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10. THE WORK OF SEEING MATHEMATICALLY[1]

The reigning epistemologies in mathematics education take (visual) perception as an unproblematic phenomenon, assuming that students see (i.e., perceive and understand) the curriculum materials presented in the way that a knowing mathematics teacher/educator intends them to be seen (i.e., perceived and understood). Moreover, all current epistemologies – including not only all forms of constructivism but also all embodiment and enactivist theories – miss a fundamental contradiction: Students cannot see and therefore intend the object of learning precisely because they do not yet know it and therefore are asked to learn it. There is therefore an essentially passive dimension in knowing and learning that current epistemologies do not theorize: If I cannot intend the mathematical learning object, it somehow has to be given to me, or reveal itself to me, so that I come to see and understand (Roth & Radford, 2011). This leads to inherent contradictions that the existing epistemologies cannot overcome; attempts on the part of (radical) constructivists to overcome this phenomenon, which has come to be known as the *learning paradox* simply reiterate the position of the individual as the source of all knowing.[2]

In this chapter, I present an investigation of visual perception concerning real-world and ideal-mathematical objects. I show that new, previously unknown objects are not simply seen (intentionally) but that they are given to the subject of mathematical activity as the movements of the eyes are shaped by structures in the world. I show how mathematical seeing (perceiving and understanding) is grounded in immanent – i.e., immediate and unmediated – processes, which constitute initially immanent forms of knowing that are not subject to sign mediation. Moreover, the objects of experience that the children of today encounter, much as the ancient Greek encountered them, *are not* the mathematical objects that they subsequently learn (learned) about. These objects will be geometrical and ideal, whereas their sensuous experiences that children encounter initially through the senses of touch and vision inherently are real objects that only in the limit (of engineering) approximate the ideal objects that they denote. I offer a radical re-theorizing of mathematics along the lines of my recent work on mathematical cognition.[3]

Throughout this chapter, I insist on the difference between the lived experience of mathematically seeing and the *accounts* of experience of seeing in mathematics that societal actors – children, teachers, or lay and professional mathematicians – provide when asked about what they see. Almost all research, both quantitative and

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 227–245.

qualitative, is concerned with *accounts* of experiences of mathematical seeing rather than with the living/lived work of mathematical seeing.[4] I articulate the difference between the two and provide some guidance with respect to the ways of going about researching the lived work rather than accounts thereof. In this, I counter the false belief that our perceptual experiences are "constructed," and I insist that the real work (doing, seeing) that makes mathematics an objective science is actually lived and the result of our living/lived, sensuous bodies rather than that of the constructivist mind.[5] In this manner, I articulate and elaborate an approach that is an incommensurable, asymmetrical alternate approach to *formal (including constructivist) analyses* of living/lived experiences of mathematical seeing.

THE LIVING/LIVED WORK OF SEEING MATHEMATICALLY

I begin this investigation with two practical inquiries, which, when readers engage with these, allow them to *live* the experiences of mathematically *seeing* a geometrical object as a specific object (cube) and *doing/seeing* the proof of the angle sum of a triangle. The *immanent* aspects of these living and lived experiences are radically different from accounts of experiences that are articulated in one or the other way for someone else or for the person himself/herself.

Case 1: What Makes a Cube a Cube?

To start our inquiry into the difference between the lived work of seeing mathematically – the living/lived, sensuous experience of mathematical seeing – and an account of (the experience of) mathematical seeing, consider the drawing in Figure 10.1. What do you see? Take a moment to look at the figure and find an answer before you proceed reading.

The figure is known in psychological research as the Necker Cube. Although there are but a few black lines on a two-dimensional sheet of paper of white color, most research participants report something like "I see a (three-dimensional) cube," "I see a cube from below that extends from front right to back left," or "I see a cube from the top that extends from the front left to the back right." When asked further, participants may outline – by moving their fingers along certain lines salient in their perception – where they see the different surfaces of the particular cube they see. In their statements – which may be provided verbally alone or communicated using a range of semiotic resources – they provide *accounts* or reports of experience. What they have not provided access to is the actual, lived work of seeing that is obliquely referred to in their accounts/reports.

Qualitative researchers, including researchers employing phenomenography, tend to be interested in reporting all the different things that research participants have reported seeing, which, in addition to a cube, may simply be a set of lines, or an assembly of several flat geometrical figures, and so on. Constructivist mathematics educators may be tempted to say that these participants "constructed" the particular cube or cubes that they see. In fact, should it not be strange that

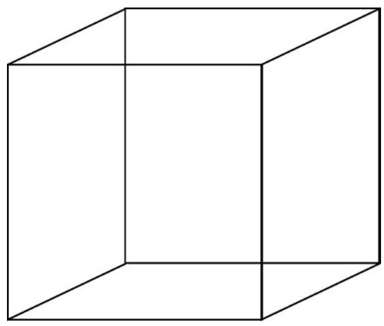

Fig. 10.1 This diagram has been used in psychological research on perception and is known as the Necker Cube

participants report seeing this or that cube given that there are only lines on a flat page? How is it possible to see something three-dimensional when there are only two dimensions? Both sets of research reports are limited, as they do not get us any closer to the real question of the lived work (experience) that is denoted in the reports/accounts that provide us with the structures that people exhibit to one another. So what more is there? Related to this question we may distinguish between *formal analysis* and *ethnomethodology* (Garfinkel, 1996). The former approaches to research report structures, here perceptual structures, whereas the latter is concerned with the living work that brings the structures about, here the perceptual structures. Ethnomethodology, as its descriptive name suggests, is concerned with the methods by means of which people (Gr. *ethnos*) produce and exhibit to each other the structures of social action, whereas formal analysis, generally having to specify particular research methods, is concerned with the identification of the structures. Phenomenological studies, too, are concerned with the conditions that produce this or that sensual experience rather than with the phenomena as they are given to us in our senses.

So what is the lived perceptual work underlying the report of seeing this or that cube? The drawing (Fig. 10.1) allows us to investigate perception and how we come to see what we see. Upon first sight, you may see a cube, if you see a cube at all, from slightly above extending from the front left to the back and right (Appendix, Fig. A1a).[6] But, if you see a cube, you might actually see one from below and extending from front right to the left back (Fig. A1b). These two perceptions are the two spatial configurations that participants report seeing in psychological experiments, where these perceptions are categorized as "cognitive illusions." Rather than wondering about illusions, let us engage in the analysis of the living/lived work of perception to find out what is at the origin of the perception of the cube in one or the other way (i.e., from below or from above). We may do so, for example, by exploring how to quickly switch back and forth from the cube seen slightly from above to the other one seen from below.

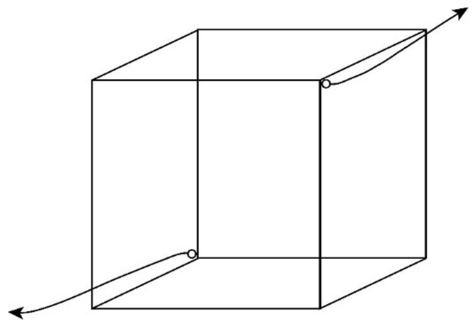

Fig. 10.2 Placing the gaze at one of the two vertices and following the trajectory toward a vanishing point *gives* rise to one or the other cube in sensuous experience

To begin with, look at the figure (Fig. 10.1) and allow the first cube to appear, for example, the one that you see from below and extending into the back toward the left, and then intend seeing the other one until you see it. Move back to see the first; return to the second. You might also do this: look at the first cube, the one seen from the bottom and extending toward the back and left. Close your eyes – but intend to see the other cube upon opening the eyes again. Practice until you can switch between the two in the rapid flicker of the eyelids. Once you achieve this, observe what is happing with your eyes during the flicker. That is, how can you generate *this* or *that* experience voluntarily and intentionally?

You may notice that if you place your eyes to the lower left corner that appears inside the set of lines and then move toward a non-present vanishing point to the left ("along the surface") – this may be along the edge leading from the "front" vertex toward the back left – then the cube-seen-from-below becomes instantly apparent (Fig. 10.2). Similarly, focusing on the equivalent vertex further up and right and then moving along the edge "backward" to a non-existing vanishing point allows you to see a cube-from-above (Fig. 10.2). That is, unbeknownst to your intellectual consciousness, the *movement* of the eye from one of the two vertices toward a non-existing vanishing point in the back to the left or right of the diagram creates one or the other perceptual experience. This, therefore, is a statement about how the work of seeing produces the cube even if we do not attend to it. If the eyes do not make these movements, then the cubes do not appear and the lines remain on a flat surface. Most importantly, therefore, this experiment shows us that the cube is not (intentionally) constructed because when you looked at the figure for the first time, the cubes appeared, you did not intentionally construct it. And for the very first time you looked at the figure, you might have not seen any cube at all or only one and not the other. That this is so can be accentuated by taking a common form of puzzle, where a person initially sees nothing but splotches (Fig. 10.3). Most people have to gaze at the image for a while and then see some figure; turning the page around, the same figure can be seen again.[7] Why do we see these figures? They arise from the movements of the eyes, which we initially *cannot*

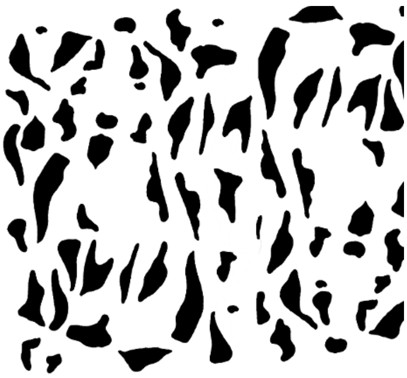

Fig. 10.3 The forms that come to be seen are not "constructions" but arise from the invisible via the ground of the image

intend because the figures are invisible. The initially arbitrary movements are shaped by the material image, making the image eventually appear. But once you have seen what there is to be seen, you can find it again, initially perhaps with some difficulty, but eventually at will. At this time, the required eye movements know themselves, they constitute "kinetic melodies" that occur again such that you may again see what there is to see. We do not know where the originary movements come from, much as we do not know where mathematical insights come from.[8] They are more or less haphazard but come to be honed in repeated execution.

How do the eyes know to move like this to make the cube appear? The answer extends the possibility of this text, but I have worked out a possible response based on the phenomenology of the flesh. Briefly stated, this knowing emerges from first uncoordinated movements during which the corporeal movement (of the eye) auto-affects itself such that it develops the capacity to move and develops an immanent memory of this capacity. In other words, during first arbitrary, random movements, corporeal-kinetic movement forms (archetypes) emerge that would be more ancient, more basic than any "image schemas" or "sensorimotor schemas," if they exist at all. Nothing is constructed at that point because there are no tools available for the construction; in fact, this capacity, the self-knowledge producing the movement precedes any intentional movement, any intention to act, and any intentional thought. Before I can *intentionally* move the eyes, these have to *immanently* know that they can move.

It is clear in the preceding account of the perceptual work that different movements of the eyes underlie the different visuo-sensual experiences; that is, if there is a different movement, a different experience is produced. If our living flesh does not produce some movement, this form of experience is not available to the person. The source of the movements underlying our disciplinary visions and *di*visions can be located in the *habitus*, sets of structured structuring dispositions that we cannot ever access but the results of which express themselves in praxis. There is a dialectical process at work, because habitus is shaped by the social and material field that it inhabits, but habitus itself allows the social and material field to appear in specific ways. Ultimately, this mutual dependence leads to the fact that habitus and field are homologous.

We can push this analysis further – but this is difficult and requires considerable practice. The question we attempt to answer is this: How do we see one and the same cube over an extended time? Or, equivalently, is the eye movement from the vertex to the corresponding vanishing point necessary for us to see a cube? To reach an answer, fixate, for example, the lower vertex. Or, equivalently, attempt to have both cubes appear at the same time. You may not be able to achieve this feat on your first few attempts – psychologists generally use equipment that allows them to fix an image to a specific location on the retina. But as soon as you achieve this feat – that is, as soon as your eye is fixed so that the parts of the image continuously fall onto the same equivalent spots on the retina – you notice that the figure dissolves completely and you do not experience anything except a dark grey perceptual field. This graying tends to start at the periphery of your perceptual field and move inward. You no longer see lines. That is, as soon as the eye no longer moves, you cannot see the lines and even less a cube. To see a line or cube, the eye needs to move back and forth between the cube and some other place that constitutes the ground against which the cube appears as the figure. The eye does work to produce the sensuous experience that you have. Once the movements exist, we may speak of the construction of the cube, as the eye now knows how to move to bring forth the cube. *But initially*, the eye was not in the position to construct anything because it did not know how to move or that some movement would produce anything. In one sense, the cube is a cube because the eye finds it again upon moving away, and to generate the cube, my eye has to move from the vertex to its corresponding vanishing point.

The upshot of this investigation is this: We do not just see or recognize a cube because of some mirror image that is produced on the retina. Rather, our eyes have to do work, and associated with this work there are changes on the retina. Based on the changing images, and based on prior experience, we have learned to see cubes. This is not the outcome of a conscious construction, but rather, it is the result of a shaping of our eyes' movements as they follow the structures that they find in the world. The visible *is given to* the person, a recognition that is widely shared among artists and philosophers. Artists *find* what there is to see after having finished the painting rather than communicating what is already visible to them in and through the painting. We can see cubes because our eyes know what they have to do to make a cube appear. It is in the non-perceived movement of the eye that the distension and dehiscence between the cubical figure and the ground occurs and that the former comes to detach itself from the latter. But we should not think of the image as something standing before the ground, as if projected against a screen; rather, in the image the ground is rising to us. It is not merely, as enactivist theorists would say, that the organism is bringing forth a world – *initially* the world gives itself to the organism, which learns how to make any figure reappear, at

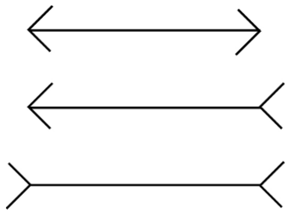

Fig. 10.4 The Müller-Lyer "illusion" makes line segments of equal length to be of different length

which point we might describe the process as a bringing forth. That is, the movements of the eyes are not random, not constructed, but they are entrained by the structures of the material world in which the organism is embedded – movement patterns and the structures of the world are homologous, as I note above. The eyes *follow* lines and thereby are entrained into certain movement patterns that are not their own but that arise from the structures in the world. This then leads to the fact that "it is in reference to my flesh that I apprehend the objects in the world" so that "in my desiring perception I discover something like a flesh of objects" (Sartre, 1943, p. 432). It is in reference to my flesh that I apprehend the objects of the world, which means "that I make myself passive in relation to them and that they are revealed to me from the point of view of this passivity, in it and through it" (p. 432). There is therefore a fundamentally passive component to perception that tends to be obliterated in the (social, radical) constructivist literature but that is essential to understand the dual, subjective/objective nature of mathematics that has become the point of unresolved contention between formal and constructivist accounts of mathematics.

We can enact further phenomenological investigations relevant to geometry by, for example, investigating the conditions for seeing an angle or seeing two lines as equal or unequal. Thus, in *Geometry as Objective Science in Elementary Classrooms* (Roth, 2011a), I exhibit how the movements of the eyes make us see two line segments of demonstrably equal length appear to have different lengths (Fig. 10.4). The Müller-Lyer illusion is produced as the eye follows the inward and outward pointing arrows at their ends in a different way. At its very heart the phenomenon is based on the same movement processes that allow us to see a drawing as a cube. Thus, such a perception of equality of lengths important to perception in geometry is explained by the movements of the eyes in the context of particular configurations of lines. This illusion is sustained even when we have measured the two lines and therefore know that the two lines are of equal length. That is, we are passive with respect to our perception even when "we know better." There is therefore nothing constructive about the originary experience, it is happening to us. We come to see what we see because of the movement of the eyes, movements that our eyes, as an aspect of our living/lived bodily selves, are given as originary, archetypal corporeal-kinetic forms.

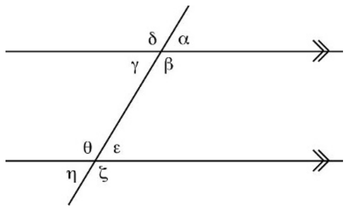

Fig. 10.5 The angles produced when a line crosses two parallel (») lines

We can sum up this first part of our investigation by saying that there are two parts to perception: (a) the account or gloss of what is mathematically seen and (b) the living/lived work of mathematical seeing that underlies the account. Qualitative research generally and phenomenographically oriented qualitative research specifically investigate and report on these accounts; this kind of research presents us with the structures that either the participants or the researchers report. It is our phenomenological analysis that actually leads us to an understanding the living/lived work that produces the different experiences that people report.

Case 2: Mathematical Seeing while Proving the Angle Sum of a Triangle

In the following description of mathematical practices, using proving as an example, I follow the kind of studies produced in the field of ethnomethodology of mathematics. This work is concerned with the irreducible relation of living/lived work and accounts of this work. These descriptions are consistent with the phenomenological studies of the foundation of mathematics (geometry), which recognize the co-presence of lived (subjective) and formal (objective) dimensions of mathematics. Accordingly, there are records and accounts of mathematical proofs, on the one hand, and the living/lived labor of doing a proof, on the other hand.

The Proof Account The proof that the internal angle sum of a triangle is 180° involves a drawing (Fig. 10.5) and the following. In a first step, we note the relationships between angles that are produced when a line crosses two parallel lines (marked by the sign "»").

– The pairs (α, ε), (β, ζ), (η, γ), and (θ, δ) are known as corresponding angles; corresponding angles are equal (i.e., $\alpha = \varepsilon$, etc.).
– The pairs (α, γ), (β, δ), (ε, η), and (ζ, θ) are known as vertically opposite angles; vertically opposite angles are equal (i.e., $\alpha = \gamma$, etc.).
– The pairs (ε, γ) and (θ, β) are alternate angles. Alternate angles are equal (i.e., $\varepsilon = \gamma$) – because of (a) and (b).

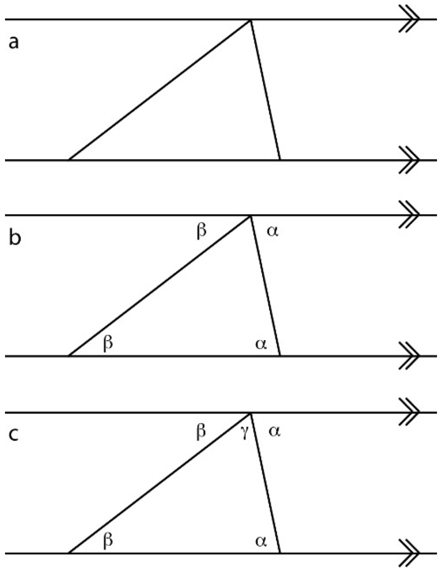

Fig. 10.6 Steps in and part of the account for the proof that the interior angle sum of a triangle is 180°

With these identities in place, we can prove that in the Euclidean plane, the angle sum in a triangle is 180° – if the total angle around a point is defined as 360°. This proof includes the following steps together with three diagrams (Fig. 10.6).

- Any triangle can be drawn such that the base lies on one of two parallel lines and the opposing vertex on the other (Fig. 10.6a).
- We know that alternate angles are equal, as marked (Fig. 10.6b).
- Hence, because of configuration of lines at the upper parallel, that α, β and γ add up to 180°, that is, $\alpha + \beta + \gamma = 180°$. Therefore three angles in a triangle add up to 180°.

The preceding steps and figures do not constitute the entirety of the proof; rather, they constitute what we know to be the proof account. These are the parts that one might find in a textbook on geometry, on a website, or, in the case of new mathematical discoveries, in relevant journals. This is the part, therefore, that allows us to re-do the proof over and over again, which certainly has been done so since antiquity, when the proof was done for a first time. For example, the reviewers of an article take the submitted proof as instructions for doing the proof, checking whether there are "no holes" in the proof procedure. When they get the same result, their own subjective work has reproduced the objective account. The proof becomes a fact. In written form, this account suffices to be able to hand the proof procedure down – initially, to share it with others in the prover's community.

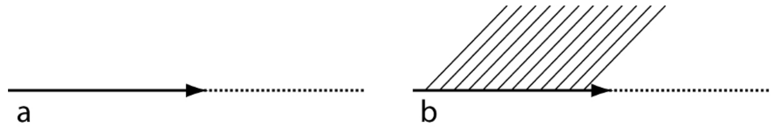

Fig. 10.7 In the dynamic of drawing a line, the plane becomes bisected, here denoted by a hatched and an unhatched part

Ordinarily, newcomers to a discipline learn these practices in face-to-face work with others who monitor and give feedback to correct actions; but the written accounts are such that they allow others to re-discover the proof in their own praxis.[9] That is, as initially arbitrary and tentative actions are marked as subject to correction, the student tries again. Once such actions receive approval, then the immanent generating mechanism, the self-affected movement, can now or after some trials reproduce the action intentionally. This possibility for the rediscovery of the proof in fact constitutes the objective and tradable nature of geometry as objective science. Thus, "the important function of *writing* is to enable the *continual objectivity* of ideal sense entities in the curious form of virtuality" (Husserl, 1939, p. 212). The ideal (subjective) objects exist virtually in the world in written form, and they therefore can be actually produced at any time. The lived praxis (labor) within which this written account *counts* as the proof, however, is not contained in the written account. *It is precisely this lived work that we are interested in here and in ways of capturing it*. We already see some of what is involved in the preceding inquiry concerning what makes a cube a cube. To bring this proof to life we actually need to do it in and as of living/lived labor for which the written record has to provide sufficient resources.

The Living/Lived Work of Mathematical Seeing in Proving I am interested in the living/lived work within which such accounts constitute the resources that allows us to count what is happening as a proof. Part of the kind of work involved is articulated in the first subsection, that is, the lived work of seeing. In the present instance, for example, this living/lived work includes the re/cognition that pairs of corresponding, opposite, or alternate angles are equal. That these pairs of angles are equal presupposes the seeing of each angle – where the work of *seeing* is described above. Such *seeing* is related to the living/lived work of drawing multiple lines, each of which bisects the plane (Fig. 10.7). This work involves particular movements, kinesthetic structures or kinetic trajectories, which are inscribed in the living/lived body (the flesh) where it constitutes an immanent form of knowing. From the perspective of the living/lived work, the writing gesture produces the divisions of our pre-geometrical perceptual experience of left/right, up/down, and so on. Even if the movements initially are arbitrary and random, they constitute traces that mark differences in space, and thereby shape the perceptual experiences that follow.

When, after the completion of the first line (involving a complete bisection of the plane), a second line is added, it, too, bisects the plane. Four sectors are thereby

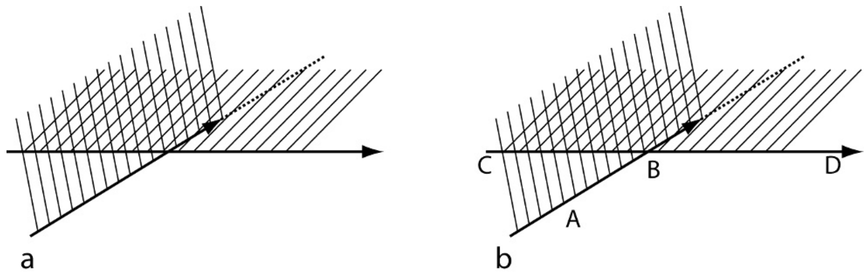

Fig. 10.8 Two intersecting lines produce four sectors

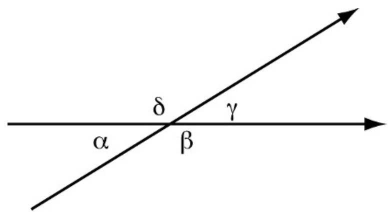

Fig. 10.9 The placement of the labels a and b is apparently disengaged from the temporal practice of drawing the figure

produced, which appear in three different hatchings: not hatched, once-hatched, and twice-hatched (Fig. 10.8a).

I could have also drawn the second line in the reverse and produced the same account. For this reason, the angles enclosing the single-hatched areas are the same. What is in the first drawing the angle forming first to the left and then to the right will be, upon beginning the diagram from the other side, again first to the left and then to the right. In this very act of drawing, we also produce an order that goes with the naming of locations (Fig. 10.8b). In this way, the unfolding from drawing the AB line with respect to CD forms angles ABC and ABD, which we may also name, following the tradition, by the Greek letters α and β (as well as the equivalent angles γ and δ) (Fig. 10.9). Here, the order in the actual making constitutes a conceptual order: "The temporally placed label of an angle or its apparently disengaged placement in a finished figure exhibits this seen relationship as a proof-specific relevance" (Livingston, 1987, p. 96). The conceptual order is *in* and *arises from* the movement rather than from the constructive mind, if there indeed should exist something of that kind. Mind and sensorimotor schema are *postkinetic*, as are all accounts of mathematical experience.

The relationships between the lines, angles, bisectors, and sectors have to be seen; this seeing, as shown above, is based on the movements of the eyes, movements that we are not in conscious control of. Not surprisingly, phenomenological philosophers have recognized the fundamental passivity that is associated with a first cognition that such seeing involves. Any first formation of sense therefore has two passive moments: the first existing in the first cognition and the second in the fact of the retention of this first cognition. Thus, "the passivity of the initially darkly awakened (insight) and the eventually increasing clarity of that which appears is accompanied by the possibility of a change in the activity of a *remembrance*, in which the past experience is lived again actively and quasi anew" (Husserl, 1939, p. 211). The memory is awakened passively but can be transformed back into corresponding activity. The *re*cognized relationship may therefore be maintained throughout the proof procedure, which leaves as its end result a sequence of diagrams (Figs. 10.5, 10.6). In the drawing, we do not specify a particular angle to be produced. Any work that produces two, non-parallel lines suffices to get us to this point. This fact produces the generality of the proof procedure.

This immemorial, subjective memory is important in the constitution of geometry as an *objective* science in and through the subjective, living/lived, sensuous work of the geometer. A sense-forming act that came about spontaneously can be actively/passively remembered, and therefore reproduced not only by the original individual but by any other individual as well. It is in the reproduction of the living/lived work that the evidence of the identity between original and subsequent act arises: "That which now is originally reconstituted is the same as what was evident before" (Husserl, 1939, p. 211). That is, together with the original sense formation comes the possibility of an arbitrary number of repetitions that are identical in the chain of repetitions. That is, the very subjective, living/lived work of doing and seeing geometry that allows me to recognize relationships again make for the social nature of geometry and its historicity as objective science.

Interestingly, the very generality of the proof derives from the way in which the sensuous work generally and the sensuous work of seeing specifically unfolds. For example, in the drawing of a line that crosses two parallel lines and labeling alternate angles using the same letter, the proof makes available that any such line could have been drawn, which in fact occurs when the second line between the two parallels is drawn such as to form a triangle. The very possibility to have one line between parallel lines with alternate angles enables all other lines. The relations between the angles in configurations of parallel lines crossed by a third thereby imply the angle sum of the triangle to be 180°. From the way in which living/lived work draws parallel lines and sees the equivalent angles that follow from (the idea of) parallelism simultaneously constitute the angle sum to be 180°. That this is so can be discovered over and over again because (necessarily written) proof-accounts describe, like a recipe, their own work. It is precisely "in this particularistic way, the generality of our proof-account's description was evinced in and as the lived, seen, material details of the proof" (Livingston, 1987, p. 108). The very nature of geometry as objectivity science arises from the demonstrability and visibility of its procedures in the living/lived (subjective) work of proving, including the living/lived work of mathematical seeing. Anyone may reproduce the living/lived work anywhere. In sum, therefore, we realize that the "generality of our proof both is in and not in the proof-account; it is in that proof-account through the pairing of that account with its lived-work" (p. 108).

In this brief description, we can see how the living/lived work of producing, seeing, and labeling the angles is actually accomplished. This drawing, seeing, and labeling is available to those present; this drawing, seeing, and labeling makes the work objectively available to those present. But this sensuous work does not (and cannot) appear in the proof account proper, where the lines and labels appear disengaged from the actual movements of drawing, seeing, and labeling. All of these involve our living/lived, sensuous body in the manner described in the first section above for the eyes' work that makes a cube from a set of lines. Seeing an angle involves fewer lines, but nevertheless requires the movement of the eye that puts into relation the two unfolding lines, the half planes, and the seeing of the intersecting planes against the background (generally white). Even imagining an angle or a line in our minds or recognizing someone else drawing an angle or a line *requires* the activation of the same immanent movements in us that operate when we actually see or draw a line. This fact has been recognized over 200 years ago through phenomenological analysis and has been recently substantiated by neuroscientific studies on the function of mirror neurons. The account, as we might find in textbooks, is disengaged from this living/lived work, but it may serve as a resource on the part of the learner, as an instruction for reliving the sensuous work of proving in and through his/her own living praxis of drawing, seeing, and labeling. The relation between accounts and the lived work can be stated in this way (Husserl, 1939): In textbooks the actual production of the primal geometrical idealities is surreptitiously substituted by means of drawn figures that render concepts visual-sensibly intuitable. It is up to the students to find in their own subjective sensuous work the practical relevance of the instruction, which in the present example would be the proof-specific relevance of the lines, markings, naming, and so forth.

We can see that in this pairing of proof account and lived, sensuous work of proving there is the possibility of a pedagogy. In fact it has been said that the proof account is "completely and hopelessly a pedagogic object – it teaches the lived-work that it itself described" (Livingston, 1987, p. 104). This is so because we can see in it a *formulation* of the work that is described, much like an instruction that presents both what is to be done and what will be found as an outcome of the actions. However, this condition still does not solve the ultimate problem of the difference between the account and the lived work: the students have to find in their own living/lived corporeal actions the relevance of this or that definition, this or that instruction, this or that description of an outcome. There is a surplus in the transitivity of the living/lived action over its ideation that constitutes the difference between living/lived work and any account thereof.

In this section, I articulate but the beginning of an analysis that indicates the nature of the lived work as distinct from the objective accounts produced and handed down for millennia from the ancient Greek to the present day. The *accounts* of mathematical seeing and doing – though not the subjective work of perceiving – are objectively available to all the generations; the lived (subjective) work of mathematical seeing and doing has to be enacted each and every time by the person actually doing or following (observing) the proof. In this way, the subjective enactment of geometrical seeing and the objectively available account have to be intertwined to make geometry the objective historical science that it is. The sensuous work has nothing to do with a mental construction, as the movements underlying the (intentional) drawing of a line emerge from experiences that have nothing at all to do with intentions. These are originary movements that have nothing do with the "(embodied) image schematas" of cognitive science and embodiment/enactivist accounts but may be thought of as *archetypal corporeal-kinetic* forms or as kinetic melodies that would enable any such schemata, if they were to exist at all.

OF PERCEPTUAL WORK AND ACCOUNTS OF PERCEPTION

In a text on the formal structures of practical action, Garfinkel and Sacks (1986) propose a way of theorizing the ways in which accounts of structures and the generally invisible work that brings these structures about are related (Fig. 10.10). Thus, the expression "doing [proving the sum of the internal angles of a triangle is 180°]" consists of two parts. The text between brackets "[]" topicalizes a particular practice that social scientists and educational researchers might be interested in; the text is a gloss of what a researcher or lay participant might say that is happening. For example, observing a student, a teacher might explain to the researcher visiting the classroom that the former is "proving the sum of the internal angles of a triangle is 180°." This text is the *account* for what is currently happening. Similarly, if asked by the researcher what she has been doing, the student might gloss, "I was proving that the sum of the internal angles of a triangle is 180°." Almost all research in the social sciences and education is of this kind; ethnomethodologists refer to this kind of research as *formal analysis*. Research methods are provided in articles to articulate how the researchers arrived at identifying the structures that appear between the gloss marks (i.e., between "[" and "]"). But formal analysis does not capture the first part of the expression: it misses the "doing." This moment of the expression allows us to ask a pertinent research question, paraphrasing Garfinkel and Sacks: "What is the work for which 'proving the sum of the internal angles of a triangle is 180°' is that work's accountable text?" or "What is the work for which 'proving the sum of the internal angles of a triangle is 180°' is that work's proper gloss?"

In contrast to constructive formal analysis, ethnomethodology is interested in specifying *the work* by means of which the structures are produced that are accounted for and glossed by the bracketed texts. In other words, the question ethnomethodology pursues is that in the living/lived work, for example, of proving that the internal sum of a triangle (on the Euclidean plane) is 180°. Once we know the organization of the living/lived work, we are able to predict the kinds of results people produce in the same manner as we can predict what kind of entities people will see when looking at the diagram known as the Necker Cube. However, from knowing the accounts, we cannot infer the nature of the lived work. For this

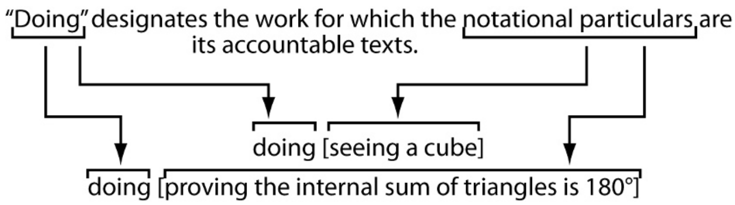

Fig. 10.10 Conceptualization of the difference between the *work* ('doing') that produces a phenomenon and the description of the experience (seeing a cube, proving the internal sum of a triangle

reason, phenomenological and ethnomethodological accounts of mathematics are related to formal analyses – whether quantitative or qualitative – in asymmetrically alternate ways. This is not to say that ethnomethodology disputes the accounts provided by formal analysis; those achievements can be demonstrated and are demonstrated in and as the outcomes of the living/lived work of doing mathematics. This asymmetry is radical and incommensurable, but nevertheless obtains to related aspects of mathematics. Ethnomethodology (as phenomenology) is not in the business of "interpreting" signs that people produce. Rather, its "fundamental phenomenon and its standing technical preoccupation in its studies is to find, collect, specify, and make instructably observable the endogenous production and natural accountability of immortal familiar society's most ordinary organizational things in the world, *and to provide for them both and simultaneously as objects and procedurally, as alternate methodologies*" (Garfinkel, 1996, p. 6). The two examples I use here constitute such materials that allow readers, in and through engaging the work specified, to experience the living/lived, worksite-specific (inherent lived) praxis of *doing* and *seeing* mathematically.

IMPLICATIONS FOR PEDAGOGY

The upshot of this approach is that no account can get us closer to the actual living/lived experience of seeing and doing mathematically, even when, and precisely because, persons retrospectively *talk about* their living/lived mathematical experiences. Therefore, no textbook paragraph or professor utterance can *tell* us to see mathematically. This is so because these accounts inherently involve *re*presentations of the sensuous experience, that is, means of making some past experience present again. We do not get in this way at the sensuous experiences themselves. In any instance imaginable, these representations – the means of making a past presence present again – are different from the sensuous work in the living present. Only metaphysics will make a claim to the contrary, because it has not recognized that ever since the Greek antiquity, scholars have attempted to access living/lived *Being* in and through externalities, that is, beings (representations). Being (capital B) and beings are not the same thing, though in metaphysical accounts of knowing and learning (which includes all forms of constructivism from Kant to the present day), the latter are freely substituted for the former. Therefore, the dehiscence of Being and beings is never recognized – but this is precisely the divide that I see between all forms of formal analysis and ethnomethodology, the former being concerned with beings (identifiable, identified structures) and the latter with Being, the never-ending living/lived labor of producing the structures identified in the asymmetrically alternate way in formal analyses. By their very *re*presentational nature, therefore, *pedagogical instructions* are radically different from what they intend to instruct: seeing, doing mathematically.

It should be clear, therefore, that mathematical seeing cannot be taught explicitly, because the forces and movements underlying seeing are invisible and inaccessible to consciousness. I can notice what my eyes do, but only because they already have developed the competency. I can voluntarily move my eyes from one vertex to another, from a vertex to a vanishing point (Fig. 10.2) because my eyes already master these movements and therefore give me the capacity to intend particular movements. Before that – e.g., during the initial look at an image (e.g. Fig. 10.3) – my movements are inherently arbitrary. That is, mathematical seeing requires particular forms of movement capacities that emerge from initial arbitrary and random movements. If my flesh does not yet know them, I cannot intend these movements precisely because I do not know how to enact the movements required for seeing mathematically. In a praxis framework, this impossibility to teach disciplinary perception may be attributed to the invisibility of habitus:

> Given that what is to be communicated consists essentially of a *modus operandi*, a mode of scientific production which presupposes a definite mode of perception, a set of principles of vision and di-vision, there is no way to acquire it other than to make people see it in practical operation or to observe how this *scientific habitus*) we might as well call it by its name) “reacts” in the face of practical choices – a type of sampling, a questionnaire, a coding dilemma, etc. – without necessarily explicating them in the form of formal precepts. (Bourdieu, 1992, p. 222)

That is, one can observe only the *effect* of habitus, which is a particular form of vision and *di*vision, never habitus itself, because it is immanent in the movements. The only way of support that can be offered is by having the student participate in the actual praxis where disciplinary seeing, the forms of vision and *di*vision are in action. This allows students, as I suggest above, to identify in their own, initially unintended movements those that yield results similar to those that they can see brought about by the more experienced person.

DIRECTIONS FOR RESEARCH

Readers will notice that in my approach to lived experience of mathematical seeing, I am not interested in asking people what they have seen while engaging in this or that mathematical task. Any response I might receive is only a *re*presentation of the sensuous work of seeing filtered through the particular perspectives of the person. It has been noticed that what a practitioner has to say retrospectively about what s/he has done does not get us any closer to the lived praxis than what a theoretician says. Accounts of experience are as far from experience as any other description including the accounts a theoretician might provide; they constitute but *rationalizations* of an originary event given everything else that we have experienced and learned since then. We know very well – as the popular adage goes – that hindsight always has 20/20 vision. Retrospective accounts always and continuously are subject to change; what I get from people when I ask for accounts of experience, therefore, depends on when and under what conditions I ask. What I am interested in instead is this: (a) the enabling of a situation whereby the interested reader experiences the living/lived work of seeing mathematically that is the focus of my research (e.g., while doing the entire proof, including the drawing, seeing, concluding) and (b) an understanding of the fundamental living/lived processes that enable this or that sensuous experience (e.g., how we come to see a cube as a cube, a line as a line, etc.).

The kind of distinctions I make in the preceding sections allow us to move from accounts of doing and seeing mathematically to the actual sensuous labor (work) of mathematical doing and seeing. The two stand in an incommensurably and asymmetrically alternate relation. I am not interested in the interpretation of signs people produce but in the sensuous labor of doing mathematics. That is, I am not interested in local practices as texts that are interpreted for their "meaning." Rather, I am interested in accessing the sensuous labor of mathematics as events that are "in detail identical with themselves, and not representative of something else" (Garfinkel, 1996, p. 8). This requires attention to the "witnessably recurrent details of ordinary everyday practices," which literally "constitute their own reality" (p. 8). We see above that knowing the work allows us to specify the structures that formal analytic procedures identify. This means, that "you can use ethnomethodology to recover in phenomenal ordered details – in a phenomenal field of ordered details the work that makes up, at the worksite, the design, administration, and carrying off of investigations with the use of formal analytic practices. You can't do it the other way around" (p. 10).

Much of the living/lived work goes unnoticed – not in the least discoverable in the disattention that formal analysts pay to the living/lived work of doing mathematics. In fact, phenomenological analyses that focus on *Life* show that it remains invisible, especially to the so-called sciences of life, *bio*logy. However, under special circumstances, parts of this work are to be exhibited: in situations of trouble, for example, when experienced scientists struggle with the classification of a specimen or when scientists struggle with providing an expert reading of a graph even though it was taken from an introductory course of their own domain. With respect to research method, what really matters in, and to, praxis is made available and perceivable only in the actual living/lived work of doing research – one has to experience it to be able to see it. To allow readers to re/live the work in and through their own living/lived bodies, reading/seeing or hearing accounts are insufficient. What research of the living/lived mathematics experience can do is:

– to provide for situations that make the phenomenon instructably observable such that in doing what the instructions say, the reader experiences in and through his/her living/lived labor the relevant mathematics; the phenomenological investigations of seeing a cube and proving the angle sum (the work is only partially detailed) would be of that kind.
– to provide something like a musical score, which, when readers actually "play the tune," allows them to live the mathematical conversations presented in the same way as musicians live the music written by some classical composer who, in most cases, no longer lives (e.g., Roth & Bautista, 2011).

In summary, therefore, to get at the living/lived work, we need research to go about differently than what formal analysis allows us to do. There is no difference whether formal analysis denotes itself as qualitative or as quantitative. Distinctly different are phenomenological and ethnomethodological approaches, because they are concerned with the living/lived work of doing mathematics. No retrospective account can get at this because of the inherent, unavoidable dehiscence between Being and beings, presence and the making present of the present (representation). But we have to inquire into the living/lived work, because this is the only way accessible to the "inner-historical," nature of mathematics, the very problem of its objectivity continually re/produced living/lived (subjective) sense-building and sense-producing work of everyone in the culture doing/seeing mathematics. We cannot understand mathematical seeing as a living/lived form of life unless we gain access to the very engine that keeps it alive, produces and transforms it across generations: the sensuous work of doing/seeing mathematics.

NOTES

1 The work described in this chapter was made possible by several grants from the Social Sciences and Humanities Research Council of Canada. Aspects of this work were presented as part of the lecture series "Alterative Forms of Knowing (in) Mathematics" at Portland State University and at the WISDOM[e] conference at the University of Wyoming. Some parts of the text were initially published in *Forum Qualitative Social Research*.

2 In a chapter devoted to the *learning paradox*, von Glasersfeld (2001) states that "far from being *given*, what is called 'data' can be seen as the result of the experiencer's own construction" (p. 143). In this, he simply reiterates the (neo-)Kantian position of conceptions preceding sensuous experience. This flies in the face of many 20th-century philosophical analyses, affirmed by the neurosciences, according to which an essential aspect of knowing *is* given to the subject.

3 See, for example, Roth, 2010, 2011a.

4 Marx/Engels (1962) frequently use the adjective "lebendig [living]" and the corresponding noun "Lebendigkeit [vivacity]" of things that are *alive* and changing in contrast to things that are dead and unchanging. In his focus on the living person, Marx explicitly situates all economic phenomena in the phenomenological life of the individual. I pair the "living" with "lived" in the expression living/lived, because at a number of very different levels, human beings are not only alive but also live (experience) this state of being alive.

5 During the WISDOM[e] conference, Pat Thompson and Les Steffe suggested that I did not understand (radical) constructivism. But all they were doing was reiterate the subjectivist idealist position that von Glasersfeld has laid out, a position that many philosophers have shown to be untenable in the face of real data. I deconstruct this position in several recent works (Roth, 2011a, 2011b).

[6] The two cubes that participants in psychological studies tend to report seeing in Figure 10.1

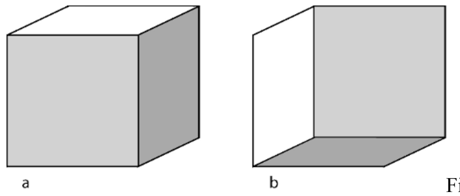

Fig. A1

[7] There is a small Dalmatian dog on the left. When the page is rotated through 180˚, there is the same Dalmatian dog, again on the left.

[8] Arguing that these structures come from the unconscious only gets us deeper into Western metaphysics, as the unknown and unknowable now are explained in terms of cognitive structures currently not available to consciousness. Thus, "[t]he non-presence always has been thought in the form of the present . . . or as a modalisation of the present" (Derrida, 1972, p. 36–37). This is also the fundamental point and problem both in de Saussurian semiology and Freudian analysis.

[9] *Praxis* denotes the real situation where the living/lived work occurs; it generally is not characterized by thematization and "metacognition." *Practices* refer to the patterned action and therefore denote something apparent to a theoretical gaze rather than to the regard of the practitioner.

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