induces another strategy to tackle a problem. They differ from field or registers because they can occur inside of the same field or the same register.

APPENDIX 2
SOME HYPOTHESIS FOR THE TEACHERS

We have list general hypothesis, concerning only the cognitive point of view, that may help the teachers when choosing their scenario and their classroom's management but that each of them has to adapt to the precise content he wants to teach, to his students, etc.

1) Related to the contents' choices

Apart from conceiving an appropriate scenario, more local decisions may occur.

a) To introduce a new notion
Depending on the very type of the concept (cf. supra), one can or cannot find a "good" problem making the students easier to apprehend the meaning of the concept.

b) To work on a notion
Solve some SIT seems indispensable. But conceptualizing depends also in particular on the variety of tasks that the teacher suggests solving during the lesson. If there are only simple and isolated tasks (SIT), one may guess that the students will lack some tools to adapt their knowledge.

The order of the tasks is another "variable" on which the teacher may play. Working on complex problems, involving not only one notion in one field, may also increase the level of available knowledge so as mixing new and old knowledge.

c) The obligation of writing
It allows an important and useful students' work on rigor and precision.

For instance writing completely a proof let students realize that they had not consider particular cases, that their notations were incomplete, etc. … It helps to understand precisely what is involved in reasoning.

d) The presentations of the lesson
In every case, the moments of teacher's presentation of knowledge are indispensable, to define and formalize what the students have to know. And it is yet more important when students have work by themselves before, so that they need to be informed of the corresponding knowledge.

Of course if there is nothing else during the classroom that this kind of teacher's presentation, one may guess that many students will switch off.

2) Related to the class management's choices

a) Various tasks with time to solve them
The general idea is, in relation with the corresponding tasks, to choose an appropriate management. For instance when the tasks are complex, the idea is to associate a management that let students work alone, without the teacher during some time, so that he can afterwards lean on the students' actual work to make them go up.

b) Autonomous work in class (or in small groups)
It is important to give students occasions to work by themselves, to discuss between them, eventually to work in small groups, and to give individual appropriate assistance when there is a need of it. But it is also important to detect what the students have done when working alone and to make the most of it.

c) Habits
It is when repeating sometimes a way of unusual work that the students may benefit of it.

d) Home work
It is important to give homework that all the students may realize, to improve them.

e) Appropriate assistance
There are many types of assistance and it is important to choose the moment to deliver them – before the work on a task, during it or after it. They may be general or particular, direct or not, they may take the shape of questions, or explanations … The important thing is to adapt the assistance to the students' question and knowledge.

NOTES

[1] This corresponds to what Rogalski refers to under the term "open dynamic environment management" in the previous chapter.
[2] See previous section.
[3] Knowledge presentation phase
[4] The importance and variability of the relationship with knowledge, in terms of the student's socio-cultural origin, and the potential weight of emotional factors tied to the parents' level of schooling (Charlot et al., 1992 ; Bautier, 2006; Bautier & Rochex, 1998).
[5] The Zone of Educational Priority includes institutions attended primarily by underprivileged students.
[6] For the most part. In some studies, the teacher's activity is examined in terms of its effects on the teacher (cf. regulation loop).
[7] In no case do we consider the (nonetheless important) domain of the unconscious.
[8] The idea of "availability" of these aspects is our way of translating the characteristic invariance of acquisitions under Vergnaud's models.
[9] Less and less, if we look at the current direction of French curricula.
[10] Despite what might be implied by the word "level."
[11] In-class lessons and/or textbook lessons.
[12] In certain studies, we speak of complex tasks.
[13] This is less frequent in general outside of university level classes.
[14] "Concept" should be understood broadly, and includes some important theorems that are the object of a chapter.
[15] As defined by Brousseau (1990): The respective expectations of teachers and students.
[16] A chronology of class periods is established, based on the *a priori* task analysis and the effective unfolding of in-class events.
[17] Studies on the environment should be inserted here.
[18] Cf. Vygotsky, tied to his social analysis of knowledge under which "the collective appropriation may precede individual appropriation" (Vygotsky, thought and language).
[19] They studied the type of interactions beyond mathematical content.
[20] Proximal development zone (chapter 1)
[21] In the sense of the double regulation schematic from part 0.
[22] We find again here the idea of assistance with a constructive function.
[23] Establishments for educating future primary school teachers.
[24] Professional establishments for training future teachers.
[25] Short for "didactic and ergonomic double approach" for the analyses of teaching practices (Robert & Rogalski, 2002).
[26] In our initial findings, we described the way activities are set into motion, as well as the maintenance of students in the activity, called the student investment or more broadly, "keeping the class going."
[27] With a restriction: They place themselves within the approach they are helping to legitimize.
[28] This result comes from one of the OPEN (Observation of educational and teaching practices, 2008) subgroups concerning "practice organizers." In this subgroup, researchers in professional and other forms of didactics worked together with sociologists. Researchers were invited to respond to the question "For you, how should the term 'practice organizer' be defined?"
[29] What ergonomists call "discretionary tasks."
[30] There is a very important discussion here on the information supplied by evaluations.
[31] The levels of organization introduced in the appendix, which also take into account the flow of activities and the timeframe of the action, should also be distinguished.
[32] In previous studies, we have used the terms "lines of action" and "singularization" in discussing the macro and local levels.

[33] Subdivisions on the order of several minutes, punctuated by activities organized for students.
[34] We have established these assessments of 9th and 10th grade (predominantly algebra) class periods after careful study. The problems given to students were not exercises of immediate application, but were introduced just before or after a lesson, and did not stray far from the lesson.
[35] This is, however, one of the strongest assessments made of the knowledge CAPES students developed at the university.
[36] A theory that to us does not seem to be contradictory but complementary to what we do, but which we would not be able to summarize briefly.
[37] Also this term is impossible to summarize briefly.
[38] In short, there should be no surprises.

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3. DIVERSITY, VARIABILITY AND COMMONALITIES AMONG TEACHING PRACTICES

INTRODUCTION

Researchers in mathematical didactics aim to understand and improve the teaching and learning of the discipline. However, the weak diffusion of research results into teaching practices prompts us to look closer at various teaching practices. Do institutional constraints and professional norms render these practices mostly homogenous? Do teachers have some amount of leeway, resulting in individual differences in styles? Are students' classroom activities completely determined by their teacher, or are teachers reciprocally affected by their students? And could this mean that students are themselves responsible for variation in their teachers' practices?

We will address these questions through the case of teaching decimal multiplication to French sixth graders (age 11), beginning with a study of the regularity and variability of mathematics teachers' practices. The "double approach" presented in chapter 2 consists of understanding teachers' work as involving goals beyond student learning, taking into account their own professional objectives as well.

We will analyze the practices of four teachers who work under similar professional conditions. By examining commonalities in their practices, we will analyze the constraints under which these teachers work. This will allow us to both determine if all the originally anticipated scenarios are feasible, and to understand teachers' pre-class and in-class constraints. By examining the variability in individual practices, we intend to present coherences in teaching practices. It is the internal coherence in a teacher's practice that forbids the spontaneous adoption of another way of operating.

After specifying the research topic and the methodology adopted for the "double approach," we will present our results regarding the originally anticipated scenarios, the institutional constraints in place, and finally the scenarios deemed realizable under these constraints. We will then describe our observations of teachers in terms of the regularity, variability, and coherence of their teaching practices.

A METHODOLOGY BASED ON THE DOUBLE APPROACH

We will specify the research topic and present the methodology used, developed under the framework of the "double approach."

F. Vandebrouck (ed.), Mathematics Classrooms: Students' Activities and Teachers' Practices, 75–89.

A research topic aimed at interpreting the constant and variable aspects of practices, in terms of the constraints and flexibility afforded teachers

We have subdivided the overall investigation of regularity and variability in teaching practices into three subtopics. We will detail each topic and indicate briefly for each the approach used to resolve the issues involved.

The first subtopic concerns the various ways a class can be taught, in terms of the institutional and social components of teaching practices. This subtopic also includes the choices made by teachers from among these various possibilities. After evaluating the issues at play in teaching decimal multiplication, we will investigate the possible didactic transpositions in light of the numerous publications on this topic. We will then compare the observed practices to those that were originally anticipated.

The second subtopic, which includes personal, cognitive, and mediatory components, concerns the development of lessons, focusing on student work as a function of their teachers' activities. Our goal was to compare students' effective activities to the tasks as envisioned in the scenario, while also examining classroom interactions and assistance provided by teachers during the completion of the tasks.

Teacher constraints, the amount of freedom allowed within these constraints, and overall practice coherence constitute the third subtopic. Through a survey of official documents, we determine the constraints of the school system regarding the number of hours of class time as well as curriculum topics. Through interviews with teachers, we attempt to evaluate the weight of the constraints tied to the expectations of the school system, and of those tied to practicing the profession, in the classroom, with students. The variability of practices can be explained by the fact that teachers make different choices while operating within the leeways afforded them under these constraints. We attempt to define the limits of the leeway afforded teachers in order to specify the space of possible professional activities. Finally, between constraints and leeway lies the question of the coherence of teachers' choices. Even if, from the theoretical standpoint, this coherence of practice is a given, we are still interested in understanding how it is manifested. We look for indicators of this coherence by examining differences between the choices the teachers made during the preparation stage and the ones made during the actual classroom practice.

A corpus of published sources and classroom observations

The first subtopic involves determining the scenarios that are actually realizable in the classroom. This determination was conducted using studies on the mathematical topic and examining them in light of the institutional requirements and the constraints that stem from students' prior knowledge and their difficulties learning the topic. These studies rely on published sources such as curricula, manuals, evaluations of student competencies, publications intended for teachers, and research conducted in mathematical didactics.

We begin with an analysis of the mathematical concept at hand, decimal multiplication. The meaning of the multiplication must be understood in reference to the theory of the conceptual fields (Vergnaud, 1990). Studies of the mathematical issues involved in teaching decimal multiplication include the work of Brousseau (1987, 1998) and Douady and Perrin-Glorian (1986) on decimal numbers, and the work of Vergnaud (1979, 1981, 1983), Rogalski (1985) and Butlen (1985) on multiplication. Data extracted from these studies touch on multiplicative situations, properties of the multiplicative operation, calculation techniques, multiplicative written expressions and their potential transformations, and the connections between situations, properties, and their written forms. These data were then used to analyze the possible and observed teaching scenarios.

These possible scenarios were determined by assessing two constraints that have a strong influence on teachers' choices: The didactic transposition from the concept to the lesson, and students' difficulties in learning the subject. We first analyzed the diverse lesson plans proposed in didactic research, official curricula, and teachers' manuals. We also analyzed the results of various evaluations conducted by the Ministry of National Education and by the *Association des professeurs des mathématiques* (Mathematics Teachers Association) in order to better understand the difficulties on the part of students that teachers confront and that they therefore may keep in mind while planning their course.

The teachers whose lessons were observed were chosen according to precise criteria derived in accordance with the research topic. All variables concerning the lesson, except those tied to the teacher as an individual, were fixed. All the observed lessons involved experienced teachers using the same manual to teach the same topic to sixth grade classes who were at the same overall level, of similar size, and for similar lengths of time. In order to neutralize the time factor, each teacher was observed during all class periods dedicated to decimal multiplication. The term "sequence" refers to the set of these class periods.

The observable factors used for collecting data on teachers' lesson plans and class period activities are described in the two following sections. These factors were defined so as to be neither so fine that they hide commonalities, nor so broad that they mask differences.

The observable factors in scenario analysis

As indicated in chapter 2, the planning of a lesson is called a "scenario," both to acknowledge the fact that teachers picture themselves in class during lesson planning, and to differentiate the planned lesson from students' actual activities. Three observable factors are used to analyze scenarios: the mathematical field, the teaching strategy, and the mathematical tasks assigned to students. The *mathematical field* describes the set of content introduced during the sequence: concepts, situations, symbolic representations and their transformations, properties, and theorems. The *teaching strategy* consists of the organization of a sequence's mathematical content along a path chosen for mathematical or cognitive reasons. These reasons can vary with the teacher. Some teachers begin by providing the information to be learned before giving students mathematical problems to solve, while others choose the reverse strategy. We can also differentiate teachers by whether or not they institutionalize the mathematical knowledge that may or may not have been constructed by students through the in-class problems. Finally, the *mathematical tasks* are analyzed in reference to the criteria presented in chapter 2.

The observable factors in lesson analysis

In order to analyze the events of a class period, three observable factors were defined: students' effective activities, the assistance provided by teachers, and the order and organization of the lesson.

Recall that once a task has been assigned to the class, the *potential activity* is what the student ought to do to complete the task, the *real activity* is what the student does, and the *effective activity* is the reconstruction by the teacher of the probable real activity, as a function of the potential activity and of productions by the student (such as what the student says).

Below are three examples of tasks, together with the corresponding potential activities. All three can lead to the same effective activity: determining the product of two decimal numbers using a calculator.

– Task 1: Calculate 3.14 × 2.08. Potential activity: Apply the standard solving technique for calculating the product of two decimal numbers.
– Task 2: True or false? 3.14 × 3 = 9.43. Potential activity: Determine the last digit of the product of two decimal numbers.
– Task 3: Place the decimal point in the result of the equation 3.4 × 2.5 = 85. Potential activity: Determine the order of magnitude of the product of two decimal numbers.

The assistance provided to students by the observed teachers was primarily procedural, responding to what we call *didactic incidents*. As a result, this assistance was assimilated into incident management methods. The incidents considered here are not breaches of discipline, but actions that do not correspond to the possible correct responses. Four types of incidents were identified: Questions, errors, incomplete answers, and silences (when a student does not respond to a question asked by the teacher). Below are examples of the most common incidents. All are in reference to Task 4: Place the missing decimal point in the equation 1.35 × 42 = 5.67.

– Question. Raphael asks, "Can we say there is no missing decimal point?" Clearly, Raphael is counting digits after the decimal point. His question shows negative progress towards activity that would lead to the correct answer.
– Error. Maud says, "To place the decimal point, I added a zero. I wrote, '1.35 × 0.42 = 5.67.'"[1] Maud's error is most likely a carry-over from decimal addition.
– Incomplete response. If Maud had only said, "To place the decimal point, I added a zero," her incomplete response would have been an incident. The class could then have wondered if Maud was thinking of .42, 4.02, 4.20, or 42.0, all of which could have corresponded to possible attempts to solve the problem.

An incident is managed through the subsequent intervention of the teacher. Methods of incident management that were observed during sequences led by the participating teachers were classified into two groups, depending on whether they tended to provoke students into re-tackling the task. We understand teachers' reception and management of incidents as factors that influence students' work, as well as, we hypothesize, student learning.

Class periods were divided into episodes characterized by the teacher's specific goals. This provided a chronology to the sequence. At a global level, this chronology allows us to analyze the organization of learning moments, as well as the dynamics between the class and the solving of problems. At a local level, this chronology feeds into the analysis of incident management, particularly regarding the influence of the passing of time on the interactions between students and teachers.

FROM POSSIBLE SCENARIOS TO REALIZABLE SCENARIOS

Using the previously referenced studies, we identified the possible ways to teach decimal multiplication. After evaluating the constraints, and examining teaching manuals, we determined the realizable scenarios.

A typology of possible scenarios

In the research literature, strategies for teaching decimal multiplication are differentiated by their representations of decimal numbers and by their global didactic choices. In terms of representations, decimal numbers can be considered as particular cases of rational numbers, or considered independently of fractions. This decision has consequences on the proposed tasks, particularly regarding rewriting and the available methods of justifying the solving technique. In terms of students' planned cognitive itinerary, three types of scenarios can be identified. In the first type of scenario, the solving technique is first introduced by the teacher, and then applied by students to calculate products. These products may serve as answers to problems in which the multiplication is contextualized. In the second type of scenario, the teacher first presents an introductory problem. The solving technique is partially determined by students, and may be defined in terms of the example problem. The technique is then applied. In the third type of scenario, problems arising from multiplicative situations are given to students. The solving of these problems leads to the determination of the solving technique, which will be reinforced and reapplied to new problems.

All teaching manuals propose scenarios of the first two types, and consider decimals independently of fractions. The algebraic properties of the operation on which the solving technique relies always remain implicit. The study of multiplicative situations is largely neglected: The multiplication is always decontextualized, except when the problems involve price calculations. By contrast, literature aimed at teachers (generally written by teacher educators or researchers), as well as research in mathematical didactics, suggest only scenarios of the third type. Our analysis also shows that authors writing directly to teachers connect fractional and decimal representations, but do not always connect the meaning of the multiplication to the solving technique.

From possible scenarios to realizable scenarios: The effect of constraints

To develop a teaching scenario, teachers use published sources and their own mathematical knowledge. They also keep in mind certain constraints, with the most important being official requirements, students' current knowledge, and known difficulties in learning the specific topic.

In France, fractions are introduced in elementary school, but are not studied at much depth until later. Calculating with fractions is taught in secondary school. At the time this study was conducted, multiplication in elementary school was limited to multiplication of a decimal number by an integer. Multiplication of two decimal numbers was not taught until secondary school. The specific mathematical content to be taught was prescribed: Exploration of different methods of calculation (written, mental, reasoned, approximate, or with calculation tools) and a number of multiplicative situations. There was a strong time constraint. Considering the entire curriculum, we can estimate that overall 4 to 6 hours were spent on a sequence covering decimal multiplication (including solving problems arising from multiplicative situations).

The evaluations of student competency conducted at the end of elementary school or the beginning of sixth grade provide precise information on students' mastery of decimal numbers and solving techniques, but less information on their recognition of the multiplicative model within these problems.

Decimal numbers remain, for some students, two integers of possibly different status separated by a decimal point. In French, to read the number 3.14 aloud, we do not say "three-point-one-four" but "three-point-fourteen," without reference to units and subunits.[2] The proportion of errors corresponding to the misconception that a decimal number is composed of two integers varies between 10% and 50%, depending on the problem. Problems involving multiplication of a decimal by a power of ten (10 and 0.1, 100 and 0.01, etc.) are solved correctly by 50% to 70% of students.

Integer multiplication problems are solved correctly by approximately three out of four students, depending on variables such as the presence of a zero in the multiplier, or the necessity of using a product from the multiplication table of two factors larger than five. This proportion remains approximately constant for the multiplication of a decimal by an integer. Exam questions given after the unit that involve multiplication of two decimals show certain difficulties in learning. The questions are solved correctly by only 35% to 55% of students. Twenty percent of the errors are in the placement of the decimal point.

We find few multiplicative situations on exams. Does this represent the actual intentions of the school system, or the assumptions of test creators as to teaching practices? In any case, the only situations covered on exams are size isomorphisms and finding the area of a rectangle. Otherwise, the results of the multiplication are largely unused. Situations involving a unit price and a quantity are recognized as multiplicative by 80% of students. Finding the area of a rectangle is a source of difficulties for more than half of students, who confuse the concepts of area and perimeter or their respective formulas.

Such results cannot help but affect teachers' choices. The task is considerable (effect students' acquisition of the concept of a decimal number, broaden the meaning of multiplication, and teach a solving technique which causes many students to stumble) and the teaching time is limited. It is therefore unlikely that a teacher will develop a scenario where multiplication is contextualized, where fractions and decimals are connected, and where students construct and justify their solving technique with reference to a multiplicative situation.

OVERALL SIMILARITIES IN SCENARIOS

The sequences of four teachers were compared, from the outline of their scenarios to classroom activities, in order to respond to the central research question concerning commonalities and variance among teaching practices. Results are presented in the two following paragraphs. The first discusses scenarios, and the second, activities in class.

The teachers are given names of mathematicians in order to distinguish them and to refer to them throughout the analyses. We call them Ms. Germain, Ms. Agnesi, Ms. Theano, and Mr. Bombelli. The reader should be aware that mathematics teachers at this level of schooling teach only this subject. They have studied mathematics for at least three years at the university level, and have received training analogous to that of future engineers or mathematics researchers.

Analysis of the mathematical field

Teachers' choices did not diverge widely. Their scenarios were all of the first or second type, as defined in the publications cited above, and decimal numbers were always treated independently of fractions.

The mathematical field is composed of the content studied: Calculation techniques, properties of the operation, symbolic representations of numbers, multiplicative situations, etc. Table 1 summarizes the comparative analysis of the mathematical fields.

All teachers taught the solving technique, justified it, and presented alternate calculation methods to students, such as mental, reasoned, or approximate calculation. All teachers also treated the case of multiplication by a factor less than one. This case is crucial, as it challenges the idea that multiplication results in a larger number. This property, carried over from working with integers, is the source of numerous difficulties. The teachers were also unanimous in not discussing multiplication by zero or one. This unanimity disappeared, however, for the algebraic properties of multiplication and its effects on the order of magnitude. As for symbolic representations, all of the teachers covered the signification of decimal notation, but none made the link with fractional representations. Ms.

Agnesi was the only one to propose a connection between decimal writing and changes in units of measurement. Teachers were completely unanimous in neglecting the study of multiplicative situations. The only problems in which decimal multiplication was contextualized were price problems within a numeric framework. No other situation was studied, and no other framework was called upon, even in the sixth grade classes. Teachers preferred to introduce these subjects later on in the school year, without specifically discussing multiplication.

Table 1. Mathematical fields of the observed sequences.

Mathematical content	**Ms. Germain**	**Mr. Bombelli**	**Ms. Agnesi**	**Ms. Theano**
Technique et properties				
Solving technique	♦	♦	♦	♦
Justification of the technique	♦	♦	♦	♦
Mental, reasoned, or approximate calculation	♦	♦	♦	♦
Multiplication by zero or one				
Algebraic properties of the operation	♦	♦	♦	
Effect of the multiplication on the order of magnitude	♦		♦	
Multiplication by a factor less than one	♦	♦	♦	♦
Representation of decimals				
Decimal notation	♦	♦	♦	♦
Fraction notation of decimals				
Representation using units of measure			♦	
Multiplicative situations				
Size isomorphisms	♦	♦	♦	♦
Product of lengths				
Operation on a length				
Composition of operators				

Analysis of teaching strategies

A certain pattern emerges in terms of teaching strategies, particularly regarding the construction of new knowledge: There was no non-didactic situation, no change of framework, and no tool/object dialectic. Thus, our assessment of the lack of didactic engineering in everyday teaching is confirmed. The teachers, like the authors of teaching manuals, have never imagined scenarios of the third type. We will see that one of the teachers designed a scenario of the first type, and the other three designed scenarios of the second type.

Despite these overall commonalities, we note different dynamics between the course and the exercises, exercises which are sometimes problems aiming at the introduction of new knowledge. For example, Ms. Germain introduced the topic by asking her students the question "How can we calculate the product of two decimals?" She let them produce rules that were effective for certain particular cases. At the end of the sequence, all of these rules allowed students to construct the usual technique. Mr. Bombelli, by contrast, began by presenting the solving technique, which he justified with the help of multiplicative operators. He then gave students exercises on which to apply the technique. Ms. Agnesi began with price problems in which the products of decimal factors could be calculated through conversions. These examples allowed students to infer the solving technique, and the rest of the sequence was dedicated to application problems and the systematic examination of multiplication properties. Ms. Theano introduced the calculation of the product of two decimals using orders of magnitude, allowing students to again infer the solving technique. Students could then check their conjectures with a calculator. Next were application problems and mental exercises that helped students begin to question the solving technique.

Overall, there was general homogeneity as to the content taught, and diversity as to the dynamics between constructing new knowledge and putting it into use for solving problems. What, then, can we say about the mathematical tasks presented to students?

Analysis of mathematical tasks proposed to students

Among tasks proposed by teachers, we distinguish those that aim to introduce new knowledge, and those that lead to applications, to theoretical questioning, or to solving problems arising from mathematical situations. Table 2 summarizes our results.

Table 2. Tasks proposed to students as a function of the intended activity.

Mathematical tasks	**Ms. Germain**	**Mr. Bombelli**	**Ms. Agnesi**	**Ms. Theano**
Introduction of new knowledge				
Non-didactic situation				
Frameworks mobilized	Numeric	Numeric	Numeric	Numeric
Multiplication as a knowledge object	♦	♦	♦	♦
Multiplication as a tool				
Multiplicative situation			♦	
Potential activities				
Determination of a product	**75%**	**71%**	**50%**	**64%**
By written calculation or with tools	*17%*	*14%*	*17%*	*09%*
By mental, reasoned, or approximate calculation	*58%*	*57%*	*33%*	*55%*
Theoretical questions	**25%**	**29%**	**33%**	**18%**
Multiplicative situation	**00%**	**00%**	**17%**	**18%**

The analysis of proposed tasks confirms the uniformity in teachers' choices regarding the introduction of new knowledge: No non-didactic situations, no change of framework, and no tool/object dialectic. Only Ms. Agnesi proposed problems relying on a multiplicative situation. However, the solving technique was not constructed with reference to this situation.

We also noted a certain homogeneity concerning the exercises given to students. However, this result was not statistically significant due to the small population size. Many of these exercises (50% to 75%) led to a potential activity of calculating the product of two decimals, but applications of the solving technique (9% to 17%) were less frequent than mental, reasoned, or approximate calculations (33 to 58%). Other exercises led to theoretical questions (18% to 33%) or to solving problems arising from multiplicative situations (0% to 18%).

Overall, the lessons of the observed teachers were convergent in terms of content introduced and tasks prescribed, but were distinguished in part by the teaching strategies used. Following this assessment, we will attempt to determine if, despite the similarity of tasks, students' activities will differ, particularly regarding knowledge construction. The analysis of classroom activities will allow us to evaluate this prediction.

DIFFERENCES IN CLASSROOM ACTIVITIES

The study of classroom activities has two parts: Analyzing the effective activities of students and analyzing the assistance given by teachers.

Before analyzing the activities, we should note that observed sequences lasted between 2.5 and 5 hours, not including evaluation. The estimated timespans from lesson plans were thus respected. Presumably, no teacher spent longer on these multiplication lessons as a result of being included in this study.

A larger variety of effective activities than potential activities

The passage from potential to effective activities requires some methodological explanations. Once students have difficulties with a task, teachers can provide assistance that will guide students to different activities. For example, a teacher who has assigned Task 4, "Place the missing decimal point in the equation 1.35 × 42 = 5.67," may ask students to find the result of 1.35 × 42. This will provoke student activity, leading to a response of 56.70. The teacher can then prompt students to apply this result to the original task, which will this time lead to using a reasoned calculation to deduce that 1.35 × 4.2 = 5.67. But a teacher who asks students to think of orders of magnitude will provoke very different activities. This example demonstrates why the effective activities arising during the observed class periods were both more numerous than, and different than, the potential activities identified during task analysis. It is exactly the effect of the teacher on this transformation that we aim to evaluate and interpret. Our results are given in Table 3.

Table 3. Classification of potential and effective activities.

Potential and effective activities	**Ms. Germain**	**Mr. Bombelli**	**Ms. Agnesi**	**Ms. Theano**
Potential activities				
Written or tool-based calculation	17%	14%	17%	09%
Mental, reasoned, or approximate calculation	58%	57%	33%	55%
Theoretical questions	25%	29%	33%	18%
Problem (contextualized multiplication)	00%	00%	17%	18%
Effective activities				
Written or tool-based calculation	9%	62%	27%	40%
Mental, reasoned, or approximate calculation	58%	25%	50%	44%
Theoretical questions	33%	13%	16%	13%
Problem (contextualized multiplication)	00%	00%	07%	03%

The table reveals differences between the scenario and the in-class activities of each teacher, as well as between the in-class activities of the four teachers. These differences are confirmed through statistical analysis: A number of chi-squared tests of independence with a 1% threshold were performed on the raw data that produced the above table. These tests confirmed both a significant difference for each teacher between the potential and effective activities, as well as a teacher effect on the effective activities.

Examining the respective teaching strategies allows us to interpret these results. Ms. Germain's strategy was to let students develop rules for calculating products, with the intention that these rules would lead them to the solving technique. For her students, technical exercises were often enriched by complementary questions favoring reasoned strategies or student introspection. Mr. Bombelli's strategy, by contrast, was to present the solving technique and have the students apply it. This teacher reinforced written calculation activities over mental, reasoned, or approximate calculation activities, and over theoretical questions. Ms. Agnesi chose to introduce the solving technique through price problems, leading her students to reasoned calculation activities. Finally, Ms. Theano asked her students to place the decimal point by determining the order of magnitude of the product, and then to check this result with a calculator, leading to approximate and tool-based calculation activities.

To conclude our analysis, we note that students' effective activities show a wider variety of practices that the potential activities would have allowed us to predict. A teacher's in-class work therefore seems to determine students' activities. During the lesson, the teacher modifies the proposed tasks appropriately in accordance with a teaching strategy.

Didactic incidents and teacher assistance

To consider students' actions in class, and their management by teachers, we will present didactic incidents observed in class, and the assistance provided by teachers in response to these incidents.

The number of incidents per class hour varied as a function of the teacher. Overall, incidents were frequent. Mr. Bombelli, who had the fewest incidents, had an average of one incident every three minutes. Ms. Agnesi, who had the most, had double this incident rate. Incident classifications are given in Table 4. While the effect of the teacher on incident classifications was not significant, we observe four values that are noticeably different from the average values (highlighted in bold). These values will inform our interpretations of teaching practices.

In Mr. Bombelli's class, questions predominated, while in Ms. Agnesi's class incomplete answers were the most common. This difference provides evidence of a pedagogical divergence: While Ms. Agnesi values student participation, Mr. Bombelli's students are expected to answer completely and correctly. Thus, when Mr. Bombelli's students are unsure, they prefer to ask questions rather than answer incompletely. We note also numerous student answers to questions posed by Ms. Theano that indicated that the questions were completely out of reach for students.

Ms. Theano focused predominantly on orders of magnitude, despite this concept posing numerous theoretical problems.

Table 4. Classification of incidents in the observed sequences.

Didactic incidents	Total	Ms. Germain	Mr. Bombelli	Ms. Agnesi	Ms. Theano
Error	25%	27%	28%	21%	26%
Question	18%	16%	**32%**	15%	20%
Incomplete response	38%	36%	**16%**	**49%**	36%
Silence	9%	12%	8%	6%	7%
Question out of reach	4%	1%	0%	4%	**11%**
Disagreement	6%	7%	16%	5%	0%

Examining the assistance provided by teachers in response to in-class incidents reveals both their practices and the effect on students' activities. Table 5 shows the classification of each teacher's incident management methods into those that provoke students to further activity and those that do not.

Table 5. Incident management by teachers.

Incident management	Ms. Germain	Mr. Bombelli	Ms. Agnesi	Ms. Theano
Provokes further student activity	72%	21%	42%	50%
Does not provoke further activity	28%	79%	58%	50%

Ms. Germain's incident management provokes further activity in students in more than 70% of cases. By contrast, Mr. Bombelli prefers, almost 80 times out of 100, to not pass the activity back to students but instead to complete the proposed task himself. The management methods of Ms. Agnesi and Ms. Theano fall between these two extremes. The substantial differences that appear between teachers in Table 5 are confirmed by statistical analysis: A chi-squared test of independence at a 1% threshold was conducted, which allowed us to conclude that teachers have a significant effect on incident management. Incident management methods therefore appear to be a personal aspect of teaching practices.

In conclusion, our analyses show that teaching scenarios are overall constrained, particularly by institutional factors, but that there remains a certain amount of leeway that teachers use as much for designing a cognitive path for students as for managing in-class interactions. Their choices conform to their conceptions of teaching and learning.

SOCIAL, PERSONAL, AND COHERENT PRACTICES

The ergonomic approach, by considering teachers' practices as simultaneously personal and as taking part in a professional arena, allows us to propose several hypotheses for interpreting the results discussed above, in terms of the overall commonalities of practices, as well as their local variations.

Between the scenario and the in-class activities: Results in the form of hypotheses

Whenever teachers make the same choices in their work, we must ask what professional necessities their choices are reflecting. Our analyses and interviews have prompted several hypotheses. As teachers act as if they were all respecting principles of professional necessity, we will describe these hypotheses in terms of principles.

The observed teachers respected the curriculum's content, as well as its rhythm. They thus responded to a "principle of conformity to official curricula," which assures them professional legitimacy in encounters with students and their parents, with colleagues who will teach the same students in the following year, and with inspectors who are charged with implementing instructions from the ministry.

Two other principles allow us to better understand commonalities between teachers in terms of the field of mathematical content to be taught. The "principle of pedagogical efficacy" reflects the fact that teachers do not introduce mathematical content with which students show difficulties unless it is indispensible to the sequence. We can see this principle at work in the omission of problems involving fractions and finding the area of a rectangle. In addition, the "principle of an enclosed mathematical field" leads teachers to avoid teaching content that is too directly tied to the omitted concepts. As a result, the mathematical objects that remain within the field of the sequence connect to each other, but do not depend (or depend only slightly) on non-integrated objects. These principles are surprising, as they apparently lead to excluding from instruction those topics that students find the most difficult! In fact, these two principles lead to a hierarchy of content, and to avoiding subjects that threaten to pose difficulties that the teacher cannot handle without deviating from the intended path and risking confusion that will not be beneficial to student learning. This guarantees a strong guideline that keeps teachers within what Rogalski (2003) calls "the envelope of acceptable trajectories."

Finally, the "Principle of the necessity of success by steps" explains how teachers segment their instruction in such a way that students regularly engage in the activity of applying what they have just learned. Without making use of any complete model of learning dynamics, teachers use isolated simple technical tasks to evaluate the impact of their instruction as they go along.

How coherent are teaching practices?

The assessment of commonalities and variance of teaching practices raises the question of the coherence of these practices for each teacher. Analyzing each sequence, and comparing the different results obtained, allows us to identify levels of coherence of practices. These levels may seem unjustified, as they are based on only four examples of practices. We mention these results due to their confirmation by other research presented in this volume.

We have repeatedly remarked on the contrast between Mr. Bombelli's sequence and that of Ms. Germain. The factor dividing these teachers seems to be tied to their conceptions of the classroom. For Mr. Bombelli, it is a place for demonstration and application of knowledge. For Ms. Germain, the classroom is a place for construction of knowledge by students. These conceptions give coherence to their respective practices. In a classroom that is conceived as a place of demonstration and application of knowledge, the demonstration of knowledge takes place very early. The effective activities are primarily applications. Incidents are mainly questions or errors, and their management rarely provokes further activity from students, as the teacher can, if necessary, show an example. In a classroom conceived as a place of knowledge construction, knowledge is established fairly late. Research activities predominate, and incident management provokes further activity by students.

Ms. Agnesi's practice does not fall under one of these extremes. This is a teacher who would like her students to express themselves easily. She tries to involve them as much as possible in the class, and encourages their activity. Her conceptions of teaching and learning lead her to expect her classroom to be above all a place of exchange between teacher and students. Her students respond to this expectation. The number of didactic incidents in her class is substantial, particularly for incomplete responses, for which the rate is markedly higher than those found in the other classes.

Before concluding, we should note that this study also shows the variability of each teacher's practice. Despite the constraints and the conceptions that organize their instruction, teachers are continuously adapting their actions in class. One result particularly concerns the effect of time pressure on the practices of certain teachers. The classroom conceptions of Ms. Germain and Ms. Agnesi, as a place of knowledge construction or of exchange, require giving plenty of time to students. However, to respect the rhythm imposed by the principle of conformity to official programs, once the first half of the sequence is over, teachers find themselves obligated to adopt a more closed style of student interaction.

CONCLUSION

This study of the teaching practices of mathematics teachers is a clinical study. The results refer only to the work of four teachers, which limits the applicability. Nevertheless, these results have not been invalidated by a large number of studies on teaching practices, several of which are presented in this volume.

The observed regularities show that the school system, in fixing the content to teach and the length of the lesson, constrains teaching practices from initial lesson preparation to the eventual in-class activities with students. Other research shows that it is often gaps in the curriculum that constrain teaching practices. The study presented in this volume by Julie Horoks gives one example, in the case of similar triangles. Our research on histograms (Roditi, 2009) provides another. In addition, the conditions of the profession lead teachers to share several general principles, and, consequently, to make overall analogous choices as to content and the organization chosen to transmit it. These invariants outline an envelope that contains the observed teaching practices, but that does not contain all the scenarios that would be *a priori* imaginable, if only criteria tied to student learning were taken into account. These results have implications for teacher training.

Nevertheless, practices are varied. Teachers use the leeway available to them beyond the constraints, and the range of observed differences includes the inferred activities of students as much as the assistance provided by teachers. The observed diversity can be explained by the personal component of practices, whose connection to conceptions of teaching and learning was shown above. The research presented in chapter 11 shows the greater or lesser influence on practices of other personal characteristics of teachers, such as age, gender, and initial training. In addition, linguistic analyses of teachers' speech, such as the study presented in chapter 4 show the specificity and global stability of a teacher's speech patterns. Hence, not everything is possible for a single teacher, and the numerous choices a teacher makes seem to center around a pre-determined logic, while constantly adapting during every instant of class.

Overall, this research has highlighted elements related to individuals that explain the diversity of teaching practices. It has also shown that teachers share certain elements, and that this commonality homogenizes their practices. These elements are undoubtedly tied to institutional constraints, but also, more largely, to their profession.

NOTES

[1] In France, we always write the ones digit, and thus write 0.42 and not .42.

[2] Translator's note: In France, the decimal separator is actually a comma. The number is written 3,14 and pronounced "three-comma-fourteen" (*trois-virgule-quatorze*).

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4. STABILITY OF PRACTICES: WHAT 8TH AND 9TH GRADE STUDENTS WITH THE SAME TEACHER DO DURING A GEOMETRY CLASS PERIOD?

In this chapter, we will present a comparative study examining excerpts from two geometry classes[1] taught by the same teacher.[2] The classes involve students in two different grades at the same junior high school. The first class contains 8th graders (*quatrième* in France, students age 13-14) and the second 9th graders (*troisième*, age 14-15). The study focuses on ordinary teaching practices at a relatively privileged establishment. The class periods we will examine cover the first non-self-evident exercises given to students after lessons (in the previous class period) on two of the most important theorems studied in junior high: The Pythagorean theorem (8th grade), and Thales' intercept theorem (9th grade). In both cases, these exercises are given to students as in-class problems, and follow the in-class correction of a simpler exercise that was given as homework.

Our goal is to make progress on two research topics. The first concerns teaching practices, their stability for a given teacher, and, more specifically, the identification of intra-personal regularities, or "practice invariants." To understand these invariants, imagine if we were to enter another class taught by the same teacher. What, beyond personal characteristics (voice, gestures, etc.), could tell us that this was the same teacher?

The second topic, which we will touch upon only briefly, is that of the ultimate consequences of these invariants on students' activities.

As discussed previously, these analyses of in-class teaching practices fall within the framework of studying the five identified components of teaching practices, which can lead to several levels of work. This study is primarily focused on directly observable components (cognitive, mediative – cf. Robert & Rogalski 2005, and to a certain extent, personal), which are tied to in-class actions and which we will examine on local and micro levels.

Our study examines excerpts from two classes of similar makeup led by the same teacher. By studying two classes with similar student populations, we intend to neutralize any social or personal components. To lessen as much as possible the influence of the "institutional" parameter, we are focusing on two geometry class periods, and specifically on students' in-class work on two exercises that are given almost immediately following the corresponding lesson. In both cases, these exercises are the second given during the class period, and the first exercises on the subject to be somewhat complex. Students work on solving these problems during the classes' second half-hour.

F. Vandebrouck (ed.), Mathematics Classrooms: Students' Activities and Teachers' Practices, 91–116.

Our work is based on (transcribed) videos of class periods and on an interview with the teacher, called D.

We will define three types of progressively finer analyses:

– An analysis of assigned tasks, of how these tasks unfold in class, and of possible student activities, at a local level (first section);
– An analysis of selected periods of interaction between the teacher and students and their corresponding linguistic actions (second section);
– An analysis of linguistic markers (third section).

The analysis of tasks and of their realizations in class enables us to determine students' possible activities and to identify areas of potential regularities in teachers' practices.

The analysis of interactions reveals the manner in which the teacher, sentence by sentence, guides the progress of the didactic project while simultaneously guiding students' understanding. This analysis also allows us to more precisely define the invariants discussed above.

The analysis of linguistic markers allows us to identify patterns in teacher interactions with students (Robert & Rogalski, 2005). We will compare the nature and classification of these patterns in the two classes to complete the analyses.

This study, focusing on excerpts from just two class periods, can clearly serve only as an introduction to practice stability analysis. However, our hypothesis is that our results, though based on only a few class periods, can be taken as representative of the stability we seek.

ANALYSIS OF ASSIGNED TASKS, THEIR UNFOLDING IN CLASS, AND POSSIBLE STUDENT ACTIVITIES DURING THE TWO EXCERPTS

We will describe and compare the tasks assigned to students, the realization of these tasks, and student activities, following the methodology given below. Next, we will conduct a global analysis to identify possible invariants.

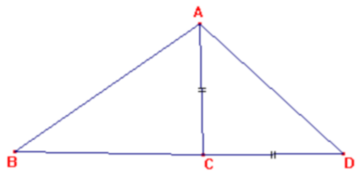
Analysis of the two exercise tasks

The 8th grade exercise

The lessons preceding this class period discussed the Pythagorean theorem, as well as the converse property (that any triangle for which the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs is a right triangle).

After a simple exercise on applying the theorem's converse, the teacher assigns the following exercise:

1) Construct the following figure, with BC=6,4cm
AC= CD=4,8cm
AD=6,8cm and BA=8cm
2) Are points *B*, *C*, and *D* collinear? Justify your answer.



The given figure is ambiguous, as we cannot tell if *C* is on *BD*. It is not explicitly stated that *B* and *D* are on either side of *C*, but all students will take it for granted that they are. We note that a figure in this type of geometry problem should be accurate enough to inspire the strategy to follow, but it should also inspire doubt, so that students will begin to work on the problem. This double role, described by Perrin-Glorian, is found in figures illustrating other geometry exercises.

Table 1. Tasks and task analysis in the 8th grade "Pythagoras" exercise.

A1. (Partial) recognition of the knowledge to be used and the way to do it A2. Introduction of intermediates. A3. Combining of multiple strategies or concepts. A4. Introduction of steps. A5. Introducing results from previous questions. A6. Choice.	Students are asked to complete a multistep reasoning process (A4) with multiple successive changes of viewpoint (A3), including: Going from looking for collinearity to identifying a 180-degree angle, from investigating one angle to seeing it as the sum of two, and from finding the measure of these angles to finding whether the triangles are right triangles. Note that there are three possible cases, as neither, one, or both of the triangles could have a right angle. If neither is a right triangle, students at this level will not be able to answer the questions. We can, however, count on the didactic contract to exclude this possibility. Note also that no comparison of the length of *BD* to the sum of the lengths of *BC* and *CD* is mentioned or implied. Two isolated tasks are included in these steps: the respective investigations of the natures of triangles *ABC* and *ABD*. The sides opposite the potential right angles are easily identifiable as the longest sides, so choosing the strategy of applying the Pythagorean theorem (or its converse) requires only a single adaptation: the choice of the "legitimate" theorem (A1). The required calculations involve integers or decimals with one digit after the decimal point, so calculator use is appropriate.

We note that the order of tasks as given in the exercise may actually be the reverse of the order in which they are completed by students.

The 9th grade exercise (ages 14-15)

The lesson that preceded this class covered Thales' intercept theorem. This theorem concerns the parallelism of two lines that cross a pair of intersecting lines, and equates this potential parallelism to the presence of equal length ratios.

After a simple exercise on applying the converse of the theorem, the teacher assigned the following exercise:

Triangle *EFG* has EF = 5, EG = 7, and FG = 9 (all units in cm). Point *M* lies on *EF* with EM = *x*. Point *N* is placed on *EG* such that lines *MN* and *FG* are parallel.
Express EN and MN in terms of *x*.
Find *x* such that the perimeter of trapezoid *MNGF* is equal to 19.8.

Below is a list of tasks corresponding to this exercise, taking into account the curriculum and the level of the class. Tasks marked in italics will not be analyzed.

Table 2. List of tasks and analysis of tasks in the 9th grade "Thales" exercise.

A1. (Partial) recognition of the knowledge to be used and the way to do it. A2. Introduction of intermediaries. A3. Combining of multiple strategies or concepts. A4. Introduction of steps. A5. Introducing results from previous questions. A6. Choice.	Create a figure with a variable point. This first step is not explicitly indicated. The numeric data provided do not preclude a construction based on true lengths and proceeding by measuring. Placing *M* needs an adaptation (A1). Recognize that Thales' theorem must be used with the given figure. To use it, adapt the statement of the theorem as given in 8th grade. In effect, the length EM must be replaced by the variable *x*. This is not a simple use of the theorem (A1). *Perform an algebraic transformation on quotients involving numbers and letters to set in fractions. This task must be performed twice, independently, constituting work in a second framework (A3).* *Express the perimeter of a trapezoid, the definition of which is not given (but assumed to be known), by an algebraic expression derived from previous steps (A2).* *Write and solve an equation in* x *(unknown) of the type* cx = ax +b *(algebraic work).* *Verify that the solution is geometrically acceptable (not explicitly indicated).*

Some ambiguities may appear. The figure is not described strictly according to the order of its construction: *N* is given to be on side *EG* before *MN* and *FG* are revealed to be parallel. Students are accustomed to this type of text; nevertheless, could this result in differentiation among students? In addition, students at this level have not yet begun to frequently encounter the word "express."

The steps are mostly indicated in the problem statement, with the exception of the first and the last. The questions are not completely independent, but there is no preliminary conjecture or intermediary to introduce. Work can begin quickly.

Comparison of problem statements of exercises on using the Pythagorean theorem (8th grade) and Thales' theorem (9th grade) in terms of intended tasks.

The two problem statements differ with respect to ways the students have to adapt the relevant theorems to solve them. In the 9th grade exercise, adaptations are tied to contextualizing Thales' theorem through the recognition of the appropriate solving strategies and the integration of an algebraic work in a geometric task. The change of framework is indicated, with the algebraic portion treated almost independently. The only intermediate calculation is very guided. The theorem, while unmentioned in the problem statement, serves as a tool for a calculation that is directly tied to a figure emblematic of the theorem. The theorem itself was previously encountered in 8th grade.

By contrast, for the 9th grade exercise, students must first undergo an adaptation tied to a complete reasoning process, with multiple steps and numerous changes of viewpoint. Only then can students place the Pythagorean theorem (and its converse) in context. In this case, the context involves a figure that clearly features at least two triangles. At this point, the only adaptation required is to twice choose the appropriate theorem from three possible choices. These choices were given immediately prior to this exercise,[3] and were used in isolation in the previous exercise. Again, the theorems function as tools, and are not cited in the problem statement. We see that their use is not obvious for students, despite the possible effects of the didactic contract.

The 8th grade exercise thus involves levels of action that require more initiative from students than is required by the 9th grade exercise. We can predict that few students will be able to solve the exercise by themselves.

How will the teacher organize the lesson to take these differences into account? And how will this translate into the class structure and into patterns of interactions with students?

Comparison of classroom events, teacher assistance, and student activities during the two excerpts

General characteristics of work

With the exception of board writing, the work done by students in the two classes is analogous. Students work at their desks, sitting by themselves. There, they work individually, or discuss strategies and share results as a group. Students raise their fingers to answer a question, and recopy correct answers written on the board. In 8th grade, these answers are dictated by students to the teacher to write. In 9th grade, students write at the board under strict supervision (except for the calculations). The board plays the same role in both classes (as a model).

Chronology and nature of work (overall)

Again, these exercises lasted approximately a half-hour and concluded the class period.

Table 3 compares the respective lengths of time allotted by the teacher to the "Pythagoras" and "Thales" exercises. (This table is only a rough indication of the lengths of time. It does not give exact seconds, and does not take into account transitions between activities, which lasted up to around 20 seconds.)

Despite the differences in tasks, we can note substantial structural similarities in the 8th and 9th grade classes in terms of organization, the breakdown of work, and the length of different subtasks assigned by the teacher.

Table 3. Time allocated to activities in the two classes. Percentages are calculated out of the total.

	Pythagoras (8th)	**Thales (9th)**
Organization of the work		First question : in two steps (1) beginning, (2) end
Work on the figure	6 min. (37%)	Construction: 2′30″ (17%)
Finding a strategy	Individual work followed by group work: 8′30″ (30%)	Group work (1): 5′30″ (37%)
Finding the solution	Individual work: 4′30″ (16%)	Individual work, in two separate periods: 2 min. (13%)
Recopying the solution	9 min. (32%)	(1) 2 min. (2) 3 min. (33%)
Total	28 min.	15 min.

The teacher first requires students to recopy the problem statement and the figure (if given). In both cases, he requires them to in some way "enter" the problem statement. In 8th grade, the required description of the associated figure prolongs the length of this stage.

In both cases, the teacher then assigns as a subtask the determination, as a group, of the strategy to follow. In the 8th grade class, students first try to find a strategy by themselves, and then share their ideas during the group phase. The percent of time spent on the group strategy stage is comparable in the two classes. The teacher then lets students work individually on solving the problems. In the 9th grade class, there are two separate periods of individual work. Again, the amount of time allotted to the activities is similar in the two classes. The teacher concludes both classes by providing a model of a correct answer, which is either dictated by students and written by the teacher, or written by a student under strict guidance. The drafting of this model answer occupies a third of the exercise time in each class.

Assistance

In both classes, the assistance provided to students is primarily procedural, and usually consists of identifying a step of the task to be solved. Assistance is particularly common at the beginning of each question or sub-question. For example, the teacher begins by asking students to construct the figure, and then divides the first and second questions into subtasks: "*Now then, of course, you should first draw the figure*," or "*Now then, what is it like? Arthur, describe the figure for us*." The teacher then engages students in finding a strategy (which consists of *dividing* the main task into subtasks). During the strategy finding stage, the teacher uses incomplete responses from students, and throws them back, slightly modified, to ask finally the precise question with the waited answer. In this way, he constructs a path for the strategy. He then summarizes the strategy, and lists the steps on the board.

Some informational assistance is given in response to questions on the definition of the perimeter and on recognizing an equation.

Finally, the teacher's assistance on the writing up of the solution is both procedural and constructive, as he explains in a somewhat general manner what to justify and how to do so. In particular, he notes how to contextualize the theorem and where to place the justifications.

In both classes, constructive assistance is present during exercise correction, with a link between old and new knowledge established through reminders and/or repetitions. This type of assistance is reserved for partial results, such as why fractional representations are preferred over decimal representations, when to use the Pythagorean theorem or its converse, how to explain the answer, remembering this type of exercise that mixes algebra and geometry, etc. By contrast, in both classes, no constructive help is given concerning the global solving method of beginning by drawing the figure, finding a strategy, etc.

Possible activities – A minima

We find two types of possible activities during the exercises: *a minima* activities, for students who wait for indications from the teacher before beginning, and *a maxima* activities, for those who can directly embark on the strategy suggested by the teacher.

We assume that all students draw the figure and then try to find a strategy. For 8th graders, this search may beunclear, while for 9th graders, it may be incomplete. We cannot know for sure that they proceed in this sequence, nor even that they begin under the suggested method. Some of the intended adaptations may have escaped them, without this omission having any perceptible effect on their final work. Thus, at the moment of solving, many students may have completed the calculations (but nothing more).

They have, however, been able to recopy a completely solved example from the board and hear the teacher's explanations.

They will then have had access to isolated activities, each involving a single mathematical concept, but will not have been able to link them.

Comparison of proposed mathematical activities and possible activities according to the a priori task analysis

The commonalities found in Table 6 are more closely linked to the nature of the work provoked by the teacher and to the sequence of activities than to the mathematical content involved.

The consistency thus comes from the organization of the series of activities proposed to students, in which only the overall nature of the work (form and type, length of approximately five minutes) is imposed by the teacher.

Table 4. Tasks and activities for both exercises (with the less studied elements in italics).

	"Pythagoras" (8th grade)	**"Thales" (9th grade)**
***A priori* tasks given in the problem statement**	Use the Pythagorean theorem and its converse as steps in an overall reasoning process to determine if an angle is 180 degrees and if points are collinear. Recognize the methods of applying each property.	Use Thales' theorem to complete algebraic calculations (combining geometry and algebra). Recognize the methods of applying each theorem.
Mathematical activities proposed by the teacher[4]	Understand the problem statement and draw the figure. Find a global strategy with viewpoint changes (alignment → 180° angle → two right angles → two right triangles) (A4, A3). Solve. Show that *ABC* has a right angle at *C* (Pythagorean converse) (A1). Show that *ACD* does not have a right angle at *C* (contrapositive) (A1). Conclude by evaluating angle *BCD*.	Understand the problem statement and draw the figure. Find a strategy. Recognize that Thales' theorem is required and that EM should be replaced by *x* (A1). Solve. Complete two independent phases of algebraic work involving numbers and letters (A3). *Find the perimeter of a trapezoid (presumably a known task). Use the previous calculations to express it as an algebraic expression (A2).* *Write and solve a first-order equation (algebraic work). Verify that the solution is geometrically acceptable (not explicitly indicated).*
Possible student activities	Draw the required figure (SIT): – Describe as a group. – Draw individually. Try to solve the problem, possibly without success. (Individual and group work.) Listen to the correct method for completing all three steps. Note the three steps and treat them successively as simple, isolated tasks. Individual work. *Calculate AB^2 and $BC^2 + AC^2$ (SIT).* *Recopy the completed example of the above.* *Calculate $AC^2 + CD^2$ and AD^2 (SIT)* *Recopy the completed example of the above.* *Calculate angle BCD.* *Recopy the final example.*	Draw the required figure (A1). Try to solve the problem, possibly without success. (Group work.) Listen to the described strategy. Use Thales' theorem geometrically (A1). Individual work (1). Recopy the example. Begin the algebraic work (A3). Individual work (2). Recopy the example. *Listen to the group strategy discussion and begin to calculate the perimeter (find the missing lengths).* *Individual work.* *Correction, recopying.*

By contrast, the mathematical subtasks that determine the specific possible activities (notably the *a minima* activities) differ in nature, with the order of adaptations inversed between the two classes. In 8th grade, students pass from A4 to A1, while in 9th grade the sequence is from A1 to A3 and A2. Once the strategies are established, the 8th grade students are more likely than the 9th graders to continue to work on the sequence of simple, isolated tasks (SIT) that follow from their theorems. This can lead to variations in the students' knowledge development.

We realize that finer analyses will be necessary for evaluating this hypothesis. How, in particular, does a teacher's speech contribute to this consistency?

In terms of the mediative component of the teacher

At the very beginning, the teacher adds a subtask to the list of intended tasks: constructing the figure – we cannot know how long it would have taken students to come up with this step without help. This allows students to enter the exercise and the teacher to explain the concepts in play.

A period of trying to find a strategy is immediately imposed by the teacher as a "general" strategy. The teacher ensures that the presentation of the strategy to be followed is relevant to all students. This presentation occurs after a period of individual work. By having students share their thoughts, the teacher can reconcile a wider variety of ideas, holding onto ones that can help make progress toward a strategy. Any viewpoint changes are mentioned as part of the reasoning process, but, unlike changes of framework, are not highlighted. Students complete the (indicated) calculations during an additional period of individual research, and a detailed correct example on the board (the model) concludes each question.

We can say that this teacher introduces a number of systematic work patterns. The words "habit, habitual" appear frequently, both as actual words spoken repeatedly and as aspects of students' activities, which repeat. Thus, in this class it is habitual to draw the figure, to identify hypotheses and a conclusion before beginning, and to work as a group at the teacher's request to find the specific methods to use before beginning. This, we have seen, can take a variable amount of time. Each time, the teacher provides a corrected model on the board, possibly written by a student.

The teacher also provides substantial guidance to students. He does not let them follow their own initiative for long. Nor, with the exception of two or three students, does he make use of their ideas for finding a strategy. Only the quickest students will be able to develop their own methods before beginning to solve the problem. By contrast, D gives all students a certain amount of autonomy once the tasks have been laid out.

First assessment

The analyses above are associated with what we call the cognitive and mediative components of D's practice, during each of the excerpts studied. We noted important similarities in classroom activities, despite differences in the tasks' possible student activities. To what extent can consistency hypotheses based on only two excerpts be valid?

To answer, we will examine the personal component of D's practices (for certain elements).

We obtained some supplementary information on this component though a questionnaire completed by the teacher on the use of the board in 9th grade. The questionnaire was completed after the teacher watched a video of his own class (Beziaud et al., 2003).[5]

The responses to the questionnaire showed that the teacher considered the 9th grade class period under study to be a typical one. He described his practices as fairly stable, and did not imagine there to be possible alternatives to the choices he had made.

The teacher stated that his goal was to supervise students fairly closely and to encourage student-teacher interactions, both with the class as a whole and with individual students. He also stated that he chose what was to appear on the board carefully, preferring the amount of time spent on writing on the board to remain short.

These elements, apart from the two class periods under study, support the existence of practice invariants.

The questionnaire also allows us to partly deduce the manner in which D claims overall to manage his constraints and leeway. The time spent on an activity is dictated by the progression of the curriculum, which must be completed. The teacher is there to help students, to reassure them, to encourage them, and to allow them certain autonomy, but within a framework that is defined strictly enough that even the most "fragile" students can find something to do.

It thus seems valid to us to identify this teacher's "intervention logic" as a kind of recombination of the mediative and personal components. The fact that we can engage in this reconstitution is proof of the desired stability and explains the consistency suggested above, in the case of the first non-trivial exercises given after a geometry lesson on one of the curriculum's "big theorems." The teacher chooses to give problem statements that are different in terms of how they call upon their theorems, and analogous in terms of the management system they enable. From the teacher's point of view, the exercises allow some students to take initiative and others to work on simple isolated tasks.

In this type of class period, regardless of the task details, students' work is first established as a group. This process consists of listing at least the first subtasks, which then become isolated if not simple (cf. SIT, chapter 2). This listing of subtasks more or less transforms the activity on the corresponding tasks. To develop the list of subtasks, the teacher modifies and completes students' responses to open-ended questions. The students do not have control of the preliminary investigation.

Next, the time given to students for individual work allows them to attempt and even complete at least the first of these subtasks. The teacher circulates among students and occasionally publically uses volunteers' indirect assistance.

Finally, once a certain number of students have finished working, a carefully completed example solution on the board gives students who recopy it into their notes a model to follow.

During the development of the example solution, there is little reference made to individual work. There is no overall assessment of strategies, or reference to subtasks or to methods used. There are comments on how this exercise differs from others or on how to write the example. "Constructive" assistance does not involve the global strategy.

The 8th grade/9th grade comparison leads us to ask if the consistency in the in-class activities might cause a more difficult task to be even more divided into isolated simple tasks. Perhaps the teacher compensates for the difficulty of a task by subdividing activities? Would students who work *a minima* have sufficient information to allow them to return on their own to the exercises completed in class?

There are alternatives: choosing other ways of working, or a different organization of the sequence of activities to engage weaker students in developing (even partially) the overall reasoning processes (A4).

Furthermore, we wonder if, implicitly, the teacher is delegating certain aspects of learning geometry to these strict procedural habits. The teacher behaves almost as though these types of routines could be transferred to students without being explicitly taught. The stages of drawing the figure, determining the hypotheses and the conclusion, finding a strategy and/or method, and writing out the answer are each distinct, and are always completed in the same order, with the same process. This is correlated to the reduced role of constructive assistance.

Question to pose at this stage:

– Can we find other invariants in teachers' speech? How do they fit into the already noted invariant organization of the sequence of activities proposed to students?
– What influence do these invariants have on student activities?

We have seen, for example, that more complex tasks lead to a more substantial subdivision, and that a certain number of elements remain implicit or absent. All students appear to be working, with some even reporting "success" on the mathematical task. Are there, nevertheless, misunderstandings or missing links in some students' mathematical work?

ANALYSIS OF STUDENT/TEACHER INTERACTIONS

We will first focus on the teacher's speech during interactions with students, to look for potential similarities.

This third analysis will supplement our *a priori* analysis of tasks and possible student activities, and enrich our detailed understanding of the way the teacher considers his students, interprets their work, and keeps them working on the mathematical activity, stage by stage. These local analyses therefore have global goals.

Note that in this study we are only analyzing interactions aimed at the whole class (which may nevertheless involve only a single student directly). We are also only considering interactions that involve more than two exchanges. Each interaction studied is initiated by a question from the teacher or from a student, and each interaction ends once the desired response has been given and the teacher is satisfied that everyone has heard it.

This choice of which interactions to study is supported by the fact that only these interactions represent a true negotiation between the teacher and the students, and only they are indispensible, from a didactic point of view, to in-class events. In particular, student/teacher interactions initiated by the teacher are opportunities for him to mark a step in the progress of his plan for the class period[6]. These interactions help us to understand what information students receive. They also provide us with information about the division of work between the students and the teacher. In other words, we can identify regularities or similarities in the manner in which the teacher, sentence by sentence, guides the progress of the didactic project as well as students' understanding, or in how he contributes to students' mathematical activities during these interactions. How does the teacher enlist students into the activity, and then keep them there? What autonomy do students have? How does the teacher contribute, at different moments, to the knowledge adaptations that were expected based on the *a priori* analysis? How does he intervene into the difficulties encountered by students, or keep track of what students have done, particularly during the presentation of the correct response? How does he handle contributions from students, and particularly from strong students? Does he revisit the methods and potential choices? Does he reassemble the subdivided steps? What type of help (procedural, constructive) does he provide?

The identification of linguistic actions from the transcripts will reveal the role of the teacher's speech during in-class interactions. It will also enrich our analysis of student and teacher activities. We will first present our methodology, and then the comparisons we found by using this methodology in the two class periods studied.

Methodology: Tools for analyzing linguistic actions in the teacher's speech

The teacher's linguistic actions allow us to identify the choices in speech that may contribute to the development of students' activities.

We use the term "linguistic actions," with its connections to language and context, to indicate various considerations. These considerations cause us to attribute a different linguistic action to a phrase depending on the circumstances of its utterance. A single phrase can also correspond to multiple linguistic actions.

For us, a linguistic action is a quadruple with four components. These components are the episode, the syntax type (question or statement), the content (mathematics, meta, etc.), and the speech's function.

The first component, the episode, is identified following the *a priori* analysis, and is characterized by students' work on a task or subtask. The linguistic interactions defined above are analyzed within the context of this episode.

For the syntax type, we identify questions posed by the teacher. These questions contribute to students' participation in the task, and provide information regarding how the teacher takes students into consideration.

When the content of the teacher's speech is "meta," it concerns his own interventions. This can include indications of method, elements of structuring class time or reminders, or placing work in a larger context. Meta speech helps us reconstruct the teacher's intentions. For example: "*Remember, you can find the sum of two squares directly with the calculator. You can write down the intermediate results, or you can do it directly*."

Finally, we analyze the functions of speech, as identified by Bruner for the processes of tutoring and the support provided by an adult when helping a child (Bruner, 1983; chapter 1). For Bruner, these functions define the manner in which the teacher contributes to students' work step-by-step, while trying to support their activities. Does he talk to them about what they have done, correct them, encourage them, or do something else?

Our functions are defined in terms of the work involved in leading a mathematics class. They are designed to allow characterization of the multiple forms of support possible in a class.

Below are the functions of speech that we have identified:

– Participation functions:

Engagement: "*Let's go*."

Repeating information.

Calling for attention: "*Now then, pay attention*."

Encouragement.

Sharing student responses. (Student: "*Variable*." Teacher: "*Variable*, *x is a variable*. *The point M varies, then x varies from what to what?*")

– Other functions, identified by comparing adaptations that students must make of their knowledge, the state of their work, and teacher remarks:

Identification of student work. The teacher considers student productions or questions: "*Now then, to answer Raphael, who just asked if we should write the hypotheses or the conclusion…*"

Information. The teacher provides or requests information regarding the knowledge in play. For example, he may ask for or provide results, theorems, etc.: "*EFG is a triangle such that, then I'll give you…EF = 5, EG = 7, FG = 9, and all units are in centimeters*."

Evaluation. The teacher gives his opinion only on the validity of students' responses, without other commentary.

Structure. The teacher punctuates students' work by placing them in a larger context: "*Now then, J. B., for the second step, tell me what should be done*."

Orientation. The teacher orients students' work without giving everything away: "*We don't really know its true location on EG, huh. In other words, it's a point?*"

Justification. The teacher engages in the justification process: "*Now then, why do we begin with AB squared? Why not one of the others?*"

Assessment. This indicator can refer to a recap or a reflection: "*We put the point M somewhere, and MN is parallel to FG. Now we apply Thales' theorem and write it up like we did in the earlier exercise*."

The teacher can express multiple functions in a single discussion, as in the example below:

Student: "*Well, we'll say we're using Thales' theorem*."

Teacher: "*There you go. We're going to use Thales' theorem because we evidently have straight lines?*"

The teacher evaluates the student's contribution while sharing it. He poses a question that orients the student toward a mathematical justification. The word "evidently" adds meta content to this discussion because it may refer to a habit or remind students of something.

For each episode, we use this labeling system to characterize the functions in play in each linguistic interaction and in the set of interactions. We deduce from our analyses elements that involve taking students into account: questions, assistance, or support for students' work. This enables an initial approach to the class period excerpts, particularly in terms of "internal" regularities. It can also lead to other comparisons.

We used this methodology on the two phases analyzed below.

A comparison of linguistic actions in the two classes

To study the linguistic actions (and attribute the quadruples described above), we chose three episodes to study from the 8th grade class (Pythagoras). The first involves the description of the figure. The second, which includes two disjoint periods of time, focuses on the group efforts to find a solving strategy. The last is the presentation of the correct solution. The episodes are sufficiently long (more than two exchanges) to allow a true dialogue to be established in which the didactic stakes are perceptible.

For the 9th grade class (Thales) we analyzed interactions involving more than two exchanges, of which there were four: Finding a solving strategy for the first question, correcting the first question, finding a solving strategy for the second question, and finally correcting the second question.

In the appendix, we provide an extract of the complete analysis and the results for each exercise. These results provide the basis for what follows.

We are only comparing the linguistic actions in two analogous episodes in the two classes: development of a solving strategy and correction. For the 9th grade class, we are only considering the first exercise. We will try to identify similarities and differences in the episodes.

Comparison of functions of speech

Within the strategy development phases. In both classes, the teacher speaks much more often than the students. This is clear from the transcript.

With that said, students' **participation** is substantial, instigated through the questions asked or due to participation functions. The teacher systematically shares student results. This sharing often involves validation, which can then be modified with a commentary or question from the teacher, leading students to the intended results:

Teacher: "*So, this situation is fairly banal, huh. Given all that we've done, what is the only new thing, Bertrand?*"

Student: "*Uh ... x*."

Teacher: "*x, that is, the point M. What do you say about point M?*"

Student: "*Well, we don't know its real place on EG*."

Teacher: "*We don't know its real place on segment EG. In other words, it's a point?*"

The **structuration function** is used frequently in both classes, but more in 8th grade than in 9th grade: "*For the first step, what will I do?" "For the second step, continue, Alexander*."

The **information** provided by the teacher in the 8th grade class primarily concerns mathematics: "*So, what, what are we going to look at? Well, if each is a right angle, and then we'll look at angle BCD. Okay? And it will either be flat or not*."

However, in the 9th grade class, there is also "meta" information that situates the proposed exercise in terms of students' knowledge ("*that theorem of Thales*") and in terms of old and new ("*Now then, the only thing that's going to be a little different from usual is?*").

In both classes, the mathematical information constitutes help that is apparently procedural.

In the correction phases. In this phase, the teacher again speaks much more than students.

In 8th grade, the exchange is marked by strong **participation** by students. This translates into questions posed at each teacher contribution, which engage students ("*Kurdis, can you give us the first step in detail?*") while **structuring** their reasoning ("*Now then, J. B., for the second step, tell me what we should do?*" "*Now then, how will we finish?*"). The questions can **orient** students toward the intended response ("*What can we conclude about BCD? That is isn't …?*"). This student participation through constant questioning is reinforced by the **sharing** function, which allows the teacher to share with the class the dialogue that he has established with the student at the board. The teacher frequently uses **structuration**. Different steps in the reasoning process are explicitly identified and are reassembled at the end. The teacher leads the process, and students' autonomy is weak. The teacher leads them step-by-step towards the intended answer while orienting their reasoning process.

Assistance is therefore more procedural in nature. However, the assessment that marks the end of the exchange can represent constructive assistance for some students: "*Now then, what is interesting in this exercise is that we have a single question and, ultimately, we applied the converse of the Pythagorean theorem: the result where the formula doesn't hold but we can conclude that the triangle is not a right triangle, and with some help after adding two angles we were able to get a conclusion*."

The teacher relies on students' work before indicating the desired write-up of the demonstrations: "*But I just explained, J. B., that the converse wasn't applicable when the equation was true*." The teacher's speech has mainly mathematics as its object; the meta content concerns possible solving methods.

In the 9th grade class, **participation** is less expressed by the speech's functions than in 8th grade. However, there are still numerous questions posed. The **information** in the exchanges primarily concerns mathematics; there is little meta information. Considerations of students take place orally, but are also based on what the teacher is able to observe in written work: "*Now then, remember when* *we're writing it up, that to apply Thales' theorem there is no 'if then.' Some, not many but two or three, wrote the exercise on the sheet with 'If the lines are parallel … '.*"

An explicit allusion to the exercise and a reminder on the write-ups can constitute constructive help for some students: "Remember that this is the general problem statement and we're applying it. So we know whether or not the lines are parallel." Reminders of new information were featured in the teacher's speech during the entire period and revealed his goals during the class.

Global results In almost every discussion in each episode of both classes, the teacher used questions to mobilize students.

During both classes, the teacher assessed, evaluated, and shared with students. Students were required to engage in tasks that were often subdivided. They were also required to mobilize their attention, and to collaborate with the teacher on justification and structure.

Student participation differed in the two classes, but this difference was in large part due to the task. Again, the 8th grade exercise required more adaptations than the 9th grade problem, and the 8th graders were perhaps less comfortable with solving such an exercise. The teacher needed to ensure that students understood the solving strategy. He therefore gave them time to put it in their own words after describing the three required steps. In 9th grade more than in 8th grade, a single word was often sufficient for students to complete the teacher's sentence. Furthermore, we note again that the correct answer was not presented in the same way in 8th grade as in 9th grade. In 9th grade, a student wrote on the board (occasionally prompted by questions by the teacher). By contrast, in the 8th grade class, the teacher wrote as dictated by students, who were therefore required to express themselves orally.

In comparison, we noted that this teacher had an overall stable use of functions during the strategy development phase, with little variation in details. We can note, however, more information, justification, and sharing functions in this phase in 9th grade, and more structuring in 8th grade. We hypothesize that these variations stem from the unfolding of events that are tied more or less strongly to the task, but that are certainly tied to students. The greater difficulty and lesser subdivision of the initial 8th grade task explains the greater presence of structuring, while their lower response frequency led to less sharing and assessment than for the 9th graders, who had more propositions to create.

For the correction phase, during which a correctly solved example was written on the board, we note stability in the use of structuration functions and in the direct involvement of students (beyond sharing). The sharing function is used more in 8th grade. We note again that the 8th grade exercise was more difficult, that students were less accustomed to it, and that the majority of them did not solve it. For 8th graders, the presentation of the correct response was also a time for students to solve and work on the problem. The teacher again engaged students during this phase and used their answers. In the 9th grade class, where many students were able to solve the problem, the correction phase served more as a time for evaluation or assessment.

Finally, a close study of the functions of speech allows us to identify another invariant: The use of sharing/evaluation/orientation functions that correspond to a light modification of student responses by the teacher to get closer to the desired response:

8th grade:

Student: "*We have a triangle ABD …*"
Teacher: "*We see a triangle … ABD …*"
Student: "*See if it's a right triangle*."
Teacher: "*Ah, we could know if it's a right triangle*."

And in 9th grade:

Teacher: "*What is the only new thing, Bertrand?*"
Student: "*Uhh … x*."
Teacher: "*x, that is, point M*."

Taking students into account

The teacher takes students into account at several levels:

– Directly in exchanges. The teacher may take students into account by varying his responses depending on whether they give the desired response. He may also take them into account by choosing the answers according to the moment the students give them.
– In answering students' questions.
– In reference to their work during the individual work time.

Some examples:

– In 8th grade, during the correction phase: "*Now then, I would like to insist on the placement of this sentence.* ***Corentin*** *did the same thing, but he put this sentence a little earlier. He stated right away that he had, that he was going to apply the converse of the Pythagorean theorem. Now then* ***Corentin****, what did I say to you? Did you understand what I said?*"
– In the same phase: "*There you go. We're not at all sure that we're going to apply the converse of the Pythagorean theorem, because at the beginning you don't know if the equality will hold or not. If it does, you'll say that it follows from the converse of the Pythagorean theorem, sure; but if it doesn't, we can't justifiably apply it. So it's really important that you do this in this order. Do* ***you*** *understand?*"
– Or in 9th grade: "***Fanny****, you have 2x/5 – 5x/5. That makes -3x/5*."
– 9th grade: "*It's not clearly false. Well, now then, here's the first question. So we have answers in function of x. Re member that 7x over 5, then, that can be written in different ways. There are some …* ***raise a finger those of you who wrote a decimal****, like* ***Ludovic****. What did you write?*"

Assessment: What analogues are there in the speech during interactions in both classes?

In both classes, the overall use of speech functions is similar, with a few variations. Most notably, there were more participation functions during the 8th grade episode than in 9th grade.

In particular, we see large analogues, adapted to the level of the class, in the teacher's responses to students' answers. For example, during interactions with students, the teacher always takes their responses and shares them, validates them, and modifies them in ways adapted to the class and the students. The teacher negotiates the desired response while remaining careful to maintain communication.

The teacher's role in class is to evaluate, share, and assess, while the student's role is primarily to resolve subdivided tasks. Students are all asked to participate and are encouraged by participation functions, which are more frequent in 8th grade than in 9th grade.

The teacher's speech thus adapts itself to the class and to the type of task: more calling for student involvement when they've been working on their own, more maintaining student attention in 8th grade when the solving work is taking place in real time during the correction phase (for example), and more controlled by the teacher in 9th grade when he is validating a model solution at the board.

This analysis is still missing elements that could further reinforce (or weaken) the mark of the teacher on the speech. For instance, we have not examined the use of personal pronouns (*we*, *one*), which could help the teacher place himself on the same side of the task as students. In fact, such a study, as yet done in Chappet-Pariès (2008) shows an analogous usage of the use of personal pronouns in the two classes

We borrow several practical tools used in research led by Trognon at the Ecole de Nancy (Gilly et al., 1999) to describe more precisely the illocutionary goals that indicate what speech content is trying to produce. Here, again, the choices of goals manifested during the exchanges are very close in the two classes (Chappet-Pariès, 2008).

Possible next steps include the treatment of other class periods to see what analogies prove persistent and to understand the impact on students. What do students understand, with which potential effects?

LINGUISTIC MARKERS IN THE TEACHER'S SPEECH

We will now compare the above analyses with a different approach to the teacher's speech. This approach relates verbal formulations to the organization of the teacher's contributions to students' in-class work. This organization is identified through verbal indicators we will call "speech markers." Speech markers are "particles," such as "good!" or "so" (when not used as a logical connector). They are grammatically optional and do not change a statement's truth-value. They have been studied in teaching activities, with initial work done in the teaching of English as a foreign language (Sinclair & Coulthard, 1975).

These markers have two functions. First, they mark the organization of verbalized content, and thereby play a role in speech coherence. Particles such as “now then” (*alors*) and “so” (*donc*) play this role. The second function of markers is to ensure the pragmatic structure of the interaction and mark the roles of the speaker. This function is played by terms such as “good” (*bien*) and “Okay” (*d'accord*). The markers are therefore a signal of the relationship between the student's statement and the teacher's reaction, between the teacher's statement and the desired student response (statement or action), or between the teacher's units of speech. They can also simply “punctuate” the teacher's public activity, such as writing on the board. They can mark the introduction of a new element in the teacher's speech, or return to a previous line of speech after an interruption by a student's action or by an observed action to which the teacher responds.

We will first present these markers as evidence of the organization of the teacher's speech. We will examine their use in the initial “draw the figure” episode in D's classes in the 9th grade “Thales” exercise and the 8th grade “Pythagoras” exercise. We will see how they constitute indicators of invariants in the organization of the speech. We will also identify variations, which we can then interpret in terms of the relationship between the mathematical content in play and students' ability levels.

Markers: a diversity of contributions in speech

Markers can introduce acts of speech (analyzed above) that place students in their role as students by using imperatively tensed verbs (or present tenses or infinitives with the same intent as imperatives). These acts of speech can also involve posing questions requiring a response. The teacher uses these markers to call for student participation.

– “***Now then****, listen closely to what he says …*” (Pythagoras, 8th grade.)
– “***Now then****, this says you draw a figure*.” (Thales, 9th grade; present tense functioning as an imperative.)
– “*… and* ***then****, it will vary from what to what?*” (Thales, 9th grade.)
– Markers also punctuate the progress of the class activity. They ensure that students are all working on the same goal at the same time.
– “***Now then****, I'm going to write the third step here …*” (Pythagoras, 8th grade; announcement of an activity.)
– “***Now then****, there are some who have finished …*” (Pythagoras, 8th grade; state of activity in the class).
– “***So****, here we'll pick, sure, the first and the last relationship*.” (Thales, 9th grade; commentary on current activity at the board.)

Different markers may specifically signal the end of an activity and the completion of a (sub) task. For example, “*There you go!*” (*voilà*), “*Okay*” (*d'accord*), and “*Good*” (*bon*) are examples of considering a result proposed by students. We see them function here within an interaction:

Teacher: "***Now then,*** *we're at our main exercise. How many results are there in this chapter?* […]"
Student: "*The Pythagorean theorem*."
Teacher: "*The Pythagorean theorem*."
Student: "*The converse of the Pythagorean theorem*."
Teacher: "*The converse of the Pythagorean theorem, and then, the one that doesn't really have an official name. Well, it's when the equality doesn't hold and we can conclude that the triangle isn't a right triangle.* ***Okay****?*" (Pythagoras, 8th grade.)

And this other one interaction:
Teacher: "*And these, these are what?*"
Student: "*Hypotheses*."
Teacher: "*Hypotheses.* ***Now, then,*** *pay attention.* ***So,*** *we will put the conclusion here,* ***okay****?*" (Thales, 9th grade.)

"So" (*donc*) as a marker can have a "conclusive" function, or can function by connecting previous activities to those that will follow, appearing in the introduction of a new unit of interaction.

We see in these last examples that the markers indicate the boundaries of units in which the teacher and students take turns speaking. These units are not necessarily limited to the well known triplet of "question from the teacher," "answer from the student," "evaluation by the teacher."

Finally, the markers can "punctuate" the continuous speech of a teacher: "***Now, then****, remember when we're writing it up, that to apply Thales' theorem there is no 'if then.' Some, not many but two or three, wrote the exercise on the sheet with 'If the lines are parallel…'* ***Now then,*** *remember that this is the general problem statement and we're applying it, so we know whether or not the lines are parallel ...*" (Thales, 9th grade). We note here that "so" is used as a logical connector relating to the current mathematical activity: "… *so* we know …"

"***Now, then****, what is interesting in this exercise is that we have a single question and, ultimately, we applied the converse of the Pythagorean theorem: the result where the equality doesn't hold but we can conclude that the triangle is not a right triangle, and with some help after adding two angles we were able to get a conclusion.* ***Okay****?* (Pythagoras, 8th grade.)

Speech markers as traces of the organization of the teacher's activity

The analysis of markers leads to defining "interaction units" bounded by introductory markers (particularly "Now, then") and conclusive markers (particularly "There you go," "Okay"). These units include a variable set of turns at speaking by the teacher and the students. The teacher's turns at speaking (and occasionally the students' as well) themselves contain one or more semantic units (the equivalent, for oral speech, of multiple clauses on the same content).

An initial analysis of speech markers for a 10th grade algebra teacher (Robert & Rogalski, 2005) has revealed the existence of patterns of interaction that are identifiable by markers. These interaction patterns contain an introductory marker, a set of treatments of the task object, an assessment of the activity, and a conclusive marker. (The assessment phase is optional, and can also take place after the conclusive marker). When such patterns are recurrent in a teacher's speech, they indicate an invariant organization of the teacher's activity.

Our analysis of D's speech markers during the Thales (9th grade) and Pythagoras (8th grade) problems shows the same organization for the initial phases, when the students need to make a figure according to the problem statement.

After the statement is read, an interaction unit begins:

"***Good, then***, *this says you draw a figure*" (Thales, 9th grade).

"***Then*, *so,*** *you go in the exercise book; you paste this little piece of paper and you draw the figure*" (Pythagoras, 8th grade).

In both cases, the link with previous activities is marked, by "Good … this says" and by "so," respectively. The "then" marker introduces the task to complete: "*Draw a figure*." In both classes, there was an analogous closing marker several minutes later, with a number of contributions in between:

"*Everyone has had time to draw a figure?* ***It's going okay****?*" (Thales, 9th grade.)

"***Now that*** *everyone has had time to draw a figure* …" (Pythagoras, 8th grade.)

Inside this interaction unit are several subunits of interaction. The first involves analyzing the situation through questioning a selected student.

"***So***, *this situation is fairly banal, huh. Given all that we've done, what is the only new thing? Bertrand!*" (Thales, 9th grade.)

"***Now then***, *what is it like? Arthur, describe the figure for us!*" (Pythagoras, 8th grade.)

These interaction units are themselves concluded by once the answer to the question is given. This conclusion happens after several exchanges and a number of semantic units (analogues of clauses) from D. The interaction subunit remains enclosed within the main unit.

In 9th grade, the teacher wanted to explicitly highlight the presence of a variable (*x* as a number, *M* as a point), which constituted the introduction of an important new factor. This factor is made explicit in the closure of the main interaction unit: "***So*** *x is a variable, M varies*."

In 8th grade, it is important that the figure be described as composed of two separate triangles, and not as one "big" triangle (as the drawing in the problem statement could imply). The closure of the interaction unit is strongly marked: "***So*** *we describe it [the figure] as you said afterward; that is, two triangles.* ***So*** *you have ABC and you have ACD.* ***There you go****!*"

Beyond these invariants, the study of markers during this figure-constructing episode reveals a difference that is linked to students' ability levels in terms of the mathematical content in play. We therefore find in 8th grade a long, argumentative contribution from D that is aimed at involving students in analyzing the figure by taking "what we see" (in the figure, the sides of the two smaller triangles seem to form a side of another triangle) and distinguishing it from what we can deduce from the problem statement (which raises the question of the collinearity of three points on these sides). Involving Arthur (a student) in this analysis will require an interaction between what Arthur sees in the figure, and what is really there. The teacher's remark at this point is organized with multiple clauses, using both argumentative connectors (*because, if, yet, so*) and markers (*now then, so, there you go*), until the conclusion: "*We do not define the figure like that*." After delivering this long argumentative thread, the teacher restates the initial task: "***Now then**, first job, **so**, you're told to draw the figure*" (where the "so" marker brings us back to the figure).

We have also compared the role of the particles "now then" (*alors*) and "so" (*donc*) in D's remarks, as well as in the comments of four other 9th grade teachers, in class periods devoted to exercises or discussion sessions.

In general, "now then" predominates over "so" as a speech marker (appearing twice as frequently, with some variability). One teacher, however, used "so" 80% of the time. The particle "now then" appears most often functioning as a marker, and only very rarely in its function as a logical connector. "So," however, is consistently present as a logical connector, but with wide variability between teachers in the same grade. For teacher D, "so" is as much a logical connector as a marker, and is the connector used in approximately half of all logical connections, in both grades.

These data, though "surface" data, indicate a larger variability between teachers than within a single teacher's practice. The stability of a given teacher's practice is tied to the teacher's style, and is not only reflected in general invariants (the genre) of the mathematical activity.

DISCUSSION AND CONCLUSION

Given these three analyses, we ask: If we were to enter a class taught by this teacher, with his body hidden from view and his voice distorted, what would allow us to say, "This is the same teacher"?

A first type of teacher invariance concerns the global organization of in-class events (first analysis; first section).

The types of work are the same in both classes. Students' activities take analogous amounts of time, and the speech that accompanies these activities is also managed in the same way.

After an initial period spent on the figure and on the question in play, a second phase, which may take place immediately, is dedicated to listing the solving strategies. The teacher responds to the choices of the students who are called upon, and moderates the sharing of their answers. The third phase is more directed, and gives students time to work on their own, according to the plan that was designed in the previous phases. This third phase is followed by a very structured correction period in which a model solution is written on the board. In both classes, there is little constructive assistance from the teacher. There is more or less a kind of procedural assistance, actually of the same nature when it occurs.

A second type of teacher invariance relates to certain characteristics of the teacher's speech during interactions with students (second section). The functions of the speech are relatively stable. Some of these functions are more variable than others, and are associated with adaptations that the teacher implements as part of his goals for the class, while taking the task and the students into account. Broadly, for example, the role played by direct participation in a more difficult exercise appears to be compensated by sharing for in an easier exercise.

Moreover, consideration of students' work and student questions is very similar in the two classes. During the strategy-finding period, contributions from all students are regularly and systematically considered for evaluation and sharing. During the correction phase, the teacher refers to his observations of students during the individual work period. He worries about showing the details of the calculations to students who were not able to start working on the problem. However, students called upon in class are never allowed to describe a complete reasoning strategy other than the one intended. It is as if only one course of reasoning is acceptable and only one could lead to a correct result. In each class, the differentiation between students is apparent from the moment they are called upon, in the form and length of their exchanges. Occasionally, exchanges are initiated by students. In addition, the teacher addresses some "meta" responses in an aside to certain students.

A third invariance concerns the similarity in the use of linguistic markers that structure speech (third section).

There are differences between grades in the phases of teacher interaction with the class. These differences indicate adaptations by the teacher based on students' reactions. We note variations in the number of students who are called upon at their desk or to write on the board. The length of each phase also varies between the two grades. In addition, the greater difficulty of the 8th grade exercise led the teacher to divide the problem into more simple and isolated tasks than in 9th grade. The teacher also included students more frequently in the correction phases and encouraged more sharing.

Our analyses have thus allowed us to show a real stability in the mediative component of this teacher's practice, both at the most global level (in-class events) and at the most micro level (linguistic markers). A local analysis reveals more variations (in procedural assistance and functions) that are determined by students' reactions, but no modification of the sequence of planned activities associated to the different tasks assigned in the two classes.

If we suppose that students' activities presumably occur multiple times in the year in analogous unfolding initiated by the teacher, there can be repetition effects that differ for different students. Based on the results above, we present some examples of possible such effects over the long term.

If the teacher suggests every time that students begin a geometry exercise by trying to find a strategy, will students all appropriate this step without constructive assistance?

Some types of tasks given by the teacher for short individual work periods encourage *a maxima* activities, which are visible in the work of some students. Will others students be always excluded?

The highly structured correction phase does not allow for questions from students who are still very far from solving the problem after working on their own. Will these students always have doubts as to the validity of their resolution? Will they be able to use the calculations they have recopied?

Local assistance is provided by the teacher to all students. Is this sufficient for them to learn?

Finally, some intended activities are only possible as long as the teacher's usual management style does not contradict the necessary course of action. For example, if there were never any long individual or group work periods, we might wonder if students were capable of coming up with the steps of a complex exercise by themselves.

In other words, does the stability of this teacher's practice contribute to all students' learning in the same way? And does this always play out in the same way for each student?

In addition, the invariant linguistic characteristics shown in the analyses of communication and speech (with priority given to the use of certain associations of functions) are tied, to a certain extent, to the personal component of the teacher. We can investigate more generally the relationships between the cognitive, mediative, and personal components of a single teacher.

The possibility that certain student activities are incompatible with certain practices has not been ruled out. The teacher cannot develop these activities without making changes that are all the more costly since his practices are stable. We can then wonder if this stability of practices, as studied for experienced teachers, can be modified, and how much expensive it may be.

A comment made by D at the end of the 8th grade class period led to a glimpse of the difficulty involved: "*Now, then, what is interesting in this exercise is that we have a single question and, ultimately, we applied the converse of the Pythagorean theorem: the result where the equality doesn't hold but we can conclude that the triangle is not a right triangle, and with some help after adding two angles we were able to get a conclusion. Okay? Now then, we could certainly imagine this exercise with intermediate questions. It would definitely have been simpler. You were all completely capable of finding what needed to be done each time*."

We can also ask what invariants are shared between teachers. This question leads to an examination of constraints and personal choices.

Learning more about the stability of experienced teachers' practices may allow us to better adapt professional development trainings, by more clearly outlining the links between tasks, intended activities, and adapted management.

APPENDIX

Example of the speech methodology analysis

Finding an overall strategy (9th grade)

Teacher's and students' speech	**Linguistic actions**
So, this situation is fairly banal, huh. Given all that we've done, what is the only new thing, Bertrand? *Uhh…x*	Meta. Information, question, participation
X, that is, the point *M*. What do you say about point *M*? Well, we don't know its real place on *EG*.	Sharing. Question, math, orientation
We don't know it's real place on segment *EG*. So in other words it's a point? *Unknown.*	Sharing. Question, math, orientation
Unknown. What other word could we… *We don't know where it is*.	Sharing. Question, math. Orientation
We don't know where it is, Marc? *Variable.*	Sharing. Question, math
Variable, *x* is a variable. The point *M* varies, so *x* varies from what to what*?* *From…well from 0 to 7.*	Sharing. Math, orientation, question, math, information
From 0 to 7, we can even write that at the beginning. They don't ask for that, huh. One time we did a problem where they asked for that. But we'll write right away that *x* goes between? *Zero and 7.*	Sharing. Math, information, structure, question, math
Zero and 7. Okay? *Zero and 5.*	Sharing. Question, getting attention.
Zero and 5? I wasn't paying attention to… *M* is on *EF* and EF, look, it's 5. Ah you switched them, pay attention. Everyone has had time to draw a figure? It's going okay?	Evaluation. Getting attention, information. Question. Other, getting attention.

This episode is dominated by participation, with a strong sharing component and numerous questions. The information from the teacher primarily concerns mathematics.

All episodes were analyzed in this way, which allows for rough quantitative evaluations. The overall results are presented below.

In the 9th grade class, in the first episode (work on the problem statement) we find a mix of structure, mathematical information, and meta speech. Participation is fairly week and is dispersed during the course of the exchange through a few questions and through contributions from students.

In the next episode, which takes place before the individual work phase, we note strong participation. The first part of this episode concerns primarily mathematical information, and the second integrates more meta content that situates the exercise relative to new and old elements.

At the beginning of the correction phase, there is less participation. Most of the information relates to mathematical justification. The teacher refers frequently to students' work.

In the 8th grade class, the teacher encourages student participation through questions and participation functions, which include a strong sharing element. The teacher speaks after students' responses to share their comments with the class, and then orients students' work towards a path that will more effectively lead them to the desired response. The episodes we analyzed are also marked by their strict structure. Different steps in the reasoning process are first identified in the search for a strategy, and then elaborated explicitly during the correction phase. Justifications are requested: "*Now then, why do we begin by AB squared? Why don't we begin with the others?*"

Above all, the speech concerns mathematics. Nevertheless, the teacher does pose several questions concerning the reasoning process: "*Now then, we will try to put several ideas on the board, without writing them out in full. Dominique, do you have an idea? What could we look at?*") and comments on the calculation: "*Remember, you can find the sum of two squares directly with the calculator. You can write down the intermediate results, or you can do it directly*."

NOTES

1 Each class had approximately 30 students.
2 The name of the teacher has been changed (we refer to him as "D").
3 Immediately before presenting the problem statement, the teacher had a student list the possible three theorems to be used: The Pythagorean theorem, its converse, and its contrapositive (which does not have a specific name in this class).
4 Key: SIT = Simple isolated task; A1 = recognition of methods of application; A2 = introduction of intermediary; A3 = combination of multiple frameworks; A4 = introduction of steps.
5 This was completed through an interview that strictly followed the questionnaire (private oral communication).
6 Interactions initiated by the student (which were rare in the observed class periods), if aimed at the whole class, were also indispensible for determining the progress of students' activities and were analyzed with this in mind.

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5. THE CASE OF TEXTBOOKS IN MATHEMATICS TEACHING

INTRODUCTION

The mathematics textbook is, in France, a central object to teachers' professional practices (and more markedly among beginners), and in students' daily work, whether in class or in their personal work. Paradoxically though, it is rarely studied for itself in the education research in France (Bruillard, 2005) and especially in the mathematics education[1] research.

We first need to point out that in France, the design of the textbooks is the task of private publishers who, each, suggest their textbooks to the teachers. Hence, the choice of the textbook given to the students is done by the team of teachers in each school, the financing is provided by the administration.

A textbook is the result of a complex interaction between multiple constraints. We are interested as a first step in describing the perception of that interaction which we have developed through an experience of writing and editing a textbook. We make it a point to underline the resulting leeways.

Taking into account this description, it seemed interesting to evaluate the homogeneity of the textbook, to research whether the authors benefit from this leeway while writing, mainly when writing exercise panels which they have suggested. We have thus analyzed the exercises of four textbooks (covering the same theme) in terms of the knowledge at stake and the activities that they would generate for the students.

CONSTRAINTS AND LEEWAYS IN THE CONCEPTION OF A TEXTBOOK, A POINT OF VUE BASED ON EXPERIENCE

This first section attempts to describe the interactions of the constraints that weigh on the writing and in a less central manner on the use of textbooks. It is based on the experience of Christophe Hache as the person in charge of the Domino collection at the time of the publishing of the textbooks for grades six[2] (Hache, 2005) and seven[3] (Hache, 2006).

The textbook in question is that "of the student." This textbook is given to the student at the beginning of the year and then taken from him at the end of the year.

The constraints imposed by the fact that, materially speaking the textbook is a book, concern not only the users, of course, but the designers of the textbook as well.

F. Vandebrouck (ed.), Mathematics Classrooms: Students' Activities and Teachers' Practices, 117–133.

As a first example, the fact that the student has at disposition the whole textbook complicates matters, and affects the statement of any discover activity of a notion (the entire chapter about a certain notion is in fact not very far), having the entire set of questions of a problem can interfere in the genuine research process while solving the first few questions. Similarly, the position of an exercise in the textbook is an indication de facto with respect to its solving.

The teachers can have access to a “teachers’ textbook.” However, if we exclude the re-transcription of the official programs and the exercise corrections from the textbook, their content turns out to be poor in general. In middle school,[4] in mathematics, teachers seldom use this teachers’ textbook[5] (things are less clear at the primary level[6] for example, where the content is much more developed, and where teachers are generally not specialized in mathematics).

Therefore, group work sessions, long research sessions, or sessions where arrangements for special modalities need to be made (exchanges between students, debates, etc.) are hardly accounted for in the textbooks.

The paper format makes the appropriation or modification of the text by the teachers less natural (adaptation to the chosen approach, adaptation to students, etc.). Regarding this point, we can mention the novelty of the Sésamath textbook (Sésamath, 2006): the textbook is sold in paper form but is also entirely available in digital form with a format that allows all modifications (Sésamath website).

A third constraint is related to the stability of the practices related to textbooks (writing, use, manipulation…). The expectations of the teachers (who choose the textbooks), the representations of the authors or publishers regarding what should be included in a textbook, and the way to present things are strongly influenced by the omnipresence and homogeneity of the existing or previous textbooks. We can, for example, refer to the Sésamath textbook experience (Sésamath, 2006) which was designed by a significantly large group of volunteer teachers without any intervention from a publisher. We will show that the non-commercial aim of the project and the absence of publisher during the writing eliminate a large number of constraints; the textbook however remains mostly classical. The project was not to produce a revolutionary textbook, but the proximity of the results and the standards shows well an important stability in the representations and expectations.

Last but not least, there is a time constraint to the writing of the textbook. The changes in official programs usually follow an age group: the programs are for instance changed for grade six at the beginning of the 2005 school year, then for grade seven at the beginning of the 2006 school year, and so on until 2008 when changes are applied for grade nine. These official program changes go hand in hand with renewing the textbooks. The author teams are usually not overloaded[7] and hence work on one textbook at a time. They therefore have around a year to conceive the textbooks based on the official program. This one-year deadline is one the heaviest constraint applied to the conception of a textbook: not much back stepping or possibilities to go backwards, not much proofreading, very few experimentations, a lot of urgent work, external collaborations and consultations are difficult to set up, etc.

The relationships between publishers of a textbook[8] and the authors are continuous: the publisher is present from the start of the work and follows the writing process in an uninterrupted[9] manner. The structuring of the book, the layout of the chapters, the exercise sections, and so on, are also elements which interfere at a very early stage in the conception of the textbooks. They link content (which is mainly the responsibility of the authors) and form (which is mainly the responsibility of the publisher). Thus the work of the authors cannot be done without taking into account the constraints of publishing (and vice versa). The publisher has, in a central way, a commercial objective: s/he designs products, looks for clients (tries to seduce them despite the competition) and wants to sell. The quantity of sold products determines the profit made. We can for example refer to the fact that in 2008, there were no textbooks for the grade 12[10] humanities track because the market was “too small.” From a commercial point of view, the “client” is easily defined: it’s the teacher who, within his/her team and once having received all the published textbooks, must choose the one that will be bought by the students of his/her school the next year. The “client” expectations are however difficult to define, the choice criteria not being explicit nor totally rational. “The publishers naturally seek, as a priority, the support of prescribing teachers; the textbooks are hence conceived in terms of the wishes of the teachers more than those of the institutions or the needs of the students. (…) If the publishers continue to propose, as they constantly claim, the product desired by the teachers, and if these same teachers prefer a safe product which reassures them in their habits, then how can pedagogical innovation be made? (…) Reflecting on the textbook and its necessary evolutions must start from the needs of the students and not the wishes of the teachers” (Borne, IGEN,[11] 1998).

Appeal to a maximum number of teachers (regardless of their practices) and eventually get the students interested (which is reflected by “show the teacher that the textbook will be interesting to the students”) goes through a certain number of rules, as far as the publisher is concerned. An example: the principle of elaborating the most omnipresent textbook, and this for all levels, is that the principe of regularity. In order for the textbook to be “well done,” in order for it to be “user-friendly to the students,” in order to reassure the teachers, all chapters “must” be structured in the same way (sections, number of pages, etc.). The irregularity of the types of notions considered is not really taken into account.

The mathematical content of the textbooks is framed by the official programs. These are commented and underlined through the “program accompanying documents” or the “program implementation documents.” Note that, in general, the official programs are issued one year before being implemented, and these accompanying or implementation documents[12] are issued at a later stage, commonly after the publishing of the textbooks. These are not the only constraints linked to contents of teaching. In fact, the mathematical contents impose layout and organization constraints; this is one subject of research in mathematics didactics. The choice of the type of task suggested, their declination into exercises, and the balance in volume between the different types of expected activities are also elements constrained by the addressed content.

The work on the mathematic contents is mainly the task of the authors. Usually, the publisher's influence is indirect, in particular through the rigidity of the layout or the time constraint. It could happen that the publisher interferes at the level of a exercise, or less commonly a chapter, refusing a certain way of presenting a notion (the official instructions argument is then negligible compared to, for example, that of the teachers being used to a certain way of doing, or being ready, or supposedly ready, or not, to accept certain changes).

Leeway is given to authors (and then users) of textbooks. During the elaboration of the textbook, the setting up of the team of authors (and the team of reviewers), and macroscopic editorial choices (ways of approaching programs, chapter division, etc.) are aspects for which constraints seem less pressing (excluded, always, the time constraints for reflection, experimentation, etc.).

Leeway during the writing phase is also important, especially when it comes to the "exercises" section of the book. In fact, the publisher focuses on the first parts of the chapters: e.g. introductory activities, lesson pages, solved exercises. We can put forward the following explanations:

– The publishers refer to the teachers' declarations, who say that a significant attention is given to the lesson pages and solved exercise (rigor, accessibility for the students, etc.) in the choice of their textbooks,[13]
– The publishers (and teachers) have very few tools to analyze and evaluate an exercise or set of exercises.

Writing the exercises is hence essentially dictated by quantitative constraints: size of exercise given, number of exercises, and sometimes the possibility of illustration.

Of course, beyond the making-of of the textbook, teachers are allowed significant leeway. The work of Arditi (2011) explores precisely the variability of practices of teachers using the same textbook. The job habits don't leave much choice with respect to buying or not a textbook, but it can be noted that certain teachers don't use a textbook; they are self-produce the pedagogical material needed, or they use equality different textbooks, or other resources at their disposal, and the photocopier (Ben Salah Breigeat, 2001). The choice of textbook is an important aspect for which the criteria are rarely explicitly stated (there is no research about this topic nor any training, or only little). This is where the teacher plays his/her role of client for the publisher.

The main freedom for the teacher with respect of the textbook remains of course that of organizing his/her teaching. The way of using a textbook is beyond any doubt very varied. During the design of a teaching session or a sequence of sessions, the teacher takes ownership of the textbook, and reconstructs what s/he needs from what is suggested. During the teaching process, when s/he uses the textbook, the teacher also adapts the content of the textbook to the flow of the session, whether consciously or not.

Several fields of questions come to mind based on the notions discussed above. We can first wonder about the different effects of these constraints: is there a strong uniformity in the content of textbooks? Can we see some variations? We can wonder about the things that do not appear in the textbooks because of these constraints. Reading between the lines is always difficult, but it is always possible to confront, from this point of view, the content of textbooks with the official programs. Identifying and describing a potential uniformity, or at least similarity, of the textbooks brings out the question of how to describe, or characterize the textbooks.

Considering what precedes, we can question in what way the textbook can be an efficient tool to diffuse research results and new teaching approaches: what textbook allows such transposition? What type of organization is required while writing?

Several questions are laid forward in what precedes: how to organize teaching contents over the year (as the author of a textbook, but also in a broader sense, to teach in class)? How to analyze teachers' expectations while they choose a textbook, and then while they use one? Are teachers' practices linked to the used textbook? How do students use the textbooks inside and outside the classroom?

The idea that mathematics textbooks are very similar to each other for a same class level and a specific time period is widely spread. The layout, number of chapters (typically between 15 and 20 chapters in middle school, each tackling a specific notion), and the division of each chapters into sections (typically "activities,"[14] "lesson," "solved exercises," "direct application exercises," "challenging/practice exercises," all on around 20 pages) reinforce that impression. Moreover, we saw in the previous paragraph that several elements seem to concur with this standardization.

These declarations can be opposed by the fact that teachers mainly use the textbooks as a source of exercises, and that the elements mentioned above (layout, number of chapters, division of chapters, etc.) are essentially linked to the form: the choice of textbook can keep its meaning if we only rely on the content (choice of approach to notions, suggested exercises, etc.). It is important to note that the form is of course not without any link to the content: the division of chapters and the allocation of weight to a notion in a chapter does not allow for example, and very generally, the valuing of the work done on multiple notions (except in the last chapters, if the notions have been discussed in the order suggested by the textbook). Changing frames becomes rather rare, and when it happens, they seldom link two new notions. It is not rare that once a chapter is covered, the studied notion is only evoked once or twice in the rest of the textbook. What can be said then about the structure and organization of the student knowledge?

ANALYSIS AND CHARACTERIZATION OF EXERCISES IN TEXTBOOKS

In what follows, we propose to present a work methodology to tackle the question of similarity of textbooks, in a second time we present some of the results obtained (a more detailed account of the corresponding research can be read in Hache, 2008).

We wondered, while limiting ourselves to the set of exercises suggested in comparable chapters, to what extent do the constraints described in the first section of our chapter have a unifying effect on the content of the exercises in the textbooks.

Our choice to focus on the exercises has several justifications:

– The teachers use several textbooks as a database for exercises (the textbook distributed to students as well as other textbooks that they have at their disposal).
– We have seen that writing the exercises is one of the parts of the textbook where constraints imposed to authors are the least weighing hence this is where most of the diversity should be found, a priori.

We studied a given chapter in four textbooks. The methodological choices (choice of textbooks, choice of studied chapter, and as mentioned earlier choice of analyzing exercises in this chapter) have been made such that we increase the chances of finding differences between the textbooks. The notion studied through the analyzed exercises, proportionality, was chosen according to the following criteria. It is one of the notions traditionally studied in middle school, but the new programs give it a much more central position than before. Its approach has evolved with respect to the old programs, its study starts now more explicitly as of grade six. The organization of the chapter (the grouping of exercises for example, as well as their content) had to be "reinvented" by the authors: for example, the cross multiplication is not studied until grade eight although it used to be seen in grade seven. Since the position of the notion changing in the programs, the way it is handled in the textbooks cannot conform to pre-established canons.

The chosen textbooks are the following: Phare 5ème (Brault et al., 2006), Domino 5ème (Hache, 2006), Transmath 5ème (Malaval, 2006) and Sésamath 5ème (Sésamath Ass., 2006). The choice was also made here so that it favors heterogeneity. Phare is a recent collection, for which the grade six edition was a major success among the teachers in 2005. Domino is also a recent collection written by a team of authors, among which some are researchers in mathematics didactics (but it had little success among grade six teachers). Transmath is one of the oldest existing collections, the grade six textbook was also a big success in 2005. Sésamath is a new collection (no grade six textbook in 2005). The textbook is written by a team of around 30 teachers, it is available online for free (and sold in paper form, the publisher is not a school publisher but a publisher of educational and cultural CDs and software, the paper form of the textbook sold well in 2006).

At this point of our work, we have four complete sets of exercises from the chapter about proportionality, from four textbooks of grade seven edited in 2006. We will hence present the analysis method that has been set up in order to better understand the similarities, or on the contrary the differences, between these textbooks.

We analyze the proposed tasks in the textbooks with respect to the activities that we believe they generate for the students. In order to do so, we place ourselves in a theoretical situation where the student masters his/her lesson (in case of hesitation about the content of the lesson, we can't refer to the teacher's lesson, we refer to the lesson in the studied textbook), where he has taken ownership of the question, and where he wants to solve the exercise.

In order to describe the possible student activities from the exercise given, we have used the classification of adaptations introduced by Robert (chapter 2). For each studied given, we have detailed the different necessary adaptations for the expected answer, while specifying the knowledge at stake. Hence, it seems important to specify and classify the different types of knowledge related to proportionality used in grade seven (whether they are directly related to this notion or used in parallel).

Analysis of tasks: Mathematical content

In this paragraph, we detail the way we analyze the mathematical content to be taught. This step is a prerequisite to the analysis of exercises.

Excerpts for national programs:

Programs of grade six 2004 (Official bulletin special issue n°5 on Sept 9th 2004):

"Solving proportionality problems has already been done in elementary school. It is carried on in grade six, with new tools

Handling "proportionality" problems, using appropriate reasoning, in particular:

- using de image of one unity;
- using a linearity ratio expressed, if needed, in the form of a quotient;
- using a proportionality coefficient expressed, if needed, in the form of a quotient.

Recognizing situations which are related to proportionality and those that are not related. Applying a percentage rate."

Programs of grade seven 2005 (Official bulletin No. 5 on Aug. 15th 2005):

"Proportionality still has a central position;

Completing a table of numbers whose values are only partially provided, which represents a proportionality relation. In particular, determine a fourth proportional;

Recognize if a complete table of numbers is, or isn't, a proportionality table.

- Use proportionality in the following case: Comparer proportions,
- Calculate and use a percentage,
- Calculate and use the scale of a map or a sketch,Recognize a uniform movement from the existence of a proportionality relation between duration and distance, use this proportionality."

The skills studied in grade seven, through the filter of the textbooks, have been classified as follows.

Recognizing a situation/ a proportionality table

Many things in the textbooks remain as hidden knowledge of this domain. When this is the case, the fact that the situation is a proportionality situation by choice, by modeling, in order to make calculations, is always implicit. This will be a problem in statements where proportionality is assumed whereas it could also be questioned (which is in general done in one or two exercises in each textbook).

An example of a situation where proportionality doesn't go without saying:

> What distance does a snail travel in 2 hours 30 minutes at a speed of 14 meter per hour? (Transmath, No. 56, p. 99)[15]

An example of calling into question:

> "Currently, the length of a day (between sunrise and sunset) decreases by 3 minutes every day. What would be the total decrease within 30 days?" What is implied by this statement? Is it correct or wrong and why? (Transmath, No. 46, p. 99)

Similarly, the distinction between proportionality situation and proportionality table is sometimes blurry:

> A snail is moving on a branch. The total duration of its trip, in minutes, and the distance covered, in centimeters, are given in the table below.

Duration (in min)	4	6	9
Distance (in cm)	26	39	58.5

> 1) a. Show that the movement of the snail appears to be uniform.
> b. Specify the proportionality coefficient of this situation.
> 2) (...)
> b. What distance can the snail travel in one hour if its movement remains uniform? (…). (Phare, No. 70, p. 130)

The use of the expression "appears to be" seems to indicate that the statement is emphasizing the fact that the proportional character of the table does not allow to prove the fact that the movement is uniform. The 2b) questions seem to imply the opposite.

Similarly:

> For exercises 14 through 19, state whether the table corresponds to a proportionality situation or not. If it does, specify the proportionality coefficient. (…). (Phare, p. 124)

We have labeled five skills related to identifying a proportionality situation (or table):

– Being able to recognize the existence of a proportionality situation, or the strong presumption of existence (coded Rr);
– Being able to identify (or propose) proportional magnitudes (coded Rg);
– Being able to admit the proof of proportionality (it's the case in many concrete situations) (coded Ra);
– Being able to prove proportionality (calculation of quotients for a table, existence of a formula in a situation, existence of a uniform movement, of a scale …) (coded Rp);
– Being able to prove non-proportionality (coded Rnp).

Example:

On the label of a one liter fruit juice bottle we read:

Proteins:	0.4 g / 100 mL
Average energetic value:	50 Kcal

Copy then complete the following table:

Proteins				
Lipids				
Energetic value				

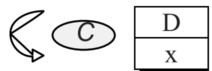
(Sésamath, No. 5, p. 76)

This exercise requires, among other things, recognizing in this situation several proportionality situations (Rr), the case of lipids (upper bound) and of energetic value (50 Kcal / 100mL or 50 Kcal / L ?) are particularly delicate, they could be coded Rg. In all cases, the proportionality is eventually admitted (Ra). The last two categories are linked to proofs.

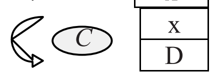
Exploring a proportionality situation

Proportionality whether given, admitted or proven, several explorations can be considered. We have distinguished the following cases:

– The coefficient (or scale, or speed, or percentage…) being known, as well as one more value given, calculate the missing value
  - ° [Cmu] We multiply by the coefficient,



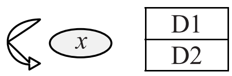
  - ° [Cdi] We divide by the coefficient



– Similarly, given a formula, knowing how to use it:
  - ° [Fmu] directly,
  - ° [Fdi] begin by inversing it.

The use of the coefficient alone rarely appears in the textbooks.
The use of formulas is not considered. Therefore, the work is focused on the following three skills:

– [Coef] Two values are known, knowing how to calculate the coefficient



– [QP] Fourth proportional. Three values are known, knowing how to calculate the fourth. Usualy, several methods are possible: calculation (then use) of the coefficient, using the image of one unity, multiplication of one "column" to obtain the other (the method can be imposed or not).

D1	D13
D2	

– [DonInc] Values to be completed: the known values are not enough to calculate the missing value (either they are “too many” values, or there are several missing values). A new skill emerges, for example: being able to use addition of “columns” (the method can be imposed or not).

Other practiced skills

In the chapters where the notion of proportionality is practiced, other skills are used or practiced. We retain the following groups of skills:

– Graphs, tables

Proportionality is practiced in grade 6 (and before) through the concept of proportionality between two magnitudes. The tables and graphs are also tools which are practiced in grade six (reading and interpretation, some constructions are considered). In grade seven, the proportionality table is introduced as well as how it is related to previously studied proportionality situations.

The link between the alignment of points with the origin of a coordinate system and proportionality can be seen without being required, it will be explicitely studied in grade eight. However, there exists a strong link between graphical representations of data and proportionality (proportionality between frequency and height in a bar diagram, between frequency and angles in a circular diagram, etc.).

– Skills related to magnitudes

We have distinguished between the skills related to converting different magnitudes (lengths, areas, weights, volumes, durations, speed), from the skills related to the notion of magnitude itself (recognize the magnitude at play, compare magnitudes without measurement or with an ad hoc unit).

Regarding this last point, it should be noted that the first magnitudes and their measures are studied starting elementary school (length, weight, capacity and duration are for example studied in cycle 2), angles and volumes are studied in cycle 3, and their measure is studied in grade six. The conversions of volume measures are skills in the process of being acquired during grade seven.

Speed has a special status: it is not studied on its own, only the idea of “uniform speed” is covered (as an example of a situation of proportionality relation between duration and distance).

The quotient magnitudes are introduced by the program in grade eight.

– Skills related to context

Many skills are used in solving exercises linked to proportionality. In addition to those mentioned above (related to magnitudes, graphs and tables) several old skills related to geometry are used (mainly drawings, measures, formulas, eventually in space geometry). In addition, manipulations of rounding, orders of magnitude, fraction notion, and percentages are carried out (notions studied in grade six).

– Others

Some exercises attempt to link the work on proportionality to the beginning of algebraic manipulations in grade seven. A textbook suggests some exercises using the scientific notation of numbers, which is a notion studied in grade eight.

Analysis of tasks: adaptations

These adaptations are grouped into eight categories (see chapter 2):

- SIT: simple and isolated tasks, these are tasks for which the student uses a skill without adaptation while solving. Ref. examples 1, 2 and 3.
- A1: the student must recognize the modalities of the application of a skill in order to be able to use it. Ref. examples 2, 6 and 8.
- A2: the student must introduce an intermediate item to be able to progress in his/her work. It could be placing a new point in geometry (or simply naming an existing point), doing intermediate calculation, etc. Ref. examples 3, 4 and 5.
- A3: it's about being able to relate several skills to each other, make a change of frame, link the work on new skills with old ones. Ref. examples 3 and 6.
- A4: the given leaves it up to the students to introduce a new reasoning step. It is not uncommon that, for example, in exercises related to proportionality using a context, the hypotheses of proportionality be kept hidden, hence the student must theoretically decide or admit it. Ref. example 7.
- A5: the answer to a question requires the use of a result established in the previous questions.
- A6: choosing one method among many. Ref. example 8.
- A7: working with skills that have not yet been introduced as is at this level, not yet formalized. Ref. example 9.

Example 1 (it's a Multiple Choice Question):

Voici un tableau de proportionnalité. 3 | 1,5 4 | 2 x… Le coefficient de proportionnalité manquant est…	$\frac{4}{3}$	$\frac{3}{4}$	0,5

(Transmath, n°60 p100)

Proportionality is explicitely stated in the given. The question is specific

Here is a proportionality array. The missing coefficient is…

and refers to a notion in the lesson. We can think that the absence of context allows an application of the lesson without any particular difficulty.
The question was coded SIT-Coef.

Example 2:

The price of a pair of sunglasses has increased by 3.2€. Its initial price was 40€. What percentage of the initial price does this increase represent? (Phare, No. 3, p. 122)

The necessary numerical values are provided in the given, the student must however adapt his/her knowledge to the situation. The question was coded A1-QP (in fact, it has been decided to coded percentage calculations QP: they are introduced in grade seven as a calculation of the fourth proportional).

The distinction between SIT and A1 is done according to the context. The existence of a casing, the existence of examples close to the exercise situation in the lesson, the position of the exercise in the chapter are all elements which can be taken into account while deciding of the necessary qualification and adaptation for the student.

Example 3:

> 1st example: "Give a possible scale for a doll house which is 50 cm high." (Domino, No. 39, p. 63)

It is necessary at this point to introduce the order of magnitude of the size of a house (as we imagine it), and this was coded A2-Order of magnitude. Two other activities are coded for this exercise: the change of unit (A3-Length conversion) and the scale calculation once the real height is chosen and the conversions computed (SIT-Coef).

Example 4:

> 1. Represent by a circular diagram the distribution of the population by continent in 2003
>
> 2. What complimentary information would allow you to complete the study?
>
> (Domino, No. 45, p. 64)

Distribution of the population by continent in 2003

Continent	Percentage
Africa	13.6%
America	13.7%
Asia	60.7%
Europe	11.5%
Antarctica	0.5%

To solve question 1), the student must introduce numerical data: in one way or another, s/he must introduce a line for the "Total." This part of the question is coded A2-Total.

Just like the below exercise, s/he must decide of which measure to make in question 1) (This part of the question is coded A2-Measures).

Example 5:

> The sketch of an apartment is given, dimensions *a*, *b*, *c* and *d* are coded. Length *a* is in reality equal to 17.2m. Determine the scale of this sketch. Determine the real lengths *b*, *c* and *d*. (Phare, No. 8, p. 123)

Example 6:

> During a cross Atlantic sailing trip, the skipper has noted the caps in order to trace back the route he sailed. A broken line is drawn (6 segments, "Boston" on the left, "Lisbon" on the right).

a. What is, rounded to the nearest millimeter, the length of the broken line representing his route?
b. His drawing is done at a scale of 1/600 000 000. What distance did he sail?
(Sésamath, No. 14, p. 77)

The student will probably realize the usefulness of converting the length unit in order to answer question b., conversions of lengths being old skills (reviewed in grade 6). This part of the question is coded A3-Length conversion. The use of a scale in question b. was coded A1-Cdi. Note that this is one of the rare cases in the studied set of exercises where the coefficient is used with only one value (Cdi or Cmu).

Example 7:

In a salted swamp, for each 1000 g of evaporated water we get 32 g of salt.
a. Calculate the mass of salt obtained from 5000Kg of sea water.
b. How much water must be evaporated in order to obtain one ton of salt?
(Transmath, No. 17, p. 96)

Even if undoubtedly the chosen model here is proportionality, it is still part of the student's job to indicate that (and eventually be convinced of it). This part of the work was coded A4-Rr.

Example 8:

Below are the US Metric conversions:

For length: 1 yard = 0.9144 m 1 foot = 12 inches 1 yard = 3 feet 1 mile = 1760 yard	For mass: 1 pound = 16 ounces 1 pound = 0.4536 kg

Express the height and weight of the students of the class using these units.
(Domino, No. 69, p. 67)

It is the student's job to set up his work method in order to find the adapted units to the situation, and to make the corresponding calculations. This part of the question was coded A6-Magnitude.

It must be noted, that here also, the qualification as A6 can depend on the context. It is possible that while reasoning, a student might a have to make method choices, the adaptation can hence be classified as A1 for instance if the choice is very local and is made explicit and detailed in the lesson. When an incomplete proportionality table is given, there exist, for example, several methods to complete it (calculating a coefficient, adding columns, etc.). This type is generally classified as A1.

Example 9:

The speed of sound is 340 meters per second and that of light is 299 792 480 meters per second.

a. Express these speeds in kilometers per hour (…) (Sésamath, No. 28, p. 78)

The notion of speed and corresponding conversions of measurement units are studied in garde eight (coded A7-Speed).

CONCLUSIONS

In the work discussed above (Hache, 2008), we have tested, in the context of research in mathematics didactics, constraints and leeway perceived throughout the conception of a school textbook. We described in particular the methodology that was put in place to test the similarity between sets of exercises proposed in four French textbooks (for the chapter about proportionality in grade seven). For this purpose, we have analyzed each exercise in the textbook, detailing the different adaptations needed for solving it, linking them to the mathematical skills at work.

The conclusions can be of different orders. We can first ask ourselves whether the analyses made reinforce the impression of uniformity in the content of the textbooks for the reader, and thus by this means, question the effectiveness of the constraints perceived during the design. We can furthermore draw conclusions regarding the processing of proportionality in the textbooks: the analyses conducted allow us to distinguish the types of activities proposed to students… and certain activities which are not proposed.

Before concluding about these issues, we need to highlight the gap that exists between the analyses of the exercise statements in a textbook and the student activities in the classroom. Three important filters intervene between the two:

– - The teacher, when preparing his session, assuming that s/he picks exercises for the students only from the designated textbook, will not solve all of them. Quantitative analyses including the set of all exercises are thus left with only a restricted meaning, an indicative value: this set of exercises corresponds to the offer proposed to teachers, the offer would be more interesting for certain types of activities possible for the students, and less interesting for others. The teacher could (or not) differentiate, in a certain way, these relative important notions while selecting the exercises.
– - On the other hand, the teacher can modify the exercise statement. S/he can do it consciously and a priori, before proposing it to the students, but s/he will above all (and necessarily) do it during the teaching session. We can see in other chapters of this book that these modifications of the proposed tasks are usually in the sense of a simplification and a reduction of the adaptations which are left for the students. Refer to the work for Sara Arditi about this last point (Arditi, 2011).
– - Lastly, the study of the gap between the possible activity inferred from the analysis of an exercise statement and the actual activity of a student in the

classroom while working on this statement is a research filed still under development.

The analyzed exercises show that nevertheless, there are heavy and recurring tendencies while processing a chosen notion. We could allow ourselves to deduce some elements with respect to the teaching of proportionality. The objective of the study was more restricted and consisted in studying the proposed content in the textbooks, and estimate the similarity between textbooks.

Globally, even if the global "portrait" of the textbooks leaves some room for fluctuations (the perceived tendencies are well underlined), we can still observe some specificities for each textbook.

We can first detect that an important part (usually more than half[16]) of the possible student activities are about simple and isolated tasks (SIT): no adaptation is needed for the use of the intended skill. If we include the uses where students need to "simply" recognize the application modalities, we realize that in almost three quarters of the situations the skills are used without leaving any real initiative to the student: the work necessary for learning is thus mainly perceived as using skills without really taking any step back. This phenomenon is very stable in three of the textbooks and less clear in one of them (Sésamath), where the reasoning steps are, for example, left to the students (in fact, it's mostly about introducing the idea that's there's actually proportionality).

Moreover, it can be noted that the skills related to proportionality and used in the exercises are mainly linked to the exploration of a proportionality situation (calculating missing values), and are barely linked to the questioning about the proportional character of the situation (the ratio is almost two thirds to one third). In this case also one of the textbook (Domino) is shifted compared to the others: more than half of the items are linked to the question of proportionality and not to the exploration of the situation of proportionality.

In these chapters about proportionality, many skills not directly related to proportionality are used: skills related to measures of magnitudes, to graphs, to tables, to geometry, to writing numbers, and so on. Globally, the proportion of skills linked to proportionality represents the majority, this phenomenon is found in three textbooks. In one of the textbooks (Domino), we find a more significant proportion (66% instead of 38% for the other three textbooks) of possible activities which use skills not directly related to proportionality.

The skills not directly related to proportionality vary from one textbook to the other depending on the explored situations. The contexts of exercises, the recommended tools, the skills used differ (scales, uniform speeds, percentages, graphical representation of data, etc.).

Beyond these variations, which lead us to believe that the leeway given to authors is only used in minimal way,[17] the global portrait of the work about proportionality in the textbooks is very particular: using mainly calculations instead of explorations and exercises which are almost systematically guided (the student only having to take punctual initiatives). Even we globally notice a relative variety of the tackled notions accompanying work done on proportionality, for each textbook the skills practiced are not many (even very few for Phare and

Sésamath). We detect through these analyses a relative absence of exercises allowing the student to take ownership of the question of proportionality for a given situation, to take the initiative in choosing the calculations to be done, the method to be followed, or the other skills that need to be mobilized.

Certain questions are asked with more acuity stemming from this work. Are the results found generally applicable? In particular, the small number of situations allowing the student to take the initiative for some adaptations would be reason to worry about the reality and nature of the possible learning from exercises. A study about several contents would allow us to make progress regarding this point.

It seems fundamental on the other hand to complete the above analyses in two directions: do the choices made by the teachers and the modifications made change the conclusions drawn above? In what sense? In what way, if need be, does the work of students on exercise statements, whether in the classroom or at home, compensate for the evoked gaps?

NOTES

1 We can refer to the thesis of Ben Salah Breigeat (2001) which focuses on the implementation of the practices of young teachers, and analyzes among other things the similarity between their discourse and that of the textbook used. The work of Robert and Robinet (1989) studies the meta-cognitive representations of authors of textbooks through exercises they propose. In parallel, the APMEP (association of mathematics teachers in public teaching) has guided several reflections about the textbook as an object.

2 Mainly 11 year old students.

3 Mainly 12 year old students.

4 School grades six, seven, eight and nine (mainly 11 to 14 years old students).

5 We can for example consider the revenue from selling these textbooks when they are proposed separately from the student textbook.

6 Mainly 6 to 10 years old students.

7 The Sésamath textbook is distinguished by the fact that is was designed by a group of around 50 teachers (regarding the grade seven textbook). Several teams work in parallel and the deadline for elaborating a textbook is so far around two years. It should also be noted that the e-version of the textbook (can be downloaded) can theoretically be modified even after the paper form is published.

8 The word "publisher" remains relatively blurry here. It designates at the same time a company, the people working in it, including but not exclusively those who work on a mathematics textbook.

9 Note that nevertheless among current textbooks commercial marketed in middle school, two are distinguished with respect to this point of view: the Sésamath textbook, the editor participates a bit in the design. And the Aventure math textbook (Busser et Massot, 2006) designed and edited by the team of a magazine (Tangente).

10 Mainly 17 year old students

11 General Inspection of National Education.

12 Official documents which complete, explain and enrich the texts of the programs.

13 Paradoxically the teachers claim at the same time using exclusively lists of exercise in their daily practice.

14 The word "activity" in the textbooks very generally refers to exercises that are to be done before the lesson to introduce it: revision, discovery and introductory exercises, etc. This meaning is different than that of the theoretical framework of this book.

15 For more simplicity, we will name the textbooks by the collection names (and not by the author name), we will only indicate the grade level and year when it's not a grade seven class in 2006.

[16] For more quantitative data, see Hache (2008).
[17] The notion and textbooks were chosen in a way that maximizes differences between the textbooks (see above, the paragraph about methodology).

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6. TEACHING PRACTICES AND STUDENT LEARNING: A CASE STUDY

INTRODUCTION

The following study is an example of the analysis of the relationship between teaching practices and student learning of a mathematical notion. In this research, we tried to describe in what way what is proposed by a teacher in class can influence student learning, using different tools to analyze tasks and mathematical activities previously exposed.

Our aim is to understand the effect of teaching practices on student learning of mathematics, for a given content.

In an attempt to specify this influence, we can discover more specifically, by comparing several possible class management choices, if there are conditions of student work that seem to trigger learning differently, while favoring for example the acquisition of a targeted property by a larger number of students. Are there certain ways of organizing class work which turn out to be more beneficial for student success? In case of failure, we can wonder about what was "lacking" for the students in class, among everything that was proposed by the teacher, which helped initiate their learning, and we can try to distinguish these types of lacks.

Naturally, we are not questioning the teacher's work, and we will consider the different components of his/her job to explain certain choices made for his/her class.

THEORETICAL HYPOTHESES OF OUR METHODOLOGY

We focus our study on what actually takes place in the classroom and what remains accessible to research. We build our hypotheses on the activity theory, which explains our interest in activities that students have had the chance to carry out in class, and hence apprehend the subsequent student learning from the actual observed resolution of tasks in the classroom.

Student activities as a way to describe teaching practices and link them to learning

In order to describe teaching practices in the classroom, and consider them in light of the "progress" observed among the students, we systematically analyze what is proposed by the teacher (in particular the mathematical contents and real class management), and we attempt to evaluate the knowledge that could follow from it for the students, and then control this potential learning for all the students. Hence,

F. Vandebrouck (ed.), Mathematics Classrooms: Students' Activities and Teachers' Practices, 135–149.

we can understand how these acquisitions (even if provisional or partial) depend on what has been previously done in class.

Actually, we only determine the possible activities (a minima and a maxima depending on the students), or presumably impossible for the class, but we can't really tell what happens precisely with each student. We do not go as far as the actual activity of each student, and hence maybe not as far as what lacks individually in class. Nevertheless, we can link these possible activities with the success of each student through the evaluations conducted by the teacher and prepared by all that has been done in class. As a witness of the learning prompted by the observed teaching, we choose to retain the results of the students in the exam sanctioning the end of the chapter.

This study supposes being able to observe a whole set of sessions taking place in an ordinary classroom, while comparing if possible several analyses in different classrooms. In order to enable a comparison between several administered teachings, we have preferred to limit our observations to sessions covering the same mathematical notion. We have chosen to focus on similar triangles (in grade 10), we explain this choice further below.

In order to bring to light the relationships between teaching practices and student learning, we seek to *characterize* what has been proposed by the teacher in class regarding similar triangles and measure its impact on students' results in an exam covering this notion. By conducting an analysis of all the sessions of this chapter, we can, for each skill evaluated in the exam, rebuild – with a certain scale – the entire work proposed beforehand regarding this topic, as well as the work conditions. This will eventually allow us to identify the conditions that lead to the success of a certain number of students, by considering their exam results.

The limits of the framework

Obviously, learning does not occur only through what happens in the classroom, and we should particularly take into account the influence of a longer time span on student knowledge, by studying for example the exams given some time after in-class sessions, which was not possible here. We must take into consideration other factors which could also influence student learning, such as external help which certain students resort to in their personal work, or the state of their previous mathematical knowledge, but also the level of the school and class, the social origin of students, etc. However, it would be impossible for us to measure the influence of these different variables, or even to observe them in order to analyze them. Through our methodological choices, we have tried to limit external factors, by choosing whenever possible, appropriate fields of observation. In particular, the choice of the notion of similar triangles allowed us to limit as much as possible the knowledge taught to the observed sessions.

A BRIEF ANALYSIS OF THE NOTION OF SIMILAR TRIANGLES

In order to establish, from the observations, links between what happens in class and potentially generated learning, we needed to observe all the sessions about a whole chapter of the mathematics program, and then find a notion whose teaching can be limited to a reasonable number of classroom sessions. This choice also needed to allow us to minimize the contribution of previous knowledge, thus favoring a new notion for which the learning is essentially happening in the observed classrooms. Therefore, we have chosen to focus on similar triangles: triangles with identical angles and proportional sides.

Nevertheless, this choice is not without consequence. We can ask ourselves if the specificity of the notion – which has only been recently introduced in the programs at the time the observations took place – would have an influence on the observed teaching practices, and consequently on what could be lacking to students in class – in terms of justification or proposed tasks for example. An analysis of the chosen notion, as a mathematical object as well as a teaching object, appears to be indispensable to better understand the choices made by the teachers, but it does not exempt us from a discussion about the possible generalization of our results to other types of notions.

Introducing the notion of similar triangles

Similar triangles are defined as two triangles having two respectively equal angles, and with the property of having proportional sides: if two triangles are similar, then their sides are proportional. This definition and the corresponding property, respectively noted D and P, will be used subsequently in the task analysis of similar triangles.

In the supporting documents for the 2000 programs, similar triangles are associated to isometric triangles, for which several introductions are proposed, through the Euclidean equality, or using transformations. For similar triangles, which can be considered as an extension of this first notion, only the introduction using the cases of similitude is possible, the students not having the necessary knowledge about transformations at that stage (such as homothetic transformations). In fact, it is specified that no new transformation should be introduced to students during that year.

How, in light of these considerations, can we choose an introduction to the notion of similar triangles in grade 10? The absence of non isometric transformations in the programs does not allow other introductions than the one using the cases of Euclidean similitude, but we will see that this absence can as well cause some problems in the processing of the notion, particularly in what concerns identifying corresponding vertices.

The problem of identifying corresponding vertices

In most exercises about similar triangles, it is necessary to associate corresponding vertices in geometrical figures, in order to be able to deduce the ratios between the different side lengths, or inversely, to associate corresponding sides in order to deduce angle equality. Yet, this approach, which often requires a preliminary identification, is not always obvious, particularly in certain geometrical configurations.

In the programs and accompanying documents, nothing is specified as to this issue, but it can be found in the textbooks as well as in the classrooms a form of writing that is supposed to make this approach easier, by placing the letters which refer to the corresponding vertices on top of each other. This actually allows a fast enough deduction of the ratio equalities between corresponding lengths. By contrast, in the other direction, to go from lengths ratios to angle equalities, the difficulty remains, if we want to give a method to identify vertices from the lengths ratios. The theorem that states that this correspondence exists does not give a technique to automate this identification. How can we thus give an algorithmic method of identification without using similitude?

This suspected obstacle will be detected in the classes that we have observed through the students' difficulty in solving the problems which involve identification. Moreover, the teachers do not dispose of a technique which they can suggest to students. In fact, we did not identify any discourse about this topic in the observed classrooms, except for the use of the particular form of writing the vertices found in some textbooks. Here, we can question the possible link of this lack with the novelty of the notion in the curriculum.

SOME METHODOLOGY ELEMENTS AND ANALYSIS EXAMPLES

We now present the data that we have collected for our research, define the different variables retained for the analysis, and finally apply this methodology to an example from our work, to illustrate our methodology. For more details, see Horoks (2006).

The data collected and their analysis

We have filmed three teachers in their grade 10 classroom, throughout a chapter about similar triangles, in different schools. These observations were sometimes made without needing the presence of the researcher in the classroom (two out of the three observed classes), which allowed us to minimize the occasional disturbance, but given that we did not receive certain videotapes, we weren't always able to draw precise information for all sessions.

We collected exam papers sanctioning the end of the chapter about similar triangles, after their correction by the teacher. We got some complementary information from the teachers, in particular about the level of the students in their class, in order to refine our interpretations of the exam results. We use these results in order to evaluate the knowledge of the students following the observed classroom sessions.

Afterwards, two similar observations were conducted as part of a master's thesis study (Cisse & Corlay, 2006). We took these into consideration to confirm the results that we were able to obtain with the other three teachers.

Methodology to analyze the sessions

Using the videotapes, we have noted and categorized all the tasks proposed by the teachers, through the exercises, about similar triangles. By looking at the details of the resolution of these tasks as led by the teacher in class, we try to define what parts of the tasks that are potentially carried out by the students in class.

We then compare the exam tasks with similar tasks proposed beforehand in class, and try to relate the success of students in some of these tasks with the way they took place in class. Therefore, we try to highlight certain ways of organizing the work which favor learning, for a big part of the students in the class, or of some of them only.

In order to be able to synthesize this large amount of data (we have collected several dozens of film hours in total), we have built analysis grids which we will present below through an example.

Methodology to analyze tasks

To analyze the tasks proposed in every exercise of the chapter, we will consider:

- The complexity of the tasks, i.e. determining in particular if it is simple, isolated, or on the contrary if it requires adaptations of theorems from the lesson to be used;
- The type of adaptation of the notion of similar triangles (ref. chapter 2), needed to solve the task;

  A1. Recognizing (partially) knowledge application modalities (notions, theorems, methods, formulas, etc.): typical in geometry, recognizing the configuration(s) where Thales is used. This can include recognizing variables, notations, or recognizing formulas or theorem application conditions, etc.

  A2. Introducing intermediates – notations, points, expressions: typical in geometry, introducing a parallel line, or naming a point using Thales.

  A3. Mixing several frames or notions, changing points of views, frame changes or interplays, connections or interpretations, etc.: typical in geometry, using algebraic calculation to reach a result (for example, solving $x^2 = 1$ in the middle of a geometry problem). Problem texts that play on the graph/function automatically contain this adaptation.

  A4. Introducing steps, organizing calculations or reasoning (after repetitive (in)dependent use of the same theorem for a reductio ad absurdum argument invoking the theorem): typical in geometry, using four times Thales' theorem in a non-independent way then its reciprocal. The steps can be classical (studying a function) or must be imagined.

A5. Using previous questions in a problem.

A6. Having choices – imposed (only one way to do things after all) or not.

We have taken into account other variables related to the studied notion, in particular previous knowledge that could interfere (geometrical or algebraic) and the geometric configuration within which is situated the exercise. We have this taken into consideration everything that seemed to have a possible influence in terms of student activities out of the suggested exercise texts.

Analyzing tasks and activities through an example

Below is an exercise proposed by one of the observed teachers to his class:

> (C) is a circle with center O and radius r, [AB] is a diameter of (C) and P is the point of [AB] such that AP=1/3r. A line d distinct from (AB) passes though P and cuts the circle at two points M and N.
> 1) Show that the triangles APM and NPB are similar triangles.
> 2) Deduce that PM xPN = 5/9 r^2

To solve the first question of this exercise, we show that the two triangles have two equal angles, using vertically opposite angles, and the inscribed angle theorem. They are therefore similar.

Since two similar triangles have respectively proportional sides, after identifying the corresponding vertices we get: MP/BP = PA/PN, and so prove the intended equality.

Below is the task analysis that we conducted, using previously presented variables:

Table 1. Analysis of the tasks of the exercise completed in class.

	Configuration	**Knowledge**		**Adaptation**
		old	**new**	
1)	Circle	inscribed angle	D	A1
2)	Circle	algebra “without x”	P	A2

However, this first analysis does not precisely inform us about the tasks that are potentially completed by the students. Therefore, we should look at the actual resolution of the exercise in the classroom, to understand, for example, what the teacher has left for the students to do, and what s/he has taken in charge him(her)self. To conduct an analysis of the actual exercise resolution, we have taken into account:

– The time spent on each task, and on the different steps of the same task.

– The forms of work adopted (whole class, group work or individual).
– The help provided by the teacher, and the way it modifies the prescribed task (at what time of the activity does it occur, does it change or not its nature).

Finally, we are interested by the adaptations, autonomies, and initiatives left to the students, and which allow us to determine more closely their possible activities in class. For example, taking into account the moments of silence of the teacher indicates possible individual search by the students. Naturally, we consider them more as activities a minima and a maxima, since we suspect that the completed activities are not the same for all students.

Therefore, we present in Table 2, in the same grid as before, the information gathered about the actual resolution of the previous exercise in class, and we specify the moments of silence of the teacher, the nature of help provided (what it is about), the moment of intervention, and the more or less directive form they have for the students (when we can determine it).

Table 2. Analysis of the actual resolution of the exercise in class.

Task	**Actual resolution**			
	Silence moment	**Help**		
		nature	**moment**	**Form**
9 1)	29 min from the start	About the method	After search	Individual
9 2)	15 min from the start	About the method	After search	Individual

We conduct the same analysis for each exercise that was given in the three observed classes, and we complete a report of all the sessions about similar triangles.

Considering the exam

We then analyze the exercise tasks given in an exam, without analyzing the course of the exam this time, since for these exercises, students do not receive any help from the teacher. We then compare these exercises with those given in class and which might have better prepared the students for the final evaluation. Below is an example of an exercise given in the exam of the same class as before, followed by the solution:

We consider a circle (C) and a triangle MNP inscribed in the circle.
The bisector of angle NMP cuts [NP] at D and (C) at E.
1) Show that triangles MNE and END are similar.

2) Deduce that $EN^2 = EM \times ED$

To solve the first question, we can show that the two triangles are similar because they have two respectively equal angles: one common angle and two angles equal by transitivity (using the bisector and the inscribed angle theorem), which allows us to say that by definition, the two triangles, having two equal angles, are similar.

For the second question, we identify the corresponding vertices (E, N, D and E, M, N) and we apply the proportionality of sides: EN/EM = ED/EN, which yields the desired equality, using the property of proportional sides in similar triangles.

Comparison with similar exercises done in class

We compared this exercise to all the similar exercises done in class, in particular to the "closest" one in terms of the retained variables (configuration, old and new knowledge, adaptation), and which could have – among other things – prepared the students for this exam exercise. Naturally, everything that was given in class could have prepared the students for the exam, and we try to take into account the work habits of a given class in order to better interpret the autonomies left to the students.

In fact, we will see in the next section that teaching practices are stable and coherent, and that we find the same type of choices in the actual resolutions of the tasks all along the chapter for a same teacher.

Regarding the exam exercise, it's actually the exercise presented above which seems to be the closest to what was asked in the exam. In fact, the configuration is the same, though the one that was worked in class is bit simpler, since in the exam the triangles are "nested," which does not make it easy to identify corresponding vertices. The associated old knowledge is also the same in both cases. Moreover, it's an exercise for which the students have been let quite a long time to look for the answer. How will this work organization in class influence the results of the students for a similar exercise in the exam? We will look at the results of the students for this exercise and how to interpret them.

Interpreting the exam results of the students

In order to push further this interpretation, we wanted to check whether certain organizations were only beneficial to certain students. Therefore, we have divided the students into two categories, "good" and "weak," based on their answers in the analyzed exam, as well as their results over the year, the assessment of their teacher, and their orientation for grade 11. We can hence, in this example, split the class into two groups of almost same size (16 "good" and 17 "weak" students). We remain however vigilant in the use of this categorization, since we do not know if the good students are the ones benefiting from certain choices made by the teacher, or if those who benefit from these choices become good students!

For the first question of the exam exercise, Table 3 sums up the comparison with the exercises completed in class.

Table 3. Comparison with the exercises completed in class – question 1 of the exam exercise.

Exercises completed in class	3 exercises
Configuration	same as in class
Previous knowledge	same as in class
New knowledge	Identical
Adaptation to the notion in the exam compared to the one encountered in class	Harder
Type of class work	long time to search for the answer

Out of the 25 students who have answered this question, only 10 have correctly applied the definition D, and these 10 are part of the group that we have named "good students." So it seems the difficulty of the adaptation, which was higher in the exam compared to the class work, was an obstacle for a large number of students, and was only overcome by few good students.

By contrast, for the second question, where this time the type of adaptation was simpler in the exam (ref. Table 4), the results of the students are far better than those obtained in the previous exercise: out of the 25 students who answered this

Table 4. Comparison with the exercises completed in class – question 2 of the exam exercise

Exercises completed in class	3 exercises
Configuration	same as in class
Previous knowledge	same as in class
New knowledge	identical
Adaptation to the notion in the exam compared to the one encountered in class	easier
Type of class work	long time to search for answer

question, 23 applied correctly the property P, among those were all the good students (16) and some of the weaker ones (7). It happens that the adaptation of the property expected during the exam was prepared by applications about the proportionality of the sides of similar triangles of higher level in class than in the exam. Moreover, in both cases, the students had longer time to search for the answer, so we can assume it could have prompted real activities about the new properties in question.

Another important observation that can be made, and that can’t be directly read from these first two results, is the difficulty of the sequencing between these two questions. In fact, it requires identifying the corresponding vertices in the triangles in order to correctly associate the lengths of the corresponding sides.

This offers us the chance to detect whether the students have assimilated this inescapable step. Many students here did not succeed in showing the similitude of the two triangles (question 1), yet they were able to answer successfully the next question, which followed from it. We can thus notice, by observing the adopted techniques in the students’ papers, that many rely on a certain type of “cheating:” certain students deduce the corresponding sides from the expected final answer about the lengths, and not by recognizing the corresponding vertices. This might be linked to the fact that the teacher has taken in charge this systematic step while solving the exercises in class. On the other hand, the distinction between the two steps - showing the triangles' similitude and then deducing the application of the proportionality of sides that follows from it - is systematically indicated, by usually being cut out into two different questions, in the exercise texts given to students. Hence, using similar triangles as a non-explicitly-indicated tool to calculate or compare lengths is never entirely left for the students to do on their own. This might be explained by the absence of elements about this topic in the school mathematics programs.

According to this example, it seems that analyzing tasks in terms of adaptation types is very appropriate to evaluate student learning. The work organization in the classroom, in this case a long individual time of search for the solution, with little or no intervention from the teacher, nor sharing, seems to have been only beneficial to few “good” students, allowing them to overcome, during the exam, the difficulty of a functioning level higher than the one completed in class. Finally, the difficulty linked to correspondence detected in our analysis of the notion seems to be proven for the students.

By conducting this same analysis for every exercise in the exam, we were able, by comparison, to draw conclusions about the relationship between the system {tasks + actual resolution} in class and the student learning. Moreover, by extending the same analysis to several grade 10 classes, with different teachers, we were able to confirm some of our results about the links that exist between what happens in class and the learning, as well as draw conclusions about the regularity of observed teaching practices.

OUR RESULTS ABOUT LEARNING

Our results cover the links between what took place in the classroom and student learning, which was our initial concern, as well as – and it's not negligible if we place ourselves in the perspective of analyzing teaching practices – the diversity and regularity of these on the notion of similar triangles. We present below both types of results.

The success in exams linked to what took place in the three classes.

We have chosen to compare the exams given in the three observed classes in an attempt to deduce the influence of the choices made by the teachers in terms of tasks and actual resolution, on the success of the students.

The results of the students are quite different in the three classes, and this is not surprising, given that the exams are not the same and our observation fields are not identical. In fact, it was intentional to conduct observations in schools with different levels, going from "average" to "excellent" – according to the observed teachers. Therefore, we cannot settle, for example, for looking for the class in which the exam was most successful, since this would not allow us to deduce the types of proposals, by the teachers, that contribute to student learning.

Instead, we can be interested, for each exam, in the exercises that are associated with the most success among the students, or on the contrary, the most failure, in order to link them in each case with what has been previously proposed in class. It is this relationship that we can efficiently describe in the three classes.

The exam exercise that was mostly successful in the three classes

Regarding the exam exercise which the students most successfully completed in each of the three classes we were able to note that it was in each case an exercise which was previously prepared in class, through applications more difficult than or as difficult as the one in the exam, and sometimes through a repetition of similar applications about new notions. In the three cases, it's an exercise that requires applying the definition of similar triangles in its simplified version (two equal angles are enough for the triangles to be similar); in fact it's the property that was most frequently applied in the classrooms in the proposed tasks. It is also the one that we find most commonly in manual exercises. In all cases, these are the exercises for which the students took advantage at least once from an individual work time.

Thus, it seems that the students are capable of carrying out on their own certain tasks when they have been previously exposed to them, and when they had to solve them alone, while they were associated to the same knowledge and in similar configurations as those of the exam.

The exam exercise that was least successful

Regarding the exam exercise which the students least successfully completed, the comparison is not straightforward. These exercises more or less “failed” by the students are about different property applications. In Mrs. B’s class, it was an exercise for which no similar application was done in class, and the students’ failure seems to confirm our previous comment. We actually observe in the three classes that when the previous work is not as difficult as the one expected in the exam, the student fail significantly, regardless of the form of work adopted in class. This seems to indicate that students’ knowledge is not transferable to adaptations of higher difficulty, which takes us back to an observation made by Crahay (2000): at the end of the day, the students learn … what we teach them. However, by digging further, we realize that the exercises in the three exams that were least successful included questions that required a non trivial identification of corresponding items in the figure. This difficulty seems to be detected among students. It is not linked to the organization of the work in classroom by the teacher, but it’s still related to an important element of the choice made by the teacher, since we didn’t detect in any class a discourse or specific work about this identification. We need to look more precisely at how, in the exercises where such identification is essential, the task is dealt with. Do the students take the initiative? Or is it always indicated by the teacher, or by the ordering of the letters in the exercise's text? And how is the task corrected? During the analysis of the observations conducted as part of the DEA thesis of Cisse and Corlay previously mentioned, the obtained results regarding the identification are very interesting: the two teachers had only proposed in the classroom exercises where the corresponding vertices were in the correct order in the text of the tasks, but not in the exam, composed by the researchers, and this identification was a major source of difficulty for the majority of the students, particularly one exercise where the letters were given for the first time in a different order.

The exam exercise that had the most heterogeneous results

We were finally able to consider the exam exercise for which the students obtained the least homogeneous results in each class, i.e. the ones for which there was a notable difference between the answers of the “good” and “weak” students, as we labeled them. In each case, it is an exercise which was prepared only through activities giving the students more autonomy: a work completed at home, or in class, but most importantly without intervention from the teacher during or after the activity. This seems to indicate that the autonomous student work is not equally beneficial for all students, which converges with the results obtained by Felix (2004).

In one of the three classes (Mrs. B), the exam exercise where the differentiation between the students was the most distinctive was prepared during a small group work session, so it was impossible to observe precisely for each student, or even each group, how the occasional and personalized interventions of the teacher could have modified the nature of the activities. Nevertheless, we were still able to notice during this session the absence of collective intervention during and after the student work, and link this fact credibly to the heterogeneous results of the students.

SOME OF OUR RESULTS ABOUT TEACHING PRACTICES

In all the cases detailed above, the choices made by teachers regarding the organization of their teaching of similar triangles – whether related to the content and the variety of suggested tasks, or the course of the activities that follow from it – have had probable influences on the students learning of this notion. However, these choices are also subjected to certain constraints including those imposed by the programs, which we have already mentioned, and which should take into consideration to better understand the offer given to the students of these three classes about this chapter.

Teaching practices about similar triangles: common points

We find common points to the three observations conducted: for example, the time spent to cover the chapter, or even each new notion of the lesson. For the three observed teachers, we don't find any discourse about the identification of corresponding elements, and little work about this difficulty, which does not seem surprising giving the absence of instructions in the scholarly programs, and the lack of exercises corresponding to this specific task in the manuals. This reflects in particular the importance of the institutional constraint which weighs on the teachers.

The management of moments of task solving by the three teachers

To refine our comparison, we have tried to look precisely, for a given task proposed by the teacher, at how his/her interventions redefined the initial task – by simplifying it or modifying it through the introduction of new elements – which could have therefore lead – again a minima and a maxima – to constituting the activities of the students about this set of sub-tasks. This allowed us to determine in particular what has been taken in charge by the teacher, as well as what has been left as autonomous work for the students. By conducting this analysis for each of the proposed tasks, we can get a more precise idea of the possible activities of the students in a class, as well as an insight of the teaching practices about this notion.

In the three classes, such an analysis for all the tasks shows us that the simplification of an initial task is systematic, but does not interfere at the same rhythm, nor with the same frequency, depending on the teachers and the proposed tasks. However, for a same teacher, this modification of the task is always carried out following the same model, all along the chapter.

For example, for Mrs. B, this simplification interferes quite early in the course of the student activities, and they have enough time, especially after the shoring of the exercise or the correction, to complete simple and isolated tasks, or to write the correct answer. Conversely, Mrs. P doesn't interfere except after an autonomous time of search for the students, and thus potentially leaves complex tasks to the students, with a much higher level of adaptation than in the other classes. Finally for Mrs. M, the tasks proposed in class, although more complex initially, are quickly simplified, and the students only get a time of search for the most difficult questions while working at home.

The limits of our study

It isn't easy to interpret the influence of these choices on the results of the students, since they also depend on external factors, such as the level of the school. The results of Mrs. F's students are very good, despite the little actual work done in class concerning the mathematical difficulties of the chapter. How can we thus explain that the exam was successful despite all?

Therefore, we compile the report for the complexity of the proposed tasks in each class, as well as the tasks that are left for the students to do in class after the simplification by the teacher, or to be done at home, and finally the tasks of the exam. This report is summarized in table 12.

We notice that in Mrs. F's class, the possible activities of the students correspond least to the tasks initially proposed – and this reinforces more the legitimacy of the fact that not only the tasks should be taken into account, but their actual resolution as well to infer consequences about the potential learning. The gap between what is proposed and what is potentially completed by the students in class has a very likely influence on their aptitude to solve complex tasks, but this influence cannot be evaluated by the exam proposed by the teacher, since the latter only includes simple tasks.

Table 5. Complexity of the tasks proposed by the three teachers.

	Mme B	**Mme P**	**Mme F**
School level	weak	good	very good
Tasks in class	simple	complex	complex
Part left up to the students	simple tasks	complex tasks	simple tasks
At home	simple	simple or complex	complex
In the exam	complex	complex	simple
Results of the students	average and heterogeneous	good	good

This is one limit of our study, the fact of not having participated in the elaboration of the exam text, but the time constraints that weigh at the same time on the researcher's work as well as that of the teacher did not allow us to analyze all the observed sessions before the exam, fast enough so that the latter would take place soon after the end of the chapter.

The stability and coherence of the observed practices still allow us to draw conclusions about the links with student learning, even if we can only rely on this one chapter. Obviously, it would be necessary to extend this study to other mathematical notions, to determine in what way our results are specific to the chosen notion.

CONCLUSION ABOUT THE PRACTICES AND THE LEARNING

A first lack in the teaching of similar triangles in these three classes – a lack that we had detected as early as the analysis of the notion – is the one linked to the scholarly programs and manuals, and this is regardless of the choices of the teachers concerning class management. In fact, the problem of corresponding elements is above all linked to the gap in the programs (relayed by the manuals) which does not allow the teacher to offer the students a systematic method of identification, and even less a mathematical justification of the technique potentially indicated. Perhaps there would be one using transformations?

A prospective limitation is the reduced variety of tasks proposed to the students compared to the possibilities. In the classrooms, certain applications are little, or not at all worked on, and this can be related to the incapacity of the students to complete them on their own afterwards, during the exam. This can be once again related to the institutional constraints: at the same time the necessity to respect the advised times, and also the impossibility to pick adequate exercises in the manuals, where we find, to a more or less large extent in the textbooks, this same lack of diversity.

By considering what was really allotted to the student, in terms of entering the task, as well as adaptation of the theorems, we notice that the possible student activities do not always reflect the proposed tasks, considering the teacher interventions. In fact, two of the three teachers reduce, for certainly different reasons, the activities of the students to working on simple and/or isolated tasks. This dimension actually allowed us to interpret the exam results, even though we didn't choose its text.

Last but not least, we were able to identify absent elements in the discourse held in each class, for example the validation discourse: the lack of institutionalization for Mrs. B, following a small group work session, could eventually act as a brake for the learning of the less good students.

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7. THE STUDY OF A SCENARIO AND ITS IMPLEMENTATION IN THE CLASSES OF TWO DIFFERENT TEACHERS

Comparison of the in-class events
and the effects on the work of students in exams

INTRODUCTION

In this chapter, we compare the implementation of a same teaching scenario about orthogonal symmetry by two experienced teachers (we will name them Martine and Denis in what follows) in their respective grade six[1] classes, Denis being a teacher in an Education Action Zone[2] (a.k.a. ZEP).

The study we present here fits into the general theoretical framework exposed at the beginning of this book. Our specific aim is to contribute to the study of regularities and variability of practices between teachers, as well as to the one regarding the relationship between teaching practices and student learning. We also try to link these elements with the logic underlying the teachers' practices from traces that we perceive in the analyses.

We begin by specifying the methodological elements that we have retained for our study. Next, we present the characteristics of orthogonal symmetry that we have retained for the study of the scenario, before we present the latter. We then compare the events in the classrooms of Denis and Martine, during the implementation of this scenario. Analyzing student works in exams and linking them to the analysis of the class activities allows us to explore the effects of the practices on the learning of students.

ELEMENTS OF METHODOLOGY

In accordance with the theoretical framework described in the first chapters of this book, student activities are central to our analyses, as intermediates between teachers' practices – of which they are partly the consequence – and students' learning – which they are likely to induce.

Studying the object of teaching

Exploring student activities requires a preliminary study of the targeted object of teaching – here, orthogonal symmetry. It implies considering epistemological, didactical and curricula elements. This study allows us to determine the different

F. Vandebrouck (ed.), Mathematics Classrooms: Students' Activities and Teachers' Practices, 151–166.

dimensions that permit characterizing a teaching scenario of orthogonal symmetry in grade six in France.

Analyzing the scenario

The scenario designates here, like in previous chapters, the project conceived by the teacher before the classroom sessions. It includes the set of exercises and the lesson the teacher intends on presenting to students.

The *a priori* analysis of exercises and lessons as well as their articulation aims at determining activities which they are likely to induce for students. We "convert" the text of the exercises into mathematical tasks and establish solving procedures that the students may implement to complete these tasks. A possible solving procedure is characterized by the mathematical knowledge it mobilizes, as well as the adaptations (chapter 2) needed for their functioning, the geometrical paradigm (Houdement & Kuzniak, 2003) to which it refers, the misconceptions about symmetry that are likely to interfere in the solving, and finally the means of control of the validity of their procedures at the disposal of the students. Hence, we take into consideration the position of the tasks in the scenario from two perspectives: the knowledge already available to students at this point and the function of the task within the scenario.

The tasks of the scenario are therefore classified into four categories:[3] tasks for recognizing properties of a figure (axes of symmetry, etc.), construction tasks (symmetrical of figures, axes of symmetry, perpendicular bisector of a segment, etc.), proving tasks (justifying a statement using properties of a figure), and drawing tasks (for example freehand drawing).

This *a priori* analysis of the whole scenario allows us to make a first prevision of the set of activities it is likely to induce for the students, thus giving access to the cognitive itinerary elaborated by the teacher.

Analyzing the in-class events

The analysis of the observed class period is also guided by the search for the characterization of possible student activities. It should be reminded that real activities are not accessible, mainly because part of them happens inside one's head, but we can rebuild possible activities *a minima* and *a maxima* using observable traces of real activities (oral interventions of the students and the teachers, written traces in notebooks, etc.).

Our analyses of classroom sessions are therefore centered on identifying traces of student activities, as well as anything that could influence these activities: modifications – reductions or enrichments – of initially prescribed tasks. For example, a reduction of a task may come from a hint given by the teacher when presenting the task thus triggering a procedure strongly; similarly, if the teacher adds a justification request for an answer, this can enrich the task.

We proceed from videotapes of the sessions, which are transcribed and completed by the teacher documents (course preparation, exercises, etc.). Using the transcriptions and videotapes, each session is divided into episodes. We distinguish five types of episodes: exercises, lessons, corrections – of works completed at home or during a previous session – lesson recitation/recalling, other; the division matches teacher interventions (such as: “we correct the exercise that was assigned”, “now, we move to the lesson notebook”).

We then divide each episode into phases depending on the form of the work – individual, small groups, whole class – and its nature (a certain task, a certain lesson statement, answering a given student question if it produces a development, etc.) determined through the teacher’s questions and the students’ interventions.

The retained indicators for the analysis are: time allocation in terms of the forms of work, the nature of the work and its distribution in terms of the forms of work, and finally the part of the work which is left for the students to complete.

Analyzing students’ works in exams

Exams are graded evaluations conducted in class. We consider students’ works during these evaluations as traces – definitely imperfect – of the attained learning following the received teaching.

Each work leads to a coding of 0 and 1. It is considered as a success (coded 1) if the work contains a trace of the knowledge which is expected to be mobilized to solve the task, even if the solution is incomplete or wrong. For instance, at this school level, for tasks requiring to justify the length of a segment knowing that of its symmetrical segment, the main purpose is to evaluate if the student mobilizes the property of conservation of lengths by orthogonal symmetry rather than actually measuring the length: we hence coded 1 any work that mentions this property, even if the answer is not well written or if the student doesn’t explicitly mention the symmetry of the segment extremes (an element required by certain teachers). We coded 0 when there was no trace of the expected knowledge, including when the task was not started.

This coding for each student and each task thus allows us to calculate success rates, as percentages of 1s with respect to the whole or to groups of tasks, for a given student, or for the whole class for a given task, or group of tasks. Hence, the success rate for a group of tasks represents at the same time the average number of students having succeeded in the tasks and the average number of tasks achieved by student.

The data of the study

The scenario that we present here is the one elaborated and implemented without any intervention from the researcher during the school year 2006-2007 by Martine. As part of a particular experimental system (Chesnais, 2009), it was transmitted to Denis who implemented it in his own class during the following school year 2007-2008, with the support of the researcher.

We have videotapes of all the sessions devoted to orthogonal symmetry by both teachers, so for each teacher a dozen of sessions each around one hour long.

As for student works in exams, we have those for a round twenty tasks in each class, spread over three exams for Denis (during the teaching, at the end of the teaching, and few weeks after the end of the teaching of the notion), and two exams for Martine (during and at the end of the teaching).

THE CARACTERISTICS OF THE TAUGHT NOTION

Epistemological aspects

Orthogonal symmetry is not only a mathematical concept; it is also an object of shared culture in the society. According to us, in the second case, this covers the perception of a bilateral 'approximate' symmetry with a vertical axis, without however associating it to the underlying mathematical notion; we thus speak using Vygotski's terminology of the everyday concept of symmetry, as opposed to the scientific (mathematical) concept.

The scientific concept is nowadays defined as a geometrical transformation; however a study of the historical genesis of the concept shows that it is first and foremost associated with the harmony, the equilibrium of a figure or object. We thus distinguish two aspects of symmetry, which we describe as static and dynamic.[4] The first refers to the properties of a figure, the second to the transformation, i.e. the relationship between two sets of points (eventually overlapping). In the everyday concept, the static aspect is omnipresent, whereas the idea of transformation, movement or displacement, is almost absent (it is limited to an association with the folding which allows superimposing two parts of a figure). However, in the mathematical concept, ever since orthogonal symmetry was defined as an element of the group of plane isometries, the transformation pre-exists somehow, since the existence of axes of symmetry in a figure is nothing but the corollary of the existence of an orthogonal symmetry which preserves it globally.

What is targeted by the teaching is, eventually, the scientific teaching and the syntheses of the static and dynamic aspects. However, the everyday concept can play the role of a lever or an obstacle. For example, the idea of an axis of symmetry as splitting a figure into two identical matching parts, linked to folding, prevents from understanding that the image of a figure can be the figure itself. This makes it impossible to access the second level of apprehension of transformations (Grenier & Laborde, 1987) that is the symmetry as an application of the plane onto itself, an involutional bijection whose axis points are invariant points – the first level consists in considering the transformation through its action on the figures.

Didactical aspects: Alternative conceptions

Grenier (1988) has identified several conceptions of symmetry that are expressed through theorems-in-action (Vergnaud, 1990) with limited validity domain. We describe them as alternative conceptions and we retain three of them:

– Orthogonal symmetry as transformation from a half-plane onto another. This conception is linked to the everyday concept (mainly through the folding and mirroring). It is associated with the idea that this transformation only works in one direction (most commonly, downwards or left to right). It represents an obstacle to the conception of the symmetrical of a figure cut by an axis as well as the definition of the symmetrical of a point using the perpendicular bisector of a segment, given that the latter is "meaningless" through this conception.
– The confusion with other geometrical transformations. Students sometimes build the symmetrical of a figure touching the axis by central symmetry around the contact point or by translation; or even recognize an axis of symmetry on a figure that does not have one but has a center of symmetry instead (such as a non rectangular parallelogram for example).
– The conceptions linked to special cases of vertical and horizontal axes. In particular, they lead to the construction of the symmetrical of a point on a same horizontal line as the point, despite an oblique axis, or by only noticing the vertical and horizontal axes of symmetry on a figure. Together with the confusion with the central symmetry, this also implies theorems-in-action such as "a segment and its symmetrical are supported by the same line."

Overcoming most of the alternative conceptions and completing a synthesis of the static and dynamic aspects requires a (re-)definition of the symmetry without the folding tool. Furthermore, the work of Grenier (ibid.) gives evidence of the resistance of these conceptions despite the teaching. Moreover, the organization of a specific work seems essential to allow the students to overcome them and favor a scientific conceptualization of the notion.

The teaching curriculum

The current curricula of grade six in France, at the time of our study,[5] regarding geometry, are organized around the progressive introduction of geometrical transformations and the objective of transiting from an essentially perceptive and instrumented geometry to a deductive geometry – which we reformulate using geometrical paradigms defined by Houdement and Kuzniak (2003) as a transition from a geometry labeled natural (geometry 1) to a geometry labeled natural axiomatic (geometry 2).

These curricula stem from those of 1985, with some modifications; in particular, regarding orthogonal symmetry, it was clearly indicated that the static aspect must be (re-)defined as the invariance under the transformation:

> Depending on the situations, [orthogonal symmetry] will appear under the form: of the action of a given axial symmetry on a figure; of the presence of an axis of symmetry in a figure, that is an axial symmetry which preserves it.

However, this indication has disappeared, leaving uncertainty regarding the way the two aspects should be approached. The competencies about orthogonal symmetry targeted in the 2005 curricula are essentially linked to the constructions: this seems to indicate at first sight that the focus is on an instrumented geometry, but the construction of the symmetrical of a figure on plain paper without folding or using tracing paper requires, mainly for polygons, constructing the symmetrical of the vertices using the straightedge and compass – or only the compass –, then linking the obtained points: these constructions rely on the definition of the symmetrical of a point via the perpendicular bisector and on the conservation properties (of lengths and angles).

It seems possible to interpret these curricula in different ways. In fact, they recommend on one hand an "experimental work (folding, tracing paper)", which places the problem in the geometry 1 frame, and on the other hand a work on the properties of symmetry and the definition of the symmetrical of a point, which pertains to geometry 2. The articulation to be done between the two and the objective in terms of the apprehension level of transformations (Grenier and Laborde, ibid.) that must be targeted remain vague.

Overview of the study of the notion

There are several possible scenarios and the curriculum does not allow making a definite choice among them. A scenario is then characterized by the choices related to: taking into consideration the articulation between everyday and scientific concept – the second one must progressively replace the first; the share of perceptive, instrumented and deductive work; the link to be established between the static and dynamic aspects – aiming at the articulation of the two; and finally the way it deals with the alternative conceptions – aiming at their disappearance *in fine*.

PRESENTATION OF MARTINE'S SCENARIO

Due to lack of space, we cannot present the scenario in a comprehensive way. That is why we only present its chronological structure, followed by the way the different dimensions – mentioned in the above overview – are updated throughout the scenario.

Chronological structure

Martine's scenario is organized into three parts:

– The first introduces the symmetry in its dynamic aspect and includes the characterization of symmetrical figures as superimposed through folding, then the definition of a symmetrical of a point,[6] the construction of symmetrical of figures (points, segments, half-lines, lines, circles) and the conservation properties (of alignment, lengths, areas, angles, parallelism, and orthogonality).
– The second part is about the static aspect of symmetry and includes looking for symmetry axes of geometrical figures on one hand (segment, line, angle, special triangles, special quadrilaterals, circles), and of figurative drawings (for example a house) on the other hand; it includes a "method to determine if a figure has an

axis of symmetry" and ends with an exercise about finding common figures having a given number of axes of symmetry.
– The last part covers the notion of perpendicular bisector of a segment and links the two aspects of symmetry. It includes the definition of the perpendicular bisector as a line perpendicular to a segment at its midpoint as well as the fact that it is an axis of symmetry to the segment. We also find the property of points of the perpendicular bisector being equidistant from the endpoints of the segment (direct and reciprocal), used mainly to carry out and justify the construction of a perpendicular bisector of a segment and of a symmetrical of a point using a compass. It ends with the reformulation, using the perpendicular bisector, of the definition of the symmetrical of a point and with exercises involving simultaneously several elements of the chapter as well as previous knowledge.

The scenario is thus clearly organized around the dynamic and static aspects of orthogonal symmetry, introduced successively. Starting with the dynamic aspect allows the redefinition of the static aspect using global invariance (however this does not appear explicitly *a priori*). Lastly, the fact that the third part is designed to link the two aspects, in particular through the notion of perpendicular bisector, shows that the scenario is guided by the requirement of a certain mathematical and didactical coherence.

Mathematical concept and everyday concept

The scenario revolves mainly around the mathematical concept. The place of the everyday concept is not very important, and the transition from one to another is partially supported. The first exercise of the chapter refers to the everyday concept of movement (via the displacement of a tracing paper) in order to characterize the orthogonal symmetry by the folding and/or the turning over of the tracing paper. The introduction of the definition of a symmetrical of a point with respect to an axis (as a point such that the axis cuts the segment perpendicularly at its midpoint) is based on the above characterization: we believe this contributes to articulating the everyday concept and the mathematical concept. However, in the work about the axes of symmetry (second part of the scenario), the recognition and drawing tasks include figurative drawings, usually in one block: the intervention or not of the mathematical concept in solving these tasks seems very dependent on how the course is led (for example if the teacher asks for justification or not), as the axes can be identified by mobilizing only the everyday concept.

Dynamic and static aspects

The structure of the scenario shows that the articulation of the two aspects is one of its main objectives, and that it is handled in a very organized way.

To start with, certain exercises allow working on the articulation between the two aspects. For example, one of the tasks consisting in "complet[ing] a figure so that it creates an axis of symmetry" allows linking the notion of an axis of symmetry with the construction of the symmetrical of a figure. Next, we find exercises where a figure and its symmetrical are partially overlapping (for example, one of the tasks is to construct the symmetrical of a segment with respect to a line which is perpendicular to it at a point distinct from the midpoint). Finally, certain tasks, after having constructed a symmetrical, consist in proving a property of a figure formed by two elements situated on both sides of the axis (for example, students are asked to justify the nature of a triangle ABA', where A' is constructed as the symmetrical of A with respect to a line passing through B). Yet, as we have specified previously, these tasks are designed to alter the conception of transformation from a half-plane onto another: they thus favor the mathematical conceptualization of the symmetry and prepare the transition to the second level of apprehension of transformations.

Levels of apprehension of the transformation

The symmetry is introduced as an action on figures that is at the level 1 of Grenier and Laborde (1987). The symmetrical of a point is then studied as the symmetrical of a particular figure which is used to establish a construction method for a more complex figure. The scenario thus mobilizes the transformation between the first and second levels of Grenier and Laborde (1987), in compliance with curricula indications. For example, the conservation properties (mainly the conservation of lengths and angles) are evoked since the very first exercise, that is in the case where the apprehension of the figures remains global, then it is worked out again with the symmetrical constructions and the proving tasks which mobilize them in a more local apprehension of the figures.

The interplay between geometrical paradigms

The structure of the scenario itself seems to be underlined by a theoretical coherence related to the mathematical concept – thus to geometry 2 – in the sense that the set of symmetrical constructions and proving tasks, for example, are based on the definition of the symmetrical of a point and the properties of symmetry established in the lesson.

Moreover, the initiation to deductive reasoning constitutes an important object of the scenario. In fact, first of all, the tasks of proving (requiring precisely, most of the time, such a type of reasoning) represent 20% of the tasks of the scenario; then, this initiation is explicitly taken in charge via a sequence of exercises, in the first part of the scenario: it is about leading students to justify a statement by using the properties of a figure and to thus clarify the "rules of the mathematical game" – in particular the invalidity of measurement or perception in this type of tasks. Certain construction tasks also contribute to the transition towards geometry 2: for example, the students must develop themselves the method of construction of the symmetrical of a point, starting from the definition. This type of work seems to pertain precisely to geometry 2, whereas the construction tasks often consist, in grade six, in applying a procedure, the main challenge being the mastery of the instruments – a work that is more related to geometry 1. In addition, what is at stake in most of the construction tasks of the scenario is the mastery of the construction of the symmetrical of a point in complex configurations (i.e. where the figure is not limited to the point and the axis of symmetry, where the construction eventually requires involving a reasoning or is likely to involve alternative conceptions): the construction of the symmetrical of a point is thus not targeted for itself, but is rather the means to a more elaborate work on the concept.

Alternative conceptions

They are taken into consideration in two ways.

On one hand, certain tasks have a clear objective to bring out and question some conceptions. Thus, an exercise consists in “finding the mistakes” in the drawing of symmetrical or non symmetrical figures; for example, when the drawing is made up of two parts such that one of them is obtained by a transformation other than orthogonal symmetry, then the work focuses on the confusion with other transformations. Similarly, certain constructions of symmetrical of figures cut by the axis can be an opportunity to question the conception of transforming from a half plane onto another.

On the other hand, particular cases – mainly those related to vertical and horizontal axes – are not overrepresented in the first and third parts of the scenario. In particular, in the third part, they are almost absent: we believe this can either have as a consequence overcoming these alternative conceptions, or it can allow alternative conceptions relating to special cases to coexist with correct conceptions, their respective application domains being different. In fact, we can think that the effect varies depending on the students, on the particular alternative conceptions and on the tasks. As for the transformation of a half plane onto another, it seems that the focus of the work on the mathematical concept and the important number of tasks through which this conception is questioned would allow overcoming it. Nevertheless, in the second part of the scenario (the one that handles axes of symmetry of figures), the geometrical figures are often presented in stereotyped positions (for example, the rectangle with parallel sides on the edge of the paper) and the figurative drawings are such that almost all the axes are vertical or horizontal. Yet, Grenier had pointed out in her work that it is precisely in the case of figures that the associated alternative conceptions arise the most and not in the case of points. This part of the scenario thus exposes to the risk previously mentioned: dealing solely with the static aspect from the perceptive point of view (related to the everyday concept) and reinforcing the alternative conceptions linked to these particular cases.

Organization of the exercises and their links to the lessons

Martine’s scenario includes a significant part of work on exercises (overall 75 tasks) most of which is planned as class work, with a part of individual work. Homework consists essentially in solving tasks similar to the ones handled in class.

Exercises in each part are organized in a very progressive manner: the number of tasks per exercise increases progressively, as well as the variety and complexity of tasks (quantity of knowledge involved, mixing progressively new knowledge with old ones, with more and more adaptations, also increasing in difficulty).

Finally, the exercises are systematically linked to the lesson, whether they constitute an application (more or less direct, depending on the adaptations at stake), or they allow the introduction of new knowledge which then become the object of a lesson.

Overview of the analysis of Martine's scenario

The scenario thus appears to be organized around the change of geometrical paradigm and the initiation to deductive reasoning. The learning objectives are clearly oriented towards the mathematical concept of orthogonal symmetry – even though some ambiguity remains in the second part – and to the link between its dynamic and static aspects.

COMPARISON OF THE IN-CLASS EVENTS

Due to lack of space, we will only present the common points and the main differences that have emerged during the analysis of the in-class events in Denis's and Martine's classrooms.

Time allocation

Studying the class period activities from a quantitative and global perspective, we essentially observe convergences: Denis and Martine take noticeably the same time to cover the chapter (a dozen hours). Moreover, the proportion of time corresponding to the same type of episodes (lesson, exercises, corrections, etc.) is nearly identical in both classrooms. In particular, around 60% of the time is allocated by both teachers to solving exercises.

However, while we observe the progression throughout the chapter, we notice that Denis spends more time on the construction of symmetrical of figures and on the methods to find axes of symmetry or construct the perpendicular bisector of a segment, while Martine lingers longer on the definition of the symmetrical of a point and the properties of the perpendicular bisector of a segment.

We also notice that on average, the length of episodes according to their nature is similar in both classrooms: exercises, lesson and correction episodes last respectively around 15 minutes, 6 minutes, and 13 minutes on average in Martine's classroom, 15 minutes, 8 minutes, and 11 minutes in Denis's classroom. Moreover, the distribution of time between individual work and collective work is in the ratio of around one third to two thirds in both classrooms.

The nature of the work: Mainly differences

During the exercises episodes, Denis intervenes in general collectively at a very early stage, sometimes even from the beginning, to reformulate the task, simplify it or divide it and he usually takes in charge part of the work – mainly the adaptations – whereas Martine almost always lets the students work directly on the tasks, even when they require adaptations. Consequently, while Martine's students often face exercises for which they must develop a solving procedure based on the lesson content on their own, Denis's students are less often put in such situations: for the constructions of symmetrical of points or figures, no method is given to the students prior to the individual work in Martine's classroom (for example, for the symmetrical of a point, the only help provided is a reminder of the definition of two symmetrical points); on the contrary, in Denis's classroom, procedural elements are sometimes provided from the beginning (for the construction of the symmetrical of a point, the instruments to be used are indicated after only 2 minutes). As for proving tasks, Denis could go as far as solving them entirely collectively, whereas part of the work is individual in Martine's classroom.

Both teachers seldom provide individual help during the phases of individual work.

Collective work on the exercises is very different, in its own nature and also because of the contributions of the two teachers, which exceed the simple correction of the task. In fact, for Martine, this collective work often involves the targeted knowledge – related to the function of the task in the scenario – whereas for Denis, it is either limited to the task, or it covers knowledge other than those targeted; for example, in the first exercise of the chapter, the collective work phases' subject in Martine's classroom is not only the correction of the exercise (that is indicating the movement of the tracing paper required to go from one figure to another), but also naming and characterizing the geometrical transformations involved, in order to identify the axial symmetry as being the one that corresponds to the folding or the turning over of the tracing paper. During the corresponding collective work phase in Denis's classroom, the aim is not only to find how the tracing paper must be moved: his aim is clearly to identify and characterize the transformations, and yet all of the involved transformations are mentioned but "axial symmetry"! Similarly, in solving the exercise targeting the introduction of the definition of a symmetrical of a point, the nature of the individual work and collective work is very different between the two teachers: for Denis, the individual work, at the start of the task, consists in carrying out a manipulation with tracing paper in order to obtain the symmetrical of a point, then drawing a segment and a line, the collective work which follows aims at completing sentences with gaps by the words "perpendicular," "midpoint," "symmetrical" and "perpendicular bisector." As for Martine, she alternates individual and collective work throughout the episode: when she realizes that the students are taking time to carry out the material manipulation, she handles it collectively, and then comments this manipulation trying to make the students realize that they have constructed the symmetrical of a point (which was not explicit in the text of the exercise). She then asks the students:

> Martine: "*Now, observe the figure, and try to tell what you see, what you notice, […] what can you say about this line AA', compared to the line d?*"

This question raising difficulties for the students, Martine alternates individual reflection time and collective synthesis until the students reach explicitly the characterization of the symmetrical of a point. In Denis's classroom, the fact that the axis is perpendicular to the segment and goes through its midpoint is not explicitly presented as the characterization of a figure, which would allow defining symmetrical points: the only challenge seems to be completing sentences with "the right word," but arguments such as "we already mentioned this word" are used to invalidate students' answers. On the other hand, in both classrooms, the case of one point belonging to the axis is mentioned – which was indicated in the scenario for Denis – then both teachers come back to the result of the observations – however with diverging perspectives: Denis asks the students to place another point in the same way and to place the coding that indicate perpendicular lines and midpoint, whereas Martine asks:

> Martine: "*What conditions make the points A and A' symmetrical with respect to a line d?*"

This difference in the approach is typical of the teaching of Denis and Martine: the latter often makes the students reformulate ideas and reasoning, while Denis is more keen on getting them into action.

As for proving tasks, they are entirely solved collectively in Denis's classroom, whereas in Martine's case, only the correction is done collectively after an individual work completed by the students. Furthermore, Martine leaves the validation and justification of reasoning to the class during the collective phases, which Denis only does exceptionally.

The contributions during the exercise phases are also more systematic and more focused on the targeted knowledge in the case of Martine.

Lastly, Martine's interventions always seem to be adapted to the students' interventions, thanks to a good interpretation of their works and their errors. She can even go as far as resuming at length, during lesson episodes, students' reasoning about given tasks. Such interventions, much more limited and rare in Denis's case, are, above all, sometimes less well focused.

As for the lesson episodes, the responsibility of the contents is better shared between Martine and her students than Denis and his students. In the first case, lesson statements are always generated as a synthesis of the exercises, and are grounded in the students' activities, whereas this articulation, although present in Denis's classroom, seems to be artificial sometimes.

Overview of the comparison of the in-class events between Denis and Martins

On the largest scale, we observe similarities between the developments of lessons of Martine and Denis. Is it by chance or are the in-class events, or at least what is related to time allocation, entirely determined by the content of the scenario? It seems unlikely that the time distribution between individual and collective work for an exercise only depends on the handled task. On the other hand, the function of the task in the scenario with respect to its global coherence could play a role given that, if the objective is to bring out such or such content, this would require collective time.

What appears as a main difference is that, other than the tasks effectively prescribed to students, the content and modalities of the collective phases can be very different between the two classes. Martine often leaves up to the students a part of the responsibility in the wording of the answers, whether to let them develop this competency, to evaluate their degree of comprehension, or even to facilitate their comprehension.

EFFECTS ON STUDENTS' WORKS

The exams given in the classrooms of Denis and Martine include identical exercises as well as different exercises but with certain similar comparable tasks.

The results (success rate in percent of the students of Martine and Denis for 11 identical tasks, ref. Figure 1) reveal relatively significant differences for certain tasks, but mostly similarities in the success rates depending on the tasks. It should be noted that rates of success were significantly different between the students of Denis and Martine for almost all the types of tasks during the first year of the experiment, in the situation where each teacher implemented his own scenario (Chesnais, 2009).

In particular, for the six tasks about recognizing and drawing axes of symmetry (labeled "rec. axes fig. x" in Figure 1), the success rates are only slightly lower in Denis's class, except for the third one (where the difference is more important) which is discussed below. As for the unique proving task, the rates are also similar.

In certain cases, the differences are more important, in particular in the tasks that consist in citing the definition of the symmetrical of a point: the rate is 70% for Martine and barely 15% for Denis. These differences could be explained by the fact that Denis's students are not quite used to cite decontextualised knowledge during the exams, and also by the fact that this type of tasks causes particular problems related to language (ZEP students being more exposed to this kind of difficulties). However, this result should also be linked to what took place during the in-class events.

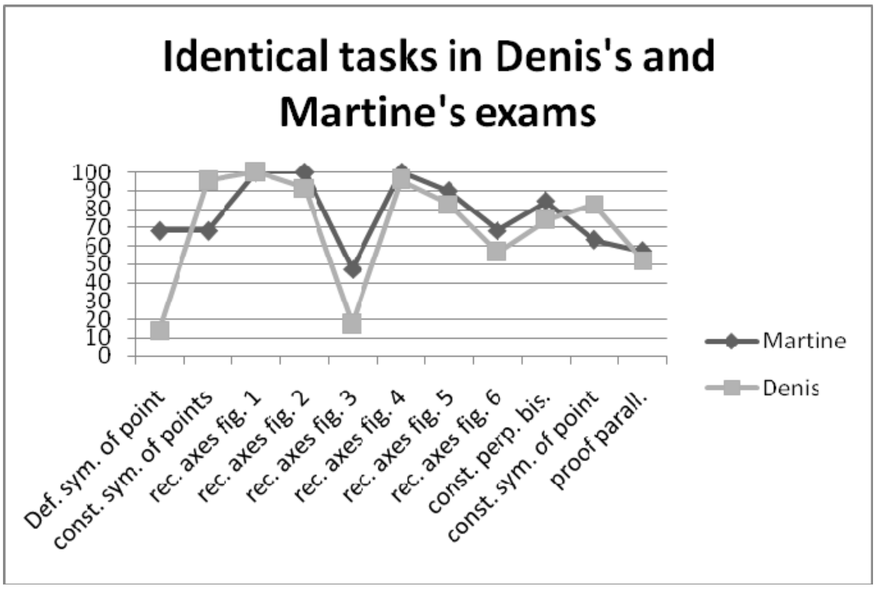

Figure 1. Comparison of exam success rates (in %) between the students of Martine and those of Denis on identical tasks.

The task "rec. axes fig. 3" has a success rate of 50% in Martine's class and 20% in that of Denis. This was the most difficult task among those of the same type, since it included other axes than vertical and horizontal ones. In this case, relating things to what has happened during the chapter did not allow us to identify a factor that could explain this difference, but we put forward the hypothesis that this is not exclusive to the chapter on axial symmetry: unusual and particularly difficult tasks cause more problems to ZEP students.

We also compared success rates relative to other proving tasks, not identical, but similar: for two elementary proving tasks (involving conservation of lengths and conservation of angles), the success rate is 80% for both in Martine's class and respectively 50% and 43% for Denis's students. In other words, an important difference persists between the students of the two teachers for this type of tasks.

To sum up, the success rates are in general very close in Denis's class to those obtained in the class of Martine, except for the lesson question, a particularly complex task and certain proving tasks.

CONCLUSION AND EXPLANATION HYPOTHESES

The implementation of the same scenario by two different teachers in their respective classrooms has allowed us to identify common points as well as important differences both in the classroom events as well as in the effects on the students' works in exams. Despite the obviously very limiting factor of the small number of observations (mainly the fact that the experiments only included two teachers), we propose in this last part an interpretation of these results based on both the scenario properties and the question of the logic underlying the practices.

The study of Martine's scenario highlighted its qualities, in terms of the potential for student activities. The analysis of its implementation in Martine's classroom then Denis's allowed us to confirm to a large extent this *a priori* analysis, and it even suggested a certain "sturdiness" of the scenario to transmission. In fact, the research on didactical engineering has shown a long time ago that the transmission of a scenario in classrooms does not necessarily guaranty the reproducibility of the in-class events nor the results of the students. This scenario definitely deserves to be more tested, but we still put forward the hypothesis that its strong coherence from a mathematical and didactical point of view as well as the presence of certain "sensitive" tasks, i.e. playing a key role to reach the targeted learning (for example, the list of exercises allowing the initiation of deductive reasoning rules) contribute to its efficiency.

As for the differences observed between the implementations in the two classes, we first need to indicate that the modalities of the experiment probably play a role. Certain variations in the in-class events can thus certainly be explained by the fact that the scenario was designed by Martine: since she masters the task objectives and the global organization of the scenario, she's able to establish links more easily and put the tasks into perspective during the collective phases. It should also be reminded that Denis was supported by the researcher during the implementation of the scenario: this support essentially consisted in a discussion during which the scenario was presented to him along with explanations concerning its structure and some elements of the didactical study of the concept (in particular about alternative conceptions). It seems important to indicate also that Denis was observed – without intervention from the researcher – the year that preceded the experimentation of the scenario of Martine. The comparison between Denis's practices during the first and the second year (Chesnais, 2009) shows a development towards practices likely to induce richer activities for the students, as well as a clear progress in the students' works in exams.[7] The respective parts of the effect of the change in scenario and the support of the researcher on this evolution remain however unknown.

The theoretical framework of the double approach allows us to bring forward hypotheses regarding the differences in the observed practices. In fact, the differences of career paths and even of age between Denis and Martine – elements accounted for by the personal component of the practices – suggest that the two teachers do not have the same resources at their disposal. For example, Martine being older than Denis has taught while the previous curriculum was implemented: at the time, the fact that the static aspect needed to be redefined as global invariance in the transformation appeared explicitly. Likewise, it must be reminded that Denis teaches in a ZEP, with students who come in majority from disadvantaged social environment and whose level in mathematics at the beginning of grade six – as measured by national tests conducted by the ministry – was clearly below that of Martine's students. We can assume that these elements – accounted for by the social component of the practices – influence Denis's practices: he "adapts" his practices to his audience. However, our analyses do not allow us to know whether these differences between Denis and Martine are responsible for the differences in the students performances, or on the contrary for the similarities: in fact, the question remains to know whether in-class events closer to those observed in Martine's classroom would have allowed Denis's students to obtain even better results, or if the characteristics of Denis's in-class events are a necessary adaptation to his students and partly explain that they got results close to those of Martine's students.

We therefore hope that we have been able to show in this study the fertility of the theoretical framework presented at the beginning of this book, when it is applied to study teaching practices and their effects on student learning.

NOTES

1. Grade six, in France, stands for the first class in secondary teaching (10-11 years old students).
2. The Education Action Zones (a.k.a ZEP) designate, in France, clusters of schools that take in a big proportion of students from underprivileged social backgrounds.
3. The first three categories are stem from a categorization proposed by Lima (2006) regarding exercises about orthogonal symmetry in grade six textbooks; we have added the last category.
4. We believe to be the first to attribute these qualifiers to orthogonal symmetry, but the distinction between dynamic and static conceptions for a given concept is not new (see for example, for the circle, Artigue and Robinet (1982) or for the convergence of sequences, Robert (1982)).
5. They are defined by the Official Bulletin Special Issue N°5 of September 9th 2004. Reforms in 2007and 2008 have changed them since.
6. The definition is stated as follows in the lesson:

   > Point A' is the symmetrical of point A with respect to a line (d) means that: if A belongs to (d), A and A' are overlapping; if A doesn't belong to (d), the line (d) is perpendicular to the line (AA') and (d) passes through the midpoint of segment [AA'].

7. Since the observations took place over two consecutive years, the students are not the same, but their characteristics (mainly in terms of social origin and level measured by the results of national tests at the beginning of grade six) are similar.

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8. STUDENT ACTIVTIES WITH E-EXERCISE BASES

INTRODUCTION

In this chapter, we examine the use of specific Internet resources: electronic exercises bases. We call an electronic exercises bases (EEB) for mathematics an Internet resource developed for mathematics teaching and learning purposes, consisting of mathematics exercises following a certain classification, and such that each exercises is associated to an environment, which includes different types of suggestions, aids, tools (graphs, calculators, etc.), lesson reminders, as well as explanations, answer analyses or complete solutions. In most of these products, the exercises have parameters which are randomly generated. This allows students to work on the same exercise several times. In such a case, the structure of the newly proposed exercise remains the same but the variables (e.g. numerical values, functions) differ.

Such resources, whether free or not, exist for all school levels and are more and more common. They can differ largely depending on their didactical structure, the type of accepted answer, or their type of implemented interactivity. Nevertheless, although the study of the use of technologies in mathematics learning is a fertile and expanding field of study, only few studies are specific to the use of EEB. Most of the articles dedicated to mathematics and digital technologies deal with “open environments” (microworlds or Computer Algebra Systems). These studies usually aim at conceiving and testing several didactical engineering, in which the “antagonistic milieu,” within the meaning of Brousseau (1997), includes the technological tool and is resistant to students’ actions, producing retroactions which help them to construct new knowledge. On the contrary, EEB constitute “allied milieu” designed to help learners. Moreover, using an EEB does not present any major technical difficulty. Many EEB have originally been designed for private use by students. Therefore, the question of handling technological tools is less complex in the cases of EEB and arouses less questions of instrumental genesis developed in the case of open environments (Artigue, 2002). The results of research are thus not transferable from one technology to the other. The question for the researcher studying the use of EEB in the classroom is to qualitatively analyze the use of resources in ordinary classrooms and to derive information about the activity of the students and teachers using these tools. This concern corresponds to the general question of this book, in which we try to analyze the teaching and learning of mathematics as they are, and not as they could or should be.

F. Vandebrouck (ed.), Mathematics Classrooms: Students’ Activities and Teachers’ Practices, 167–183.

Positive consequences of the use of EEB have already been observed in certain research studies. Ruthven and Henessy (2002) have for example carried out an extensive study about the use of technologies in mathematics teaching in England. They observe that drill and practice products, which are particular EEB, allow a work adapted to the rhythm of each student, as well as an increase of the motivation of these students. However, it seems important to pursue more precise investigations about student activity using these tools, in order to determine the contributions, limits and constraints of using EEB in mathematics classrooms.

We directly import the model of double regulation of activity introduced in chapter 1 to analyze students' activity on EEB. In fact, the scenarios of use of these resources expect students to repeat several times the same exercise, with variant exercise statements, or to solve a series of similar exercises. The actual activity of the students thus produces results, mainly feedbacks from the software, which modify the initial situation on the EEB: we can talk about productive[1] activity of the students and functional regulations of their activity. The student actions and the software retroactions are particularly observable with EEB because students repeat several times the exercises. The evolution of the productive activity results for a given exercise, in the course of regulation loops, can hence be observed and interpreted or not in terms of constructive students' activity (during the average time of action). The question of the long-term learning, and the study of the effects of learning through the EEB in a paper-pencil environment, is more complex and therefore our results are necessarily limited.

Based on our observations, we conduct *a priori* analyses of the situations proposed to students, and in regards to chosen episodes, we analyze, in particular, the tasks prescribed to the observed students. For the mathematical analyses of tasks, we retain the tools developed in chapter 2. In particular, we wonder whether the tasks are direct applications of explicit mathematical knowledge or, on the contrary, if there are adaptations and/or recognitions of knowledge to be made. We also take into consideration the software environment of the tasks, that is all the external hints or instrumental factors that could be of help, or not, in completing the tasks. We finally specify personal data about the observed students even if we often only have few elements on that matter. The results of the productive activity of the subjects are observed through the answers entered by the students into the computers. In particular, the software retroactions, as well as the aids given by teacher, if any, provide us with data regarding the modifications of the situations in the regulation loops (chapter 1).

In the second section, we give a first example of situation analysis that consists of analyses of tasks and software environments for EEB exercises suggested to students. In the third and fourth section, we study, using examples, how the situations influence the students' activity. In particular, we show that the expected activity is not always the activity developed by the students. We also show how difficult it is for the students to regulate their activity while facing the software without teacher intervention. Finally, in the fifth section, we conclude about the favorable conditions for a reasoned use of the EEB with the students.

EXAMPLES OF TASKS AND SOFTWARE ENVIRONMENTS ANALYSES

From the point of view of the mathematical knowledge at stake, the methodology to analyze these tasks is the one presented in chapter 2. We distinguish in particular the task of direct application of knowledge from all the other tasks which are described as complex.[2] The analysis of the tasks depends from the scenario in which these tasks intervene. For example, the fact that the implemented tasks are old or new for the student, with a level of knowledge which is "available" or "indicated" (see chapter 2), is an information which must be taken into account. Some elements of the scenario are implemented in the resources whereas other items are left at the discretion of the teacher.

From the point of view of the interface with the software, the characteristic elements of the tasks are the type of expected answer (multiple choice, numerical value, geometrical drawing, and so on), the aids proposed by the software (in particular the occasional corrections), and more generally the software environment of the exercise which can facilitate or complicate the solving of the tasks. This makes the analysis of the tasks more complex for the researcher than in the traditional environment. Below is an illustration of this complexity through an exercise from EEB Euler.[3]

It's an exercise which involves old knowledge from grade 9 at the indicated level; no recognition of knowledge at stake is necessary. Knowledge is explicit with the statement of the exercise. The exercise given is "*Given an orthonormal coordinate system, move the points A and B such that the line (AB) represents the function defined for all x by f(x)= 7/2 – x/8*." Hence, the task consists in moving two points on the gridline so that the line passing through those two points becomes the curve representing a linear function randomly given. So the work is on the transition from the algebraic registry to the graphical registry (Duval, 1995).

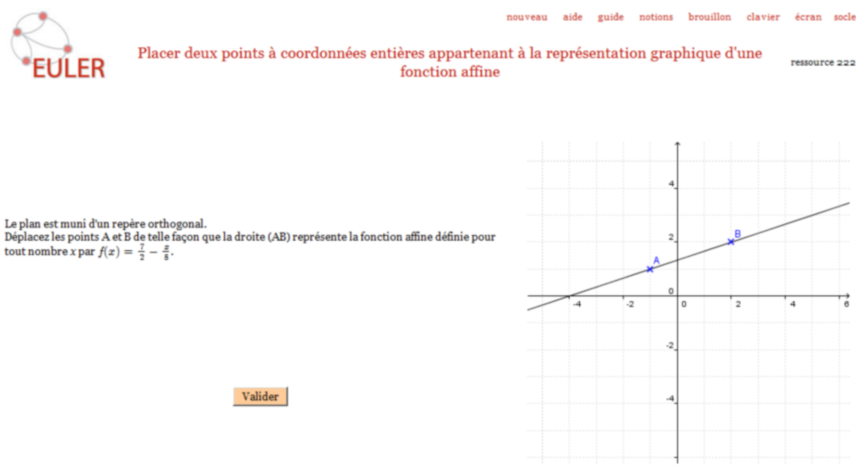

Figure 1. Exercise from the EEB Euler.

The task is however not immediate since there are two sub-tasks for a grade 9 student: the first is finding the coordinates of the two points of line *(AB)* using its equation stemming from the algebraic expression of the given function *f.* The second sub-task is to move the points *A* and *B* on the screen until they reach the correct position.

The software environment brings difficulties because the presence of the line *(AB)* on the graph, from the beginning of the exercise, disrupts the student's perception of the expected task. Indeed, the task is to move points *A* and *B* which are already given, whereas in a traditional environment, the task consists in placing these points. Moreover, we can only move these points onto positions with integer coordinates. This constitutes a major difficulty related to the task environment and this complicates it since it doesn't allow the student to place all the points found by calculation. Students must test their calculation in order to find points *A* and *B* with integer coordinates. Finally, the points that can be placed must have abscissas and ordinates of values between -5 and 5, which is another difficulty for the students while they look for the points. In a paper-pencil environment, we can extend the graphical representation to place the points with abscissas or ordinates outside [-5, 5]. Here, this is not possible. The interest of the software is yet considerable. On one hand, it offers the possibility of repeating the exercise with random variables. The students can thus repeat several times the exercise with new lines and practice until they succeed. On the other hand, the retroaction in the case of a mistake indicates that the student's answer is wrong and gives the function represented by the line *(AB)* suggested by the student, as is shown in the example in Figure 2.

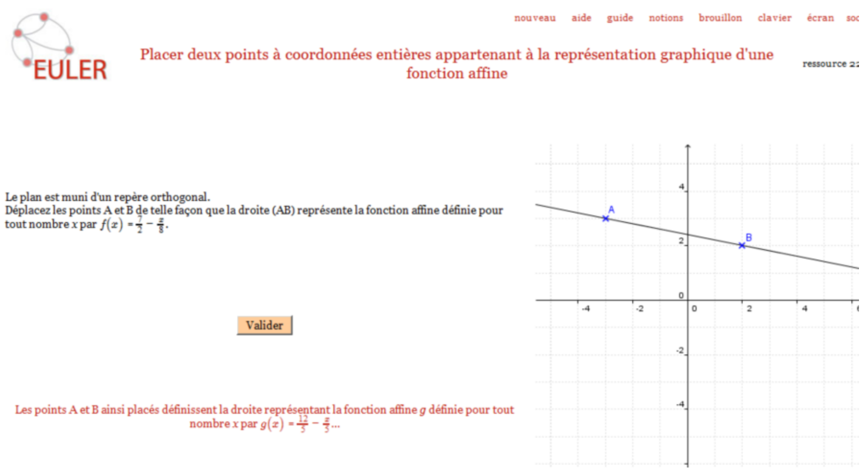

Figure 2. Graphical feedback for the exercise.

The software message is "*The points A and B that you placed define the line representing the function g defined by g(x)=12/5-x/5*." This feedback allows students to reflect on their propositions and understand their mistakes. This could allow rectifying their answers but they are unfortunately not entitled to a second trial.

Analyses of teaching practices with EEB developed by Cazes, Gueudet, Hersant, and Vandebrouck (2006) show that such quite immediate exercises seem to be necessary for the learning of the students. However, due to their over simplicity when they are solved in a paper-pencil environment, such exercises are rarely proposed in class, particularly in classical solving sessions at university level. This simple example gives an idea of the work possible through EEB, but only a thorough exploration of each of the websites would allow us to discover the numerous possibilities offered by these resources. In particular, we have observed that tasks are made possible thanks to the work on the EEB, with new associated activities; whereas new activities about tasks similar to the paper-pencil ones can be created. The sections below are dedicated to the study of examples of the actual activity of students on EEB, in a grade 10 classroom on one hand, and in higher education on the other hand.

While many tasks, which are well represented in EEB, seem adapted to be easily solved using a computer, definitely not all tasks can be completed using a computer. The teacher can, for example, choose to leave certain immediate applications for computerized work, so that the paper-pencil environment activity is centered on more complex tasks. We will come back to the work of the teacher in chapter 9.

EXAMPLES OF ACTIVITIES IN GRADE 10

The examples presented in this paragraph stem from observations conducted during the 2004-2005 and 2005-2006 school years in general and professional high schools. In each case, an observer is placed behind a student and notes all his/her visible actions. The methodology is the one detailed in chapter 2, with specificities related to the computerized work of the students. Only few episodes which are significant for our chapter are analyzed below, in order to directly access interesting results. These are observations of sessions organized in half-group, supervised by the teacher who is responsible of the class, who provides the students with individualized help. Each student–works on a computer to solve a series of exercises selected by the teacher. The two students observed during one session are called Alice and Fanny. These two students are good tenth graders, and they work during one session on the functions theme on the EEB MathEnPoche.[4] The lesson has already been covered during the year but some new knowledge about functions is still ongoing learning. For each proposed situation, and for each student, we examine the expected activity, then their actual activity.

First situation proposed to Alice and Fanny, expected activity

The first exercise that they come across is a multiple choice questions type. It is a series of 5 questions which are immediate applications of knowledge about images and pre-images. The environment facilitates the activity since there are only two possible choices, like in the following question 1:

> Question 1: Complete
>
> We know that
> 2 has for image 1 by function f
> Therefore
>
> The point of coordinate (☐ ☐) is on the graph of f.

Figure 3. First multiple choice questions with two possible choices (with English translation).

For example, Alice answers correctly 3 out of 5 questions. For the other two questions, she mixes up "image" and "pre-image," receives a simple error message and rectifies her answer during the second trial: "*it's enough to invert the answers!*" The same exercise is followed by 5 other analogous questions where now there are more choices, as shown in the following question 6:

> Question 6: Complete
>
> We know that
> -5 has for image 2 by function f
> Therefore
>
> f(☐) = ☐
>
> ☐ has for pre-image ☐ by function f
>
> The point of coordinate (☐ ☐) is on the graph of f.

Figure 4. Other multiple choice questions with numerous possible choices (with English translation).

The questions are the same as before, but now there are six blanks to complete. The expected activity is not the same as in the previous question, elsewhere more than two writing registries are mixed, which constitutes an additional adaptation.

The strategy consisting of answering sort of randomly then eventually rectifying during the second trial does not work anymore, since the software does not indicate the error locations.

Alice's actual activity

Alice understands that she cannot simply rectify her answer during the second trial if needed. She thus looks at her lesson book before each answering and reads in a low voice the explanation about "pre-image" and "image." Hence, she answers question 6 correctly. Again, she still gets mixes the two terms in questions 7 and 8, looks at her notebook again, and then corrects her answer. She makes the same mistake in question 9 and so she decides to call the teacher. He explains immediately. She solves the last question without any mistake.

Second situation proposed to Alice and Fanny, expected activity

During the rest of the session, Alice and Fanny work on finding graphical images and pre-images (exercise 8) and graphical solutions of equations of the type $f(x)= a$ (exercise 9). This is new knowledge for grade 10 but it has been studied previously in traditional sessions (knowledge in the process of acquisition).

In exercise 8, given a function defined through its algebraic expression and a representative curve, students must determine the image of a number and the pre-image(s) of another number if any.

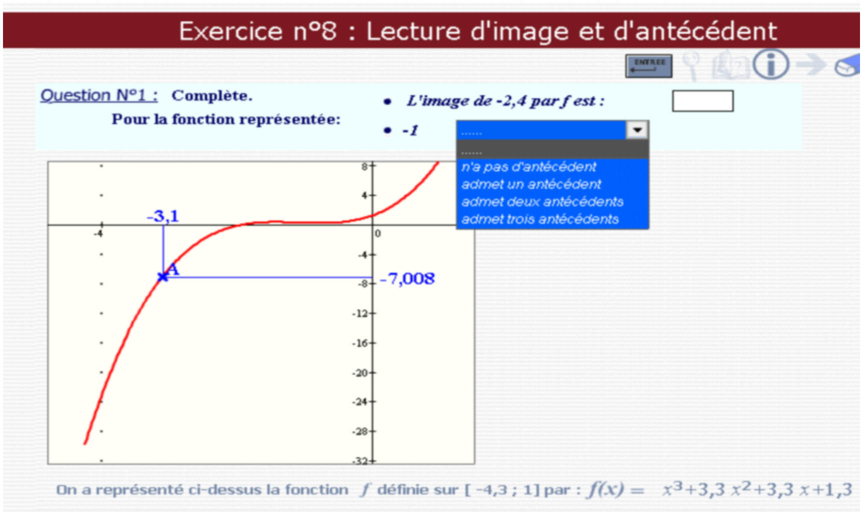

Figure 5. Exercice 8: Reading the image and the pre-image(s).

A series of 5 consecutive questions of the same type is given. The considered functions are random polynomial functions of first, second and third degree, with decimal coefficients having at most one digit after the decimal point, defined on an interval. The value of the image should be typed in a box. For the pre-image(s), a rolling menu (see figure 5) allows the selection of the answer. In this example above, the algebraic expression of f is $f(x) = x^3+3.3x^2+3.3x+1.3$. The two questions are "*The image of -2.4 by f is: …*" and "*-1: doesn't have any pre-image; has one pre-image; has two pre-images; has three pre-images*." Depending on the choice made, one or several boxes are displayed to enter the value(s) of the pre-image(s). A point *A* which moves on the curve allows students to read the requested values. We note that when several pre-images exist, the tool does not allow visualizing them simultaneously. Students should not stop at the first found value. The expected activity is to move the cursor in order to graphically read the images and pre-images. Hence, the work only covers the graphical registry and immediately applies the knowledge about this registry. The choice of work registry is thus imposed and this registry, for this kind of questions, does not belong to the usual didactical contract of the students. Moreover, the environment which imposes the cursor manipulation complicates the proposed task as we will see.

Alice's actual activity

The first curve proposed to Alice in exercise 8 is that of the function $f(x) = x^2-4$. Alice must determine the image of 1.2 and the pre-image(s) of -7, if any. In compliance with the usual didactical contract, Alice calculates algebraically the image of 1.2. This approach is reinforced by the fact that the second degree polynomial is a polynomial that she can easily handle algebraically. In order to find the pre-image, Alice notices right away that there isn't any on the graph, since the value -7 is relatively far from the minimum of f. Alice validates her answer, and receives a congratulations message. She does not find the exercise very interesting and moves to exercise 9.

Fanny's actual activity

The first curve proposed to Fanny in exercise 8 is that of the function $f(x) = x^2-5{,}19$. She is asked to find the image of 0.2 and the pre-image(s) of -4.7, if any. Fanny also calculates algebraically the image of 0.2 but finds it hard to complete her calculation to find the pre-image of -4.7. Some adaptations emerge and are linked to the presence of a decimal number. The teacher walks nearby Fanny who calls her: "*it's not clear to find the pre-image!*" The teacher shows Fanny how to move point *A* to obtain a display of the coordinates of the points on the curve. Fanny immediately applies this instrumented method, answers correctly, and completes successfully the 5 exercises of the series, proceeding in the same manner, graphically looking for the pre-images as well as the images and using the cursor.

Third situation proposed to Alice and Fanny, expected activity

In exercise 9, the aim is to solve graphically a series of five equations of the form $f(x) = \alpha$, α being a decimal number with at most one digit after the decimal point, and *f* defined by its graphical representation on an interval. Depending on the cases, there are 0, 1, 2 or 3 solutions.

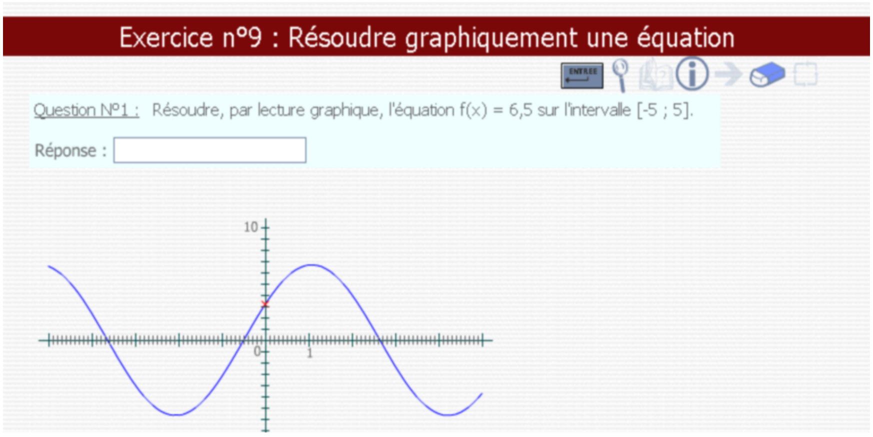

Figure 6. Exercice 9: Solve equation graphically.

In the example in Figure 6, the equation to solve graphically is $f(x) = 6.5$ in [-5,5]. Hence, students' activity is always to immediately apply knowledge still in acquisition. Note that the task is again made complicated by the computer environment. In fact, for each question, students must move the cursor on the curve, the cursor being originally placed at the origin. Moreover, the cursor coordinates are not displayed, and students have to read those coordinates over the axis.

Alice's real activity

In exercise 9, Alice does not figure out that she must use the movable cursor since she did not have the chance to do so during exercise 8. Moreover, in this exercise, it is not possible to proceed algebraically. She hence tries to estimate the answer and gives successively two coherent answers but not precise enough. So she comes across the aid window which does not help her. Indeed, the aid explains how to find the solution graphically whereas Alice's problem is that of handling the cursor. So she calls the teacher who shows her how to use the cursor. Alice then engages correctly in the expected activity. But she still can't validate exercise 9 because of three mistakes in three consecutive statements. These three mistakes have different causes:

– Alice forgets a solution;
– Alice doesn't see one solution which is at the border of the frame of the graph;
– Alice reads the coordinates incorrectly.

In the latter, Alice thinks that the precision of her answer is insufficient. She doesn't see that in fact she made a mistake while reading the given. The observer explains to her this mistake. In the two other cases, the mistake is more serious, but Alice does not have the possibility to rectify it because of a poor manipulation of the software. Alice's activity is therefore not regulated since she does not understand her mistakes or she cannot correct them due to her manipulation errors. However, Alice seems to be able to globally complete the requested task.

Fanny's real activity

In exercise 9, Fanny spots the red cursor and manages to move it, like in exercise 8. Then, she starts the expected activity. In general, she finds the correct answers using the cursor. However, for question 3, she comes across a sinusoidal curve and the equation "$f(x) = 4.5$" which has three solutions. Yet, Fanny forgets one of them and provides successively two wrong answers: -1.5 and 4.8 then -0.5 and 4.8. She is directed to the aid window, which gives her, like Alice, the method to graphically find an image or a pre-image. Since she knows how to solve the exercise, she says: "*I don't want these aids*!" The teacher passing by, she asks her: "*how can I delete this*?" The teacher points out the red button to close the aid window. Fanny clicks on it and the correction is displayed (something that Alice was not able to find). She looks at the displayed values in the correction having two decimals and this attracts her attention. She then attributes her mistakes not to the fact that one solution was missing, but to the fact that she did not find all the decimals since she did not use the magnifying glass.

Analysis of Alice and Fanny's actual activity

In the first questions proposed to Alice (the questions with two choices, Figure 3), the students can reach the expected result as of the second trial. All seems to be happening as if information allows them to develop a "two shots strategy" based on the feedback. This strategy is fostered by multiple choice questions with two options. It is only when this "two shots strategy" stops working and when the task gets complicated that Alice tries more in-depth work to rectify her mistakes. These examples illustrate the idea that students take into consideration the feedback only when they feel the need to do so. We can nevertheless hypothesize that Alice's confusion of "image" and "pre-image" is well corrected since she correctly tackles exercises 8 and 9 afterwards. The learning does not however happen on a short cycle of regulations: the correction is long and progressive. In particular, Alice needs to take several initiatives, the situation must become problematic for her (Alice only tried to understand when more than two registries were mixed, that is when there were 6 blanks to fill), and the teacher must intervene fast and advisedly.

In exercise 8, there is, since the beginning and for both students, a gap between the expected activity and the activity developed by the observed students. The task is the same as in the paper-pencil environment, but the expected activity is not the same. The students do not develop the expected activity since they do not understand that the EEB expects them to manipulate the cursor. Moreover, the algebraic expressions are provided by the software and this does not favor graphical work. Finally, the random expressions which Alice and Fanny come across are of second degree, which does not discourage them from looking for answers algebraically. This would probably not be the case if the expressions had been systematically more complex or of third degree.

Lastly, the examples illustrate the software feedback, and if though they are taken into account by the students, they do not allow them to easily regulate their activity correctly. In the multiple choice exercise, the feedback is not enough to let Alice regulate her activity on her own. She needs her lesson notebook, then a conversation with the teacher, in order to successfully complete the task. The work which consists in interpreting information provided by the software is generally hard to do. In exercise 8, Alice suggests a correct answer and receive a retroaction validating her answer but without any explanation of the expected procedure. Thus, she has no indication allowing her to detect the gap that exists between the work she completed and what was expected from her. We can wonder about what made Alice gives up that exercise. She probably, subconsciously, has the impression to be missing something, but not enough to ask for the teacher's help. The activity in exercises 8 is hence not at all regulated. As a result, Alice cannot tackle correctly the activity in exercise 9, since she does not know that she can manipulate the cursor. Here as well, the aids proposed, reminding her that the mathematical method to solve equations, does not allow her to regulate her activity since her problem is now the manipulation of the cursor. Once more the teacher comes to the rescue and shows her how to use the cursor.

On the other hand, Fanny, possibly thanks to her critical look at the software work, calls the teacher as early as exercise 8 which quickly regulates her activity. So Fanny adapts to the situation in exercise 9 on her own. However, like Alice, she cannot regulate correctly alone her activity in exercise 9 and gets angry at the aids ("*I don't want these aids*!"). Reading the correction is not enough for her to understand that one solution is missing. This seems to be an ambiguity of the software: the answers are accepted with a 0.1 error margin, but in the correction, they are given with a 0.01 precision. In this situation also, because of her critical look at the software, Fanny prefers to say that the software is hard to use rather than question her own work.

The examples of Fanny and Alice illustrate the importance of taking into consideration the instrumental aspect in the proposed situations. The more the resource environment gets sophisticated, the more the students have to articulate the software manipulation with the learning of mathematics. The examples always show that the role of the teacher is fundamental.

EXAMPLES OF UNIVERSITY ACTIVITIES (L1)

The example developed in this paragraph stems for observations conducted over the course of the school year 2004-2005 in a university. The student is called Charles, and we study his activity during a computerized solving session during which the students work on the EEB Wims[5] and the teacher walks around providing individual help. Charles works alone on his PC and the session is focused on practicing previous knowledge or ones in the process of acquisition. The observation methodology is different from the one used in the previous paragraph as students working with EEB Wims are logged, which allow to recover their activity traces through log-files. No one directly observes Charles' activity, but we can note the times of work of the student on a Wims exercise called *Joint*. Then, we examine a posteriori Charles' work on an exercise similar to the exercise *Joint*, proposed in a paper-pencil exam.

Situation proposed to Charles, expected activity

The exercise in Figure 7 deals with knowledge about continuity and differentiability of functions of a real variable. An adaptation of knowledge is required from the students since they have to recognize that the given functions are of class C^1 on the considered intervals. Then, it is enough to compute the limits and the derivatives of the two given restrictions and to equal the results, which is a direct application of the algebraic knowledge about limits and derivations. They must write that the two limits and the two derivatives should be equal, which allows calculating a_1 and a_2 through an immediate calculation also. When the student gives an incorrect answer, Wims provides a retroaction of the type as presented in Figure 8.

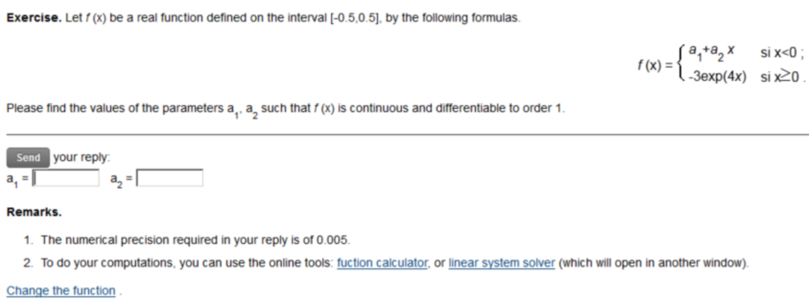

Figure 7. Wims Joint exercise.

Software retroaction for the Joint exercise

It is not possible to solve this exercise by trial and error since only one answer is accepted. After the first student answer, Wims provides a retroaction. This retroaction is in the graphical register whereas the exercise statement is in the analytical register. It does not necessarily help the student in finding the correct answer but it does suggest another point of view. Nevertheless, the student cannot go back to propose another solution. Wims proposes directly another analytic function, built from a panel of reference functions and linear combinations.

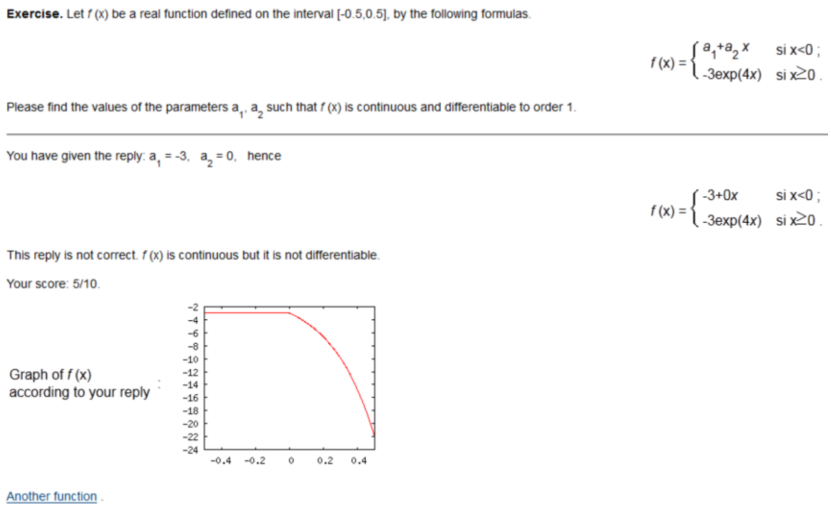

Figure 8. Example of feedback for the Joint exercise.

Charles' actual activity

The log-file shows that Charles worked for 34 min on this exercise. He also worked on four consecutives statements of this exercise. In his first trial, he worked for 9 min and got the score of 5/10; this means that he only found the missing value a_1 (like in the previous error example, Figure 8). After reading the correct answer, that is the missing value a_2, he did not try to understand where this value came from. Indeed, the log-file shows that he quickly restarted the exercise with a new statement. This attitude is not abnormal; students almost always restart the next exercise immediately. He worked for 13min and obtained the score 10/10. Then he restarted this exercise two additional times, worked each time for 5min, and obtained in both cases the score 10/10. So we can see that Charles did solve the exercise correctly, that is he regulated correctly his activity a prioiri. However, we can wonder why his calculations last 5 long minutes each time despite the fact that they should be immediate algebraic calculations? What does he do for the same exercise in the exam? Below is his exam paper. The statement is "$f(x)$ *is a real function defined on* [-0.5,0.5] *by the following formulas* : $f(x) = -5\exp(-5x)$ *if* $x < 0$ *and* $f(x) = a_1+a_2x$ *if* $x \geq 0$. *Find the values of the two parameters* a_1 *and* a_2 *such as* $f(x)$ *is continuous and derivable of order* 1."

We notice that Charles develops well the expected activity to find the missing value a_1. On the other hand, to calculate the derivative to the left then to the right, he uses the limit of the rate of change, whereas it would be enough to apply the classical derivation formulas in the definition intervals. In other words, Charles uses a correct procedure but it isn't the fastest and most suitable one. This explains the considerably long time spent on this exercise for each trial during the Wims session and which certainly penalizes Charles during the exam (since he is left with less time to solve the rest of the exam than what the examiner had scheduled).

Exercice. Soit f (x) une fonction réelle définie sur l'intervalle [-0.5,0.5], par les formules suivantes

$$f(x) = -5\exp(-5x) \text{ si } x < 0$$

$$f(x) = a_1 + a_2 x \text{ si } x \geq 0$$

Veuillez trouver les valeurs des paramètres a_1, a_2 telles que f (x) soit continue et dérivable d'ordre 1.

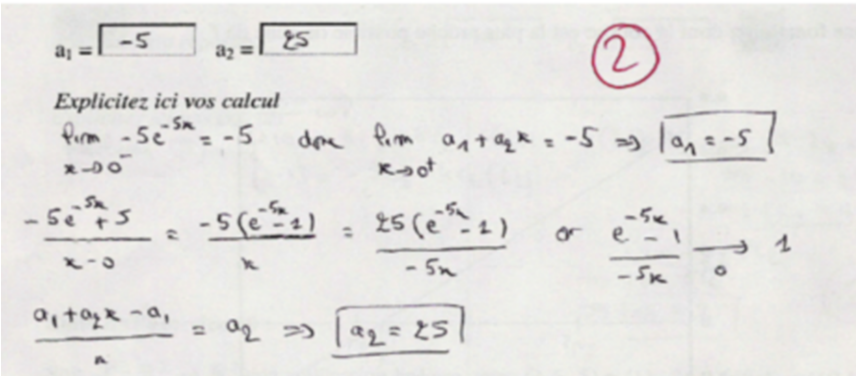

Figure 9. Charles exam, exercise Joint.

Analysis of Charles' activity

In this case, Charles is able to solve the exercise. A regulation of the activity has been made after the first trial where he was able to solve only partially the exercise. It is the software retroaction in the form of a note which allows Charles to regulate his activity. The log-file indicates that this regulation happens in a very short time. During the second trial, he provides a complete answer for the exercise. His only problem is that his solving strategy is not optimal. Unfortunately, at this time, neither the software retroactions, which connect two points of view, nor the presence of the teacher during the Wims session, allow Charles to improve his strategy. In particular, the software regulation "*correct answer*" as of the second trial, does not allow Charles to realize that his procedure is not optimal. He cannot regulate best, autonomously, his activity during the Wims session. This leads to two new activity loops, where Charles applies the same solving strategy without having any doubt. He always succeeds, but does not visibly increase his speed between the last two trials and by using a relatively long solving method (5 minutes).

CONCLUSIONS

We have described in this chapter results of the students' activity using EEB, which complete those developed in Cazes, Gueudet, Hersant, and Vandebrouck (2006). In the observed situations of EEB use, whether in high school or university, the EEB allow a strong individualization of the students' activity with respect to their work in sessions of traditional exercises (mainly in university solving sessions), even though the student always follow a work plan proposed at the beginning of the session par the teacher. The model of double regulation of the activity (chapter 1, Leplat 1997) allows us to analyze precisely the activity of the students with the EEB and to highlight in particular the regularities and differences between the students with respect to the organization of this autonomous activity.

Introducing task analyses reveals that the latter are often very close to the tasks that can be proposed by the teachers in traditional sessions. Nevertheless, the observations show that the students work much longer on a same exercise during the EEB sessions than during the traditional sessions. For example, in Charles' case, the log-file tracking the EEB activity shows that even on a task with immediate application of knowledge, Charles works for several minutes, restarting the exercise as many times as needed. In a paper-pencil environment or during a classical solving session, he would have only solved one example. This same log-file shows that in other situations, the students are not easily discouraged, in general, by exercises which require adaptations of knowledge. The situation is thus different from the traditional paper and pencil situation since the students have more responsibilities in their activity and can follow at their own rhythm the work plan proposed by the teacher. If they do nothing, then nothing happens, and so they are somehow obliged to work. In particular, there are rare moments of collective corrections where the students can just wait for the answers. However, managing the progress of the path, repeating or changing the exercise, activating an aid or a correction, choosing to take notes, all definitely contribute to the empowerment of the students but also seem to be a source of difficulty, especially for weak high school students. Furthermore, in certain cases, some of them prefer to continue succeeding in competing simple exercises rather than facing more difficult exercises, an observation we had previously noted in Cazes, Gueudet, Hersant, and Vandebrouck (2006).

Here, the results illustrate the valuation of the occasional productive activity using these tools, with often gaps between the expected activity and the students' observed activity. It could be that the task is not a direct application task, that the knowledge to be used is not explicit or sometimes the software environment complicates the task compared to the traditional environment. From the start of the exercise flow, a modification of the initial situation must be applied in order for the actual student activity to be in compliance with the expected activity. This modification is made thanks to the teacher, if s/he is present at the right time. This pertains to the problem of working in total autonomy with these tools, whenever we want to tackle slightly more complex tasks. Charles' case is a good example of the teacher not intervening at all, the completed activity is then deferred with respect to the expected one. This gap can also be due to the denaturation of the task by the software environment, which can permit for example obtaining the correct answer without developing the expected mathematical activity (mainly in the case of the multiple choice with two choices, but we did identify more complex examples), or it can favor a more economical activity, of the type trial and error in particular. As for the results of the activity, we also found that the software retroactions to the students' actions are often not enough, too difficult to understand by the students to allow them to regulate alone and correctly their activity, and even not adapted to the actual activity. In fact, the retroactions can only be generated for the result of the activity and not the activity itself. It is therefore very hard to implement a priori retroactions which are relevant and adapted to the diversity of the students.

These difficulties, as soon as the tasks are not easy for the students, can generate ineffective activity loops (see chapter 1). The students can for example be satisfied by inadequate procedures by repeating several times an exercise, since these lead to a correct result, or even "very often" to the correct result. The allied environment can also reinforce certain "action logic" at the expense of a "learning logic": in the "action logic," the aim of the students is exclusively to obtain the answer expected by the software (valuing the productive activity at the expense of the constructive activity). It is the case of Alice when she responds to her two-options multiple choice questions. In other examples, the students can intentionally identify regularities in the correct answers displayed by the EEB, after several unfruitful attempts. These regularities can allow them to gradually infer the correct answer without fail (misappropriation of the EEB), without being able to know if it is based or not on a learning. In certain extreme cases, the gaps between the activities are not intentional and result in undesired learning.

The EEB seem to be at first well adapted for a students' work on technical exercises, that is to say exercises of immediate application of knowledge. This wasn't the case in exercises 8 and 9 for Alice and Fanny, nor for the Joint exercise proposed to Charles. The task analyses thus appear to be important in determining the exercises that are proposed to students with an EEB. In this sense, the EEB re-emphasize the importance of the technical exercises, which are important for the learning and which are often neglected in work sessions. The observations or the activity log-files provided by certain EEB show that this work is important, that students need time to successfully solve these technical exercises, and that the work is less repellent thanks to the technological potentialities of the EEB.

As soon as the tasks move away from the technical level (whether the application is not direct, or whether the knowledge are assumed to be available for students), it is more difficult to get from the students autonomous activity loops (the activity and its regulations) which are mathematically acceptable, in other words that the produced or returned mathematics be correct and consistent. This does not however mean that the EEB cannot be used in other ways than for simple or direct tasks. This means that there is a margin phenomena which, if placed too high in terms of task complexity, excludes weak students. The notion of ZPD (Vygotski, 1978, chapter 1) is particularly useful here in the sense that the students essentially work autonomously. In particular, the EEB can emphasize the differentiation between students if the teacher is not specifically vigilant to the student difficulties. Learning is observed, on an average term, for students confronted to tasks which include adaptations that are accessible and for which they find immediate resources. It is the case of Alice who, after the multiple choice, and using her notes and with the help of the teacher, seems to have well understood the difference between image and pre-image. She was obviously able to reinvest this knowledge during exercises 8 and 9. However, we saw how she went from an "action logic" to a "learning logic" as soon as the exercise was not a questionnaire with two blanks anymore, but had six blanks, that is as soon as her "two shots strategy" stopped working and the task mixed more than two writing registries. This example thus shows that the distinction that we introduce between "action logic" and "learning logic" depends on the student of course, and on the situation s/he faces on the EEB as well. We go back to the initial idea of a situation that must include knowledge adaptations in order to hope for an intentional constructive activity and in particular for a mathematical learning. Quite often, it is the teacher, present and vigilant, who can react to the actual activity of the students and emphasize this learning. The teachers must hence develop a specific work for help and for the integration of the EEB use in the usual classroom practice. We will examine the activity of the teachers who use EEB in the next chapter.

NOTES

[1] The terminology of productive activity comes from the field of professional didactics but could be understood in a more naive way, in the sense that the student activity produces results numerical answers, implementation...

[2] For which we can specify the types of adaptations of knowledge at stake (A1 à A7). See chapter 2.

[3] http://euler.ac-versailles.fr/baseeuler/recherche_fiche.jsp

[4] http://mathenpoche.sesamath.net/

[5] http://wims.unice.fr/wims/

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9. TEACHERS' PRACTICES USING E-EXERCISE BASES IN THEIR CLASROOMS

INTRODUCTION

This chapter examines teachers' uses of specific computational tools: electronic exercise bases (EEBs, see chapter 8). The data come from a project tracking the use of online resources in grade 9 mathematics classes (ages 15-16) in the Paris region. Here the use of EEBs constituted a case of special innovation and should for this reason be distinguished from ordinary classroom situations, where the emphasis is placed on coherence and stability of practices. The stability factor in the use of EEBs is not therefore discussed. Instead we raise questions concerning the way in which these new tools are appropriated by teachers, the different uses in terms of student activity, the changes they bring about in the day-to-day activities of teachers, and, more generally, their impact on the evolution of teachers' overall classroom practices.

The theoretical framework adopted in addressing these questions is the double approach. Special importance is given to concepts drawn from Activity Theory. The activity of the acting subject depends in part on the subject him/herself and his/her conceptions and representations; but it also depends on the situation[1] in which the subject is placed. In terms of the data referred to here, the subject is the teacher, and the situation refers to the particular computational tool used (EEB), though it also takes into account an array of other factors, including the teaching establishment, the social context, and the specific teachers and students involved. Via his/her activities, the subject "transforms the real and transforms himself" (Pastré and Rabardel, 2005). The results of the activity can therefore be divided into two categories: transformations of the real – that is, of the subject's environment – and transformations of the subject him/herself. In order to distinguish these two categories of result, Samurçay and Rabardel (2004) introduce the notions of "productive activity," which refers to changes in the subject's environment, and "constructive activity," which refers to changes in the subject him/herself. In terms of our data, productive activity refers to the effects of the teacher's activity on the students, whilst constructive activity refers to the transformations of the teacher's personal representations and, more particularly, the transformations of his/her teaching practice. Although distinct, productive activity and constructive activity are nonetheless closely connected. More particularly, constructive activity cannot take place without productive activity. Additionally,

F. Vandebrouck (ed.), Mathematics Classrooms: Students' Activities and Teachers' Practices, 185–198.

we should note that productive activity ends when the execution of a task is complete, whereas constructive activity can continue over a longer period. Thus the results of teachers' activities are analyzed both in terms of direct observation of their implementation of student activities (productive activity), and in terms of the transformations of the teachers themselves, in so far as such transformations can be determined (constructive activity).

By analysing both the teacher's productive and constructive activity, we are therefore better able to assess the cognitive and mediative components of his/her practice – even if we cannot describe them in full. We should recall that the results of the activity of the subject (the teacher) have a retroactive effect on the three other components of his/her practice by means of a 'double regulation' that is both 'external' (affecting the institutional and social components of practice) and 'internal' (affecting the personal component of practice). This double regulation functions according to different timeframes (see chapter 2):

– In the short term – that is, limited to an "episode" within a given lesson. From the point of view of the teacher, we define an "episode" as an instance of exchange/dialogue with one or more students or indeed the whole class. We do not discuss this short-term action in detail here. This is because, firstly, the effects of the teacher's activity on students are in this case too limited to be interpreted in terms of learning outcomes. Secondly, the regulations resulting from the teacher's activity take place in real time. The effects of constructive activity on the teacher him/herself seem to be too limited to be consciously recognised and reinvested by the teacher (Rogalski, 2003)
– In the medium term – that is, within the space of a lesson or a lesson sequence (a series of lessons on the same topic). Within this larger timeframe, regulation begins to have an effect on the teaching subject. For example, the teacher might, over the course of the lesson, become aware of persistent difficulties for the students that s/he had not expected; as a result s/he will modify the structure of the lesson in an impromptu manner, going back over certain material and establishing connections with previous topics. From the point of view of the students, the consistencies in the teacher's activity can also allow us to presuppose certain effects on their learning.
– In the long term – that is, in the course of planning, preparations etc. This is the most relevant timescale in terms of the effects on students and the effects of constructive activity on the teacher him/herself. Here we draw on the notion of "geneses of the technology uses" (Abboud-Blanchard & Vandebrouck, 2012) to designate the effects of the teacher's constructive activity. We then attempt to understand, in terms of the five components of the teacher's practices, how the geneses of the EEB uses can be integrated into the global practice of teachers.

In the next section, we introduce the data collected and the methodology employed. This is then followed by an overview of the results of our study of teachers' activity in terms of lesson preparation and class proceedings (third section). Finally, the last, fourth, section examines three detailed case studies that supplement the more generalized findings set out in the third section.

DATA COLLECTED AND METHODOLOGY

The data come from two different samples. The first broader sample consists of thirty or so teachers who were involved in the project from 2004-2006. These teachers were observed either directly or indirectly (i.e. via observations of their students' activity). They also filled out questionnaires about their use of EEBs; these questionnaires are especially useful as a supplement to the observations made during lessons. The second smaller sample consists of six teachers whose practices were studied more extensively.

The data collected for each teacher can be organised chronologically in accordance with the timeframes described above – that is, lesson preparation and proceedings (short and medium-term timeframes) and teachers' critical re-viewing of their lesson(s) (medium and long-term timeframes).

As far as concerns lesson preparation, teacher activity is analysed first of all in terms of his/her choice of EEB, lesson content and specific exercises. That is, the initial focus is on the classroom scenario (incorporation of the lesson in a progressive, global learning agenda) and the lesson's learning trajectories, as set out by the class teacher.[2] Such features provide information on the cognitive component of the teachers' practices. However, such details as, for example, the complexity of the tasks set, which are important when studying student activity, are not taken into account here. Indeed, such details are meaningless when examining the activity of several teachers teaching different topics and at different periods in the school year.

The lesson proceedings are analysed by means of the transcripts of the recordings made and the notes taken by the observers. They provide information relative to class management and teacher-student interactions. Particular attention is paid to the way in which different media are used (EEBs, but also hand-outs, the black/whiteboard, exercise books, project notebooks) and to the teacher's interventions. The latter permits a description of the ways in which teachers help students to complete the tasks they have been set using particular resources. This second phase of the study provides access essentially to the mediative component of the teachers' practices, though this may also corroborate and reinforce certain considerations relating to the cognitive components of practices.

Finally, the third observation phase, in which the teacher "learns from," so to speak, his/her classroom experience, is very difficult to observe because it takes place over a protracted period of time. Often the teacher will not give a similar lesson until the following year. Furthermore, the modifications to be made to future lessons are not always made explicit; when they are, it is usually during questionnaires or interviews conducted after the lesson itself. Changes may also become visible over the course of a sequence of several lessons on the same topic. The information collected here aids understanding of the possible long-term evolution of the cognitive, mediative and personal components of practice. Moreover, it helps us to consider the geneses of EEB uses in relation to the evolutions of all five components of teaching practices.

RESULTS

The results obtained provide information on those features of the cognitive and mediative components of teaching practices that are visible during observation; some features of the other three components were also visible, but to a lesser extent. We will present the various actions carried out by the teachers in chronological order, beginning with lesson preparation, moving onto the proceedings of the lesson itself and ending with subsequent modifications.

Lesson preparation

All of the teachers whose lessons were observed chose a single EEB to be used throughout the year and for all of his/her classes. Some teachers used different digital resources, such as spread sheets or interactive geometry software, but none of them used more than one EEB. The first of the teacher's actions is therefore to select "his"/"her" EEB. In some cases this choice was the same for all the teachers in a given establishment.

The teacher's second action is to choose the type of lesson s/he will give. Lessons using EEBs are nearly always made up of a small number of students working alone or two on a computer, with the teacher offering individual help. EEB-based lessons can even sometimes be supplementary sessions in which students participate on a voluntary basis. The frequency of EEB usage varied considerably from teacher to teacher. In the questionnaires, the majority stated that one EEB-based session per week would be desirable but that, for practical reasons, many of them had actually given fewer than this.

After the initial sessions in which the EEB was rapidly explored, all of the teachers produced what we will later refer to as "work plans" – that is, a structured sequence of exercises chosen from the EEB. Most often the work plan was the same for all the students. However, some inclination to adapt it to different students' needs was observed. For example, in a remedial session following a test, the students were given exercises addressing the mistakes they had made in the test. Or again, in the same establishment, several teachers grouped together and decided that each would offer a workshop on a different topic; their students were then redistributed, each being sent to the session that best corresponded to his/her needs. When work plans were used, many teachers inserted some personal remarks at the beginning. The work plans were quite precise and generally too long to be completed in a single session.

The exercises set by teachers always related to topics that had either already been studied or else were currently being studied. Their format and title were similar to those of traditional exercises. No exercises introducing new topics were used. Furthermore, the teachers' goals for the sessions were always limited: the aim was to work on a specific topic or to revise specific concepts. Exercises drawing on several concepts at the same time were only very rarely used, with the exception, now and again, of a few exercises at the end of the work plan, intended for the strongest students. Additionally, teachers in their first year of using the EEB preferred to use simple exercises featuring a single problem to solve and closed-ended questions; they then expanded their repertoire of exercises in subsequent years. Various mathematical topics were covered in the sessions, from geometry and factorisation to functions and statistics. The topics were always chosen in relation to the class's overall progress.

Lesson proceedings

All of the teachers observed structured their lessons in the same way – in three sections of very different length: first, a very short start-up phase; second, the bulk of the lesson, or what we call the "cruise" phase; and, finally, a very short, even barely existent closing phase. We structure our findings on lesson proceedings around these three phases.

The start-up phase

In the start-up phase the teacher mostly addresses the whole class, giving practical instructions regarding the distribution of students among the workstations and how to access the exercises.

For example: "*Sit down, ideally one of you at each desk (...) When you've opened the session, I want you to pay close attention to the instructions I've given you. I've written these at the beginning of the session (...) I want you to call me when you're ready to start. I want to see some good work (...). And don't forget to show your working out for worksheets one and two (...).*"

In this example the instructions also deal with the way in which the tasks are to be completed and the arrangements for addressing the teacher. This initial phase is always very short; it may even be missed out completely once the teacher and the students have got used to the equipment and the initial lesson set-up has become "routine." On the other hand, when the technology is used for the first time extensive practical instructions will be needed.

The "cruise" phase

In all the lessons we studied, this phase made up close to the entirety of the lesson, with the teacher no longer addressing the whole class and instead speaking almost exclusively with one or two students at a time. Likewise, there is no collective review of the answers on the blackboard. These two features demonstrate important departures from traditional exercise-based lessons. A further difference is that the students cannot wait for a collective answer review because they know there will not be one. Indeed, the observations made in these lessons show that the students were "active." They also interacted with the software by, for example, reading the “help” box that accompanied the exercise or providing an answer and viewing its evaluation.

Amongst the various kinds of assistance supplied by the teachers, we can distinguish “mathematical” help (which can be procedural or constructive; see chapter 2) and “instrumental” help. We also see, though rarely, certain teacher interventions relating to the students’ application to their work; where such interventions took place, there aim was to encourage, praise or push the student. Instrumental help concerns the handling of the exercise database; at times it is difficult to distinguish this from mathematical help. We noted that, in all the lessons observed, instrumental help was often initiated by the students. Most of the mathematical help provided by the teacher was procedural: s/he helps to resolve the difficulties encountered in the exercises and checks the students’ work. However, individual assistance can be constructive. For example, if a student is making a persistent mistake, the teacher will re-explain a particular concept they have studied and its relation to the exercise. There are thus various types of teacher interventions, each adapted to a specific case. This means that the teacher is often required to assess diverse situations rapidly so as to respond accordingly.

We will now provide a few examples of different types of teacher intervention. They are all taken from the “cruise” phase and are specific to the use of EEBs. The assistance offered is usually a mixture of the mathematical and the instrumental.

Example 1: “*Yes? What is it you don’t understand? Right, where’s the question, first of all? I didn’t see your answer just now. What does this value represent? Go back …*” The teacher has to understand which exercise the student is working on, which is all the more difficult because many EEBs set the exercises in a random order, which means that the question asked at any one time can differ from one session to another. In the case of a set of questions proposed in sequence, the teacher has to grasp the student’s whole chain of reasoning.

Example 2: “*They sometimes offer you some comments on your answer; you should always click ‘yes.’ It’s not enough just to read the ‘well done,’ you need to read the rest as well (…) You don’t have to read and copy out everything …*” Here the same teacher adapts his/her advice to the specific student and the specific exercise and encourages the student to interact directly with the EEB.

Example 3: “*You will all need at the very least your exercise book and a pen or pencil. Because you won’t be able to do everything in your heads. Some of the exercises you can do mentally, but at times you’re going to need to do some working out on paper.*” Here the teacher repeatedly encourages the students not to neglect pen and paper when working out their answers.

Example 4: “*So you did the calculations, you gave all the answers … did you get them right? OK, then move on to this one. I recommend you go to ‘information’ and read exactly what they want you to do.*” In this single intervention the teacher performs three actions: s/he finds out what the student had done before s/he came over, s/he advises the student to move on with the exercises and, aware of the difficulties of the following exercise, s/he asks the student to read the information.

These examples show that the EEB is an action tool for the teacher, who finds him/herself playing a dual role: s/he is both mediator between the student and the software, explaining how to use the EEB, and monitor, checking whether the questions have been answered correctly. These two roles are not in themselves new for the teacher, since they have to be adopted whenever a new instrument is introduced, be it "traditional" (e.g. protractors in the grade 5/6 – age 11-12) or technological (e.g. a new model of calculator). The difference is that the teacher must now simultaneously keep in mind not only the kind of mathematics the EEB proposes and the automatic structure and sequencing of the exercises, but also the means of assessment and the kind of help that is to be provided. Depending on the teacher, these functions are more or less close to his/her usual practice and thereby more or less easy to perform. For example, a teacher who usually monitors students' work individually will have less difficulty conducting EEB sessions than a teacher who adopts a more collective approach, mostly teaching from the front of the classroom.

Furthermore, the teacher's interventions are always adapted to the work of a specific student, without making generalizations on the basis of the rest of the class. It is as if the task of collective correction has been taken over by the computer, with the teacher instead providing personalized comments for each student. It may be that work with EEBs provides opportunities for the furnishing of "meta" information (Rogalski, 1997). Again, the extent to which this opportunity is exploited or not depends on the individual teacher.

In the next section, we will examine a case study (the case of Diane) to see how a teacher, given the specificities of working with EEBs, adapts his/her interventions and fulfils the new roles required to manage the lesson's "cruise" phase.

The closing phase

The closing phase is very short. In most cases it begins when the bell rings to indicate the end of the lesson. The teacher announces the lesson is over and asks the students to switch off their computers. Sometimes s/he will try to gauge how far the students have advanced with the exercises because each of them is likely to be at a different stage in the work plan; this will only rarely be followed by some further comments.

Example: "*I want you to write down the last results you've got and then close everything down as usual....OK?... Who managed to get to the last exercise? Charlène, you did, didn't you? Aurélia, you were more or less there. Marianne?*"

There is thus no summing-up and very little in common with traditional lessons. On these points teachers are divided. Some do not mind ending the lesson in this way. First, as we said above, these lessons aim at the consolidation of existing knowledge; the teachers do not therefore feel the need to do a collective summing-up of the mathematical topics being practised. Second, the students and the teacher feel that they have "worked hard" in so far as they have made use of new technology and met institutional requirements; but this work ends the machines are switched off. Meanwhile for other teachers this lack of conclusion at the end of class marks a break with their usual practice and can generate frustration. In the previous example we can see how the teacher tries at the last minute to glean as much information as possible as to where exactly the individual students have got to in their work.

Proposed modifications

In the interviews conducted after the lessons, some of the teachers offered immediate reflections on their activity in the classroom and proposed some short-term modifications.

For example, in one of the lessons the teacher met with the following difficulty: for many students it is not evident that "divide by three" means the same thing as "multiply by 1/3." However, the particular setup of the software means that it is not possible to divide by three. In the interview the teacher explained that she had not anticipated this problem because she had never encountered it when setting traditional exercise work.

More generally, the teachers found it hard to integrate the EEB lessons into their overall teaching agenda. Indeed, given that they are conducted in a classroom other than the one usually used and make use of a special technological resource, these lessons can seem marginal. This is even more so in the case of sessions tailored to students' individual needs. To offset this problem, some teachers take exercises similar to those encountered in the EEB and insert them into their class tests or ask students to perform them in front of the class during the next (non-EEB) lesson. Likewise, the students are asked to work with EEBs as part of their homework, for example when revising for tests.

After the first year of experimentation, almost all of the teachers wanted to continue in the following year. They envisaged some adjustments in terms of:

- their work plans ("construct lesson plans"; "make shorter and more targeted lesson plans")
- the overall structure of the timetable ("get started from the first day of term so as to get into good habits and normalize the use of EEBs"; "try working as a team, rotating the students"; "arrange for 'maths help sessions' to coincide with those of other colleagues so the students can choose whether to work with computers or not").

Following the second year of experimentation the teachers generally envisaged reinforcing the use of EEBs ("*more regular use*") or integrating their use of EEBs in full-class sessions by using a video projector ("*use also in class*"; "*greater use in class*") or extending their use "*at home and in the library*" or indeed adapting their use of EEBs to the needs of different students ("*increased use to deal with weak points*"). We also observed the beginnings of a teamwork initiative; an example of this will be given in the following section where we discuss the case of Flora.

EVOLUTION OF TEACHERS' ACTIVITY: THREE CASE STUDIES

We now turn to the question of long-term evolutions in teaching practice as seen in three case studies. Where possible, these evolutions are interpreted in terms of constructive activity. The interpretations are particularly difficult because, firstly, the evolutions are closely linked to the personal component of teaching practices, for which the data are not always sufficient. Secondly, the long periods of observation needed to detect such evolutions are not always possible. We will present the cases of three teachers who were part of the sample taken at the end of our study.

Michel

Michel teaches in a *lycée professionel* (vocational college, ages 15-18). We collected data on the classroom proceedings of five of his lessons given within a two-year period.

Over the course of the lessons Michel enriched the work plans he used with the target age 15-16 (first year of *lycée*) section of the EEB Paraschool[3] with supplementary study sessions and exercises taken from the target age 14-15 [final year of French *collège*] section. In this way he increased the bank of exercises proposed to students so as to facilitate the transition between the two levels. Indeed, Michel was conscious of the difficulties previously encountered by his students and tried to overcome these by setting exercises from the age 14-15 section, which he began to consult in his second year of teaching with Paraschool. We can say that the modification in his use of the EEB was guided by the cognitive component of his habitual teaching practice.

Other changes Michel made suggest an evolution in the mediative component of his practice. For example, over the course of the lessons he perfected a new format for the dialogues he conducted with individual students as he walked around the classroom: he would systematically ask them to go back over their thinking to check they have understood. We see this taking place in the following dialogue: "*Yes? What is it you don't understand? Right, where's the question, first of all? I didn't see your answer just now. What does this value represent? Go back …*" Michel also devised a special strategy for addressing the whole class. When he first started using the EEB, he used a function that blocked all the computers and stopped the activity of all the students whenever he needed to give an explanation to a group of students or even to just one student. However, these explanations had no relevance to the activity of students who had advanced at a different rate and were at a different point in the work plan. After several lessons he began instead to give his explanations to groups of students on the board, leaving the rest of the class to continue working at their computers. In this way the explanations on the board remained visible to any other students who might need them. Here it seems that the evolutions in Michel's EEB activity resulted from constructive activity, which had an influence on the mediative component of his practice. Certainly we can see a learning curve in Michel's method of dialogue with students and his way of managing blackboard use in relation to the autonomous activity of his students.

Flora

Flora teaches in a *lycée d'enseignement général* [ages 15-18 and giving access to higher education]. We observed two of her exercise-based lessons, given to a group of fifteen students aged 15-16 using the EEB Euler.[4] In addition to watching Flora's classes, we also conducted an interview in which she described the changes she had made during the year and her plans for the following year.

An initial evolution in Flora's activity seems to have been the desire to integrate the EEB into her overall teaching agenda; in other words, she thought to establish connections between her traditional classroom lessons and her EEB-based lessons. For example, she noticed during an EEB session that the students frequently struggled with linear functions; she therefore decided to adapt next year's traditional classroom session so as better to deal with this topic. In addition, she aimed at a regular use of the EEB even in her traditional classroom sessions. More generally, Flora envisaged restructuring her annual lesson plan so as to begin with, for example, lessons on how to use the EEBs. Flora's progressive integration of the EEB indicates a change in the cognitive and mediative components of her practice that in turn testifies to the presence of constructive activity.

We also detected some localized evolutions in the proceedings of Flora's EEB lessons, though Flora did not seem to notice these herself. For example, her addresses to students during the second lesson we observed aimed to clarify, simplify or develop the methods used to arrive at the answer. She appeared to tailor the help she provided to each student, improving the methods of struggling students and enriching the stock of possible methods in the case of more confident students. Yet in the interview she did not seem to be aware of this "differentiation" in the help she offered. It would therefore appear that this is a stable feature of the mediative component of Flora's practice, common to all her exercise-based lessons.

Finally, Flora helped to create a community of EEB users in her establishment. At the end of the experiment she persuaded two of her colleagues to work with her as a team during special tutoring sessions (i.e. individual help sessions), which are all timetabled at the same time in her establishment. The teachers began to think collectively about how to combine the teaching of topics with the use of EEBs. They also thought of possible ways to link up the EEB-based special tutoring sessions, which are attended by only the weakest students, with their regular, whole-class lessons. For example, they might ask a student from the special tutoring session to come up to the front during an ordinary lesson and explain what they had covered in the tutoring session. What we have here is an evolution in the institutional, social and personal components of Flora's teaching practice. We can therefore affirm the presence of constructive activity. Flora developed, in particular, an inclination towards teamwork and she now participates in other projects aiming to integrate new technology in teaching.[5]

Diane

Diane had been interested in educational ICT for many years. Prior to our project she had already participated in experiments on the use of spread sheets and interactive geometry software in secondary schools. The evolutions we observed therefore stretch over a very long period, and it would seem that they too can be interpreted in terms of constructive activity. The EEB used this time was Mathenpoche.[6]

The first long-term evolution observed in Diane's practice relates to the use of paper and pencil. During the early experiments with spread sheets, she handed out a sheet for the students to fill in and stick in their exercise books. When observing her lessons, we noticed that the students had a technology's exercises book that they used throughout the lesson. Diane also gave precise instructions on what the students should write down in theses books – for example, the answer to an exercise plus each stage in their working out. Additionally, many of her addresses to the students during the lessons concerned the use of pencil and paper. This goes hand in hand with her idea (expressed in a questionnaire prior to the experiment) of "keeping track" of the students' computer-based activity so that they would have notes to refer to when working independently.

The second evolution we observed in Diane's practice relates to her desire, like that of Flora, to integrate the EEBs into her overall teaching agenda. This can be seen particularly in the questions she set for class tests. At the beginning of the experiment she systematically included in paper and pencil tests exercises taken from one of the EEB sessions and presented as a screenshot. Later on she changed this slightly, instead using exercises that were similar to those from the EEB sessions, but that she had adapted for the test. In both cases she asked the students not simply to write their answer but also to show every stage in their working out; in this way the students recognised the importance of her instructions to take notes during EEB lessons. We can therefore observe that the effective integration of EEBs into Diane's classroom work brought about, first and foremost, an evolution in the cognitive component of her practice.

Bearing in mind these two evolutions, it is clear that Diane was working in the long-term towards the inclusion of the EEB in her global teaching agenda. As in the case of Flora, this corresponded to a progressive integration of the EEB into her teaching practice. Her actions regarding the use of pencil and paper are typical of the sort of strategies we have seen employed by a number of teachers. Indeed, the aim of Diane's various actions was to "oblige" students to keep a written record of their work at the computer; afterwards they were required to employ a similar technique in class tests. If the students are conscientious and use their notes to revise the exercises they have done in class, this may well have a positive effect on their marks.

A third evolution relates to the important role that Diane accorded to the EEB in the last few lessons we observed. Indeed Diane more so than other teachers, seemed to "fade into the background" during EEB classes, her aim being to train the students to work independently with the computers and with as little assistance as possible. In her view, the teacher's role is to help the students to acquire a sound method that will allow them to deal with the exercises independently. Moreover, we noticed that when Diane did monitor or assist the students, she generally concerned herself very little with evaluation. This shows, again, that Diane was willing to delegate part of her usual activity to the EEB. In this case she allowed the software to do the evaluating for her. This contrasts with other teachers, such as Michel, who attributed more importance to the function of evaluating students' work. We should emphasise here that Diane had a thorough understanding of the EEB she was using. This meant that she could, in the course of her explanations, refer students to the appropriate help sections provided by the software. This allowed her to encourage the students as much as possible in developing their autonomy. Certainly, the effort to develop student autonomy is one of the strong points of the personal component of Diane's teaching practice. We can therefore assert the presence of constructive activity in Diane's practices: she acquired a thorough knowledge of how to use the EEB and this because the EEB was the best means of achieving her teaching objectives.

Finally we should note that, thanks to her thorough knowledge of the EEB, Diane's interventions during the lesson sometimes succeeded in promoting the construction of mathematical knowledge (constructively oriented assistance) and not simply the execution of the task in hand (procedurally oriented assistance, see chapter 2). In the final lesson we observed, for example, she explained that the concepts of images and pre-images often pose problems for students because they tend to mix them up or else do not understand how they relate to whether a point is positioned on or off the curve. Diane thus helped the students to distance themselves from the specific EEB tasks they worked on and to think back over their mathematical knowledge. This can be seen in the following conversation she had with a student:

Diane: "*When you draw a curve, what are the coordinates of point M on the curve?*
Student: *M.*
Diane: *No. When you have a curve, what do you do to find the position of M? If I ask you to draw the curve* x^2*, what do you do first? You make the table of values, and then? You've got the x values, now you need to find the y values, in other words the* x^2 *values. Then once you've got your table of values, how do you move onto the curve? ...*"

It is clear that the use of the EEB brought about changes and privileged new types of interactions with the students. Could we see this as an evolution in the mediative component of Diane's practice?

CONCLUSIONS

Let us recall that one of the conclusions of the study of students working with EEBs (chapter 8) was the primordial importance of the role played by the teacher. This was true across all class levels. Without the mathematical assistance of the teacher, it is unlikely that students would be able to complete the exercises proposed. We can therefore appreciate the importance of the present chapter in which we have looked at the practice of teachers using EEBs and addressed the underlying question of the effects of EEBs on students' progress in mathematics. We have also asked more specific questions relating to teachers' appropriation of this new tool, its possible uses in terms of student activity, and the changes it brings about in teachers' daily activities and the evolutions of their global practices more generally. We have made several important findings.

Firstly, the teacher's productive activity is determined by the circumstances pertaining to the use of EEBs in class. Indeed, we have seen that a number of features relative to the organisation of the lesson tend to be adopted fairly quickly. These include a relatively long and rigid work plan, a smaller class size, work on topics that either have already been studied or are currently being studied, student-teacher interactions that is chiefly one-to-one, the absence of a summing-up addressed to the whole class at the end of the lesson, and limited use of the black/whiteboard. These features, which are both cognitive and mediative, seem to be linked both to the short- or medium-term "external" regulations of the teacher's EEB-related activity and, especially, to the significant constraints associated with the use of EEBs in class.

Secondly, the circumstances of the use of EEBs during lessons determine a teacher's geneses of the EEBs uses. In particular, teacher's activity will unavoidably, if progressively, acquire certain specificities. Thus we have the insistence that students keep a written record of their work and the linking up of EEB-based lessons with the ordinary (traditional classroom) lessons and class tests. These evolutions seem chiefly to affect the cognitive and personal components of teachers' practice. They are the result of constructive activity ("internal" regulations, chapter 1) but can accompany (as in the case of Flora) a desire for professional development. Nonetheless, these evolutions all have their limits. We can see this, notably, in the teachers' work plans, where, even after extended use of the EEB, crossovers between different mathematical topics still tend to be avoided.

Furthermore, the disparities between teachers and the different evolutions in their activity can testify to more complex geneses of uses that are linked to the teacher him/herself. These seem to be more directly linked to the mediative and personal components of practice. However, to detect geneses of this kind would require study over a much longer period of time. It is hard to say whether Diane's relinquishment of a part of her evaluation work to the EEB was really a matter of professional development or rather a helpful adjustment when integrating the EEB into the mediative and personal components of her practice. Flora's tailoring of the assistance she offered based on the needs of individual students did not seem to be conscious and so could be a pre-existing feature of her practice. We might also ask ourselves whether Michel's methods for helping students during EEB sessions (a very specific use of the blackboard and a particular dialogue format when discussing students' work with them) really evolved because of his use of the EEB or is rather a reflection of the mediative component of his pre-existing practices.

Lastly, another conclusion of the study of students working with EEBs (chapter 8) was that students must adopt a reflexive attitude in order for their work to be profitable. Whereas we might say that such an attitude is always desirable in learners, it is actually shown to be necessary when working with EEBs. Attempts by teachers to develop this reflexive attitude in their students would therefore appear to be beneficial. According to our observations, teachers actually do this, more or less consciously and in harmony with the different components of their practices. Thus, Michel felt that he should make available a written explanation that his students will be able to use "*where necessary*." Meanwhile Flora, when helping students with a given exercise, altered her assistance, more or less consciously, to meet the needs of different students. Finally, Diane worked on developing her students' independent learning skills by, for example, leaving them to evaluate their own work with the help of the EEB. Drawing on these three examples we can thus see that, in spite of various disparities resulting from the mediative component of these teachers' practice, all of them seem to have paid attention to the development of a reflexive attitude in their students.

NOTES

1 The use here of the term 'situation' is the same as in chapter 1.
2 The precise methodology at this stage is the same as that laid out in chapter two and is consistent throughout the book. In particular, we attempt to analyse the complexity of the tasks required to complete a given set of exercises, bearing in mind the help offered by computer technology. An analysis of the difficulty of applying knowledge to specific maths problems is important in gauging the quality of students' mathematical activity.
3 http://www.paraschool.com/
4 http://euler.ac-versailles.fr/
5 http://www.edumatics.eu.
6 http://mathenpoche.sesamath.net

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10. TEACHER'S ACTIVITY IN DYNAMIC GEOMETRY ENVIRONMENTS

Comparison with a session in traditional environments

INTRODUCTION

Research about Information and Communication Technologies (ICT) has focused in priority, over the past few decades, on the potentialities and limits of these technologies for the mathematics students' learning. There were relatively few studies about teachers' practices in these environments. Over the past few years, we witness an increasing tendency in the research to take into consideration the "teacher/teaching dimension" in the use of ICT, given its influence on the students' learning. The studies related to this dimension drew frequently on theoretical frameworks and methodologies which are developed in traditional non technological environments. For example Monaghan (2004) based his investigation of teachers' practices on Saxe's cultural model of "emerging goals." Abboud-Blanchard, Cazes, and Vandebrouck (2009) use in their work the model of Engeström stemming from the activity theory. Other researchers have developed specific frames allowing the study of teachers' practices in technological environments such as the frame introduced by Ruthven (2009), defining five structuring features of classroom practices. The "instrumental orchestration" is also such a framework, defined by Trouche (2004), that extends the instrumental approach developed by Rabardel (1995) and that was itself enriched through the work of Drijvers et al. (2010). The last ICMI study (Hoyles & Lagrange, 2010) shows the variety of these frameworks and some attempts to cross-analyze them.

Our current research aims at analyzing the activity of an ordinary teacher using a dynamic geometry system. It is based on the double approach theoretical framework defined in chapter 2 of this book and the related methodological tools. We aim to cross-analyze the results of the analysis with those of a study, which uses the same theoretical framework, for a "similar" session in a paper and pencil (p&p) environment. This cross-analysis allows us to identify in teachers' practices what seems to be common and what seems to differ from one environment to the other.

Other researchers have already highlighted issues, sometimes in the form of observations made during analyses, or have brought forward results which constitute research paths that we aim to explore. We mention for example two studies that explored the practices of ordinary teachers using technological tools in their classrooms, while considering these practices in their entirety and complexity. The first is that of Monaghan (ibid.) who talking of social interactions in the

F. Vandebrouck (ed.), Mathematics Classrooms: Students' Activities and Teachers' Practices, 199–214.

classroom stresses that the teacher in a technology-based session spends more time addressing small groups (few students to a computer) than making collective interventions for the whole class. He also highlights the fact that in these environments, the teacher has a more coaching-students activity to perform the requested tasks than in a p&p environment. The second study is the one of Kendal and Stacey (2002) who state in their conclusion that mathematical knowledge/skills used during ICT-based sessions remain globally in the same range of those mobilized during p&p sessions. They add that the main contribution of the technological tool, at least as perceived by the teacher, pertains essentially to the knowledge/skills related to the use of this tool.

The study we present in this chapter focuses on studying in a more detailed way the features of a teacher's practice in technology-based sessions, particularly those related to the teacher-students interactions and their influence on the students' activities. Our methodology, in particular the study of the cognitive functions of discourse, will also allow to revisit and refine certain phenomena pointed out by Monaghan or to introduce "nuances" to the conclusions of Kendal and Stacey.

We have analyzed a space geometry session, in a grade 9 class (aged 14/15 years), with a dynamic geometry software. Generally, teachers seem to be aware of the contribution of this type of software when tackling space geometry learning in middle school. Thus the teacher was convinced about the usefulness of dynamic geometry to improve the students' "vision" of 3D geometric figures. The results of the analysis were contrasted with those of a similar analysis of a session in a traditional p&p environment about the same topic, in another grade 9 classroom.

We employed the methodological tools introduced in chapter 1 of this book to analyze how the tasks designed for ICT environments differ (or not) from those of non ICT ones. We explored the differences in classroom management, the forms of aids and the assistance discourse of the teacher. We thus tried to understand the possible impact on the students' activities.

In the first section, we present the technology-based session and in the second one, the session in a p&p environment. For each of these two sessions, we first present the analysis of the tasks proposed to the students then the study of the session's progress together with the reconstitution of the student activities triggered by the teacher. We complete this analysis by a study of the teacher's discourse, but we limit this study, as we explain further below, to the functions of the discourse. In the third section, we put forward some elements of comparison of the two sessions.

A SPACE GEOMETRY SESSION IN TECHNOLOGY ENVIRONMENT

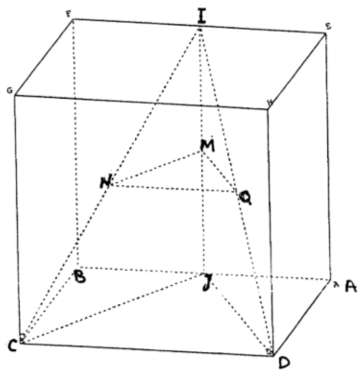
The session we study was videotaped in the classroom of a teacher that we will call Anne; it is a grade 9 class with relatively low level of achievement. The students are in a computer lab and work by groups of two to a computer; the dynamic geometry software is GeospacW.[1] The usable recordings concern only 9 groups. The objective of this space geometry session is to establish the theorem of the ratio of areas and volumes for a particular geometric reduction (the scale factor is a half) of two pyramids. The corresponding lesson was seen during a previous session on the same day.

Analyses of mathematical tasks and task pertaining to the technological tool

During this session, the students worked on the third part of a space geometry problem, the first two parts having been solved during a prior session. Anne did not design the problem herself; she took it from a document written by the software designer.

The students have first to download the folder containing the figure and the constructions completed during the previous session. They then have to perform the tasks described in Figure 1.

3ème partie



1. Soit M le milieu de [IJ]. On a coupé le solide IJCD par le plan P_1 passant par M et parallèle à la base JCD.

Pour effectuer cette section avec le logiciel :

- *Créer / solide /polyèdre convexe / défini par ses sommets (nom du polyèdre Py)*
- *Créer le plan P_1 passant par M et parallèle au plan JCD*
- *Créer /ligne / polygone convexe / section d'un polyèdre par un plan (Nom du polyèdre Py, nom du polygone p)*
- *Créer le point N, point d'intersection de la droite (IC) et du plan P_1*
- *Créer le point Q, point d'intersection de la droite (ID) et du plan P_1*

Examiner successivement avec le logiciel les plans JCD et MNQ.

Calculer les valeurs exactes de MN, NQ, et MQ.

2. Calculer les aires des triangles MNQ et JCD.
3. Calculer la valeur exacte du volume du solide IMNQ
4. Comparer les volumes de IMNQ et de IJCD

Figure 1. Student worksheet.

Through the first and second parts of the exercise, the students were able to:
- download the folder where the cube ABCDEFGH (of side 2cm) was already drawn, construct the midpoints I of [EF] and J of [AB];
- conduct a guided exploration of the figure by rotating it;
- conjecture that triangle JCD is isosceles;
- calculate the lengths: JC = JD = $\sqrt{5}$;
- calculate the area of JCD: 2 cm^2;
- identify that IJD is right at J;
- calculate the volume of IJCD: 4/3 cm^3.

In this third part, the students must first (question 1) construct using GeospacW the section of pyramid IJCD by the plane parallel to the base (JCD) and passing through the midpoint M of the altitude [IJ], thus obtaining points N and Q.

The construction task of the section using GeospacW is entirely guided. The manipulation instructions are provided step by step in the worksheet; the students only need to follow these instructions.

Then the students must inspect the planes (JCD) and (MNQ) using GeospacW. One can expect that at this point the student will perceptually identify a reduction: the two triangles are both isosceles, respectively at J and M, and MNQ seems to be a reduction (the scale factor is ½) of JCD.

Analysis of mathematical tasks (related to the end of question 1 and to questions 2, 3 and 4)

To calculate the exact values of MN, NQ and MQ, the difficulty could be that the students are tackling for the first time an exercise about the section of a pyramid by a plane parallel to the base. So they must apply the previously seen theorem in the particular case of this exercise, and use it to deduce a section of MNQ of the same nature (isosceles) as the base JCD and the parallel lines: (MN)//(JC), (MQ)//(JD), (NQ)//(JC).

The students can, for the first two calculations, choose and adjust the same reasoning: either the midpoint theorem or Thales' theorem. For the length of MQ, they can also use the same theorem or apply the theorem seen previously: the section of a pyramid by a plane parallel to the base is a reduction of the base.

To calculate the area of MNQ, students have to identify the formula and apply it to their figure then to determine a method allowing them to find the missing elements. Hence, they must take initiatives: construct the altitude of triangle MNQ from point M, place point H foot of this altitude, and calculate MH.

Finally, to calculate the volume of IMNQ, students have to adjust the formula which gives the volume of a pyramid. To compare it to the volume of IJCD, they need to shift to numerical frame by introducing the comparison of two numbers within a geometric frame, which could represent a difficult task for them.

The different mathematical tasks proposed to students are thus complex tasks since they require several adaptations of different types. In addition, repeating in this first question the application of a same theorem is also in itself difficult.

Analysis of tasks related to the technological tool

We are interested here in student tasks related to the use of the technological tool (labelled T in what follows). We will zoom in on these tasks, a zoom that will momentarily isolate them from the associated mathematical tasks. These T-tasks are exclusively related to the first question, the three following questions are to be achieved only with paper &pencil. We can identify four T-tasks:

Downloading the geometric figure
The student must first download the folder containing the figure constructed during the previous session. This technical task consisting in downloading a folder identified through its name is a simple task with no particular difficulty for the student. The students have already done this manipulation during the previous session.

Creating point M
The figure displayed on the screen is the cube ABCDEFGH, represented in cavalier perspective, with points I and J already constructed. The student must hence create the point M midpoint of [IJ]. It is a simple task that consists in opening the menu "create," selecting the item "point," choosing "midpoint of a segment" and specifying in the window that pops up the names of the extremities of the segment and that of the midpoint, M. The students have already completed this same task in the previous session to create points I and J. Thus they only need to recognize the application modalities of the command series needed for the creation of M.

Creation of the section
Once M is created, the worksheet provides the student with a detailed list of instructions to execute for the creation of the pyramid section and of points N and Q. This task is simple and does not seem to require any adaptation; a precise execution of the given instructions would be enough to reach the requested construction.

Inspecting the planes
Ones the construction completed, the student is asked to inspect, using the software, successively the planes JCD and MNQ. This task requires two types of adaptations: s/he must recognize the application modalities of the command series which lead to visualizing only one plane (previously seen in the first parts of the problem), and then s/he must go back and forth between the planes JCD and MNQ. This successive visualization of two planes will allow, on the mathematical side, to visually identify that the triangle MNQ is isosceles, and to conjecture that it is a reduction of JCD (already proved as being isosceles in the first part).

The T-tasks are thus, as we mentioned above, concentrated in the first question. The most complex part (construction of the section) is deliberately made easy (by the designers of this scenario) by providing a list of guided instructions to be executed. The other parts call upon knowledge related to the use of the software which was already experienced in the previous session. Therefore, it is reasonable to think that these tasks will only require a limited time of the expect student activity, the mathematical tasks being more complex and requiring more time to be performed.

The lesson in progress

We analyze in the following the lesson in progress both at a mathematical side and at technology use side. For the latter, we present elements related to two levels: the global level of the session through the collective interventions of Anne; the local level of the groups (two students per computer).

To start, it is useful to emphasize the fact that the teacher can only coach each group for a very short time and that her help must be efficient enough so that the students could complete the work alone.

This prelude might explain why the aids provided by the teacher are all almost with procedural function, whether instrumental or not, and that they simplify after all the students' activities. In fact, the division of the student's work into simple and isolated tasks is obvious.

The teacher having only very little time to spend with each group uses the fastest way possible to help them, almost dictating the work to be done. Most of the time, when teacher is with the students, they have only to execute her instructions, to finish a sentence she began, knowing that when she will leave they will be alone for a long moment.

However, we notice that the teacher did not succeed in reaching the desired goal within the allocated time. In fact, the students took "too" much time to execute the construction of the pyramid section; some of them were still doing it 10 minutes before the end of the session.

Anne had planned the T-tasks as an aid in the starting phase in order to accelerate access to the mathematical task. The time allocated to this was supposed to be limited. Her collective intervention at the beginning of the sessions proves it: "*So I remind you all that for this 3rd part, you will directly examine the plane, this will help you think how to answer the following questions.*"

Realizing during the session that these T-tasks took longer than expected, she tried to speed up the execution either by doing the work herself (while involving the students verbally), or by coaching the execution of the instructions given in the student worksheet until the whole set of T-tasks is completed. She also tried to make a collective intervention to let the students realize how slow their work was.

The aids provided by the teacher are centered on the construction of the section which, a priori, only required the meticulous execution of the given list of commands. We cannot consider these aids to have a procedural function (as defined in chapter 2 of this book), since there is no modification of the expected activities. We prefer to introduce in this context a type of aids that we label: "handling aid." It consists in accompanying the student in handling the software so that the mathematical activity is achieved without modification of the planned task. This type of aid is directly dependent on the use of tools. It is present in technology-based lessons (but it also can be observed in a non-technology environment when a tool is used for the first time); especially when the students cannot all handle the software with ease.

We can also underline that the teacher did not plan to use the software to help recognizing the cases of use of Thales' theorem or the midpoint theorem, to calculate the lengths MN, NQ and MQ, although examining planes (IJD), (IJC) and (ICD) with the software provide an unequalled aid as compared to the p&p environment. Anne noticed this fact with one group but did not propose this aid to the entire class.

Possible students' activities – activities "a minima"

The groups that we have listed, numbered 1 to 9, even though they don't have the same work rhythm, have all created the pyramid section and inspected the planes (JCD) and (MNQ). Yet very few have calculated the length MN which was the object of question 1.

Below, we give several examples of the groups' work.

Group 1: This is the group with which Anne intervened several times. Following the teacher's instructions, the students create the pyramid section by the plane and name it, inspect a plane and measure the lengths of the segments using the software. Next, they look alone for the theorem to be used in order to calculate the length MN. The teacher leaves them alone for 20 minutes then returns, following their request, to help them to use the midpoint theorem that they had already chosen. She then leaves them again for 12 minutes and come back to evaluate an area which we assume to be that of triangle MNQ.

Group 5: The teacher intervenes 17min50 after the beginning of the session. The students have created the section of the pyramid by the plane, and are trying to calculate MN. The teacher points the key elements on the figure: the length of the side of the cube, the nature of certain triangles, faces and base of the pyramid… She points out elements to organize their work: the use of the software to inspect a plane, the choice of a theorem to calculate a length; she leaves the students alone to find and apply this theorem.

Group 8: The teacher intervenes with this group after 43min. The students have constructed the figure with the software. They used Pythagoras' theorem to calculate IC and are looking for MN. The teacher indicates that 2 methods are possible and encourages them to use the notebook to find the useful theorem then turns to another group.

Group 9: The teacher intervenes with this group at the end of the session, after 47 minutes. We hypothesize that the students had succeeded in constructing the section and inspecting the required planes. They calculated correctly the length MN but the other lengths were inexact since they misplaced the main vertex of the isosceles triangle MNQ.

The activities of the students are hence very different from one group to another. None of them was able to reach neither the comparison of the pyramid's volumes nor even that of the areas. Nevertheless, some students were able to carry out on their own some adaptations such as the choice of the theorem to use, in particular group 1 which unlike the other groups and contrarily to the teacher's expectation, used one of the midpoint theorems to calculate the lengths of the sides of triangle MNQ. This group succeeded in organizing a reasoning to calculate the area of MNQ.

About the mediative component of the teacher

As we previously underlined, the students are autonomous for very long moments and when she is present, the teacher divides the task into sub-tasks which the students can immediately execute, in order to "give them a boost." Her interventions with the different groups are often similar as we will see below when we examine them more closely.

The few collective interventions of the teacher which address the whole class are almost all concentrated at the beginning of the session and are about how to start the work. Anne intervenes collectively at two other moments within the session: a first time, after half an hour, to tell the students that they can use a result seen recently in class (nature of the section of a solid by a plane parallel to the base), a second time, at the end of the session, regarding the work to be finished as homework.

Analysis of the students-teacher interactions

The authors of chapter 4 have introduced tools for discourse analysis and have used them to study the linguistic interactions in the classrooms in a traditional p&p environment. These are tools which allow the analysis of linguistic actions of the teacher through the study of discourse functions, what it is about, and through the identification of the questions asked to the students. We use these tools here to analyze the linguistic interactions in Anne's classroom. We are interested in the teacher's discourse functions.

We first notice that the *enrollment* functions represent a very small percentage of all the discourse functions (7%). We attribute this fact, in ICT environment, to the management of the *mobilization* of the students' attention and of the *engagement* in the tasks by the environment itself. In fact, the students seem to be involved rapidly in the T- tasks which are proposed to them. Their attention seems to be maintained throughout the session, the software allowing them to perform manipulations (even incorrect) while waiting for the teacher to pass by. The only moments where they "switch off" are those where they encounter a technical problem and where the teacher does not come fast enough to deal with it.

This small percentage can also be a result of the fact that the teacher addresses, most of the time, students individually. Indeed, after the first phase of the activity launching (beginning of the session), we witness a split of the class group into several "mini-classes" (groups of 2 students to a computer) who function in an autonomous way and to whom the teacher addresses as such.

This is clearly attested by the fact that there has been throughout the session (except for the beginning) only two collective interventions: one to structure the time (with the hope of speeding up (see above)) and another to remind the students of the theorem seen previously.

We secondly notice that the *structuring* functions take on a considerable importance (28%). We first identify the *time structuring* which occupies an important place. This is aligned with our analysis of the lesson in progress with respect to time management (see above). The tasks execution time progresses slowly; the teacher is aware of this and tries to fix it through structuring. We also spot the presence of the *orientation* function which, associated with the *information* function, allows by introducing sub-tasks (21%) to guied the work of the students.

The *task distribution* and the *introduction of sub-tasks* concern particularly the mathematical tasks. Those relative to the T-tasks are a minority 5%, we attribute this fact to the nature of the T-tasks which are simple and well detailed in the student worksheet.

The successions of discourse functions:

We are now interested in the succession of functions within the observed groups (mini-classes). In fact, we have noticed while studying the teacher's discourse that some functions seem to alternate in the same order in all the groups.

The interventions of the teacher begin mainly by an *evaluation* (sometimes associated with a *structuring*) followed by an *orientation* towards a *division in sub-task*: 13 times. They can also begin with *information* (associated sometimes to a *structuring*) followed by an *orientation* towards a sub-task: 7 times.

We can thus establish the succession of the teacher's actions:

– she arrives into a mini-class;
– she makes an evaluation or summary of the work completed;
– she guides the student (with possibly a structuring) and directs his/her work through a division into sub-tasks;
– if the student begins the execution correctly, she leaves him/her to go to another mini-class.

A SIMILAR SPACE GEOMETRY SESSION IN A PAPER AND PENCIL ENVIRONMENT

We observed the teacher Dany (the same observed in chapter 4) during a space geometry session in a grade 9 classroom.[2] We analyzed this session and compared it with Anne's session. In fact, the two sessions are about the same notion, the section of a pyramid by a plane parallel to the base, and allow comparisons which are likely to highlight specificities of the ICT sessions.

The study of Dany's session does not cover all the points that were examined for Anne's session. Those on which we focus here are: the organization of the way that students get to work; the aids of the teacher; the teacher's discourse during the students-teacher interactions, in particular the rephrasing of the students' answers and finally how the students are "taken into account" in the course of the lesson.

Analyses of mathematical tasks

We present briefly the session in order to situate the analyzed interactions. The students are not left to search autonomously at all; we can rather speak of a "dialogued lesson."

The teacher first presents to the students the objective of the session: "*So far, we dealt with sections of solids such as the rectangular parallelepiped or the cube, today we will deal with pyramids*." He then asks them to imagine what could happen if different pyramids that he shows them are sectioned by a plane parallel to the base.

The students chosen to answer give the expected answers. The teacher proves then the expected result in a special case, that of the exercise which he dictates "*SABCD is a regular pyramid with square base; for the side we will take AB equal to 3cm, and a height; last time we specified that when the pyramid is regular, O is the center of ABCD, we will let SO be equal to 6cm*."

On the white board, he projects the drawing of the initial pyramid SABCD (Figure 2).

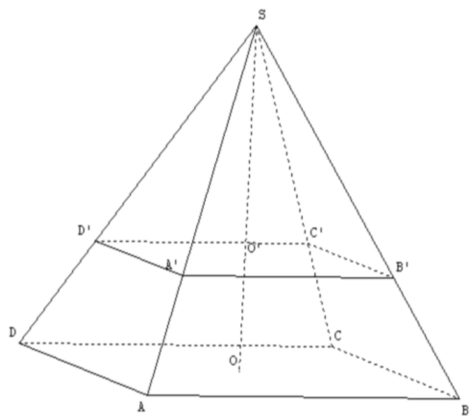


Figure 2. Drawing projected on the board.

The teacher asks the students to draw the figure while giving them very precise instructions, then he questions them about the plane parallel to the base: "*What should I specify now when we say that we are cutting the pyramid by a plane parallel to the base, this is too general, a plane parallel to the base, what should we specify to speak more concretely of the section?*" The rest of the session is about finding the nature of the section A'B'C'D' (a square), drawing it on the figure, identifying the parallel lines and then elaborating the proof that A'B'C'D' is a square. Beforehand, the teacher asks the students to draw "sub-figures" in real dimensions: the square ABCD, the line segment [AC] and its midpoint O, the height SO and so on.

A last phase is related to the ratio of areas of the bases of the "big" pyramid SABCD and the "small" one SA'B'C'D'.

The proposed tasks are not announced from the beginning of the exercise but progressively throughout the session. The only adaptations that the students need to do pertain to recognizing parallel lines and applying Thales' theorem, except in the case of finding the ratio of the areas, where the students need to shift from one purely geometrical frame to a numerical frame.

The lesson in progress

Firstly, we notice the repetition of two phases, a first one of elaboration of the reasoning orally, and a second one where a student repeats it while the teacher writes it on the board. Secondly, a very long time is dedicated to the construction of "sub-figures." It is as if the teacher wants to implicitly show the meaningfulness of these figures and the contribution that can have, in a space geometry exercise, to return to plane geometry.

Considering the aids provided by the teacher, most of them have a procedural function that contributes to structuring, organizing and to finally simplifying the activities. Dividing the work into simple tasks or even simple and isolated tasks is very obvious.

The teacher uses the students' answers which he chooses, and continues to explore them until he reaches the correct answer. He thus builds himself the thread of the very precise strategy that he has chosen to follow. What is written on the black board serves as a model to the students.

Possible students' activities – Activities a minima

During this session, the students can try to answer the questions asked by Dany or wait to copy the answers from the board. Hence, there are two types of activities, *a minima* and *a maxima.*

We can assume that also students manage to copy the drawing in cavalier perspective and the three plane figures drawn on the board. We can also assume that all the students wrote the reasoning dictated by the teacher. Some of them might have anticipated and completed successfully, on their own, the required constructions. Nevertheless, no complete proof of the nature of quadrilateral A'B'C'D' was presented at the end. Right angles are not mentioned and as for the length equality, even if it is mentioned, it is not completely written. It is true that this type of activity is time-consuming and that the work of the next session will probably draw on the drafted results.

About the mediative component of the teacher

Everything related to the organization of the work, the choice of elements to be taken into account, the tasks division and the succession of activities to be conducted is taken care of by the teacher.

He manages the changes in points of view, the transition from the perception of the nature of the section to its representation then to the calculation of length.

As we have previously indicated, he does not seem to give much importance on the proof regarding the nature of the constructed section since he does not complete it and it is only initiated in the dialogue with students.

The announced project "section of a plane parallel to the base" is maybe not really carried on to its end. We can hypothesize that it was transformed, the new project being to bring forward the theorem about the reduction and ratio of lengths, areas and volumes.

The study of two students-teacher interactions

We chose to study two episodes which take place, for the first at the beginning of the session, and for the second almost at the end. We chose them since we consider that they clearly illustrate the practice of this teacher.

The first is an exchange which accounts for Dany's strategy to lead a negotiation with the students. He starts by asking an open-ended question whose expected answer is yet very precise "*when we say we are cutting the pyramid by a plane parallel to the base, what should we specify to speak more concretely of the section?*" When the students answer "*the length AA'*," he dares not reject the answer, which is not expected, since it is relevant, but he makes a detour towards other elements of the figure and asks: "*Who sees something else, what can I specify? Juliette is thinking of A'. Charlotte, here you see O and you see O'. So another possibility is to specify the position of which point?*" The odd thing about this exchange is the lack of arguments on behalf of the teacher to assert his point of view: the choice of O' instead of A'. To be noticed also that he says "*Well which is the point that I am interested in on [SO]?*"

In this episode, the *enrollment* of the students is important. It could be identified through the large number of questions asked and the use of *mutualization* and *engagement* functions.

The second episode is surprising, because the teacher answers his own questions. Obviously, this is a way to speed up didactical time. In fact, during the proving phase, the teacher dictates the reasoning without even asking the students for some hints.

The *orientation* function is a majority in this episode. It allows the teacher to guide his students towards the expected answers. The questions calling or not answers from the students are quite present which reinforces their strong *enrollment*: "*Yes, SO' and SO. In the sequence of all these ratios, we will have SO' over SO, and we know what SO' over SO is. And SO' over SO is equal to … 4 over 6, and 4 over 6 is what ratio? Simplify. Two thirds, Ok?*"

The teacher's discourse seems to reflect a sort of paradox that he is trying to deal with as best as he can: the students should construct their representation of the section to then deduce the important results in grade 9, but they shouldn't spend too much time on it. The teacher thus guides the task by dividing it into several sub-tasks, or simply by doing the work himself. The lack of mutualization of students' answers can be a mean to let the work progress. The *enrollment* by regularly questionning the students helps in maintaining the students' attention.

COMPARISON OF THE TWO SESSIONS

The proximity of the two sessions lead to a comparison of what could be particular of a technology environment and what is common to both environments. The comparison issues pertain essentially to: the mathematical tasks proposed to students, the teacher's project, the students' activities, the teacher aids, the forms of work, as well as the teacher discourse.

The mathematical tasks proposed to the students are richer in Anne's session since they require several adaptations such as the construction of reasoning steps.

As for the tasks related to the technological tool, they are well detailed in the student's worksheet.

Dany's session is more "beaconed" i.e. the teacher does not define the global task straightaway; instead he builds it throughout the session. So the students follow the path drawn by the teacher.

The student activities *a maxima* are richer, thanks to the nature of the tasks, for some of Anne's students than those of Dany's. We cannot measure the activities *a minima* of Anne's students since we only observed part of the class. Nevertheless, it seems that all the students in Dany's class have listened to the reasoning and copied what was on the board.

By reconstructing the activities of the different groups in the ICT-based session, we note that the use of the technological tool seems to have a differentiating effect, whereas in the p&p session, the teacher tries hard to neutralize the differences between the students by providing what is on the blank board as model. We note also a differentiating effect with respect to time. For example, after 38 minutes, one group in Anne's class is still working on the first task. The groups do not have the same trajectory of performing tasks within the session. The teacher does not try to standardize the progress of the work. It is as if Anne is addressing successively several "mini-classes" that work autonomously. This functioning mode seems to be characteristic of the technology-based sessions in computer lab, even if we can also observe it, in a less-obvious way though, in a p&p group session.

The aids provided by both teachers are of the same nature, in general of procedural function, leaving even less leeway to the students during the technology-based session when Anne is present by their side. Therefore, the teacher's interventions lead to a division of the tasks into simpler tasks, or to a mere execution of a series of technical instructions.

Another element of difference is related to the forms of work adopted in class and reflecting on the students' activity. The students in Anne's sessions have longer autonomous research time: 12min, 20min … The teacher is present with each group for not more than 5 minutes. This explains in part the very detailed presentation of the T-tasks that we have explained above. In Dany's class, the students collectively follow the teacher's reasoning are never autonomous.

This students' autonomy, in a ICT environment, also implies a necessity for the teacher to adapt to their reasoning. When s/he comes to help a student, s/he has to reconstitute what s/he has already done in order to validate it or to provide the adequate aid. On the contrary, in the paper-pencil environment, the teacher carries on his project and the students adapt to it.

As for the teachers' discourse, the succession of functions for Dany ends with a summary function whereas they are extended through a sub-task for Anne. This difference is certainly due to different session objectives. Dany's session is a discovery session, during which he only proposes tasks that students should complete progressively. His aim is to let them understand the results on which he will base his lesson. Anne, on the contrary, has chosen to let the students face alone the results they find. Her aim is to engage them in autonomous research during her absence, which she obtains by boosting them.

Nonetheless, there are analogies between the two analyzed sessions. In both cases, the exercise was not completed. Moreover, the repeated use of the same theorem to deal with a space figure seems to be problematic. This might be related to the specific difficulty of the space geometry chapter. Finally, returning to the plane is a method used in both session and for very different reasons over a very long time. For Anne, we have noted difficulties of students relative to the use of the tool "inspection of a plane." For Dany, it is the duration allocated to the construction of the plane "sub-figure" that seems to be an implicit methodological aid provided to students.

CONCLUSION

To conclude, we will go over our results and extract the relations between the prescribed tasks and what really happened in the classes.

The two environments do not lead the teachers to radically different projects. In both cases, all the students have to solve the same mathematical tasks, not specific to technology environment. These findings join those of other researches addressing similar issues. They are close to what Kendal and Stacey (2002) underline about CAS[3], that is mathematical knowledge and skills stay globally within the range of those expected in non-technological environments.

However, working in a computer lab generally implies that students are divided into groups with two students working on a computer. Consequently, the class is split into several mini-classes who function in a relatively independent way. This fact also entails a quasi-disappearance of the collective phases as well as the collective time management. The teacher can in certain cases refrain from making public indications given to some students, which could be useful to others. In addition, the limited time that the teacher can spend with a mini-class leads him/her to simplify the tasks so that the students, even in him/her absence have always something to do. The two environments don't thus have the same repercussions on the organization of the progress of the didactical time.

Moreover, the succession of the discourse's functions, in technology-based session, is always the same, which leads us to describe in an orderly manner the teacher's activities:

- arrival into a mini-class;
- evaluation or assessment of the accomplished work;
- coaching the students by dividing the task into sub-tasks;

– moving on to another mini-class when the students begin the execution correctly.

We find in this functioning mode one of the observations raised by Monaghan (ibid.) namely, the teacher in ICT-based session spends more time addressing small groups than the entire class. Therefore, the teacher activity is reduced to a coaching of the students in the execution of the prescribed tasks.

We have nevertheless identified specificities of the ICT environment which are not only related to the form of work (small groups). The aids of the teacher are mostly of procedural function and are motivated by a concern to advance the work so that the students move on faster to the mathematical tasks. This leads to a division of the tasks, of which some require only the mechanical execution of a series of commands. This led us, on a methodological level, to the introduction of a new type of aid to take the latter into account: the handling aids.

Another specificity of the ICT environment seems to be the limited presence of the enrollment functions in the teacher’s discourse. The students are mobilized directly through interactions with the machine. More generally, these interactions lead to a certain autonomy of the students which gives the teacher the impression of not having to enroll throughout the session. In fact, remaining mobilized in front of the computer does not guarantee maintaining the mathematical activity. This contradiction in an ICT environment disturbs the teacher since it is rarely present in a traditional environment.

Finally, for the teacher, working in ICT environment seems to be more costly than a functioning in a p&p environment. A certain number of difficulties are related to the organization of the students’ work of in small groups. Even though globally the teacher’s interventions are almost analogous, they have to be repeated in each “mini-class” with the necessary corresponding adaptation. Trying to have collective interventions in order to make a summary and unify the knowledge of the students is even more difficult since the students have very different solving trajectories. This can contribute to explain the feeling of “unfinished work" that teachers often refer to at the end of a session in a computer lab. Other difficulties seem to be more linked to the specificities of the environment and reinforce the previous difficulties. These could be the heterogeneity of the students with respect to the use of the technological tool or the need to provide unusual handling aids. It could also be the fact of sharing with the computer certain enrollment's functions, which disrupt the usual classroom management.

Several studies using the double approach theory have confirmed the stability and coherence of the practices of experienced teachers (see first chapters of this book). In this chapter, we have showed that a teacher who is not an expert of technologies, and who uses them occasionally, sees this stability disrupted mainly with respect to the mediative component of his/her practices. In these conditions, which are the most frequent in the French educational context, this disruption experienced by the teacher can lead him to a “a minima use,” which complies with the institutional injunctions but does not exploit the potentialities of the ICT. Some teachers however modify their practices, and thus access to a new stability, like some of the teachers studied in the previous chapter.

NOTES

[1] http://www.aid-creem.org.
[2] It is not the same as the one in chapter 4.
[3] Computer Algebra Systems.

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11. QUALITATIVE AND QUANTITATIVE STUDIES ABOUT MATHEMATICS TEACHERS IN FRANCE

INTRODUCTION

The below research fits well into the research problem of this book and into the theoretical framework of the double approach, yet it presents a particularity which distinguishes it from the other works: it is based mainly on a quantitative study, carried out on a large scale, about the practices of high school mathematics teachers.

The aim of this research is also within the scope of the common objective of identifying regularities and irregularities in the practices of teachers, teaching mathematics in high school, but this time the aim is to do so on a large scale, which is not without effect on the analysis and processing of the collected data, as we will see further below.

Obviously, apprehending the real work of the teachers by finely analyzing the activities implemented in the classrooms can only be done through a limited number of observed teachers, but our choice has been to explore the practices of high school mathematics teachers, in a more global manner, by trying to express, not for an “authentic” reality of the teaching job, but to approach a certain reality.

As we saw in chapter 2, a teacher’s practices are partly the result of external components which intervene in his/her choices and which we try to apprehend through different analyses. Depending on our different research objects, we were brought to consider such or such component, more particularly allowing a better illustration of our questions. For this research, was paid to the personal component special attention, through the exploration of three variables which were studied more thoroughly: the age, the gender and the professional/academic background of the teachers. In fact, we assumed that these variables can have an influence on the practices of mathematics teachers teaching in high school, and can account for the irregularities or regularities. To carry out this study properly, we conducted several investigations: large scale questionnaires, interviews, session observations.

In the next section, we specify our research problem and the theoretical elements on which we relied to conceive our research, then in the second section, we present the global results of the questionnaire, and in the third section, we focus on the clinical study of five teachers who completed the questionnaire, then we compare the results of these two investigations before concluding on the question of the practices of high school mathematics teachers.

F. Vandebrouck (ed.), Mathematics Classrooms: Students’ Activities and Teachers’ Practices, 215–228.

RESEARCH QUESTIONS AND METHODOLOGY

The research studies presented in this book all have a specific entry to deal with the question of regularities and irregularities of mathematics teachers' practices: Roditi (chapter 3) has chosen for example to focus on the maneuver leeway intervenes by four teachers for teaching multiplication of decimals in grade 6, Abboud-Blanchard and Paries (chapter 10) have tried to analyze the influence of the integration of computer tools into the practices of a mathematics teacher.

The research presented in this chapter has the particularity of relying on two studies, one quantitative and the other qualitative. In fact, it seems important to work on a large scale in order to have a global vision of mathematics teachers' practices, and to be able to explore more particularly the personal component which intervenes in their foundation, as well as to work on a reduced scale to refine and specify the study of their practices.

Research problem

We seek to understand the practices of mathematics teachers equally from both a general and a particular point of view, in order to define the "field of possibilities," to describe a certain professional reality which best approaches the "authentic" reality, to give a complete overview of the teaching practices in light of the specificity of the individuals bring them to life.

To properly carry out this research, we will ask simple questions such as:

– Who are the mathematics teachers in high schools in France?
– What are their professional practices in their everyday routine, inside and outside their classrooms? Are there personal variables which are more discriminating than others for the comprehension of the diversity of these practices?
– Can we establish a categorization of teachers which could account for the common practices and the personal shared characteristics?

Theoretical references

Like all the works presented in this book, the theoretical framework used in this research is that of the "double approach," inspired by the activity theory with the differences specified in chapter 2.

The main difference with other works, regarding the quantitative study, is that the teachers' practices are not directly apprehended, this is rather done through fictional teaching situations proposed to the teachers in a questionnaire which includes several questions designed to explore their personal component as well. As for the qualitative study of the practices of the five teachers who completed the questionnaire, the loans are direct and compliant with what was presented in chapter 2.

All the mathematical contents presented in this research are analyzed using tools stemming from mathematics didactics, and more particularly the theory of situations by Brousseau (1997) and the tool-object dialectic of Douady (1986).

To complete these references, we will mention the concept of "ideal-type"[1] borrowed from Weber (1965), which seems quite adapted to describe, the teaching practices as a whole. In fact, this concept allows the construction of a certain reality which can be used as a framework to study the practices of high school mathematics teachers. The sociological references do not go beyond this simple loan.

Methodology

For this research, we have chosen to carry out two studies, one quantitative designed to try to comprehend the practices high school mathematics teachers, in its entirety, and the other qualitative to put to the test this entirety by comparing the results gathered indirectly to the results gathered directly, and to study some points in more detail. In fact, in order to carry out a quantitative study on the teachers' practices apprehended in a fictional manner, we found it advisable to carry out another investigation, this time qualitative and related to actual reality of the classroom, on a restricted number of teachers who had filled out the questionnaire.

At first, the quantitative study using the questionnaire will be used for a global overview of the high school mathematics teachers and to establish a typology of these teachers and their practice through the personal and professional shared characteristics.

Then, the qualitative study through the observation of sessions will allow us to specify some elements of the real practice of five teachers who are part of the quantitative study and test the typology which was drawn through the study using the questionnaire.

The global methodology of this research study uses tools and concepts presented in chapter 2, which were however adapted to a large scale study for the quantitative study.

The questionnaire

The questionnaire which was the basis for our study should at the same time enlighten us on the personal characteristics of the teachers and give us an idea about their professional practices. Hence, it was conceived in different parts, each with different aims: the first part is to gather as much objective information as possible and about the teacher and the second, to try to apprehend his/her practices "in class" based on fictional situations. The teachers were given quite a heavy questionnaire, including around thirty questions.

The first part had three sub-parts, each designed to gather specific information about the teacher filling out the questionnaire:

– "Who are you?": personal information about the teacher (age, gender, education, administrative situation, schools worked in, etc.).

- "Your training?": to explore his/her career path (training, workshops, etc.).
- "Your practice?": information about the elements of his/her professional practice, "outside the classroom" (teamwork, organization of lessons, averages, commitment to official instructions, etc.).

The second part of the questionnaire was aimed at confronting the teacher with fictional teaching situations in order to apprehend the elements revealing their practices at specific moments:

- Choice of problem subjects: the teacher had to choose his/her preference out of three subjects that were, almost equivalent in terms of mathematical content, but with different prescribed tasks and formulations. The first (subject 1) is very classical and directive in its formulation and includes questions which allow a simple and guided identification of the function-tangent link, the second (subject 2) is much more concise, leaving the linking of the two notions at stake up to the students, whereas the third (subject 3) imposes the use of a calculator even though it is not really necessary or useful for solving the problem, with questions closer to those of subject 1. The teacher also had to indicate the implementation elements.
- Types of aids recommended in case of difficulties encountered by a student: the teacher had to choose between several types of proposed aids (reference to the lesson, partial information, methodological comments, etc.). We could not at this level differentiate "procedural assistance" and "constructive assistance" (chapter 2) given that the nature of these aids could only be determined *a posteriori*.
- Reaction to an "incident": the teacher had to react to an unsuitable use of a skill used by a student (using the limit of the rate of change to determine the slope of the tangent to a curve at a given point).

We are aware that this indirect reasoning can create gaps with reality, but it was important to collect as many answers as possible in this study.

All the collected information was processed statistically, using a data processing software (SPAD). First, we carried out descriptive statistics on the two parts of the questionnaire, and then we analyzed these two parts using factorial analyses to create the desired regrouping.

Observations on the sessions

To complete this study using the questionnaire and test our typology, we studied the practices of five teachers who filled out our questionnaire very closely. We chose these teachers since they allowed us to account for the diversity of teachers in our sample, in terms of their personal characteristics. We hence observed and analyzed one "exercise" session of their choice. To analyze these sessions, we studied three dimensions related to the given couple:

- Global study of the scenario;
- Study of the tasks related to the exercises proposed;
- Study of the process and the work conditions of the students.

In this chapter, we do not describe these different elements of analysis explicitly, but we refer to them to compare the two studies.

The teachers' typology

In order to determine a typology of high school mathematics teachers based on personal characteristics and the elements of their practice, we conducted factorial analyses based on their answers to the questionnaire.

THE QUANTITATIVE STUDY

Global overview of high school mathematics teachers

We have gathered 225 questionnaires, mostly sent by teachers from the Académie de Versailles.[2] Even though our sample is not representative of all high school mathematics teachers, its non-negligible size encouraged us to believe that exploring it would allow us to get a global idea about all high school mathematics teachers in France.

Who are the high school mathematics teachers?

Almost all the high school mathematics teachers had some experience in middle school before teaching in high school (almost 80% of them, 6.6 years on average). Almost half of them had already taught in a "difficult" or "sensitive" school, on average during 7.3 years. Their involvement in professional development is relatively important, since 56% of them said to have attended at least 3 workshops since they started teaching.

If we consider their educational path, we see that there are slightly more teachers who have attended "Classes Préparatoires"[3] (56% compared to 44% who went to university exclusively), and they are generally overqualified with regard to the recruitment examination[4] they sat for.

Furthermore, almost one quarter of high school mathematics teachers are members of the APMEP,[5] and they have expressed an interest in the history of the subject matter they teach (more than 57%) and in computer science (44%), however only 23% are interested in mathematics didactics. They occasionally read professional magazines (57%), some do regularly (29%), and others never do (14%).

It should be noted that few teachers had another job before becoming teachers (almost 10%).

What are their "outside classroom" practices?

High school mathematics teachers work readily work in teams (47% "regularly," 46% "occasionally"), in particular to plan their school year progression (71%). In general, they follow the official instructions carefully (only 9% don't "always" follow them and 21% follow them "scrupulously") and finish the program in their classes (4% "rarely" finish it and 19% "not always").

To prepare their course, teachers use several manuals (71%), or only the class manual (13%), or even have a personal course plan which they adapt to the levels of the pupils in their classes (20%).

The averages are generally increased to the higher half mark (40% of the teachers), or even to the tenth of the mark (30%), very few are increased to the mark above (20%).

What are their "in- classroom" practices?

Students working in small groups is not a very common practice (62% never use it, 33% do so occasionally).

Regarding the choice of exercises which the teachers were given to work on, it seems that the first two subjects, more classical in form even though they were different in terms of the tasks prescribed to the students, were more commonly chosen (51% chose the first one, and 48% the second one). The third exercise, less conventional in form, only got 32% of the votes, but it was often selected as a unique choice (most teachers chose at least two exercise).

As for the choice of the implementation of the subjects, the teachers answered with respect to their interpretation of this question[6]: 46% of them gave us indications about classroom management, 33% made comments about the nature of the exercise, 13% commented them regarding their students.

The question of possible aids did not allow us to determine the function of these aids, whether procedural or constructive. The answers were given in terms of the suggestions that were made. Thus, almost all the teachers (84%) chose the reference to the lesson to help the students with difficulties while solving the proposed problem (this choice was most often accompanied by other propositions). Almost one out of three teachers indicated that he/she would definitely give methodological comments, other specified that they would ask their students to use a calculator (29%), or evoked the possible links with previously solved exercises (35%) or with the different questions of the exercise (42%). Some even suggested providing a partial answer (42%) or intermediate results (15%).

The reactions of the teachers confronted with the proposed fictional incident[7] were very varied, depending on whether they assumed responsibility for this event or they left it to the students. Hence, they either validated the procedure used by the student while ensuring the necessary readjustment (40%), or they left that readjustment to the students (16%). Some teachers completely rejected the used procedure (6%) or referred the student to the corresponding lesson (15%).

These results provide indications about the way high school mathematics teachers perform certain tasks inherent to their job. They give us a quantitative preview of their practices, while specifying certain elements related to their personal characteristics. They do not claim to convey the professional reality of high school mathematics teachers as it really exists, but rather simply an "approximate" reality.[8]

Results of three specifically studied variables

To explore the answers to our questionnaire more specifically, we chose to look more closely at three variables which we believe are likely to engender differences in high school mathematics teachers' practices: the teachers' gender, their age in terms of three particular age groups (under 36 years, between 36 and 36 years and over 36 years), and their academic background through the examination they sat for (Capes et Agrégation,[9] externally or internally).

The observations made in the study of these three variables were revealed in the set of teachers' answers and hence must be considered in regard to the limitations due to the relative representativeness of our sample.[10]

Gender

– Women seem to show more "professional sociability" while doing their job, through their involvement in group work and professional development. Men seem to work in more personal way. In particular, it should be noted that more women play the role of pedagogical advisors than men.
– Women seem to express more concern regarding certain conformity to their practices to institutional expectations. Men seem to worry less about this issue.
– It is also possible that women are more concerned by the transmission of know-how of mathematics, whereas men focus more on the transmission of knowledge which is more strictly mathematical.
– Women seem to be better mediators than men in the relation of the student to knowledge, mainly regarding the provided aids. Men apparently take the students into account more frequently on the level of the organization of their teaching ("outside classroom" practices), whereas women do it more during the lesson ("in classroom" practices).
– Women can be more open to pedagogical innovations than men, even though this observation does not query the professional dynamism of men. Men on the other hand have expressed more interest in computers, yet we were not able to foresee the impact of this observation on their practices.

Age

The age of the teachers is generally related to their professional experience, most of the observations we have established reflect this reality.

– The "students' parameter" seems hard to account for in the teachers' practices according to their age. It is very prevalent in the practices of middle-aged teachers, but less significant among younger teachers as well as older teachers, for different reasons.[11]
– The youngest teachers' lack of professional experience seems to be highly related to their practices. This is reflected on several levels: they tend to avoid dispersion and focus on learning the basic professional gestures,[12] and to compensate for this lack of experience, they look for more teamwork, unless they retreat into their shell.

- The older teachers tend to work in a more personal way, whether in elaborating their progressions or in planning their lesson. This observation is most probably the result of a combination of facts (more professional experience, detachment while exercising their job, generational choice, etc.) which lead these teachers to become more reserved.
- Age seem to also be a determining factor regarding the attitude of the teachers to official instructions. The youngest teachers are more concerned with their practices being in compliance with the institutional expectations, whereas for older teachers this concern seems to be less present and this, even though we noticed furthermore that more teachers among the older ones "always" finish their curricula, while more teachers among the young ones do not "always" do so.
- Most youngest teachers show a significant interest in computers. These teachers however show little interest in didactics, while the older teachers seemed to be more sensitive to that. The latter are also regular readers of professional magazines, while the younger ones rarely read any. Moreover, the APMEP members are usually older teachers.

Academic background

- The practices of teachers qualified through internal examination differ depending on whether their specific path led them to open out retreat into their shell for diverse reasons. Moreover, it must be noted that teachers qualified through internal examination have a wider professional experience in "difficult" or "sensitive" schools, and many of them did not receive any initial training.
- Regarding the different paths followed by the teachers, it seems that teachers who passed an Agrégation internally followed a more similar educational path (or even identical) to that of teachers who passed an Agrégation externally than qualified teachers.
- There do not seem to be major differences in the way teachers elaborate the annual progression of their teaching in terms of the examination they passed. To plan their course, it seems that qualified teachers focus more on one manual whereas this practice is quite uncommon among teachers who passed an teachers' Agrégation.
- Teachers who passed an Agrégation externally may tend more to position themselves as privileged arbitrators of knowledge in their practices.
- Teachers who passed an Agrégation externally are different in several aspects.[13] It seems that the preparation of their examination led them to be positioned differently from a professional point of view. We also noted that teachers who passed an Agrégation internally create more methodological aids character for their students.

Typology of high school mathematics teachers

Through our questioning about the practices of teachers studied on a large scale on one hand, and through the common personal characteristics and the shared elements of practices on the other hand, we tried to establish a typology of the teachers. We hence created partitions (in 3, 4 and 5 classes), based on the answers to the questionnaire, by projecting elements related to the practices on elements related to the personal characteristics. By examining closely the work in the partitions, we finally were able to draw a typology composed of four types of non-equivalent up importance[14], defined by individual and professional criteria.

Here are some characteristic elements of these different types:

Type 1: Teachers who are quite resistant to official injunctions, who do not always follow the official instructions, who are hostile to pedagogical innovations, who organize their course in a personal way, who believe that their practices do not differ according to the classes which they teach, who have not or barely taught in middle school, and who have started their job after passing external Agrégation.

Teachers of this type resist institutional constraints and social adaptations, they definitely believe that their level of mathematical knowledge allows them to teach legitimately.

Type 2: Teachers whose practices can be described as ordinary, since the modalities retained to characterize them show "unexceptional" practices.

What we mean by "unexceptional" is the fact of occasionally reading magazines, generally finishing the curriculum, producing term mark averages that they push up to highest decimal, not being trainers and not being interested in didactics. However, in this group we distinguish teachers who teach in "normal" schools and those who teach in schools labeled "difficult."

– "Normal" schools: annual progression elaborated rather in a personal way, occasional teamwork with mathematics colleagues.
– "Difficult" or "sensitive" schools: annual progression elaborated in collaboration, regular teamwork with mathematics colleagues.

The differentiation of the schools is reflected on the level of the social exposure of the teachers teaching in "difficult" or "sensitive" schools. The pressure they are subjected to incites them to social adaptation which is not really necessary in "normal" schools.

Type 3: Rather young teachers, who benefited from an initial training in a IUFM[15], they are not trainers, never read magazines, and have not had any professional development training. They do not show particular interest in the history of mathematics, or didactics, some of them are interested in computers. These teachers, in general, did not choose subject 3. Their lack of experience (related to their young age) forced them to avoid dispersion and focus on teaching practices, which allow them to fulfill their duties "properly."

Type 4: Teachers with some professional experience who have benefited from training in a CPR[16] (hence are not found among the youngest), who have take part in many professional development trainings, who read professional magazines regularly, who are interested in didactics and the history of mathematics. In this category, we find pedagogical advisors and a large number of teachers who passed the internal Agrégation. When these teachers have to choose and exercise for their students, they adapt their choice according to the possible management or the content (based on their answers to question a) and do not hesitate to choose non classical subjects (such as subject 3). In order to help their students, they recommend both referring to the lesson and other pedagogical tools (mainly links).

One should remember that this typology aims at providing an "abstractive synthesis of several concrete phenomena" (Weber, 1965, p. 179) and does not account for the "authentic" reality of mathematics teachers and their practices.

QUANTITATIVE STUDY OF FIVE TEACHERS

We then chose to examine more closely the practices of five teachers who filled out our questionnaire and accepted our offered to observe their sessions and interview them. These teachers were chosen because, considering their personal characteristics, they allow us to account for the diversity of the teachers in our sample as best as possible.

The qualitative study

Hence we chose for our study:

- a woman over 46, qualified externally: Mrs. CE1.
- a woman over 46, having passed an Agrégation internally: Mrs. AI1.
- a woman aged between 36 and 46, having passed an Agrégation externally: Mrs. AE2.
- a man aged between 36 and 46, qualified externally: Mr. CE2.
- a man younger under 36, qualified through external examination: Mr. CE3.

The main objective of our visits to their classes was examining more directly the practices of the teachers chosen among those who filled in the questionnaire. The visits also aimed at evaluating the contingent gaps between the answers to the questionnaire and the practices, which are observed directly, and thus evaluate the reliability of our large scale study.

By comparing the results found during the visits and those of the questionnaires of the teachers, we noticed that they were quite similar, and that they did not show any major contradictions. This observation is satisfactory, given that it implies that the questionnaire data is somehow reliable, and can be used to support a quantitative research on the practices.

Some variables are more visible than others when we compare these two surveys. Hence, the weight of professional experience is a datum that we easily identified during our visits. Many of the different aids selected by the teachers in their questionnaires also coincide with those we were able to observe during our visits. We were also able to detect in the practices of the visited teachers traces, which correspond to complaints expressed by the teachers about their students. The choice of subjects also matched what the teachers had offered to their students during our visit.

However, we were not able to master some variables in one session. These are mainly the variables related to the teamwork of the teachers, the group work, or the elaboration of progressions, which is not surprising since these variables account for practices, which are difficult to perceive in one single visit. Similarly, it was impossible to find traces of the academic level of the teachers (degrees) or the path they followed, except for the two teachers who passed aggregations. In fact, the practices of these teachers reflect rigorousness or even a rigidity, which can stem from their previous experience.

Comparison of the two studies

We will now compare the results of the two studies to test their pertinence, even though we are aware that the study of few cases cannot validate them.

Gender

The three women in our study all showed a rather high professional sociability, each in their own way. Mrs. CE1 works equally well in teams, both with mathematics teachers and with teachers of other subjects in her high school. Mrs. AI1 regularly works in a team with her colleagues and acts as pedagogical advisor. Mrs. AE2, although she believes she only works "occasionally" in a team, is in charge of the "Kangaroo 19" club of her school. As for the two men of our study, Mr. CE2 said he seldom worked in a team (which goes against the type 2 reference "difficult teaching"), whereas Mr. CE3 does it more willingly (which might be related to the fact that he feels the need to do so as a "young" teacher).

Regarding the fact that women seem to be better mediators than men in the relation of the student to knowledge and mainly on the level of the aids, we can simply indicate that Mrs. AE2 and Mr. CE2 were the only ones who only used procedural aids during their sessions, while the three other teachers provided with their students constructive aids as well.

Age

The practices of the "youngest" teacher in our study do not seem to correspond with the characteristics of his age group, since he does not seem to be constrained by his lack of experience (rich proposed problem and dynamic classroom management) nor is he detached from the contingencies related to his students (varied nature of oral exchanges and aids provided during the session). This difference can be explained by the fact that, at the very beginning of his career, Mr. CE3 benefited from a particular coaching, thanks to a team project in a "difficult" school.

Mrs. AE2 and Mr. CE2 only used procedural aids during their session. They were also the only ones to manage their session exclusively with collective or individual (to a lesser extent) exchanges. This observation could be related to the particular consideration of the "students" parameter by the teachers of this age group.

The two teachers of the oldest age group revealed undeniable professional experience during their session, on the level of the global organization of their session, as well as the nature of the proposed tasks, or the level of the process.

Academic background
The two teachers who passed an aggregation (one internally and the other externally) have a similar academic backgrounds, which reinforces our observation on similarities in the paths followed by teachers in these two categories.

The professional positioning of the only teacher who passed an Agrégation externally also corresponds to the one we indicated. The nature of the exchanges (mainly collective) and the organization of the session (short research phases, strictly procedural aids) allow us to assume that Mrs. AE2 considers herself to be a privileged arbitrator of knowledge in her practice.

We can also report that the only pedagogical advisor in this qualitative study is the teacher who passed an Agrégation internally, which corresponds to what we had noticed in our quantitative investigation.

The teachers' typology
It was possible to link the association of each teacher to one of the four types of our typology thanks to the information collected during our visits.

We had indicated that many of the teachers belonging to type 1 had passed an Agrégation externally, while many of the teachers belonging to type 4 had passed an Agrégation internally, or were pedagogical advisors, which coincides with the types of Mrs. AE2 and Mrs. AI1; yet, we do not attribute a type to a teacher *a priori,* only because of the examination he/she passed. Several parameters must be taken into account in order to be able to class a teacher in one of the four determined types.

Furthermore, a teacher can belong to one type at a given moment of his/her career and then to another type at another moment. Type 3 is mainly a transitional type which can evolve into any of the other three types; this is the cases of Mr. CE3 who is associated to types 3 and 4, since some of his while the others tend more towards type 4. Different teaching conditions and personal elements can lead a teacher to move from one type to another.

Three of the studied teachers were classed in type 4 based on their answers to the questionnaire. The sessions that we observed allow us to enrich and illustrate the reality of this type.

Thus, it appears that these three teachers communicated with their students by varying the nature of their exchanges (collective, individual, semi-collective) and by using both procedural and collective aids, which was not the case of the two other teachers associated with other types. These characteristics can be emblematic of this type, even though they are only linked to elements of practice and not to personal characteristics of the teachers.

Mr. CE2 was classed in type 2, since his practices did not reflect the specificities in the other types. Nevertheless, the differentiation that we made the level of practices linked to the nature of the school ("ordinary" or "difficult") does not seem to be relevant for this teacher. It is possible that Mr. CE2 situates the adaptations he must make due to his teaching in a "sensitive" school on another level (than the one we were able to observe).

Mrs. AE2's session illustrates the practices linked to type 1 of our categorization in an instructive manner, even though she only gives only one visible example. Hence, the fact of proposing complex tasks to the students, while systematically dividing them into sub-tasks during the class and of only providing procedural aids could reveal a practice associated with this type. Similarly, the singular choice of subject (Von Koch flake) worked through a problem personally prepared (which is a characteristic of this type) by the teacher allows us to consider the relationship to teaching of mathematics teachers of this type.

CONCLUSION

The two studies that we have carried out allowed us to get to know high school mathematics teachers better, to describe a certain professional reality and approach some elements of their practices. In accordance with the adopted approach, the quantitative studied provided global information about high school mathematics teachers and allowed us to catch a glimpse of a "field of possibilities" on their "outside classroom" practices (teamwork, following official instructions, finishing curricula, using manuals, etc.) as well as their "in-classroom" practices (choice of subject, reaction to a mistake, aids) even though the latter were only approached in a fictional manner. The qualitative study allowed us to examine these practices more closely, to stress some of them, and shed light on a less virtual reality.

The particular study of the three retained variables (age, gender, degrees) allowed us to perceive some variations in the practices of high school mathematics teachers. Professional experience, inescapably related to the age of the teachers, is a discriminating factor, which acts either as a vector of autonomy, allowing more opening out or reserve for the older teachers, or as an inhibiting yoke for the younger teachers, who are thus constrained by a certain pragmatic caution in their profession.

The gender is also a factor to be considered in order to account for the differences at the level of the teachers' practices (different positioning with regards to the institution, mediations of learning or colleagues), even though we regret that we were not able to perceive in what way it could have any influence, on the level of the relationship with the students.

The examination passed by the teachers allowed us to apprehend some differences, both on the level of the followed track (external or internal) and the level of the type of examination (Capes or Agrégation), but this variable, related to the individual background of the teachers, is difficult to perceive and to analyze.

We should nevertheless remain very careful as far as the results of the study of these variables are concerned, that they only reflect a global vision considering, and do not lock the teachers in a very reducing characterization.

The question of the categorization of mathematics teachers based on personal characteristics and modalities of practices has been solved through the choice of the four types defined based on our quantitative study and highlighted in our qualitative study.

The choise of these four types, in the conditions that we have specified (frontier porosity, global coherence, etc.), is a tool for me, allowing a global vision of all the high school mathematics teachers' practices. This tool should not be an excuse to lock the teachers in a sterile categorization, which would not take diversity of personalities forming the mathematics teachers' body into consideration and the professional freedom to which they are entitled.

NOTES

1 For Weber, the conduct of social science depends upon the construction of hypothetical concepts *in the abstract*. The "ideal type" is therefore a subjective element in social theory and research; one of many subjective elements which necessarily distinguish sociology from natural science.

2 This corresponds to approximately 10% of the teachers working in the Academie of Versailles. With Académie of Paris and Académie of Creteil, Versailles is one of the three academies of Region Ile de France.

3 First and second year after Baccalaureat for good students devoted to prepare access to ingenious' high school

4 Among the 37% of certified teachers and 58% of teachers in our sample aggregated over 63% of teachers have at least a master's degree.

5 Association of Mathematics Teachers from french public schools

6 The question was: "Can you clarify your choice with the conditions of implementation?"

7 The teachers must react to the use, by a pupil fictitious, of a method not suitable to solve a simple task.

8 These are only declarative data and there could be a gap with reality.

9 In France, there are two types of examinations to become a teacher: the CAPES and the Aggregation (of high level). Students can pass them externally, while pursuing university studies, or internally while they are already teaching without the official title.

10 The statistical representativeness of our sample is actually not guaranteed, for several reasons (age, distribution, gender, professional corps, and so on).

11 We assume that the youngest teachers would essentially be preoccupied by mastering basic professional gestures, and that they would have to "leave aside" the "students" parameter then, after getting past this state, they would focus more on this parameter (feeling of guilt or requirement) to be detached from it later on through a professional rebalancing, the fruit of more confident experience

12 Being able to organize the progressions, manage the classroom, evaluate the students, etc.

13 Less group work than other teachers, an implementation more focused on management, using, etc.

14 For information only, ours ample is divided in the following way: type1 = 20%, type2 = 24% and 20%, type3 = 16% and type4 = 20%.

15 IUFM: Institut Universitaire de Formation des Maîtres that is University department for teacher training.

16 Centre Pédagogique de Formation is Pedagogical Training Center (training center prior to the IUFM).

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12. STRATEGIES FOR TRAINING MATHEMATICS TEACHERS

The first step: Training the trainers

INTRODUCTION

Here we build on the preceding chapters to provide an overview – in keeping with the core theoretical framework set out in chapter 2 – of research into teaching practices. In the first part of the chapter, we reiterate the book's shared methodological elements and we reconsider its findings, drawing on other studies where relevant. We focus in particular on those findings that offer useful insight when thinking about how to train mathematics teachers. Our approach here is speculative: although we make some suggestions, we do not, for the moment, provide extensive research evidence. That will be for another study.

In the second part of the chapter we discuss teacher training, but we do so from a very specific angle: the training of teacher trainers (or teachers' educators). Indeed, the large amount of research into teaching methods that we now have before us is, first and foremost, an invitation to consider the possible channels of communication between such research and the actual practice of teachers in classrooms, including its impact on students' learning. One such channel of communication is the teacher trainer, who is a messenger between two worlds: the world of the school and the classroom, and another, broader world encompassing, on the one hand, the diversity of theoretical research relating to schools, learning, and teaching and, on the other hand, educational institutions and schooling policy. The training (both initial and on-going) of teachers in France is subject to well-defined institutional frameworks and most of the flexibility in terms of methods and goals lies with the trainer. Moreover, recent developments have resulted in a school curriculum that is increasingly complex and that encourages innovation per se without clearly defining the aims and methods of this innovation (technological resources, competence-based learning, personal mentoring, etc.). The question then arises of how a teacher training programme can take on board the dictates of this curriculum without neglecting its central task, which is the training of teachers to teach mathematics to students. Teacher trainers occupy a key place in the answer to this question and how they are trained is thus a serious issue that deserves our attention. It will be addressed in the second half of this chapter by means of some general considerations, followed by a specific example – how teacher trainers

F. Vandebrouck (ed.), Mathematics Classrooms: Students' Activities and Teachers' Practices, 229–245.

could be trained relative to the use of technology in mathematics teaching and learning.

TEACHING PRACTICE IN MATHEMATICS: AN OVERVIEW OF EXISTING RESEARCH

Most of the studies of this book referred to below adopted 'double approach' methodology. According to the associated theory, students' learning objectives cannot be abstracted from factors pertaining to the teaching profession as a whole – this can be understood in terms of the interconnection of the five components that constitute practices (see chapter 2). When analysing a given teaching practice, we can draw on these five components to study and to establish the connections between, on the one hand, the choices made in preparing for class and in class – observable during the lesson itself – and, on the other hand, the impact of various constraints. Such constraints might be linked to the nature of the teaching profession, the conduct and composition of the class, the curriculum and timetabling, the various expectations of parents, of colleagues, and of the administration, and the teacher's individual conceptions of his/her work. In addition, the complex structure of teaching practices per se can be analysed by distinguishing several levels: implementation in the classroom (the local level) intersects with the teacher's projects and conceptions (the global level) and with routines and automatisms (the 'micro' level).

Having provided this methodological background, we will now present a selection of research findings.

Difficulties faced by beginner teachers

Chesné's (2006) study serves to reinforce, and to transpose into the terms of our core theoretical framework, what teacher trainers have already been reporting for some time. Namely, beginning teachers seem to experience a work overload at the local level (that is, in their day-to-day work in the classroom). This, added to their temporary lack of resources at the global level, means that such teachers often have difficulty in conceiving of a complete academic agenda inscribed within a global vision. Given that at the beginning of their careers they also suffer from a lack of resources at the micro and local levels, these teachers tend to struggle with time management during lessons and lack the required routines and automatisms of more experienced teachers. They have difficulty managing the class and, above all, they fail to pay sufficient attention to their students' needs.

As Chesné notes, when faced with such difficulties, beginning teachers tend to overreact in one of two contrasting ways:

- by paying excessive, often highly individualized, attention to students and becoming overly concerned by their reactions, resulting in a mathematics programme that remains highly localized and lacks overall coherence. Such an approach has been described and criticised by previous studies (Bloch, 2005; Margolinas & Rivière, 2005);

– by, on the contrary, overemphasising mathematics in itself at the expense of attention to students' needs.

In other words, such teachers have not yet succeeded in balancing instruction in mathematics with attention to students' needs.

As Abboud-Blanchard et al. (2008) have shown the first of these two (over)reactions tends to be exacerbated by the use of technology in the classroom. Indeed, because lessons using technology are often conducted in a computer room with one or two students per machine interacting directly with the software, the inexperienced teacher is likely to favour exclusively individual assistance, providing chiefly practical help in completing the exercises and thereby neglecting the lesson's global mathematics agenda.

Furthermore, for many beginning teachers, some mathematical concepts are already 'second nature.' As a result, when teaching these concepts they forget the difficulties,[1] skipping over explanations, intermediary steps and provisional simplifications and dedicating insufficient time both to the concept itself and its associated techniques.

The illusion of simplicity and, more generally, the lack of awareness of the possibility of any gap between what the teacher says and what the student understands,[2] are particularly apparent amongst beginning teachers who work with socially underprivileged students. For this reason, such problems are all the more damaging (Chesnais 2006, Coulange 2012).

Consistencies in teachers' global choices of content

Content choices – especially those that bring global content decisions to bear on the teaching of a particular concept or topic – are, owing to institutional and social constraints, made in a similar way by most teachers of mathematics and associated disciplines. This can be illustrated by an example provided in chapter 3. Institutional constraints at the global level – timetabling, curricula and the (in)availability of resources – lead to identical, 'forced' choices regarding student progress during the year and the amount of time allocated to each topic. There is also frequently a feeling of 'working against the clock' to 'get through' the curriculum.

Social constraints, necessary to some extent for the smooth functioning of the class, can be what lies behind the creation of 'shared management principles' (see Roditi, ibid.). Similarities in content choice are also addressed in chapter 9. Here the authors emphasise that previous constraints, to which the use of E-exercise Bases are now added, mean that teachers using this new tool focus on previous knowledge or knowledge they are still in the process of acquiring, thus failing to explore more profoundly the potential of computational tools to provide a way into new topics. The authors also show that teachers restrict themselves to exercises that do not mix different mathematical fields (e.g. algebra and geometry), their aim being to avoid complications in lessons where time will already be used up in explaining the software.

The stability of individual teachers and differences between teachers owing to management and lesson content choices

Analyses of the practices of experienced teachers teaching different groups of students (chapter 4) have shown that, in comparable social and institutional conditions, the classroom practices of an experienced teacher show relatively little variation. Further, this consistency is to be found, above all, in the stabilized, consistent management choices made by the teacher as s/he teaches. That is to say, such choices are more consistent than the content decisions made in advance when preparing the lesson.[3] The balance between attention to students, coverage of the curriculum and professional compliance seems to be the result of a stabilized adaptation on the part of experienced teachers and is difficult to undermine. This means that changes in content choice could be introduced into these teachers' lesson preparations without repercussions for the teaching process and the class events.

By way of contrast, where the use of technology by teachers is sporadic,[4] the result is disruption of the teaching process (see chapters 9 and 10). Abboud-Blanchard and Paries (chapter 10) show, in particular, that teachers without sufficient training and using technology intermittently find the consistency of their teaching practices perturbed as a result, especially in terms of the mediative component of these practices. Teachers experiencing this kind of disruption will often minimize their use of technology as a result, meeting institutional requirements but failing to exploit the full potential of such resources. Nonetheless, Abboud-Blanchard and Vandebrouck (2012, 2013) show that repeated use, even intermittently, of the same technological resource will sometimes lead teachers to modify their teaching practices, whereby they re-establish a new stability.

The question can also be asked whether, because of such changes in practice, some students – particularly those from underprivileged backgrounds – might not find themselves frequently excluded from the mathematics activities of a given class: given the teacher's class management choices, such students might never have the opportunity to develop their knowledge to the expected level. The assistance offered may be of limited and purely procedural use, merely enabling them to carry out the task in hand (see chapter 7).

At the same time, as many of the earlier chapters have shown, differences between teachers are to be seen in the way in which they realise the details of their teaching within the confines of the required framework; their choice, at the local level, of a variety of exercises (specific problems set) and of evaluation methods and homework; and in the way they organize their class events (lesson structure, independent work and how such work is set up and built on by the teacher). As we can see, these differences largely concern the local level. In other words, the diversity is due to the varying extent to which different teachers exploit the freedom they are given in terms of specific class exercises and lesson structure. The same differences can be seen when we analyse the detail of teacher's language, which reinforces their other choices (questions to students, register, speech functions, use of linking words, lexical flags) (see chapter 4).

Finally, there are some teachers who consistently share some of the characteristics laid out in chapter 2. It seems that because of time constraints, made ever tighter by current timetable restrictions,[5] when teaching some of their 'average-ability' classes and particularly those of upper secondary school (students aged 15+), these teachers like to move speedily from topic to topic, emphasising the acquisition of new knowledge. But in doing so they provide little (especially qualitative) exploration, little revision and little maintenance of existing knowledge, and little accommodation of the new material in terms of what has already been learned. Rather, we see a univocal reorientation of class activities towards the new material, which has to be speedily (even instantly) and accurately grasped. All of this is in marked contradiction to the new directives recently announced for French upper secondary school (students aged 15+).

As is shown by the studies to be cited below, all of this has an effect on student activity.

Relations between teachers' choices and students' learning

A number of recent studies compare students' results in class tests and the teaching they have received (see chapter 6 on the study of similar triangles; Dumail (2007) on the teaching of square roots to 14-15-year-olds; and chapter 7, on orthogonal symmetry taught to 11-12-year olds). Such research has confirmed the importance for students' assimilation of knowledge of the collective work done in class. However, if classroom exercises do offer some guarantees, they can also in some cases inhibit adaptability in the application of knowledge, which is also indispensable to assimilation. Similarly, some working styles, such as work in small groups, are not necessarily of benefit to all students; here the quality of the teacher's supervision plays an important role.

Nonetheless, as we have noted, the degree of variation in teachers' choices is limited by the constraints on individual practice that were mentioned above.

Discussion of consistency

Now we come to learning and the activities most conducive to it. Here a number of wide-ranging questions can be posed regarding consistencies in teaching practice, tackling the subject from several different angles and revealing the extent to which teaching is a matter of choices, uncertainties, and even gambles and losses. We believe that teachers understand better than anyone else the reasons that lie behind the choices they make. These reasons need to be analysed.

Managing diversity and successfully engaging students are a priority in any classroom, whether or not it is composed of 'difficult' students. This applies equally to work done outside of the classroom. More specific to the teaching of mathematics, and especially relevant to the teaching of 'difficult' classes, are decisions regarding simplifications, shortcuts, reductions in the complexity of the material to match students' abilities, requirements for written work, and, more generally, the question of how to interpret and adhere to the curriculum.[6] Such questions force us to reflect on what it is that is asked of teachers, how the teacher understands these requirements, and to what extent s/he puts them into practice, bearing in mind the special context in which s/he works.

More broadly speaking, the balance between allowing a large number of students, including the socially disadvantaged, to participate actively in class – this being necessary for good class management – and the learning benefits that come from setting a sufficient number of difficult exercises, is a delicate and controversial one that entails both content-related and management choices. Indeed, the issue is often openly debated. Should the teacher work through the curriculum (too) quickly, or should s/he devote extra time to difficult concepts in order to enable more students to grasp them, even if this means that not everything will be ‘covered’? Undoubtedly there exist thresholds, different for different classes, that can help the teacher make an informed practical decision. Such a decision will be something of a compromise, with varying knock-on effects.

Further research is needed to understand more precisely what is at stake here at the various levels.

All of these findings and the questions they raise constitute the initial groundwork necessary to any discussion of teacher training, both initial and on-going. It should be added that there have been very few studies to date focussing on the training of secondary school teachers.

A few pioneering studies of the training of school teachers (Masselot, 2000; Vergnes, 2001) have suggested that initial and on-going training does not achieve its goals in all cases – far from it. Moreover, and in accordance with the findings of another useful precursor (Mangiante, 2012), it would seem that the personality of trainees plays an important role in determining his/her future success (or otherwise) as a teacher.

Grugeon’s (2008) study of an initial teacher training course given at the IUFM (University Institute of Teacher Training) in Amiens is another isolated study that has not been followed up. However, Grugeon’s evaluation of the course, which was inspired by principles similar to those put forth in this book, is encouragingly positive.

As we have seen, then, studies that attempt to evaluate teacher training courses and their effects on the methods of beginning teachers are still very rare. This is no doubt because of the difficulties they entail.

DIFFERENT APPROACHES TO THE TRAINING OF (MATHEMATICS) TEACHERS

Before describing our own approach, it will be useful to set out different types of training that exist. These are as follows:

- Training that emphasises students’ learning. These may be prescriptive (focus on the curriculum, learning activities, lessons, etc.) or didactic (class content, class management, difficult students) in nature.
- Training that emphasises the teacher (and his/her conceptions); in initial teacher training this often involves the use of reflexive practice techniques.

– Training that emphasises the day-to-day realities of the teacher's work, offering guidance on social and institutional constraints, margins for individual initiative and classroom techniques.
– Training 'in action' that takes place as part of collaborative research projects between teachers and academics (Bednarz, 2004).
– Training that concentrates on the acquisition of particular skills selected by the institution (mostly during initial training) or even on precise named tasks (e.g. in France, the *Plan Académique de Formation* [Acadamic Training Plan]).

In our view, this last type of training is the most questionable because it is too limited: it is insufficient to analyse tasks simply in terms of their outcome. Indeed, understanding an activity requires an appreciation of the context in which it takes place. In other words: "To act always means to choose, to prioritize, to diagnose, to assess, to judge, to anticipate, to adapt, to contrast, to construct meanings – all of which are eliminated by a breakdown of the task. Destructuring the task results in the dissolution of its processes, … [leaving] the dregs of the action from which the subject has vanished" (Astier, 2006).

We will not go further into the distinctions between these different approaches, which are, doubtless, to some extent complementary. Nonetheless, it is worth taking a closer look at the third approach mentioned (emphasis on the day-to-day realities of teaching) as it has a special relevance in the light of advancements in educational research over the past few years.[7] Indeed, whereas previously the tendency was to blame the lack of application of pedagogical research in the classroom on the persistence of teachers' own conceptions (Robert & Robinet, 1992), the inadequacy of this argument was later recognised: the cleavage between theory and practice cannot be attributed to people alone but must be seen in the broader context of teachers' overall working conditions. We therefore turned to the 'double approach' theory of professional practice (Robert & Rogalski, 2002), which argues that the analysis of teaching methods only in terms of students' learning outcomes (i.e. the end goal) contributes neither to an understanding of the details of effective practice nor to the creation of new methods. From this point of view, the aim of teacher training is to equip every trainee teacher with means that would allow him or her to attempt to implement "ideal" teaching, etc. A style of training that emphasises the actual practice of teaching is therefore an attractive choice.

But who should be trained first – the teacher or the teacher's trainer?

A TRAINING PROGRAMME FOR THE TRAINERS OF MATHEMATICS TEACHERS

In chapter 2 we said that we have two aims. On the one hand, we strive to supply researchers with information about students' learning, focussing on a specific area of study and bearing in mind the type of teaching that was received and the particulars of the schooling system. On the other hand, and from a more long-term perspective, we hope to assist and develop teacher training strategies, drawing chiefly on conclusions made from the analyses above and the formulation of hypotheses guided by our chosen theoretical framework (see above).

We have chosen to approach the teacher training process from the perspective of one of its chief participants: the trainer him/herself and, in particular, how s/he is trained. Actually, trainers can have very different statuses and some have not even been trained (other than on the job) – this in spite of the fact that their work requires the mastery of a certain number of non-intuitive skills. The least of these is the ability to read teaching publications critically and perhaps resume their content for the trainees; to assimilate research findings and translate or 'transpose' these for teachers to use; to observe their own practice with critical distance; to recognise constraints on teachers and opportunities for initiative; to expand and adapt the professional 'toolkit' proposed to each trainee; to create effective training situations, and so on.

We envisage a training programme that would arm teacher trainers with the tools needed to train secondary school maths teachers both initially and throughout their careers. Such a training programme would centre squarely on the tasks of teachers in and around the classroom and would, where relevant, be compatible with other necessary training components. Within our programme, the trainer is an educational researcher. After all, it is researchers who have the knowledge necessary to construct, run and, ultimately, evaluate such a programme. The training strategy we propose is based on hypotheses developed through the core theoretical framework adopted in this book and on data obtained through empirical studies that demonstrate the complexity of teaching practices and their impact on students' learning. We also integrate and build on a number of analytical tools taken from education theory.

A general framework for 'training the trainers'

The framework for training teacher trainers that we have put together over the last ten years consists of a relatively long period of training (a minimum of several months) and is addressed to experienced teachers whose participation in the programme is voluntary. The training sessions generally begin with an analysis of the teacher's role in class using, for example, video footage; the rest of the session evolves on the basis of participants' contributions, with new themes and ideas being introduced where relevant. In this way, a thorough coverage of a range of issues is achieved over the course of several sessions.

In what follows we will set out the theoretical basis behind the training programme; as we will see, it promotes and is itself structured around three main principles.

Student activities at the heart of the training programme: an interface between teaching and learning[8]

Our study of teaching practices – begun in 1990 – was part of an effort to clarify the relationship between the way a particular topic is taught and the resulting learning outcomes. That is, we sought to gain a better understanding of the impact teachers have on their students' learning. From our particular theoretical angle we measure ‘learning’ in terms of students’ handling of activities relating to the topic taught, even if, admittedly, other relevant criteria could be considered.

The activities analysed are those proposed in class by the teacher him/herself. They are ‘rated’ in terms of the degree of conceptualisation expected, which is determined on the basis of the curriculum and the difficulty of the concepts being tested.

As such, it is the activity of students – mostly in class – that serves as a potential means of ‘decoding’ the teacher’s work inside and outside the classroom. Learning to decode in this way is in fact one of the main goals of the teacher training programme for which we wish to prepare teacher trainers. Indeed, the ultimate goal is to train teachers who will teach mathematics to a very high standard (in both a conceptual and technical sense), notwithstanding, and indeed accommodating, institutional (curriculum, timetabling) and social (class makeup, behaviour, pressure) difficulties. In our view, the achievement of this goal depends chiefly on the teacher’s work inside and outside the classroom. Teaching trainees to analysis this work is thus the principle aim of our proposed training programme. It would be followed by specific guidance relating to the teaching of mathematics.

Indeed, the first of the three principles underpinning the programme is the use of analytical tools from pedagogical research to decode the relationship between a teacher’s work and the activity of students. These tools are explained to the trainees where necessary[9] and are combined and developed where appropriate by means of analysing video footage of teachers at work in the classroom. We believe that the appropriation of such tools will supplement trainers’ existing experience and provide a broad range of training material. Their use will allow trainers to construct training sessions around student activity, which is of key importance in our theoretical framework, engaging the teachers-in-training in a discussion of the possible strategies that could be used in a given situation.

Mainly these tools facilitate the localized study of selected tasks within the confines of each training session. In the course of such study, the trainer activates (and where necessary adapts) the teachers’ existing knowledge on the basis of their spoken contributions. The analytical tools also permit the study of class sequencing and of classroom activities appropriate to each task. Thus the trainees might think about the exercises used by the example teacher, the various help s/he offers, the questions s/he asks, his/her reaction to students’ questions, how s/he handles corrections and how she sums up at the end of class. In a more global sense, the tools given to teacher trainers will allow them to appreciate the overall, carefully planned package of lessons and exercises needed for the teaching of a particular concept, from the initial introductions right through to testing (i.e. concrete classroom scenarios). They will also help the trainers to consider the need for changes in classroom dynamics, such as the presentation of material in new forms, changes in the context of teaching, and variation in the volume and type of learning tasks.

Finally, in order to guide the creation of these in-class teaching scenarios, we introduce a more global analytic approach that would serve to inform ‘strategic’ choices. The idea is to identify what we call the ‘relief’ [*relièf*] of a given mathematical concept. This 'relief' is a function of the concept's specific mathematical features and the difficulties frequently encountered by students when studying it. Thus, for example, might the initial presentation of the concept in question include reference to another, simpler concept with which students are already familiar? Or could the mathematical application of the concept be approached through the lens of a topic already studied in a subject other than mathematics? Lastly, in so far as the teacher's day-to-day work entails, for the teaching of each concept, several factors, including lesson preparation and choice of material, the organisation of students' classwork, and the channelling of students activities, the analytical tools we propose will help trainers both to appreciate and enrich teachers' choices.

All of this helps the trainers and trainees to express themselves precisely; it aids in the construction of an all-important, shared professional vocabulary base inspired by pedagogy research. This vocabulary will continue to expand throughout the training programme.

The necessity of appreciating the teacher's full role in all its complexity

Nonetheless, a number of studies (including several chapters in this book) have shown the inadequacy of considering learning goals alone when addressing the variations and consistencies found in teaching practice. Such studies reveal the extent to which various professional constraints, surmountable or otherwise, such as institutional requirements, student diversity, plus the teacher's own, individual conceptions and routines, impact on teacher's choices. These studies and the theoretical framework they adopt (the double approach theory), have informed the second principle underpinning our programme for training teacher trainers, that is, that the complexity of the teacher's role must be explicitly acknowledged and directly addressed during training. This complexity means that teachers must learn, for example, to anticipate potential in-class difficulties when preparing their lessons and that during the lessons themselves they must be able to improvise according to students' reactions. For this reason, pre-class preparation and in-class implementation are to be approached as a complementary whole and not studied in isolation. Likewise, the reality of professional constraints (the curriculum etc.), the real scope for individual initiative, and the consistency of practice between teachers should be openly discussed, and their key elements should be explained.

The complex nature of teaching will be illustrated by analyses of actual lessons (as mentioned above), mostly in video format. The analyses consist of a systematic confrontation of the work to be done in class – which is discussed at the start of the session – and the related student activities that the teacher can bring into play via his/her explanations and class management. What we have in mind when training teacher trainers is not so much the analysis of the teacher's words, nor the analysis of his/her class management (such as dealing with student diversity, for example), but rather the meeting point between these two types of analysis and all the other, more global issues that they imply.

Training methods that aim at the development of new practices

The third principle behind our training programme is shaped by our hypotheses regarding the development of teaching practices. This influences in particular the methods of training the programme uses. Working with activity theory (which is explained to the trainees), we make very schematic use of a model representing collective intervention in what we might call a 'practice-oriented zone of proximal development.' The idea is to adopt a collective style of working in training sessions that might influence actual practices – whilst also respecting the complexity of those practices – and that remains focussed on teachers' choices of student activities (i.e. the didactic element). At this point our approach is somewhat 'homologic': the training programme we propose for teacher trainers is structured in the same way as might be the training programme that our participants will later provide to trainee teachers. For the trainers, however, numerous supplementary explanations are provided that would be of little use to beginning teachers. The most important aspect of our training method can be summed up by the slogan 'to enrich teaching practices together, we should first consider existing practices.' In other words, we create training sessions that move from analysis at the local level (classroom work and corresponding choices) to the global level (inscription of local choices in larger decisions and projects). This means that, in order to respect the inherent complexity of the future, real-time teaching practices that we ultimately hope to enrich, we must start by bringing the trainees as close as possible during training sessions to the teacher's own position (especially in the classroom), therein encouraging a constructive, non-judgemental understanding (via the analytical tools described above) of the teacher's role. To a certain extent this means working with 'existing knowledge' (i.e. the 'tried and tested' practices of the future trainers or of experienced teachers undergoing mid-career training; the envisioned practices of beginning teachers), which has to be brain-stormed during sessions. Providing that the all-important rules of alternation are respected, beginning teachers will develop their own practices over the course of the programme. Within this framework, special roles are accorded to the collective body of participants and to the trainer him/herself, facilitating permanent toing and froing between the inter-individual and the intra-individual perspectives. The aim is to promote, through discussion, the emergence of questions and the forming of awareness that are prerequisite to the knowledge we wish the trainees to develop. Moreover, this allows the trainer to ensure that the knowledge transmitted is relevant to the needs expressed by the participants. The trainer will thus carefully choose and control his/her interventions during discussions, focussing, when s/he does intervene, on the possibilities for exercising individual initiative within the teaching profession (bearing in mind the constraints). By working in this way, the trainer promotes a move towards questions of a more global nature – that is, towards the decontextualisation that is necessary for the future development and application of the ideas discussed in training. We can now see why the long duration of the training programme is so important: it ensures that a sufficient number of subjects are discussed.

The three principles we have named as the basis for the training programme, and especially the third one regarding methods, require a constant adaptation on the part of the researcher-trainer in order to reduce the gap between his/her own project and knowledge, and the needs expressed by his/her trainees. As such, s/he must both learn from the participants by carefully scrutinizing their contributions during sessions and, at the same time, rework the content of these same group discussions so that the participants can learn from it. In other words, the trainer tries to understand as thoroughly as possible the participants' own perceptions of their needs and their learning potential; in this way s/he can adapt his/her own assessment of their needs accordingly, ensuring that the information and suggestions provided during training sessions are pitched appropriately.

Specific training program for teacher trainers in the use of technology

Several studies have demonstrated the complexity engendered by the use of technological tools in the teaching of mathematics; chapter 8 on the use of online exercise bases (EEB) is an example. This complexity translates as an increase in the complexity of the overall work of the teacher employing technological tools as seen in chapters 9 and 10. By now a large amount of research addressing the integration of technology in the teaching of mathematics has improved understanding of the corresponding teaching practices. Nonetheless, most teachers still receive an inadequate training in the use of technology.

Our previous research on the in-service teacher education in the use of technology has shown that, firstly, the training sessions are usually specific to the trainer's personal teaching methods (as a teacher himself) and, secondly, that there is an excessive focus on the artifact at the expense of cognitive and mediative considerations (Abboud-Blanchard & Emprin, 2009). Furthermore, we showed the absence of knowledge identified by trainers as the main training objective. Finally, little guidance was given in terms of how to integrate the technology into concrete teaching practice.

On the basis of these findings we propose a training programme for teacher trainers that builds awareness of the complexity of teaching with technology and that helps trainers to construct a training strategy that is distinct from their own, personal practices. Such a programme envisages the use of technology to enrich students' learning whilst maintaining a balance with methods in traditional learning environments. Specificities relating to the use of diverse technologies are to be provided as the need arises throughout the training program.

The technological training programme we propose fits into the overall training framework described above; the general approach and the theoretically-informed principles are the same. The programme consists in the creation of a collaborative environment for the training of teachers in the use of technology that is based on movement in two directions:

– A bottom-up movement, whereby teachers share their experiences, practices, visions, and knowledge regarding the integration of ICT.

– A top-down movement, whereby, in accordance with their needs, the trainers-to-be get access to research findings on the subject and are given the chance to consider critically the issues being debated in light of their own teaching experience.

The trainer-researcher has here a dual role. In his/her role of trainer s/he adapts the form and content of the training programme to the existing knowledge of the participating teachers (the trainers-to-be). In his/her role of researcher, s/he not only communicates and explains research findings and analytical tools but also, by observing their use and reception, strives to revise and adapt them. It should be noted, however, that this dual role is flexible and varies according to specific training activities.

Let us now clarify what we mean by 'collaborative environment' and 'collaboration' in this context. It seems evident that collaboration in itself offers no 'automatic' guarantee that the training programme will be successful. Generally speaking, the use of collaborative methods in teacher training brings a number of possible benefits: teamwork, shared goals, different perspectives and a range of professional knowledge that can be applied to the study of teaching practices, and the opening up to new possibilities (Bednarz et al., 2011). In our training programme, 'collaboration' means both the sharing of professional experience (between trainees themselves and between trainees and the trainer-researcher), and the sharing of theoretical and empirical experience amongst all the members of the group. With the help of the trainer, the aim is to clarify existing views and to tease out ideas that are still in their early stages. In this way, each group member collaborates by sharing knowledge and experience from his/her own practice, the aim being the co-construction of new knowledge that is both theoretically aware and rooted in teaching practice.

To begin with, the training programme is divided into modules, each one dealing with a different technological tool that is familiar at least to some of the participants. Each module consists of three stages. In the first stage, a group of trainees discuss their own uses of the tool in class and provide an overview of relevant online resources. In the second stage, the participants consider useful findings and relevant analytical tools drawn from educational research; this information is provided by both the trainer-researcher and the trainees themselves via selected theoretical readings. In the third stage there is a group discussion of the principle issues that need to be raised and the specific aspects that need to be emphasised when training teachers in the use of the technological tool in question. In terms of the training methods employed, selected key activities are introduced at appropriate stages in the training. This includes the analysis of video extracts of teachers using technological tools in class. These videos provide a shared experience around which activities – such as discussions, theoretical and professional readings, and the writing of reports giving the participants' ideas on appropriate use of the technology – can be organised. The reviewing of online resources is on-going throughout the programme. This is because, firstly, online resources are currently the principal source of material for teachers wanting to use new technology in the classroom and, as such, they cannot be overlooked during training. Identifying the right resource for use in a particular class and for teaching a specific mathematical concept is in itself a complex and time-consuming activity for teachers. Moreover, many resources are of limited use to teachers who did not participate to their design because of a lack of information regarding their specific benefits for mathematics learning and their insufficient indications about key features of their use in class. We can take the example of a teacher wishing to use dynamic geometry software. He or she must first of all choose (if the choice has not already been made by the school team) an appropriate software package amongst those available on the market. The choice might be made in terms of product accessibility (open access or otherwise) or in terms of the ergonomic and/or didactic properties of a particular product. The teacher must then learn him/herself and teach the students how to use the software. Finally, s/he must single out amongst the range of possible activities offered those that are the most useful and relevant for his/her teaching agenda. In doing so, s/he must be mindful of an appropriate balance between the time invested in preparations and setting up of software, and the amount of effective learning time that is achieved. The second reason why online resources are studied throughout the training programme is because the range of such resources is vast (ready-to-use lessons, online exercise databases, study plans for a particular topic, etc.), as are their sources (educational websites, resources constructed by groups of teachers or individuals, etc.) and their format (straight-forward student activities using an technological tool, lesson plans, with or without instructions, etc.). The study of such resources allows trainees to understand what is available and to identify the relative advantages and disadvantages that make them effective or ineffective as teaching aids.

By means of the various activities mentioned, we hope to achieve two principal results. Firstly, the future teacher trainers will develop a strong awareness of both the basic and more intricate particularities of teaching with technology. These particularities can be viewed from several perspectives: the student's perspective (e.g. an increase in the variety of class activities; the risk that the computer-based activity reduces mathematical reflection); the teacher's perspective (e.g. greater than usual class preparation and planning; changes in the nature and timing of assistance offered to students or to the whole class); the perspective of the subject matter itself (e.g. provision of new perspectives on geometrical figures with dynamic geometry software or spread sheets; provision of multiple approaches to concepts thanks to the ease and frequency of changes in presentation); or, finally, from the perspective of the special conditions and material constraints that come into play when conducting lessons based around technology. The raising of awareness of such issues among future teacher trainers will encourage them to develop teacher training programmes with a greater focus on didactic aspects rather than on the technology itself. Moreover, it will lead to the construction of a shared professional vocabulary that will facilitate the description of situations, experienced or observed, in less personal, more objective terms that will increase the usefulness of such descriptions for other teachers, especially those at the start of their careers. The overall aim of the training is thus the progressive guiding of trainees towards the ability to identify, as a group, the issues surrounding the use of

particular types of technology that would require special mention and/or discussion when training teachers. Such issues are various and might include:
- material conditions and constraints such as the types of software available at school, the technical features of resources, the type of classroom available (computer room, ordinary classroom with projecting equipment, an interactive whiteboard, etc.)
- differences between traditional and technological working environments and their implications for student and teacher activity
- the gains and the limitations in terms of students' learning progress in mathematics and the range of tasks that could be set to students (these should be customised to particular mathematical concepts and particular types of software, e.g. arithmetic-algebraic articulation for spread sheets; experiential testing when making conjectures and finding proofs with dynamic geometry, etc.)
- the nature of the teacher's activity in class, the kinds of interaction used (see, for example, chapters 8 et 9 of this book)

We now turn to the second result we hope to achieve via the methods used in the training programme. This builds on the identification of pertinent teaching issues just mentioned. The trainees, working in groups, will 'embody' their new insights by designing a resource that could be used during teacher training; they will then present it and discuss it with the other trainees. The resource is based on an activity that was really used in the class(es) of the designer(s). We hypothesize that creating a resource based on the real experience of the teacher in question, together with the writing up and collective discussion of a report that explains the creation of that resource, will encourage reflection and result in an improved understanding of the more implicit issues encountered when designing the resource. We thus see the creation of the resource not so much as the goal of the exercise in itself[10] but rather as a useful training method. Furthermore, the exercise heightens awareness of the complexity of teaching with technology and increases the participants' understanding of the process by which a teacher becomes a teacher trainer. And finally, the thinking through and writing up the written report entails a detailed analysis of the tasks to be used and an understanding of the impact of teacher-student interaction on the latter's learning. It thus helps to develop a metacognitive approach to the potential benefits of technology in the teaching of mathematics.

Discussion

In our training programme, the trainees are practising teachers. The question therefore arises whether the training process is not twofold: does it develop professionally a teacher, a trainer, or both? Moreover, are there differences between the two processes of training a teacher and training a teacher trainer and, if so, what are they?

Comparing the training of teachers and of teacher trainers, Llinares and Krainer (2006) identify the main difference as lying with the different constraints encountered by the two groups in the exercise of their profession. Whereas teachers face serious institutional constraints linked to the curriculum, the different class levels and the specific topics they have to teach, trainers have the freedom to choose the content of their programme and its methods. Trainers are also free to make their own choices regarding the means by which they develop their professional skills.

We agree with these observations. We also believe that both teachers and trainers learn via the 'constructive activity' of their professions (see chapter 1). That is, their productive activity generates knowledge linked to the practice of their profession, and this knowledge is itself a source of training. When a teacher is training to become a teacher trainer, there is a meeting of the different skills and knowledge required by each. We might therefore ask ourselves the following questions: what connections will take place between these different types of knowledge? Will some play a more important role than others? Might different types of shared knowledge emerge? If so, how, as researchers, can we detect this emergence, and how, as trainers, can we promote it?

In terms of the current theoretical perspective of mathematics pedagogy, the same frameworks are used to study the two types of practice (that of the teacher and that of the trainer) and the corresponding training. Given what we have said above, is this approach satisfactory?

In fact, if we turn to the question of how to evaluate and compare training programmes, we are faced with a complex situation composed of four different levels: that of the specific training given; that of the trainers themselves in all their diversity; that of the practice of the teachers trained; and that of the 'corresponding' learning outcomes of students. This gives us an appreciation of the enormity of the subject and of the need to devise theoretical and practical working solutions.

NOTES

1. They are simply not aware of such difficulties. Thus, for example, Lenfant's (2002) study shows that many beginning teachers teaching basic algebra fail to draw on arithmetic solutions, even though these are often more accessible to students.
2. This can sometimes be seen to a lesser extent amongst other teachers, notably those working in the so-called 'priority education zones' (ZEP) [socially disadvantaged districts receiving special investment in education].
3. This has been confirmed by a study of blackboard usage (Vandebrouck, 2002). Analogous findings have been found in studies of primary education (Maurice & Allègre, 2002).
4. At present time this is usually the case in France.
5. Timetable restrictions are always evoked when discussing this issue.
6. The introduction of the *socle commun* [the minimum knowledge base that all students are expected to acquire on completing compulsory schooling in France] has led to even more questions regarding the curriculum.
7. A study of the interaction between these different forms of training remains to be done.
8. This is a quotation from a paper ('Les didactiques en questions – État des lieux et perspectives pour la recherche et la formation') published in the proceedings of a conference held at the University of Cergy-Pontoise in 2011.
9. In practical teaching, such tools are not used but transposed; that is they are applied to tasks other than those for which they were initially conceived.

[10] As it is the case with, for example, the pairform@nce training programme (http://national.pairformance.education.fr/).

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