NEW DIRECTIONS IN MATHEMATICS AND SCIENCE EDUCATION

Alternative Forms of Knowing (in) Mathematics

Celebrations of Diversity of Mathematical Practices

Swapna Mukhopadhyay and
Wolff-Michael Roth (Eds.)

SensePublishers

ALTERNATIVE FORMS OF KNOWING (IN) MATHEMATICS

NEW DIRECTIONS IN MATHEMATICS AND SCIENCE EDUCATION
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Scope

Mathematics and science education are in a state of change. Received models of teaching, curriculum, and researching in the two fields are adopting and developing new ways of thinking about how people of all ages know, learn, and develop. The recent literature in both fields includes contributions focusing on issues and using theoretical frames that were unthinkable a decade ago. For example, we see an increase in the use of conceptual and methodological tools from anthropology and semiotics to understand how different forms of knowledge are interconnected, how students learn, how textbooks are written, etcetera. Science and mathematics educators also have turned to issues such as identity and emotion as salient to the way in which people of all ages display and develop knowledge and skills. And they use dialectical or phenomenological approaches to answer ever arising questions about learning and development in science and mathematics.

The purpose of this series is to encourage the publication of books that are close to the cutting edge of both fields. The series aims at becoming a leader in providing refreshing and bold new work—rather than out-of-date reproductions of past states of the art—shaping both fields more than reproducing them, thereby closing the traditional gap that exists between journal articles and books in terms of their salience about what is new. The series is intended not only to foster books concerned with knowing, learning, and teaching in school but also with doing and learning mathematics and science across the whole lifespan (e.g., science in kindergarten; mathematics at work); and it is to be a vehicle for publishing books that fall between the two domains—such as when scientists learn about graphs and graphing as part of their work.

Alternative Forms of Knowing (in) Mathematics

Celebrations of Diversity of Mathematical Practices

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PREFACE

Mathematics education frequently is theorized and organized according to standard practices, the purpose of which is to educate adapted citizens. Even though constructivist theory emphasizes the personal construction of knowledge, actual mathematics education practices generally aim at making students construct the "right," that is, the canonical practices of mathematics – not realizing that for many, this may mean symbolic violence to the forms of mathematical knowledge they are familiar with, and that the standard processes typical of mathematics education contribute to the reproduction of social inequities. Those who do not complete academic mathematics courses at the high school level are systematically biased against when it comes to university entry and other career choices.

This book grew out of an ongoing public lecture series concerned with thinking about mathematics education differently. The series was titled *Alternative Forms of Knowledge Construction in Mathematics* and has taken place every Spring term since 2006 at Portland State University, Oregon, and also, since 2009, Portland Community College. The impetus for the lectures came from the fundamental belief that mathematics is a human construction, thus mathematics cannot be separated from its historical, cultural, social, and political contexts. To date, nineteen scholars from North America and beyond have participated in the series. All of the lectures were videotaped and then streamed for global dissemination at: http://www.media.pdx.edu/dlcmedia/events/AFK/. Readers are encouraged to access these talks, each about an hour long, that supplement the chapters in this collection. These chapters have been contributed by sixteen speakers from the first five years of the series, based either on the original talks or new writing. Although contributions from Rochelle Gutierrez, Cyril Julie, Lionel LaCroix, and Arthur Powell are absent from this collection, we encourage the readers to access their lectures in the series from the live link cited above.

In mentioning the cultural aspects of mathematics, and especially their educational implications, we always remember the impact of Claudia Zaslavsky (1917–2006). As an activist-scholar, Claudia was a pioneer in documenting and valorizing mathematical activities of many cultures, particularly in her seminal book *Africa Counts,* published in 1973. As well as being a leader in ethnomathematics, Claudia served as guide and mentor to many. Fittingly, the first year's set of lectures was dedicated in memory of her contribution to the exploration of alternative ways of thinking in mathematics. In continuing the struggle to recognize the knowledge and voices of others, our aim in the lecture series was to reach out to many who, even within the mainstream, do not realize their own voices.

As a public lecture series, and now this book, the project grew out of collaboration of various kinds. We would like to take an opportunity to thank many without whose support it would not have happened. Firstly, we were funded through annual awards from 2006 to 2011 from Portland State University's Diversity Action Council. Human diversity is mostly recognized in relation to

race/ethnicity, linguistic capabilities, religions, or sexual orientation. Extending its boundaries to the cognitive/cultural realm is not only helping us broaden our understanding of diversity, it also potentiates a discourse in higher education on diversity in mathematical practices, which is a radically different perspective.

We thank Portland Community College, Cascade campus, for its unwavering support since 2009. This is mostly due to Ann Sitomer of the mathematics department at PCC to whom we owe a great deal of gratitude for her intelligence, keen interest, and sense of partnership in collaboration. We believe that a strong and stable partnership between a community college and a four-year college is essential not just for enhancing STEM participation, but also for the fundamental reason that education, in its all aspects, should be easily accessible to every member of society.

We thank Provost Roy Koch, Portland State University, Dean Randy Hitz of the Graduate School of Education, and Professor Christine Chaille, the chair of Curriculum and Instruction, for their ongoing interest in, and support for, the lecture series. Despite their busy schedules, they attended the lectures and often welcomed the speakers to Portland by introducing them to the participating audience.

Our heartfelt thanks got to the participants – first as speakers, now as authors – and the audience. The talks were deeply engaging and further invigorated by the participation of audiences as diverse as one could imagine, spanning a wide range of age and interests, many of whom travelled for hours to attend the lectures. Without their interest and support, the series would not have lasted. The speakers were, simply put, assets of an extraordinary quality. They adapted to the low-cost operation of the series by staying in Swapna's modest guestroom, and provided not only stimulating public talks but also remarkable professional development opportunities at her own house!

Big thanks go to the technology crews of Portland State University (Rick Arnold and his team) and Portland Community College (Michael Annus and his team). They worked tirelessly in videotaping, streaming, and storing the lectures on their servers. Not only were they a vital part of the logistical team, they enjoyed each lecture and had thoughtful comments on each of the topics presented.

Last but not least, we thank Brian Greer for his ongoing cognitive and emotional support. Not only is he generous in terms of offering his expertise, he also has been a major architect in designing the project.

There are many others to whom we owe gratitude for their support – we thank you all.

In solidarity

Portland, OR
Mt. Gravatt, QLD
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Brian Greer worked for most of his career in the School of Psychology, Queen's University, Belfast, before moving to a position in mathematics education at San Diego State University in 2000. He now lives in Oregon where he is an Adjunct Professor at Portland State University and works independently. His work has evolved from psychological studies of mathematical cognition, through work on aspects of mathematics teaching/learning, towards a critical stance and an interest in the cultural and political contexts in which mathematical education takes place.

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Chris Jordan is an internationally acclaimed artist and cultural activist based in Seattle, WA. His work explores contemporary mass culture from a variety of photographic and conceptual perspectives, connecting the viewer viscerally to the enormity and power of humanity's collective behaviors. Edge-walking the lines between beauty and horror, abstraction and representation, the near and the far, the visible and the invisible, his work asks us to consider our own role in the incomprehensibly complex world we find ourselves part of. Jordan's works have been exhibited and published worldwide.

John Kellermeier is a professor of mathematics at Tacoma Community College. His research interests have been in gender issues, feminist pedagogy, and curriculum inclusion in the mathematics and statistics classrooms, and in ethnomathematics. John has published articles on curriculum inclusion in *Transformations*, *Feminist Teacher*, and *NWSA Journal*. He has led numerous workshops on teaching mathematics and science from a feminist and multicultural perspective and has developed ethnomathematics courses for the undergraduate student.

Jerry Lipka is a professor of education at University of Alaska, Fairbanks, and a long-term member of *Math in a Cultural Context*. He has worked with Dora Andrew-Ihrke, Evelyn Yanez, and many elders for numerous years. Slowly Lipka, Andrew-Ihrke, and Yanez have developed mathematics modules for elementary school students/teachers based on elders' knowledge. More recently, they have concentrated on professional development as they connect elders' knowledge from everyday practice as a way to teach mathematics cohesively and while connecting to the community. Lipka has published extensively on these topics. He, Dora Andrew-Ihrke, and Evelyn Yanez have conducted workshops in Norway, Sweden, and Guam as well as with many Alaskan school districts.

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Swapna Mukhopadhyay is an associate professor at Portland State University. As a mathematics educator, her scholarly interests focus on issues of critical mathematics education and cultural diversity. The main thrust of her work is in realizing that mathematics is a socially constructed mental tool that is accessible to all. Using the framework of ethnomathematics, she works towards unifying research and curriculum design, an act that is synonymous with activism. She co-edited *Cultural Responsive Mathematics Education* (2009). She is also a part-time potter.

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BRIAN GREER, SWAPNA MUKHOPADHAY, &
WOLFF-MICHAEL ROTH

CELEBRATING DIVERSITY, REALIZING[1] ALTERNATIVES

An Introduction

> Remember that our basic message is: We are allowed to think about alternatives. (Slavoj Žižek, speaking to the Occupy Wall Street protesters, October, 2011)

This book is about the celebration of diversity in all its human forms, specifically in relation to mathematics and mathematics education: culture, ethnicity, gender, forms of life, worldviews, cognition, language, value systems, perceptions of what education is *for*. All of which are reflections of the unavoidable (yet often denied) reality that mathematics education *is* politics.

There are obvious and direct manifestations of political involvement in education. Governments, through their bureaucracies, set policies and control curricula, testing, teacher education requirements, research bodies, and so on. They also, increasingly and with more and more guile, control the discourse – while no child is to be left behind (a deliberate echo of the ethos of the US Marines), a student who cannot jump through the hoops of algebra (the intense study of the last three letters of the alphabet) is now framed not just as stupid, but also as undeserving of educational and economic opportunities, and even as unpatriotic. For studying these processes in action, the National Mathematics Advisory Panel (NMAP) set up by George Bush constitutes an invaluable case study (Greer, 2012; Roth, 2008).

As one manifestation of the increasingly nationalistic rhetoric surrounding mathematics education, national egos are bound up with international comparative exercises such as PISA and TIMMS. Poor performance by the US in such "beauty contests" is exploitable for political leverage – to find scapegoats (whether the increasing cultural and linguistic diversity of the US population, or the teachers, students, parents), and to create the perception of crisis so that radical deformation can be pushed through. Students, teachers, and school communities (i.e., people) are invisible inside a black box that can be manipulated by external levers of tests, carrots and sticks (many more of the latter than the former), in the context of a set of hysterical demands, such that all children be at grade level in reading and mathematics by 2014. Meanwhile, back on planet Earth, the differences in test scores between the White majority and other ethnic groups – particularly African American, Latino/a, and Native American – stubbornly persist. Meanwhile, under

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President Obama's embrace of privatization, the public school system faces an existential threat (Ravitch, 2010).

Less obviously, there is an pervasive political influence in applications of mathematics in ways that impact most aspects of our lives, but are generally outside our control or even our awareness, that has been characterized and extensively analyzed as "mathematics in action" (e.g., Skovsmose, 2005). Mathematics education does little to prepare people to be aware of, and to deal with, this formatting of their lives by educating them about the nature of mathematical modeling (Greer & Mukhopadhyay, 2012). To the contrary, mathematics education largely provides training in simplistic argumentation (the mathematical concept of "function" corresponds to a single cause producing a single effect, which is a good model for essentially nothing, even in the physical sciences), blind faith in numbers and mathematical models, and slavishly following rules, the rationale for which is not questioned and that absolve people from making human judgments. It also encourages the attitude that simple technical solutions can be applied to complex human problems (including mathematics education). Both forms of thinking in fact belong together and are the pinnacles of metaphysical thinking that today expresses itself in technology gone wild (Heidegger, 2006).

The world is in a mess. Nearly a century ago, H. G. Wells (1920) commented that "human history becomes more and more a race between education and catastrophe" (p. 594). We need to ask what responsibilities we bear as mathematicians and mathematics educators for bringing this situation about and for trying to change it:

> It is clear that mathematics provides the foundation of the technological, industrial, military, economic and political systems and that in turn mathematics relies on these systems for the material bases of its continuing progress. It is important to question the role of mathematics and mathematics education in arriving at the present global predicaments of humankind. (D'Ambrosio, 2010, p. 51)

Whereas mathematics has been used in the creation of both "wonders" and "horrors" it is neither good nor bad in itself – at least when considered in a decontextualized manner.[2] Mathematicians have a particular responsibility to avoid contributing to the horrors, in particular through participation in the military-industrial-academic complex.[3]

Prominent in educational-political consciousness and media coverage, in the United States and elsewhere, is an ongoing hegemonic struggle that goes by the term "Math Wars"[4] that has intensified since the publication of *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) but has a much longer history and clear parallels in other disciplines. At the risk of simplism, the difference lies between those who pre-eminently value fluent and error-free performance of decontextualized mathematical procedures and those who attach more importance to conceptual understanding. The former group comprises an unholy alliance of certain mathematicians, certain mathematics educators, many experimental psychologists who address mathematics education, the corporatocracy, politicians, and policy-makers. As an example of this coalition in action, they dominated any traces of progressivism among the mathematics educators on the National Mathematics Advisory Panel.

NCTM represents what we could call the "enlightened mainstream." For example, in a section of *Principles and Standards* (NCTM, 1989) that has since undergone sustained criticism, it declared in favor of giving increased attention to number sense, meaning of fractions and decimals, use of calculators for complex computation, actual measuring, problem-solving strategies, and justification of thinking (selected from a list of 37 recommendations), and decreased attention to isolated treatment of division facts, paper-and-pencil fraction computation, use of clue words to determine which operation to use [in word problems], and rote memorization of rules (selected from 18 recommendations) (NCTM, 1989, pp. 20–21).

NCTM also proclaims a strong commitment to equity, but, upon examination, this seems to mean essentially that non-majority students should have access to unexamined mathematics education, not that it be examined in relation to its relevance to, and value for, such students – as Piaget might have put it, assimilation without accommodation. Within the statement of the Equity Principle (NCTM, 1989) the section headed "Equity requires accommodating differences to help everyone learn mathematics" in no way addresses the nature of the mathematics to be learned. We need to "[make] problematic the *there* in *How do we get There*?" (Martin, 2003, p. 18). The very considerable body of writing on equity and mathematics education is fundamentally flawed because of its internal gaze, mostly ignoring the systemic problems in capitalistic society (Roth, 2008).[5]

Likewise, it is necessary to deconstruct the superficially appealing (and intentionally so) slogan "Mathematics for all" (Martin, 2003) which underlies a project predominantly aimed at economic competitiveness – to whose benefit? (Gutstein, 2009). In official rhetoric, mathematics and science education are seen as essential to the competitive accumulation of human capital, which is really about how people can be exploited by the wealth-making class. The Nobel laureate in economics, Amartya Sen, has proposed an alternative that he terms "human capability" by which he means "[focusing] on the ability of human beings to lead lives they have reason to value and to enhance the substantive choices they have" (Sen, 1997, p. 35).

In general, mathematics education suffers from the same morbidity as education as a whole, in which the forces for the status quo have the upper hand. Thus, "[t]he more educational research finds out, the less educational policy changes, as it plays up to the powerful who tend to desire the reproduction of the status quo rather than to bring about changes of life conditions that lead to differences that make a difference" (Roth, 2008, p. 371). Critical surveyors of the scene (Pais, 2012) comment on the apparent lack of progress; little accumulates. As an example, the treatment of fractions may be taken as paradigmatic of the failure of research in mathematics education to accumulate wisdom that can be cashed at the educational bank, except insofar as it underpins many career trajectories. How many studies of children struggling with fractions can be done? How much more do we now know about teaching/learning fractions than we did 10, 20, 30 … years ago? Many totally functional adults "who could never do mathematics" first hit "the wall" with fractions. Why don't we ask why carpenters and others, when measuring length, use fractions of an inch such as ½, ¼, $^1/_8$ (Roth, 2008) so that, for example, the question of finding least common denominators does not arise?

How often does anyone in mathematics education ask a fundamental question such as why do people need to know how to compute with fractions? Division of a fraction by a fraction is notoriously difficult to illustrate in a meaningful context, as is illustrated by an example in the Common Core Standards for Mathematics for sixth grade[6]. Under the heading "Apply and extend previous understandings of multiplication and division to divide fractions by fractions" we find the example: "How many $^3/_4$-cup servings are in $^2/_3$ of a cup of yogurt?" And the reader might ask herself/himself: Why should I now (or a young student) be able to compute $^3/_7 + {}^{13}/_{19}$? Note that we are *not* saying there are no justifications for the need to have this ability, but rather that we should articulate more carefully what those reasons are, *and talk to the students about them*.

There are alternative approaches to the study of the conceptual field of multiplicative structures that are grounded in students' experience. The phenomenology of fractions and the diversity of situations that they model are extremely complex, a complexity typically ignored in standard pedagogies (Freudenthal, 1983). Culturally based approaches are possible, as illustrated at many points in this book. The position taken in this book, and by the emerging groups of practitioners, researchers, and activists who self-identify as critical mathematics educators, goes well beyond that of "the enlightened mainstream" in insisting on the historical, social, cultural, and political situatedness of mathematics education, and the diversity that characterizes mathematical practices as much as any other human activity. Arguably the most pervasive and damaging aspect of mathematics education as it is typically practiced in schools is the lack of relevance and connection to students' lived experience. One mathematics educator who had lived through four very different political regimes in Palestine commented on this state of affairs in this way:

> What is startling about the math curriculum is – with the exception of some changes at the technical level – how stubborn and unchanging it has remained under the four completely different realities in which I have lived, studied, and taught; how insensitive and unresponsive it has been to the drastic changes that were taking place in the immediate environment! When something like this is noticed, it is only natural to ask whether this is due to the fact that math is neutral or that it is actually dead! (Fasheh, 1997, p. 24)

Mathematics education as a research field predominantly shows a similar insensitivity to the circumstances in which students live. This is apparent in the following reflection on a visit to a school in a South African township where the physical learning obstacles were obvious:

> How is it that the research in mathematics education has not noticed this hole in the roof? … Black children are simply treated completely differently, and their future has been spoiled by the apartheid regime. To ignore this fact is a political act. (Skovsmose, 2005, p. 20)

The same willful ignoring is apparent in educational research in the US on children living in poverty (Berliner, 2006). On the basis of extensive data analysis, the author concludes that "the most powerful policy for improving our nation's school achievement is a reduction in family and youth poverty" (p. 949).

In sketching a program for critical mathematics education research, one fundamental form of diversity that demands greater attention is the variety of sites for learning mathematics (Skovsmose, 2012). Skovsmose points out that the discourse of the field has been dominated by what he calls the "prototype mathematics classroom," an idealization that ignores the global diversity of circumstances in which people learn mathematics in schools.[7]

For critical mathematics educators, equity is not a matter of merely "giving" people access to unreconstructed mathematics education, but rather a matter of valorizing the diversity of mathematical practices that are intimately bound up with forms of life. Particularizing the declaration that "the intellectual activity of those without power is always characterized as non-intellectual" (Freire & Macedo, 1987, p. 122), the position that we seek to undermine is that the mathematical activity of those without power is always characterized as non-mathematical.

In positive vein, it is increasingly possible to point to manifestations of cultural resilience and resistance, and assertions of agency and identity, of which the ethnomathematics program is an important part. To adapt Spivak's famous phrase, the subaltern can speak mathematics. An essential form of this resistance comes in the form of alternative practices. As illustrated by several of the contributions to this book, serious attempts are being made to integrate knowledge of cultural mathematical practices into school mathematics, not as a peripheral activity, and with no implication of inferiority (Pinxten & François, 2011), illustrating another form of diversity, namely the variety of educational possibilities (Skovsmose, 2012). Serious work is being done to actualize Freirean principles of emancipatory education and advance social justice through mathematics education (Gutstein, 2006). Indeed, in that Gutstein and his students work around generative themes that come from their lived experience and the political reality of their milieu, this work could be considered a manifestation of ethnomathematics, in its wider sense, being integrated into mathematics education.

Meaningful integration of culturally based knowledge into school mathematics inevitably creates a strong tension. Acknowledging that "an understanding of [academic] mathematics and a world-language such as English … [represent] access to communication, further educational opportunities, employment, and development' (Barton, 2008, pp. 167–168), the author points to the dilemma of what and how to teach mathematics to students who "learn mathematics in a distinct cultural-linguistic context – how can they study an international subject while retaining the integrity of a minority world view?" (p. 142).

A possible way out of this dilemma already has been proposed (Pinxten & François, 2011). These authors embrace a characterization of ethnomathematics as "the generic category of all mathematical practices, with academic mathematics as a particular case" (p. 264). They also invoke the Freirean principle of the oppressed learning the language of the oppressor, hence that "everyone is entitled to 'access' to academic mathematics because it is the best position from which you can criticize the Master discourse" (p. 264).[8] On these foundations, they propose a concept of "multimathemacy" that reconciles the honoring of alternative forms of mathematical knowledge and practices with pursuing academic mathematics as a choice of the student, and taking into account his or her circumstances.

An overarching theme that we suggest the reader should be attuned to when reading the book is that of humanization (a consistent theme in the work of Paulo Freire).[9] Mathematics and mathematics education continue to be dehumanizing in many respects, including the following:

– A pervasive thread in mathematics-as-a-discipline, historically, has been the search for the Holy Grail of absolute certainty and precision. Even though results of Gödel and others have shown this to be an illusion, there is still a powerful desire to perfect a formal architecture of mathematics – which really becomes pernicious when the attempt is made to force mathematics education into that mold.
– There is no essential reason why mathematics-as-a-school-subject should be taught in a fashion that inflicts psychological damage on students, but that is, too often, the case. Taking such positions as that there is only one right answer (untrue as soon as mathematics is applied to reality) or only one right way to carry out a computation or express a proof (totally untrue) affords authoritarianism.
– Mathematics is often presented as existing independently of the people who do it, and independent of their bodies, senses, desires, emotions, and aesthetics – everything that makes a person flesh and blood. Thus, "mathematicians … have increasingly chosen to flee from nature by devising theories unrelated to anything we can see or feel" (Mandelbrot, 1983, p. 1).
– In terms of mathematics education as a research domain, we can simply point to research that reduces people to values on a few variables (the methodological straitjacket that forces everything to be a factor so that statistical rituals can be performed) or scores on (generally ill-conceived) tests. Likewise, when carrying out interviews, the pervasive image of brains as containers of knowledge from which dumps can be made (the ever-present brain-as-computer metaphor).

To (re)humanize mathematics and mathematics education it is necessary to:

– Connect with students' lived experience, their bodies, their immediate experiences, their emotions, needs, and desires. Which implies activity with hands and eyes, interacting directly with our physical and social worlds, not just through symbolic mediations on pages and computer screens.
– Celebrate mathematics as a pan-cultural activity, acknowledging the whole of humanity and its diversity.

– Understand that mathematics, like any human activity, is inherently social. Education is, fundamentally, about interpersonal relations between students and teachers.

Let us also make clear that we do not reject the glories of mathematics as intellectual achievements of humankind – giving appropriate acknowledgment to the contributions of all cultures by deconstructing the Eurocentric narrative of the history of academic mathematics – just as much as literature, music, or art (which are also pan-cultural activities).[10] Although mathematicians and teachers often appear to go to extraordinary lengths to disguise the fact, mathematics is creative and aesthetically deep. Learning mathematics in school, instead of too often being a form of intellectual child abuse, should be an intellectually exhilarating experience.

To return to how we began, we *are* allowed to think about alternatives; the world *can* be other than what is the case.

NOTES

[1] "Realizing" is deliberately ambiguous, as it can mean both "becoming aware of" and "making happen."

[2] The essence of a thing does not reveal itself when subject to the theoretical gaze that isolates it from everything else; rather, the essence reveals itself in practical use (Heidegger, 1927/1977). In praxis, mathematics is part and parcel of politics and therefore inherently bound up with value.

[3] Giroux (2007, pp. 14–15) points out that this was the original formulation in the retirement speech of President Eisenhower in which he warned of the dangers of the military-industrial complex.

[4] This is a pernicious metaphor for several reasons. It diminishes suffering in actual wars, potentiates symbolic violence by invoking nationalism, and encourages the media to frame the discussion as a confrontation between extremes.

[5] This crucial point has been argued very forcefully in a very penetrating critique of research on equity within mathematics education (Pais, 2012).

[6] www.corestandards.org/assets/CCSSI_Math%20Standards.pdf (p. 42)

[7] A parallel point can be made about research in psychology being most often restricted to middle-class groups from the richer part of the world (where students are available and obedient) yet still claiming that its results constitute scientific truths (Pinxten & François, 2011).

[8] The civil rights activist Bob Moses, who now works in mathematics education, characterizes such access as a civil right.

[9] Not addressed in this book, yet a matter of extreme importance, is the generally impoverished nature, from our point of view, of mathematics education at the university level.

[10] It is worth noting that many mathematicians have urged that alternative epistemologies, such as Navajo conceptions of space, could be a rich source for suggesting innovative extensions to academic mathematics.

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PART I

MATHEMATICS AND POLITICS OF KNOWLEDGE

It is unfortunate but true that there is not a long tradition within the mainstream of mathematics education of both critically and rigorously examining the connections between mathematics as an area of study and the larger relations of unequal economic, political, and cultural power. (Apple, 2000, p. 243)

The chapters in this section of the book interrogate political dimensions of mathematics as a domain of knowledge. Relating to both historical and contemporary contexts (and not forgetting the historical continuity that links these contexts), the authors here highlight how mathematics has been used as a weapon of cultural violence in the service of oppression[1].

The introductory quotation remains true for the mainstream. However, increasingly perturbing the mainstream is an emerging group of self-identifying critical mathematics educators (for an overview, see Greer & Skovsmose, 2012). In the same spirit, the chapters in this section deal with central issues within the nexus of knowledge and power in relation to mathematics and mathematics education, namely: the role of mathematics in cultural hegemony in the colonial past and in the neo-colonial present; curricular hegemony that negates alternative world-views, such as those of American Indians in the United States; the discourse of contemporary educational "reform" that perpetuates deficiency models under elaborate disguises; and the role of language, both in parallel, and in interaction with, mathematics education as a tool of cultural violence. There is a fundamental ideological fault-line, slicing through all aspects of intellectual and political life, between those who promote a single dominant model of humanity and see diversity as a problem, and those who celebrate human diversity in all its forms.

Despite attempts to protect them, mathematics and mathematics education are part of, indeed in many important respects central to, this hegemonic debate. A fundamental principle opening up mathematics to critique is that it is "a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context" (Hersh, 1997, p. xi). That characterization of mathematics implies recognition of the pan-cultural nature of mathematical practices beyond the academic (Bishop, 1988). In a subsequent paper, the author examined the deployment of mathematics in the project of European colonization, not in the most obvious sense of underpinning technological advances offering military supremacy, but in the less tangible, but ultimately more pervasive, form of contributing to a worldview, one based on particular notions of rationality, thus "the secret weapon of cultural imperialism" (Bishop, 1990, p. 51).

In chapter 1, Gary Urton begins by summarizing the key points of Bishop's paper. Urton then exemplifies the "weapon" in action by describing how Spanish systems for accountancy were imposed upon the Inkas, replacing the sophisticated and functional system that they had already developed. Beyond describing the forced replacement of one cultural system by another, Urton makes the point that mathematics, particularly in the form of accounting, has been used for millennia, and in many civilizations, as a weapon of statecraft to control populations.[2]

Urton makes the point that European colonizers did not impose political and economic systems – and, most generally, worldviews – on blank slates, but rather strove to suppress and replace existing systems. In his book *Culture and Imperialism*, Edward Said (1993) analyzed how European culture (in particular, literature) both reflected, and contributed to, the framing of minds of both colonizers and colonized. To our knowledge, no parallel treatment in depth of mathematics in this respect exists, but the works of Bishop and Urton certainly represent an important contribution in this direction.

The next two chapters deal with the contemporary educational context of the USA, against the background of the history of education as a weapon for oppression. To illustrate how essential is a perspective of historical continuity, consider the following analysis (Lomawaima, 1999). Having identified four tenets common to colonial education of Native Americans, including the assumption that "specific pedagogical methods were needed to overcome deficits in mental, moral, and physical characteristics", she proceeded to show how these tenets are still far from eliminated in contemporary practices. As with the imposition of the Spanish system on the Inkas, the "unnatural history of American Indian education" (p. 3) demonstrates an implicit belief in "the white man's burden" of granting civilization to peoples that did not have any. This fallacy is exposed in chapter 2 by Greg Cajete, in which he illuminates the worldview, cultural richness, and forms of social organization of North American indigenous peoples. In this chapter, Cajete focuses on three main issues. The first is that, after centuries of oppression in which education (notably through linguicide) has played a major part, indigenous Americans deserve an education that respects and reflects their traditional values, beliefs, and worldview – hardly an outrageous demand, but one that is not being adequately addressed. Secondly, American Indian students need to have access to the educational and economic opportunities afforded by mainstream education; integrating that access with the point just made involves a complex balance. Third, there is emerging resistance, and indeed, solidarity among indigenous peoples worldwide. There is pushback against the pedagogy of the oppressed – Cajete refers to Paulo Freire as an inspiring example. Increasingly, scholars from indigenous groups are prepared to take on the hegemony on its own ground (e.g. Deloria, Foehner, & Scinta, 1999).

The history of education of African Americans in the USA is also an indispensable part of any analysis of the current situation (Ladson-Billings, 2006). In place of standard ahistorical rhetoric about "achievement gaps" interpreted simplistically as a static descriptor of differences in standardized test scores among ethnic groups, there really are many factors contributing to the "educational debt" and to many other gaps, gaps in resources in particular.[3]

In chapter 3, Delaina Washington, Zayoni Torres, Maisie Gholson, and Danny Martin present an analysis of the discourse around achievement gaps, specifically the way in which the phenomenon is framed in terms of "crisis". As George Lakoff, in particular, has argued, the framing of popular conceptions through language has become a highly skilled art/business, one mastered much better in general by the right than by the left in American politics. Since Orwell's seminal essay "Politics and the English language" the expertise of persuasion has grown enormously. The discourse around "achievement gaps" affords a clear example. It is arguable that even using that term is already conceding a major part of the argument, since the term frames the discourse within a connotation of deficiency. Starkly put, "Quantitative tests of aptitude and achievement have given U.S. education a way to sort children by race and social class, just like the old days, but without the words 'race' and 'class' front and center" (McDermott & Hall, 2007, p. 11). The requirement of *No Child Left Behind* to report test scores by ethnic groups had a superficial logic that many educators bought into. In conjunction with requirements for "adequate yearly progress," it was supposed to leverage schools into improving performance. But what happened? At the time of writing, no substantial improvement in performance has been reported, and 82% of schools are failing to meet the required standard (not to mention the hysterical demand that all students be proficient in mathematics and reading by 2014).

A particularly interesting example of crisis rhetoric was presented at an AERA meeting (Winerip, 2003). Using a model of the "issue attention cycle", the paper summarized by Winerip documented coverage in *The Ann Arbor News* over the period 1984–2001 of the differences in test scores between black and white students. The analysis shows that the

> … news media and public ignore a serious problem for years; for some reason, they suddenly notice, declare it a problem and concoct a solution; next they realize the problem will not be easily fixed and will be costly; they grow angry, then bored; finally, they resume ignoring the problem. (Winerip, 2003)

The historical analysis by Washington et al. makes it clear that crisis rhetoric in relation to education in general and mathematics education in particular has a long history, and it has been a tool for the achievement of particular political ends. A comparable thesis in relation to geopolitical economics was presented in *The Shock Doctrine: The Rise of Disaster Capitalism* (Klein, 2007). Washington et al. also point to the dangers of researchers, however motivated, accepting the framing of crisis. Researchers, however well-intentioned, need to be wary of contributing to the problems they purport to be trying to solve (McDermott & Varenne, 2006).

The use of language as a weapon of cultural violence (in particular, within education) is, in many ways, more obvious that that of mathematics. There are striking parallels between the two contexts of cultural hegemony, to the extent that Greer and Mukhopadhyay (in press) were able to take many excerpts from *The*

Hegemony of English (Macedo, Dendrinos, Guanari, & Dendrinos, 2003) and simply replace "language" by "mathematics." Consider also this point:

> [C]ountries in the postcolonial world need the "indispensable global medium" for pragmatic purposes, even for survival in the global economy. On the other hand there is the fact that the medium is not culturally or ideologically neutral, far from it, so that its users run the "apparently unavoidable risk of co-option, of acquiescing in the negation of their own understandings of reality and in the accompanying denial or even subversion of their own interests. (Kandiah, 2001, p. 112)

This quotation is about English, but a precisely similar dilemma obtains in relation to academic mathematics.

As well as such clear parallels, the politics of power relations relating to teaching of languages and of mathematics interact wherever children are taught mathematics in other than their home language, as is examined by Marta Civil and Nuria Planas in chapter 4. The contexts they work in are those of Latino/a immigrants in Arizona faced with increasingly Draconian laws of cultural violence, and immigrants in Barcelona from South America, Asia, and elsewhere faced with a strong policy of Catalan as language of instruction (and against a backdrop of English gaining dominance throughout Europe). Similar research and analyses have been carried out in many contexts worldwide. Civil and Planas describe complex situations circumscribed by legal and educational policies that are politically and ideologically situated. These circumstances have significant cognitive effects on children's learning of mathematics and represent cognitive violence through suppression of non-dominant languages and a framing in terms of bi- or multi-lingualism as a deficit rather than cultural and cognitive enrichment. On a more positive note, the authors document considerable resilience, resistance, and strategic sophistication among the students as they navigate their milieus and negotiate access to mathematical learning.

NOTES

[1] How mathematics is used is a human choice. It acquires its political color depending on whose interests it is being used to advance. So, it can be a weapon of liberation as well as of oppression.

[2] The growth of the formal theory of statistics (the word itself is etymologically close to "state") is intertwined with politics and theories of the nature of human societies).

[3] Despite an increased attention to unpacking and understanding academic achievement gaps as social construction from sociological, cultural, and political perspectives, mainstream mathematics education still tends to ignore how the cultural and economic aspects of lives of many – mostly minorities and poor – play a crucial role in school performance (Marable, 2000).

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GARY URTON

1. MATHEMATICS AND ACCOUNTING IN THE ANDES BEFORE AND AFTER THE SPANISH CONQUEST

In an article entitled "Western mathematics: The secret weapon of cultural imperialism," the author argues that Western European colonizing societies of the 15th to 19th centuries were especially effective in imposing on subordinate populations the values of rationalism and "objectivism" – defined as a way of conceiving of the world as composed of discrete objects that could be abstracted from their contexts – primarily through "mathematico-technological cultural force" embedded in institutions relating to accounting, trade, administration, and education (Bishop, 1990). Thus,

> mathematics with its clear rationalism, and cold logic, its precision, its so-called "objective" facts (seemingly culture and value free), its lack of human frailty, its power to predict and to control, its encouragement to challenge and to question, and its thrust towards yet more secure knowledge, was a most powerful weapon indeed. (p. 59)

The question to be addressed in this article is whether or not Western societies were unique in the use of mathematics, especially when employed in state accounting, as what Bishop terms a "weapon" of statecraft. A wealth of literature produced by critical accounting historians over the past several decades has elucidated the role of accounting as a technology of and a rationale for, governance in state societies. Accounting and its specialized notational techniques are some of the principal instruments employed by states in their attempts to control and manage subjects. It is suggested that

> [r]ather than two independent entities, accounting and the state can be viewed as interdependent and mutually supportive sets of practices, whose linkages and boundaries were constructed at least in their early stages out of concerns to elaborate the art of statecraft. (Miller, 1990, p. 332)

In this chapter I examine the policies and procedures of political arithmetic as employed in Western Europe in the early Renaissance period and in the contemporary, Inka Empire of Pre-Columbian South America. I argue that state accounting, as realized in the practices of alphanumeric, double-entry bookkeeping in Europe and in khipu (knotted-string) record-keeping in the Inka empire constituted highly effective strategies for the exercise of social control in the two

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 17–32.

settings. In addition to examining pre-conquest mathematical practices in the two separate settings, I consider the encounter between Spanish written (alphanumeric) record-keeping practices and Inka knotted-string record keeping following the European invasion and conquest of the Inka empire, in the sixteenth century. I begin with a brief overview of the rise of double-entry bookkeeping and the use of Hindu-Arabic numerals in accounting systems that emerged in the early Renaissance mercantile states of Western Europe.

ACCOUNTING AND NUMERATION IN EUROPEAN DOUBLE-ENTRY BOOKKEEPING

Two almost simultaneous developments in European mathematics and commercialism during the fourteenth-fifteenth centuries are critical to the picture of accounting and recordkeeping practices of Spanish colonial administrators in the sixteenth century. These developments were the invention of double-entry bookkeeping and the replacement of Roman numerals by Hindu-Arabic numerals. The earliest evidence for double-entry bookkeeping dates from the 13th century when the method was put to use by merchants in northern Italy. The first extended explanation of double-entry bookkeeping appeared in a treatise on arithmetic and mathematics written by the Franciscan monk Frater Lucas Pacioli in 1494. The invention and implementation of double entry went hand-in-hand with the replacement of Roman numerals by Hindu-Arabic numerals, which had been introduced into Western Europe almost five hundred years before their eventual acceptance into accounting practice, in the 15th century.

The cities of northern Italy that were the centers of commercial activities from the 14th to the 16th centuries also became centers of learning in arithmetic and mathematics. It was in these cities – Venice, Bologna, Milan – that Hindu-Arabic numerals were first linked with double entry to form the basis of modern accounting science. It was here as well that abacus or "reckoning" schools grew up that were patronized by the sons and apprentices of merchants throughout Europe. The masters of those schools, the *maestri d'abbaco*, taught the new arithmetic, or *arte dela mercadanta*, "the mercantile art" (Swetz, 1989). It was in northern Italy as well where, a couple of decades prior to the publication of Pacioli's exposition of double-entry bookkeeping, the first arithmetic textbook, the so-called *Treviso Arithmetic*, was published in 1478. While not discussing the double-entry method itself, the *Treviso Arithmetic* proclaimed itself from the opening passage as intended for study by those with an interest in commercial pursuits. Double-entry bookkeeping employing Hindu-Arabic numerals spread throughout Western Europe in the century or so leading up to Spanish adventures in the New World.

From virtually the earliest years following the invasion of the Andes, European administrators – toting accounting ledgers filled with columns of Hindu-Arabic numerals and alphabetically written words and organized in complex formats – came into contact with Inka administrative officials wielding bundles of colorful knotted cords. These local administrators – known as *khipukamayuqs* "knot-keepers/makers/organizers" – were, oddly enough, speaking the language (in

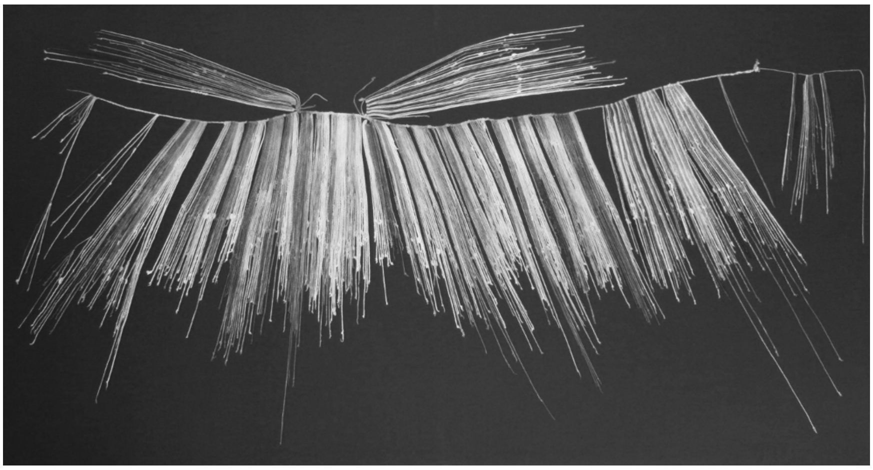

Fig. 1.1 Khipu (Museum for World Culture, Göteborg, Sweden; #1931.37.0001 *[UR113])*

Quechua) of decimal numeration and practicing what may have looked for all the world, to any Spaniard trained by the reckoning masters of northern Italy, like double-entry bookkeeping.

THE KHIPU AND ITS METHODS OF INFORMATION REGISTRY

Khipus are knotted-string devices made of spun and plied cotton or camelid fibers (Fig. 1.1).[1] The colors displayed in khipus are the result of the natural colors of cotton or camelid fibers or of the dyeing of these materials with natural dyes. The "backbone" of a khipu is the so-called primary cord – usually around 0.5 cm in diameter – to which are attached a variable number of thinner strings, called pendant cords. Khipus contain from as few as one up to as many as 1,500 pendants (the average of some 420 samples studied by the Harvard Khipu Database project is 84 cords). Top cords are pendant-like strings that leave the primary cord opposite the pendants, often after being passed through the attachments of a group of pendant strings. Top cords frequently contain the sum of values knotted on the set of pendant cords to which they are attached. About one-quarter of all pendant cords have second-order cords attached to them; these are called subsidiaries. Subsidiaries may themselves bear subsidiaries, and there are examples of khipus that contain up to six levels of subsidiaries, making the khipu a highly efficient device for the display of hierarchically organized information.[2]

The majority of khipus have knots tied into their pendant, subsidiary and top strings. The most common knots are of three different types, which are usually tied in clusters at different levels in a decimal place system of numerical registry (Fig. 1.2).[3] The most thorough treatment to date of the numerical, arithmetic, and mathematical properties of the khipus is *Mathematics of the Incas: Code of the*

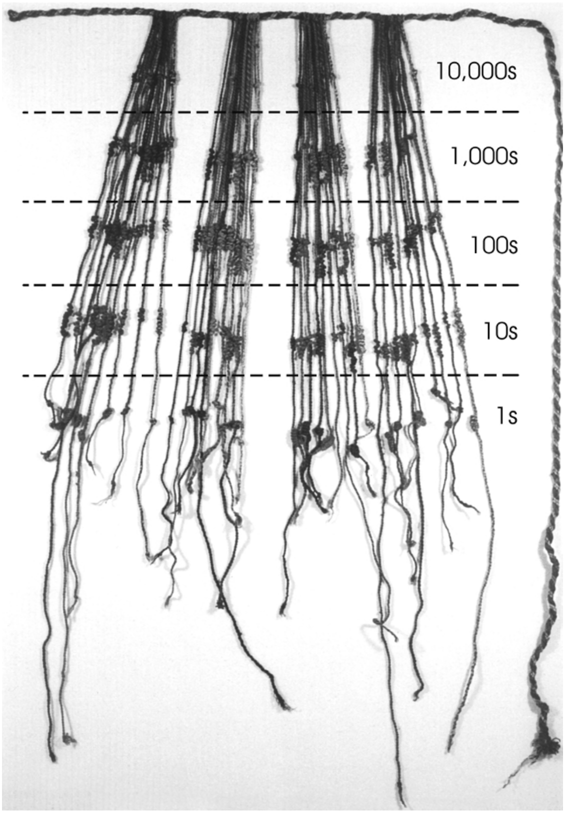

Fig. 1.2 Clustering of knots on a khipu in decimal hierarchy

Quipus (Ascher & Ascher, 1997). The Aschers have shown that the arithmetic and mathematical operations used by Inka accountants included, at a minimum, addition, subtraction, multiplication, and division; division into unequal fractional parts and into proportional parts; and multiplication of integers by fractions.

What kinds of information were registered on the khipus? In addressing this question, it is important to stress that – although we are able to interpret the quantitative data recorded in knots on the khipus – we are not yet able to read the accompanying nominative labels, which appear to have been encoded in the colors, twist, and other structural features of the cords. The latter would, were we able to read them, presumably inform us as to the identities of the items that were being enumerated by the knots. Thus, in discussing the identities of objects accounted for in the khipus, we are forced to rely on the Spanish documents from the early years following the European invasion.

According to the Spanish accounts, records were kept of censuses, tribute assessed and performed, goods stored in the Inka storehouses, astronomical periodicities and calendrical calculations, royal genealogies, or historical events.[4] The overriding interest in the recording, manipulation, and eventual archiving of quantitative data in the khipus was the attempt to control subject peoples throughout the empire. This meant to be able to enumerate, classify, and retain records on each subject group. The most immediate use to which this information was put was in the implementation of the labor-based system of tribute. Tribute in the Inka state took the form of a labor tax, which was levied on all married, able-bodied men (and some chroniclers say women as well) between the ages of 18 and 50. In its conception and application to society, Inka mathematics appears to have taken a form remarkably like the political arithmetic of seventeenth-century Europeans. In sum, the decimal place system of recording values – including zero – of the Inka knotted-cords was as precise and complex a system of recording quantitative data as the written Hindu-Arabic numeral-based recording system of Europeans at the time of the conquest, although the records of the former were not as rapidly produced, nor as easily changeable, as those of the latter.

Accounting has long been one of the principal institutions and administrative practices involved in maintaining and legitimizing the status quo in western European nation-states. Can this be said of khipu accounting in the pre-Hispanic Andes as well? We gain a perspective on this question by looking at two accounts of how censuses were carried out in the Inka state. As in other ancient societies, census taking was a vital practice in the Inka strategy of population control, as well as serving as the basis for the assessment and eventual assignment of laborers in the *mit'a* (taxation by labor) system (Julien, 1988; Murra, 1982). The first account of census taking is from a famed mid-sixteenth century soldier and traveler:

> the nobles in Cuzco told me that in olden times, in the time of the Inka kings, it was ordained of all the towns and provinces of Peru that the head men [*señores principales*] and their delegates should [record] every year the men and women who had died and those who had been born; they agreed to make this count for the payment of tribute, as well as in order to know the quantity of people available to go to war and the number that could remain for the defense of the town; they could know this easily because each province, at the end of the year, was ordered to put down in their *quipos*, in the count of its knots, all the people who had died that year in the province, and all those that had been born. (Cieza de León, 1967 [1551], p. 62, my translation)

Some forty years after Cieza wrote down the information cited above, Martín de Murúa gave an account of Inka census taking that varies somewhat from Cieza's understanding of this process and that contains interesting details concerning the actual procedures involved in local population counts.

> They sent every five years *quipucamayos* [khipu-keepers], who are accountants and overseers, whom they call *tucuyricuc*. These came to the provinces as governors and visitors, each one to the province for which he was responsible and, upon arriving at the town he had all the people brought together, from the decrepit old people to the newborn nursing babies, in a

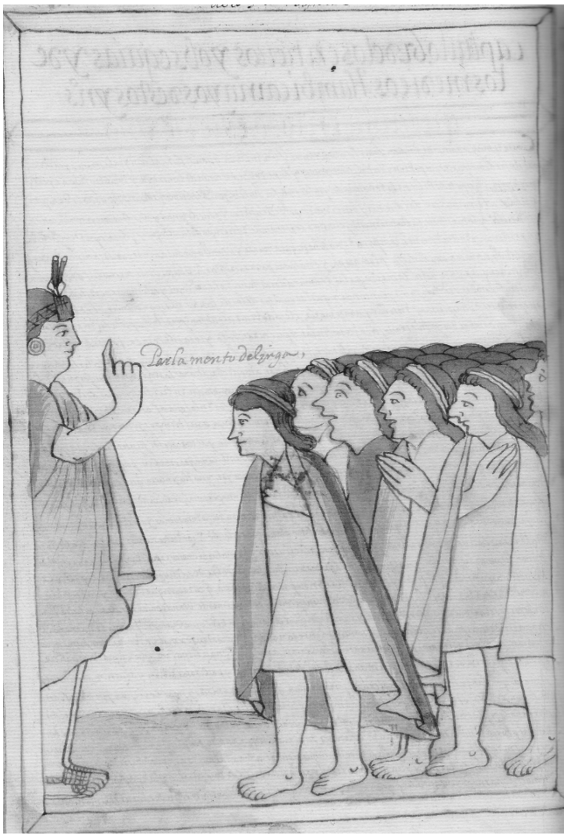

Fig. 1.3 Conducting census count of men, by Age-Grade (Murúa, 2004, p. 114v)

> field outside town, or within the town, if there was a plaza large enough to accommodate all of them; the *tucuyricuc* organized them into ten rows ["streets"] for the men and another ten for the women. They were seated by ages, and in this way they proceeded [with the count]. (Murúa, 2004 [1590], p. 204, my translation)

Late sixteenth-century drawings of these male and female census accounting events from the chronicle of Martín de Murúa are shown in Figures 1.3 and 1.4.

In Inka census taking, people were ordered into public spaces to be counted and classified. Although resistance and evasion may have been common in such proceedings, from what the Spanish chroniclers and administrative officials tell us, Inka censuses were accomplished using non-coercive measures – that is, local people apparently were compliant with the claims of authority coming from local officials and state administrators. Thus, as much as an accounting tool, the census khipu was an instrument for the performance and display of state authority and

Fig. 1.4 Conducting census count of women, by Age-Grade (Murúa, 2004, p. 116v)

power within local communities.[5] The census data collected by local record-keepers were knotted into khipus, copies were made of each record, and the data were subsequently reported to higher-level accountants in regional and provincial administrative centers. Two issues arise with respect to these procedures: one concerns the practice of making one or more copies of khipu records, the other concerns the training and education of state record-keepers.

Whereas there are a number of references in the Spanish chronicles to khipu copies, the study of such copies in the corpus of extant khipus has proceeded slowly. Recent advances have come about, however, following the development of a searchable database – the Khipu Database (KDB).[6] From searches of 420 or so samples included in the KDB, some 12–15 examples of copies of accounts have been identified. While referred to as duplicate, or "matching" khipus, we could also consider "pairs" of khipus to represent an original and a copy.

Copies (or matching) khipus occur in three different forms. First, there are examples in which the numerical values on a sequence of strings on one sample are repeated exactly on another khipu. In some samples of this type, we find that while the pair of khipus bears the same knot values, the colors of the strings may vary. The second type of matching khipus, which I have termed "close matches," involves instances in which two different samples contain not exactly matching sequences of numbers, but rather ones in which the values are similar (e.g., those of one sample varying a small amount from those on another sample). And, finally, we have examples in which a numerical sequence recorded on one cord section of a khipu are repeated exactly, or closely, on another section of cords of that same khipu.

Duplicate khipus may have been produced as a part of a system of "checks and balances." However, duplicates seem as well to possess most of the requisite elements of double-entry bookkeeping in which "all transactions were entered twice, once as a debit and once as a credit … [T]he debit side pertained to debtors, while the credit side pertained to creditors" (Carruthers & Espeland, 1991, p. 37). Close matches would be accounts in which the debits and credits sides of the ledger were not in balance. On pairs of khipus having identical numerical values on sequences of strings but in which string colors vary (Urton, 2005), color could have been used to signal the statuses of credits and debits in the matching accounts.[7] In the Inka state, debit/credit accounting would have been employed primarily in relation to the levying of labor tribute on subject populations.

The principal information that we lack in order to be able to confirm whether or not duplicate khipus might have been produced and used as double-entry-like accounts are the identities of the objects recorded on the khipus. Since we still cannot read the code of the khipus, we are unable to determine whether paired accounts were simply copies made for the purposes of checks-and-balances or if they might represent a relationship between a debit for an item on one account and the credit for that same item on another account. Research into this matter is on-going.[8]

What can we say about the individuals who became khipu-keepers for the state? How were these individuals recruited and trained? What role did they play in exercising authority and maintaining social and political control in the Inka state? A late 16th-century chronicler provided the following account of a school that was set up in the Inka capital of Cusco for the training of khipu-keepers.

> The Inca … he set up in his house [palace] a school, in which there presided a wise old man, who was among the most discreet among the nobility, over four teachers who were put in charge of the students for different subjects and at different times. The first teacher taught the language of the Inca … and upon gaining facility and the ability to speak and understand it, they entered under the instruction of the next [second] teacher who taught them to worship the idols and the sacred objects [*huacas*]. … In the third year the

> next teacher entered and taught them, by use of *quipus*, the business of good government and authority, and the laws and the obedience they had to have for the Inca and his governors. … The fourth and last year, they learned from the other [fourth] teacher on the cords and *quipus* many histories and deeds of the past. (Murúa, 2001, p. 364, my translation)

The curriculum for these young administrators-in-training aimed at engendering loyalty and adherence to the values, policies and institutions of the Inka state. The khipu studies component of the administrative curriculum fulfilled the common objective of accounting education, which is producing "governable persons" who themselves would go on to serve as provincial administrators. The curriculum and examination promoted discipline and proper accounting practices in a way that rationalized institutional arrangements in the interest of the state and, ultimately, transformed the bodies and minds of the apprentice administrators.

The situation outlined above was not to last for long, however, as less than half a century after the school of administration was set up a cataclysmic event brought the school, not to mention the entire imperial infrastructure, crashing down; this event was the Spanish conquest.

CONQUEST, COLONIZATION, AND THE CONFRONTATION BETWEEN KNOT- AND SCRIPT-BASED TEXTS

The story of the conquest of the Inka empire by the Spaniards, which was undertaken by Francisco Pizarro and his small force of around 164 battle-hardened *conquistadores*, beginning in 1532, has been told too many times – in all its astonishing, entrancing and appalling details – for me to add much to the telling in the space available here (Hemming, 1970). The events of the conquest and the processes of colonization that are relevant for our discussion here are the following. The initial battle of conquest, which occurred in November 1532 in the Inka provincial center of Cajamarca, in the northern highlands of what is today Peru, resulted in the defeat of the Inka army and the capture and execution of Atahualpa, one of two contenders for succession to the Inka throne. Pizarro then led his small force southward arriving in the Inka capital city of Cuzco in 1534. The Spaniards and their native allies were soon forced to defend Cuzco against a rebellion led by the Spanish-installed puppet-king Manco Inca. This gave rise to a decades-long war of pacification of the rebels, which finally came to an end in 1572 with the execution of the then rebel leader, Tupac Amaru.

Three years prior to the capture and execution of Tupac Amaru, a new – the fifth – Viceroy of Peru Francisco de Toledo had arrived in Peru with a mandate to put down the rebellion and to transform the war- and disease-ravaged land of the former Inka empire into an orderly and productive colony for the benefit of the king of Spain Philip II. Viceroy Toledo instituted a set of reforms that were in some respects a continuation of certain of the processes of pacification, reorganization and transformation that had been on-going since the earliest days following the initial conquest. In other ways, Toledo's reforms represented something completely new, different and profoundly transformative in their effects on Andean life-ways (Stern, 1993, pp. 51–79).

The end result of the Toledan reforms, the clear shape of which became manifest by the mid-to late 1570s, included, most centrally, the following institutions: *encomiendas* – grants of groups of Indians to Spanish *encomenderos* "overseers" who were charged with the care and religious indoctrination of the natives and who, in exchange, had the right to direct native labor for their personal benefit but without the right (after the Toledan reforms) to levy tribute demands on them; *corregimientos* – territorial divisions for the management and control of civil affairs, including (theoretically) oversight of the *encomenderos*; *reducciones* – newly-formed towns that were laid out in grid-like ground plans to which the formerly dispersed natives were transferred for their surveillance, control and indoctrination; *doctrinas* – parish districts staffed by clergy who attended to the religious indoctrination of the natives within the reducciones and who received a portion of the tribute for their own maintenance; and *mita* – a form of labor tax based on the Inka-era *mit'a*, which supplemented what was, for Andeans, a new kind of tribute imposed on them by Toledo: specified quantities of agricultural produce, manufactured goods (textiles, sandals, blankets), and coinage.

The census was a critical institution for reorganizing Andean communities. Spanish censuses were carried out by administrative *visitadores "*visitors" who produced documents, known as a *visitas*, which were detailed enumerations of the population in the *reducciones* broken down (usually) into household groupings. Each household member was identified by name, age and – in the case of adult males – *ayllu* "social group" affiliation (Guevar-Gil & Salomon, 1994; Urton, 2006). The *visitadores* were usually joined in their rounds by the *kurakas* "local lords" and often by the local *khipukamayuqs*. The khipu-keepers could supply historical, corroborating information on population figures and household composition (Loza, 1998). It is important to stress that participation by the native record-keepers was not primarily for the benefit of the Spaniards, rather, it was to ensure that the natives would have their own, khipu-based accounts of the enumeration in the event – which seems always and everywhere to have come to pass – that disputes arose over the population count, the amount of tribute levied, or other administrative questions.

There are two contexts in which Andeans encountered Old World mathematical principles and practices: the manner of collecting information for the censuses, and the striking and circulation of coinage. These practices were closely linked to new forms of tribute, as well as to what was, for Andean peoples, a completely new form of communication: writing – that is, the inscribing of marks in ordered, linear arrangements on paper, parchment, or some other two-dimensional surface. Such a medium and associated recording technology were unprecedented in the Andean world.

Central to the Spanish attempt to establish an orderly colony in the former Inka territories, from the 1540s through the 1570s, was a program of enumerating the native population, investigating its form(s) of organization, and beginning to sketch out its history. One form that this process took was to call the khipu-keepers before colonial officials and have them read the contents of their cords (Loza, 1998). These recitations were made before a *lengua* "translator"; the Spanish words spoken by the translator were written down by a scribe. This activity resulted in the production of written transcriptions in Spanish alphanumeric script of the census data and other information previously jealously guarded by the khipu-keepers in their cords.

Many of the khipu transcriptions discovered to date have been assembled in an important collection, entitled *Textos Andinos* (Pärssinen & Kiviharju, 2004). The Spaniards were at least initially respectful of the khipus and their keepers, as the khipus were the primary sources of information on the basis of which Spanish officials began to erect the colonial administration. However, once the information was transferred from khipus to written texts, the locus of textual authority, legitimacy and power began to shift toward the written documents.

Whereas many native Andeans learned how to read and write alphabetic script and how to manipulate Hindu-Arabic number signs, only a handful of Spaniards appear to have achieved any degree of familiarity with the khipus (Pärssinen, 1992); it appears that no Spaniard became truly proficient at manipulating and interpreting the cords. What this meant was that, rather than contests over interpretations of information contained in the two sets of documents coming down to reciprocal readings of the two sets of texts, what emerged between the 1540s and the 1570s were separate, contested readings by the keepers of the two different text types before a Spanish judge. As disputes intensified, and as more and more original data were recorded uniquely in the written documents, the khipu texts became both redundant and increasingly troublesome for the Spaniards (Platt, 2002). By the end of the tumultuous sixteenth century, khipus had been declared to be idolatrous objects – instruments of the devil – and were all but banned from official use.[9]

The circulation of coins is another area in which Andeans were confronted with a completely new and unfamiliar terrain of political relations, economic activity and shifting relations of authority over the course of the early colonial period. The first mint in South America was formally established in Lima in 1568, just 36 years after the events of Cajamarca. The royal decree that controlled the weights, fineness, and the fractional components of the coins to be struck in Lima were issued by Ferdinand and Isabella in 1479, which was amended by Charles V in 1537. The first coins struck in Lima bore a rendering of the Hapsburg coat of arms on the obverse and a cross with castles and lions on the quartered face on the reverse.

The two initial coin types were the *real*, a silver coin, and the gold *escudo*. Each of these coin types was broken down into subunits valued in relation to a general, unified standard of valuation known as the *maravedí*. The *maravedí* was used to coordinate values between different types of coins as determined by material differences and subdivisions of standard units (for example, the silver *real* = 34 *maravedís*; the gold *escudo* = 350 [from 1537–1566] or 400 [after 1566] *maravedís*). From this primary coordinating function, the *maravedí* served as a common denominator that permitted the interrelating of heterogeneous monetary values pertaining to gold and silver (Craig, 1989).

From almost the earliest years following the conquest, Spanish officials in the countryside (the *encomenderos*) had been levying tribute in kind, which in some places included a demand for plates of silver and bars of gold, and translating the value of these items into Spanish currency values. Spanish officials regularly produced documents translating the quantities of items of tribute in kind into values in *pesos ensayados* (a unit of value in silver currency). This was the main context within which the *kurakas* "local lords" in communities would have begun to encounter translations of the use-value of objects, which they were familiar with in their local non-monetized economies, into exchange-values stated in terms of currency equivalents (Spalding, 1973). Furthermore, the Viceroy Francisco de Toledo introduced in the mid-1570s a new tribute system, which included not only produce and manufactured goods but also coins; the sum to be given yearly by each tributary was four-to-five *pesos ensayados* (that is, coinage in *plata ensayada* "assayed silver"). Tribute payers were designated as male heads of households between the ages of 18–50. The native chronicler Guaman Poma de Ayala (1980) drew several images of native people paying their tribute using what appears to be coinage bearing the quartered reverse face of the *cuatro reales* (Fig. 1.5).[10]

People in the newly built *reducciones* were able to acquire coins to pay their tribute from forced work in the mines (a component of the Toledan tributary system), as well as from marketing and wage labor. The engagements with currency that resulted from these activities required people to begin to think about the different units of coinage, shifting equivalencies between coinage units, as well as to accommodate themselves to fluctuations in currency values in the periodic currency devaluations and the debasement of coinage that took place during the colonial period. The act of "devaluing" currency is a claim of authority on the part of some entity (e.g., state) over the exchange-value of the coinage one holds in one's own purse. One's subsequent use of that same coinage according to the newly announced rate of exchange represents compliance with the claim by the entity in question to control the value of one's currency. Although we have almost no data on the basis of which to consider how Andean peoples responded to such changes, these were some of the processes that were transpiring on the front lines of the confrontation between Old and New World mathematics entailing new and transformed notions of legitimacy, rationality, and authority in the early colonial Andes.

CONCLUSIONS

I began this exploration by asking about the relevance and salience of a characterization of mathematics as "the secret weapon of cultural imperialism" (Bishop, 1990). Having now looked at several aspects of arithmetic, mathematics, and accounting in Western Europe and the Andes during the period leading up to and a century or so beyond the fateful encounter between Pizarro and Atahualpa in Cajamarca in 1532, we return to ask: In what sense was mathematics linked to state

Fig. 1.5 Paying tribute with a coin bearing quartered design (Guaman Poma de Ayala, 1980, p. 521 [525])

power and governmental legitimacy in this historical conjuncture? I argue that much as critical historians writing near the end of the last century found in terms of the writing of history in colonial contexts, truth in history is usually the preserve of the conqueror. This is not necessarily because the conqueror knows what is, in fact, true; rather, it is because the conqueror possesses the power to speak, to produce conventionalized written accounts, and to represent and establish the rules of globalized statecraft. This is the case not only in terms of narrating and writing the events of history and explaining their causes, but also in taking the measure of the world and accounting for those measurements – geographic, demographic, economic and so on – for as long as the dominant group holds power.

Power and the exercise of authority take many forms. In its most extreme and, paradoxically, weakest form, power is maintained by force. As Foucault has shown more clearly than any recent political theorist, the most effective species of power is that which takes shape as individuals and groups become complicit with and participate in institutions of the state, such as in censuses, regulatory and corrective institutions, and accounting. What is the place of mathematics in this Foucauldian, 'genealogical' conception of power and authority? I think that here we must return to the question of the certainty of mathematics, and of how that certainty relates to truth and, ultimately, to power. I suggest that the critical observation on these matters for our purposes here is that mathematics may be made to serve, although it itself is not responsible for giving rise to, regimes of power. A "regime of power" may be a particular method of bookkeeping, an accounting procedure, or a state militia.

NOTES

[1] According to my own inventory, there are some 850+/- khipu samples in museums and private collections in Europe, North America and South America. While many samples are too fragile to permit study, almost 450 samples have been closely studied to date. Observations may be viewed at http://khipukamayuq.fas.harvard.edu/ and http://instruct1.cit.cornell.edu/resear4ch/quipu~ascher/.

[2] For general works on khipu structures and recording principles, see Arellano, 1999; Ascher & Ascher, 1997; Conklin, 2002; Radicati di Primeglio, 2006; Urton, 1994; 2003.

[3] Approximately one-third of khipu studied to date do not have knots tied in (decimal-based) tiered arrangements. I have referred to these as 'anomalous khipu' and have suggested that their contents may be more narrative than statistical in nature (Urton, 2003).

[4] I summarize here from a range of my studies. Readers find more detailed information in Urton, 2006 and 2009.

[5] Guevara-Gil and Salomon (1994) have discussed what were similar procedures, and results, in the censuses undertaken by Spanish *visitadores* (administrative 'visitors') who were responsible for counting, classifying and (re-)organizing local populations in the early colonial Andes.

[6] The Khipu Database project (KDB), located in the Department of Anthropology, Harvard University, is described fully on the project website <http://khipukamayuq.fas.harvard.edu/>. I gratefully acknowledge the following research grants from the National Science Foundation, which made the creation of the KDB possible: #SBR-9221737, BCS-0228038, and BCS-0408324. Thanks also to Carrie J. Brezine, who served as Khipu Database Manager from 2002 to 2005.

[7] It is interesting to note that in early Chinese bookkeeping, red rods signified positive numbers while black rods were used for negative numbers. As Boyer noted, "[f]or commercial purposes, red rods were used to record what others owed to you and black rods recorded what you owed to others" (cited in Peters & Emery, 1978, p. 425).

[8] Three articles published in the 1960s-70s by economists and accounting historians contain a lively debate not only about whether or not the khipus contained double-entry bookkeeping, but about the claim made by one of the disputants (Jacobsen) to the effect that the Inkas may in fact have invented the technique (Buckmaster, 1974; Forrester, 1968; Jacobsen, 1964). There is not space here to review the arguments made in these three articles. Suffice it to say that, while interesting for historical purposes, these articles are all poorly informed about the nature of the khipus, about what the Spanish documents say about their use, as well as about Inka political and economic organisations.

[9] The khipus were declared idolatrous objects and their use was severely proscribed by the Third Council of Lima, in 1583 (Vargas Ugarte, 1959). However, the khipus continued to be used for local recordkeeping purposes – in some cases down to the present day (see Mackey, 1970; Salomon, 2004).

[10] See the fascinating study by Salomon (1991) of one of the few references we have in the colonial literature to the engagement with coinage (*la moneda de cuatro reales*) by a native Andean during the colonial period. Salomon argues that the story, which appears in a well-known manuscript from Huarochirí (Salomon & Urioste, 1991), is concerned with the internal conflicts of a man due to the competing religious sentiments he experiences over loyalty to a local deity (*huaca*) and the Christian deity. The narrative plays on the precise symbolism of images, as well as the lettering, on a quartered Spanish coin.

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GREGORY A. CAJETE

2. CONTEMPORARY INDIGENOUS EDUCATION[1]

Thoughts for American Indian Education in a 21st-Century World

The perspectives, orientations, ideas, models, interpretations, and belief presented in this chapter are a *personal synthesis* based on my own creative process as an Indian educator. This work is a reflection of my particular understanding of the "shared metaphors" which American Indians hold in common with regard to tribal education. It is a general exploration of the *nature* of Indigenous education and the creative possibilities inherent in the introduction of an Indigenous frame of reference toward the development of a contemporary philosophy of American Indian education. As a whole, this work illustrates a dimension of the body of understanding that underlies American Indian tribal orientations to learning and teaching.

This work is also an open letter to Indian educators and those involved with Indian education issues. My approach has been that of a teacher exploring the dimensions of Indigenous teaching and learning in creative ways. The description of this journey, this "curriculum," has taken the form described here. Teachers create "curricula" (circles of learning and teaching) through constantly creating models and applying them to actual teaching situations. Ideally, teachers constantly adjust their models to fit their students and the constantly changing realities of educating. Through such constant and creative adjustment, teachers and students engage in a symbiotic relationship and constantly form feedback looks around what is being learned. In this way, teachers are always creating their stories even as they are telling them. This work explores a "culturally informed alternative" for thinking about and enabling the contemporary education of American Indian People. It is a translation of traditional Indian concepts and foundational principles into a contemporary framework of thought and description. It advocates the development of a contemporized, community-based education process that is founded upon traditional tribal values, orientations and principles, but simultaneously utilizes the most appropriate concepts and technologies of modern education.

The content presented represents only a small portion of what is available in the vast sea of research related to American Indian cultures. Indeed, American Indian cultures are among the most studied anywhere in the world. Access to this vast sea of content, facilitated by Indian educators and scholars, is an essential step to the

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 33–51.

creation of a contemporary epistemology of Indian education. This access to, and revitalization of, the Indigenous bases of education must occur not only in the contemporary classroom, but must include all dimensions of Indian communities as well. All Indian People, young and old, professional and grassroots, must consider themselves participants in a process of moving forward to the Indigenous basics of education. Indian People themselves must introduce contemporary expressions of tribal education to their own people. Ultimately, it is up to each community of Indian People, whether they live in an urban setting or reservation, to decide how their needs regarding cultural maintenance or re-vitalization may be addressed through education. It is up to each community of Indian People to decide what is appropriate to introduce through the vehicle of modern education and what should be imparted within the context of appropriate traditional mechanisms in the community.

Modern education and traditional education can no longer afford to remain as historically- and contextually-separate entities. Every community must learn to integrate the learning occurring through modern education with the cultural bases of knowledge and value orientations essential to the perpetuation of a community and its way of life. A balanced integration must be created. Over time, the emphasis on only modern education and Western-oriented curricula will by their nature and predisposition tend to erode an Indigenous way of life. In their embracing of modern education, Indian educators and tribal leaders must understand that the unexamined application of modern education and its models essentially conditions People away from their cultural roots not toward them. Modern education provides tools essential to the survival of Indian People and communities, but this education must be contextualized in a greater whole. In support of cultural preservation, Indian educators and tribal leaders need to advocate culturally-based education as one of the foundational goals of self-determination, self-governance, and tribal sovereignty. Indigenous education offers a highly creative vehicle for thinking about the evolving expressions of American Indian cultures as they enter the 21st century.

THE CONTEMPORARY DILEMMA OF AMERICAN INDIAN EDUCATION

The exploration of Indigenous education presented in this chapter is a "culturally-informed alternative" which includes the expression of the "universals" of the educational process as viewed from the perspective of traditional American Indian thought. Its foundation of credibility lies in the applicability of the perspectives and models presented to the whole process of teaching and learning, not just that of American Indians. The universals, which are explored, may be viewed as "archetypes" of human learning and as part of the Indigenous psyche of all Peoples and cultural traditions, including those of Western civilization. All relevant sources of thought and research and educational philosophy, regardless of cultural source, have been considered to fully illuminate the future possibilities of a contemporary education that mirrors Indigenous thought and its primary orientation of relationship with the natural world.

A pervasive problem affecting the contemporary vision of American Indian education stems from the fact that its contemporary definition and evolution has always been largely dependent on the prevailing winds of American politics. Much of that which characterizes Indian-education policy is not the result of research predicated upon American Indian philosophical orientations, but the result of "Acts of Congress," the history of treaty-rights' interpretation through the courts, and the historic Indian/White relations unique to each tribal group or geographic region. Historically, the views guiding the evolution of modern Indian education have been predicated upon assumptions that are anything but representative of Indian cultural mindsets (Deloria, 1990). In spite of such policy orientations, traditional educational processes paralleling mainstream education have continued to take place within the context of many Indian families and communities. While there has been progress in the last thirty years, the integration of these two approaches to education has been practically non-existent.

The basis of contemporary American education is the transfer of academic skills and content that prepares the student to compete in the social, economic, and organizational infrastructure of American society as it has been defined by the prevailing political, social, and economic order of vested interests. American educational theory is, therefore, devoid of substantial ethical or moral content regarding the means that are used to achieve its ends. The ideal curriculum espoused by American education ends up being significantly different from the experienced curriculum internalized by students and the real workings of much of American society. The American society that many minority students experience is wrought with contradictions, prejudice, hypocrisy, narcissism, and unethical pre-dispositions at all levels including the schools. As a result, there have been educational conflicts, frustration, and varying levels of alienation experienced by many Indian People due to their encounters with mainstream education.

A fundamental obstacle to cross-cultural communication continues to revolve around significant differences in cultural orientations to the world and to the fact that Indian People have been forced to adapt to an educational process that is essentially not of their own making. Traditionally, Indians view life through a different cultural metaphor than that of mainstream America. It is this different cultural metaphor that frames the exploration of the Indigenous educational philosophy that is presented in this chapter.

Traditional Indian education represents an anomaly for the prevailing theory and methodology of Western education since what is implied in the application of "objectivism" is the assumption that there is one correct way of understanding the dynamics of Indian education, one correct methodology, one way of understanding the reality of Indigenous educational philosophy, and that there is one correct policy for Indian education. And that one way is the way of mainstream America. The mindset of "objectivism," when applied to the field of Indian education, excludes serious consideration of the "relational" reality of Indian People, the variations in tribal and social contexts, and the processes of perception and understanding which characterize and actually form its expressions (Peroff, 1989).

Objectivist research has contributed a dimension of insight, but it has substantial limitations in the multi-dimensional, holistic, and relational reality of the education of Indian People. It is the affective elements – the subjective experience and observations, the communal relationships, the artistic and mythical dimensions, the ritual and ceremony, the sacred ecology, the psychological and spiritual orientations – that have characterized and formed Indigenous Education since time immemorial. These dimensions and their inherent meanings are not readily quantifiable, observable or easily verbalized, and as a result, have been given little credence in mainstream approaches to education and research. Yet, it is these very aspects which form a profound orientation for learning through exploring and understanding the multi-dimensional relationships between humans and their inner and outer worlds.

For Indian educators, a key to dealing with the conflict between the objective and relational orientations, the cultural bias, and the cultural differences in perception lies in the kind of open communication and creative dialogue that challenge the "tacit infrastructure" of ideas that have guided contemporary Indian education.

Education is essentially a communal social activity. Educational research that produces the most creatively productive insights involves communication within the whole educational community, not just the "authorities" recognized by mainstream educational interests. Education is a communication process and plays an essential role in every act of educational perception. There must be a "flow" of communication regarding the educational process among all educators as a result of individual internal dialogue, interactions among educators, publication, and discussion of ideas. Unfortunately, a serious blockage of communication and fragmentation of educational thought continues to be the rule rather than the exception, and communication related to Indian education is no exception (Bohm & Peat, 1987).

Most educators have embraced many ideas based on the established "tacit infrastructure" of mainstream American education religiously. This situation, as it pertains to Indian education, limits creative acts of perception. A free play of thought and opening up of the field – one that is not restricted by unconsciously determined social pressures and the inherent limitations of the currently established paradigms of Indian education – needs to occur. It is only in realizing that there is a "tacit infrastructure" and then questioning it that a high level of creative thought regarding the possibilities and potentials of Indigenous educational philosophy can become possible. And only in realizing that American Indian perceptions of education have traditionally been informed by a different "metaphor" of teaching and learning can more productive insights into contemporary Indian education be developed. These traditional "metaphors" of education derived their meaning from unique cultural contexts and interactions with natural environments. In turn, the collective experience of Indian People and their elegant expressions of cultural adaptations have culminated in a body of shared metaphors and understandings regarding the nature of education and its "essential ecology."

In this exploration of Indigenous education, I attempt to develop insights into the community of shared metaphors and understandings that are specific to Indian cultures, yet reflective of the nature of human learning as a whole. Ultimately, an exploration of traditional Indian education is an exploration of nature-centered philosophy. Traditional Indian education is an expression of environmental education par excellence. It is a environmental education process which can have a profound meaning for the kind of modern education required to face the challenges of living in the world of the 21st Century. It has the potential to create deeper understanding of the collective role as "caretakers" of a world that Americans have been largely responsible for throwing out of balance.

The thesis presented here is essentially a continuation of my dissertation, *Science: A Native American Perspective* (A Culturally Based Science Education Curriculum Model) produced under the auspices of the New Philosophy Program of the International College. The perceived needs that motivated the writing of my dissertation continue to form the impetus for this work. These needs can be summarized as follows:

- The need for a contemporary perspective of American Indian education, which is principally derived and informed by the thoughts, orientations, and cultural philosophies of Indian People themselves. The articulation and fulfillment of this need is, I believe, an essential step in Indian educational self-determination.
- The need for exploration of alternative approaches to education that more directly and successfully address the needs of Indian populations during this time of "educational and ecological crisis." During such a time of "crisis," it is essential to open up the field and to entertain the possibilities of new approaches in a creative quest for more viable and complete educational processes.
- The need to integrate, synthesize, organize, and give focus to the enormous amount of accumulated materials from a wide range of disciplines about Indian cultures and Indian education toward the evolution of a contemporary philosophy for American Indian education that is Indigenously inspired and ecologically based.

The purpose of contemporary American Indian education, as it is currently interpreted, has been to assure that Indian People learn the skills necessary to be productive – or at least survive – in the midst of the post-industrial American society. American Indians have been taught to be consumers in the tradition of the "American dream" and all that that entails. We have been encouraged to use modern education to "progress" by being participants in the "system." We have been conditioned to seek the rewards and benefits that success in the world modern education purportedly provides. We are enticed from every direction to pursue careers in law, medicine, business, and the sciences, which form the pillars of Western thought and conditioning. Yet, in spite of the many Indian People that have succeeded by embracing Western education, Indian People must question the effects modern education has had on our collective cultural, psychological, and ecological viability. What has been lost and what has been gained by participating in a system of education that does not stem from, or really honor, our unique Indigenous perspectives? How far can we go in adapting to such a system before that system literally educates us out of cultural existence? Have we reached the limits of what we can do with mainstream educational orientations? How can we re-vision and establish once again the "ecology of education" that formed and maintained our tribal societies?

Ironically, a number of the most creative Western thinkers have embraced what are essentially Indigenous environmental-education views and are vigorously appropriating Indigenous concepts to support the development of their own alternative models. For example, cultural historian and philosopher Thomas Berry (1999) proposes a new context for education which is essentially a re-invention of the roles and contexts that are inherent to Indigenous education:

> The primary educator as well as the primary lawgiver and primary healer would be the natural world itself. The integral earth community would be a self-educating community within the context of a self-educating universe. Education at the human level would be the conscious sensitizing of humans to the profound communications made by the universe about us, by the sun, the moon, and the stars, the clouds, the rain, the contours of the earth and all its living forms. All music and poetry of the universe would flow into the student, the revelatory presence of the divine as well as insight into the architectural structures of the continents and the engineering skills whereby the great hydrological cycle functions in moderating the temperature of the earth, in providing habitat for aquatic life, in nourishing the multitude of living creatures would be as natural to the educational process. The earth would also be our primary teacher of sciences, especially biological sciences, and of industry and economics. It would teach us a system in which we would create a minimum of entropy, a system in which there is no unusable or unfruitful junk. Only in such an integral system is the future viability of humans assured. (p. 64)

Berry's comments mirror what might be termed a contemporized exposition of the Indigenous education processes of tribal societies. It is exactly within the light of such a vision that *this story* must unfold for Native and non-Native alike. If our collective future is to be one of harmony and wholeness, *or* if we are to even have a viable future to pass to our children's children, it is imperative that we actively envision and implement new ways of educating for ecological thinking and sustainability. The choice is ours, yet paradoxically we may have no choice.

AMERICAN EDUCATION FROM A TRIBAL PERSPECTIVE

Learning is always a creative act. We are continuously engaged in the art of making meaning and creating our world through the unique processes of human learning. Learning for humans is instinctual, continuous, and simultaneously the most complex of our natural traits. Learning is also a key to our ability to survive in the environments that we create and that create us.

Throughout history, human societies have attempted to guide, facilitate, and even coerce the human instinct for learning toward socially defined ends. The complex of activities for "forming" human learning is what we call "education" today. To this end, human societies have evolved a multitude of educational forms to maintain their survival and as vehicles for expressing their unique cultural mythos. This cultural mythos also forms the foundation for each culture's "guiding vision," that is, a culture's story of itself and its perceived relationship to the world. In its guiding vision, a culture sets forth a set of "ideals" which guide and form the learning processes inherent in its educational systems. In turn, these ideals reflect what that culture values as the most important qualities, behaviors, and value structures to instill in its members. Generally, this set of values is predicated on those things it considers central to its survival.

This narrative is a journey into the realm of cultural ideals from which the learning, teaching, and systems of education of Native America evolved. As such, these ideals present a mirror for reflecting on the critical dilemma of American education. For, while the legacy of American education is one of spectacular scientific and technological achievement, resulting in abundant material prosperity, the cost has been inexorably high. American prosperity has come at the expense of the environment's degradation and has resulted in unprecedented exploitation of human and material resources worldwide.

American education is in crisis as America finds itself faced with unprecedented challenges in a global community of nations desperately struggling with massive and profound social, economic, and cultural change. American education must find new ways of helping Americans learn and adapt in a multi-cultural, 21st-Century world. It must come to terms with the conditioning inherent in its processes and systems of educating which contribute to the loss of a shared integrative metaphor of Life. The loss of such a metaphor, which may ultimately lead to a social/cultural/ecological catastrophe, should be a key concern of every American.

The orchestrated "bottom-line, real world" chorus sung by many in business and government has become the all-too-common refrain of those who announce they lead the world. Yet, what underlies the crisis of American education is the crisis of modern man's identity and his collective cosmological disconnection from the natural world. Those who identify most with the "bottom line" more often than not suffer from image without substance, technique without soul, and knowledge without context: the cumulative psychological results of which are usually unabridged alienation, loss of community, and a deep sense of incompleteness.

In contrast, traditional American Indian education historically occurred in a holistic social context that developed a sense of the importance for each individual as a contributing member of the social group. Essentially, tribally contextualized education worked at sustaining a life process. It was a process of education that unfolded through mutual, reciprocal, relationships between one's social group and the natural world. This relationship involved all dimensions of one's being while providing both personal development and technical skills through participation in the life of the community. It was essentially an integrated expression of environmental education.

Understanding the depth of relationships and the significance of participation in all aspects of life are the keys to traditional American Indian education. "*Mitakuye*

Oyasin" (we are all related) is a Lakota phrase that captures an essence of tribal education because it reflects the understanding that our lives are truly and profoundly connected to other People and the physical world. Likewise, in tribal education, knowledge is gained from first-hand experience in the world and then transmitted or explored through ritual, ceremony, art, and appropriate technology. Knowledge gained through these vehicles is then used in the context of everyday living. Education, in this context, becomes education for "life's sake." Education is, at its very essence, learning about life through participation and relationship to community, including people as well as plants, animals, and the whole of Nature.

This ideal of education directly contrasts with the predominant orientation of American education that continues to emphasize "objective" content and experience detached from primary sources and community. This conditioning for being a marginal participant and perpetual observer, involved with only objective content, is a foundational element of the crisis of American education and the alienation of modern man from his own being and the natural world. In response to such a monumental crisis, American education must forge educational processes that *are* for *life's sake* and honor the Native roots of America. A true transition of today's American educational orientations to more sustainable and connected foundations requires serious consideration of other cultural, life-enhancing, and ecologically viable forms of education.

Traditional American Indian forms of education must be given serious consideration as conceptual wellsprings for the "new" kinds of educational thought capable of addressing the tremendous challenges of the 21st Century. Tribal education presents examples of models and universal foundations for the transformation of American education and the development of a "new" paradigm for curricula that will make a difference for *life's sake* in the world of 21st Century.

To begin such a process, orientations of American education must begin to move from a focus on only specialization, to holistically contextualized knowledge; from a focus solely on structures, to understanding of processes; from objective science, to systemic science; and from building, to "networking" as a metaphor for knowledge (Capra, 1982).

American education must rededicate its efforts to assist Americans in their understanding and appreciation of "spirituality" as it relates to the Earth and the "place" in which we live. It must engender a commitment to "service" rather than competition as an espoused social value. It must promote practiced respect for individual cultural and biological diversity. It must engage students in learning processes that fully facilitate the development of their human potentials through creative transformation.

American Indians have struggled to adapt to an educational process that is not their own with its inherent social, political, and cultural baggage. Yet, American Indian cultural forms of education contain seeds for new models of educating which can enliven American education as a whole, as well as allow American Indians to evolve contemporary expressions of education tied to their cultural roots.

For American Indians, a new circle of education must begin which is founded on the roots of tribal education and reflective of the needs, values, and sociopolitical issues as Indian People themselves perceive them.

Such a new circle must encompass the importance Indian People place on the continuance of their ancestral traditions; emphasize a respect for individual uniqueness in the diversity of expressions of spirituality; facilitate a strong and well-contextualized understanding of history and culture; develop a strong sense of place and service to community; and forge a commitment to educational and social transformation which recognizes and further empowers the inherent strength of Indian People and their respective cultures (Hampton, 1988).

To understand how to accomplish this, Indian People must begin to exploit all avenues of communication open to them and establish a reflective dialogue about a contemporary theory for Indian education that evolves from *them* and *their* collective experience.

In the past, Indian education has been defined largely by non-Indian educators, politicians, and institutions through a huge volume of legislative acts at the state and federal levels, which for decades have entangled Indian leaders, educators and whole communities in the morass of the federal government's social/political bureaucracy.

Indeed, Indian education stems more from the U.S. Government's self-serving political/bureaucratic relationship with Indian tribes than any truly culturally contextualized process rooted in tribal philosophies and social values. In fact, no contemporary theory of Indian education exists which can be said to guide the implementation or direction of educational curriculum development. Instead what is called "Indian education" today is really a "compendium of models, methodologies and techniques gleaned from various sources in mainstream American education and adapted to American Indian circumstances, usually with the underlying aim of cultural assimilation" (Deloria, 1990).

It is time for Indian People to define Indian education in their own voice and in their own terms. It is time for Indian People to allow themselves to explore and express the richness of their collective history in education. Among American Indians, education has always included a visionary expression of life. Education has been, and continues to be, a grand story, a search for meaning, and an essential food for the soul.

INDIGENOUS APPROACHES TO EDUCATION ARE VIABLE ALTERNATIVES

Alienation from mainstream approaches to education has been one of the consistent criticisms leveled against modern education by Indian students. They have been given relatively few choices of school curricula that truly address their alienation beyond compensatory programs, remediation, and programs that attempt to bridge the social orientations of students with those of the school. Rather, most of the attempts at addressing such issues have revolved around refitting the problematic Indian student to the very "system" that caused their alienation and failure in the first place. Too often, the Indian student is viewed as the problem rather than the inherent and unquestioned approaches, attitudes, perspectives, and curricula of the educational system. The knowledge, values, skills, and interests that Indian students possess are largely ignored in favor of strategies aimed at enticing them to conform to mainstream education. Few comprehensive attempts have been made to create a body of content and teaching models that are founded upon contemporized expressions of American Indian educational philosophy. The inherent worth and creative potential of Indian students and Indian perspectives of education have not been given serious consideration by mainstream education. Many of the brightest and most creative Indian students continue to be alienated from modern education.

The alienation of Indian students from education and the resultant loss of their potentially positive service to their communities need not continue if we revitalize and reclaim our own deep heritage of education. Indigenous approaches to education can work if we are open to their creative message and apply a bit of "Gadugi" – a Cherokee way to say working together – to find ways to revitalize and reintroduce their inherently universal processes of teaching and learning. Indigenous educational principles are viable whether one is learning leadership skills through community service, learning about one's cultural roots through creating a photographic exhibit, or learning from Nature by exploring its concentric rings of relationship.

The creative potential of building upon and enhancing what students bring with them culturally has been explored at a number of Indian educational institutions. The development of tribal community colleges and the evolution of "contract" schools governed by tribes offers one of the most plausible areas for the ongoing development of this nature.

INDIGENOUS EDUCATION AND ITS ROLE IN INDIVIDUAL TRANSFORMATION

In the context of development of a basic conceptual framework of a viable Indigenous educational philosophy, it is essential that the relationship of Indigenous education to establishing and maintaining individual and community wholeness be seriously considered. Much of Indigenous education can be called "endogenous" education in that it revolves around a transformational process of learning by bringing forth illumination from one's ego center. Educating and enlivening the inner self is the life-centered imperative of Indigenous education embodied in the metaphor, "seeking life" or for "life's sake." Inherent in this metaphor is the realization that ritual, myth, vision, art, and learning the "art" of relationship in a particular environmental context facilitates the health and wholeness of individual, family, and community. Education for wholeness, by striving for a level of harmony between individuals and their world, is an ancient foundation of the educational process of all cultures. In its most natural dimension, all true education is transformative and Nature centered. Indeed, the Latin root *educare*, meaning, "to draw out," embodies the spirit of the transformative quality of education.

> A transformational approach to education is distinctly universal, integrative and cross-cultural because it is referenced to the deepest human drives. From this viewpoint all human beings concern themselves with self-empowerment and with whatever enables them to transform their lives and the conditions in which they live; such a viewpoint engenders the intent of people striving to create whole, happy, prosperous, and fulfilling lives. (Faal, 2010)

The goals of wholeness, self-knowledge and wisdom are held in common by all the traditional educational philosophies around the world. Indeed, even through medieval times all forms of European education were tied to some sort of spiritual training. Education was considered important in inducing or otherwise facilitating harmony between a person and the world. The goal was to produce a person with a well-integrated relationship between thought and action. This idealized outcome was anticipated as following naturally from the "right education."

The "right education" is, of course, a culturally defined construct, one of whose main criteria is socializing the individual to the collective culture of a group. However, this sort of socialization is only one dimension of education, a first step in a lifelong path of learning. In reality, "right" education causes change, which in time creates a profound transformation of self. This transformation is a dynamic creative process, which brings anything but peace of mind, tranquility and harmonious adaptation. The exploration of self, and relationships to inner and outer entities, requires a tearing apart in order to create a new order and higher level of consciousness. Harmony is achieved through such a process but it lasts for only a short period of time before it again has to be revised as people and their circumstances change. This is the endogenous dynamic of tribal education (Fig. 2.1).

The process begins with a deep and abiding respect for the "spirit" of each child from before the moment of birth. The first stage of Indigenous education therefore revolves around learning within the family, learning the first aspects of culture, and learning how to adapt and integrate one's unique personality in a family context. The first stage ends with gaining an orientation to place.

Education in the second stage revolves around social learning, being introduced to tribal society, and learning how to live in the natural environment. The second stage ends with the gaining of a sense of tribal history and learning how to apply tribal knowledge to day-to-day living.

The third stage revolves around melding individual needs with group needs through the processes of initiation, the learning of guiding myths, and participation in ritual and ceremony. This stage ends with the development of a profound and deep connection to tradition.

The fourth stage is a midpoint in which the individual achieves a high level of integration with the culture and attains a certain degree of peace of mind. It brings the individual a certain level of empowerment and personal vitality and maturity. But it is only the middle place of life.

Fig. 2.1 The Indigenous stages of developmental learning

The fifth stage is a period of searching for a life vision, a time of pronounced individuation and the development of "mythical" thinking. This stage concludes with the development of a deep understanding of relationship and diversity.

The sixth stage ushers in a period of major transformation characterized by deep learning about what is subconscious. It is also a time of great travail, disintegration, wounding, and pain, which paves the way for an equally great reintegration and healing process to begin in the final stage. The pain, wound, and conflict act as a bridge to the seventh stage.

In the seventh stage deep healing occurs in which the self "mutualizes" with body, mind, and spirit. In this stage deep understanding, enlightenment, and wisdom are gained. This stage ends with the attainment of a high level of spiritual understanding which acts as a bridge to the finding of one's true center and the transformation to "being a complete man or woman in that place that Indian People talk about."

These stages of inter-relationship form a kind of creative continuum, "life way," which helps us to become more fully human as we move through the stages of our life. Indigenous education traditionally recognized each of the most important inter-relationships through formal and informal learning situations, rites of passage, and initiations.

Inherent in Indigenous education is the recognition that there is a knowing Center in all human beings that reflects the knowing Center of the Earth and other living things. Indian elders knew that coming into contact with one's inner Center was not always a pleasant or easily attainable experience. This recognition led to the development of a variety of ceremonies, rituals, songs, dances, works of art, stories, and traditions to assist individual access and utilize the potential healing and whole-making power in each person. The connecting to that knowing Center was choreographed through specific ritual preparation to help each individual on their journey to their own source of knowledge. Through this process the potential for learning inherent in each of the major stages of a person's life was engaged and set about the task of connecting to one's knowing Center. This was the essential reason for the various rites of passage associated with Indian tribes and various societies within each tribe.

Since the highest goal of Indigenous education was to help each person to "find life" and thereby realize a level of completeness in their life, the exploration of many different vehicles and approaches to learning was encouraged. This was done with the understanding that each individual would find the right one for itself in its own time. But the process of finding one's self and inner peace with its usual implications of being "adjusted," as it is called in modern circles today, was not the central focus of Indigenous education. Seeking peace and finding self was seen to be a by-product of following a path of life, which presented significant personal and environmental challenges, obstacles, and tests at every turn. This "individuation," as Jung called it, did not come easy. It had to be earned every step of the way. But in the process of earning it, one learned to put forward the best that one had, one learned the nature of humility, self-sacrifice, courage, service and determination. Indian People understood that the path to individuation is riddled with doubt and many trials. They understood that it was a path of evolution and transformation.

> Individuation is a work, a life opus, a task that calls upon us not to avoid life's difficulties and dangers, but to perceive the meaning in the pattern of events that form our lives. Life's supreme achievement may be to see the thread that connects together the event dreams, and relationships that have made up the fabric of our existence. Individuation is a search for and discovery of meaning, not a meaning we consciously devise but the meaning embedded in life itself. It will confront us with many demands, for the unconscious, as Jung wrote, "always tries to produce an impossible situation in order to force the individual to bring out his very best. (Sanford, 1977, p. 22)

There are elemental characteristics, which exemplified the transformational nature of Indigenous education. The following are a few of the most important elements that may provide points of reference for learning goals and the development of content areas.

First, was the idea that learning happens of its own accord if the individual has learned how to relate with his/her inner Center and the natural world. Coming to learn about one's own nature and acting with accord to that understanding was a necessary preconditioning which prepared the individual for deep learning.

Second, there was the acceptance that, at times, experiences of significant hardship were a necessary part of an individual's education and that such circumstances provided ideal moments for creative teaching. A "wounding" or memory of a traumatic event and the learning associated with such events provide a constant source for renewal and transformation which enlarged the consciousness if individuals were helped in understanding the meaning of such events in their lives.

Third, that empathy and affection were key elements in learning. Also, direct subjective experience combined with affective reflection is an essential element of "right" education. Therefore mirroring behavior back to learners became a way that they might come to understand for themselves their own behavior and how to use direct experience to the best advantage.

Fourth, an innate respect for the individual uniqueness of each person which gave rise to the understanding that ultimately each person was their own teacher as far as understanding and realization of their process of individuation. Indigenous education integrated the notion that there are many ways to learn, many ways to educate, many kinds of learners, many kinds of teachers, each of which had to be honored for their uniqueness and their contribution to education.

Fifth, that each learning situation is unique and innately tied to the creative capacity of the learner. When this connection to creative learning and illumination is thwarted, frustration and rigidity follow. Learning therefore had to be connected to the life process of each individual. The idea of life-long learning was therefore a natural consideration.

Sixth, that teaching and learning are a collaborative cooperative contract between the "teacher" and learner. In this sense the teacher was not always human but could be an animal, a plant, or other natural entity or force. Also, based on this perception, the "teachable" moment was recognized through synchronistic timing or creative use of distractions and analogies to define the context for an important lesson. The tactic of distract-to-attract-to-react was a common strategy of Indigenous teachers.

Seventh, learners need to see, feel, and visualize a teaching through their own and other People's perspectives. Therefore, telling and retelling a story from various perspectives and at various stages of life enriched learning, emphasized key thoughts and mirrored ideas, attitudes or perspectives back to learners for impact. Re-teaching and re-learning, are integral parts of complete learning. Hence, the saying, "every story is retold in a new day's light."

Eighth, that there are basic developmental orientations involved with learning through which we must pass toward more complete understanding. Learning through each orientation involves the finding of personal meaning through direct experience. The meaning that we each find is always subjective and interpretive based on our relative level of maturity, self-knowledge, wisdom, and perspective.

Ninth, that Life itself is the greatest teacher and that each must accept the hard realities of life with those that are joyous and pleasing. Living and learning through the trials and pains of life are equally important as learning through good times. Indeed, life is never understood fully until it is seen through difficulty and hardship. It is only through experiencing and learning through all life's conditions that one begins to understand how all that we do is connected and all the lesson that we must learn are related.

Tenth, that learning through reflection and sharing of experience in community allows us to understand our learning in the context of greater wholes. In a group there are as many ways of seeing, hearing, feeling, and understanding as there are members. In a group we come to understand that we can learn from another's experience and perspective. We also become aware of our own and other's bias and lack of understanding through the process the group. We see that sometimes People do not know how to take or use real innovation and that many times People do not know how to recognize the real teachers or the real lessons. We see that a community can reinforce an important teaching or pose obstacles to realizing its true message. It is not until, as the Tohono Odum phrase it, "when all the People see the light shining at the same time and in the same way," that a group can truly progress on the path of knowledge.

SOCIAL CONSCIOUSNESS AND INDIGENOUS EDUCATION

Paulo Freire, a Brazilian social reformer and educator, introduced a notion of education which closely parallels the role of Indigenous education in the transformation of the social consciousness of American Indians as they strive to "self determine" themselves in the face of the challenges of the 21st century. His thesis is founded on the notion that critical consciousness of cultural-historical roots of a people – as expressed and understood from the perspective the people themselves – is the foundation of a people's cultural emancipation. The modern struggle of Indigenous peoples throughout the world has been characterized by an attempt to maintain the most cherished aspects of their ways of life, their relationship to their lands, their consciousness of themselves as a distinct People. They are constantly engaged in a dynamic struggle to retain "the freedom to be who they are" in the midst of subtle and at times overt oppression by modern societies (Freire, 1970).

Freire's central message about education is that one can only learn and understand to the extent that one can establish a direct and participatory relationship with the natural, cultural, and historical reality in which one lives. This is not the same as the Western schooled authoritarian style of problem solving, where schooled "experts" observe a reality or situation from the outside and at a distance, then develop a solution or dictate an action or policy. This approach decontextualizes the problem from the totality of human experience and leads to a distorted perspective of the problem as an event that has relationship only to itself and to nothing else. This form of ultra-objectification denies the reality of interrelationship and reduces participation and learning to only an intellectual exercise of applying a preconceived objective method or model. The result is a perpetuation of dependence on an "outside" authority and the maintenance of the political power brokers behind such authority. Indigenous People who are "administered" education, extension services, and economic development in these terms usually remain oppressed and gradually become dependent on the "authority." In these circumstances, Indigenous Peoples' ability to revitalize and maintain themselves culturally, socially, and economically through a self-determined process of education is significantly diminished, if not outright destroyed.

Freire's approach is to begin with the way a group communicates about their world and their experiences in their social contexts. Then "generative" words, metaphors, or proverbs are identified which evoke thought, feelings, or reveal a historical perspective that have intrinsic meaning to people and their cultural way of life. These words or phrases are then translated into a variety of meaningful images and discussed with the People themselves to "unpack" their meaning. This process evolves through various stages of dialogue through structures called "culture circles." In the "culture circle" a group reflects on key generative words and symbols facilitated by a coordinator who helps form the dialogue. Since the words and symbols being used come from the language, cultural, or historical experience of the group, the people begin to reflect on their own collective stories in ways that stimulate new insights about themselves, their situations, and solutions to problems, which they face. Motivation, meaning, and "re-searching" of their cultural roots for possible models for viewing their problems are built into the "culture circle." The group learns by telling and retelling their stories, reflecting on there meaning, and reinforcing the vital elements of their cultural orientation. This process of learning stimulates the thinking of "People submerged in a culture of silence to emerge as conscious makers of their own cultures." The group learns how to create new meanings and apply insights derived directly from their own culture, history, and social experience to their contemporary life. What they learn about themselves through themselves forms the basis for authentic empowerment and the beginning of release from imposed authority through a process of education that has become their own. Through such a process the group can truly cease being "objects" for outside political, economic or educative manipulation. Instead, they become subjects in the making of their own stories for the future and controllers of their own destiny.

Freire's method has had a profound effect on increasing the literacy and the social consciousness of not only rural peoples in Brazil, but also millions of people in third-world nations. It works primarily because it acts to release what is essentially an Indigenous response to learning by fostering authentic dialogue about what is important to people in contexts of social and political situations that directly affect them. Relevancy of what is being learned and why it is being learned becomes readily apparent because it is connected to the cultural orientations, as the people themselves perceive them. The democratization of knowledge and the educational process perpetuated by Freire's approach mirrors that which occurs in Indigenous education. A new relationship between Indigenous

People and modern education and knowledge bases is made possible. The knowledge and educational orientation of modern educators is changed from an expert-recipient relationship to one of mutually reciprocal learning and co-creation. What is established is essentially a more ecologically sound and sustainable process of education. A kind of education is engendered which frees teachers, learners, and community to become partners in a mutual learning and becoming process.

Freire's method mirrors, at a social level, the ecologically inspired orientation of Indigenous education, which I have called "natural democracy." There is a direct communication between all individuals engaged in the educative process. The implicit paternalism, social control, and non-reciprocal orientation between experts and recipients of education give way to authentic dialogue which generates a high level of critical consciousness and the kind of educational empowerment that allows Indigenous People to become agents of transformation in their own social and cultural contexts.

The history of American Indian education has largely been characterized by a policy of assimilation combined with covert attempts at modernization of American Indian communities to "fit" them into the mainstream profile of American life. This has been, for the most part, a technical process of development, combined with intense indoctrination in the political and bureaucratic ways of the federal government. Educational development, like other extensions of "federal aid," has occurred through the actions of technicians, bureaucrats, and political manipulators who act to keep real decision-making power outside the parameters of the tribes and individuals affected. Many Indian educators, social reformers, businessmen and politicians continue to perpetuate this federal and mainstream paradigm either because they have never questioned their own educational conditioning within this system or because they have not found or explored alternatives. This situation has largely prevented Indian People from being the subject and beneficiaries of the exploration of their own transformative vision and educational process. As a result, Indian tribes are still relegated to having to "react" to "their" administration by the federal government because of continued dependence on federal aid and extension services. Rather than being "proactive" and truly self-determined in their efforts to educate themselves through themselves, Indian People continue to struggle with modern educational structures which are not of their own making and are separated from, and compete with, their traditional forms of education. There continues to be a kind of educational "schizophrenia" in the reality of Indian education today. Indian People continue to be one of the most educationally disadvantaged and "at risk" minorities in America today. This reality exists in spite of the enormously profound and elegant expressions of traditional education and philosophy. The essential question is what needs to happen to reclaim and rename this enormously important heritage not only for Indian People but also as a contribution to the educational development of all future generations?

The next phase of the development of Indian education requires the collective development of transformative vision and educational process based on authentic dialogue. This kind of development requires that “new structures” and “practices” emerge from old ones through a collective process of creative thought and research. An ongoing and unbiased process of critical exchange between modern educational thought and practice and the traditional philosophy and orientations of Indian People can only generate these kinds of new structures and practices.

A new kind of educational consciousness, an “ecology of Indigenous education,” must be forged which allows Indian People to explore and express their collective heritage in education and to make the kinds of contributions to global education that stem from such deep ecological orientations. The exploration of traditional Indian education and its projection into a contemporary context is much more than just an academic exercised. It illuminates the true nature of the ecological connection of human learning and helps to liberate the experience of being human and being related at all its levels.

From this perspective, education takes on the quality of a social and political struggle to open up the possibilities for a way of education that comes from the very “soul” of Indian People. It also brings to the surface the extent and the various dimensions of the conditioning of modern educational processes that have been “interjected” into the deepest levels of their consciousness. They become critical observers of the modern education to which they have had to adapt and which demands conformity to a certain way of education that more often than not has been manipulated to serve only certain “vested interests” of American society. Through the exploration of Indigenous education they learn how to demystify the techniques and orientations of modern education. This understanding allows them to use such education in accord with their needs and combine the best that it has to offer with that of Indigenous orientations and knowledge. They cease to be “recipients” of modern education and become active creators of their own education.

At a more inclusive level, exploration of Indigenous education liberates the Indian learner and educator to participate in the kind of creative and transforming dialogue that is inherently based on equality and mutual reciprocity. This is a way of learning, communicating, and working of relationship that mirrors those ways found in Nature. It also destigmatizes the Indian learner as being “disadvantaged” and the educator as the “provider of aid.” Rather, it allows both the learner and educator to co-create a learning experience and mutually undertake a pilgrimage to a new level of self-knowledge. The educator enters the “cultural universe” of the learner and no longer remains an outside authority. By being allowed to co-create a learning experience, everyone involved generates a kind of critical consciousness and enters into a process of empowering one another. And with such empowerment, Indian People become significantly “enabled” to alter a negative relationship with their learning process. Ultimately, with the reassertion, contemporary development, and implementation of such an Indigenous process at all levels of Indian Education, Indian People may truly take control of their own history by becoming the transforming agents of their own social reality.

In the final analysis, Indian People must determine the future of Indian education. That future must be rooted in a transformational revitalization of our own expressions of education. As we collectively "Look to the Mountain" we must truly think of that seventh generation of Indian children for it is they who judge whether we were as true to our responsibility to them as our relatives were for us seven generations before. It is time for an authentic dialogue to begin to collectively explore where we have been, where we are now, and where we need to go as we collectively embark on our continuing journey "to that place that Indian People talk about." I hope that this work will stimulate that kind of dialogue.

NOTES

1 This chapter is a reprint of and excerpts from the introduction and the final chapter of Cajete (1994).

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3. CRISIS AS A DISCURSIVE FRAME IN MATHEMATICS EDUCATION RESEARCH AND REFORM

Implications for Educating Black Children

Various groups, ranging from politicians to educational researchers to everyday citizens, have called for a reform of the U.S. educational system, particularly in the area of mathematics education. Due to the perception of a worsening system – evidenced by stagnating test scores and lags in international comparisons – many have moved to treat mathematics education research and reform in terms of crisis management. Instances of crisis management are seen in efforts to close the so-called racial achievement gaps, in efforts to educate and judge pre-service and in-service teachers, and in the standardized testing movement. This crisis management frame not only emerges in mainstream mathematics education discourse but also within critical perspectives on mathematics education, where there is an attempt to fight back hegemony, neoliberalism, and neoconservative forces working against democratic ideals and social justice.

The metaphor of crisis is powerful – articulating the need for immediate action that most, if not all, of the research and policy communities must undertake. There is a particular lure in discussing crises in education because they typically involve the needs of *children*. In this chapter, we examine the implications of employing a frame of crisis within mathematics education research and reform. What does it mean to have the power to name a phenomenon a crisis? When crisis management is emphasized, what myths and storylines emerge? We give particular attention to the implications of a crisis management frame for marginalized student populations, giving focus attention to Black children. We begin by making sense of the crisis narrative in mathematics education and inspecting who has the power to invoke crisis discourse, referencing specific historical events. Next, we consider *framing* and *discursive frames* as means to understand crisis narrative. Further, critical mathematics education, itself, is interrogated with respect to its relationship to frames of crisis. We conclude by discussing alternative framings for the aims and goals of mathematics education.

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 53–69.

THE CRISIS NARRATIVE IN MATHEMATICS EDUCATION

The prevailing discourse in narrative in mainstream mathematics education is plagued with crisis talk. It can be argued that every major turn in the mathematics education reform has been spurred by some impending crisis purportedly relating to threats to the U.S. economy and national security. In many of these instances, the crises have been imposed on mathematics education. In other instances, such as the math wars of the 1990's, the debates reflected ideological and political differences. As such, we assert the manufactured crises in mathematics education function more in the service of political, economic, and organizational agendas than the collective, best interests of children, teachers, schools, and communities. Lived crises emanating from human experience are often relegated to secondary status and recognized only to the extent in which individual interests converge with larger political and economic agendas.

Consider the "new math" reform movement of the 1950s and 1960s. Stemming most notably from the panic of Americans to the "missile gap" and the launching of Sputnik by the Soviets in 1957, the new math quickly became necessary as a way to help prepare the nation for a looming international crisis. This crisis threatened the economic and political standing of the US as a global super power. As a scapegoat, the blame for this failure and burgeoning doom was placed on American schools, particularly mathematics and science education. Others echoed the sentiment that in times of perceived national crisis, mathematics (and science) education receives special attention:

> That event [Sputnik] came amidst the cold war and Soviet threats of world domination. (It was more than 40 years ago, but I still remember Nikita Khrushchev banging his shoe on a table at the United Nations, and his famous words "We will bury you.") Sputnik spurred the American scientific community into action. (Schoenfeld, 2004, p. 257)

The responses to Sputnik, as well as larger narratives in the media and government, reinforced the perceived link between mathematics education and national defense. This commentary on the skills of American learners was not a new phenomenon. During World War II, military recruits were tested and found lacking in mathematical skills. Lappan described how Dr. Vannevar Bush, organizer of the Manhattan Project, which was responsible for the development of the atomic bomb, lobbied for federal support of science, mathematics, and technology, predicated upon the argument that the U.S. victory during World War II would be due to the nation's technological superiority. More specifically the argument rested on the idea that "without federal support in peacetime, any future conflicts might result in defeat due to scientific deficiencies" (Lappan, 1997, p. 1).

Subsequent to arguments such as that made by Vannevar Bush, the National Science Foundation (NSF) was created in 1950 as a way to support and strengthen science and mathematics education. Eight years later, as the press used Sputnik to call attention to the low quality of math and science instruction in the public schools, Congress responded by passing the 1958 National Defense Education Act to increase the number of science, math, and foreign language majors, and to contribute to school construction. With support from the National Science Foundation (NSF), a range of curricula with "modern" content – which emphasized concepts that had not previously been included in school mathematics such as set theory, modular arithmetic, and symbolic logic – were developed, giving birth to the new math. This movement was also marked by the collaboration of collegiate mathematicians and mathematics teachers in efforts to bridge collegiate mathematics and high school mathematics.

Although the new math movement was launched using a discourse of *national* crisis and as a reform to benefit all Americans, the larger political context showed that this was not the reality of the situation. Three years earlier, the U.S. Supreme court had finally overturned legalized segregation in the *Brown vs. Board of Education of Topeka* decision, which was brought before the Court by the National Association for the Advancement of Colored People (NAACP). The Supreme Court rejected the "separate but equal" doctrine that had been in effect since 1850, and that kept Blacks from attending white schools. Desegregation evolved slowly over the 1960s and racial conflict persisted. The pre- and post-Brown eras were certainly times of progress for Black Americans but the prevailing politics made it clear that their advancement in society was still restricted, even in the articulated crisis following Sputnik. By and large, Black children were not the intended beneficiaries of the new math rhetoric and reform. By the 1970s, Black children continued to attend predominantly minority and under-resourced schools and this trend continued until the late 1980s.

By the early 1970s, the new math movement was dead due to its alienating aspects – mathematics teachers were uncomfortable with the curriculum and parents were unable to help their children with homework. However, the new math movement essentially reinforced the idea that our educational system and accompanying reforms, particularly related to mathematics, would continue to be driven by perceptions of crisis. Following discontinued funding for the new math, there was an urgent call for "back to basics" in mathematics and other subjects due to the perceived sacrifice of procedural skills for conceptually based learning. However, the back-to-basics reform for education did not go unchallenged. Soon there was a new approach to the educational schooling system – *open education*. The open education movement was nothing new: it was a mere repetition of progressivist programs promoted in the 1920s (Klein, 2003). Some scholars argued that the effects of this movement were particularly devastating to underserved children, who were overwhelmingly minority and low-income children, because they had been denied opportunities to appropriate fundamental skills. This had negative effects:

> I have come to believe that the "open-classroom movement," despite its progressive intentions, faded in large part because it was not able to come to terms with the concerns of poor and minority communities. I truly hope that those who advocate other potentially important programs will do a better job. (Delpit, 1987, p. 185)

The idea of an educational crisis reached new heights by the early 1980s. This was in large part due to various reports and commissions that investigated the failures and shortcomings of K–12 education, including *An Agenda for Action: Recommendations for School Mathematics of the 1980s* (National Council of Teachers of Mathematics, 1980) and *A Nation at Risk: The Imperative for Educational Reform* (National Commission on Excellence in Education, 1983). In the *An Agenda for Action*, NCTM called for new directions in mathematics education, focusing more on problem solving through the use of manipulatives and calculators. These ideas would later be codified with the publication of the *Curriculum and Evaluation Standards* (NCTM, 1989). *Agenda for Action* received little public attention, mainly because, in 1983, the nation was called to action through the report *A Nation at Risk*. The report, which warned "if an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might well have viewed it as an act of war," helped to manufacture the threat of the next educational crisis (e.g., Gándara & Contreras, 2009). Thirteen "indicators of risk" within the report served as a basis for demonstrating how the poor educational system was putting America at risk. Furthermore, the document proclaimed that students from other countries were outperforming U.S. students, especially in mathematics and science. Failure was attributed to public schools and teachers. The document claimed that if measures were not taken to address mathematics and science in the school system, then the country would cease to be an economic global leader. Although it did not result in immediate federal action, the report has been acknowledged as the most important document in education in the 20th century, laying the foundation for a newly formed federal commission which launched a generation of attempts to reform and improve American public schools.

Ronald Reagan, claiming credit for the report, pushed forth the notion of competition and standards stating, "Just as more incentives are needed within our schools, greater competition is needed among our schools. Without standards and competition, there can be no champions, no records broken, no excellence in education or any other walk of life." Around the same time, Secretary of Education Terrel Bell created a "wall chart" ranking states by their educational attainment using ACT and SAT scores, and Secretary of Education William Bennett claimed a decline in student achievement, emphasizing the "crisis" at hand. Both contributed to the persistent idea that the nation was in grave danger due to the poor performance of public schools. Competition and standards among and within schools became a focal point and test scores became the prime indicator for the quality of schools. As the public opinion was in support of a strong focus on basic skills and clear, high standards, NCTM worked to recast its own agenda. In 1986, NCTM established the *Commission on Standards for School Mathematics*. The *Curriculum and Evaluation Standards for School Mathematics* was developed.

The *Standards* took a progressive stance to mathematics education, reiterating the need for student centered discovery learning, during a time in which "constructivism" was the favored idea around student learning. However, the NCTM Standards, despite the humanistic and inclusive approach, gained traction within an overarching conservative project operating within the U.S. educational system (Apple, 1992). To make a finer point, the NCTM Standards earned legitimacy and, indeed, profited from the crisis rhetoric of the day. Further, the NCTM Standards could potentially "*increase*, not decrease, the wide disparities that now exist in class, gender, and race in education" (Apple, 1992, p. 417). For example, the *Standards* called for increased use of technology and computers, yet access to such teaching aids are necessarily cost-prohibitive to underserved, minority communities. Here again, the prescriptions for reform appeared to be inclusive of particular communities and lack the required sensitivity to have relevance for other communities.

By the 1990s, much of the attention in education was on international comparisons of students' mathematics achievement. This shift of focus was influenced by the release of results of the Third International Mathematics and Science Study (TIMSS). In particular, U.S. rankings on international exams were touted as proof of the need for an increased focus on proficiency in mathematics education. As the trend continued into the new millennium, the National Mathematics Advisory Panel, for example, not only stressed the importance of mathematics and economic competitiveness, but also stated that the safety and quality of life for the nation are at risk if we do not develop a technical workforce of highly skilled laborers. The "looming" crisis in education constitutes the preparation of our youth for "commodity jobs" in a global economy that privileges "innovative work" (Shaffer & Gee, 2005). By invoking examples of India and China as countries that are willing to work for less money and also as educating their youth to be innovative, the crisis narrative was articulated with great clarity – foreign villains who could be fended off with greater strength in mathematics education. However, the campaign to bring more students into science, technology, engineering, and mathematics (STEM)-related fields is contradicted by labor statistics that actually show that low-skilled jobs will be more prevalent in the U.S. labor market, not highly skilled technical labor.

In the 21st century, *accountability* and *choice* have become the catchwords for education reform, emanating especially from *No Child Left Behind* program, ushered in by President George W. Bush. Through this initiative, school quality began to be measured primarily by standardized test scores. Whereas *A Nation at Risk* was a report, not a legal mandate, *No Child Left Behind* became a federal law where states or districts not in compliance with its mandates were in jeopardy of losing funds used for their most disadvantaged students. More recently, President Obama introduced the *Race to the Top* grants program, stating "if you set and enforce rigorous and challenging standards and assessments; if you put outstanding teachers at the front of the classroom; if you turn around failing schools – your state can win a Race to the Top grant that will not only help students outcompete workers around the world, but let them fulfill their God-given potential" (Obama, 2010). Despite these top-down proclamations of national educational crisis, the needs of Black and other marginalized children continue to be unmet. Thus,

> [d]uring the 1990s, segregation increased further across both schools and classrooms. Classroom-based segregation increased as a function of additional tracking within schools, a strategy that created largely segregated experiences for many within "integrated" schools. By 2000, 72% of the nation's Black students attended predominantly minority schools, up significantly from the low point of 63% in 1980. The proportion of students of color in intensely segregated schools also increased. Nearly 40% of African American and Latino students attend schools with a minority enrolment of 90 to 100% … Thus, with respect to school segregation, America stood at the gateway of the 21st century, almost exactly where it stood 30 years earlier – having lost in a giant tug-of-war much of the ground it gained during the 1970s. (Darling-Hammond, 2010, p. 35)

The brief accounting of educational reform presented above reveals that the invocation of crisis stands at the root of each effort, ranging from new math, to back to basics, to progressive movements, and culminating in the confluence of NCTM *Standards*, No Child Left Behind, and Race to the Top.

FRAMES AND FRAMING IN RESEARCH PARADIGMS

Crisis frames a problem and demands an *immediate* solution. However, both crisis and immediacy can position education research as an endeavor that views children, teachers, and schools as entities that need to be fixed, hence reinforcing deficit views, especially for certain groups. Metaphors like that of an education crisis do not manifest spontaneously into public consciousness, but are the product of prolonged use of discursive frames. Those who strategize about how best to handle or frame crises within the public eye are *crisis managers,* to use a term introduced by Cho and Gower (2006). Crisis managers have the benefit of naming – parsing out the facts and elaborating on particular narratives – crises for public consideration. Said simply, crisis managers *frame* crises.

Framing has two important features – selection and salience (Entman, 1993). Salience refers to making an idea notable or meaningful through placement or repetition. Framing "*select[s] some aspects of a perceived reality and make[s] them more salient in a communicating text, in such a way as to promote a particular problem definition, causal interpretation, moral evaluation, and/or treatment recommendation* for the item described" (p. 52, original emphasis). Frames then can be used to (a) define problems, (b) diagnose causes, (c) make moral judgments, and (d) suggest remedies. One or more of the uses can be at play when a frame is invoked. Consider an excerpt from Obama's State of the Union address in 2011:

> Half a century ago, when the Soviets beat us into space with the launch of a satellite called Sputnik, we had no idea how we would beat them to the moon. The science wasn't even there yet. NASA didn't exist. But after investing in better research and education, we didn't just surpass the Soviets; we unleashed a wave of innovation that created new industries and millions

> of new jobs. Think about it. Over the next 10 years, nearly half of all new jobs will require education that goes beyond a high school education. And yet, as many as a quarter of our students aren't even finishing high school. The quality of our math and science education lags behind many other nations. America has fallen to ninth in the proportion of young people with a college degree. And so the question is whether all of us – as citizens, and as parents – are willing to do what's necessary to give every child a chance to succeed.

The quotation works to frame education through definition, diagnosis, judgment, and remedy. In fact, the frame is not new but a regurgitation of the mathematics education crisis frame. The repetition of the "old" mathematics education narrative and its juxtaposition within the speech to the Great Recession works to create a sense of urgency and public resolve around educational reform.

Frames also have "locations": (a) the communicator, who creates the frame consciously or unconsciously (e.g., crises managers); (b) text in which the frame is used (e.g., news, speeches, policy documents such as those published by the NDEA, or organizations such as NSF); (c) the receiver who perceives the frame (e.g., generally the greater public); and (d) the culture wherein a stock of common frames is employed (Entman, 1993). In this sense, the text is a tool of the communicator that serves to deflect attention away from particular meanings and understandings, like the inequity within the public education system, and refocuses attention towards others, like the state of the economy.

Frames are particularly influential in guiding collective sense making. Powerful players, like the President of the US, use frames to set their own agendas before the receiving public such that they garner support regardless whether the solution benefits the public or not. By bringing to the fore certain thought patterns, commonsense knowledge is reshaped and the target audience is told how to think, feel, and decide. Whereas frames may be perceived in various ways by the receiver, it is important to acknowledge the *dominant meaning* – how framing is most likely interpreted. For instance, while the viewer of the State of Union address may be attuned to the described crisis in terms of the high school dropout rates, this interpretation of crisis is swallowed by the dominant meaning, which points to issues relating to the national economy and security. It is political elites that control the framing of issues and, consequently, attempt to control public opinion through the frames that they employ (e.g., Dutta-Bergman, 2005).

Frames and framing are not restricted to public or political discourse. Frames can offer a construct for understanding research paradigms – "*a research paradigm is defined here as a general theory that informs most scholarship on the operation and outcomes of any particular system of thought and action*" (Entman, 1993, p. 56, emphasis added). Certain questions drive the core of the meaning of mathematics education research (Sierpinska, Kilpatrick, Balacheff, Howson, Sfard, & Steinbring, 1993): (a) What is the specific object of study in mathematics education?, (b) What are the aims of research in mathematics education?, (c) What are the specific research questions or problematics of research in mathematics education?, (d) What are the results of research in mathematics education?, and (e) What criteria should be used to evaluate the results of research in mathematics education? Given a frame of crisis, the answers to such questions are constrained. The specific object of study under a frame of crisis is that which is *broken*, be it children, teachers, or curricula, and the aims of research are to *fix* these broken people and texts. Specific research questions then inquire about the *brokenness* of whatever is under study. The results of research in mathematics education are then measured against the extent to which the broken object is repaired; the criteria used to evaluate the results becomes a checklist or numeric of the idealized mathematics student, teacher, or text.

THE IMPLICATIONS OF THE CRISIS FRAME IN MATHEMATICS EDUCATION

The prevalence of particular frames within a research context gives rise to the creation of a *project;* such projects can be social, market-oriented, neo-conservative, liberal, social justice-oriented, and so on. Critical mathematics education researchers, for example, have characterized mathematics education research as projects organized around social justice, ethnomathematics, civil rights, and equity. Others have described the current state of mathematics education research as a neoliberal project subject to “market forces, market-driven goals, and increased globalization” (Martin, 2010). Martin has applied the construct of a “project” to mathematics education research and has argued that mathematics education has, historically, been put in service to the prevailing racial projects in society and is, itself, a type of racial project. To this list, we suggest that mathematics education is also a *crisis-management project*, wherein mathematics education research and policy are shaped by manufactured crises through frames generated by political elites (but generally unmoved by crises emanating from human experience). Martin notes, “It is clear, depending on how the aims and goals of mathematics education are conceptualized and framed, that the enterprise simultaneously represents and serves a host of competing projects, each of which calls for a preferred structuring of mathematics teaching, learning, curriculum, assessment, research, policy, and reform” (Martin, 2010, p. 3).

Additionally, projects (and frames) can work in concert with one another. To illustrate the integrated nature of projects in mathematics education and expand the idea of a crisis management project in mathematics education, we explore how neoliberal market-focused projects and racial projects are intimately tied to crisis management projects. Critical mathematics education scholars have argued that crisis framing serves to rationalize the development of human capital for corporate interests (e.g., Gutstein, 2009). In this way, ideology and interests are linked. In other words, frames necessitate a particular ideology, which is linked to the interests of certain groups. *A Nation at Risk* is considered by some an example of a crisis management tool representative of the Reagan era, which framed the discourse surrounding public education to advance a neoliberal agenda. This view competes with the philosophy that education should prepare students to be critical citizens. Thus, *No Child Left Behind* (NCLB) is a political platform that invited the testing industry and marketplace ideals into educational systems as a means to monitor upward and downward progress of a failing system – in the same way a EKG monitors the heart of an at-risk patient (Giroux & Schmidt, 2004). This move towards testing was also touted as a means to attend to issues of equity, so that underserved schools and children would be identified and held accountable for annual improvement.

For those schools and districts that perform well, testing has served as a mechanism of social mobility, particularly for those with power and privilege. On the other hand, testing has served primarily as a punitive stick for those students, schools, districts that have been historically and systematically denied educational resources. Further, high stakes tests privilege those of higher social class and those with "white" cultural capital, often curtailing opportunities for the poor and students of color. In this way,

> [t]esting has also become an ideological weapon in developing standardized curricula that ignore cultural diversity by defining knowledge narrowly in terms of discrete skills and decontextualized bodies of information and ruthlessly expunging the language of ethics from the broader purpose of teaching and schooling. (Giroux & Schmidt, 2004, p. 220)

In a sense, the crisis management tool of testing has served to recreate and exacerbate, instead of disrupt, the inequities that it was intended to eradicate. Herein lies the untenable paradox of a crisis management project in education and education research: the tools that are used to solve various crises often produce frames that give rise to additional crises. For example, testing, proposed as a monitor of educational inequity, has created yet another crisis, namely the so-called racial achievement gap, to legitimize its continued existence, (i.e., testing must continue until the "gap" is closed). The gaps of today that are cause for alarm have been at play since Blacks entered the US through the Trans-Atlantic Slave Trade. However, it was not until these academic "gaps" converged with larger goals of perceived national economic vitality and profit-making forces that an academic gap for Blacks rose to the level of crisis.

Black children are historically viewed under an *achievement* lens via the testing regime rather than an *experiential* lens (Martin, 2007). The achievement lens "has contributed to the social construction and representation of Black children as less than ideal learners" (p. 8). Given this lens, achievement framings locate Blacks at the bottom of the racial hierarchy and this hierarchy has also been imposed on students' mathematics learning, yielding their subsequent position in a *racial hierarchy of mathematical ability*. Here, it is important to note that crisis frames (e.g. the "racial achievement gap") do not operate independently of larger social frames. There are four discursive frames used in a colorblind racial ideology: (a) abstract liberalism, which relies on equal opportunity, choice, and individualism, basically, meritocracy; (b) naturalization, which suggests racial segregation (including hierarchies) are a natural phenomena and do not originate out of inequitable opportunity structures; (c) cultural racism, which relies on typically racist notions of cultural deficits like "Blacks are lazy"; and (d) minimization of racism, which argues that race is no longer a salient feature of U.S. life (Bonilla-Silva, 2003). As such, the crisis frame is reinforced by racial frames, which seek to rationalize the nature of the Black education crisis – the location of Black children at the bottom of the hierarchy of mathematical ability. Moreover, the achievement lens instantiates two larger narratives of meritocracy and naturalization, wherein some groups simply have natural math ability and rise within educational milieus due to their natural talents.

Mathematics as a crisis management project reduces children to objects in need of repair. Black children are objectified by a ranking or trend line that measures their progress relative to that of white children. Black children are considered "fixed" once there is parity between these groups. This contemporary example of the "racial achievement gap" in mathematics should not distract from larger historical frames that objectify Blacks and position them as deficient. The analysis of the impact of *Brown v. Board of Education* revealed that Jeffersonian views about the inherent inferiority of the African were foundational to the nation's beginnings, slavery, and segregation (Tate, Ladson-Billings, & Grant, 1993).

Test scores or achievement data were the most prevalent areas of focus and, to the extent that issues of race and ethnicity are even considered, psychological discourse was primarily used and race and ethnicity are considered as independent variables (e.g., Parks & Schmeichel, 2011). The treatment of race and ethnicity as variables in mathematics education research is further evidence of the objectification of children, insofar as racial and ethnic *experiences* are reduced to an undifferentiated singular variable.

CRISIS RHETORIC IN CRITICAL PERSPECTIVES

The frame of crisis also emerges within critical perspectives on mathematics education, where there is an attempt to fight back hegemony, neoliberalism, and neoconservative forces that work against democratic ideals and social justice. We argue that, without care, critical perspectives can serve to reinforce deficit-oriented crisis frames or become co-opted by crisis managers. We further note that critical mathematics education scholars when challenging various projects within research often use the dominant frames and, unfortunately, miss the opportunity to reframe the issues. By examining responses to National Mathematics Advisory Panel (NMAP), we discuss the advantages and challenges of reframing crisis discourse.

NMAP was commissioned by President George W. Bush, as part of a larger national conversation regarding mathematics education. The Panel was charged with reporting to the President and the Secretary of Education "on the conduct, evaluation, and effective use of the results of research relating to proven-effective and evidence-based mathematics instruction" (National Mathematics Advisory Panel, 2008). The Panel asserts:

> This Panel, diverse in experience, expertise, and philosophy, agrees broadly that the delivery system in mathematics education – the system that translates mathematical knowledge into value and ability for the next generation – is

> broken and must be fixed. This is not a conclusion about any single element in the system. It is about how the many parts do not now work together to achieve a result worthy of this country's values and ambitions. (p. xiii)

The idea that the answer to crises is to get more students to go into STEM fields can be challenged (Gutstein, 2008). Only the wealthy benefit from technological and productivity increases. We may indeed analyze the NMAP and the American Competitiveness Initiative anchored in sociopolitical contextual factors and criticize the NMAP final report for the exclusion of proposals that teachers should know their students, their students' culture. Moreover, there is an absence of multidisciplinary perspectives, the inclusion of only rigorous *scientific* research, exclusive focus on Algebra to the detriment of other topics, the lack of recognition of school constraints such as NCLB, lack of appreciation of math as a human activity, and the lack of intellectual excitement (Greer, 2008). Finally, the panel formation as an instantiation of white institutional space where white superiority and white privilege and power are protected and maintained through the exclusion of research perspectives of Black math scholars and through their notable absence as members of the panel (Martin, 2008). This effectively e(race)es race from the conversation such that the word "race" rarely appears in the document and is only invoked "for the purpose of reifying the notion of a racial achievement gap in mathematics achievement" (p. 392). These are but three responses to different aspects of the crisis framework promulgated by NMAP. In hindsight, these critiques are necessary but insufficient as a response to crises framings in mathematics education. Each response uses critical perspectives to analyze crises frames, but does not address the issue of mathematics education crisis directly. We can summarize these arguments in this way: (a) there may or may not be a mathematics education crisis, but it is not a function of an economic crisis; (b) there may or may not be a mathematics education crisis, but there is a crisis in theoretical and methodological approaches to what is valued as the "best available scientific evidence"; (c) and there may or may not be a mathematics education crisis, but the Panel lacks credibility for any assertion due to a lack of racial and ethnic diversity.

Whereas each argument is forceful on its merits and highlights a crucial perspective in mathematics education research, these arguments do not attend to the dominant frame of mathematics education being in a state of crisis – a dysfunctional delivery system that "translates knowledge into value and ability" (p. xiii). If the frame is not repositioned or disrupted in the context of the argument, i.e., if the critique works within the dominant frame, the *back-story* has been accepted, the crises narrative takes primacy, and the critical perspective is essentially relegated to the background. Critical math researchers therefore need to be in the business of reframing. Reframing is equivalent to social change in that it causes one to seek different goals; it changes the questions we ask, and it structures "our social policies and the institutions we form to carry out policies" (Lakoff, 2004, p. xv). The author provides a poignant example as to why reframing is so important:

> On the day that George W. Bush arrived in the White House, the phrase *tax relief* started coming out of the White House. … Think of the framing for relief. For there to be relief there must be an affliction, an afflicted party, and a reliever who removes the affliction and is therefore a hero. And if people try to stop the hero, those people are the villains for trying to prevent relief. When the word tax is added to relief, the result is a metaphor: Taxation is an affliction. And the person who takes it away is a hero, and anyone who tries to stop him is a bad guy … the Democratic senators … had their own version of the tax plan, and it was their version of tax relief. They were accepting the conservative frame. The conservatives had set a trap: The words draw you into their worldview. (pp. 3–4)

Even if the responses were considered and taken up by NMAP, then what? Still remaining are the sentiments that led to narratives of crises in the first place. In other words, the nature of framing makes it impossible for the dominant crisis metaphors and critical perspectives to coexist without substantively changing the debate. Without reframing, critical mathematics arguments and projects can be taken up as commodities. *Mathematics for All* is an empty rhetoric in reform documents such as the *Curriculum and Evaluation Standards for School Mathematics,* which are coupled with top-down and school-focused notions of equity rather than on reflective community-based notions of equity that focus on the ties between mathematics learning and the lived realities of marginalized students (Martin, 2003). Although well intentioned – the rhetoric suggests an inclusive goal – these documents did not account for students' experiences in their ideas of equity. Thus, "Rather than responding directly to the needs of marginalized students … policy makers and mathematics educators have decided what (valued) mathematics should be learned, who should learn this mathematics, and for what purposes" (p. 12). Without reframing, equity reforms that have the potential to benefit Black children are vulnerable to being co-opted and reshaped to serve outside interests. Critical scholars hold the key to equity in mathematics education research. Therefore, the intent is to push the conversation about reframing arguments, particularly as they relate to crisis. In the following section, we propose three alternative frames in mathematics education research that push back against mathematics education research as a crisis management project.

ALTERNATIVE FRAMINGS IN MATHEMATICS EDUCATION RESEARCH: IMPLICATIONS FOR BLACK LEARNERS

Accepting the crisis metaphor in the discourse of mathematics education research necessarily limits our ways of knowing about Black children. Issues regarding who gets to frame an educational phenomena as a crisis and how these phenomena are framed as crises are critical in setting a responsible agenda within mathematics education research. We propose three ways in which the crisis frame and research as a crisis management project can be treated. First, we consider the stance of leveraging the existing crisis frame in service to the needs of Black children. After inspecting the limitations of working within the crisis frame, we provide an alternative frame that focuses on the experiential reality of Black children, where crises are named by children and communities based on their day-to-day social realities. Finally, we consider the abandonment of the crisis frame and call for accepting that crisis does not accurately depict education within the Black community.

Marginalized children may be empowered in the educational system yet doing so within a crisis discursive frame (Lubienski, 2008). "Gap gazing" relative to achievement results may be a means of promoting educational equity in mathematics. Gap analyses of racial achievement are critical for shaping public opinion and policy, in addition to informing mathematics education practice and research. Without such audits, the veneer of a quality education can in fact mask systemic and long-standing inequities. Further, new statistical methods (i.e., hierarchical linear modeling, cross-classified models, and propensity score matching) can provide more nuanced analyses that inform issues of equity. However, as previously mentioned, the untenable reality of leveraging existing frames is that we must accept the "baggage" of those frames; in this case, the racial hierarchy of mathematical ability. Further, a focus on achievement outcomes exclusively can disguise inequities of *experience* that serve as inputs. With the onslaught of testing that creates achievement data, teachers of Black children, however well-intentioned, disproportionately employ remedial and test-preparatory strategies to the detriment of the students in their classes (Davis & Martin, 2008). Thus,

> [t]he mathematics instruction that these [Black] students are exposed to emphasizes repetition, drill, right-answer thinking that often focuses on memorization and rote learning, out-of-context mathematical computations, and test taking strategies. … This type of instruction leaves African American students disengaged and viewing mathematics as irrelevant and decontextualized from their everyday experiences. (p. 20)

To account for the qualitative input of student experiences as well as the quantitative output in the form of test scores, we need an experiential lens in the research of Black children in mathematics education. The experiential lens modifies the operative crisis frame. An experiential lens does not exclude crisis as a reality – experiences and a crisis narrative are not mutually exclusive.

When evaluating the lived realities of certain groups, *crisis* may be the only way to describe the nature of their experiences. For example, the "pedagogy of poverty" is certainly a crisis for the children who experience a menu of teaching practices, including: "giving information, asking closed questions, giving directions, making assignments, monitoring seatwork, reviewing assignments, giving tests, reviewing tests, assigning homework, reviewing homework, settling disputes, punishing noncompliance, marking papers, and giving grades" (Haberman, 2010, p. 82). A *prima facie* case can be made that these practices are common. However, when these practices are taken in sum "to the systemic exclusion of other acts" (p. 82) an oppressive pedagogical formula supplants worthwhile teaching and learning.

Minority and poor students alike are often thought to be inferior by their teachers and, thus, are taught mathematics in low-level, procedural ways. Such teaching and learning could be considered a crisis. A single mother of three children, Tina, who was 23 years of age suggested:

> Math was a drag because our math teacher. … He made us write sentences. More sentences than math. … He told us to go up to the board and do a math problem and if we didn't get the problem all the way correct, then we had so many sentences to write. So, all I could [remember] was me writing more sentences than doing math. (Martin, 2000, p. 60)

Tina's mathematics socialization was strongly influenced by the pedagogy of poverty. We find that Tina experienced remedial course work very negatively, which resulted in her eventual disconnection from mathematics. It could be argued that Tina is articulating very clearly an educational crisis, however, voices like hers are often muted from larger, national conversations, excluded from the general framings of crisis unless such crises align with master narratives and frames or used as sound bites in politicized reform efforts whose goals are not intended to ameliorate the experiences of learners like Tina. Therefore, an inherent problem with reframing crisis through the experiential lens is that the dominant frames of crisis can overtake nuanced and personal-level accounts of crisis.

Framings that grow from an experiential lens are also subject to commodification and misinterpretation. Social justice in mathematics education, for example, has become en vogue, and used by some as code to signal pedagogical treatments for youth of color. Without care, social justice educational movements in mathematics can manifest as another instantiation of a crisis management project, where Black children's experiences, for example, are used as legitimating objects. Specifically, advocates of social justice in mathematics can fall prey to packaging marginalized children's experiences to substantiate their arguments within the research community and make the case of immediate reform. Gutstein (2012) warns against such constructions of social justice and notes that "[social justice] [c]urriculum, then, is not imported, but is particularized to students' lived realities" (p. 7). Gutstein calls for the use of generative themes in curriculum construction. By using generative themes, children's experiences take primacy in the fight against neoliberalism.

Perhaps the only way out of the *crisis frame* is to deny its existence altogether. We maintain that the frame of crisis presupposes several misconceptions: (a) at some point the system worked well and has suddenly and unexpectedly been untracked, (b) there is an immediate solution, (c) there is an easily identifiable cause, and (d) there is someone or something concrete to blame. These misconceptions play to our commonsense understanding of crisis, but are insidious when applied to field of mathematics education research. That is, instead of attempting to modify the discourse of crisis by allowing Black children to name the nature of the crisis, it is best to simply acknowledge that the invocation of crisis is in fact the modus operandi of the U.S. educational system. Rather that being in a constant state of crisis, mathematics education functions exactly as it is intended: to help maintain the social and economic status quo.

Racism and institutions designed to control Blacks in the U.S. are not static but transform to maintain racial oppression (Alexander, 2010). As slavery ended, Jim Crow was adopted. As the Civil Rights Movement gained ground and the old racial caste system began to crumble, “a new system of control” (p. 22) emerged – the mass incarceration of Black people. Trickling down to the schooling level, we may include the miseducation of Black children as an institutional goal and norm that attempts to maintain control of Blacks as second-class citizens keeping white supremacy and power protected. An examination of the effects of gender and racial stereotyping on Black school boys reveals that “just as children were tracked into futures as doctors, scientists, engineers, word processors, and fast-food workers, there were also tracks for some children, predominantly African American and male, that led to prison” (Ferguson, 2000, p. 581). In this sense, schooling for Black children functions, to produce mathematically illiterate, marginalized citizens.

Eliminating the frame discourse of crisis does not mean that the conditions for Black boys described above, for example, are not alarming. Rather, the discourse of crisis may not be accurate. A crisis is a matter of (a) perceived value of possible loss, (b) perceived probability of loss, and (c) perceived time pressure. This seminal crisis model on all three metrics points to the denial of crisis within mathematics education for Black children (Billings, Milburn, & Schaalman, 1980). The decades over which the loss of Black human capital has occurred indicates a perception that time is of no consequence to Black children. The non-monetary costs to society have remained largely ignored and are inconsequential to white interests (Freeman, 2005). Negative outcomes for Blacks simply serve as evidence of the natural racial order to society, instead of an indicator of dysfunction within the U.S. By resisting the crisis discursive frame, the mathematics education research community may be forced to interrogate a system that is designed for the detriment of Black children.

CONCLUSION

Crisis has been a longstanding metaphor used to refer to mathematics education. When using the crisis metaphor to frame the way we think about mathematics education, the narrative forms the basis for the types of questions that researchers ask and the types of answers that are deemed acceptable. This necessarily limits knowing about Black children in ways that tied to their experiences with mathematics *as Black children*. We call for critical mathematics researchers to continue the work of reframing the so-called crisis discourse. Our goal is to create discussion among the mathematics education researchers about the frames that they are using and whether such frames are aligned with crisis rhetoric.

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MARTA CIVIL & NÚRIA PLANAS

4. WHOSE LANGUAGE IS IT?

Reflections on Mathematics Education and Language Diversity from Two Contexts

In the last decade we have intensified our work on mathematics education and language diversity in our two contexts of research: Tucson, AZ, and Barcelona, Catalonia. The two of us are interested in the role and use of languages in scenarios of mathematical teaching and learning. From the perspective of our investigations, this interest entails different interpretive methods and large collections of data. In this chapter, we focus on classroom data to draw on similarities and differences coming from the analysis of English and Spanish bilingual communities in Tucson, and Catalan and Spanish bilingual communities in Barcelona.

A vast body of literature in the area draws on the basic understanding of the relationship between attention to language diversity in multilingual classrooms and opportunities for mathematical learning. Many researchers have documented that for multilingual people the choice of which language to use is not arbitrary, but guided by social and political aspects that influence which language to use and how (e.g., Clarkson, 2009; Moschkovich, 2002; Setati, 2005). In multilingual classroom settings, there is a complex relationship between, on the one hand, the different value given to different languages, and on the other, the students' language choices in their mathematical practices. Following this assumption, it seems clear that the valorization of languages has a role in determining forms of communicating and knowing (in) mathematics. Thus, we consider the topic of this book, *Alternative Forms of Knowing (in) Mathematics,* from the perspective of how certain interpretations of language diversity intervene in the construction of processes of (having access to) knowing school mathematics.

We begin with a brief discussion of the language policies and dominant language ideologies in our contexts of research. We then develop two themes to better understand the phenomenon of students learning mathematics in a classroom with a language that is not their home language: (1) dynamics of power in the bilingual mathematics classroom, and (2) bilingual students' views of access to mathematics.

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 71–89.

OUR CONTEXTS OF RESEARCH: ARIZONA AND CATALONIA

In 2000, Proposition 203 was passed in Arizona. This Proposition severely limited bilingual education and placed English Language Learners (ELLs) in Structured English Immersion (SEI) programs, where instruction was in English with minimal clarification in the students' home language if needed. In 2006, the Arizona legislature enacted HB 2064 by which ELLs were to receive four hours per day of English language instruction. This means that for at least four hours a day ELLs are kept apart from non-ELLs to receive instruction in reading, writing, grammar, vocabulary, not attached to specific content areas but geared only to developing their English language. In schools where there are large numbers of ELLs (which is the case where our research takes place), this results in the segregation of these students as they are often kept together for most of the school day (beyond the four-hour block). In July 2010, The Arizona Educational Equity Project under the auspices of The Civil Rights Project at University of California, Los Angeles, published nine papers that analyze the educational conditions of ELLs in Arizona with the implementation of the four-hour block separation. The authors of one of these papers write:

> In the case of English learners in Arizona, these students are typically *triply* segregated in the schools to which they are assigned: by ethnicity, by poverty, and by language. Linguistic segregation at the classroom level for much of the day intensifies all the negative impacts of school segregation. (Gándara & Orfield, 2010, p. 4)

These authors further ask, "is the four-hour Structured English Immersion block that is being implemented today in Arizona a return to the Mexican room?" (p. 9). The Mexican room refers to the segregations of Mexican American students in the 1940s, a segregation that resulted in their receiving a lower quality education. And in their conclusion, they write, "[the segregated four-hour block] is stigmatizing, marginalizing, and putting these students at high risk for school failure and drop-out" (p. 20). Language policies reflect the current political environment. In the case of Arizona, language policies that restrict the use of a language other than English in schools are closely associated with an anti-immigration movement in the State. One compelling analysis of Proposition 203 suggests that there:

> … is also collateral damage as the result of the political spectacle surrounding Proposition 203. The issue was promoted as pro-immigrant and supposedly only dealt with the narrow issue of the language of classroom instruction. However, it sparked widespread debate about immigration and immigrant communities as a whole, stirring up strong emotions about illegal immigrants and directed attacks on the Hispanic community in particular. (Wright, 2005, p. 690)

As educators, to ignore the political underpinnings of school language policies would be irresponsible. In this chapter, we explore this topic in two contexts with language policies that privilege one language over any other in schools, but with different political and historical situations.[1]

We now turn to briefly explain the language situation of Catalan and Spanish in Catalonia, an autonomous region in North Eastern Spain. Similarly to what happens with English in Arizona, Catalan is the official language of teaching in Catalonia.[2] Several political scenarios, including the reaction to Franco's dictatorship period (1939–1975), led to Catalan becoming the official language in schools. Catalan was a forbidden language during the dictatorship. The only official language in all parts of Spain was Spanish, although during later stages of the Franco regime, certain uses of Catalan were "tolerated." This same language is now being politically affirmed as a consequence of processes of Catalan nation-building that focus on differences between Catalonia and the rest of Spain. At present, discourses that point to Spanish as the national common language for all people in the country are gaining force. However, the Catalan nationalist government is committed to the continuity of the current language policies in Catalonia, primarily due to the idea of a differential identity that they sustain.

In a provocative text titled "We don't speak Catalan because we are marginalized" (Woolard, 2003), the author argues that there are some users of Catalan who are perceived to own this linguistic capital more than others due to both language and social origins:

> As the Catalan language has become more of a necessity for getting work done in formal institutions and public spheres and for high achievement in institutions like schools, it has also become a social resource more predictably acquired and used by middle-class Castilian speakers, whose interests are more often identified with these institutions. Now Catalan has higher class connotations not just because native Catalan speakers tend to be from the higher classes, but because middle and upper-middle class individuals from non-native ethnolinguistic backgrounds also tend to have good control of and make more extensive use of Catalan. (pp. 100–101)

In the Catalonian context, all students are somehow in paradoxical positions. Those who are Catalan-dominant speakers are also Spanish and Catalan bilingual, but at the same time are expected to behave as monolingual in their classrooms. On the other hand, many immigrant students from South America speak (a version of) Spanish, which makes them, in a way, language-privileged in comparison to immigrant students from South Asia. Nevertheless, at school they are also expected to use only one language, Catalan. Finally, those who are neither Spanish- nor Catalan-dominant speakers tend to first learn Spanish in their neighborhoods, and even in their families with their siblings, and then are expected to use Catalan when they enter school. It is clear that the two languages – Catalan and Spanish – are very visible in educational and institutional discourses, and they ideologically serve for much more than just talk (Adler, 2001).

SOCIO-POLITICAL ISSUES IN MULTILINGUAL MATHEMATICS CLASSROOMS

The work in Tucson is part of the research agenda of the Center for the Mathematics Education of Latinos/as (CEMELA).[3] This Center aims to understand the interplay of mathematics education and the language, social, and political issues that affect Latino/a communities. CEMELA's theoretical framework is grounded on the view of cultural and language diversity as resources towards the learning of mathematics for all students. With respect to language we include two excerpts from CEMELA researchers that help illustrate our position:

> Education for subordinated groups can mean self-determination, and this is intertwined with empowerment, self-respect, respect for one's history and community. From this perspective, understanding development in mathematics is to understand the relationship of a constellation of sociocontextual factors. Within this constellation is the nature of language use, the resultant discourse community in mathematics classrooms, and students' participation in this discourse community, especially when there is more than one cultural language. (Khisty, 2006, p. 438)

> A crucial pitfall to avoid when examining language and mathematics learning for students who are bilingual, multilingual, or learning English is using deficit models of language minority learners and their communities … [A]ny time we use monolingual learners (or classrooms) as the norm, we are imposing a deficit model on bilingual learners. Bilinguals learning mathematics need to be described and understood on their own terms and not only by comparison to monolinguals. (Moschkovich, 2010, p. 11)

CEMELA's research as well as that of other researchers working with non-dominant students in mathematics education point to several sociopolitical issues intervening in the unequal distribution of access to mathematical knowledge. Partial explanations for this unequal distribution have to do with classroom practices that do not include diversity. But there are other structural conditions that need to be considered to understand the whole picture. A sociopolitical approach to the understanding of what happens in multilingual mathematics classrooms, and why, requires complementing discussions on topics of language proficiency, teaching strategies, learning difficulties, and inclusive curricula. Far from viewing language as a neutral object in the classroom, it is necessary to address questions concerning the several visible and invisible messages that are sent to learners (who are, in particular, language users) through the differing representations and valorizations of languages (and language uses).

It is a complex issue to know whether language uses cause valorizations or valorizations cause language uses. Both directions of influence are at the heart of sociopolitical debates concerning multilingual mathematics classrooms: Do students facilitate particular positions in the classroom by the mere fact of using *a* language in their talks with others at certain moments? Is it that talking with some of the other participants in the classroom leads to the use of a certain language together with the creation of particular positions? In our sociopolitical approach, we do not pretend to situate the debate on what comes first or what is caused by what. We argue that language policies, multilingual classroom practices, and students' language uses are part of broader social and political debates that cannot be easily deconstructed. When viewing the teaching and learning of mathematics in multilingual classrooms, explanations cannot be reduced to individual and professional conditions of the teacher and the student. We need to include considerations from several other possible issues of influence, such as the situational circumstances that make one language more appropriate to use than another. We illustrate some of this complexity in this chapter.

DYNAMICS OF POWER IN THE BILINGUAL MATHEMATICS CLASSROOM

In this first section we discuss findings from our studies in Barcelona and Tucson with a focus on understanding some of the dynamics of power in the bilingual mathematics classroom. A search for dynamics in the classroom allows going beyond tensions between language groups and concentrating on how students from these groups effectively gain and share power. We assume that power is any expression of influence and control. It can be exercised by everybody, and not only by those who belong to the majority group. Under this assumption, our main question is how language minority students succeed in gaining power, resisting certain influences, and sharing language resources with other people in the class through mathematical interaction.

Data from Catalonia

To illustrate the case of Catalonia, we draw on observations of several mathematics lessons in a secondary classroom with Spanish and Catalan bilingual students – some had immigrant origins while some were immigrant themselves. There were eight students from South America who were Spanish-dominant bilingual, whereas the other 16 students from Catalonia – mostly from Barcelona – were Catalan-dominant bilingual, except for one, who was a second-generation immigrant from a Colombian family. The lessons were planned so that the students spent most of the time working in linguistically homogeneous small groups determined in terms of students' dominant language. There was also a whole-class discussion at the end of the session when the students got a chance to share their different approaches to the task. The classroom teacher, who was bilingual in Catalan and Spanish herself, encouraged students to use their first language by grouping them according to their dominant language. Our research focused on students who spoke Spanish at home. They all had similar working-class backgrounds. Most of their parents had not completed high school, had limited Catalan proficiency, and had immigrated to Catalonia for work reasons.

The data come from five videotaped 50-minute lessons that focused on geometrical transformations, mainly translation, rotation, homothecy, and symmetry. These concepts were part of a unit called "Our dynamic planet," which included a variety of mathematical activities that encouraged students to pose questions and solve problems in real contexts. This unit had been designed the year before by a group of teachers in the school as part of the development of innovative teaching materials in support of the students' mathematical learning. In the third lesson, the teacher wanted the students to think about "How can you mathematically represent a tornado?" The following excerpt shows part of the interaction in one of the Spanish-dominant groups, with Máximo (a second-generation Colombian boy), Luna (a girl born in Peru), and Nicolás and Eliseo (two boys born in Colombia) (Planas & Setati, 2009):

Máximo: [Catalan] Hem de deicidir les fletxes que dibuixem i ja està (We need to decide the arrows that we draw and that's all.)
Eliseo: [Catalan] Primer pensem les fletxes, després les dibuixem i després en parlem. (First we think about the arrows, then we draw them and then we talk about it.)
Máximo: [Spanish] Esta idea de las flechas no es fácil. Tenemos que imaginar los diferentes movimientos que existen dentro del tornado. (This idea of the arrows is not easy. We have to imagine the different movements that exist within the tornado.)
Eliseo: [Spanish] Una flecha tiene que ser una línea recta para que el tornado baje. Tenemos la t para la translación. (An arrow needs to be a straight line for the tornado to go down. We have the t for the translation.)…
Luna: [Spanish] La pregunta pide representar un tornado, ¿no? (The question asks to represent a tornado, doesn't it?)
Nicolás: [Catalan] Sí, diu que s'ha de representar matemàticament un tornado. (Yes, it says that we need to mathematically represent a tornado.)
Luna: [Catalan] No és parlar d'un tornado, és representar-lo matemàticament. (It is not to talk about a tornado, it is to mathematically represent it.)
Eliseo: [Spanish] Nos puede ser útil representar un tornado antes de dibujarlo. (The drawing of a tornado can be useful before its representation.)
Nicolás: [Spanish] Está claro que con una sola flecha no basta, porque un tornado es más que una translación. (It may be useful to represent the tornado before drawing it.)
Eliseo: [Spanish] Hay que pensar en cómo dibujaríamos una espiral. Dibujaríamos curvas. (We need to think about how we would draw a spiral. We should draw curves.)

In looking at data from this group, it can be observed that the students use their two languages for different purposes. They use Catalan when getting familiar with new vocabulary, when situating the use of this vocabulary in the context of the given task, and when beginning to organize approaches to solving the task. However, they use Spanish, their dominant language and the language that they share with their peers, when arguing and counter-arguing with various degrees of specificity and developing more complex comprehension processes that are not centered on the repetition of the teacher's ideas. The use of the dominant language when elaborating on an argumentation has been observed among Latina/o students, who use Spanish to justify an answer or elaborate on an explanation, and return to English to give priority to the acquisition of new vocabulary (Moschkovich, 2007).

From the perspective of power, the students in the group exercise considerable sense of agency. By the mere fact of alternating between their two languages in a classroom context (although Catalan is the official language), they are producing strategies of resistance. They develop autonomous actions based on their ability to change from one language to another in their interaction with one another. Power is often understood as a property of social structures and institutions; but it also has to do with particular actions that people, who participate in these structures, do to maintain active participation. Although data from these lessons show that the immigrant students from Latin America do not participate in whole class discussions as much as their Catalan-dominant peers, they have been agentive enough to alternatively use Spanish and Catalan in their small group.

The alternative use of Catalan and Spanish in peer interaction appears as a powerful feature of communication in the development of the mathematical task. All the students in that group are Spanish-dominant speakers. Thus, by switching between languages, they may not be signaling to any of the peers in the group, but, instead, may be unconsciously reacting to what they anticipate will happen in the whole class discussion with the teacher and other Catalan-dominant peers. It is our argument that the alternative use of languages leads students to better face the requirements of multilingual schools and classrooms. Conversely, avoiding the use of the home language or not getting really involved in improving the knowledge of the language of teaching is detrimental to learning. Particular ways of alternating languages can be advantageous from the perspective of language minority students' learning in linguistically homogeneous group work and at the same time can be detrimental from the perspective of these students' learning in whole-class discussion.

Data from Arizona

During the academic year 2006–07, Marta conducted 26 classroom observations, with 13 of those being videotaped between October and April. Most of the 27 students had varying degrees of bilingualism (English-Spanish) with the exception of 2 students who were monolingual English speakers. Two other students were relatively recent arrivals from Mexico and were Spanish-dominant. The teacher was bilingual but the instruction followed the language policy and was all in English, except for occasional clarification in Spanish with these two students. The vignette we present focuses on the interactions between Dania (fully bilingual) and Carolina (Spanish-dominant, arrived to the US in fifth grade) and Albert and Adam (both English-dominant but with some knowledge of Spanish). The students are working in pairs, the two boys sitting near the two girls, but they are not in a group of four. They are working on a fraction problem that shows two sections of land partitioned into several plots of different sizes, each with a person's name as the owner. The students need to figure out what fraction of a section each person owns[4]. Dania and Carolina are speaking in Spanish all throughout the problem and they are making good progress towards finding the fractions. Adam comes over from time to time to check with Dania. In the excerpt below, Adam has 1/8 + 1/16 for one of the plots, while Dania has 3/16. Marta asks them if they have the same answer.

Marta: You have, you have one eighth and one sixteenth over there.
Adam: Yeah, because this is just one, one little square, and the other square.
Marta: Yeah, but she has three sixteenths and you have one eight and one sixteenth.
Adam: How did you get three sixteenths?
Dania: Porque. (*Because*) ¡Ay! (Makes a gesture of frustration) OK.
Carolina: Explícale. Explícale... porque a mí no me entiende.
[*Explain it to him… because he doesn't understand me.*]
Dania: Adam! Look, it's because this... If you put this part...
Carolina: Explícale en una [hoja] limpia. Explícale en una limpia. (*Explain it to him on a new one [on a clean diagram]. Explain it to him in a new one.*)
Dania: (Starts drawing lines on a new diagram) No. Aquí está. ¿verdad? (*No. Here it is, right?*)
Adam: Come on, come on...
Dania: Wait!
Adam: Come on, come on, come on, come on, come on... What are you doing with that square? I'm talking about that square.
Dania: No. I know! I know what I'm doing.
Adam: Alright, alright, alright, alright. (Pause)

It is hard to capture in a few lines of transcript the non-verbal expressions – their gestures and the overall dynamics of this exchange. Dania and Adam have been going back and forth arguing about the different fractions. By the time Adam asks Dania about the three sixteenths, Dania is somewhat exasperated with Adam and his questions because she feels he does not understand her. Although Carolina does not use any English at all in her communication, she is part of the exchange. She tells Dania to explain it to him because he does not understand her (Carolina); she suggests that Dania use a new diagram and hands her one. She appears to be a full participant of this small group interaction, even though the communication is in English. The excerpt that follows further supports Carolina's full participation. The students continue discussing back and forth but they do not seem to resolve the dilemma of 3/16 vs. the 1/8 + 1/16 since they are mostly bickering. Adam goes back to his seat and a few seconds later he says that yes, the answer is 3/16. Marta asks him how he knows that.

1 Adam: I just.... I knew it was going to be three sixteenths.
2 Marta: What do you mean? How did you know that? What if they're not right? How do you know it?
3 Dania: Yeah!
4 Albert: How do you know, Adam? Huh? How do you know?
5 Adam: Yeah, because I added (mumbles – inaudible).
6 Marta: Well, I'll ask you the same question, Mr. Albert. How do you know that one eight and one sixteenth is, is three sixteenths?

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7 Carolina: Porque, porque... (Because, because…)
8 Marta: A ver, un octavo más un dieciseisavo. (Let's see, one
       eighth plus one sixteenth.)
9 Carolina: Porque un octavo son dos dieciséis, más otro, es tres
       dieciséis. (Because one eighth is two sixteen, plus
       another one, it's three sixteen.)
10 Dania: So, un dieciséis más dos dieciséis es tres dieciséis.
       (So, one sixteen plus two sixteen is three sixteen.)
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In line 6 Marta is trying to get Albert involved since he usually does not participate in mathematical discussions, but Carolina starts answering (line 7). Marta then repeats part of the question in Spanish, which may have indicated to her to go ahead and continue her explanation of why 1/8 + 1/16 is 3/16. She does this confidently.[5] She has been working on this problem with Dania and they both have a good understanding of how the section is partitioned and what to do to find the corresponding fraction for each plot. There are several factors that support Carolina's participation. The school is in a primarily Latino/a community with a strong affiliation to Spanish language. The neighborhood has stores with signs in Spanish and is essentially a bilingual neighborhood. Most students are bilingual or have some understanding of Spanish, as many have a relative (usually a grandparent or a parent) who is Spanish-dominant. Thus, despite the restrictive language policy, Spanish is very present during the school day. Furthermore, Carolina has a strong mathematics background from her school in Mexico. During an interview, speaking Spanish, she says:

> Well, I remember that in Mexico what, some of the things that we are barely learning here, I was taught over there in fourth grade. Like, the fractions, the, the, what do you call them? The fractions of figures, I was taught that in fourth grade, third and fourth, that I do remember, I even have books from Mexico. … Things that we are barely being taught here in sixth grade, were taught to me in third and fourth over there. … In fifth grade they were teaching us, and I knew it already, I knew everything already because they had already taught me that in fourth grade [in Mexico].

Carolina's participation, however, is limited. Although many students in her class did understand Spanish and could speak it, only a few had an academic command of Spanish strong enough to carry on mathematical conversations. Dania was one such student and Carolina tended to sit with her. But another constraint to Carolina's full participation is at the level of whole class discussion. Her teacher in 5th grade as well as the one in sixth grade were bilingual, but they both had to make an conscious effort to remember to encourage Carolina's participation in Spanish. Carolina's fifth-grade teacher felt that she did not always give the student the attention she needed because she was the only one who did not speak English in the classroom. This teacher also shared her frustration at not being able to fully use her expertise as a bilingual teacher given the current language policy in schools. Similarly the sixth grade teacher also shared with us her views on the language policy:

> Well, I am really against them throwing kids into English classrooms when their language is, when their first language is Spanish. I'm really (with emphasis), really against that. I have never liked this law that came out where you know we're forcing these kids and throwing them into, into an English classroom. I can just feel for them, you know that they're just not understanding anything that is going on. I used to teach bilingually myself … and to me I feel like it was working because I was using both languages throughout the day and it really, it really, I thought was very beneficial. Now I am not supposed to use Spanish because I am an SEI teacher; I am not supposed to use it but it, it makes me angry and I still use it.

Several teachers at this school (and others in our research) had taught bilingual classrooms prior to the passing of Proposition 203 in 2000 and were now required to leave this approach behind. Although Spanish was heard in the small group discussions and in social talk in the school, it was not the language of whole-class communication. Students like Carolina were able to participate in many aspects of the mathematics classroom but for her to participate in the whole-class discussion she had to be singled out by the teacher explicitly switching to Spanish and inviting her to participate. In our observations, we did not witness any instances of Spanish-dominant students, in classrooms where they were in a minority, speaking up in Spanish in front of the whole class, unless the teacher had directed the question to them. Although we have reason to believe that this singling out was probably well received in this school due to the strong affiliation to Spanish and the overall bilingual environment, still we wonder about how students in this age group perceive being treated differently from their peers through a language switch. We turn our attention to students' views on language in our next section.

BILINGUAL STUDENTS' VIEWS OF ACCESS TO MATHEMATICS

In this second section with empirical data, we discuss findings from our studies in Barcelona and Tucson with a focus on understanding how access to mathematics is perceived by bilingual students. The exploration of access to school knowledge implies multiple and often simultaneous social, cultural, political, and technical dimensions such as teachers' strategies or the support from the institutions (Tate & Rousseau, 2002). In this chapter we pay special attention to the dimension given by students' views. We assume that accessing knowledge requires knowledge: students' access to school mathematics requires knowledge on the part of students regarding what school mathematics is about. The students' views can themselves contribute to create obstacles to knowledge or, instead, enable access.

Data from Catalonia

Inspired by the research conducted with multilingual students in South Africa (e.g., Setati, Chitera, & Essien, 2009), Núria examined students' responses to a writing prompt "What language do you use in your mathematics classroom when working in a small group? What makes you choose the language?" In Barcelona, all ten students who responded to the writing prompt were working-class teenagers – about 13 years old – with good knowledge of both Catalan and Spanish (the two official languages in Catalonia). Six of the students were either born in Latin America or come from Latin American families. The other four students had Catalan as their home language and had always attended Catalonian schools. It was expected that the students addressed their processes of negotiation as a part of their own language identities in their bilingual mathematics classroom. The students could write either formally or informally, and either in Catalan or in Spanish. Despite the offered choice of languages, all students wrote in Catalan. This was probably because both the classroom dominant language and the writing prompt were in Catalan and Núria tended to speak in Catalan as well.

Addressing their language choice during group work, most of the ten students wrote about the complexity of their language repertoires and their flexibility with the two languages. Despite the monolingual language policy in the Catalonian schools, students pointed to the effective existence of two languages in their classroom. In their use of Catalan and Spanish, it is not that the students did not know a specific word, or could not say a sentence in one of the languages and were then forced to switch the language; nor is it that the switching is attributed to external impositions, at least according to their views as expressed in the writings. Changing language is shown rather as a consequence of interest in including all speakers. For example, Paola and Victor clearly indicate that in their narrative below:

> I can speak [both] Catalan and Spanish. I use Catalan when writing and reading, and Spanish when discussing with my peers. (Paola)

> I use Catalan but groups are not always the same. Sometimes in my group I have peers who prefer to speak Spanish and so do I. (Victor)

Paola and Victor's use of Spanish represents a form of recognition of the Spanish-dominant language identity of some of their peers, but when changing language, they also signal their own position as individuals who are allowed and willing to speak more than one language. Carla, for instance, refers to the facilitation of talk in small bilingual groups as a pragmatic reason for changing language:

> I choose the language depending on the group each time. If it is a group with all Catalan speakers, I always speak Catalan. If the group has other students, the conversation is easier if we use Spanish. (Carla)

Carla and Paola give another pragmatic reason for language change in relation to language choice in group work and whole class discussion. They reflect on their experiences in two different social contexts of the classroom – small group and whole class discussion – that relate it to the notion of hybridity.

> I like speaking [both] Catalan and Spanish but I prefer Catalan for final discussions. This is not a problem because we all know Catalan. … Every month at the assemblies with other classes I prefer Catalan too. (Carla)

> I like sharing a group with my friends from Colombia, and I also like speaking Catalan. When we work in small groups, I use Catalan. Sometimes it is difficult because I have been speaking Spanish when working with my peers, but I like making the effort. (Paola)

The students' flexibility in their use of two languages points to significant agency: Catalan- and Spanish-dominant speakers are willing to linguistically accommodate each other. They believe that it is important to do so because issues of communication are at stake (e.g. "I like making the effort," or "I have peers who prefer to speak Spanish and so do I"). From the perspectives of these students, bilingualism appears as an accepted fact of life in the classroom. The pragmatic reasons indicated by the students point to an interpretation of their languages as tools for social interaction and communication.

Both Catalan- and Spanish-dominant speakers show distinct degrees of agency as they attempt to describe their language identities that are far more complex than those politically ascribed to them. They use their knowledge of two languages as a resource that opens up various options, some of which are characterized in terms of responsibility (e.g. "the conversation is easier if we use Spanish"). Diana and Norma, the two students from Latin America, wrote about the special nature of learning mathematics as a relevant outcome for their language practices. For them, some limits to the use of their own two languages come from the perceived higher value of learning mathematics in Catalan. This language was seen to constitute a more privileged resource to learn technical mathematical vocabulary and grammar:

> I prefer to speak Spanish at home, but in the mathematics class, I cannot get distracted with the language because there are some words that need to be learned and they are in Catalan. (Diana)

> Books are written in Catalan and sometimes we need to read a few pages before starting the task. We concentrate on the mathematics more if we all speak the same language when talking about the book. (Norma)

It is interesting to notice how students produce their language identities and, at the same time, consider the social and political negotiations that are needed to gain legitimacy as learners of mathematics. For example, to "make herself understandable," Diana emphasizes the fact that she speaks Spanish at home, and then she mentions the needs to learn technical words in Catalan. Because (language) identities are negotiated in relation to others, Diana's responses may be informative of personal and social limits posed to her language experience. More generally, the students address forms of resistance that work to maintain the use of their two languages together with their access to school mathematics. Their discursive attempts are orientated to not damaging their opportunities for learning mathematics.

Although the analysis of video data points to the existence of less "positive" oral discourses in the class with these same students (some of the Catalan-dominant students mark subtle language boundaries during their work with

Spanish-dominant peers, and vice versa), it is still relevant to pay attention to how they raise their voices for flexible bilingual practices in group work.

Data from Arizona

In the context of Arizona, we have data from individual interviews with 45 students at three different schools and 4 focus groups, with 4 students each, at one of these schools. All students were between 9 and 13 years of age. Through the interviews and focus groups we sought to learn about the students' activities outside school, language use, general perceptions about the school and specifically the mathematics class (in particular with the students who had had some schooling in Mexico). In this section we use excerpts from the interviews and focus groups to illustrate some of the students' comments on language use.

All students mentioned that they had some knowledge of Spanish, with many of them speaking it regularly with family members, usually their parents or grandparents. When asked what language they used in the mathematics classroom, students generally answered "English." There were some exceptions to this with students who were in the segregated section, which we will discuss later. Although English was clearly seen as the language of instruction, when probed, many students said that they used Spanish in their small-group discussions or that they used either language depending on who was in the group.

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Interviewer: When do you use English in the mathematics class?
Dania: When the teacher asks us if we can explain how we solved
       the problem.
Interviewer: And that is always in English?
Dania: Yes.
Interviewer: And when you work in groups?
Dania: It depends, if everybody in the group speaks Spanish and
       English.
Interviewer: If you could choose, would you have your classes in
       English, Spanish, or both?
Dania: Both, because sometimes they tell me words that I don't
       understand in English and sometimes when they tell me in
       Spanish, I understand. So, sometimes I don't quite
       understand things in English.
[. . .]
Dania: If I have to explain something to the teacher, I'd rather
       use English because she hardly understands, well she
       speaks it and understands it, but she hardly understands
       Spanish.
Interviewer: Would you prefer to explain in English or in
       Spanish, if you could choose?
Dania: In Spanish.
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Dania was one of the few students, we noticed, who had a good command of both academic English and Spanish. She was a good resource for students like Carolina who, as a recent arrival, was Spanish-dominant, as the excerpt from a focus group shows:

Marta: At home do you speak Spanish, English or both?
[Juanita, Alice, and Melinda answer "both"]
Carolina: Since I speak Spanish more, Spanish… I only say a couple of words [in English] when I get a little crazy
Marta: Carolina, are you understanding your classes in English well now?
Carolina: Yeah, I don't ask the teacher to translate. Only when they are words that I've never heard, that I don't know yet, that's when I ask her.
Juanita: But she doesn't want to read.
Carolina: But I do read, right Melinda? (Melinda nods her head)
Juanita: She does read, but she's, she doesn't want to.
Carolina: I read with Melinda yesterday.
Melinda: She was reading with me yesterday.
Carolina: I'm embarrassed sometimes because I don't pronounce some words well.
Juanita: But if you don't practice you aren't going to learn.
Melinda: That's what I did. I would take books home.
Carolina: Well I'm embarrassed that they are going to make fun of me because I don't know how to pronounce them.

Carolina is not alone expressing her concern for not being able to pronounce English well. Several other older students in the group (those in grades 6 through 8) brought up the same concern. In the case of Carolina, although she was the only Spanish-dominant student at the time in her classroom, the school's atmosphere overall was very welcoming to students who were English language learners; most students had either experienced it themselves, or had family members who had struggled with the pronunciation in English when they first started to speak it. Yet, this does not mean that Carolina was comfortable speaking in English as the excerpt above shows.

We now turn to a different environment, a middle school that responded to the 2006 law requiring a 4-hour block of English instruction by implementing a segregation model. In this model, all ELLs attended most of their classes with other ELLs in a section of the school that we will refer to as Section A. This arrangement that segregates students based on language proficiency highlights the complexity of language identity. In the mathematics class, these students were able to use Spanish regularly because the teacher was Spanish-dominant too. This gave us access to very rich mathematical discussions as the students used Spanish to explain their reasoning. However, these students were aware of the segregation and several of them wish that there was more emphasis on English:

Marta: What would you like the teacher to know about you that you think would help you to a better classroom experience in math?
Simón: That she wouldn't, that she would put more effort in speaking English because everything she explains, almost everything she explains is mostly in Spanish.
Marta: And you would prefer that it was in English?
Simón: Yes, to learn more.

It is worth noting that the teacher used English in much of her instruction, but she used Spanish to clarify. There was, however, a lot of Spanish being used in the class since students spoke in Spanish most of the time in their groups and even in their presentations to the whole class. In Section A, there was stigma associated with being in that section and that getting out of that section would signify an upward move. However, moving out of Section A was not unproblematic, as students were then in a more English-dominant environment but issues related to their comfort with the English language came up. This was the case for Larissa, who, while in Section A, had expressed not liking Spanish thus resenting that in Section A there was too much Spanish used. Yet, in her first year out of Section A, as an eighth grader, she said:

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Larissa: I don’t like speaking English all that much.
Marta: How come?
Larissa: I don’t know
Marta: Hmm, it’s funny because last year you told me when I
       interviewed you that you only wanted to speak in English.
Larissa: It’s because I wanted to practice it.
Marta: So now, it’s like you like speaking both but it’s almost
       like you prefer speaking Spanish a little more.
Larissa: No, it’s not that I prefer…
Marta: No?
Larissa: it’s that I almost don’t like [it]
Marta: You don’t like what?
Larissa: English
[...]
Marta: What is it that you don’t like about English?
Larissa: That I’m still not learning it well, that’s how I see
       it. ... So, there are times that I stay quiet because I
       feel embarrassed if I don’t pronounce something well.
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Similarly, Carlos, also when he was in eighth grade, out of Section A, said:

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Marta: Is there something that has been hard for you this year?
Carlos: Communicating with people.
Marta: What do you mean?
Carlos: Well, like sometimes I don’t like to speak in English,
       and so that’s why
Marta: And why do you think you don’t like speaking in English?
Carlos: Because, it’s just that I don’t like to talk, or I think
       that sometimes, I think that I’m going to say it wrong.
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While being in Section A, ELLs were perhaps more sheltered. Thus, from a point of view of the mathematics classroom, we could witness their participation in discussions. Yet the students’ desire was to move out of that Section as an indication of progress. However, once outside Section A, their opportunities for participation may have been compromised in part by their being self-conscious about their command of English. This perception was shared by Matilde, the mathematics teacher for students in Section A:

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Matilde: I work only with ELL students ... Our kids feel afraid
       to be in the regular classroom because they feel the other
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> kids have the power. So, even if I have a very brilliant kid, he goes to a nor- class, a regular classroom, and he is going to be one X student [meaning anonymous]. Because he is not going to be that brilliant because they're going to ask them questions in English so they don't know how to explain themselves and they're going to be quiet. So they're going to be relegated to the back of the class. So they are afraid to go to a regular class.

We close this discussion on Section A with Cecilia, who, as an eighth grader, was in Section A for some of her classes but in a different Section for others:

Cecilia: [answering about what language she uses in the mathematics class] But like, more Spanish. I understand more in, in Spanish with her [the teacher].
Marta: You understand more in Spanish ... You understand her more when she speaks in Spanish?
Cecilia: Yes.
Marta: OK. So, what do you usually, when you ask a question to her, what do you usually ...?
Cecilia: In Spanish.
Marta: And then does she answer in Spanish?
Cecilia: Yes, or sometimes English.
Marta: When you do mathematics, in class, when you work in the groups ... what language do you like to speak in?
Cecilia: In Spanish.
[...]
Cecilia: I like Section A because everybody is Mexican like me and we talk, and yeah, I like it.
Marta: You like being in Section A?
Cecilia: No, I, I like the people in Section A, the persons in ...
Marta: Got you! The students?
Cecilia: Yeah.
Marta: The students in Section A. Got you. But if you could choose, where would you be?
Cecilia: In Section B [a different set of classrooms for non-ELLs].
Marta: In Section B. If now you were to start eighth grade, if this was August instead of April ...
Cecilia: Section B.
Marta: In Section B. OK. And why? Why do you think that, that Section B ...
Cecilia: I would learn more.
Marta: And why do you think you'd learn more in Section B?
Cecilia: Like I said, ... all the people speak English and ... I have to speak English too.
Marta: OK, so in Section B you think that you would be using your English more.
Cecilia: Yeah.
Marta: And in Section A ...
Cecilia: In Spanish.

As we can see in these short excerpts, students whose first language is not the language of instruction face several dilemmas as they try to negotiate their identities as students who also know more than one language. They want to fit in as speakers of the official language of instruction, but they also experience the difficulties often associated with speaking a second language; they are in an environment where they can use their first language quite often socially and academically (at least in group work). We end this section with excerpts from interviews with two of the youngest students (9-year-olds). We chose these excerpts to illustrate the adult influence on language identity formation:

Penny: When in Mexico my dad says "No hables inglés, hablas español." ("*Don't speak English, speak Spanish*")
Marta: OK, why do you think he says that?
Penny: My tío (*uncle*) also says that.
Marta: OK. Why do you think they say that?
Penny: Because you are at a specific place to talk one language, like in school. If you have a friend that talks Spanish you should talk Spanish to them but in school you talk English and, and … and at your house some people talk English or Spanish
Marta: Ok is that what your parents tell you that or your dad tells you that?
Penny: Mostly my tío. He tells my cousins and me. He says don't talk English at this house.

Interviewer: What languages do you speak at home?
Denise: Spanish with my mom and English with my dad.
Interviewer: What languages do you usually speak with your friends?
Denise: English.
Interviewer: Do you like to speak Spanish or English more?
Denise: I like to speak English more.
Interviewer: Why?
Denise: Because when I was little, like in preschool, I knew everything in Spanish, but I forgot because my teachers told me to speak English only.

Teachers like Denise's who told her "to speak English only" and the politicians who push for laws that eliminate bilingual education seem unaware that "programs fostering bilingualism among children in immigrant families could provide a valuable competitive edge as the US seeks to position itself in the increasingly competitive global economy" (Hernandez, Denton, & Macartney, 2010, p. 11).

FINAL REMARKS

There is no doubt that language ideologies have an impact on the students' daily lives at multilingual schools and, in particular, on their mathematical learning. Learning is often judged from the way it is communicated, and communication has a lot to do with language(s). Such ideologies are instilled so deep inside a society that students sometimes anticipate what will be the effects of certain uses of language(s), and thus rearrange their own opportunities of communication. We have argued that research carried out on this topic has clearly stated this sociopolitical dimension of learning mathematics in multilingual classrooms. However, in this chapter we have commented on how bilingual students partially overcome sociopolitical constraints to go on with their mathematical learning in classrooms in which the language of teaching is not their home language. Using a range of examples from Barcelona and Tucson, we explored some of the ways in which language awareness influences these students' practices and views without critically interrupting their expectations of participation and access. To a large extent, our own studies from the last decade show that learning opportunities have a lot to do with social differences, which at the same time become mediated by structures of agency.

The study of bilingual and multilingual learners of mathematics, however, still deserves more emphasis as a crucial part of mathematics education research. With the consideration of social and political issues of influence in the mathematics classroom, the understanding of language competence and language use appears much more complex: it can happen that certain positions concerning language are already set before students begin the school year. If this is the case, we need to look for social relationships among groups together with the way particular positions are (re)built by the children themselves within the mathematics classroom. From this perspective, it is meaningful to explore how students behave through, and react to, mathematical conversations in the classroom.

NOTES

1 We have discussed elsewhere some of the effects of these restrictive language policies on Latino/a parents' participation in schools and on students' learning of mathematics (Acosta-Iriqui, Civil, Díez-Palomar, Marshall, & Quintos, 2011; Civil, 2011, 2012; Civil & Menéndez, 2011; Civil & Planas, 2010).

2 We have also discussed elsewhere some similarities and differences in the two contexts (Civil & Planas, 2004; Civil, Planas & Quintos, in press; Planas & Civil, 2010).

3 CEMELA (Center for the Mathematics Education of Latinos/as) is funded by the National Science Foundation – ESI 0424983.The views expressed here are those of the authors and do not necessarily reflect the views of NSF.

4 Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (2006). Connected Mathematics 2 – Grade Six. Boston, MA: Pearson Prentice Hall.

5 In line 9 Carolina does not say "sixteenths" in Spanish when referring to 2/16, but instead she says "sixteen" and Dania also uses "sixteen" instead of "sixteenth" (line 10). Whereas the two terms sound quite close in English, that is not the case in Spanish.

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PART II

ETHNOMATHEMATICS

> Much of the research in Ethnomathematics today has been directed at uncovering small achievements and practices in non-Western cultures that resemble Western mathematics. Western mathematics remains the standard of rationality. It is even suggested that if other cultures had a few more centuries of development, they might reach higher stages of rationality! The key issue, which seems to be omitted from most developments of Ethnomathematics, is that mathematical developments in other cultures follow different tracks of intellectual inquiry, hold different concepts of truth, different sets of values, different visions of the self, of the Other, of mankind of mature and the planet, and of the cosmos. All these visions belong together and cannot be isolated from each other. (D'Ambrosio, 1997, p. 15)

As is already apparent in the first part of this book, mathematics and mathematics education are being drawn within the ambit of the ideological split that divides those who believe in a single dominant worldview and those who celebrate diversity. In the context of mathematics, those in the former group see mathematics as universal, "the same for everyone", and existing independently of human activity, leading to a natural tendency for homogenization of curricula and pedagogy. For the latter group, mathematics is a human activity, consequently enriched by the intellectual diversity relating to forms of life, and necessitating diversity in curricula and pedagogy – and raising the question "What is mathematics education *for*?"

There is a philosophical debate on the status of mathematical knowledge and practice that may be characterized in terms of universalism versus relativism (Pinxten & François, 2011). These philosophical differences spill over when mathematics education is considered. It could be argued that mathematicians, individually and collectively, are entitled to define the boundaries of their discipline as they wish. What they are not entitled to do is to assume that their conception of mathematics should, or even could, be mapped simply and uncontroversially onto a program for school mathematics education.

One of the most important contributions to the emergence of resistance to forms of cultural hegemony within mathematics and mathematics education is the field of ethnomathematics. Albeit there were important earlier contributions, the ethnomathematics program was primarily founded by Ubiratan D'Ambrosio (1985). Its fundamental tenet is that all cultures have significant mathematical practices beyond academic mathematics. A distinction may be made between "near-universal, conventional mathematics" (NUC-mathematics) and "[systems] for dealing with quantitative, relational, or spatial aspects of human experience" (QRS-systems) (Barton, 2008, p. 10).

Ethnomathemics remains a disputed concept, arguably undertheorized, with educational implications that are vigorously debated (Mukhopadhyay & Greer, 2012). As emphasized in one pioneering collection, there is a historical aspect involving the construction of a counter-narrative to the Eurocentric view of the historical development of academic mathematics (Powell & Frankenstein, 1997). One position, for example, is that mathematics education should valorize the full range of mathematical practices, of which academic mathematics is just one (albeit very special and important). At the same time, in a Freirean spirit, access, the opportunity to learn academic mathematics, is important in order to be in a position to challenge the "Master discourse" (Pinxten & François, 2011).

In chapter 4, John Kellermeier describes a number of systems that have been developed in various cultures in order to try to foretell the future. Given the prevalence of such systems, we may posit something like a "divinatory imperative" as a characteristic of human societies, reflecting a wish to see into the future, and, more generally, to explain and control[1]. Science might be viewed as, at least in part, developing from these central cultural drives. From a historically and culturally comparative perspective, we can make the point that, in writing the intellectual history of mathematics, there is a strong tendency towards a strict dichotomy between what is spiritual and irrational and what is scientific and rational. No doubt it would be going too far to suggest that faith in the kind of divinatory systems described by Kellermeier is not totally different from faith in certain mathematical models. However, it is reasonable to suggest that a common desire for predictability underlies both. Historically, the origins of abstract mathematics were not separate from metaphysical considerations that today would be considered irrelevant. By way of example, Leibniz (who had a great interest in Chinese culture) was fascinated by the I Ching. It is true that Leibnitz did not focus on the interpretations by the Chinese and concentrated on the internal structure and the combinatorial properties, in particular their relation to binary arithmetic (the 64 I Ching elements map naturally onto the numbers 0–63 in binary notation). On the other hand, Leibitz also saw in the binary system a representation of the relationship between God and nothingness.

The remaining four chapters in this second part of the book all are concerned, in various ways, with integrating traditional knowledge of indigenous Americans into the system of schooling. This kind of effort is one of the major political activities within ethnomathematics. The acknowledgment of alternative forms of (mathematical) knowledge that are part of students' cultures is a matter of social justice (see Cajete, chapter 2, this volume), addressing the statement that "the intellectual activity of those without power is always characterized as non-intellectual" (Freire & Macedo, 1987, p. 122). Such efforts encounter considerable difficulties and tensions. For example, the perplexing dilemma of what and how to teach mathematics to indigenous groups who "learn mathematics in a distinct cultural-linguistic context – how can they study an international subject while retaining the integrity of a minority world view?" (Barton, 2008, p. 142).

Not surprisingly, given the leadership of D'Ambrosio in the field, and its history, Brazil is the center of much important work in ethnomathematics. Moreover, the work there is also typically infused with the spirit of Paulo Freire. In chapter 5, Mariana Ferreira describes, with powerful examples of subalterns speaking, and speaking mathematics, a workshop that was held in São Paulo to commemorate the First Decade of Indigenous Peoples. What is most clear from the voices we hear is how, in the course of the workshop, the disposition towards, and conception of, mathematics changed for the participants, whether indigenous or not. This example also brings to the fore the importance of teacher training in relation to both teachers that are culturally aligned with their students, and those that are not.

Jerry Lipka and his team have been working in a similar spirit with the Yup'ik people in Alaska for more than twenty years, sharing the same sensitivity to establishing rapport and honoring cultural activities. In chapter 6, they report on a developing project in which a very deliberate (and ambitious) attempt is being made to interface aspects of Yup'ik culture with a particular approach to teaching curriculum of multiplicative structures. In this program, the role of the body in performing mathematics is very prominent (as in many examples cited by Swetz, chapter 12, this volume). For example, it is clear that the concept of a square to a Yu'pik is bound up with the physical activity of constructing a square (echoing Roth, chapter 10, this volume), beginning at the center. In a very important sense, the Yup'ik's square is not the same mathematical object as the square with which most readers will be familiar from their learning of Euclidean geometry. Their square is practice-based, as are all the "squares" most people in Western culture interact with. None of these squares has the properties of the squares that geometers are dealing in. However, shaped by their specific cultural histories, European peoples grow up in a context where the mathematical ideas of squares and the everyday use of the term square are confounded, leading to interesting forms of cognitive development with respect to mathematics (Roth, 2011).

In chapter 7, Daniel Clark and Milton Rosa relate ethnomathematics to mathematical modeling, coining the portmanteau word "ethnomodeling". In recent years, to varying degrees, mathematical modeling has been accorded more prominence in curricula. It may be argued that "If a decision is made to mathematize situations and issues that connect with students' lived experience, then it brings a further commitment to respect the diversity of that experience across genders, classes, and ethnicity" (Greer, Verschaffel, & Mukhopadhyay, 2007, p. 96). Mathematical modeling is inherently social, in terms of the motivations for undertaking a modeling act, the physical and mental tools available to the modeler, debate about the merits and implications of alternative models, and communication of outcomes relative to the original motivations.

One pioneering field is termed "ethnocomputing." In chapter 8, Bill Babbitt, Dan Lyles, and Ron Eglash describe the use of software tools to facilitate the modeling of mathematics embedded in cultural practices such as weaving. In so doing, they are very sensitive to, and work hard to avoid, merely imposing ideas from academic mathematics upon artefacts. In their description of students working with these tools, the authors emphasize the motivational force of activities that the students can recognize as making contact with their lived experiences.

In each of the four examples discussed (chapters 5–8) we see attempts to create culturally responsive mathematics education that both acknowledges the intellectual activity in mathematical practices that are culturally embedded, and integrates such activity into school mathematics classrooms, and not merely as a peripheral activity (Pinxten & François, 2011). These efforts to realize, in both senses, diversity within mathematics classrooms are happening at a time when, in the USA, a Common Core State Standards (CCSS) is being developed that offers practically no room for cultural responsiveness and no acknowledgment of intellectual diversity.

NOTES

[1] Another fundamental human drive from which mathematics is developed is the "decorative imperative" (Mukhopadhyay, 2009) that is very clearly represented at many points in this section and elsewhere in the book.

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JOHN KELLERMEIER

5. CONSULTING THE DIVINE

The (Ethno)mathematics of Divination

What has happened? What is happening? What will happen? These are questions that humans have always sought to answer, in order to interpret and live in the world around them. It is not so much a matter of what is real but what we think is real and what model of reality helps us to survive and thrive. Mathematical thinking has always been a part of this.

SYNCHRONICITY AND DIVINATION

In today's world, divination is often thought of as attempting to predict the future. It conjures up images of fortune-tellers gazing into crystal balls. In fact, however, the root of the word divination is the same as the root of the word "divine." This implies that divination is about seeking knowledge of what is happening in the "divine," in other words, what is happening in the hidden "living world" around us. Divination is an attempt to answer the questions: "What has happened? What is happening? What will happen?" But divination seeks to answers these questions not only for the physical world but also for the nonphysical world of "spirit."

Divination has been practiced by practically every human culture. The ways in which the "divine" have been conceived, and hence explored, have varied considerably. However, for the most part, divination consists of consulting some form of oracular symbol or symbols. In particular, many of these divination systems involved using a randomly generated divinatory device or set of symbols which are then "read" by the diviner according to some, mostly preset, convention of general meanings.

In the first half of the 20th century, the psychologist Carl Jung developed the concept of synchronicity or acausal connection to explain the "meaningful coincidences" that he saw in his study of divinatory systems such as the *I Ching* and Tarot. His idea was that there is a connection between the psychological world of the collective unconscious and the physical world around us. Certain events in each of these worlds tend to happen together even though there is no cause and effect relationship between them.

Synchronicity is the model of reality upon which divination is based. That it is a useful model of reality is evidenced by the many years of use of divination to guide actions. Divination is applied synchronicity in that the querent (or person who

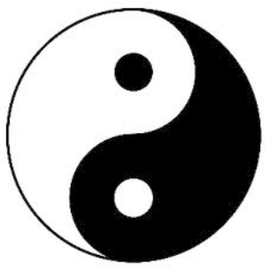
S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 97–113.

seeks advice from some kind of divinatory system) or the diviner (the person who reads divination symbols) randomly generates a divinatory device or symbol to be read. The principle of synchronicity says that this divinatory device, while randomly generated, will necessarily reflect what is happening in the querent's physical/psychological/spiritual life and thus will give the querent access to knowledge that might not otherwise be available. This is using applied synchronicity to answer the questions: "What has happened? What is happening? What will happen?"

THE I CHING

Yin, Yang and Change

The *I Ching* or *Classic of Change* is a Chinese book (classic) that is over 3000 years old. It has its origins in the same oracular fire rituals of the priests of the Shang dynasty kings, as do the ancient Chinese numerals. It was developed as a way to divine, that is, talk with the gods and the spirits. At its core are the principles of yin, yang, and change. Most Westerners are familiar with the yin-yang symbol or in Chinese *t'ai chi t'u* which roughly translates as the Diagram of the Great Reversion or Drawing of Ultimate Power.



T'ai Chi T'u

The yin and yang in this symbol are often associated with a list of opposite qualities such as the following

Yin	**Yang**
Dark	Light
Cold	Hot
Low	High
Night	Day
Rest	Action
Female/Feminine	Male/Masculine

The problem with this list is that, in Western thinking, it tends to imply static and separated qualities. That implies that yin *is* dark and yang *is* light. But in the

Chinese conception of these ideas yin and yang are not static but in motion. Thus yin is *becoming* dark and yang is *becoming* light. And this becoming does not stop. Notice that in the *t'ai chi t'u* diagram, yin (white) has a seed of yang (black) within it and similarly, yang has a seed of yin within it. This implies motion and change from yin into yang and yang into yin.

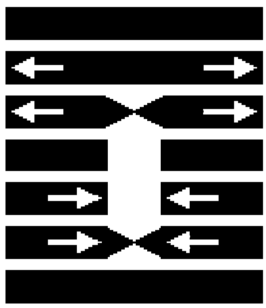
Before the words yin and yang came into existence, lines were used to symbolize the same ideas. Yin was symbolized by a broken or open line and yang by a solid or closed line.

yin *yang*

The broken line symbolizing the yin has an increasing inward tension to closing the line, while the solid line symbolizing the yang has an increasing outward tension toward opening the line. These lines are thought of as old yin and old yang and can be represented as follows.

Old yin - inward tension *Old yang - outward tension*

Now starting with the yang line, we can visualize a movement from yang to yin and back to yang, opening the line and closing the line again.



This continual movement from yin to yang to yin to yang is at the center of the Chinese conception of divination. The *I Ching*, after all, is the *Classic of Change*. It is designed to help one understand how the world is changing and to bring one's actions into line with that change.

The Trigrams and the Hexagrams

The divinatory structure of the *I Ching* is composed of sixty-four divinatory figures called hexagrams. Each hexagram is comprised of two trigrams. Each trigram consists of three lines each of which may be solid (yang) or broken (yin). Since there are two possibilities for each line and three lines the number of trigrams is given by $2^3 = 8$:

The Eight Trigrams of the I Ching

Trigram	**Name**	**Translation**	**Symbol**	**Meaning**
☰	Ch'ien	Force	Heaven	Strength, firmness, persisting
☷	K'un	Field	Earth	Yielding, flexibility, receptivity
☳	Chen	Shake	Thunder	Awaken, activity, initiative
☵	Kan	Gorge	Water	Venturing, passion, falling
☶	Ken	Bound	Mountain	Stopping, stabilization, stillness
☴	Sun	Ground	Wind	Entering, penetrating, initiation
☲	Li	Radiance	Fire	Intelligence, awareness, clinging
☱	Tui	Open	Lake	Stimulating, pleasing, joy

Since each hexagram is composed of two trigrams, one placed on top of another, there are 8 x 8 = 64 hexagrams (see table to the right). Actually, the trigrams evolved later than the hexagrams. Nevertheless, they came to be used to organize the hexagrams. The following table gives the number of the hexagram (as traditionally ordered) that is made up of each pair of trigrams.

The Coin Method of Using the I Ching

When the *I Ching* is used for divination, coins or yarrow stalks are used to randomly generate a hexagram. The coin method is more recent, being about 900 years old. Three coins are used. The coins are tossed for each of the six lines of the hexagram starting from the bottom line and working up. Heads are yang and count as the value of 3 while tails are yin and count as 2. The values of the three coins are added up and the results are turned into a yin or yang line using the following table.

Coins	Number	Line	Image
3 Tails	6	Old (or transforming) Yin	
2 Tails, 1 Head	7	Young (or stable) Yang	
1 Tail, 2 Heads	8	Young (or stable) Yin	
3 Heads	9	Old (or transforming) Yang	

The Key to the Hexagrams								
Upper Trigram / Lower Trigram	☰ Ch'ien	☷ K'un	☳ Chen	☵ K'an	☶ Ken	☴ Sun	☲ Li	☱ Tui
☰ Ch'ien	1	11	34	5	26	9	14	43
☷ K'un	12	2	16	8 Pi	23	20	35	45
☳ Chen	25	24	51	3	27	42	21	17
☵ K'an	6	7	40	29	4	59	64	47
☶ Ken	33	15	62	39	52	53	56	31
☴ Sun	44	46	32	48	18	57	50	28
☲ Li	13	36	55	63	22	37	30	49
☱ Tui	10	19	54	60	41	61	38	58

Remember that the Chinese concept of yin and yang is not static but in motion. Thus there is a distinction between the newly become yin (young yin) and the yin that is ready to transform (old yin) and similarly for young yang and old yang. When a hexagram is generated it will consists of six lines of yin or yang with some of them possibly being transforming lines. The transforming lines are special in that they represent the part of one's life that is changing. Once a hexagram has been generated (called the Primary Hexagram), its divinatory meaning can be consulted to determine what it has to say to the querent. These meanings are summarized in the link given above.

In addition, the text of the *I Ching* contains a discussion of each of the six lines of the hexagram. These are further consulted for those lines that are transforming (either old yin or old yang). These lines may indicate aspects of the situation that are changing and how to deal with them. If there are no transforming lines in the Primary Hexagram, the situation appears to have on visible change at this time.

Finally, if all the transforming lines of the Primary Hexagram are changed, it creates another hexagram called the Relating Hexagram. This hexagram further illustrates the querent's question. It may, for example, indicate the perspective to take, a possible outcome, a warning or challenge, or your goals, depending on the question asked and the interplay between the Primary and Relating Hexagrams.

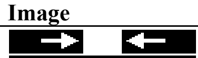
The two Primary and Relating Hexagrams and their divinatory meanings are then used to address the querent's questions. A further reading of each of the changing lines and the meaning attached to them as changing lines is found in the *I Ching*.

The Yarrow Stalk Method

An older and more traditional method of generating hexagrams is to use yarrow stalks, that is, the stalks from the yarrow plant. Fifty yarrow stalks are required for this method. Generate the hexagram as follows.

- Starting with the 50 stalks, select one and set it aside.
- Divide the remaining stalks into two piles. One on the left and one on the right.
- From the right pile, remove one stalk and place it between the fourth and fifth fingers of your left hand.
- From the left pile, remove groups of four stalks until you have a remainder of one, two, three, or four stalks. Place the stalks in this remainder between the third and fourth fingers of your left hand.
- From the right pile, remove groups of four stalks until you have a remainder of one, two, three, or four stalks. Place the stalks in this remainder between the second and third fingers of your left hand.
- You should now have either 5 or 9 stalks in your left hand. Lay these stalks aside into a first pile.
- Gather the remaining stalks and repeat steps 2 through 6 twice. After each repetition, you should have 4 or 8 stalks in your left hand which you lay aside into a second and a third pile.
- Now, using the three groups of stalks laid aside, remove one from the first pile.

– Each of these three groups should have 4 or 8 stalks. A pile of 4 stalks represents a yang and has a value of 3, while a pile of 8 stalks represents a yin and has a value of 2. Assign the corresponding value to each pile of stalks and add the three values to get a six, seven, eight, or nine. As in the case of the coin method these correspond to a line as given in the following table.

Yarrow stalk piles	Number	Line	Image
3 piles of 8	6	Old (or transforming) Yin	
2 piles of 8, 1 pile of 4	7	Young (or stable) Yang	
1 pile of 8, 2 piles of 4	8	Young (or stable) Yin	
3 piles of 4	9	Old (or transforming) Yang	

– Write this line down as the bottom line in the hexagram.
– Picking up all the stalks except for the stalk set aside in step 1, repeat this process until all six lines of the hexagram are generated.

This method requires much more time than the coin method and much more attention to detail. For that reason, it is felt to be a more meditative method with the idea that the querent will be thinking about his or her question during the process.

The Probabilities

The coin method and the yarrow stalk method each have a 50% chance of producing a yin line versus a yang line. However, the chances that a yin or a yang line is an old or transforming line differ for the two methods. Starting with the coin method, we will calculate the probabilities of each of the four types of lines. First, when tossing three coins there will be $2^3 = 8$ possible combinations of heads and tails, each of which is equally likely. The following table gives the combinations of coins, their corresponding number and line type, and the probability of each:

Coins	Number/Line	Probability
TTT	6 (Old yin)	1/8
TTH, THT, HTT	7 (Young yang)	3/8
HHT, HTH, THH	8 (Young yin)	3/8
HHH	9 (Old yang)	1/8

From this we can see that yin and yang lines are equally likely with a 50% chance of each. Furthermore we can see that if a line is a yin line then the probability that it is old is 1 out of 4 or 25%. Similarly a yang line has a 25% chance of being an old line.

With the yarrow stalk method, the calculation of probabilities is more complex. In the generation of each line, there are three steps in which the number 2 or 3 is generated. These are then added to get one of the numbers 6, 7, 8, or 9 yielding the corresponding type of line. However in each generation of 2 or 3 by the yarrow stalk method the probability of getting a 2 or 3 is not always 1/2. When the first 2 or 3 is generated the querent starts with a pile of 49 stalks. After one is reserved, the remaining 48 are split into two piles, each of which is reduced modulo 4 to a pile with 1, 2, 3, or 4 stalks. These two piles are then combined with the one reserved stalk to get a sum of either 5 or 9 stalks. Now, since the reduction is modulo 4 and the total number of stalks split into two piles is divisible by 4, there are four possible outcomes of this:

First Pile	Second Pile	Sum plus reserved stalk	Number (2 or 3)
4	4	9	2
3	1	5	3
2	2	5	3
1	3	5	3

If we assume that each of these possibilities is equally likely, then it can be seen that the probability of a 2 is 1/4 while the probability of a 3 is 3/4. However, for the second and third generation of a 2 or 3, the number of stalks with which the querent begins is divisible by 4. After reserving one stalk, dividing the remaining into two piles and reducing modulo 4, the four possible outcomes are then as summarized in the following:

First Pile	Second Pile	Sum plus reserved stalk	Number (2 or 3)
4	3	8	2
3	4	8	2
2	1	4	3
1	2	4	3

Here, the numbers 2 and 3 are equally likely, each having probability 1/2.

Summarizing all possible generations of a line we have the following:

First 2 or 3	Second 2 or 3	Third 2 or 3	Number/Line	Probability	Probability of Line Type
2	2	2	6 (Old yin)	1/4 * 1/2 * 1/2 = 1/16	1/16
2	2	3	7 (Young yang)	1/4 * 1/2 * 1/2 = 1/16	
2	3	2	7 (Young yang)	1/4 * 1/2 * 1/2 = 1/16	5/16
3	2	2	7 (Young yang)	3/4 * 1/2 * 1/2 = 3/16	
2	3	3	8 (Young yin)	1/4 * 1/2 * 1/2 = 1/16	
3	2	3	8 (Young yin)	3/4 * 1/2 * 1/2 = 3/16	7/16
3	3	2	8 (Young yin)	3/4 * 1/2 * 1/2 = 3/16	
3	3	3	9 (Old yang)	3/4 * 1/2 * 1/2 = 3/16	3/16

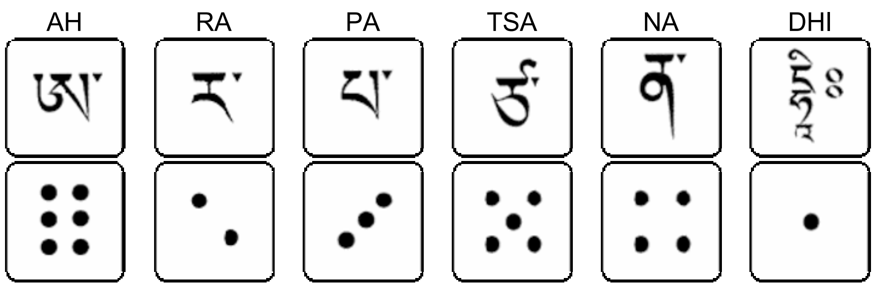
Thus, the probability of a yin line is 1/16 + 7/16 = 8/16 = 1/2, and the probability of a yang line is 5/16 + 3/16 = 8/16 = 1/2. However, the probability that a yin line is old is only 1 out of 8 or 12.5%, while the probability that a yang line is old is 3 out of 8 or 37.5%. Thus a yin line is less likely to be old and ready to transform than a yang line. Some users of the *I Ching* suggest that this is appropriate since a yin line represents rest while a yang line represents action. Thus, a yin line would be more likely to be stable and less likely to be old than a yang line.

Complexity

Throughout this paper we look at the complexity of each divination system by considering how many different ways there are to generate a divinatory device within the system. The idea is that the more possibilities within a divinatory system the more specific a divinatory message may be gained from the system. In the I Ching each hexagram consists of six lines. As previously stated there are 64 hexagrams and hence in an I Ching divination there are 64 Primary Hexagrams. Each of the lines in a Primary Hexagram can be either young or old. Each line in the Relating Hexagram can either be stable or transformed. Thus for each Primary Hexagram the number of possible Relating Hexagrams is given by $2^6 = 64$. Thus the total number of combinations of Primary and Relating Hexagrams is given by 64 x 64 = 4096. Hence there are a total number of 4096 I Ching divinations.

MO: TIBETAN DIVINATION

Tibetan culture has a long history of developing sophisticated methods of dice divination generally called *Mo*. The specific form developed by the Buddhist master Jamgon Miphram (1846-1912) is associated with the Wisdom-God *Manjushri* and uses a six-sided die. He developed this system from studying various sacred Buddhist texts. On each side of the die is one of the six syllables of the mantra (or sacred phrase) of *Manjushri*, AH RA PA TSA NA DHI. This mantra has no specific translation but represents the wisdom experienced by enlightened beings. Each of the syllables can be associated with a number on a standard six-sided die as illustrated in the following table.



There are then eleven main categories for questions from a querent: (1) family, property and life; (2) intentions and aims; (3) friends and wealth; (4) enemies; (5) guests; (6) illness; (7) evil spirits; (8) spiritual practice; (9) lost article; (10) will they come, and will the task be accomplished; and (11) all remaining matters. To perform the ritual for divination with the *Mo* die, the diviner may meditate and then recite the mantra of *Manjushri* as well as the mantra of Interdependent Origination. She or he then blows on the *Mo* die and, keeping in mind the name and question of the querent, casts the die twice and examines the results. Since there are six sides to the die, casting it twice will yield an array of 6 x 6 = 36 possibilities. Jamgon Miphram (1990) developed a book in which each of these possibilities is associated with a phrase. The book of divination interprets the associated phrase for each of the eleven question categories. The diviner then uses this as a guide in answering the question of the querent.

Complexity

There are 6 x 6 = 36 basic combinations of throws of the die. Each of the 36 combinations of two throws of the die in a *Mo* divination can be applied to one of eleven main categories of questions from a querent. Thus a total of 11 x 36 = 396 types of readings are possible in the *Mo* divination system.

RUNES

The Elder Futhark

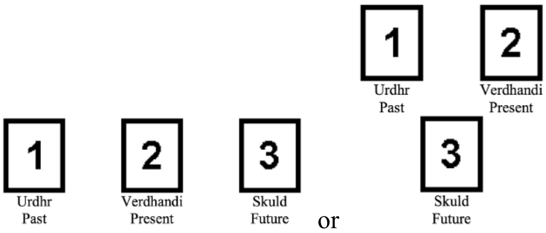
Runes are the alphabet of the Germanic peoples of northern and central Europe. The root of the word rune means “secret” or “mystery.” The Elder Futhark is the set of twenty-four characters or staves used from as early as 200 BCE to around 800 CE. It is this set of twenty-four staves that is used for divinatory purposes. The Elder Futhark is usually arranged into three groups, or *ætt* (plural *ættir*) of eight each. The following table gives the twenty-four staves along with their Germanic names and the English equivalent. Each of these staves has a basic divinatory meaning which is used in rune casting.

The First *Ætt*							
Fehu Cattle ᚠ	Uruz Wild Ox ᚢ	Thurisaz Giant ᚦ	Ansuz Mouth ᚨ	Raidho Sun Chariot ᚱ	Kenaz Fire ᚲ	Gebo Gifts ᚷ	Wunjo Joy ᚹ
The Second *Ætt*							
Hagalaz Hail ᚺ	Naudhiz Needfire ᚾ	Isa Ice ᛁ	Jera Harvest ᛃ	Eihwaz Yew ᛇ	Perthro Lots ᛈ	Elhaz Elk ᛉ	Sowilo Sun ᛊ
The Third *Ætt*							
Tiwaz Spindle ᛏ	Berkano Birch ᛒ	Ehwaz Horse ᛖ	Mannaz Human ᛗ	Laguz Water ᛚ	Ingwaz Earth God ᛜ	Dagaz Day ᛞ	Othala Homeland ᛟ

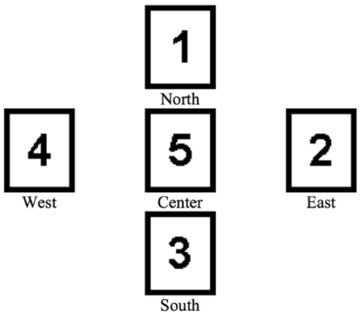
Rune Casting

For use in divination each stave is carved or drawn on wood or stone. These are usually kept in a bag or box from which they can be drawn. To use the runes for divination, the diviner chooses a layout to use to answer the querent's question. Then she or he randomly picks the number of runes need for the layout. This is often done by casting all the runes on a white cloth with the eyes closed or diverted and then randomly picking from the cast runes. The chosen runes are then laid out in order.

The Three Sisters Readings This layout consists of laying out three runes in positions representing past, present and future. It is called the Three Sisters after the Great Norns or Goddesses of Fate, Urdhr (that which has turned), Verdhandi (that which is turning), and Skuld (that which shall be). The runes should be laid out according to either of the following diagrams.



The Four Directions In the Four Directions layout five runes are chosen, four representing each of the four directions and one for the center. They are laid out in according to the following diagram.



Each of the positions, then, has a meaning showing how it relates to the querent's questions.

Position Meanings for the Four Directions Layout

North	East	South	West	Center
Destiny, outside forces that may oppose you.	Where things are moving and changing. The place of transformation.	The vital driving power in the situation.	The realm of feeling, imagination, love, well-being and family.	The outcome and synthesis of these forces.

For both of these layouts, once the runes are selected and laid out by the runecaster, the reading consists of merging the divinatory meaning of each rune to the meaning of the position it occupies and determining what it has to say about the querent's question.

Complexity

For this type of divinatory reading, the number of possible readings is given by the number of ways to permute or order a subset of a larger set of items. In general, if we have n items in a set and want to order r of them, the number of such permutations is given by $n!/(n - r)!$ (Here $n!$ means the product of all integers from 1 to n) Now, when considering the runes we have $n = 24$. For the Three Sisters Readings, $r = 3$. Hence the number of Three Sisters Readings is given by

$$24!/(24 - 3)! = 24!/21! = 24 \times 23 \times 22 = 12144.$$

For the Four Directions Readings, $r = 4$, so that the number of possible readings is

$$24!/(24 - 4)! = 24!/20! = 24 \times 23 \times 22 \times 21 = 255024.$$

TAROT

Tarot cards are probably the most well-known Western form of divination. A Tarot deck consists of 78 cards. Fifty-six of them are very like the standard deck of playing cards used in games like poker and bridge. Like a standard deck of playing cards, a Tarot deck has four suits usually called wands, cups, pentacles, and swords. These correspond to the playing card suits of clubs, hearts, diamonds, and spades. Also, like a deck of playing cards, a Tarot deck has number cards from 1 (or ace) to 10. But a Tarot deck has four face cards (Page, Knight, Queen, and King) whereas a deck of playing cards has only three (Jack, Queen, and King). These suits in a Tarot deck are called the Minor Arcana (meaning minor suits).

The remaining 22 cards make up the Major Arcana (or major suit) also called the Trumps. These cards are numbered from 0 to 21 and represent major factors in human lives. The joker that comes with a deck of playing cards is a descendant of the Fool, the zero card of the Major Arcana.

Playing cards predate Tarot cards, entering Europe around 1200 CE, probably from Arabic or other Asian sources. The earliest known Tarot decks date from the 1400's in Renaissance Italy. With the inclusion of the Trumps, or Major Arcana, reflecting major themes of magic and myth from the Renaissance, the Tarot deck was born essentially as it is used today. The most well-known Tarot deck is the Rider-Waite-Smith deck. This deck was first published by the Rider Company of England in 1909. It was mostly designed by the academic and mystic Arthur Waite, who commissioned the artist Pamela Colman Smith to draw the pictures for the cards. Unlike most early decks, the Waite-Smith version included pictures on the number cards of the Minor Arcana. Most early decks only had the pips or suit markings on these cards, just as playing cards do.

The people depicted in the images designed by Waite and Smith reflect Western culture and are all white, thin, able-bodied (with minor exceptions). Nevertheless, in more recent years a great variety of decks have been designed and produced many of which use the imagery designed by Waite and illustrated by Smith as a basis. Many others, however, diverge from the Waite-Smith imagery and even from the names of the suits and the people cards. For example, a round deck called the Motherpeace deck was designed by Vicki Noble and Karen Vogel and published in 1981. This deck essentially uses the same names for the suits only changing pentacles into discs, but renames the four people cards to Daughter, Son, Priestess and Shaman. Noble and Vogel use a variety of images of all races, cultures, sizes, shapes. There is usually an emphasis on images of women and women's experiences with much of their research for the deck based on the study of goddess-centered cultures from around the world.

The Major Arcana

The Major Arcana consists of 22 cards numbered 0 through 21. These represent major themes in human lives. The following list gives the traditional names and order of the Major Arcana: (1) Fool; (2) Magician; (3) High Priestess; (4) Empress; (5) Emperor; (6) Hierophant; (7) Lovers; (8) Chariot; (9) Strength; (10) Hermit; (11) Wheel Of Fortune; (12) Justice; (13) Hanged Man; (14) Death; (15) Temperance; (16) Devil; (17) Tower; (18) Star; (19) Moon; (20) Sun; (21) Judgment; and (22) World. Some contemporary decks rename some or all of the Major Arcana. The Motherpeace Deck, for example, renames the Hermit as the Crone and renames the Hanged Man as the Hanged One. Each of the Major Arcana cards has a divinatory meaning. They represent major events or principles in human lives.

The Minor Arcana

The Minor Arcana consists of four suits of cards, with each suit numbered one (or ace) through 10 and including four court or people cards. The four court cards are traditionally called King, Queen, Knight (or Prince), and Page (or Princess). Each of the suits, numbers, and court cards have meanings as given below.

The Suits

Name	Element	Meaning	Playing Card Suit
Wands	Fire	Spirit, passion	Clubs
Cups	Water	Emotions	Hearts
Pentacles	Earth	Material concerns	Diamonds
Swords	Air	Intellect	Spades

The Numbers

Number	*Meaning*	*Number*	*Meaning*
Ace	Beginning, gift	6	Contentment
2	Balance	7	Options
3	Planning, synthesis	8	Organizing
4	Stability	9	Integrating
5	Challenge, struggle	10	Transformation

The People Cards	
Name	*Meaning*
Page/Princess/Daughter	Risking, setting out alone
Knight/Son	Focus, moving in a fixed direction
Queen/Priestess	Maturity, security
King/Shaman	Completion, mastery

The divinatory meaning of each Minor Arcana card is a combination of the meanings of the suit and the number or people card. Many Tarot readers interpret a card as having a negative meaning if it is upside-down when laid out while other readers always lay out the cards upright.

The Readings

To use the Tarot deck for divination, a variety of rituals is used. They can be as elaborate as creating a special place, lighting candles, invoking whatever powers one wishes to call upon before shuffling the cards a specific number of times. Or they can be as simple as shuffling the deck while thinking about the querent's question. Some Tarot readers insist that only they touch the cards, while others have the querent shuffle the deck, and perhaps others present cut the cards along with the querent. Then the cards are dealt out in a predetermined layout. This layout can range from a single card or a three card reading or, as is usually done as a full reading, some version of what is called a Celtic Cross.

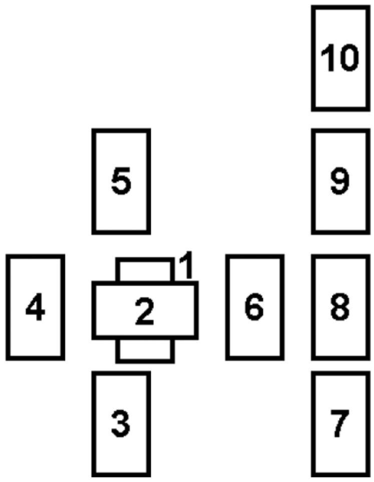
The Three Card Reading Similar to the Three Sisters Reading used in runecasting, the three cards in this reading represent the ground, the path, and the fruition.

The meanings of these three positions are given by the following

Position	**Meaning**
Ground	What the querent is grounded in. Where the question arises from.
Path	How the querent is proceeding.
Fruition	What the outcome is or will be.

Whereas these three are very similar to the Three Sisters Reading of past, present, and future used for the runes, they are not identical.

The Celtic Cross Whereas there are minor variations on how the Celtic Cross reading is laid out and whether 10 or 11 positions should be used, the following is a typical version.



The positions and their meanings are given in the following table:

Position	*Meaning*
1. Center	The central issue of situation to be examined.
2. Crossing Card	An obstacle or counter influence
3. Ground	What has helped to create the current situation
4. Recent Past	Recent developments
5. Possible Outcome	The potential of the situation
6. Near Future	What will soon happen to influence the situation
7. Self	How the querent is contributing to the current situation
8. Environment	How other people or things influence the situation
9. Hopes and Fears	What the querent expects from the situation
10. Outcome	The most likely outcome of the situation

Sometimes, before shuffling and laying out the cards, the reader asks the querent to look through the Tarot deck and choose a card that they feel represents them. This card is called the Significator and is laid down before the first card above is laid on top of it.

Complexity

Of all the divination systems discussed herein, the tarot has the most complex structure. Its divisions into major and minor arcana, four suits, number, and people cards within the minor arcana all are much more highly structured than the other systems discussed above. This structure allows the practitioner a much more nuanced approach to divination and the complexity of meanings. Furthermore, the use of 78 different divination symbols allows a significantly greater number of possible readings.

Like the rune readings, the number of possible tarot readings is given by $n!/(n - r)!$ Here $n = 78$, the number of different tarot cards. For the Three Card Reading, $r = 3$. Thus the number of Three Card Readings is

$$78!/(78 - 3)! = 78!/75! = 78 \times 77 \times 76 = 456456.$$

For the Celtic Cross reading, $r = 11$. Thus the number of possible readings is

$$78!/(78 - 11)! = 78!/67! = 78 \times 77 \times 76 \times 75 \times 74 \times 73 \times 72 \times 71 \times 70 \times 69 \times 68.$$

This number is extraordinarily large, yielding 3.1×10^{20} possible readings.

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MARIANA LEAL FERREIRA

6. MAP-MAKING IN SÃO PAULO, SOUTHERN BRAZIL

History, Social Diversity, and Indigenous Peoples' Rights

Área Indígena Guarani Boa Vista, Município de Ubatuba
Boa Vista Indigenous Area, Municipality of Ubatuba
<u>Location</u>: – The Boa Vista Area is located in the municipality of Ubatuba (north coast of the state of São Paulo), 20 km from Ubatuba city along the Rio-Santos Highway.
<u>Population</u>: 135 inhabitants, living in 25 houses.
<u>River</u>: The Promirim River cuts across the Village.
<u>School</u>: Our school is called Tembiguai Indigenous School. It works in two shifts: from 11:20 am to 4:00 pm it's the Kindergarten through 4th grade. From 4:00 pm to 6:00 pm it's the adults' turn. Kids that finish 4th grade can go on to the local public school to continue studying. This is a victory the Guarani achieved in education: the Conselho Escolar Indígena [Indigenous

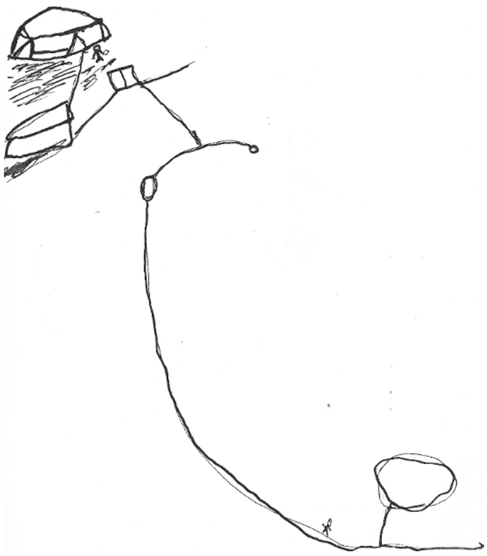


Fig. 6.1 "Boa Vista Village of the Guarani People." By Odílio Warã

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 115–158.

Schooling Council] was created by our community leaders and family members. The Council is responsible for the functioning of our entire Indigenous school.
Language: Our teaching system is bi-lingual. Children in 1st and 2nd grade learn how to read and write in Guarani, and move on to Portuguese in 3rd grade.
Food: We have two meals, lunch and a snack. There are three employees paid by the Municipality: a cook, a teacher and a bilingual aid.
Health: We have a Health Center with a FUNAI [Fundação Nacional do Índio - National Indian Foundation] nurse taking care of everybody.
This is how our community works:
Activities: Hearts of palm plantation is our main source of subsistence. The work is done by both men and women. The hearts of palm are sold at the Ubatuba street market on weekends.
Arts and Crafts: Also a main source of subsistence. Baskets, necklaces, bows, and arrows, etc. These items are sold at IBAMA [Brazilian Institute of the Environment], TAMAR Project [Sea Turtle Project] and at the street market.
Hunting: People go out hunting in groups or not. The animals that we still find around here are wild boar, agouti, skunk, armadillo, rabbit, fish (only far away), monkey, deer, capybara, sloth, anteater, and coati.
Prayer: The pajé [shaman] calls the community to sing, while giving out advice to the elders, children, and adults. He speaks about a better future for all peoples. After his advice, the dance and singing ceremony begins. It goes from 6:00 pm until 2:00 am. The Baptism-festival takes place once a year, on January 29, when the children born that year receive their Guarani names from the shamans. After a lengthy ritual period, the shaman calls in the women with children to be baptized.
Music: We have a musical group called "Nhandereko Arandu" formed by children, the coordinators, and the musicians. The songs are old, remembered by the elders especially for the future of the children as a form of preservation. A CD has already been made.
Food: Sources of food include manioc, sweet potato, banana, sugar cane, orange, guava, beans, rice, and beef.
Area: Our reservation comprises 920.66 hectares of demarcated land, part of an environmental preservation area in the Mata Atlântica [Altantic Forest]. We have several water falls and mountains. Our land is very beautiful.
(Submitted by Guarani teachers Odílio, Santa, and Daisy, Área Indígena Boa Vista.)

In this chapter I show how themes of social diversity, historical situation, and human rights can arouse powerful mathematical ideas for map-making activities in teacher training programs for ethnic minorities. This was the major feat of the two-week long (80 hours total) "Teacher Training Workshop for Indigenous Peoples in São Paulo," which culminated in 1999 with the presentation of mathematical ideas of the Guarani, Terena, Kaingang, Krenak, and Pankararu nations. The event commemorated the First Decade of Indigenous Peoples (1995–2004) with the production of the "Book of Maps of São Paulo" (Ferreira, 1999), elaborated collectively by all 60 teachers-in-training and dedicated to making the United Nations Declaration on the Rights of Indigenous Peoples (UNDRIP), then still in its draft form, a reality. It became quite obvious to all workshop participants that mathematical knowledge is needed for the construction and analysis of information about the current situation of Indigenous Peoples in Brazil. In particular, knowledge of mathematics is essential to guarantee their rights to ancestral lands according to the 1988 Brazilian Constitution, and since 2007 to the UNDRIP, when this international document was finally adopted by the United Nations. Here, I argue that developing a sensibility towards socio-cultural diversity, historical situation, and Indigenous Peoples' rights as human rights opens up a space for indigenous mathematics to be known, respected, and developed.

MATHEMATICS IS IMPORTANT FOR THE AUTONOMY OF INDIGENOUS PEOPLES

The idea that "mathematics is important for the autonomy of indigenous peoples," advanced by the Guarani Mbyá of São Paulo, opened proceedings of the Teacher Training Workshop to the relevance of a politicized mathematics education for teachers, indigenous or not, working directly for indigenous communities. In March 1999, over the course of an 80-hour workshop held at the Instituto Cajamar of São Paulo (co-founded in 1988 by Paulo Freire, former President Lula, among others), 60 indigenous and non-indigenous teachers, school directors and educators working in public schools on and around indigenous lands came together to plan the future of mathematics education for southern native Brazilians. The somewhat dry and abstract goal of the workshop was to "to train human resources in mathematic education," following the mandate of the state's Secretary of Education, promoting the event. I was hired to lead the mathematics workshop, given my prior experience as a mathematics teacher for the Xavante, Kayabi, Juruna, Suyá, and Kayapó nations of central Brazil in the 1980s and 90s (Ferreira, 1994, 1997, 1998; Lopes da Silva & Ferreira, 2000).

It was a huge challenge but nevertheless an exhilarating experience to lead a discussion about mathematical knowledge amongst 60 individuals from distinct ethnicities – 40 Guarani Mbyá, Guarani Nhandeva, Terena, Krenak, Kaingang, and Pankaru teachers, as well as 20 non-indigenous Portuguese speaking participants – with varying levels of understanding and expectations about the power of mathematics to promote social change. There was a broad range of participants, from mathematics teachers with college degrees (all non-indigenous) to others with varying levels of elementary and high school education (all indigenous). Most striking was the lack of information on the part of most educators about the overall situation of indigenous peoples in São Paulo, and in Brazil, broadly speaking. Very few *professores brancos* or White teachers, as they are known in the area, knew details about Brazil's history of colonization and oppression of Indigenous Peoples, including the process of encapsulation of the Kaingang and Terena on diminutive reservations (Brazil followed United States' "reservation system"); the dislocation of the Pankararu and Fulni-ô nations from the Brazilian northeast to shantytowns inside São Paulo city; the imprisonment of Guarani children in missionary boarding schools – all issues pointing directly to the question of human rights. Most of the participants were unaware of the fact that Indigenous peoples are the most discriminated against of all global populations, living in poverty in diminutive lands or shanty towns not only in Brazil, but worldwide. Neither did the teachers fully recognize at the time that special measures are required to protect the world's 370 million indigenous persons – most urgently those adopted in September 2007 by the United Nations Declaration on the Rights of Indigenous Peoples (UNDRIP).

Poty Poram Carlos, a young and energetic Guarani teacher at the Jaraguá Indigenous Area, located within São Paulo city limits, reminded participants that she only gained the *right* to be Guarani in 1988, in the latest Brazilian Constitution. Discarding the traditional notion that Brazilian native peoples should assimilate into the broader national society, Brazil's 1988 Constitution recognized its original inhabitants' right to be "culturally different" and to reclaim their ancestral home lands. In fact, Article 8 of the UNDRIP states that "Indigenous peoples and individuals have the right not to be subjected to forced assimilation or destruction of their culture." The document further states in Article 15 that "Indigenous peoples have the right to dignity and diversity of their cultures, traditions, histories and aspirations which shall be appropriately reflected in education and public information."

The new Constitution of Brazil in 1988 helped further empower the Organized Indigenous Movement in the country, reflecting a victory of its own making. The original draft of the UNDRIP had just been put together in 1985 by the Working Group on Indigenous Populations, the world's largest human rights forum. The right to cultural diversity, to quality education and health care, and the fundamental right to occupy ancestral territories, also featured in the new Brazilian Constitution, are main themes addressed in the narratives here presented, which originally accompanied the maps featured in the *Livro de Mapas de São Paulo*, produced during the workshop. However, despite the provisions of the new Brazilian Constitution and the UN declaration, the concrete implementation of such rights both nationally and worldwide was, and still is, far from reality. Following a trend in Latin America, indigenous communities in Brazil have become gradually more vocal in defense of their rights. However, expansion of agricultural and extractive industries and infrastructure development projects such as dams and roads into traditional lands still represent a significant and growing danger to indigenous peoples. Take, for instance the proposed Belo Monte dam, a plan to build the world's third-largest hydroelectric plant, in Brazil's Amazon forest on the Xingu river, which will displace 30 to 40,000 indigenous persons in the Brazilian Amazon. Native Brazilians are still seen as standing in the way of commercial interests and therefore threatened, harassed, forcibly evicted, and killed. Though Brazil and other states in the Americas voted in favor of the 2007 UNDRIP, by the end of 2010 none had enacted legislation for its actual implementation.

Indigenous Peoples in South and North America, Africa, Europe, Asia, and Australia still live in deep poverty and ill health, and face tremendous racial discrimination in their daily lives. Like other Indigenous organizations across the globe, the Organized Indigenous Movement in Brazil, including indigenous teachers of São Paulo, helped push for the final approval of the UNDRIP in 2007. With this goal in mind, reflecting how important it would be to move the declaration from its draft to final form, Indigenous leaders at the workshop immediately proposed the collective elaboration of the *Livro de Mapas de São Paulo*, "recognizing that respect for indigenous knowledge, cultures and traditional practices contributes to sustainable and equitable development and proper management of the environment" (UNDRIP).

Back to the workshop. The eclectic nature of the group – both in terms of the ethnicity and formal education of its members – was initially interpreted by the non-indigenous teachers as an impediment to the success of the event. How could teachers with college degrees learn about math education side-by-side with instructors holding, in some cases, only middle school diplomas? But the different ethnic backgrounds and diverse forms of knowing turned out to work in our favor. Each and every participant contributed his or her expertise and wisdom in the production of mathematical knowledge and in the making of the collective atlas. During the course of 80 hours, all teachers-in-training used their research, drawing, writing, and interviewing talents to produce a unique collection of cultural and historical narratives, illustrated by various forms of landscape mapping. In the collective introduction to the volume, the link between indigenous sovereignty and mathematics education was clearly brought up by Guarani Mbyá teachers: "Mathematics is important for the autonomy of indigenous peoples."

This essay, thus, conveys the collective intellectual and emotional process that paved the way to the production to the *Book of Maps of São Paulo*, illustrated by landscape drawings, cartographic images, historical narratives, and personal trajectories – some of which are reproduced here. The idea that mathematics is a human creation, rather than a universal achievement used as a measure of sheer brilliance – some "get it" while others usually don't – was key in bringing workshop participants together in an open and respectful environment.

The ideas and images here portrayed lead to the key realization informing our work from a Freirian perspective: mathematics, history and socio-cultural identity are intrinsically bound together, part of one and the same critical process of knowledge construction. Mathematics is the product of human creation, that is, *all* peoples fashion their own mathematical knowledge forms and practices, which help inform their own cosmologies, that is, their unique position in the universe. Combining a cross-cultural research program with a mathematics schooling practice is what gives meaning to the field of *ethnomathematics* (D'Ambrosio, 1990). *Mathema* denotes knowledge, understanding, and explanation, while *tic* comes from *techne*, the same root of art or technique. In this respect, ethnomathematics is the art or technique of explaining, knowing, or understanding in various cultural contexts. Ethnomathematics can thus be conceived as a theory of knowledge that privileges our feelings, thoughts, actions, embodied skills, and taxonomies (Barth, 1995), and a theory of cognition that allows for the interplay between culture and mathematical cognition (Bishop, 2004; D'Ambrosio, 1990; Ferreira, 1997; Zaslavsky, 1998).

First Week – Identity, Space, and Time: Who Are We and Where Are We Located?

> **Área Indígena Guarani do Krukutu**
> Guarani of the Krukutu Indigenous Area
> The Krukutu Village has 25.88 hectares. It is located on the border of the city of São Paulo. The area is demarcated. There are 23 families and the total population is 68 individuals, mostly children. The schoolhouse was built by the community. The roof is made of bamboo and wood. The Krukutu School has 20 students. There are 6 students who study at the Barragem School nearby.
> Here in Krukutu there is also a dispensary built by the community. There is corn growing which also belongs to us. There is an artesian well built by the National health Foundation in 1988. There are water pumps that take water to the houses.
> (Submitted by Marcio Werá Mirim Rodrigues, Guarani teacher at the Krukutu Village.)

A careful discussion about the identity, and location in time and space, of each one of the 60 workshop participants brought into focus an interesting phenomenon: non-indigenous teachers heard, in many cases for the first time, reports from indigenous educators about the importance of mathematics for native Brazilians' autonomy and self-determination. *Sovereignty* was a concept that most non-indigenous teachers had never heard of, while those familiar with the concept did not know how to define it. Why does the legal history of "tribal sovereignty" start with colonialism? Edevaldo Catuí, a Kaingang mathematics teacher at the Terra Indígena Vanuíre, reminded everyone that Portugal, France, England, and other colonial regimes explicitly based their sovereignty claims on religious doctrines decreed by the Catholic Church, which had the power to grant titles to portions of Brazilian land for purposes of Christian civilization of all "heathen" Brazilian Indians. "When we measure our land," he said, "we have to keep in mind that most of it was taken away in the process of Western colonization." The result of colonial assertions of sovereignty was that indigenous nations were legally stripped of their independent status. Their existence was oftentimes not recognized at all and their lands treated as *terra nullius* – legally "vacant" or "unowned." In Brazil, Indigenous Peoples are still declared to have a "right of occupancy" or "possession," but not ownership of their lands. The fundamental principle of the 1988 Constitution of the Federal Republic of Brazil is that supreme legal authority over the lands and lives of its original inhabitants lay outside indigenous nations altogether. Article 26 of the UNDRIP, however, claims that:

> Indigenous peoples have the right to the lands, territories and resources which they have traditionally owned, occupied or otherwise used or acquired.

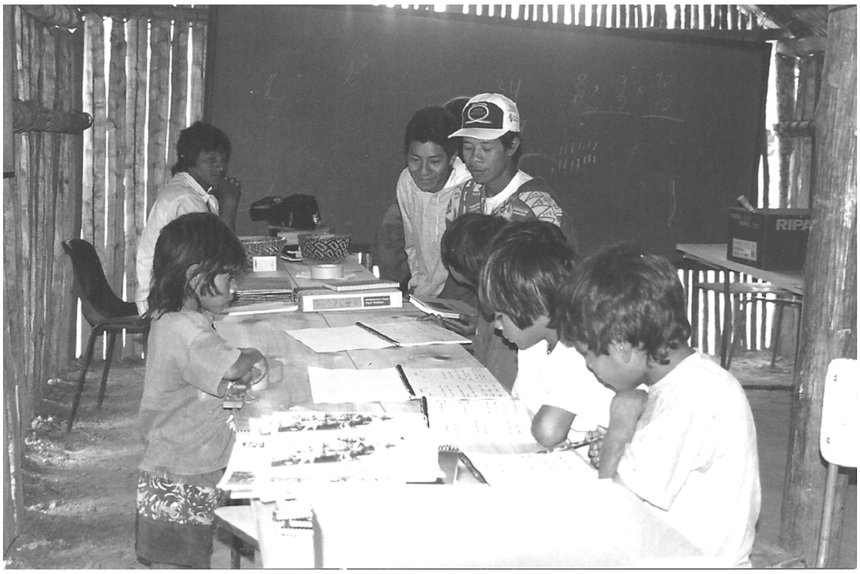


Fig. 6.2 Guarani students at school on the Área Indígena Guarani do Krukutu, Municipality of São Paulo, 2000

– Indigenous peoples have the right to own, use, develop and control the lands, territories and resources that they possess by reason of traditional ownership or other traditional occupation or use, as well as those which they have otherwise acquired.
– States shall give legal recognition and protection to these lands, territories, and resources. Such recognition shall be conducted with due respect to the customs, traditions, and land tenure systems of the indigenous peoples concerned.

It was startling for most teachers-in-training to realize that nearly every math instructor, indigenous or not, believed that mathematics is a universal "fact" that one learns in school, period. In other words, most workshop participants regarded mathematics as both culture- and value-free knowledge. In general, non-indigenous teachers talked insistently about their students' "low scores" in standard math tests. The overall feeling, expressed in many ways throughout the event, was that the indigenous instructors did not know much "real" math, let alone have their own mathematics. However, the various activities here described revealed rich mathematical knowledge forms and practices developed by each nation, and within each community, to face with dignity and sovereignty the current post-contact situation with the broader Brazilian society. Indigenous teachers pointed out the generalized lack of information about native peoples and their social, cultural, and economic rights in textbooks used throughout the country's public and private schools. In various publications, native Brazilians – invariably called *índios*, are portrayed as primitive and backwards, needy of Western schooling to become "educated," which in schooling situations inevitably results in strong prejudice about the "savage mind" (Lopes da Silva & Ferreira, 2000). The idea that "Indians don't learn math" as easily as the *homem branco* is still widespread in the general Brazilian population and in the world at large. Only recently has there been a systematic attempt to show the variety of complex mathematical skills and ideas elaborated by indigenous peoples worldwide (see Ascher, 2004; Eglash, 1999; Gerdes, 2007, 2008; Joseph, 2010/1990; Izard, Pica, Spelke, & Dehaene, 2011).

Broadly speaking, ignorance about the historical reality faced by the 600 thousand Indigenous individuals from 200 different nations living in Brazil today (pib.socioambiental.org) often results in a perverse form of racial discrimination, where Indianness and mental disability – in particular difficulty to learn math, are readily equated by many educators to this very day. Most workshop participants were stunned to discover that it is precisely a profound variety of indigenous knowledge forms and practices that help constitute the incredibly rich Brazilian socio-cultural diversity. The astonishing variety of corresponding worldviews or cosmologies, and therefore the various forms of generating mathematical (and other forms) of knowledge was clearly illustrated at the event with information about the historical specificity and cultural authenticity of the six indigenous nations from the state of São Paulo – the richest state in Brazil carrying one of the worse scenarios of violations of indigenous peoples' rights in the country (Clastres, 1995; Ferreira, 2002a; Ladeira, 2000).

This amazing cultural and ethnic diversity in Brazil at large, and in the state of São Paulo – where most indigenous peoples are primarily identified as "peasants" due to the absence of stereotypical indigenous markers – was, in fact, the driving force of the workshop. All teachers, indigenous or not, realized from the start that workshop participants represented Brazil's socio-cultural diversity at its best, and as such felt compelled to research amongst themselves and learn more about the current situation of all six indigenous peoples there represented, and the mathematical ideas of each one. It is thus that each nation or indigenous sub-group (Guarani Mbyá and Guarani Nhandeva), or those representing a particular community or village, dedicated themselves to documenting their basic concerns in terms of historical situation, identity, socio-cultural diversity, and human rights. Flourishing mathematical ideas and concepts were then identified, discussed, and documented by all participants collectively.

Fig. 6.3 Teacher Mauro Karaí, from the Área Indígena Guarani do Ribeirão Silveira, northern coast of the state of São Paulo, explains to his colleagues the situation of his people at the teacher training Program, Cajamar Institute, 1999.

INDIGENOUS PEOPLES IN SÃO PAULO: FROM CULTURAL TRADITIONS TO THE QUESTION OF AUTONOMY

Favela Real Parque, cidade de São Paulo

Favela Real Parque, São Paulo city
We, the Pankararu people, make up about 90 families all living in the Favela [shantytown] Real Parque in São Paulo city. There are also 28 families in the Favela Madalena, also in São Paulo. Others are located in different neighborhoods. Altogether, there are 950 Pankararu in São Paulo city. We all live here in São Paulo as workers. Most men work in security services and women work as cleaners either for businesses or family homes.

Our community has been living here in São Paulo city for nearly 50 years. We came from our village in Pernambuco, from the Tacaratu region. In Pernambuco there are around 7,000 Pankararu, a very large population.

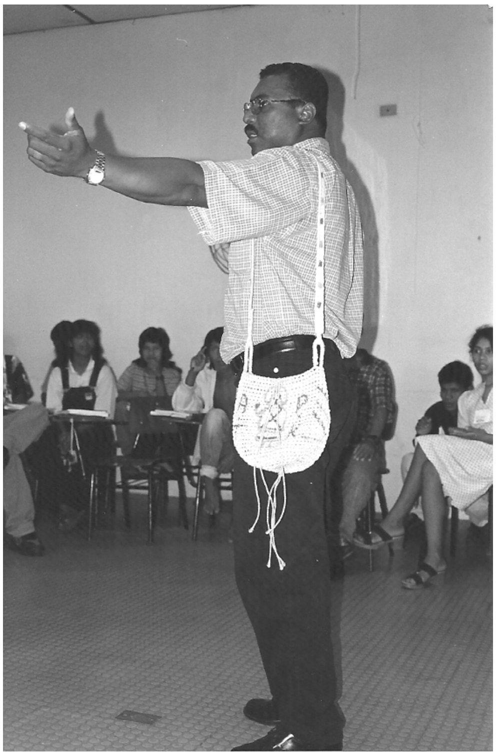


Fig. 6.4 Pankararu teacher Dimas Nascimento, at the Teacher Training Program at the Cajamar Institute, 1999.

We need to define our lives better here in São Paulo. We have the Pankararu Association, which commands our people here in the state of São Paulo. We await recognition [as Brazilian Indians] from FUNAI, because until now there's been none.

We are counting on FUNAI's recognition because it is their obligation to help us.

We await the collaboration of some organization or institution, which we will greatly appreciate.

This is what the history and life of the Pankararu people is like.

(Submitted by Dimas Joaquim do Nascimento. Pankararu school-teacher in São Paulo.)

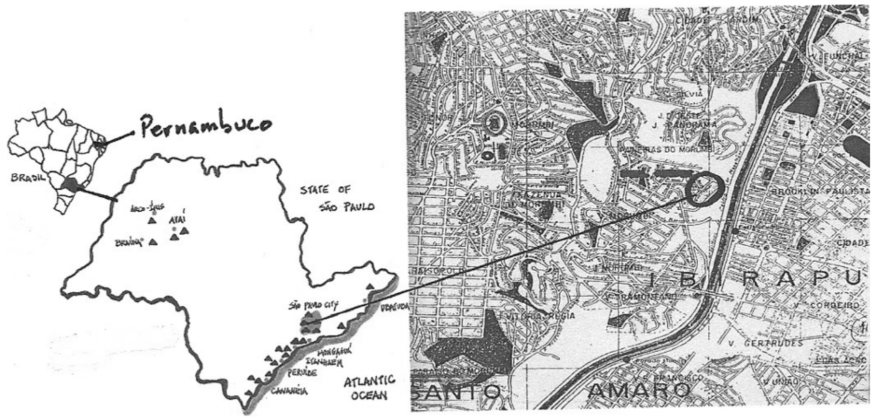


Fig. 6.5 A Pankararu community of 800 people strong live today at the Favela Real Parque, inside the city of São Paulo, originally immigrating from Pernambuco, Northeastern Brazil

In order to formulate a general idea about the current situation of Indigenous peoples in the state of São Paulo, workshop participants drew information mostly from two books: *Madikauku – Os Dez Dedos Das Mãos. Matemática e Povos Indígenas no Brasil* (Madikauku – The Ten Fingers of the Hands. Mathematics and Indigenous Peoples in Brazil) (Ferreira, 1998), and *Povos Indígenas no Brasil* 1991-1995 (Indigenous Peoples in Brazil 1991/1995) (Ricardo, 1995). In addition, some teachers had already brought along with them different maps of their lands, whether officially demarcated or not. Landless communities relied on city and state of São Paulo official government maps to locate shantytowns and other urban and rural areas inhabited by indigenous communities.

A detailed analysis of the information in these books and maps conducted by all participants during the first day of the training program brought forth the creation of a table that organized information in columns and lines about the indigenous peoples of the state of São Paulo: land demarcation, borders, population, number of houses, distribution of teachers and health workers, languages spoken and traditions practiced. In the last column, Alicio Terena sugested we make note of aspects that best characterized the ethnic identity of each group in a cross-cultural situation. For example, the Guarani baptism of their children, the beat-stick dance of the Terena, and the Krenak wedding were included in the last column of the table, while the Kaingang and the Pankararu decided to further research (shown as "researching") the subject among their elders to best convey important cultural attributes to a wider audience.

Fig. 6.6 Poty Poram Carlos and her students in front of the Guarani school at the Jaraguá Indigenous Land

Besides the information provided by workshop participants and the books mentioned above, a wide array of documents were consulted, including various National Indian Foundation (FUNAI) maps in order to verify demarcation status of indigenous areas, their sizes and borders. The research process involved a stimulating cooperation among all teachers in search for complementary information. Some of the non-indigenous teachers had never interacted one-on-one with indigenous professionals or community members at indigenous schools or more generally on and around indigenous lands. This lack of interaction between indigenous and non-indigenous teachers was soon identified as the main cause for the widespread misconceptions about the "Indians' incapacity to learn Western materials."

> **Área Indígena Guarani da Aldeia do Jaraguá**
> Indigenous Area Guarani of the Jaraguá Village
>
> My grandfather Joaquim Augusto Martins Quarayr used to tell me that a long time ago he lived in [the state of] Mato Grosso do Sul. At that time a marshal went to our village and asked the people who wanted to come to São Paulo. My grandfather saw others raising their hands and without really understanding what was happening, he raised his hand as well. So the soldiers and the marshal brought some Guarani – those who raised their hands – to São Paulo.

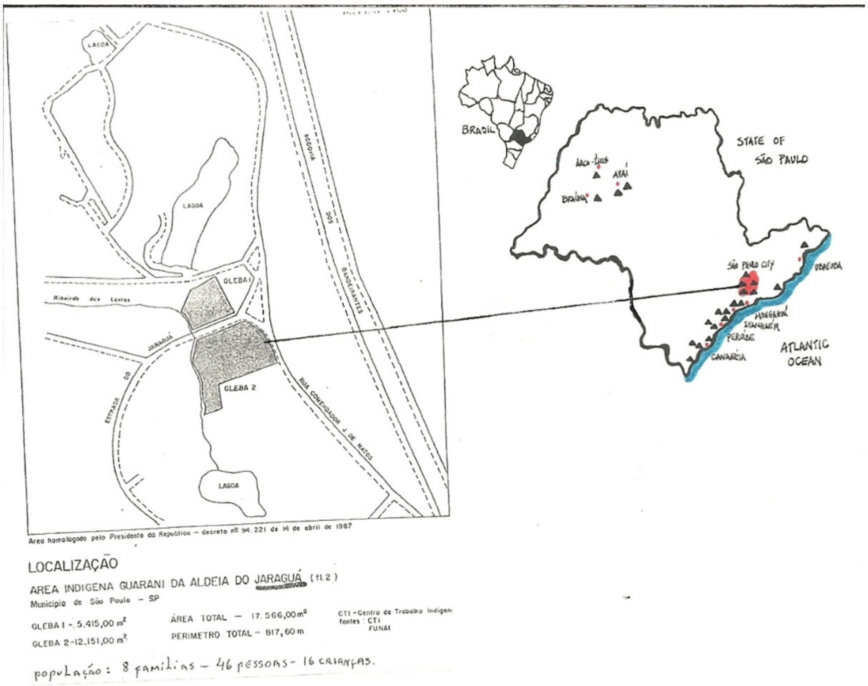


Fig. 6.7 Map: Área Indígena Guarani da Aldeia do Jaraguá, within São Paulo city limits

When my grandfather arrived in São Paulo, he felt like a fish out of water: he didn't know how to speak Portuguese and had no idea of what he would do in such a huge city. I don't know exactly how, but a German couple ended up adopting him. This couple gave him all kinds of education, a white man's education, and even found funny the existence of an Indian who didn't speak his language anymore, but only Portuguese and German!

When he was about 40 years old, my grandfather married my grandmother, who was only 12 years old. When my grandfather's adoptive parents died, he received a small piece of land as an inheritance. Because of lots of land disputes among the Guarani, because their land was not demarcated, my grandfather decided to move to the small piece of land that his German parents had given him.

Around 1984 my father decided to move to the city, near my grandfather. He invaded a neighboring track of private land. That was the beginning of the struggle for a stretch of demarcated land. My uncle followed my father in the struggle, and one by one the struggle became stronger and the land claim grew bigger and bigger. In 1986 the Pico do Jaraguá Indigenous Area was demarcated and homologated [by the Brazilian federal government].

> Today we are more or less 16 families and about 58 people. While we still lived on indigenous land, we were not practicing our culture anymore – our religion, dances, language, art craft, and cooking lore. Since 1996, with the construction of the Opy, our prayer house, we started rebuilding our Guarani culture. First the religion, and then came the dances, the cooking and afterwards the artistic creations. And now the most difficult part is to revitalize the Guarani language. I am part of the third generation and I fight hard so that the future generations keep alive the Guarani culture.
>
> (Submitted by Poty Poram Turiba Carlos, Guarani Nhandeva teacher at the Jaraguá Indigenous School)

The elaboration of the table (Indigenous Peoples of São Paulo State) generated multiple exercises involving quantitative and qualitative data, such as the standardization of all land measurements into hectares (a unit of area equal to 10.000 square meters); the sum of numerical columns; estimates of salaries; and the total number of individuals living in urban areas, among many others. The comparison of data generated by individuals from different ethnic backgrounds, varying historical situations, and geographical locations generated challenging comparisons. Take, for instance, the following land disparities: while the 66-member Guarani-Mbyá community of Aguapeú, living in the municipality of Mongaguá, live on 4,398 hectares, the 529 Guarani-Mbyá of Barragem, in São Paulo city live on 24 hectares. The 50 Guarani-Nhandeva of Jaraguá, (a small reservation in the heart of São Paulo city) hold onto a mere 7,000 square meters (0.7 of a hectare). The fact that several indigenous communities represented at the workshop (and in Brazil at large) do not live on "demarcated land" – generally speaking, they are landless – caused widespread indignation amongst all teachers. Instead of registering the sum of all indigenous territories in the "total" of column 2, participants opted to conclude there were, in fact, "6 landless peoples." Another cause for uproar was the fact that many indigenous persons worked as healthcare agents and/or as teachers without being hired or receiving any pay, while all non-indigenous participants at the workshop received regular wages.

But it was the combination of knowledge produced collectively, including information that was not mathematics specific, such as "traditions" and "languages spoken," that stimulated debate, already on the second day of the workshop, about the importance of mathematics for the autonomy of indigenous peoples. This approach to the collective production of knowledge is well aligned with Paulo Freire's philosophy of popular education whose innovative approach to literacy emphasized peasants' ability to generate knowledge collectively, using "generative terms" – such as land, water, food, transportation - that conveyed their life conditions and worldviews. *Pedagogy of the Oppressed* (Freire, 1970) enabled people to see themselves as historical actors, capable of organizing on their own and creating social change. His work provided substantial insights for the development of a system of popular education in Brazilian Indigenous schools during the oppressive military dictatorship of 1964 to 1985. Following Freire, the teachers-in-training posited that learning should be viewed as an act of culture and freedom through "conscientization" – developing consciousness, but consciousness understood to have the power to transform reality. The knowledge produced at the workshop, including the tables, maps, and narratives reproduced in this essay reflect just that - an attempt on the part of workshop participants to use mathematics to help define what it takes to liberate themselves from oppression, humanize the world they live in and, ultimately, defend their human rights. Such were the final thoughts that wrapped up Week 1 of the workshop, as shown next.

Final Thoughts, Week 1: Autonomy, Self-determination, and Indigenous Peoples' Rights

Divided into 10 working groups, the teachers-in-training reflected on the knowledge produced collectively during the first half of the workshop, in order to answer the driving question and reflect about their human rights: *Why is mathematics important for the autonomy of indigenous peoples?* Here are the answers:

– With autonomy, indigenous peoples in Brazil are going to gain techniques for bettering their life conditions and to guarantee the demarcation of their lands, defend their rights, and develop planting techniques.
– The situation of Guarani mathematics today: our parents and grandparents already had their own mathematics in the Guarani language. Brazilians are not able to see this. But these days we need to learn Western mathematics as well to have autonomy in the greater national society. I learned that this is a human right.
– Indigenous and non-indigenous peoples need mathematics. We all need to learn how to do certain kinds of calculations, to use a plus or a minus, to divide or multiply. We ourselves need to make our own calculations. This is mathematical autonomy, and helps our self-determination.
– If indigenous peoples are equal to all other peoples, then we all know mathematics. But if we also have the right in the Constitution to be different, then our mathematics can also be different. This is important to understand, because if you don't know your rights, how can you defend yourself?
– Now we know we know: even living in a shantytown, our mathematical ideas are important. I want to draw a map of our village in Pernambuco and how far away that is. Then I want to show that the Pankararu are also very close to everybody here because we all have the same problems. I want to know more about our rights to our ancestral lands, because I am a teacher and I want to teach my community in the Favela Real Parque about self-determination. That means we can make our own decisions about our future.
– We need a lot of mathematics—calculations for the plantations and livestock, calculations about where to build fences, calculations about where to spend the money we earn, calculations about what medicines to buy, calculations about what schools to have, calculations about everything! Mathematics is for our cultural survival.

- To defend ourselves from the Whites, we need to learn many things about the White society. This we've known for a long time. But to realize that it is my responsibility to educate my children and all the youth about our own Kaingang mathematics, well, that's a different thing. Even better to learn that children also have human rights. I am glad to know that Brazil signed that document [UN Convention on the Rights of the Child].
- Mathematics is important for everything, this is one thing we learned here, and that we'll never forget. When I walk across my village everyday to go to the river, I will remember that I know how to read a map. I know numbers! But I also know the trees, and the rocks, and where the animals hide. I can cook and use a lot of diffent foods, measure them together to make cakes, bread, stews. And I can orient myself looking at the stars! This is our own Krenak mathematics.
- Now we know that we have the right to self-determination, to deciding what's good for us, we want to make sure our mathematics is taught in our Guarani language in our own schools. This will make us strong, very strong.
- When you learn so much about mathematics, about numbers, you want to start counting everything because this is what matters out there. There are many things we can measure to show we are autonomous peoples: our original lands, that stretched all the way from Argentina and Paraguay all the way to the Amazon! Lots of documents out there show huge Guarani migrations centuries ago. Then we can also count to show how many of our people were killed, enslaved, placed in boarding schools, and now live in Favelas. This is using mathematics to document genocide.

Inspired by this debate, Marcos Tupã, headman and teacher of the Guarani-Mbyá of the Boa Vista Village within the municipality of Ubatuba, took the initiative and started reporting on the mathematical knowledge of his own people:

> The Guarani use mathematics every day. For the construction of houses, we use our own mathematics. To construct a house, the measurement that we use is the palm. We also count with steps. If a family wants a house they say: "I want a house that has this width and this length." They wouldn't know that a palm has 20 centimeters, but they know that it is exactly the width of a straw mat. There are those responsible for forming the work teams and dividing up the work to build a house, making sure everyone is well fed during the construction. It falls to the teachers to record these different forms of mathematics.
>
> My father doesn't read or write but when he receives a basket of goods he distributes everything. Oil, for example, he would distribute approximately three cans for each house. Whatever was left over went to the prayer house or for the work teams.
>
> It falls onto our shoulders, the teachers, to conduct this survey and record Guarani Mathematics. There is a basket we make in the village using a thick type of bamboo. Half of the bamboo strips are painted and the other half are

Fig. 6.8 Guarani Pajé Cândido Ramirez and wife, at their new village near Praia Dura, municipality of Ubatuba, northern coast of São Paulo state, 2005

> not. The bottom of the basket is marked by this division between the painted parts and the non-painted parts. Therefore division is very important for us as a people.

Energized by this statement, Guarani teachers brought up a basic feature of the circulation of goods in their communities, best captured in the words of shaman Cândido Ramirez: "*poverty is having nothing to give*." Ramirez was referring to the economy of reciprocity of the Guarani people, which entails the circulation of goods, rather than their accumulation in the hands of a few, as in capitalism. Generosity and solidarity are basic principles of a gift economy – principles that helped orient the collective learning process of workshop participants.

Guarani Reciprocity and the Economy of Gift-Exchange

We then moved to discuss professor Jaime Lllulu Manchinere's statement, in the book *Madikauku*, on the power of love in mathematics: "Love is also used in mathematics; those whom you love or your relations for whom you have compassion collaborate to distribute fairly the gains of the community." Various considerations were then advanced about the division of foodstuffs that are in fact made in accordance with the principles of reciprocity that orient an economy of gift-exchange (Mauss, 1990/1950); among them relationship to family members, health, age, and prestige. Dividing up meat from a hunt, for example, involves precise estimations and calculations. Participants acknowledged that many times you don't divide up parts equally because some houses have more members or the elders are privileged. Or sometimes the family of the hunter has more of a right to certain hunting grounds. These and other multiple factors often enter into consideration, as well as previous debts, power, status, and emotions.

Educators brought up the fact that the great majority of math books used in Brazilian schools present arithmetic operations in this order: addition in the first place and division in the last. This seems to be self-explanatory as addition is the mathematical form most easy and "natural" to learn. To divide should be the last, as it is the most difficult and requires an existing understanding of all other operations. This it a technical question that does not take into consideration cultural and political meanings behind concepts of "more" and "less," and "divide" and "multiply."

The ways in which goods are distributed in basically egalitarian societies, such as the indigenous communities we are talking about, determine that when someone gives something to someone, the giver won't necessarily become dispossessed of those items or have "less" goods. On the contrary, the giver is usually put in a position of receiving "more" goods because of principles of reciprocity (Ferreira, 1997; Lévi-Strauss, 1969/1949; Mauss, 1990/1950). The apparently straightforward problem "Last night I caught ten fish, and gave three to my brother. How many fish do I have now?" presents more than one solution. The obligation to reciprocate means that the brother is going to give back the fish or another valuable item, and this, too, enters into the calculation. The relationship itself among "brothers" – in particular amidst brothers-in-law in societies where kinship relations, especially marriage prescriptions are very structured – often regulate exchange and inform calculations.

Exploring the various possible solutions to mathematical dilemmas is a stimulating activity that brings into focus relationships that aren't necessarily numerical. Discussions about the meanings of "more" and "less" during the workshop caused math teacher Alício Terena to reflect about social inequality in the west of São Paulo state. Practically all the youth and adult Terena men and women work daily as cheap laborers or peons on regional farms in exchange for a monthly minimum wage. They work 12 hours a day cutting cane, harvesting the land or picking fruit for the farmers, while their own garden plots are left unattended. The monthly minimum wage (3.68 U.S. dollars in 2011) barely covers family basic expenses: rice, beans, foodstuffs, clothes, medicine, tools, etc. The workers have no choice but accept buying goods on "credit" from the boss who sells them at inflated prices. The debt continues to grow and can only be reduced with more work, which in turn perpetuates a never-ending dependency by the Terena on the boss. This is known as "debt-peonage," a quite common form of slavery on and around indigenous lands in Brazil, and elsewhere around the planet (Ferreira, 2005). For farm-laboring Terena, *more* work does not mean more money as one would expect. On the contrary, the *more* you work as a *bóia-fria* or moonlighting worker, the *less* you are able to generate better life conditions. In the words of Alício Terena: "The more we earn, the less we get because our debts

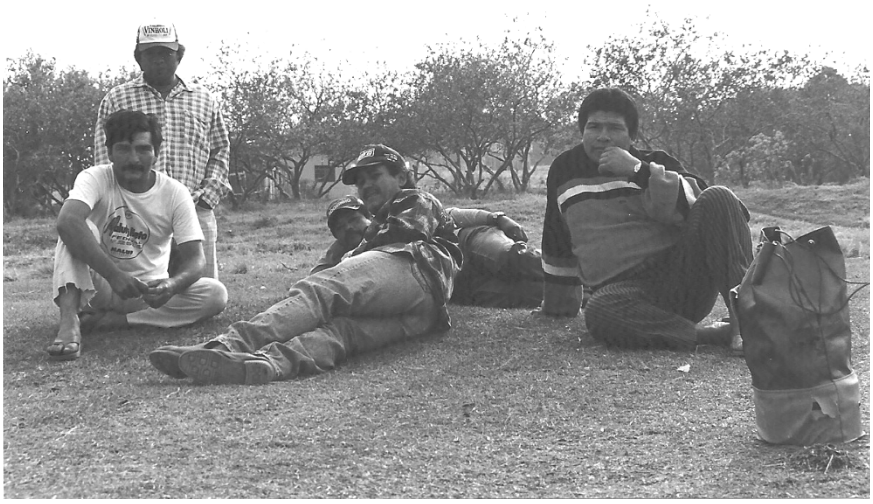


Fig. 6.9 Kaingang and Terena debt-peons in the state of São Paulo awaiting the cattle truck that will take them to the sugar cane plantations that have devastated their lives, 2000

continue to grow." Ethnographic fieldwork conducted in 1998 and 1999 on the health situation of the Terena people of the Kopenoti Indigenous Área, interior of the state of São Paulo, attested to the precarious life conditions of the community (Ferreira, 2005). Families were hungry, the children emaciated, and various ailments, including anemia, obesity, hypertension, diabetes, and alcoholism were rampant. Adults relied heavily on anti-depressants and other psychotropics, such as Valium and Prozac, to "tolerate life."

The teachers concluded as well that certain situations, like the commercialization of arts and crafts, hearts of palm, and agricultural products demanded specific types of calculations. In these situations, to sell means to disconnect yourself from these products in exchange for money. For example, if the Guarani Nhandeva shaman Francisco da Silva makes 15 arrows, sells 5 and wants to know how many are left over, the calculation is 15 minus 5. In the same way, if the price of each piece is R$10.00 then the total sale is $ 50.00 because the sum is the value of each arrow multiplied by 5.

Guarani teacher Poty Poram, introduced above, assessed that:

> Learning math in this way is important, because it is not only knowing how to count, but learning to think about life. Knowing that indigenous knowledge is important gives us the will to research Guarani mathematics among the elders.

Elementary school teacher Lidiane Krenak, who is also responsible for the Financial Council of the Vanuíre Indigenous Area in the west of São Paulo state, raised the question of gender in relation to mathematical activities on the

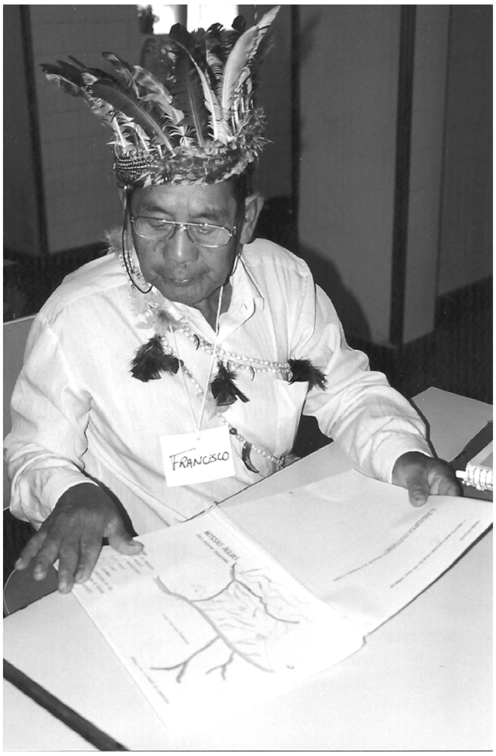


Fig. 6.10 Pajé Francisco Guarani Nhandeva studies a map in Madikauku, during the teacher-training program at the Cajamar Institute, 1999

reservation by discussing the difference between "women's math" and "men's math" for the Krenak people:

> The Krenak woman is the one who needs to be more involved with mathematics. It is she who calculates the food and firewood needed to cook and it is she who does everything, taking care of the children. So it is she that should study mathematics. For this reason I was chosen in Vanuíre to be the Financial Counselor. Men's mathematics is more about life outside of the village, such as in the cutting and selling of hearts of palms, marketplace purchases, and money.

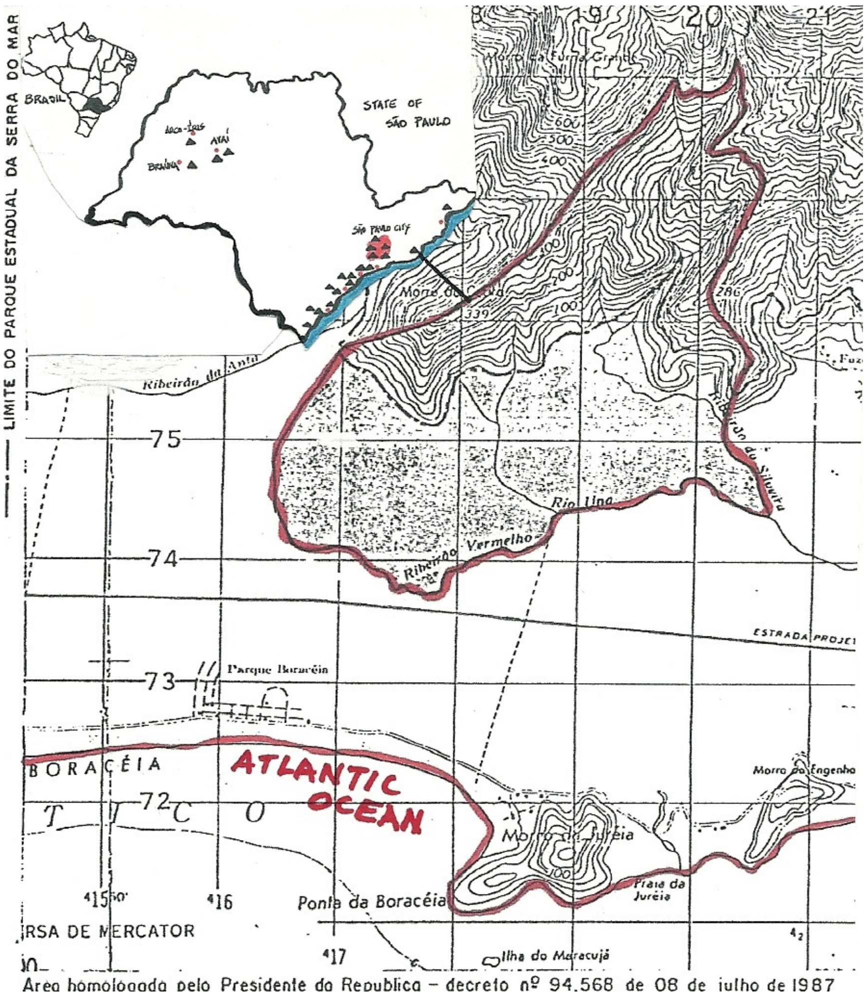


Fig. 6.11 Map: Área Indígena Guarani do Ribeirão Silveira, Municipalities of São Sebastião and Bertioga, northern coast of São Paulo

Área Indígena Guarani do Ribeirão Silveira

Guarani Indigenous Land of Ribeirão Silveira

The Ribeirão Silveira Indigenous Area is located in the northern coast of the sate of São Paulo, in between the municipalities of Bertioga and São Sebastião. It was demarcated in 1987 with a total area of 948.40 hectares. The vegetation is lush with many different plant species that are Native to the *Serra do Mar* [Costal Sierra], and a great variety of animals, such as the ant eater, sloth, armadillo, agouti, tapir, etc. The Guarani live from selling arts and crafts and from agriculture. We plant hearts of palm, banana, sugar cane, manioc, corn, sweet potato, a few fruit trees, and we also plant ornamental flowers to sell.

At the village there is a pharmacy where a practical nurse works, as well as a school house, where the teachers, the cook, and the general servant work. At the village there is also a *técnico indigenista* [a technician for indigenous issues], an *agrônomo* [agricultural technician], and a physician. As our village is located between two municipalities, Bertioga gives us educational services, and São Sebastião takes care of our health.

Today in our village there are six groups, each one with a leader who helps the headman. The *pajé* [shaman] is the great religious leader of the community.

(Submitted by Mauro Karaí Samuel dos Santos, Cristiane Martins, Elaine Maria de Souza Paião, and Weslene Pereira Santos.)

Is There Only One Mathematics? Or, Are There Different Mathematics?

By the third day of the workshop, non-indigenous teachers were already talking about "indigenous mathematics," and how much everyone could learn by studying *matemática de índio* or "Indian math." All participants thus requested that we focus on the documented contributions of indigenous peoples to the field of mathematics. The fourth day of the workshop was then dedicated to the study of documented numerical systems of indigenous Brazilians, starting with the Xavante of Mato Grosso, central Brazil, and then moving on to the Palikur of the northernmost state of Amapá. These choices reflected previous research in ethnomathematics, and my teaching of math among these groups (Ferreira, 1981, 1994, 1997, 1998; Green, 1994). As we'll see next, mathematical knowledges of the Xavante and Palikur nations spurred critical arithmetic ideas of the Guarani, Terena, Kaingang, and Krenak peoples present at the workshop.

Xavante Mathematics: "The Two of Us Together"

There was great excitement among workshop participants when we proposed looking at the current situation of the Xavante people and their mathematical knowledge. Until quite recently, the Xavante were known to be "fierce warriors," with a reputation for having fought bravely against so-called pacification fronts of the Brazilian Government – throwing clubs at encroaching airplanes – and only deciding to come into official contact with the broader Brazilian society in 1958. Compared to the Guarani, Terena, Krenak, Pankararu, and Kaingang peoples, who were stripped of most of their lands, the Xavante were able to amass large portions of their ancestral territories in Central Brazil. Questions asked included: If the Xavante were as "savage" and "primitive" as the media often portrayed them to be, how could they know any math? Wasn't mathematics only for the so-called civilized, smart and wealthy White people? Did their mathematical knowledge of the land, waterways, peoples, spirits, animals, plants, and stars in the sky help them

Fig. 6.12 Xavante young men undergoing GPS training to map environmental resources, and work towards the final demarcation of the Sangradouro Indigenous Area in the state of Mato Grosso, central Brazil, 2003

secure so much land? Do they count beyond infinity? What follows is a brief description of the Xavante numerical system, which was discussed with pleasure and in great detail by all 60 mathematics educators present at the event.

The numerical system of the Xavante people in central Brazil is of base 2, meaning that it is structured around pairings or groups of 2, characteristic of the duality of the Xavante social organization. The way in which the Xavante proceed in their logical systems is reflective of how they organize themselves socially into moieties – including a bimodal form of economy, based on a cyclical alternation between slash-and-burn agriculture, practiced in large-based villages, and the dispersion of these villages into semi-nomadic hunting-and-gathering bands (Turner 1979). Such a dualistic view of society and the universe at large – represented, for instance, in the dialectical relationship between the world of the living v. the world of the dead, and the rainy season v the dry season – informs the meaning of Xavante number words. Different worldviews – the socially constituted world and its cosmological foundations – and the everyday experience of active individuals account for the diversity of strategies of mathematical reasoning. In other words, different cultures, and individuals within any given culture, proceed differently in their logical schemes in the way they manage quantities and, consequently, numbers, geometrical shapes and relations, measurements, classifications and so forth. It is thus important to comprehend the cosmological underpinnings of a dialectical society, such as the Xavante, in order to understand how mathematics has been construed by this Gê-speaking people.

It makes sense, then, that dualism should also be the main feature of the Xavante numerical system. The difference between even and uneven numbers reflects this dialectical worldview. Number names follow the dualistic organizing principle, expressing a fundamental difference between odd and even numbers. *Mitsi* means that the element is "alone" or "on its own", and thus stands for the number 1. *Maparané*, the number 2, is the unitary base for counting, because it is the union of the "lonely halves," which form a pair. *Tsi'umdatõ* starts with the prefix *tsi*, which implies that it is an odd number (*tsi* alone, on its own), and thus stands for the number 3. *Maparané tsiuiwanã* represents 2 groups of 2, and it is the representation for the number 4. *Imrotõ*, "without a mate" (*imro* husband, mate; *tõ* without), is also an odd number, and is used to represent the number 5. *Imropö* stands for "the one who has found his mate," and stands for the number 6. And so forth it goes. When counting or expressing quantities, the Xavante express these numbers with their hands and feet, grouping fingers and toes in pairs, with great skill.

Following official contact with the broader national society in 1958, the Xavante's entry into the market economy, as well as the mandatory school-taught arithmetics in Catholic boarding schools brought about the adoption of a "natural" base 10, metric system in commercial and everyday transactions. The understanding of maps, the drawing of routes and diagrams of Xavante territories, and the balancing of checkbooks, buying medicine, ammunition, tools, clothes, and soap, and other commercial transactions have also demanded an understanding of base 10 scales, due to the widespread use of the metric system in Brazil. Demographic information that appears in official documents makes use of percentages, as well as environmental documents on the extent of deforestation and mining practices on Xavante ancestral lands and relatively nearby urban areas, such as Brasília, the country's district capital.

The absence of interest in, and eventually research about, Xavante knowledge in Catholic Salesian missionary schools (which some villagers are forced to attend to this very day) hurt the people's ability to document their ancestral mathematical ideas and make further use of them in various post-contact situations. The result was the false impression, shared among missionaries, government officials and local educators alike, that *matemática não é coisa de índio* (mathematics is not an Indian thing). The Xavante have faced enormous discrimination in the missionary and public school system because the challenge faced by the people and largely ignored by educators to transform a base 2 system automatically into a decimal one has been interpreted as "ignorance." This unnecessary type of confrontation among different numerical systems, only generates conflicts in the classroom when there is little or no understanding about, and respect for, the culturally rich ways in which mathematical knowledge helps create the world we live in. To know that there are diverse mathematical wisdoms, different ways of reckoning time and mapping space, for instance, and that it is possible to manipulate them within a worldview or to follow a global market, valorizes and enriches the process of knowledge construction – a basic tenet of the quality education that Brazilian indigenous peoples have the right to today, as expressed in the 2007 UNDRIP:

Fig. 6.13 Teacher Alício Terena, Área Indígena Kopenoti, Municipality of Avaí, interior of the state of São Paulo, 1999

> **Article 21**
> Indigenous peoples have the right, without discrimination, to the improvements of their economic and social conditions, including, inter alia, in the areas of education, employment, vocational training and retraining, housing, sanitation, health and social security.
> **Article 31**
> Indigenous peoples have the right to maintain, control, protect and develop their cultural heritage, traditional knowledge and traditional cultural expressions, as well as the manifestations of their sciences, technologies and cultures, including human and genetic resources, seeds, medicines, knowledge of the properties of fauna and flora, oral traditions, literatures, designs, sports and traditional games and visual and performing arts. They also have the right to maintain, control, protect and develop their intellectual property over such cultural heritage, traditional knowledge, and traditional cultural expressions.

"So what does it mean," asked Alício Terena, "to say that we have the right to improve our economic and social conditions, including education, employment, housing and more? Is it only in the *Estados Unidos* where these rights happen, or can it happen to us? I don't understand about the *Nações Unidas*, but I want to be a part of it. And how far away are the Xavante from us right now? Tell me! [Teachers are consulting their maps] São Paulo – Cuiabá [capital of Mato Grosso], *mil seiscentos e trinta e quarto kilômetros* [1132 miles]? I am learning more and more about the Brazilian Indians and our mathematics, and right now I feel like counting how many of us are in the same situation. We lost our lands – how much land? Our people were killed, how many? We still have our culture, our language – how many speakers? If we get together, our numbers will be bigger. What did you say, Mariana, about the Palikur people, way up there *na ponta do Brasil* [at the tip of Brazil]? Where do they live? How far are they from here? They speak French, Portuguese, and their own language? What is that? And how many families live there? I want to know this people."

Palikur Mathematics: Geometry is Everywhere

Thinking about the distance that separated us in São Paulo from the Xavante in Mato Grosso, central Brazil – 1634 km or 1132 miles, and from the Palikur in the state of Amapá, northernmost Brazil – at least 2664 km or 1655 miles to Macapá, the capital, was daunting. How could we even envision how far that is? And then how many more kilometers to the Palikur Indigenous Area, further north, bordering French Guiana? We had quite a lot of information that everyone wanted to understand about Palikur mathematics in the book *Madikauku*, stemming from the work of Green (1994, 2002). But workshop participants had something different in mind: Macapá's very popular and official soccer field – 100 meters long by 73 meters wide – is cut across midfield by the Equator, the imaginary line that divides the southern from the northern hemispheres starting at zero and going all the way up to 90 degrees latitude at both the North and South Poles. Playing soccer in Macapá (an all time favorite sport in Brazil) means that during 45 minutes, half the time of the 90 minute game, players of each team occupy either the world's northern hemisphere or the southern, and then switch fields after a 15 minute break. Some spoke of their favorite players' scores, and our chances of winning the 5th World Cup in 2002 (which we did). Others commented on their desire to be professional soccer players, or at least get to go to a few important matches, if they could save up the money.

Sônia Barbosa, a Guarani teacher at the *Área Indígena Guarani da Barragem,* located within the city limits of São Paulo, offered her map of the Barragem, highlighting the importance of both soccer and prayer to her people:

> We were walking around the village and I saw two children inside a house. We went by another house and it was already dark, but we could still see the people inside eating fish. When morning came everyone went to the soccer field to play ball, and then went to the prayer house. That's what we do everyday. The next day I noticed people working on their arts and crafts [small animals sculpted in wood] next to the fire. Here's my map.

A heated discussion ensued about how often each community played soccer, what rules they followed, and who their opponents were. "What does it all matter,"

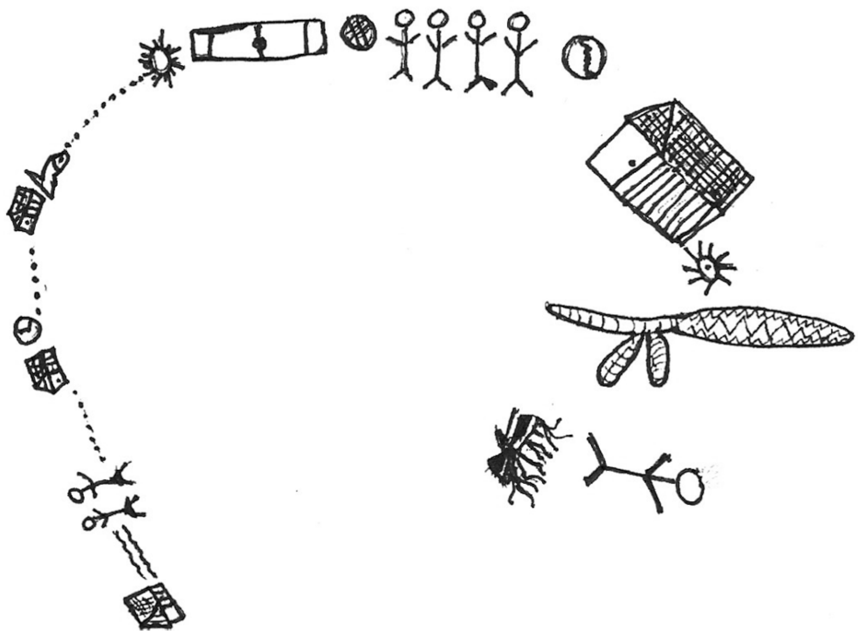


Fig. 6.14 "Guarani people pray and play soccer everyday." By Sônia Barbosa, Área Indígena Guarani da Barragem, 1999

asked Márcio Rodrigues, a Guarani teacher of the Krukutu Indigenous Area just outside of the capital, "if we are the *campeões do mundo* [world champions]?" Márcio was referring to Brazil's number 1 ranking position in the soccer World Cup. North and South, rich and poor: the idea of soccer fields questioning established dichotomies including nearby/far away, traditional/modern, and Indigenous/Brazilian begged the question, posed by Mauro Samuel, a Guarani teacher at the Rio Silveira Village: "So just how many soccer fields are we away from them?" "We are far away, more than 2664 km [1655 miles] far. Let's use division to see how close we get to them." Mauro Samuel took a pen and explained it on the whiteboard: "I'll have to multiply 2664 by 1000 to get everything into meters, so that's 2,664.000, and then divide the result by 100 meters, the length of the Macapá soccer field. That is 26,640 soccer fields all lined up from here to there!" "Twenty six thousand, six hundred and forty. That's very far, I see. Perhaps invite the Palikur to play soccer with us will make the distance smaller," said Márcio Rodrigues. Pankararu teacher Dimas Nascimento added:

> Far, yes; just as far away as Pernambuco, where we were brought from into São Paulo in the 1950s, with lots of promises about land and riches, schools and hospitals, work and leisure. If only we'd help build the *ferroviária* [train tracks] across [the state of] São Paulo! That's all done. Now where do we

> live? In shantytowns, across the huge city. no land, no garden, no clean water, no hospitals, no schools, no jobs, *nada*. You talk about human rights? We have none. And do we have our own soccer field? No. Here, I'll tell you: The Pankararu Village in the Real Parque Favela, where 800 Pankararu – men, women and children live, is the size of maybe three or four soccer fields. That's tight. Wouldn't you think the Pankararu must have more land if we want a soccer field? But where would the field go, inside our land?" "It used to be like that, and so we did have a vast amount of land in Pernambuco. Now we're here. As a teacher at the Favela Real Parque, I want to learn all about Palikur math, too. They are our brothers up there, more than anybody else around here. So I feel close to them, not that far away.

The Palikur, who traditionally called themselves Pa'ikwené ("the people of the river of the middle"), share a similar history of colonization and genocide with the Pankararu, Guarani, Kaingang, Krenak, Xavante, and most indigenous peoples in Brazil and all over the world, for that matter. Mention of thousands of Palikur, as they are known today, already featured in the travel diaries of Spanish navigators coming into the mouth of the Amazon river as early as 1513. However, their population was greatly reduced due to various epidemics, and extermination by slave-hunters. At the turn of the 20th century, after Brazil appropriated the contested territory of Amapá from France, the Palikur faced abusive treatment at the hands of Brazilian customs officers and other government officials for not speaking Portuguese at the time, for "smuggling," and ultimately for just being "Indians" (Capiberibe, 2002). Their population slowly started to recover; in 2008 there were about 900 Palikur living in Brazil, and 470 in French Guiana.

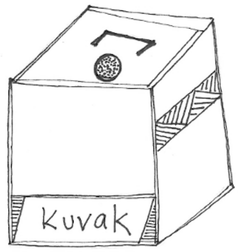
The Palikur numerical system

One of the most interesting aspects of Palikur cosmology is how their theory of the world is expressed in everyday life in terms of numbers and mathematical concepts. When this indigenous nation reckons time, measures space, and quantifies any living or non-living being of their universe, the numbers used to represent groupings and measurements do not always simply indicate quantities (Green, 1994). When we say in Portuguese that there are 18 indigenous peoples in the north of the state of Amapá, where the Palikur live, the number 18 indicates quantity within a decimal system, and nothing else. Eighteen does not provide us with information about any characteristics or traits about this indigenous people, nor about their distribution in space and time, or qualifiers applied to them, such as "indigenous," "human," "real people," etc. In this case 18 as in Indo-Arabic system, is essentially a quantifier.

Palikur numbers, to the contrary, teach us how the Palikur people think about themselves and others around them. In addition to quantifiers, numbers are also are qualifiers, adding one, two, and three dimensions and their meanings to every being, animated or not, that inhabits the cosmos at large. The Palikur number is contingent upon the following attributes: 1. Material: animated (alive) or not, human or animal, abstract or concrete; 2. Gender: feminine, masculine, or neutral; 3. Format: round, long, cylindrical, flat; 4. Position: types of clusters – bunches, hordes, packs, pairs; 5. Quantity: measurements – dimensions, collections, or plain plural form; and 6. Specificity: when the being or thing being represented does not fit within any of the above categories. For each of these attributes, there is a prefix that attaches itself to the numeral, acting like a qualifier.

Let's take, for instance, a girl or "one girl," quantified and qualified by the Palikur as a *paha-phru himano* – a single living feminine human, instead of just "a girl" or "one person." In this case, the number one for "girl" conveys three very specific attributes: material (alive and human), and gender (feminine). This qualification is quite different from "one box:" *paho-u-kiyes,* which classifies the one-masculine (*paho)* box alongside other formats: square (*u)* objects, which could include a house or book (depending on its position). "One arm," used as a length of measurement, is *paha-ti i wanti*, because the arm (*wanti*) is cylindrical (*ti i)*. "One week" is *paha-i paka*, a number one-feminine (*paha*) necessarily attached to the way in which week (*paka*) is classified within other members of an abstract group (*i*).

*Paha-**t** **l**it ahayak nunu*



*Paho-**u** bom kuvak*

"1 liter of honey" in the Pa'ikwené language spoken by the Palikur people, is:
*Paha-**t** **l**it ahayak nunu* **(**1-cylindrical liter bee honey), where "***t***" is the prefix attached to the feminine numeral one (*paha*) indicating the object is cilindrical.
One can of manioc flour is: *Paho-**u** bom kuvak* (1-square can flour), where "***u*"** is the prefix attached to the masculine numeral one (*Paho*), indicating the object is cylindrical.

The Palikur numerical decimal system, of base 10, is not simply a "counting system" as one would imagine. The way the Palikur count is intimately connected to their cosmology as people. To understand Palikur mathematics, in particular their numerical system, requires comprehension of a broader taxonomy – a cosmology, which entails the relationship of the Palikur amongst themselves and with all other beings in the universe. There isn't a way to think exclusively in "numbers" in the Palikur language, as there is in English. Rather, one must think about numbers in theory and in practice, especially in the lived experiences of this indigenous nation, as the Palikur numerical system does not exist outside of the people's conception of the world today. It is a fascinating universe that only quantifies its members insofar as it qualifies their special attributes, revealing, in its inner workings, the uniqueness of what it has meant to be Palikur or Pa'ikwené throughout, at the very least, these last five centuries of relentless colonization. The study of Palikur mathematics reveals not only how they as a people count, but a complex and intelligent system that has the ability to use geometric thought to enhance the contribution of all beings to the making and remaking of the world experienced.

The acknowledgement that each and every group of people or nation, indigenous or not, constructs its own mathematics, is not anything new to most historians of science and other scholars (D'Ambrosio, 1990; Lumpkin, 1997; Powell & Frankenstein, 1997; Zaslavsky, 1999). Further, to recognize that "Indigenous peoples have the right to revitalize, use, develop, and transmit their histories, languages, oral traditions, philosophies, writing systems and literatures, and to designate and retain their own names for communities, places and persons," now recognized in the UNDRIP Article 13, was mind blowing to most educators, indigenous or not. The teachers-in-training at the Cajamar Institute, however, reacted to these assertions with some skepticism, but mostly with great joy. "You mean, could this really be true, that we Indians have our own mathematics? How come we were always told we're ignorant and stupid and can't learn any numbers," said Poty Poram. "It has to be true we know mathematics," responded Mauro Warã, "because the Guarani can travel in the forest without getting lost, and you're saying that orienting ourselves in space is part of mathematics." "I believe it, too," added Fabiana Oliveira, "I'm really good at counting, calculating money, fabric for clothes, and measuring everything, even better than the Kaingang men." Dimas Nascimento concurred: "If we Pankararu didn't have our own mathematics, we couldn't survive this world full of numbers, and make do in a big city like São Paulo. Say some more! You say there are culturally distinct forms of working with quantities, numbers, measurements and geometric forms? Let's everybody explain our own ways of doing that." Further, the educators agreed, in principle, after discussing some of the Xavante and Palikur math contained in *Madikauku,* that the various ways of being, thinking, and acting in the world are intrinsically related to the ways in which mathematical systems are conceptualized. All agreed that it makes no sense to oppose indigenous mathematics to non-indigenous or Western mathematics – where's the evidence for that?

At the end of the fifth day of the workshop, half-way through our two-week event, there was general consensus that all human beings have the intellectual ability to develop mathematical ideas, as shown in many studies in cognitive anthropology and ethnomathematics. In addition, everyone agreed with a basic human right, now granted to indigenous peoples, too, which states:

> Indigenous peoples have the right to practice and revitalize their cultural traditions and customs. This includes the right to maintain, protect and develop the past, present and future manifestations of their cultures, such as archaeological and historical sites, artefacts, designs, ceremonies,

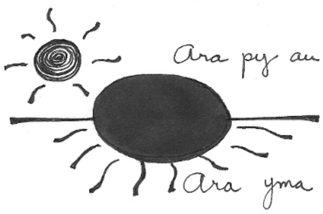
> technologies and visual and performing arts and literatures. (UNDRIP Article 11)

Mathematical systems may be as distinct as the myriad ways in which peoples worldwide order and classify the universe they live in and fashion at the same time. The little that is known about Palikur, Xavante, Guarani, Kaingang, Terena, Krenak or Pankararu mathematics is sufficient to refute prejudiced ideas about the simplicity of the "indigenous mind," and guarantee that the adoption by the UN Declaration on the Rights of Indigenous Peoples sends a clear message to the international community that the rights of Indigenous Peoples are not separate from or less than the rights of others, but are an integral part of a human rights system dedicated to the rights of all.

Discoveries from the Discussion on Indigenous Mathematics.

Presentation of Guarani mathematics by Marcos Tupã, Rio Silveira Indigenous Area Assisted by teachers Poty Poram (Jaraguá), Mariano (Rio Silveira), and Márcio Rodrigues (Krukutu), Marcos Tupã spoke about Guarani Mbyá mathematics, including ways in which his people reckons time, maps space, and conceptualizes the meaning of numbers.

> I am going to present here for you today what our group discussed last week at the meeting. We discovered that all peoples have a lot of mathematical knowledge to pass on to the children. These are old *conhecimentos* and new ones as well that we have just documented at this meeting, such as the designs that accompany Guarani numbers, as you will soon see. All of these ideas can be worked out further within our own villages, whether inside or outside of the classroom. We need to do a lot of research to document our own mathematics and to work it into the public school system. This is a way to protect our human rights.
>
> In these meetings it is important for us to have a space to discuss Indigenous knowledges, and not only White-related school lessons. It is important to produce educational materials that we can take back and use in our own schools and villages. We have a lot of work ahead of us to develop and document our knowledges and practices, showing the magnitude of our mathematics and the power that it can bring to the people.
>
> Now I will present a little about Guarani mathematics, starting with the reckoning of time [*medida do tempo*].
>
> First, I'm going to symbolize the seasons of the year, following the teachings of the *pajé*. We have two seasons, while the *juruá* [non-indigenous] have four.
>
> This is our earth, which is always moving. In the Guarani language we say:



Ara py au is like summer and spring together, when it is hot and good to plant.

Ara yma is like autumn and winter together, when it is cold and we prepare the earth for the beginning of planting.

The day is always in relation to the sun:

***Petei Porã - ****1 day*

***Petei Ara - ****1st. day of the month*

The month is always in relation to the moon:

***Petei Jaxy - ****1 month*

***Mukui Jaxy - ****2 months*

***Mboapyt Jaxy - ****3 months*

The year is related to the planting period:

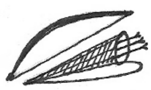
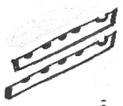
Petei Ma'ety - 1 year

We use the image of a young Indian holding a bag of seeds of corn and a planting stick to symbolize the planting season because we don't use machines to plant.

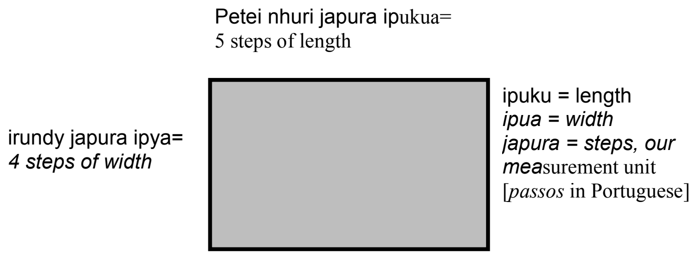
Numbers in Guarani Mbyá writing We think the following symbols can help represent numbers in Guarania Mbyá writing. We want to perfect our numerical system in the near future.

Next, Marcio Tupã presented informtion on measurements of length and width that the Guarani use when building a house.

Table 6.1. Representation of Guarani Mbyá numbers, according to the Guarani teachers present at the Institute Cajamar Workshop, 1999. By Márcio R. Guarani, Krukutu Village

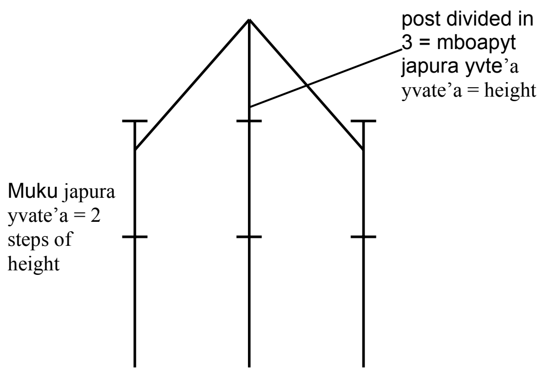
Representative Design	Meaning of the Design	Written in Guarani Mbyá	Indo-Arabic numeral system
	Stone to make a small axe	**Petei**	1
	Use of a stone and handle to make a small axe	**Mukui**	2
	Three arrows	**Mboapyt**	3
	A flute	**Irundy**	4
	The 5 fingers of the hand	**Petei Nhirui**	5
	3 doubled	**Mboa py meme**	6
	Bow and arrow	**Mboa py meme petei**	7
	4 doubled	**Irundy meme**	8
	5 plus 4	**Irundy meme petei**	9
	5 doubled	**Mukui Nhoieru**	10

The construction of a Guarani house



This is how it works. The pajé goes to the chief and says: "I want a house of such and such a length and width". The steps work as our measurement system. If it is for a family home, generally it is 5 steps of length and 4 steps of width, as is this design here. The size of a step can vary, of course, but our measurements are in accordance with the person in charge of making the house.

The steps serve to measure other things as well, like the height of the house. The posts of the house are measured as steps as well.



Presentation of Terena knowledge by Alício Terena (Kopenoti Indigenous Area)

My concern is how to take this information that we are learning here to the Terena of my village. Because we lost so much knowledge as a result of the invasion by the Whites, many Terena children do not speak their own language! This is why I am studying to become a teacher to teach the language and transform our school into a bilingual school. To teach only Portuguese and the knowledge of the Whites is not right.

I want to show the Terena the accomplishments of this workshop and everything that I've been learning to show the importance of teaching the Terena language to the children and researching about our knowledge with the elders. Before speaking about Terena mathematics here with you, I have

to do a lot of research. For example, why indigenous mathematics are so important for all peoples, and also why it is essential that we have a better understanding of the mathematics of the White men.
So, the work that I am doing in my village is like this: I design something, like a house, and I tell the children the name in our language. For instance, there are two ways of saying "house" in Terena: **peti** and **ovocuti**. Of course learning Portuguese is important for us to become an independent people. But to learn the Terena language is also important for our sovereignty.

House in the Terena language: peti or ovocuti

Fish in the Terena language: ho'e

Palm tree in the Terena language: emucaiá

In terms of numbers, we count until 3. More than that, we have lost the information.
Pohaxo – 1 Pyaxo – 2 **Mopoa'ti** ou **Mopoa'xi – 3**
These are the ideas that we are studying now. We hope to restore the language and knowledges of the Terena people very soon.

Presentation of Pankararu Knowledge, by Dimas Nascimento

I'm going to set mathematics aside for a while to speak with you about our relatives who arrived here in São Paulo in the 1950s. I do this because our history is related to our mathematics. Or to put it better, our current history is precisely the reason why the Pankararu people have lacked the necessary resources to develop our own mathematics as we should have done already.

Our relatives came from the state of Pernambuco [northeastern Brazil] on a *pau-de-arara* [cattle-truck]. It took them more than a month to arrive in São Paulo. They came fleeing the drought and starvation, hoping to find better life conditions here in São Paulo. We were promised jobs, and a good life building railroad tracks. Until recently, our presence in São Paulo city was

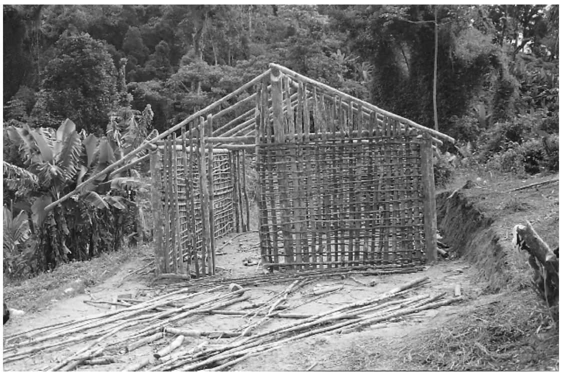


Fig. 6.15 Guarani house under construction, Área Indígena Guarani do Itaóca, 2001

not even recognized by the local administration of FUNAI in Bauru. Therefore we did not receive any support from FUNAI until very recently.

The 1,000 Pankararu that live in the city of São Paulo are divided up like this: 70% live in the Favela Real Parque, in Morumbi; 20% live in the Favela Madalena, and 10% are circulating between Jardim Ângela and the Favela Paraisópolis.

We live here in São Paulo, but as Whites and not as Indians. We were very discriminated against here in São Paulo, because our skin is black. So it's been a great struggle to be recognized as an Indian. We are gaining space very slowly. The recognition [as a federally recognized indigenous people] is already a big deal. It looks like FUNAI is going to buy an area of land for us around here, but we don't know when.

Meanwhile we have to go on with our lives passing as Whites. We have managed to convince folks we are not black, and convinced most people we are White. This is because non-Indians have this idea that Indians don't know how to work, that Indians don't know anything and therefore we don't have any rights. But this is a lie! Indians know everything!

I work in security and I've already worked in various companies. The rest of the people work in cleaning, bricklaying, and other gigs. So we have to have the White folks' documents and our own Indian documents. However, there are Whites who say: "You aren't Indian, you're black!" So the discrimination is doubled. To me, what's important isn't the color of your skin, it is blood. I feel pride to say who I am, a Pankararu Indian. This type of workshop here today empowers the people because it makes us proud to be who we are, because it values our knowledge.

Presentation of Krenak Knowledge by Lidiane Oliveira (Vanuire Indigenous Land)

How am I going to talk about mathematics if our people were massacred in the 60s and there are only a few Krenak individuals left? When you kill a people, you kill knowledge as well. So I need to explain how this happened.

As we almost went extinct, our mathematics almost disappeared, too. Many anthropologists have helped us rescue our culture, writing books about us. But it is a sad, sad history, that many elders don't like to tell because of so much suffering. And this has not ended yet, because our children suffer a lot of discrimination in the public schools today.

In the 1960s when our people still spoke the Krenak language, they were beaten down. If they didn't wear clothes, they were beaten down. If they did anything that seemed to be "Indian", they were beaten. So, how can knowledge survive? How can a people survive these conditions? We suffer until today.

Our God put us on this earth not for us to destroy it, like the White man does. Something that hurts us very much is to see a river that was clean, a forest that was green, not exist anymore. We try to pass on that the Indian was the guardian of the land. Have you ever heard that before the Whites arrived here, there were any industries? We don't have any more fish or game animals on our land. This world is ending from pollution. There are people here at this meeting that don't have land, that live in a matchbox. The real *brasileiros* are us, long before you.

Presentation of Kaingang knowledge by Ilson Iaiati (Icatu Indigenous Land)

Our case is similar to that of the Terena and the Krenak. We are trying to rescue Kaingang knowledge that was lost years ago. So I will tell a little of the history of how the Kaingang lived before, starting with the Kiki Festival.

Kiki is a drink that the Kaingang used to drink in order to dance. There wasn't *cachaça* [sugar cane rum] then. Each person had to look for firewood and coconut flowers to make kiki. It wasn't the white man that invented alcohol, but the Indians.

The Kaingang danced and sang the whole night drinking kiki to push away bad spirits. Those that couldn't keep up went to sleep and only continued the next night. But these things are of the past because the train tracks that now pass through the West in the Bauru region brought an end to my people. It was terrible, the white men electrified the road and many Indians were burned to death.

This is what I say to you teachers: our history needs to be told in the way that it actually happened. The animals have disappeared, the fish have died. Today we live in a totally dependent situation working as cheap farm laborers. Our mathematics was reduced to *um troquinho* [a little change], and then we were not able to buy anything with that *troquinho*.

Fig. 6.16 Luiz Karaí and Mariano Tupã-Mirim write letters to the mayor of Mongaguá asking for the removal of the city's garbage dump, next to the Itaóca Village, 1999

> For this, our friends and brothers – the Krenak, Pankararu, Terena, and Guarani – let us fight for a better education. Because us Indians have to have the space to restore our knowledge and be recognized as real teachers.
>
> The other day we went out to dance and the Whites asked me: "Why aren't you naked? Do you sleep naked?" And why did they ask me this question? Because of a lack of knowledge and prejudice about us. It is the same thing with our mathematics, the Whites think that we are dumb and don't know how to think or how to count. But this is not the truth and should not continue to happen.
>
> So this is my story. I've become very emotional, and I am grateful to all of you.

Ilson Iaiati wiped tears with the back of his hand, and returned to his seat amidst strong applause. "This is an important date for us," said Luiz Karaí, a Guarani teacher and headman of the village of Itaóca, southern coast of the state of São Paulo. And the headman went on:

> This is what we shall all be doing for the next 500 years! For 500 years we've been ignored, most of our people killed, our land taken away. Our languages, cultures, everything! It's very clear why our mathematics was destroyed, because it's dangerous! If we know mathematics, we can defend ourselves, protect our land, defend our rights. But not all of it has been lost. This is what we've seen here today. Thank you.

More applause, and then a standing ovation by all indigenous and non-indigenous educators alike. Luiz Karaí was referring to the fast approaching commemoration of the 500 years of the discovery of Brazil by the Portuguese navigator Pedro Álvares Cabral in the year of 1500.

A few months later in April 2000, indigenous leaders from all over the country protested against the celebration of the *500 anos de descobrimento do Brasil.* They marched to Brasília, met with President Cardoso, asking that their constitutional and human rights be respected. Amongst their urgent pleas was voting the new Brazilian Statute of Indigenous Peoples into Law, replacing its 1973 backward version. The *Estatuto dos Povos Indígenas* would help guarantee in practice their rights to land, natural resources, socio-cultural diversity, health, education, intellectual property, amongst many others. The *Estatuto* would be an important step toward the final approval of the UNDRIP, and its future implementation. Finally in 2009, two years after the UNDRIP was oficially adopted by the United Nations, the new *Estatuto* was signed into Law. Mariano Tupã-Mirim, a Guarani health agent at the Itaóca Village in southern São Paulo state, put it this way in a letter addressed to me in December 2009:

> The *Estatuto* is finally approved. Our people are really learning to live documented so we can follow all the laws. We put signs up all around our land so people know to respect. This is very good. I continue studying the medicinal plants to help the children and the community. . . . I made five new maps with all the plants, and I want to include them in the *Livro de Mapas de São Paulo*. I never thought I was good at mathematics, but I know everything about maps! How to read maps, how to draw maps, how to understand maps from other parts of the world. And my plants, where they grow and where our people can find them, even if today they have to travel far away. This is very beautiful! I understand now what Luiz Karaí [the teacher and headman of his village present at the workshop in 1999] says about mathematics being important for the autonomy of indigenous peoples.

Mathematics, Justice, and Respect for Human Rights

The maps and narratives, hand drawings and photographs shown above bring together powerful ideas about the worldviews, and in particular the mathematical knowledges of the Guarani, Terena, Kaingang, and Pankararu peoples in São Paulo, southern Brazil. It became crystal clear after a two-week intense mathematics training program, where all teachers spent 24 hours a day interacting socially and at least 60 hours of very hard work during 10 days, that in order to teach math or "talk numbers," one needs to contextualize mathematics within a broader social and historical situation, for mathematics to make sense as a product of human creation. If *Mathema* denotes knowledge, understanding, and explanation, while *tic* comes from *techne*, the same root of art or technique, then ethnomathematics, or mathematics, purely speaking, is the art or technique of explaining, knowing, or understanding in various cultural contexts (D'Ambrosio, 1990). While we scholars and educators may take this for granted, it certainly was startling for most non-Indigenous workshop participants to realize that most of São

Paulo's native population lives today in urban and suburban *favelas*, working as cheap peons and living in extreme poverty with less than one U.S. dollar a day, despite the fact that São Paulo is by far the richest state in the country, and arguably in South America.

When teacher Ilson Iaiati ended his statement presented above on Kaingang mathematical knowledge, the connection he established between nudity, ignorance, and being "Indian" in southern Brazil indicates very clearly the prevailing lack of respect for and deep ignorance about, indigenous cultures, mathematics in particular. In addition, when Ilson says he has "become very emotional" after his statement on Kaingang knowledge it is because he voiced for the first time in public, in front of 60 other math teachers, a keen awareness of the strong relationship between the history of oppression of indigenous peoples and the lack of recognition about their mathematical knowledges within the Brazilian public school system. The vast majority of Brazilian Indigenous children who have access to public education fail the system precisely because it does not acknowledge indigenous peoples' traditional mathematical ideas, and their innate ability to comprehend and make efficient use of school taught mathematics.

Theoretically speaking, Ilson recognized the important role that thoughts, emotions, and actions play in the production of knowledge, mathematical or not. Indigenous teachers' *desire* to discuss mathematics education openly in a multiethnic context, and *determination* to highlight the contributions of their own mathematics were key in setting the tone for a mathematics training workshop that otherwise may not have succeeded. In fact, most oral or graphic statements, including drawings and maps, were delivered in very emotional ways, including very moving dedications and songs that signaled the rebirth or remaking of certain concepts or ideas, as in the reconfiguration of Guarani 1 to 10 number symbols presented above. From this perspective, it is key to understand that *knowledge* is the foundational cornerstone of any cultural system, and that there are different ways of knowing depending on each and every society's ways of being in the world via their thoughts, actions, feelings, embodied skills, verbal models, and taxonomies (Barth, 1990, 1995). It was particularly important for non-indigenous teachers to realize that each and every indigenous people at the workshop had its own "culture" and "historical situation," and therefore the combination of these two factors helped shape and express the fascinating mathematical ideas being presented, for the first time in some cases, in front of our very eyes.

Making maps was a sentimental endeavor in itself from the very start. Picking up a pencil or pen and drawing the scope of your ancestral land, boundaries of your reservation, or exact location of your shack on a urban grid was a highly charged activity. Teacher Dimas Nascimento kept making bigger and bigger copies of a section of a São Paulo city map where the *Favela Real Parque* is located, as if an artificial change in scale gave the 800+ Pankararu in that shantytown the right to occupy a larger portion of land for their survival. It was challenging for other math teachers who currently dispute the official demarcated boundaries of their territories to come to an agreement of which map to use or whether to draw entirely new ones. This is why some of the narratives portrayed above are not accompanied by cartographic maps – it was very difficult to reach consensus on the "right" map to use. Other teachers figured that they would work and reproduce in the map book documents provided by FUNAI, so as to call our attention to the diminutive size of their lands, or lack thereof.

Brazil, 511 years old in the year 2011, still ignores, to a great extent, the immense socio-diversity and mathematical ideas of its contemporary indigenous peoples. To wake up educators, policy-makers, activists, and administrators worldwide to the themes of socio-cultural diversity, historical situation, and human rights has the power to generate interest in and respect for the existing multiplicity of mathematical knowledge forms and practices. In the same vein, it is true that the preservation of the biodiversity of the planet demands respect of socio-diversity, due to the incredible richness of knowledge forms among ethnically differentiated peoples (Carneiro da Cunha, 1995). It is well known today that scientific and technological development also demand knowledge of and respect for socio-cultural diversity, as mentioned before. In this respect, the richness and complexity of the Palikur numerical system, for instance – one of the most well studied and documented in the country, may contribute toward the advancement of sophisticated mathematical ideas, primarily in relation to geometric thought (Ferreira, 1998; Green, 1994, 2000).

Last but not least, Terena, Krenak, Kaingang, Pankararu, and Guarani teachers indicated a preference for the notions of "historical situation" and "indigenous peoples' rights" that include the right to self-determination, that is, the capacity of each group to define their own future goals and aspirations, eventually closing barriers to the importation of habits, symbols, and technologies of other populations (Oliveira Filho, 1998).

An understanding of school-taught mathematics, with its usual focus on algorithms and calculations, is oftentimes a fundamental tool for the establishment of more egalitarian relations between indigenous and non-indigenous peoples in Brazil. The making and interpretation of cartographic maps for the protection of one's own land, for instance, is clearly an instance where very specific mathematical knowledge is needed. And indeed, as most workshop participants concurred, "mathematics is important for the protection of indigenous peoples' rights." As Marcos Tupã clearly put it,

> Maps are important to me because they are the protection of my land. Now we live documented, and maps are the most important kind of document today. If I cannot protect my land, I am letting go of my constitutional rights to land and to cultural diversity. Didn't my people fight enough until the 1988 Constitution gave them back their right to have a piece of land and the right to be Guarani? I think the right to be yourself is a basic human right everyone should have.

Poty Poram added,

> I live on the smallest reservation there is, less than one acre for almost 20 families! What we are trying to do now is to increase the size of the Jaraguá

> Land to include a portion of the national park area that surrounds it, all inside the *capital de São Paulo*! For me and for my students, as well for all my Guarani community, everyone needs to understand your place on the map, where you are, where you are coming from and where you are headed. Especially for the Guarani people, because we are always moving, headed somewhere. Knowing maps, geography and history, helps guarantee our rights. We want the opportunity to tell our own history, to draw our own maps, to write stories in our own language. This is what I am here for, to gain all this knowledge to empower my people and help protect their human rights.

The question for Poty Poram, Marcos Tupã, and other educators then became: how to understand and transmit school-taught mathematics within the Brazilian public school system, while simultaneously respecting, researching, and documenting mathematical ideas of their own peoples.

The ideas formulated in this teacher-training workshop indicate, once again, that indigenous peoples have much to contribute to the history of mathematics education in Brazil. Respect for indigenous mathematics and other ways of knowing, however, demand that indigenous communities have access to resources that allow for the documentation, analysis, and the dissemination of this knowledge, if desired, in public and private schools, indigenous or not. It is important, then to stimulate and valorize the "invention of traditions" of indigenous peoples to the frames in which they propose – as the Guarani did in their own form of writing and the iconic representation of numbers 1 to 10 (Table 6.1). Tradition, in this way, becomes the coming together of ideas and living knowledge, created and recreated by individuals that share the same social group (Barth, 1990). Starting with this proposal, the Guarani, Krenak, Pankararu, Kaingang and Terena peoples now continue developing their own numerical systems and mathematical ideas at their villages and in their schools. Via storytelling, history, video, and map-making, we see today a growing body of indigenous multimedia literature informing the culturally specific curricula of each one of Brazil's 600+ indigenous schools. Mathematics educators welcome the fact that indigenous peoples are organizing themselves for political, economic, social and cultural enhancement, in order to bring to an end all forms of discrimination and oppression wherever they occur.

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JERRY LIPKA, MONICA WONG, DORA ANDREW-IHRKE, &
EVELYN YANEZ

7. DEVELOPING AN ALTERNATIVE LEARNING TRAJECTORY FOR RATIONAL NUMBER REASONING, GEOMETRY, AND MEASURING BASED ON INDIGENOUS KNOWLEDGE[1]

Math in a Cultural Context (MCC)[2] is a long-term curriculum and professional development project based on indigenous knowledge (IK) in Alaska. Collaborating with Yup'ik elders and teachers for approximately two decades, MCC has identified a powerful set of mathematical processes used in constructing everyday artifacts. The knowledge of Yup'ik elders in Alaska provides a unique way to develop an alternate learning trajectory to teach rational number reasoning, geometry, and measuring. The work described in this chapter represents one of the first efforts to show concretely how IK embedded in everyday activity yields a powerful set of mathematical processes – symmetry/splitting, proportional measuring, and geometrical verification. These mathematical processes can provide a basis for developing an alternative mathematical learning trajectory, providing teachers and students with fresh ways of connecting numbers/number relations, geometry, and measuring. The chapter provides an account of the first steps taken to develop a culturally valid assessment to move the IK hypothetical learning trajectory to an empirically based one.

BACKGROUND AND CONTEXT

Imagine entering school for the very first time and being told in an authoritative tone in an unfamiliar language that you cannot bring your language, experience, knowledge, and values to school. Yet this is exactly the experience that Dora Andrew-Ihrke and Evelyn Yanez[3] faced when they began their schooling in two different communities in southwest Alaska years ago. Dora was told bluntly and explicitly "to wipe her feet at the school door," symbolically wiping away her Yup'ik Eskimo culture and language and leaving behind much of her knowledge and identity and inculcating a sense of insecurity about one's place in the world. Their cultural experiences and indigenous knowledge were anathema to the neo-colonial school policies and practices, which were directed towards assimilation. Years later in the early 1980s, when Lipka was a field faculty member of a University of Alaska Fairbanks program, the Cross-Cultural Educational Development Program, aimed at increasing the number of indigenous teachers in

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Alaska, he began working with Andrew-Ihrke and Yanez and a group of Yup'ik "teachers-in-training." An action-research group was organized and formed to study and identify "what it meant to be a Yup'ik teacher" and to identify ways to make schooling more culturally relevant for indigenous teachers and students. One of the objectives of the group was to identify the cultural strengths of indigenous knowledge and practice, and how it could be applied to schooling. When Lipka discussed these plans with some school district superintendents he was met with incredulity and denigration, "what culture?"[4] By contrast, when Lipka, Yanez, and Andrew-Ihrke talked to one of the key Yup'ik elders, Henry Alakayak of Manokotak, Alaska, he was very interested in the possibility of Yup'ik knowledge going forward into schooling "so that the next generation could learn." Further, he was interested in how Yup'ik knowledge would develop in the school context and what tools and instruments would develop from a Yup'ik perspective.

In this chapter, we explore a vision of what can develop when insiders/outsiders work together over many years to apply IK to schooling, particularly mathematics education. From those early days when Evelyn and Dora began their schooling, which alienated the very ideas presented in this chapter, it has been an unusual and fortuitous journey. In part, this chapter represents our attempt to stand the neo-colonial paradigm of exclusion and assimilation on its head and proffer an alternative paradigm, in part, based on IK. The mathematical processes embedded in everyday activity, specifically symmetry/splitting, proportional measuring, and geometrical verification, generate this IK hypothetical learning trajectory for the teaching and learning of rational number reasoning, geometry, and measuring. We aim at developing an empirically based IK learning trajectory for the math curriculum as a whole.

Our aim is to bring in the lessons learned from Yup'ik elders that Dora Andrew-Ihrke and Evelyn Yanez have experienced since their earliest days and that Lipka has experienced since the 1980s. Monica Wong joined the project because of her interest in equity and assessment. Little did we know that we would be positing a way to teach the foundations of mathematics based on elders' activities, their ways of thinking, and stories. We present key lessons learned from the mathematics embedded in everyday activities of Yup'ik elders; how these lessons form an integrated foundation for teaching rational number reasoning, geometry, and measuring; and how we are simultaneously developing an assessment instrument that aligns with this alternative mathematics learning trajectory.

The legacy of colonialism is directly related to the aberrant direction schooling has taken in many parts of the colonized world from Alaska to Australia, and throughout many other parts of the world. The result of this legacy has been the underperformance of indigenous students as determined by imposed and culturally alien norms. One of the most vexing and enduring issues in education is how U.S. schools have failed to serve American Indian (AI) and Alaskan Native (AN) students and their communities. Federal reports have, for almost a century, advocated approaches that recommend educational programs connect school and community, as a way to redress the continuing lower academic performance of AI/AN students. This chapter focuses on improving students' academic performance in mathematics. To achieve this objective, the MCC project has developed a long-term curriculum and professional development project, in collaboration with Yup'ik Eskimo elders from southwest Alaska, and with Yup'ik teachers, mathematicians and mathematics educators, educational researchers, and local school districts. The project has expanded to both urban and rural school districts and has been implemented across Alaska's diverse geographical and cultural regions: Athabaskan, Inupiaq, Tlingit, and Yup'ik. This current project takes place within five diverse Alaskan school districts.

BRIEF THEORETICAL OVERVIEW

Two main theoretical positions developed during the 1970s explain the intractable and problematic underperformance of schools serving indigenous and minority students in the US. One, the cultural mismatch theory, is represented by the work of anthropologically oriented educational and psychological research (e.g., Cole, Gay, Glick, & Sharp 1971). This theory suggests that the mismatch between the culture of the community and the culture of the school – values, ways of knowing, ways of relating, language, and ways of communicating – are the primary cause for the underperformance of schools serving indigenous populations. It also provides a theoretical explanation for the type of educational experience Dora Andrew-Ihrke and Evelyn Yanez described. Instead of schooling accommodating local knowledge and values, the social and cultural organization of schooling in Alaska was based on the larger, majority society, even though, in many cases, this was an alienating experience for students. The early ethnographic studies painted a vivid picture of how cultural hegemony was systematically institutionalized, relegating cultural differences as something to be expunged – something standing in the way of indigenous students' education.

The second theoretical perspective is represented in research that suggests that culture only played a secondary role but that power associated with identity, resistance, and limited economic opportunity for employment because of discrimination played a primary role in the underperformance of AI/AN students and minority students (Ogbu, 1992). We do not argue against Ogbu's observation but we do not take an either/or position. In Alaska, both perspectives can be easily identified. However, for purposes of building approaches from IK by providing educational affordances that recognize and even celebrate elders' knowledge in the form of curriculum, we have followed the pathway established by anthropologically and psychologically oriented theories to build a culturally based mathematics learning trajectory. We are also committed to design valid assessment tools to map the learning trajectory. This program provides us with opportunities to explore the potential of IK in informing school's mathematical practices.

Schools and communities – particularly for indigenous and minority cultures – are embedded within larger socio-cultural contexts as theorized in cultural-historical activity theory (CHAT) (Roth & Lee, 2007). Further, we note the importance of practical activity, which is one of the key aspects of CHAT. Thus,

> … human psychology is concerned with the activity of concrete individuals, which takes place either in a collective – that is, jointly with other people, or in a situation in which the subject deals directly with the surrounding world of objects – for example, at the potter's wheel or the writer's desk. … With all its varied forms, the human individual's activity is a system in the system of social relations. It does not exist without these relations. The specific form in which it exists is determined by the forms and means of material and mental social interactions. (Leontiev, 1981, p. 11)

It was our work with Yup'ik elders, as they collectively shared and demonstrated their knowledge of constructing a variety of everyday artifacts as part of their system of social relations, that afforded development of our understanding of their mathematical thinking. That also led us to consider how their methods can be translated into mathematical processes and tools appropriate for schooling. Similarly, pedagogy, highly influenced by expert-apprentice interactions and relying notably on visuo-spatial ways of teaching and learning, arose from this approach.

The act of teaching and learning in schools is mediated by macro and micro factors such as classroom organization and communication. Studies in cross-cultural psychology have identified the importance of the home culture and cognition in relation to schooling (Cole & Scribner, 1974), while ethno-mathematicians have consistently argued that you cannot impose cultural knowledge from one group on another and expect meaningful understanding. Instead, culture that shapes perceptions and ways of organizing thought comes into conflict with the imposed curriculum. Likewise, curriculum, instruction, pedagogy, and values differ across cultural groups and to develop mathematics curriculum and instruction, these differences must be recognized. Clearly, we are challenged to recognize the role of out-of-school context in learning and thus apply culturally situated activities to schooling. We have to make connections between culture and cognition because

> [culture] shapes mind … it provides us with the tool kit by which to take into account the cultural setting and its resources, the very things that give mind its shape and scope. Learning and remembering, talking, imagining: all of them are made possible by culture. (Bruner, 1996, p. x–xi)

Although this particular school of thought has gained ascendency over the past few decades, it is naive to think that an ethnomathematics program can simply identify an activity, such as unschooled youngsters selling candy on the street or the way an indigenous group counts, and slot it into the curriculum and expect students to perform better in mathematics. The work described within this chapter is part of this ethnographically-oriented approach of bridging the culture of the community with the culture of the school, with the aim of improving AN students' math performance without sacrificing their identity, culture, and language. Our methods aim at mitigating the differences between the embedded mathematics of Yup'ik elders' everyday activities and school mathematics. Working with elders *in situ* –

to learn how they perform everyday tasks and embedded mathematics through their culture, language, and in their context – is necessitated by the fact that the elders are the experts or the holders of knowledge. This approach is required to develop IK based mathematics curriculum instead of more typical "clinical" work with students. We first learned activities that were taught to us by elders for the dual purpose of "valuing" their knowledge and fulfilling their dictum, "to have the next generation learn and become thinkers."

We pay close attention to how to do what was taught to us – navigate by the stars, make patterns, make clothing – and how to construct a variety of structures like fish racks and smokehouses. The elders also taught us how to identify their ways of thinking and solving everyday problems, and how those ways of thinking could be applied to build a mathematical foundation for young students. The two Yup'ik co-authors in this chapter play the critical role of being the "cultural brokers." They are able to communicate with the elders in Yup'ik, understanding not only the specific activity but also seeing the underlying connections across activities. Slowly, our team has evolved an understanding of Yup'ik cosmology and epistemology that underlie the mathematical processes across a range of everyday subsistence-related activities and have a high potential for informing the mathematics classroom.

We believe this work is part of a paradigm shift in which people in the field no longer view IK within a deficit framework, instead, it joins a small but growing body of literature that examines how IK can inform educational spaces in both indigenous and non-indigenous contexts. This change shifts the dialogue from IK as an educational hindrance to one that can inform and improve educational practice. This is a road rarely taken in curriculum and professional development and one which we attempt to navigate in our current project.

MCC's supplemental elementary school mathematics curriculum development was initially based on different aspects of the subsistence cycle, such as collecting or gathering foods, including berries and eggs. These activities typically include: locating a good place to gather berries or collect eggs, estimating distance and time (how far and how long will it take to get there), and estimating volume (how many filled buckets will be needed), sorting by attributes such as type of berry, storing (other units of measure – freezer bag full), use (recipes – measuring), and sharing/redistributing the food to elders. Integrated math modules were developed following this genre of subsistence activities. Other modules incorporated how elders processed the food gathered or caught. These included the construction of fish racks to dry salmon and smokehouses to preserve salmon. Storybooks were also developed to provide a context and support the mathematics embedded within the modules. Some narratives were traditional stories, while others were personal experience stories. Interspersed within the modules were traditional games – such as "falling sticks" – that the elders had taught us.

MCC modules provide, particularly for the rural AN students, a chance to engage in mathematics without having to work through the additional "noise" of irrelevant metaphors, unfamiliar examples, and with the possibility of learning within a context familiar to them. Indeed, positive statistically significant impacts on AN students' and other students' mathematics performance have been observed when using MCC's supplemental curriculum.

Before an in-depth discussion of traditional Yup'ik practices and how they can inform teaching and learning mathematics, an understanding of Yup'ik cosmology and epistemology is warranted. The following story was told to us[5] by Mary Active, an elder from Togiak, Alaska, at one of the many meetings held during the past two decades. This story highlights features of Yup'ik cosmology, epistemology, and ways of performing activities; we make these connections following the story.

Ground Squirrel as told by Mary Active, Togiak

Because it's Spring, I'll tell a story about a squirrel. A squirrel came out from the side of a hill and was walking along. When he got to the edge of a river, down below where there are a lot of trees, he saw three houses next to each other. The squirrel stood up on his hind legs and looked down, using his hand to shade his eyes. As he watched from up there a girl came out. She stood there looking in his direction, looking very neat and clean, wearing a belt and mittens and other things look at her (referring to a child in the room). So, then he left, no, he watched those three houses. After a while another from the house next door, it is said that the first to come out, the one who looked around, she had a lot of wood, she really had a lot of wood. And the next one who came out had some wood, but not as much as the first one. But the furthest one had absolutely nothing around her house. And when she came out she was nothing like her, and how messy her hair was.

The squirrel watched them, kept looking at them. When the last one came out, the poor girl didn't have a belt and was messy, the string that holds up her mukluks were down making her mukluks sag. Then that squirrel said to himself, although he was by himself, "If I get trapped by the one who has a lot of wood she'll take good care of me. And she has a lot of wood to cook me with. I shall get caught in her trap."

Then, he went on his way. He came upon a trap and that trap was messy. Again, he came across a trap a second time and this one was also messy and sunk into the ground. Then he came upon another trap and this one was very neat, it even had grass to keep it from melting, it was well prepared. The squirrel after doing something, entered it, got caught and lost consciousness. And there he stayed. When he came to himself, the woman, the one who was neat, was holding him, taking care of him. She was really taking good care of him. Then she took him home to her house. After taking him home she got ready to skin him. And she – he's being aware of all this, although he is dead – being very meticulous, she quickly strangled him. She took his little innards and other parts and dumped them into the river. Throwing them away, she shouted, "Come quickly back to me again!" After being unaware, after becoming unconscious he went back up to the shore and left. That woman really took care of him. She took good care of him.

The story illuminates aspects of Yup'ik cosmology and the reciprocal relations between humans and animals: how they are connected and how harmony is required to keep this balance intact. It is the squirrel that gives himself to the meticulous one – the one who will take good care of him, and the one who returns his body to the river so that the cycle of life can continue. The story also hints at important Yup'ik values, which guide how a Yupiaq should act – not to waste, to be neat, and be respectful. This can be seen when Dora Andrew-Ihrke, at workshops or presentations, or before beginning a project, such as making a ceremonial beaded headdress, places the act of producing the artifact into the context of its larger purposes – to please *Yua Ella* (the Spirit). When Dora Andrew-Ihrke tells the details surrounding the importance of her acts, she is, in effect, referencing the elders who informed her. These elders include her mother and grandmother among others. Thus, this information harkens back to a much earlier era when they were young and being instructed on how to be a "real person" (the literal meaning of a Yupiaq).

Important Yup'ik values are still extant, as described by Barker, Fienup-Riordan, and John (2010). One important action that also guides the development of an IK-based hypothetical learning trajectory is the importance of symmetry. The authors describe how in Yup'ik dance, an action on one side of the body is usually followed by the same action on the other side. Elders talk about the importance of lines of symmetry and center points in many of their processes related to constructing artifacts. Evelyn has often talked about the Yup'ik rules governing how patterns need to be balanced, black and white and along lines of symmetry, thus creating harmony. When Dora learned to make clothing for dolls, first, and then for herself without exogenous tools, she learned the importance of folding material so that this side is equal to that side. Another principle guiding this work is the importance of developing one's own cultural mathematical tools; this follows from how Yup'ik elders fashion their own clothing. Oftentimes, this begins from irregular material and transforming it into regular shapes.

TOWARDS AN INDIGENOUS KNOWLEDGE BASED MATHEMATICAL LEARNING TRAJECTORY

Current Curriculum Developments

Within MCC, we are currently developing Rational Number Reasoning (RNR), (rational numbers being numbers of the form a/b, where *a* and *b* are whole numbers, and *b* is not zero.) The RNR curriculum materials for grades 2 to 6 will intertwine the embedded mathematics of Yup'ik everyday constructions with fractions, ratios, proportional reasoning, and geometry, thus potentially establishing a robust approach to RNR, geometry, and measuring. Seven areas of research on RNR circumscribe its meaning: (a) equi-partitioning, (b) multiplication and division, (c) ratio, proportion, and rate, (d) fractions, (e) area and volume, (f) similarity and scaling, and (g) decimals and percents (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009). Repeatedly, elders have demonstrated how they use body proportional measuring and symmetry/splitting in tailoring clothing, constructing structures, and star navigating. MCC's approach has always been to work with Yup'ik Eskimo elders and Yup'ik teachers, along with mathematicians and math educators, educators, and Alaskan school districts with an aim of integrating Yup'ik and Western knowledge for the purpose of improving students' mathematics knowledge and performance. As this two-decade-old project has matured, we have increasingly recognized the mathematically laden ways that Yup'ik elders use their knowledge to solve everyday problems. Our most recent mathematics curriculum development and learning trajectory capitalizes on IK and an Indigenous worldview. Constituting mathematics curriculum from IK, that is both an authentic representation of Yup'ik cultural practice and school mathematics, is a turnaround from the not so distant colonial past.

Elders who have worked with us for many years respond, when asked what Yup'ik word or concept best describes math, with "measuring" [*cuuq* in Yup'ik]. The measuring they describe is not simply linear and two-dimensional measuring, but centers on proportional measuring. A critical aspect of Yup'ik measuring is that it is undertaken in the context of the material being used, the purpose of the materials, the wearer or the user of the end product, and the task to be performed. This situated measuring is much more akin to proportional measuring. Probably the most classic example is the construction of the kayak. The kayak is constructed with the body measures of the individual who will use it, thus creating a balance between the user, the kayak, and the particular conditions in which it will be used (e.g., riverine or ocean).

This ever so brief glimpse into Yup'ik cosmology and epistemology sets the stage for our new IK based mathematical learning trajectory and assessment instrument. Our learning trajectory encompasses:

> … a researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (Confrey et al., 2009, p. 346)

Underpinning their working definition of learning trajectories is the notion that these trajectories are models of likely paths students traverse to reach a deep understanding of a mathematical idea. Hence, it is a hypothetical learning trajectory, which requires empirical evidence to inform, validate, and refine. The remainder of this chapter focuses on describing a HLT for geometry, measuring, and rational number reasoning based on IK and we describe our initial efforts in developing an assessment to measure students' location along the trajectory.

Figure 7.1 represents the key generative aspects of this integrated mathematical approach for establishing a foundation and a pathway (horizontal and vertical integration of learning) for the mathematical strands of RNR, geometry, and measuring. This is how we envision mathematical learning from a Yup'ik world-view. This model integrates Yup'ik cosmology, epistemology, and practice,

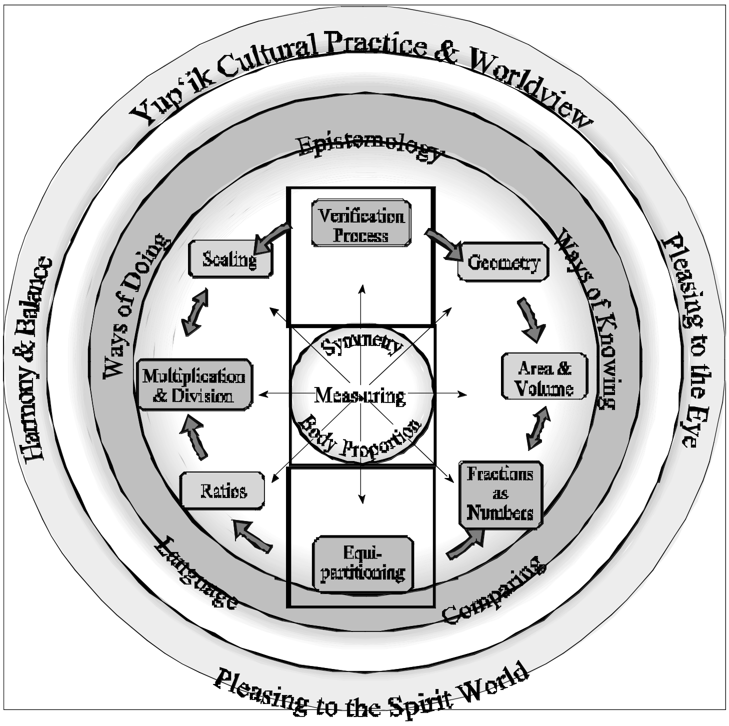


Fig. 7.1 Hypothetical learning trajectory based on IK

including a pedagogical perspective incorporated within a community of practice, expert-apprentice modeling, and the generative mathematical component. At its core, the Yup'ik values of harmony and balance translate mathematically to relate directly to symmetry and body proportionality. Implied in the model, emanating from notions of communities of practice, is the design principle of students as constructors – not consumers – of mathematics. This principle reflects elders' dictum that "students need to be thinkers," as well as a cultural imperative for those societies and subcultures in which its members must create and construct their own tools and artifacts. These principles establish a productive context for learning.

In this model the fundamental mathematical processes are located within the center of the circle.[6] The dynamic of symmetry/splitting resulting in equipartitioning, along with, as in our case, body proportional measuring, and verification, produces an approach to rational number reasoning and multiplicative thinking while connecting to geometrical thinking, the concepts of area and volume, with understanding of fractions as numbers, ratios, and scaling. We believe that this generative and integrative mathematical approach associates activities or actions with mathematical processes. The model's implication is that

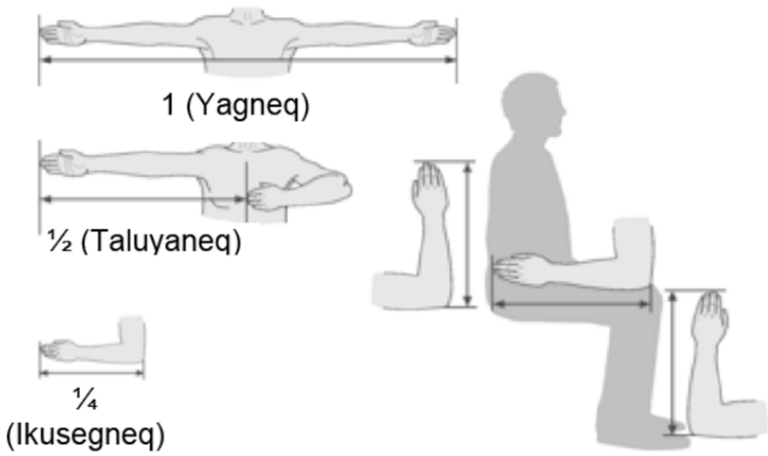


Fig. 7.2 Set of body proportional measures

one action results in multiple distinct mathematical strands and pathways. The model can be envisaged as moving along horizontal and vertical planes: the vertical planes represent the distinct mathematical strands as it moves through the grade levels, while the horizontal plane reflects the generative core processes previously enumerated. This approach can provide students and teachers alike with an efficient and integrative approach to learning a wide array of mathematical concepts. The lessons designed from this perspective contribute to student engagement, which, in turn, builds stronger connections between actions and mathematical concepts.

Dora and Evelyn say that IK-based HLT begins with the concepts of measuring, including proportional and non-proportional measures. The primary tool for measuring is one's own body. At the foundation of the IK-HLT, the emphasis is on qualitative aspects of measurement of length, area, and volume. Simple comparison of lengths, and prealgebraic notions of equality, more and less than, would be introduced; students would verify their responses by direct physical comparisons.

First and foremost, parts of one's own body are used as proportional measures. The base unit is one *Yagneq*, the distance between one's arms extended in opposite directions (Fig. 7.2). Like measuring systems that we are familiar with, when our unit is too large, smaller units are introduced and because of the proportional nature of the human body, the new measures are related: the *Taluyaneq* (half a *Yagneq*) or *Ikusegneq* (one fourth of a *Yagneq* or half a *Taluyaneq*). Elders have also shown and told us how they measure without physical instrumentation and how they use visual memory by taking the measurements of a familiar person to

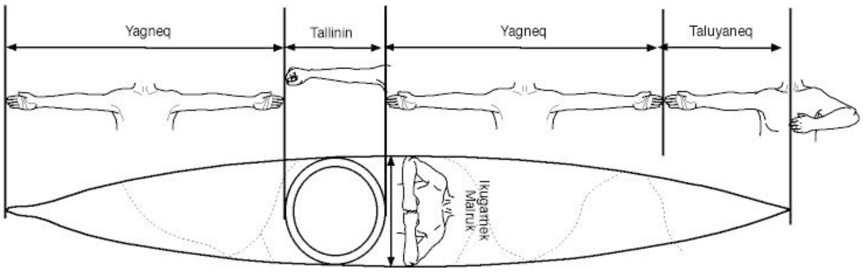


Fig. 7.3 Body proportional measures used in constructing a kayak

create a template for a person who differs in size. For example, using the familiar person as a base unit or reference for making clothing, materials may need to be added or subtracted proportionally, based on the perceived differences. Thus, body proportionality and symmetry are integral to the activity of measuring both physically and mentally.

Yup'ik measuring is body proportional measuring. In this case, body proportional measures occur in a context of purposive behavior typically related to constructing structures, or making clothing or tools. In collaboration with Yup'ik elders, we have repeatedly observed and documented how sets of proportional body measures, as illustrated in Figure 7.2, constitute one simplified set relating to a multitude of everyday tasks from constructing a variety of structures, such as kayaks, black fish trap, fish racks and smoke houses, to making patterns, star navigating, and measuring subsistence set nets.

Since Figure 7.2 illustrates how a set of body measures are related to each other establishing simple *a*:*b* ratios, students would use and explore body proportional measures without numbers as their classroom activities and projects. Further along this ratio trajectory, students in later parts of first- and second-grade would explore simple ratios, 1:2, 1:4, and so on. Further along the trajectory, students represent the body proportional measures more abstractly. The concept of ratio deepens as students construct their math tools, such as strips that are similar for making patterns; using these strips in cultural ways of pattern making, they explore fractional numbers and their products, such as ½ x ⅓, and ratios such as 2:3. Moving from the physically represented simple ratios to more complex relationships and abstract representation is part of the process of concept refinement, which leads toward a more robust understanding of ratios and proportions. The genius of Yup'ik proportional measuring in a tailoring or subsistence context is that the same process (e.g., outstretched arms or hand measure, fist-closed thumb extended) can be applied by every person to construct a particular object, such as a kayak, that fits and is in balance with the user. For example, for a kayak the Yup'ik measure is approximately 3:1, that being the ratio of the length of a kayak to *Yagneq* (arm span) (Fig. 7.3). Fish racks that hold fish for drying are constructed using a specific body measure and applied by the particular user's body, hence the rack is also in balance with the user. Thus, simple ratios (a:b = *Yagneq*:*Taluyaneq* = 2:1) are foundational in our trajectory.

The Square Is the Central Shape within IK-HLT

The square is a shape central within the IK-HLT. It is a shape from which other shapes are constructed – for example, a circle inscribed within a square. Thus, constructing a square signifies the start of the geometry strand. Since the square would be the first shape that students learn, their learning is enhanced by the use of physical material that they can easily manipulate. Posed with the question "How can you construct a square from irregular material?" Students observe an expert, prior to their experimenting by folding and unfolding, composing and decomposing squares by symmetrical folds. For the younger children, it would be necessary to mark the square into segments, such as for making a chatterbox or a paper fortune-teller (see http://en.wikipedia.org/wiki/Paper_fortune_teller) that could be created with many symmetrical folds starting from a square paper. This process mirrors how Yup'ik elders fold material. Once students are able to construct their own square they would be able to make other related shapes, such as rectangles and triangles. The construction of a square also establishes the full complement of cultural mathematical processes (in terms of splitting/symmetry, body proportional measuring, and verification) that are aligned to develop the sets of skills and concepts associated with this HLT.

Simple ratios as relationships are evident in the way Yupiaq elders perform everyday tasks. An important and common Yup'ik measure is the "knuckle" measure (Fig. 7.4), which forms a basis for constructing a square. A square is then transformed into other geometrical shapes, resulting in pleasing repeating patterns that adorn squirrel parkas or become the basis of circles used for ceremonial headdresses (Fig. 7.4). In both cases, the knuckle measure is half the length of the constructed square and ½ the length of the diameter of a circle, thus establishing a 2:1 or 1:2 relationship. With the square as a foundation, and through the splitting/symmetry verification, other shapes such as isosceles right triangles,

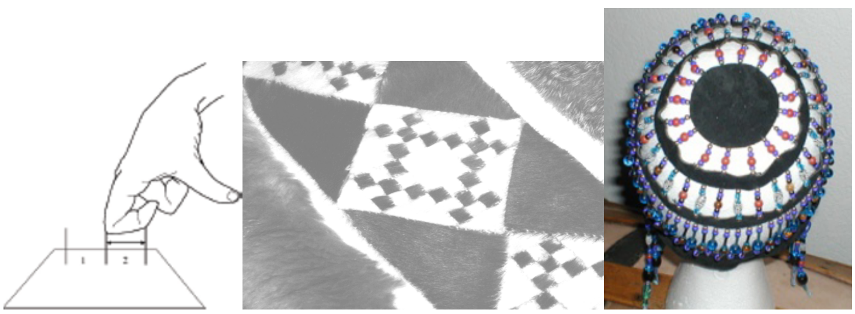


Fig. 7.4. The knuckle measure is the basis for a square and circle. Knuckle measure (left), pattern on a squirrel parka (center), and ceremonial headdress (right)

rectangles, parallelograms, smaller squares can be constructed, while “preserving specific size relationships” (Pendergast, Lipka, Watt, Gilliland, & Sharp, 2007, p. 4). The pattern-making activities, known by Yup’ik elders as *tumartat,* can be translated as “putting together the pieces to make a whole” (p. 10). They are laden with mathematical concepts – geometry, patterns, fractions and ratios.

When Dora demonstrates traditional activities using lessons learned from her mother, she explains a relationship between numbers/number theory and geometry, from which we observe a robust method of multiplicative thinking emerging via the process of fabric or paper folding. This approach differs from existing mathematics curricula in that multiplication and division occur before or concurrent with addition and subtraction and it establishes an alternative to counting as the pathway to numbers/number relations, in which multiplicative thinking is promoted in preference to additive thinking.

As mentioned earlier, elders often start projects by making a square, as the square is the critical geometric shape from which other shapes are made.[7] The following description represents the central processes of the HLT. Dora begins the construction of a square from uneven material because animal skins are irregular, and the Yup'ik make shapes with as little waste of material as possible. Using paper, she folds a crease to create a straight line, which she then cuts along. From the edge of the straight line, she then measures and marks one knuckle length from the straight edge at three points along the straight line, then marks another knuckle length from each to create a second set of points. Although Euclidean geometry axioms state that a straight line can be produced from two points, Dora marks a third one for verification and as a way to enable young learner’s accuracy. She then folds the paper along the second set of points and cuts along the fold, which represent the opposite side, or the edge of a square two knuckle-lengths wide. She then makes a symmetrical fold, creating unverified halves perpendicular to the sides, and creates a temporary center point. From this point, she measures a knuckle length in both directions to establish the outline of the square. She has envisioned the square from the center point, not from the corners. The square is then verified from a Yup’ik perspective using transformational geometry: “It is about what you do to the shape that stays the same … that is a reflection … the two sides of the mirror – the image and the original match” (Lipka & Andrew-Ihrke, 2009, p. 9). This is in contrast to employing Euclidean geometry that the four sides are equal length and all angles are at right angles. This process suggests an alternative definition of a square such as one based on the four right angles identified in the center of the square and four equal lengths radiating from the center. Producing alternative definitions and focusing on other properties of a square creates a potential inquiry by opening learning possibilities into one of the most staid concepts – the definition of a square – for students and teachers (Fig. 7.5). The same figure shows the “transformations” that occur when Dora and other Yup’ik elders’ construct a square, use recursive folding, folding through lines of symmetry which can appear as bilateral or rotational symmetry; the process of verifying if the symmetrical folds resulted in areas lends itself to defining and verifying a square from the center out. As Figure 7.6 shows some of the possible ways in which students and teachers could describe the properties of a square. It also shows how this approach supports additional inquiries. For example, would a rectangle have inscribed and circumscribed circles that touch the rectangle on four points? These types of conjectures can deepen students' mathematical understanding and increase their sense that mathematics is a subject that is open to exploration.

The mathematical richness of the elders' everyday problem-solving approach is that their actions on materials can become a source for multiple mathematical pathways that can be used in the classroom. For example, we also observe (see Fig. 7.6) that one fold creates two congruent areas, each representing half of the original square; without unfolding the paper, a second orthogonal fold creates four (2 x 2) congruent areas each being one-fourth of the original square; folding once again, a third "halving" fold creates eight (2 x 2 x 2) congruent areas each one-eighth of the original square. The number of congruent areas can be determined from the expression 2^n, where n represents the number of recursive folds without unfolding the paper. Hence, recursive folding establishes a robust understanding of multiplicative thinking for young students, with each congruent area representing the fraction $(½)^n$ of the original square. If the folding is non-recursive, fold – open – fold, then multiplication can be viewed as repeated addition, 2 parts + 2 parts + 2 parts. Thus the centrality of the square incorporates all aspects of the generative constructs of the HLT and establishes the geometry trajectory as well as the splitting/symmetry. Equipartitioning/splitting can be described as an early primitive operation of halving and doubling, or 2-splits, of either collections or wholes (Confrey et al., 2009). Equipartitioning/splitting is an operation which leads to partitive division as well as to multiplication. Figure 7.6 also embeds the beginnings of the fraction trajectory "many-as-one," starting with composite units, unit fractions, part-whole fractions, and equivalent fractions. As explained below, proto-multiplication and division occur prior to adding and subtracting fractions.

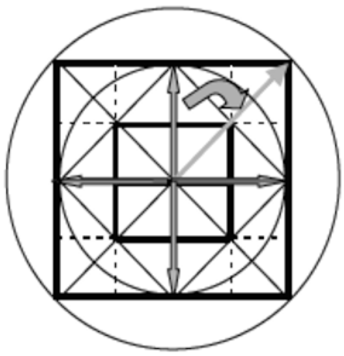


Fig. 7.5 Everyday square construction and transformational geometry

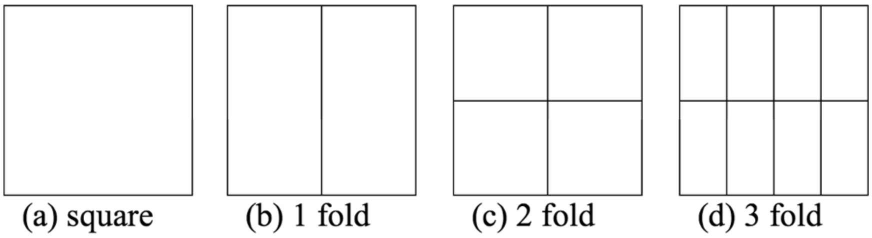


Fig. 7.6 Exponential operations through material folding, unit fractions, and equivalent fractions

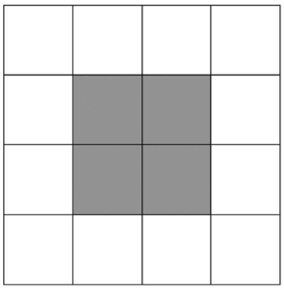


Fig. 7.7 Geometrical scaling

As students create a fourth recursive fold, for the purpose of creating pattern sets or other similar projects, then they will create Figure 7.7 shown below which introduces the concept of geometrical scaling and moving towards a deeper understanding of doubling and halving.

If students started from the innermost four squares or rectangular center and were challenged to produce a similar square or rectangle they would be learning about doubling in two dimensions and important concepts pertaining to area relationships. Conversely, if students started from the larger square and were challenged to create a square with sides half the length of the original then they produce the inner shaded square.

The square construction also provides opportunities to explore relationships between lengths, simple proportions, and early algebraic thinking. By comparing the lengths, students can evaluate their relationships and observe that the knuckle measure is shorter than the three-finger measure (see Figure 7.8), or that the three-finger measure is longer than the knuckle measure. They also observe that two knuckle measures equals one three-finger measure. By having students explore relationships prior to numeric quantities they maybe more predisposed to flexible thinking about the nature of the relationship. Elders continually stressed the importance for students to become flexible thinkers.

Since the HLT commences from a relational proportional measuring context, there are noticeable differences in the way learning is envisaged. For example, we

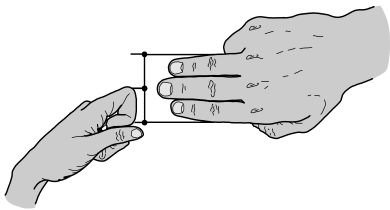


Fig. 7.8 Measuring and comparing two lengths: knuckle (left) and three finger (right)

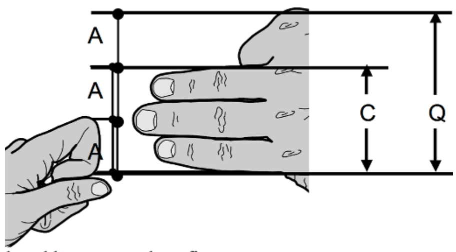


Fig. 7.9 Measuring and comparing two lengths

anticipate our HLT regarding how students measure quantity, at first without the need of formal units or an absolute measure. Further, our approach encourages students to explore measuring, without quantity, by symbolically labeling and comparing length, area, or volume. Expanding further the previous knuckle and three-finger example, we can name the knuckle length A, and the three-finger length C (Fig. 7.9). By examining the relationship between the lengths we can create the following relationships C > A, A < C, create new lengths Q = A + C, which then allows students to discover more relationships, such as A < Q and Q – C = A, and "re-examine some of the situations where they transformed two unequal quantities into equal amounts" (Dougherty & Slovin, 2004, p. 296) – e.g., C = A + A, C = 2 x A, so that A = ½C; Y = A + A + A, Q = C + A.

Comparing lengths and having students explore and symbolically represent those explorations provides students with insights into sets of relationships and early algebraic and logical thinking. This approach will build across the various explorations of developing RNR, geometry, and measuring curriculum. Our assessments need to align with this potential learning pathway. Further, comparing lengths is an important part of Yup'ik everyday life, particularly among elders who are still engaged in subsistence activities and remember the lessons learned from their elders. Dora, in particular, as well as other elders, have shown us that comparing lengths provides a basis for early algebraic reasoning and support for students to learn number theory and other relations. As far as we know, Dora was taught by her mother, and other elders, that comparing lengths was embedded in other everyday cultural activities.

Fig. 7.10 Creating thirds using rectangular paper

Dora's mother, Annie Andrews, demonstrated two folding techniques or algorithms to Dora that can provide students and teachers with a way of understanding exponents, primes, and composites. The first algorithm, she called "easy folds," could be made by folding a rectangular strip, into two, four, and eight equal pieces. This approach of "halving" is a concept that most students understand when entering school (Ball, 1990). Further, Annie Andrews demonstrated how to create primes – a folding algorithm that starts from folding a strip of fabric or a rectangular strip of paper into three equal parts. As shown in Figure 7.10, she does this by estimating a third and gently folding the estimated one third, leaving two thirds visible, with one third folded over the other. She uses equipartitioning to validate all the parts are congruent. It is an elegant and simple method, in which the top third of the two thirds is the same size as the one third, as it is easier to compare two parts instead of three. We believe that this folding procedures works for all primes up to the limits of the material and the dexterity of the folder. Further, Dora has demonstrated how her mother combined the halving fold with "thirding" to create composites such as a sixth. Once you know these two types of folds, you can create any fractional number – again to the limits of the material and the dexterity of the folder. Again, from IK and practice, we begin to observe an alternative learning trajectory for rational number reasoning, measuring, geometry, and early algebraic reasoning. The approach has the potential to provide affordances to all students to learn some of the foundations for mathematical thinking.

We also rely on the empirically established learning trajectory as a way to further describe our HLT. The learning trajectory highlights two significant differences between our work and that of others (Confrey et al., 2009). First, the use of verification of the constructed square, from its center point, is completed using reflective symmetry and congruence; the question "is this area equal to the other area" is asked, rather than paying attention to the four sides and the outer four right angles. The second difference is the direct link to geometry. The decorative pattern on a squirrel parka (Fig. 7.4 center) incorporates right isosceles triangles and squares of different sizes. It is one such pattern that is used extensively in creating clothing. The model learning trajectory we are developing from elders' knowledge includes geometry, measuring, rational number reasoning, and early algebraic thinking; all taking place within a relational measuring context. Along with development of RNR curriculum materials, assessment materials are also under development as it is expected that students taught using these new materials would improve their conceptual understanding of the targeted mathematical content. Hence the project will adopt a research design to collect empirical evidence on students' actual learning trajectories.

ASSESSMENT DEVELOPMENT

Note that this section of the chapter reflects work in progress at an early stage. This work is situated, in part, by our long-term work with rural school districts and with urban schools that have an Alaska Native orientation. These schools, in general, are under severe pressure from the State of Alaska as they struggle under the federal government's requirements. Although our developmental work over many years with Yup'ik elders and teachers, among others, takes place within a setting in which elders are comfortable and in which we can creatively explore without institutional constraints, these school districts and associated communities are threatened. Thus, over the many years of collecting data on the potential efficacy of our work we have taken a highly conservative tack to testing. Why? Work like MCC's ethnomathematics or culturally based mathematics is typically not embraced by mainstream education, and to be "relevant" to the participating school districts we needed to speak to them in a language and with evidence that they would recommend As we embark on developing assessments that can capture what students can learn under the HLT and associated classroom activities, we want to capitalize on the cultural and community strengths outlined in this paper and simultaneously be "credible" to the school districts that we work with. Thus, the interrelationship between culture, intelligence, and testing requires some thought.

Successful intelligence can be defined as "the skills and knowledge needed for success in life, according to one's own definition of success, within one's own sociocultural context" (Sternberg, 2004, p. 326). There are four models of the relationship of culture to intelligence, which we consider within this discussion of mathematical knowledge. These models shown in Table 7.1 differ along two criteria, namely the nature of mental processes and representation of knowledge, and whether differences in the instruments used to measure such knowledge exist. We replaced the term "intelligence" with "knowledge" as promoted within his definition of successful intelligence. As described in the first part of this chapter, Yup'ik cosmology, epistemology and practice, their knowledge of mathematics as represented by the HLT, is embedded within their cultural activities, thus the dimension of knowledge is culturally distinct from Western culture.

Table 7.1. Models of the relationship of culture to knowledge

		Dimensions of Knowledge	
	Relation	Same	Different
Tests of Knowledge	Same	Model 1	Model 2
	Different	Model 3	Model 4

Sternberg's model 2 and model 4 consider knowledge to differ between cultures. Model 2 proposes the use of the same instrument across cultures. when the same instrument is used, Sternberg suggests that the outcomes obtained are "structurally different as a function of the culture being investigated" (p. 326). Using the same test, a different culture approach of model 2 with Asian (Japan and Korea) and Western cultures, and using the same test can highlight different ways of thinking within cultures (Nisbett, 2003). In contrast, model 4 promotes different tests for each culture under investigation and adopts the position that intelligence is defined and distinct within a given cultural context. Ability tests presuppose a particular cultural framework and that frameworks across cultures regarding values, knowledge, and communication differ. Thus, "If a piece of research originates in one of two cultures being compared, the method and issues will bear the mark of that culture, no matter how carefully a linguistic translation is done" (Greenfield, 1997, p. 1117). Yet, we need an assessment that is credible within both cultures and allows us to measure student knowledge and understanding, which will be accepted by both school districts, proponents of ethnomathematics, state and federal funding bodies, and educational researchers. Hence, we embrace Greenfield's recommendation to have a bicultural or multicultural team to develop an instrument prior to commencement of the study to manage cultural bias.

On previous occasions, MCC has developed curriculum and assessments with a bicultural team. MCC has also used or adapted published assessment instruments. When no suitable instrument met their requirements, instruments were constructed, then piloted to assess and compare the difficulty between pre-test and post-test instruments, gauge their reliability, as well as their other psychometric properties. With the development of RNR curriculum materials, the need for a suitable assessment was an imperative. However, the approach we outline here necessitates capturing this nascent approach in a testing context that supports the strength of the model and simultaneously produces unbiased credible evidence to others.

ASSESSMENT REQUIREMENTS

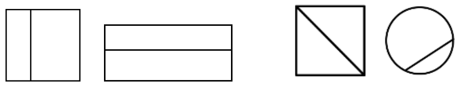
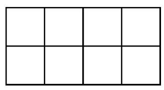
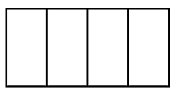
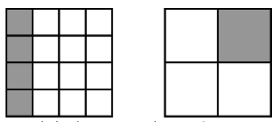
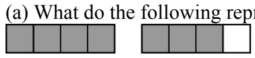
One of the unusual aspects of developing a HLT and eventually an empirically based learning trajectory is the approach that we have taken to establish the assessment. Fundamentally, the key requirement was basing the HLT on IK and, in our case, this meant working with elders/experts, as opposed to the more common Western psychological approach of child interviews and tasks. Our approach fits more comfortably within an expert-apprentice/community of practice model. Other key requirements and challenges include reflecting on both authentic IK and mathematical knowledge in the assessment. For us this is particularly challenging because we needed to include performance items along with the paper-and-pencil type. Our decision was based on the notion that performance of tasks provides a better prospect of getting closer to the mathematically embedded practices of Yup'ik elders. For example, we envision performance-based items such as students folding squares and rectangles in multiple ways to create halves and thirds, create equal partitions in both length and area models. Further, we envision students' transferring their "performance abilities" to number lines and increasingly abstract generalizations. The *Assessment of Fraction Understanding* (Wong, 2009) was identified by MCC as a pencil-and-paper instrument that could be adapted for the RNR project. Briefly, the instrument was intended to establish students' level of knowledge and understanding of fraction equivalence, which incorporates the concepts of measuring, equipartitioning, and fractions as numbers within the HLT. Assessment items incorporated area models (i.e., circular, rectangular and square), number-line models, recognition of the unit of measure, partitioning, equivalence, and fraction language. Two forms were available, one for grades 3 and 4 and another for grades 5 and 6. Common items across the two forms enabled comparison of student achievement without the need for students to attempt all items.

We started the process of developing an instrument that will reflect the elders' knowledge and be mathematically valid. Further, Wong described her work to the Teacher Leaders attending a weekend Math workshop and we began the process of integrating and transforming her *Assessment of Fraction Understanding* instrument to include other rational number meanings represented by Dora and Evelyn and other Yup'ik elders. The items listed in Table 7.2 by increasing difficulty were identified by workshop participants as those that incorporated some Yup'ik cultural ideas, however, modifications are under consideration as the embedded mathematical Yup'ik activities are hands-on (e.g., constructing the square) and knowledge of the mathematics and activity would not be accurately represented by a pencil-and-paper question. Included in the table are the skills required to answer the item correctly. This represents a preliminary look at the at assessing students rational number reasoning, whilst simultaneously considering how student assessments whether performance or pencil and paper based can support the construction of empirically supported RNR learning trajectory. These items represent an early attempt at developing a valid rational number assessment that builds from the specific ways in which elders taught us and generalizes to school mathematics. This work continues to develop.

CONCLUSION

The speculative and exploratory work reported in this chapter reflects an ethnographically-grounded approach to developing RNR, geometry, and measuring curriculum and assessment based on indigenous knowledge. Understanding the

Table 7.2. AFU items for possible inclusion in RNR assessment materials

Skill Required	*Example Items*
Recognize the quantity half, represented in a simple area model.	Circle the shapes that have been divided in half. 
Recognize a fraction quantity represented in a simple area model.	What fraction has been shaded grey?  Alternative:
Represent a fraction quantity by partitioning an area model.	As accurately as possible, shade 2/8 of the rectangle. 
Represent a fraction quantity using an equivalent representation.	In the rectangle, shade enough small squares so that 3/4 of the rectangle is shaded.  Shade in 2/2 of the shape below. 
Recognize the fraction quantity represented.	Has the same fraction of each large square been shaded?  Explain how you know?
Recognize the fraction quantity represented.	 This rectangle represents one whole. (a) What do the following represent altogether?  b) Can you think of a name for the fraction shaded?

connections between aspects of Yup'ik cosmology, epistemology, and practice and the connections between the embedded mathematics contained in everyday practice and its relationship to school mathematics represents a major step forward from earlier neo-colonial approaches to curriculum and assessment. In fact, the unique IK perspective on mathematical processes gleamed from Yup'ik elders potentially has applicability to Yup'ik and other indigenous group as well as to students and teachers in mainstream contexts. The dynamic and integrative relationship between splitting/symmetry, body proportional measuring, and verification reflects a generative model for developing school mathematics curriculum and assessment. Starting from the simplest actions such as comparing the relationship between two measures, halving, and so on forms the basis for this project's new curriculum development pathway.

School district face increasing pressure from both the state and federal government, thus we set out to establish an assessment that aligns with the strength of the new approach and one that meets more normative standards. The RNR pencil-and-paper assessments were designed to reflect IK knowledge. For doing so, it was imperative that the assessment developers and teachers understood the cultural activities and mathematics embedded within those activities. One difficulty encountered in developing a culturally valid instrument was preserving the cultural knowledge in an authentic form. The adaptation of the *AFU* instrument to another cultural context presented enormous challenges. However, the possibility of establishing an assessment instrument that reflects IK and calibrates a learning trajectory that follows the cultural activities and learning process gleamed from Yup'ik elders' knowledge represents "a first." The refinement process is expected to continue during the next few years as the project moves from a HLT based on IK to an empirically grounded learning assessment. However, there will need to conduct additional trials, increase the number of performance items to create a better fit between IK and the newly developing learning trajectory.

We believe work like this provides a positive basis upon which the contributions of indigenous people are not only recognized for how that knowledge could positively impact indigenous students' math performance but also how IK can provide insights for teachers and students from other communities. Although this is still a work in progress it affords others with ways of expanding the meaning and approaches to developing culturally relevant curriculum and culturally valid assessments.

NOTES

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This works rests on the dedication of a group of Yup'ik elders who shared their knowledge unselfishly so that the next generation could learn. We are indebted to elders such as Henry Alakayak, Annie Blue, Frederick and Mary George, Lily Gamechuk and many others too numerous to name. Unfortunately, many of the elders who have supported and contributed to this work have passed on; this work is dedicated to their memory. Lastly, we must acknowledge the steady hand of Flor Banks, project manager, who has assisted by editing multiple versions of this chapter and whose input has helped make this a stronger manuscript.

2 Math in a Cultural Context: Lessons learned from Yup'ik Eskimo Elders Project http://www.uaf.edu/mcc/

3 Dora Andrew-Ihrke and Evelyn Yanez are cousins within Yup'ik tradition and they are currently adjunct faculty/consultants to the Math in a Cultural Context program at the University of Alaska Fairbanks. They are each retired and distinguished teachers and bilingual coordinators, and authors of this chapter.

4 Many teachers and administrators were supportive, but there remains, and continues to this day, an underlying sense that all things Yup'ik are inferior. This topic and our initial response to develop more culturally relevant curriculum and pedagogy are treated in detail elsewhere (Lipka, Mohatt, & Ciulistet, 1998).

5 "Us" refers to three of the authors, Andrew-Ihrke, Yanez, Lipka, and others – Yup'ik teachers, other researchers, mathematics educators, and mathematicians. Elders have often told us that one always arrives with a story.

6 It is beyond the scope of this chapter to concentrate on the pedagogical side as there is extensive literature on this topic and very little literature on mathematical or core content area applications tied to IK. See Lipka with Mohatt & the Ciulistet Group (1998).

7 This may differ by elders.

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DANIEL CLARK OREY & MILTON ROSA

8. IN SEEKING A HOLISTIC TOOL FOR ETHNOMATHEMATICS

Reflections on Using Ethnomodeling as a Pedagogical Action for Uncovering Ethnomathematical Practices

Throughout history, people have explored other cultures and shared knowledge often hidden behind ideas, traditions, practices, and customs. This cultural dynamism has enriched these cultures, including Western culture. In this regard, the Greek foundations of European civilization are strongly influenced and impacted by Egyptian civilization (Powell & Frankenstein, 1997). Currently, however, there still exists a world-wide acceptance of supremacy of Western logical perspectives in math science in academic arenas; contemporary globalized non-Western societies place enormous value on capitalistic Western-oriented science and mathematics.

The literature on mathematics education, including texts and teaching materials and methods, is based on the scientific and mathematical concepts rooted in this dominant Western tradition that many of us are accustomed to thinking of when we *do* mathematics. Most of the examples used in teaching academic mathematics derive themselves from problems and contexts from within a Western cultural paradigm. Because mathematics is an expression of human development, culture, and thought, we have come to believe that this Western monocultural perspective in mathematics education causes problems for many members of non-Western cultures.

Mathematics forms an integral part of the greater cultural heritage of humankind and is the central premise of our work. What this heritage consists of is equally dependent upon time, sociocultural context, and place. In this regard, ethnomathematics has demonstrated how mathematics is made of many diverse and distinct cultural traditions, not just those emerging from the Mediterranean: it supports the premise that teaching/learning of mathematics should include, and place equal importance upon, those originating from indigenous and non-Western contexts.

Mathematical thinking is influenced by the vast diversity of human characteristics that include: language, religion, worldview, and economical-social-political activities. In concert with these, human beings have developed logical processes related to our universal need to quantify, measure, model, understand, comprehend, and explain. All have come to shape and operate within different socio-historical contexts. Each cultural group has developed, often unique, ways of

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incorporating mathematical knowledge; and has often come to represent given cultural systems, especially in ways that members of cultural groups quantify and use numbers, incorporate geometric forms and relationships, and measure and classify objects.

For all these reasons, each cultural group has developed unique and often distinct ways to *mathematize*[1] their own realities. Western-academic scientific arrogance often presents an overt disrespect of, and an outright refusal to acknowledge, a cultural identity (Zaslavsky, 1996). These *cultural particularities* should neither be ignored nor should they be disrespected when individuals from different cultures attend school. Equally important is the search for alternative methodological approaches. As Western mathematical practices have been adopted worldwide, it is necessary to recognize mathematical ideas from different cultures that are either ancient or traditional practices.

One alternative methodological approach is *ethnomodeling*, which may be considered a practical application of ethnomathematics that also adds the cultural perspective to modeling concepts. When justifying the need for a culturally situated view on mathematical modeling, our sources are rooted in the theory of ethnomathematics. We argue, as well, that recognizing cultural differences in mathematics reveals new perspectives on scientific questioning methods. Research of culturally situated modeling ideas addresses the problem of mathematics education in non-Western cultures by bringing the cultural background of students into the mathematics curriculum, and connects the local-cultural aspects of the school community to the teaching and learning of mathematics (Rosa & Orey, 2010a). On the other hand, local views used in mathematical modeling could be used also in global collaborations, thus potentially widening views of mathematics in others. This pedagogical approach is needed in mathematics education because it is a major factor in broadening the modeling process as well as an ethnically fairer view that can help to bridge the mathematical achievement gap of the students. This alternative approach helps in promoting intellectually innovative ideas in the area of modeling by deepening and widening the Western understanding of mathematics.

In this chapter, we present a number of arguments to justify the need to strengthen the research field of modeling by adopting a cultural perspective in problem solving methods, conceptual categories, structures, and the models used in representing mathematical ideas and practices developed by distinct cultural groups. We refer to this pedagogical approach as ethnomodeling.

ETHNOMATHEMATICS AS A HOLISTIC APPROACH TO MATHEMATICS EDUCATION

In the past decades, ethnomathematics has come to evoke a worldwide discussion that raises fervent arguments both for and against. These discussions clearly show the uncertainty, on the side of the proponents as well as the opponents, of what the definitions and goals of ethnomathematics are, and have brought forth many misconceptions and fears related to this research paradigm. In an early article entitled "Ethnomathematics and its Place in the History and Pedagogy of Mathematics", *ethnomathematics* is defined as the study of scientific and, by extension, technological phenomena in direct relation to social, economic, and cultural backgrounds (D'Ambrosio, 1985). If described in terms of the development of science in general, ethnomathematics may be considered as a corpus of knowledge established as systems of explanations and *ways of doing* mathematics, which have been accumulated through generations in distinct cultural environments (D'Ambrosio, 1998).

Ethnomathematics as a research paradigm is much wider than traditional concepts of mathematics and ethnicity as well as current definitions of multiculturalism. *Ethno* may be that which is related to distinct cultural groups identified by cultural traditions, codes, symbols, myths, and specific ways of reasoning and inferring. Accordingly, ethnomathematics may be considered as the way that various cultural groups mathematize objects or phenomena and it examines how both mathematical ideas and practices are processed and used in daily activities. In the educational context, ethnomathematics can be described as the arts or techniques developed by diverse students to explain, to understand, and to cope with their own social and cultural environments.

Ethnomathematics embraces the mathematical ideas, thoughts, and practices developed by all cultures (Barton, 1996). From this perspective, a body of anthropological research has come to focus on both the intuitive mathematical thinking and the cognitive processes that are largely developed in various cultural groups. Ethnomathematics may also be considered as a program that seeks to study how students have come to understand, comprehend, articulate, process, and ultimately use mathematical ideas, concepts, procedures, and practices that may solve problems related to their daily activity. Ethnomathematics is not only the study of mathematical ideas; it also studies anthropology and history. This form or context of study of the history of mathematics assists in identifying the cultural and mathematical contributions of different cultures across the world and translates them into academic mathematics that Western researchers and educators are able to understand. Seen in this context, the focus of ethnomathematics consists essentially of a serious and critical analysis of the generation and production of the mathematical knowledge and intellectual processes, the social mechanisms in the institutionalization of knowledge and the diffusion of this knowledge (Rosa & Orey, 2006). In this much more *holistic*[2] context of mathematics that uses an anthropological perspective to include diverse perspectives, patterns of thought, and histories, the study of the *systems*[3] taken from reality help students to come to reflect, understand, and comprehend extant relations among all of the components of the system. Additionally, ethnomathematics may be defined as the intersection of cultural anthropology, academic mathematics, and mathematical modeling, which is used to help students to translate diverse mathematical ideas and practices found in their communities (Fig. 8.1).

Characterized by our very humanness, all individuals have the capability to develop mathematical concepts that are rooted in the universal human endowments of curiosity, ability, transcendence, life, and death. An awareness and appreciation

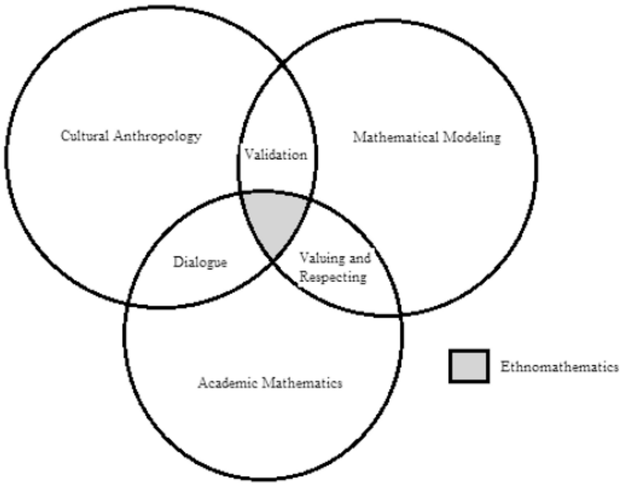


Fig. 8.1 Ethnomathematics as an intersection of three research fields

of cultural diversity can be seen for instance, in our clothing, methods of discourse, our belief systems all combining to influence our own unique worldviews that allow us to understand each aspect of daily life.

The unique cultural background of each student represents, as well, a set of values and unique ways of seeing the world as it is transmitted from one generation to another. Principles of anthropology that are relevant to the work of ethnomathematics include the essential elements of culture such as language, economy, politics, religion, art, which most certainly influence the daily mathematical practices of diverse groups of students. Since cultural anthropology gives us tools that increase our understanding of the internal logic of a given society, detailed anthropological studies of the mathematics of distinct cultural groups most certainly allow us to further our understanding of the internal mathematical logic of diverse groups of students (Rosa & Orey, 2008). On the other hand, in ethnomathematics research, the terms *primitive* and *illiterate* are used in different and inconsistent ways.[4]

Ethnomathematics places a reliance on the idea that each cultural group developed its own ways, styles, and techniques of doing certain tasks, and responses to the search of explanations, understanding, and learning, which are named *systems of knowledge* (D'Ambrosio, 1998). All of these diverse systems use forms of *inference, quantification, comparison, classification, representation, measuring,* and *modeling*. Western science is a system of knowledge, yet there are other systems of knowledge with the same aims. The other systems of knowledge use other forms of inferring, quantifying, comparing, classifying, representing, measuring, and modeling, but are not simple-minded or primitive in comparison to academic-Western-mathematics.

However, from the claim that no single scheme has absolute explanatory power, it does not follow that *all* schemes are equally valid (Code, 1991). It is our view that comparisons between different systems of knowledge are, to some extent, groundless due to the fact that they have arisen in different environments to meet different demands. In this regard, not all logics and assumptions are equally effective in understanding and dealing with a certain situation, which implies that Western logic and its associated assumptions are not always as useful as we assume in all situations (Fasheh, 1982). Western academic science owes its success to the ability of scientists to regularly select problems that can be solved with conceptual and instrumental techniques close to those already existing. This science does not aim at novelties of fact or theories and, when successful, does not expect to find them.

Clearly, when a paradigm denies the validity of a large class of problems, it leaves out a number of socially and scientifically important problems and solutions. This nature of normal science considerably limits the possibilities of multicultural research and education, and hence the rules and findings of other cultural contexts. The terms *paradigm*, *normal science*, and *scientific revolution* and defined *normal science* may be used as analysis based on past scientific achievements, and acknowledged by peers as a foundation for further practice of that particular science (Kuhn, 1962). The reluctance to adopt a new paradigm might be due to the fact that any new paradigm often requires scientists to redefine or even discard their earlier work, as reviewed by their peers. In adopting new paradigms, some older problems may in fact be declared entirely *unscientific*, whereas others that were previously nonexistent or trivial might become the very archetypes of significant and new scientific and mathematical achievement.

Research does not aim to produce major novelties, both conceptual and phenomenal. Even those projects whose goal is paradigm articulation do not aim at the *unexpected* novelty, but reinforce the existing paradigm (Kuhn, 1962). We believe that ethnomathematics may be speeding up the process of paradigm re-evaluation and finding unexpected novelties in science in general. This means that ethnomathematics may work as a catalyst, introducing and encouraging new paradigms such as ethnomodeling (Rosa & Orey, 2010a) and ethnocomputing (Eglash, Bennett, O'Donnell, Jennings, & Cintorino, 2006) to challenge the prevailing ones. Concurrently, there exist several culturally-bound paradigms within a discipline such as mathematics. None of these are better or worse than the others, they just have arisen to meet the different needs of different cultural groups.

One reason why ethnomathematics has emerged as a new paradigm in mathematics education may be because these different culturally-bound paradigms have not had the opportunity to really confront each other before and thus have not been able to interact.

For many, many years Western institutions have been educating many people from the emerging nations who return to work in their country of origin, and who integrate what they learned and experienced abroad with their own reality at home.

Presently, in a global scientific community where new publications are available on-line, where blogs and resources are universally available, paradigms from different cultures meet and interact continuously, and new ideas are emerging. It is only the all-extensive Western influence that prevents new viewpoints from emerging if they are not fully-fledged and totally revolutionary. Since science and mathematics do not, in our opinion, belong only to Europeans and North Americans, they are currently in a phase of adaptation as members of different cultures of distinct individuals participate globally. In this context, ethnomathematics was created and developed out of a non-Western reality.

ETHNOMODELING: A DEFINITION

The prefix *ethno* in ethnomathematics is today accepted as a very broad term because it refers to the social, cultural, political, economic, and environmental contexts of the members of distinct cultural groups that form our contemporary and globalized society (D'Ambrosio, 1990). It includes language, jargon, myths, symbols, and code of behaviors that were developed in these contexts. The meaning of *mathema* is to explain, to know, to understand, and to do daily activities such as ciphering, measuring, classifying, inferring, and modeling, while the meaning of the suffix *tics* is derived from *techné* and has the same root as technique.

In the context of ethnomodeling, *ethno* refers to differences in cultures that are mainly based on language, history, religion, customs, institutions, and on the subjective self-identification of individuals and professional groups. However, these cultural differences can also arise from differences based on ethnic or nationality oppressions. These are the social, economic, political, environmental, and cultural backgrounds that define a group as a cultural entity. In this regard, individuals develop a sense of cultural identity, which "is adapted and changed throughout life in response to political, economic, educational, and social experiences" (Gollnick & Chinn, 2002, p. 21). An awareness of individuals' cultural identities provides the foundation for how they define themselves in terms of how others view them. Thus, in the educational context, educators need to view students as cultural beings, embrace their diversity, acknowledge their previous knowledge, and validate their cultural identity in order to create classrooms that model tolerance and appreciation of students' differences.

For example, teachers can help students feel comfortable with their cultural identity and assist them with their learning by a using a pedagogical approach that embraces diversity and validates their cultural background and previous knowledge. Gay (2000) defined this approach as "using the cultural knowledge, previous experiences, frames of reference, and performance styles of ethnically diverse students to make learning encounters more relevant to and effective for them" (p. 1). This is one of the most important goals of ethnomodeling as pedagogical action for the teaching and learning of mathematics.

In the pedagogical action of ethnomodeling, teaching starts by teachers getting to know their students on a personal level, building teaching around their interests

when possible, and showcasing their talents and creativity in order to use them as teaching tools (Bennett, 2003). These three areas allow the creation of a classroom environment that is learner-centered and promote the academic success of the students. In our opinion, the underlying principles of ethnomodeling are:
– Students must experience academic success.
– Students must develop and maintain their cultural competence.
– Students must develop a critical consciousness through which they may challenge social injustice.

In this context, educators empower students to succeed in mathematics by providing a learning environment that respects their cultural backgrounds, embraces their diversity, and celebrates their differences.

Since individuals of different cultural groups have different views on the relationships related to spiritual values, the individual and the group, the citizen and the state, they also have differing views on the relative importance of rights and responsibilities, liberty and authority, and equality and hierarchy. These differences between cultures are a product of centuries and they will not disappear rapidly because they are far more fundamental than differences among political ideologies and political regimes. In addition to these categories, in the ethnomodeling perspective, culture is expanded to include the cultures of differing professional groups and age classes as well as social classes and gender.

In other words, *ethno* in ethnomodeling refers to members of a group within a cultural environment identified by cultural traditions, codes, symbols, myths, and specific ways used to reason and to infer. In this context, *modeling* is considered as a combination of *mathema* and *tics*, in which *mathema* means to explain and understand the world in order to transcend, manage, and cope with reality so that the members of cultural groups can survive and thrive while *tics* refers to the techniques such as *counting, ordering, sorting, measuring, weighing, ciphering, classifying, inferring,* and *modeling*. The *mathema* develops the *tics* within the context of *ethnos* because it consists of daily challenges faced by individuals, larger problems faced by humanity, and endeavors of humans to create a meaningful world. The search for solutions for specific problems that help the development of mathematics is always imbedded in a cultural context (D'Ambrosio, 2001). In this regard, in order to understand how mathematics (tic*s*) is created, it is necessary to understand the problems (*mathema*) that precipitate it by considering the cultural context (*ethnos*) that drives them. This is the main objective of ethnomodeling.

PLACING AN ETHNOMODELING PERSPECTIVE INTO THE MATHEMATICS CURRICULUM

Since we share our culture with others in our own cultural group and communicate our culture to the members of other cultural groups in an ever-evolving response to the circumstances and challenges of our worlds, we engage our world through the manufacture of artifacts, the practice of behaviors, and the development and adherence to values and beliefs. In this regard, culture is defined as the ideations, symbols, behaviors, values, knowledge, and beliefs that are shared by a community. However, the essence of a culture is not its artifacts, tools, or other tangible cultural elements, but the way or ways in which the members of cultural groups interpret, use, and perceive them. Cultural artifacts may be used in different cultures in very different ways as well as for different purposes. According to this perspective, most of the interesting problems are practically solved using different kinds of heuristics as well as different types of cultural artifacts. On the other hand, culture may be defined as a cultural group's program for surviving in, and adapting to, its environment. In this regard, the cultural program consists of knowledge, ideas, concepts, values, beliefs, symbols, and interpretations shared by the members of cultural groups through communication that help them to make sense out of the world around them. García Coll et al. (1996) propose the concept of adaptive culture to describe how distinct cultures, in response to historical and current demands such as societal racism, classism, and sexism develop their own "goals, values, and attitudes that differ from the dominant culture" (p. 1896).

In the educational context, students proceed through a series of cognitive, affective, and social processes; unique ecological circumstances faced by students from diverse cultural groups result in developmental adaptations that may be necessary for the cultural group's survival and maintenance of self-esteem. It is important to understand the context in which cultural practices, belief, technique, ability, and competency have developed because they affect the ways in which students develop cognitively, affectively, and socially. Notions of culture are always a concern of educational policymakers. It has already been stated that school curriculum has been described as one version of cultural training. This poses serious structural demands on curriculum policy as well as its practice. The central questions in educational policy and practice concerning mathematics curriculum have surrounded the development of a curriculum that caters equally for a diverse student population.

What is different from the traditional view of modeling in the mathematics curriculum is how we define ethnomodeling as a combination of:

- The organized structures and models used to represent information (data structures).
- The ways of manipulating the organized information (algorithms).
- The mechanical and linguistic realizations of these ways, models, and structures as well as their application in distinct cultural groups and society.

Rather than changing the content of science itself, one of the goals ethnomodeling is to concentrate on the form, that is, the outward appearance of modeling. In other words, the aim of ethnomodeling is not to question the foundations of modeling but concentrate on the creative way mathematical ideas and concepts are *developed, presented, acquired*, and *accumulated*. We believe that instead of being another paradigm itself, the study of ethnomodeling aims at encouraging the search for novel ideas (creativity), their examination, adoption, and application to solve problems faced daily by the members of any cultural group.

ETHNOMATHEMATICS AND MATHEMATICAL MODELING

Historically, models that arise from reality have been the first paths towards providing abstractions of mathematical concepts. Ethnomathematics that uses the manipulations of models of reality and modeling as a strategy of mathematical education uses the codifications provided by others in place of the formal language of academic mathematics. Within this context,[5] mathematical modeling is a methodology that is closer to an ethnomathematics program. Ethnomathematics may be defined as the intersection between cultural anthropology and institutional mathematics, which utilizes mathematical modeling to interpret, analyze, explain, and solve real world problems or mathematize existing phenomena.

Investigations in modeling have been found to be useful in the translation of ethnomathematical contexts by numerous scholars in Latin America[6] who study the mathematical practices and ideas found in diverse traditions and contexts. These studies also become an important tool used to describe and solve problems arising from specific systems such as cultural, economic, political, social, environmental, which brings with it numerous advantages to mathematics learning.

Considering ethnomodeling as the intersection of cultural anthropology, ethnomathematics, and mathematical modeling (Fig. 8.2) "[u]sing ethnomodeling as a tool towards pedagogical action of the ethnomathematics program, students have been shown to learn how to find and work with authentic situations and real-life problems" (Rosa & Orey, 2010b, p. 60). Outside of an ethnomathematics related research paradigm, it is known that many scientists search for mathematical models that can translate their deepening understanding of both real-world situations and diverse cultural contexts.

ETHNOMODELING

Ethnomodeling is the process of elaboration of problems and questions that grow from real situations that form an image or sense of an idealized version of the *mathema*. The focus of this perspective essentially forms a critical analysis of the generation and production of knowledge (creativity), and forms an intellectual process for its production, the social mechanisms of institutionalization of knowledge (academics), and its transmission (education). From this perspective, the holistic disposition aims at creating an understanding of all components of the system as well as the interrelationships among them.

The use of modeling as pedagogical action for an ethnomathematics program values previous knowledge and traditions by developing student capacity to assess and translate the process by elaborating a mathematical model in its different applications. This can be done by starting with the social context, reality, and interests of the students and not by enforcing a set of external values and curriculum without context or meaning for the learner. This process may be characterized as "ethnomodeling" (Bassanezi, 2002), and ethnomathematics then is "practiced and elaborated by different cultural groups … [that] involves the

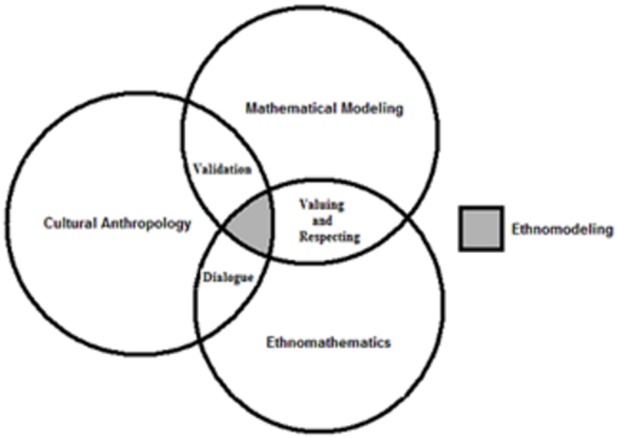


Fig. 8.2 Ethnomodeling as an intersection of three research fields

mathematical practices … present in diverse situations in the daily lives of members of these … groups" (p. 208).

In considering ethnomodeling as a tool to uncover and study ethnomathematics, teaching is much more than the transfer of knowledge because teaching becomes an activity that introduces the creation of knowledge (Freire, 1998). In our opinion, it is necessary for school curriculum to translate the interpretations and contributions of ethnomathematical knowledge into systemized mathematics because students will be able to analyze the connection between both traditional and non-traditional learning settings. Ethnomodeling offers a tool for developing a cultural approach in developing mathematics abilities in learners, and aims at developing skills in learners for observing mathematical phenomena that have their roots in distinct cultural settings.

The results may then lead to new viewpoints into mathematics education, which can be used to improve the cultural sensitivity in teaching modeling. The new viewpoints, thus generated, clearly benefit Eurocentric[7] mainstream education, in addition to promoting academic competence of learners from different culture. We characterize this approach as the transformative pedagogical nature of ethnomodeling because of its emphasis on diversity of knowledge, especially mathematics and science that are typically believed to be acultural. From our standpoint, ethnomodeling can be seen as an active force that impacts mathematics education and also studies a dynamic subject of change. In other words, ethnomodeling arises from the culture and adapts to changes in the culture.

EXAMPLES OF ETHNOMODELING

Mathematical modeling uses mathematics as a language for understanding, simplification, and resolution of real-world problems and activities. Data gleaned from these studies are used to make forecasts and modifications pertaining to the objects initially studied. In this regard, one of the traditional definitions of a mathematical model is a body of symbols and mathematical relationships that represent the studied object, which is composed of a system of equations or inequalities, algebraic expressions, etc. that are obtained through the establishment of a relationship between considered essential variables of analyzed phenomena (Bassanezzi, 2002). It is the systematic study of algorithmic processes, theory, analysis, design, efficiency, implementation, and application that describe and transform information. This definition of the Western mathematical modeling includes all data structures, which form a part of both *theory* and *design*; algorithms, which deal with analysis and efficiency; mechanical and linguistic realizations, which deal with implementation; and applications that naturally apply the mathematical ideas and concepts to solve problems.

Academic mathematics stands to benefit by coming to value a wider diversity of mathematical conceptions. For example, the importance of a non-traditional view on mathematics has a bearing on the emergence of the new types of problems related to artificial intelligence. A characteristic of these new problems is that they cannot be solved using syllogistic, that is, classical Aristotelian logic, but need multivalued logic, often called *fuzzy logic*, which is the logic that underlies inexact or approximate reasoning (Zadeh, 1984). Multivalued logic is used in attempts to formalize human-like processes that are culturally bound. The Hindu, Chinese and Japanese cultures have contributed to the development of fuzzy logic more than Western science because, in these cultures, there is a greater acceptance of a truth-value that is neither perfect truth nor perfect falsehood (Zadeh, 1984). Some ethnomathematical examples may naturally come across as having a mathematical modeling methodology (D'Ambrosio, 2002). In the 1989–1990 school year, a group of Brazilian teachers studied the cultivation of wines brought to Southern Brazil by Italian immigrants in the early twentieth century. This was investigated because the cultivation of wines is linked with the farming activity of the members of a certain cultural group from that region in Brazil. This wine case study is an excellent example of the connection between ethnomathematics and mathematical modeling through ethnomodeling.

DEFINITION OF ETHNOMODELS

In general, a model is a representation of an idea, a concept, an object, or a phenomenon (Gilbert, Boulter, & Emmer, 2000). We, however, define ethnomodels as cultural models that are pedagogical tools to facilitate comprehension of systems that are taken from the reality of the said cultural groups. Thus, ethnomodels can be considered as external representations that are precise and consistent with the scientific and mathematical knowledge that is socially constructed and shared by members of specific cultural groups. The primary objective for the elaboration of ethnomodels is then to *translate* the mathematical ideas, concepts, and practices developed by the members of distinct and diverse cultural groups. There follow a few examples of ethnomodels.

Measuring Land

Knijnik (1996) reported the mathematical thinking underlying demarcation of land by the members of the Landless Peoples' Movement[8] (Movimento dos Sem Terra – MST) of Southern Brazil. The activity on demarcation of land activity is about the method of *cubação* of the land, which is a traditional mathematical practice of the members of the movement. Flemming et al. (2005) defined the term *cubação* of the land as the solution of "problems of the measurement of land using diverse shapes" (p. 41). Thus, the practice of *cubação* of the land as a pedagogical proposal to elaborate activities for the teaching and learning of mathematics shows the importance of the contextualization of problems in the learning environment of ethnomodeling through the elaboration of ethnomodels.

Knijnik (1996), for example, presented the following problem to the landless people to calculate the area of quadrilaterals (Fig. 8.3). The mathematical knowledge of the landless people can be represented by a model that transforms "the shape of the given land into a [rectangle] of 138 meters x 102 meter with an area of 14076 square meters" (Fig. 8.4).

The model of this mathematical practice can be explained by the following ethnomodel:

– Transform the shape of the irregular quadrilateral into a rectangle whose area can be easily determined with the formula $A = b{\cdot}h$.

– Determine the dimensions of the rectangle by calculating the mean of each pair of opposite sides of the irregular quadrilateral. Base = (152 + 124)/2 = 138 m. Height = (114 + 90)/2 = 102 m.

– In order to determine the area of this irregular quadrilateral, it is necessary to determine the area of the rectangle. $A = b{\cdot}h = 138{\cdot}102 = 14076 \text{ m}^2$.

Regarding this problem, there is another ethnomodel proceeding from the mathematical knowledge of the landless people. Flemming et al. (2005) reported that the irregular shaped quadrilateral parcel presented in this example can also be transformed in to "a square with sides of 120 meters, therefore with an area of 14400 square meters" (p. 42). In this regard, it is possible to observe that adding the lengths of the sides of the quadrilateral, and then dividing it by four, the number of sides, produces the value 120.

Flemming, Flemming Luz and Collaço de Mello (2005) confirmed Bassanezi's position that a model is efficient when we realize that we are only working with approximations of reality. Mathematically speaking, both methods presented by the members of the Landless People's Movement were an approximated calculation of the area an irregular quadrilateral that fully satisfied the necessities and the life histories of the participants of this specific cultural group.

The Symmetrical Freedom Quilts

Rosa and Orey (2009) affirmed that a quilt theme is a powerful way to integrate mathematics, art, history, and reading in an interdisciplinary approach. Using the

Calculate the area of the land with a quadrilateral shape with sides 114, 152, 90, 124 meters. (p. 42)

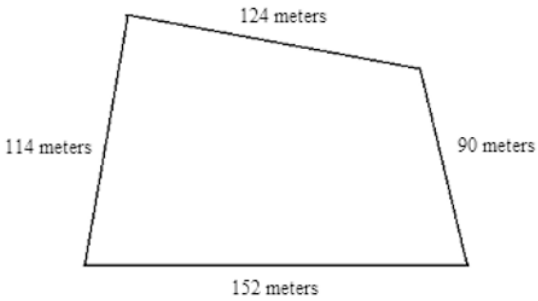


Fig. 8.3 Problem on land demaracation for the landless people

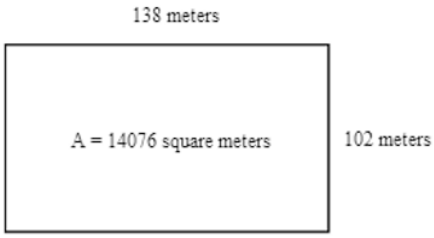


Fig. 8.4 Solution proposed by the landless people

historical context of *Underground Railroad*[9] their lesson plans combined an ethnomathematical-historical perspective that allowed teachers to develop classroom activities and projects for students to better understand geometry, especially concepts of symmetry and transformations through ethnomodeling. One of the objectives of this project is to stimulate student's creativity and interest, because quilts may be considered as cultural and mathematical expressions of student's daily life. Thus, *Symmetrical Freedom Quilts* may be considered as links between mathematics, history, ethnomathematics, and the very tactile art of quilting.

Making quilt blocks is an intellectually engaging way to explore the concepts of symmetry. As quilts are typically made by piecing a number of square blocks (usually 4 and 9 blocks patches),[10] with each smaller block made with arrangement of triangles in a particular way- the craft lends itself readily to the application of symmetry. The Freedom Shoo Fly quilt shows how its blocks are symmetrical (Fig. 8.5).

Shoo Fly[11] is one of the simplest traditional Freedom Quilt patterns. Although *Shoo Fly* is a basic pattern, its versatility provides quilters with some wonderful opportunities for creative use of colors, fabrics, and stitching. *Shoo Fly* may be adapted to a variety of sizes. Blocks often measure 9 x 9, but variations such as 10 x 10 and 12 x 12 may also be used.

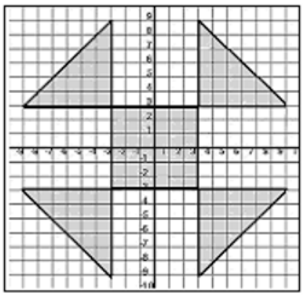


Fig. 8.5 The Freedom Shoo Fly pattern.

A rotation turns the figure through an angle about a fixed point, called the center. The center of rotation is assumed to be the origin of the x-y coordinate system. A positive angle of rotation turns the figure counterclockwise, and a negative angle of rotation turns the figure in a clockwise direction.

Rotation is a transformation that is present in the *Shoo Fly* quilt block because it moves every point 90° counterclockwise around the origin of the x-y coordinate system. The mapping of this rotation is $R_{90°}(x,y) = (-y,x)$. In so doing, the coordinates of point *A* in its rotation around the x-y coordinate system are:

$$R_{90°}A(9,3) = A'(-3,9)$$
$$R_{90°}A'(-3,9) = A''(-9,-3)$$
$$R_{90°}A''(-9,-3) = A'''(3,-9)$$
$$R_{90°}A'''(3,-9) = (9,3)$$

Fig. 8.6 shows the rotation of point *A* around the x-y coordinate system.

The other mappings for rotation are:

– Rotation of 180°, that is, $R_{180°}(x,y) = (-x,-y)$. This is the same as the reflection in the origin of the x-y coordinate system.
– Rotation of 270°, that is, $R_{270°}(x,y) = (y,-x)$.

A rotation creates a figure that is congruent to the original figure and preserves distance (isometry) and orientation (direct isometry).

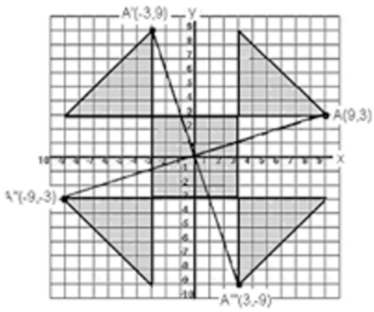


Fig. 8.6 Rotational transformation of Freedom Shoo Fly pattern

Modeling the Tipi

Spatial geometry is inherent in the shape of the tipi and it was used to recall, indeed symbolize, the universe in which the Plains Peoples lived. The word *tipi* from the Sioux language refers to a conical skin tent or dwelling common among the prairie peoples. According to Orey (2000), the majority of Sioux tribes use the tripod foundation or three-pole foundation because it is stronger and offers a more firm foundation than a

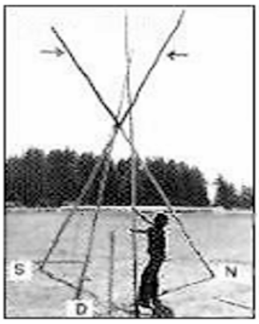
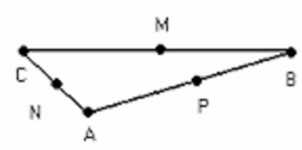


Fig. 8.7 Setting up a tipi

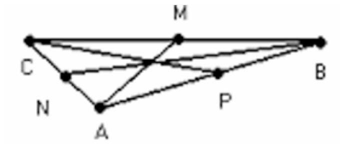
quadripodal or four-pole tip foundation. An ethnomodel explains why a tripod is more flexible than a quadripod, or a four-legged structure. In this regard, imagine three points, A, B, and C that are not collinear. There are an infinite number of planes that pass through points A and B that contain the straight line AB. Only one of these planes also passes through point C therefore we can say that three points are not collinear if they determine one plane. This means that these non-collinear points exist on one plane and that three collinear points do not determine the only plane. This means that given any three non-collinear points, there is only one plane to which exist these same three points. This can be explained using the postulate for the determination of a plane. In other words, given any three non-collinear points, there is only one plane to which exists these same three points. For example, in the 4-legged table, it has the possibility of the extremity of one of the legs that do not belong to the same plane. A table that has 3 legs, therefore, is always balanced. Similar to a three-legged table, the structure of the tipi appears to be perfectly adapted for the harsh environment in which it was used. It had the advantage of providing a stabile structure, was lightweight and portable (Fig. 8.7).

At the same time it withstood the prevailing winds and extremely variable weather of this region. Let us look at this information mathematically. The base formed by the tripod is ΔABC.

The midpoints of each of the sides of ΔABC are points M, N, and P.



It is possible to match each vertex of ΔABC to the midpoint of each opposite sides that gives us the straight lines AM, BN, and CP.



These straight lines form three medians, which are the straight lines connecting the midpoint of each opposite side of the triangle and its vertex. The medians intersect at only one point called centroid. Archimedes demonstrated that medians of a triangle meet at its balance point or center of gravity, which is the centroid of the triangle. Native Americans place their fire and altar at this point in the tipi. Cartographers call this point the geographic center. The tipi cover is folded in half and the poles are laid together before tying them to form the tripodal or quadripodal frame, which forms the foundational base for the structure.

Some Considerations about Ethnomodeling

Ethnomodeling seems to be important especially in new fields of research such as artificial intelligence and fuzzy logic. However, ethnomodeling has been given a chance only in new research or it has led to new fields of research. Current academic-Western science does not give ethnomodeling of non-Western cultures much chance to introduce new views into old themes. Different cultures can and do contribute to the development of mathematical concepts and ideas and enrich them in the field of Mathematics Education. In addition to the development of mathematical modeling and education, ethnomodeling holds another equally important objective as D'Ambrosio (1997) recognizes that ethnomathematics has the common goal of equity and dignity. In this regard, the study of ethnomodeling may encourage the ethics of respect, solidarity, and co-operation across cultures.

FINAL CONSIDERATIONS

The purpose of this chapter was to justify the research of cultural perspectives in ethnomathematics through ethnomodeling. The claim was that contemporary mathematics is dominantly Eurocentric and that this Eurocentrism has facilitated a divide that hinders the prospects of mathematics education in non-Western cultures. The motivation for a cultural approach was the assumption that adopting cultural perspectives into mathematical modeling would bring local issues into global discussion and thus help in meeting local needs.

We propose a sociocultural theory for mathematical modeling, suggesting that mathematics education is a social and cultural product and that there exist a dialectic relationship between mathematics, culture, and society. This claim is supported by the social constructivist theories in sociology and educational psychology as well as the idea of how scientific revolutions are structured. Moreover, we have presented an idea that mathematics is dominated by preferences of the West, and that this prevailing Eurocentrism poses a problem in mathematics education for non-Western cultures.

From these grounds, we have presented definitions for ethnomodeling and for the study of ethnomodeling as follows: *Ethnomodeling* stands for mathematical ideas and concepts that have their roots within a culture. The *study of* ethnomodeling is defined as the study of mathematical phenomena within a culture. Ethnomodeling differs from the traditional definition of modeling in that, whereas the traditional view considers the foundations of mathematics education as constant and applicable everywhere as such, the study of ethnomodeling takes the position that mathematics education is a social construction and thus culturally bound.

All variations of ethnomodeling have developed to meet the needs of a certain culture, which present their own consequences. First, the impact that culture has on mathematics represents a feedback loop whereby mathematics education is changing it and transforming it and culture is transformed by mathematics and science. In this process mathematics education and mathematical ideas that originate by members of cultural groups are constantly reshaping one another. This can be perceived as a transformation of both the native mathematical system and mathematics education through a cultural and global dynamic perspective, and according to D'Ambrosio (1990) this is an example of cultural dynamism. Second, there does not exist a standard for comparison between cultural groups, since every measure is subjective, and would only measure how well modeling works in a culture where it is applied. For example, Western standards such as efficiency and exactness are useful only in a very limited set of problems, and they may become insignificant if the legitimate problem field changes. Third, mathematics education can no longer ignore cultural considerations. The educators have to take into account the cultural and philosophical background of a society. Different cultures may have different perceptions of time and space, logic, problem solving methods, society, values, or which questions are considered legitimate.

Adopting ethnomodeling as pedagogical action for teaching and learning mathematics could serve several purposes. First of all, it presents a practical method where learners can learn to recognize mathematics in the context of their own local context. This would definitely produce educators, students, and researchers that have fresh, novel views on the issues of mathematics, science and engineering education.

The hardest part in the adoption of ethnomodeling is the pervasive view of the Western philosophy as the crowning jewel of scientific evolution. Current formal science is seen as good and final as such, which, in the light of history of science, is clearly an inaccurate conclusion. Much more probable is that future scientific revolutions will turn the direction of mathematics education towards directions that are unexpected, and so far even unheard of.

Ethnomodeling may be considered a young discipline and is still looking for its identity. However, this field is rapidly evolving in directions that are changing and difficult to anticipate. The boundaries of ethnomodeling are not strictly defined. Change and reform are actually a part of the nature of the scientific paradigm. This is why we believe that recognizing ethnomodeling does not give the mathematical methods of other cultural groups a Western stamp of approval, but recognizes that they offer important alternatives to problem solving and have always have been important to the development of humanity's overall mathematical knowledge.

Any study of ethnomathematics and mathematical modeling represents a powerful means for validating a student's real life experience, and gives them the tools to become critical participants in society. In so doing, educators should be empowered to analyse the role of what Borba (1990) refers to as a student's *ethnoknowledge*[12] in the mathematics classroom. There is no doubt that there exists a need to create a new role in mathematics instruction that empowers students to understand power and oppression more critically by considering the effect of culture on mathematical knowledge by working with students to uncover distorted and hidden history of mathematical knowledge.

This perspective forms the basis for significant contributions of a Freirean-based ethnomathematical perspective in re-conceiving the discipline of mathematics and in a pedagogical practice. The use of Freire's (1970) dialogical methodology is seen as essential in developing the curricular praxis of ethnomodeling by investigating the ethnomathematics of a culture in constructing a curriculum with people from other cultures to create curricula that enable the enrichment for all people's knowledge of mathematics. Seen in this context, we would like to broaden the discussion of possibilities for the inclusion of ethnomathematics and mathematical modeling perspectives that respect the social and cultural diversity of all people with guarantees for the development of understanding our differences through dialogue and respect. This is how ethnomodeling can empower students in this century against all kinds of domination and oppression.

NOTES

1. Mathematization is a process in which individuals from different cultural groups come up with different mathematical tools that can help them to organize, analyze, comprehend, understand, and solve specific problems located in the context of their real-life situation. These tools allow them to identify and describe a specific mathematical idea or practice in a general context by schematizing, formulating, and visualizing a problem in different ways, discovering relations, discovering regularities, and transferring a real world problem to a mathematical idea through mathematization.
2. D'Ambrosio (1990) stated that a holistic context consists essentially of a critical analysis of the generation (creativity) of knowledge, and the intellectual process of its production. The focus on history analyzes the social mechanism and institutionalization of knowledge (academics), and its transmission through the educational process.

[3] A system is a part of reality considered integrally. It is a set of components taken from the reality, which analyses components and interrelationships between these components (D'Ambrosio, 1990).

[4] Ascher and Ascher (1986) defined ethnomathematics as the study of the mathematical ideas of non-literate people. In this regard, we prefer to use the term *non-literate* instead of *primitive*, *illiterate* or *uneducated* for a number of important reasons. First, we use it to emphasize the idea that primitive people are not uneducated, but that they use different ways of transferring knowledge from generation to generation (Negroponte, 1995). Second, we use it to disassociate from the evolutionary-biased term *primitive*. The presumption about the abilities of non-literate people or people whose thinking does not follow the logical rules of Western science is usually that they are "simpleminded, childlike, illogical, of lesser intelligence, or incapable of analytic thought" (Ascher & Ascher, 1986).

[5] For details, see D'Ambrosio (1993) and Bassanezi (2002).

[6] For details, see Bassanezi (2002), Biembengut (2000), Rosa and Orey (2007), and Rios (2000).

[7] Eurocentrism or privileging Eurocentric ideas has been discussed by many scholars such as Banks (1999), Joseph (1997), and McCarthy (1998).

[8] The Landless Peoples Movement (Movimento dos Sem Terra – MST) is one of the most important Brazilian social movements. It is a national organization, spread throughout 23 of 27 states of the country, involving about seven hundred thousand peasants who strive to achieve land reform and social changes in a country with very deep social inequalities. For details, see Knijnik (1996) and Monteiro (1998).

[9] The term *Underground Railroad* has come to us from a story of a farmer chasing a runaway who testified that the slave vanished on some kind of Underground Railroad. It was used to describe the network of abolitionists and safe houses that helped slaves escape to Ohio and Canada. Safe houses along the way were known as *stations*, those who guided the escapees were called *conductors* and the runaways themselves were called *passengers*. For details, see Rosa and Orey (2009).

[10] The 4 and 9 patches can be added in order to make different blocks, but not 16 nor 25. It would depend upon how many blocks we use and sizes in a quilt using the basic 4 or 9 patches within the block. Every block has variations, depending on size, fabric used, as well as how the block is turned and how many extra rows are added

[11] Shoo Fly pattern. See http://www.popularpatchwork.com/news/article.asp?a=8043 and http://blackhistory.owensound.ca/quilts.php

[12] Ethnoknowledge is acquired by students in the pedagogical action process of learning mathematics in a culturally relevant educational system. In this process, the discussion between teachers and students about the efficiency and relevance of mathematics in different contexts should permeate instructional activities. The ethnoknowledge that students develop must be compared to their academic mathematical knowledge. In this process, the role of teachers is to help students to develop a critical view of the world by using mathematics.

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BILL BABBITT, DAN LYLES, & RON EGLASH

9. FROM ETHNOMATHEMATICS TO ETHNOCOMPUTING

Indigenous Algorithms in Traditional Context & Contemporary Simulation

Ethnomathematics faces two challenges: first, it must investigate the mathematical ideas in cultural practices that are often assumed to be unrelated to mathematics. Second, even if we are successful in finding this previously unrecognized mathematics, applying this to children's education may be difficult. In this essay, we will describe the use of computational media to help address both of these challenges. We refer to this approach as "ethnocomputing." Modeling is indeed an essential tool for ethnomathematics (Rosa & Orey, 2010). But when we create a model for a cultural artifact or practice, it is hard to know if we are capturing the right aspects; whether the model is accurately reflecting the mathematical ideas or practices of the artisan who made it, or imposing mathematical content external to the indigenous cognitive repertoire. If I find a village in which there is a chain hanging from posts, I can model that chain as a catenary curve. But I cannot attribute the knowledge of the catenary equation to the people who live in the village, just on the basis of that chain. Computational models are useful not only because they can simulate patterns, but also because they can provide insight into this crucial question of epistemological status.

Take, for example, our recent investigation of Navajo weaving. Figure 9.1 shows a common re-occurring angle in the weaving patterns, of about 30 degrees. When asked about this particular angle, the weaver said it is created by an "up one over one" pattern (up one weft over one warp). We could see why that would result in a 30-degree angle: because the height to width ratio of each successive stitch was about 1/3. But she then said she could do lots of other patterns (up one over two, up two over three, and so on). So we asked why this one is used so commonly. She explained that other angles gave a more jagged edge. In other words they were concerned about the aliasing problem (Fig. 9.2), a common feature in early computer graphics (and still a concern in certain situations such as bitmap images).

This implies a host of new questions: Do some weavers use anti-aliasing techniques similar to those strategies used in computer graphics, such as using an in-between color at the edges? What are the iterative algorithms for more complex shapes? Thus the ethnocomputing approach offers two advantages. First, although

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 205–219.

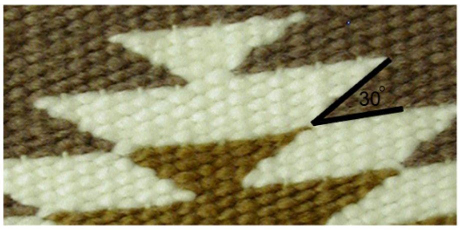

Fig. 9.1 Navajo blanket

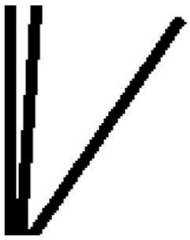

Fig. 9.2 Aliasing in computer graphics

we like to think of mathematics as being comprehensive in its ability to model patterns, some pattern generation systems are better conceptualized through the disciplinary idioms of computer science. Second, the conceptual framework of computing – the idea of information processing, algorithms, graphical user interface, etc. – allows new insight into the artisans' own perspectives in cases in which there is an analogous process. As noted above, there is a second challenge to ethnomathematics, and that is the challenge of converting these results into a pedagogical form suitable for pre-college classrooms; in particular for under-represented ethnic groups. Of course there are other applications for ethnomath, but education has been the most important. That is because of the possibility that cultural connections to math may improve the low performance and interest in mathematics that we tend to see in African American, Native American, Latino, and Pacific Islander children in the US. Several studies suggest that one factor for this low performance is that these children's identities are formed in opposition to the mainstream, such that doing well in math is considered "acting white" (Fordham, 1991; Fryer & Torelli, 2005). Another factor may be the myth of genetic determinism, in which children assume that their race lacks "the math gene" (Mathematical Sciences Education Board, 1989). Ethnomathematics could potentially offer a powerful counter to both the "acting white" myth and the genetic determinism myth. But merely pointing to a photograph of some intricate basket or monumental pyramid is not sufficient for engaging children or developing their mathematics skills. Here, too, computing can contribute.

In 2005, Anthony Lizardi, the chair of mathematics at Rough Rock, a school run by the Navajo Nation, invited us to develop a simulation for Navajo weaving. We

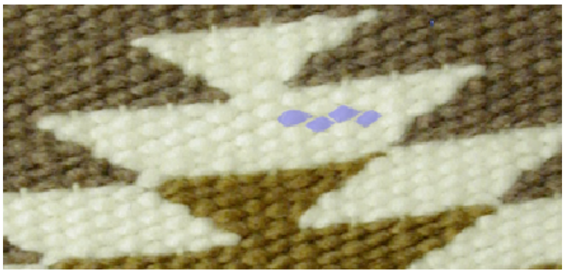

Fig. 9.3 Alternations in weft strands

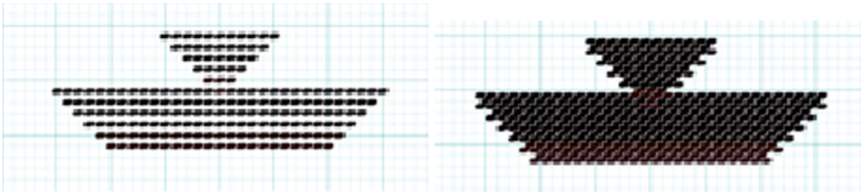

Fig. 9.4 Two steps in Navajo rug simulation

found that the weavers we spoke to were enthusiastic about this use of their work; there was some concern that future generations would lose interest in traditional weaving, and they saw this as an opportunity to maintain its relevance. They were also happy to discuss the various algorithms used to create particular shapes, which is how we arrived at the "up one over one" discussion above. However we ran into a problem with the details of the weaving process.

Upon examining the rugs we realized that would not be possible: the weft alternates up and down (see highlights in Fig. 9.3), because every other strand goes in back of the warp, and in the row above it the alternation is the opposite, with every other strand going in front of the warp. Since an important part of the weaving process compresses the rows together (the weaver pushes down with "comb"), the gaps (where the weft goes behind) from each row above and below are filled, thus producing an up and down alternation. Thus we could not simply map each individual weave to integer intersections on a Cartesian grid, which would be optimal for teaching purposes. One alternative was to simply map the weave into non-integer spaces between the grid intersections, but that would destroy much of its utility for the math teacher. Our solution was to simulate using two steps: first the user maps out a weave pattern using only the grid intersections (Fig. 9.4a), and then the user presses a comb icon, and the applet fills in the gaps (Fig. 9.4b), thus creating a completed weave simulation. Not only did this satisfy both the math teacher and our own interest in a good visual simulation, but it also better represented the actual process of weaving. Figure 9.5 shows some of the designs created by a group of Navajo high school students at the Diné College Pathways Summer Program in 2009.

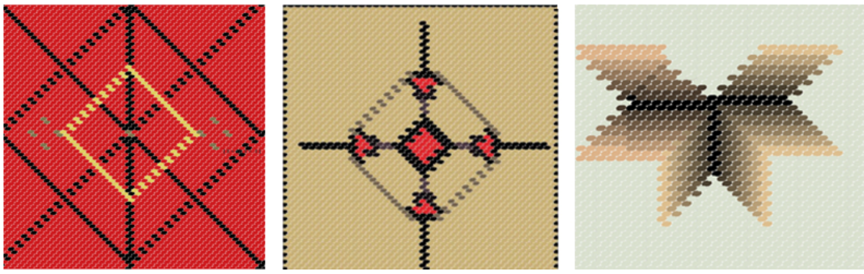

Fig. 9.5 Simulations by Navajo students

These "Culturally Situated Design Tools" (CSDTs) show statistically significant improvement in student performance, using controlled tests (Eglash, Bennett, O'Donnell, Jennings, & Cintorino, 2006; Eglash & Bennett, 2009). These web applets are available online (www.csdt.rpi.edu); the website for each "design tool" (allowing students to simulate cornrow hairstyles, Native American beadwork and weaving, Latino percussion rhythms, Mayan temples, urban graffiti and breakdance, etc.) includes some cultural background on its topic, as well as pages of lesson plans, evaluation instruments, etc.

In summary, we find the following three domains and interaction for ethnocomputing in the classroom:

– Simulations must "translate" from the particular indigenous or vernacular knowledge under investigation into the analogous knowledge forms contextualized for students in classrooms. For example, Native American traditions use the "four winds" or "four directions" as an organizing principle across many different knowledge systems: cosmology, religion, health, architecture, weaving, etc. (Eglash, 2009). This makes it an excellent candidate for teaching the Cartesian coordinate system using simulations of these artifacts such as bead work; we can clearly justify the four-quadrant coordinate system as an indigenous invention, and not merely a Western idea that is imposed on these artifacts (Barta & Eglash, 2009). At the same time, a pedagogy that introduces these artifacts should do so with the social context in which they arise. The cultural background pages for the "Virtual Beadloom" CSDT, for example, includes the use of Iroquois bead work in their US treaties and the influence of the Iroquois confederacy on the creation of the US constitution. In such instances we have attempted to steer a path between the Scylla of "white-washing" history (such that the horrors of exploitation and oppression are completely erased), and the Charybdis of a story of "victimhood" (which could demoralize students).
– These math and computing analogies are rarely exact; there is typically some negotiation between the "fidelity" of the simulation as an exact replica of the indigenous concept, and the utility of the simulation as a fit to the classroom curriculum. For example, weavers have to worry about fitting horizontal weft threads into the vertical warp without creating slack, hiding the ends when a

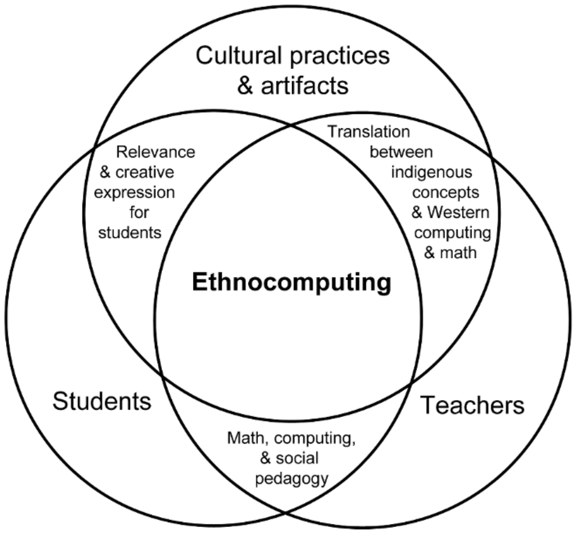

Fig. 9.6 Three domains in ethnocomputing

new color is started, etc. All of these activities might be modeled: for example, you could model the relations between two adjacent horizontal weft strands as 180 degrees out of phase. But that would complicate its use at lower grade levels where the concept of "phase" is not taught, and even at upper levels, being forced to think about phase while simultaneously using an iterative algorithm would make use potentially frustrating. In contrast to critics who complain that ethnomath adds too much external math to artifacts, the challenge in developing these simulations is to leave out much of the "high fidelity" modeling that would potentially be possible, in order to create a lower fidelity model that is both optimal for use and offers a clear translation of indigenous knowledge.

– In addition to attempting to negotiate the tension between fidelity to the indigenous conception and utility to the curriculum, a third tension exists when trying to satisfy student needs for relevance and creative initiative. For example, in our initial attempts to use fractal models of African artifacts (Eglash, 1999), we found that African American students occasionally expressed some hesitation over what were, for them, dusty museum objects. For this reason, our first simulation focused on cornrow hairstyles, which offered a compromise between African heritage and objects and practices familiar to them as part of contemporary African American culture. However, as our websites have developed, we have found that even for familiar practices (bead work in native American communities, graffiti among the urban "underclass," etc.) there is a need to teach these histories (where else are they going to learn about the history of graffiti?). Similarly, the ability to make creative use of these tools, and generate their own designs (some of which bear no resemblance to traditional examples) is critical for engaging these students, and encourages a sense of ownership over the mathematics. Moving from consumption to production, taking pride in self-efficacy and designs, learning to use math and computing as a means of self-expression rather than the disciplinary regime of "you got the wrong answer" – these are all critical components of ethnocomputing pedagogy. Figure 9.6 summarizes these three domains and their interactions.

FROM CSDTS TO PCSDTS

As noted above, our work with CSDTs made it clear that there is a component of ethnomathematics that has received little attention, because such "computational thinking" (Wing, 2006) is outside the purview of the standard math curriculum. Computing education is a key to the high-status skills and knowledge that allows a student to tap into the grid of twenty-first-century opportunities; one which under-represented students are often left out of. Would it be possible to use these CSDTs to teach computer science in primary or secondary school? In a recent publication (Eglash et al., 2011) we report on the use of our "African Fractals" CSDT (csdt.rpi.edu/african/African_Fractals/index.html) in a controlled study of two high school computer science classes with the same instructor. The control group received the same amount of instruction with a comparable fractal education website (it also used java-based applets) without any emphasis on cultural design. The results showed statistically significant advantages for the class use the African Fractals site in both performance and attitudes towards computing careers.

It was unclear, however, whether this effort to teach computing (as well as mathematics) could be applied to all CSDTs. Fractal geometry is a special case in that it is inherently mixing computing and mathematics. What about teaching conditionals, data structures, and algorithms? Such concepts were present in the CSDTs, but too deeply embedded in the tools. Take, for example, the "Cornrow Curves" simulation. Figure 9.7 shows the CSDT control panel and resulting simulation for three braids. The photo at right is one of many "goal images" that students can attempt to simulate. At left is the simulation. The left-most plait of the top braid is high-lighted to indicate that the numbers in the control panel refer to that braid. The simulation uses a recursive loop in which the original plait image is duplicated, and then geometric transformations are applied to the duplicated plait. This cycle is repeated, duplicating the previous duplication, until the desired number of plaits have been generated. However this algorithm remains invisible to the students; they only see input boxes for the parameters.

To make that algorithm visible, we would have to create a "programmable" Culturally Situated Design Tool, or pCSDT. Projects at CMU ("Alice") and MIT ("scratch") have developed programming interfaces that allow students to generate algorithms by dragging and dropping snippets of programming language ("codelets") into a "script" – thus eliminating the frustrating experience of having a program fail because you were missing a comma on line 137. But would students

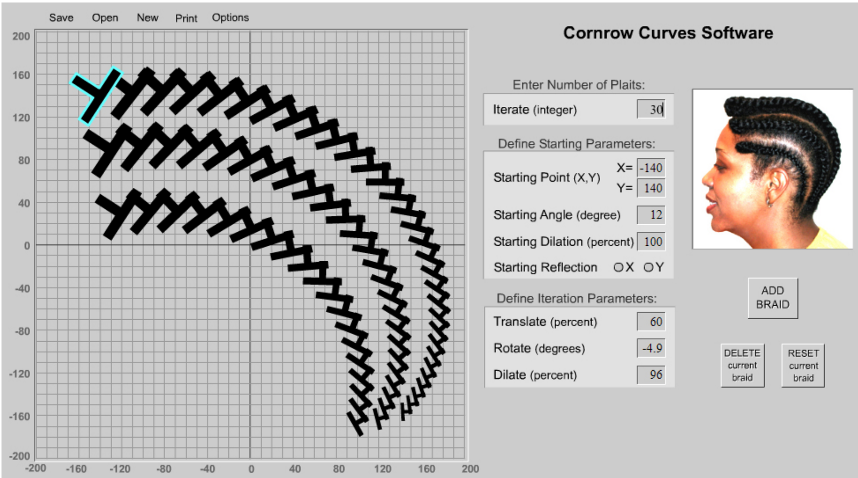

Fig. 9.7 CSDT for cornrow hairstyles

who had experienced the ease of the older CSDTs, with a purely parametric interface, be willing to create these scripts? Would we have to hide the older versions from them? Finally, we also needed to create an interface that would be easily extensible for the creation of additional pCSDT applications – we did not want to build a unique interface for each tool. And of course all this needed to happen while keeping true to the cultural connections that motivated the project in the first place.

Our design efforts crystallized around a Java applet that could be easily deployed on the web, but also brought in on physical media (CD or flash drive) in case we were in a situation with low bandwidth (or no bandwidth) internet access. To meet the requirement of being easily extensible, the program is constructed in layers. The Core layer contains the interface that is used for every programmable CSDT. The application layer contains all the code relevant to each specific tool. For example, in the case of the Cornrow Curves applet, the application class says that we want codelets for transformational geometry such as “Rotate,” a Cartesian grid for the background, a plait image for the default object, etc. Some codelets such as “Repeat While” loops are common to all pCSDTs, so they lie in the core class. Figure 9.8 shows the resulting pCSDT for cornrows.

The panel at the left contains the list of codelets, the center panel is the script created by dragging and dropping codelets, and the right-most panel is the simulation window. When users are finished with a script they can expand the simulation window to full screen before activating the script. At the top of the script, the user has declared a counting variable (called *a*) and initialized it with a value of 1. The next codelet is a control loop, to the effect of: “While $a < 20$, do the following.” Inside the loop are codelets for duplicating the plait image, and applying geometric transformations (rotation, scaling, and translation). At the end

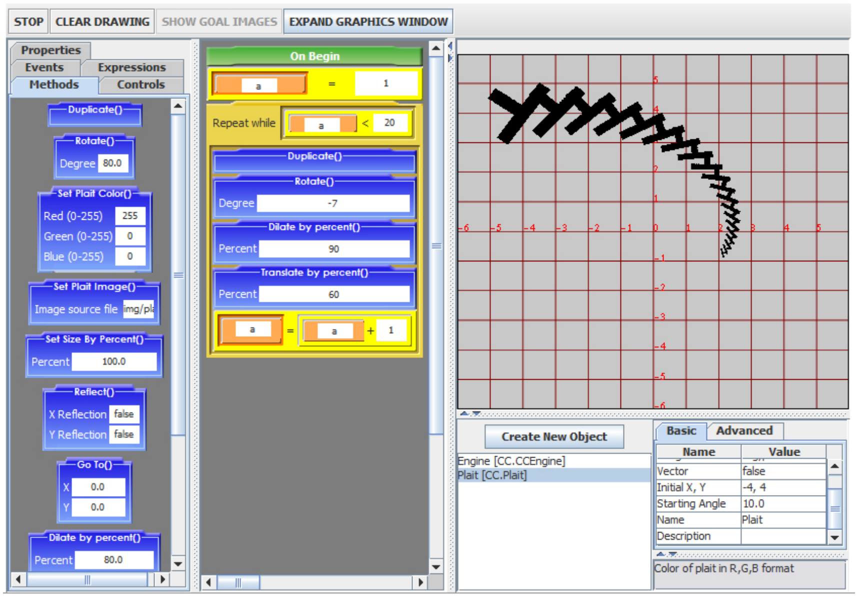

Fig. 9.8 pCSDT for cornrows

of the loop, the variable *a* is incremented by 1. Thus the script makes visible the algorithm that was invisible to users of the original CSDT. We hypothesize that a non-numeric version of something like this algorithm is also cognitively available to the stylists who create these braids.

One of the most interesting aspects of ethnomathematics simulations is that the results that they produce can surprise the software developers who create them; that is, we were not completely certain what visual patterns we would be able to produce until we actually created the simulation and began to experiment with it. We found that the new pCSDT allows many patterns that were very difficult to make with the old version. For example, in Figure 9.7 you can see three braids created on the old version: Each of those braids required a separate series of trial and error experiments. In the new pCSDT, nesting one control loop inside another allows the user to automate the process of generating a series of braids. Another problem is that real cornrow braids sometimes have rotation values that switch back and forth, like a sinusoidal waveform. On the old version, the user would have to create that effect by piecing together separate braids. The new pCSDT version allows users to introduce conditional codelets ("if-then" or "if-then-else") so that values such as rotation can be altered at any point in the braid.

The pCSDT version also allows some patterns that are impossible to make with the older version. For example, it is simple to introduce color by placing the color codelet into the script and entering R-G-B values (0-255). By inserting the counting variable (*a*) rather than a value (and this must be the value *a* multiplied by

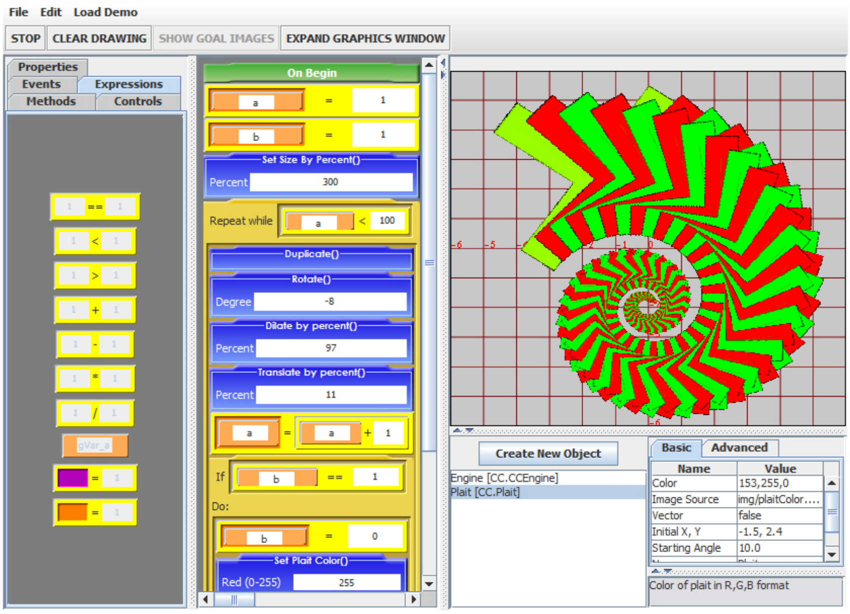

Fig. 9.9 Simulated braid with alternating colors

a constant, for which there are codelets), the color can increment with each plait, such that a braid can begin with blue and end with red, with corresponding gradients of purple in-between. By introducing a second variable (*b*) we can keep track of odd or even duplications, and thus alternate colors in the simulated braid (Fig. 9.9). Interestingly, we later realized that alternating colors are often used in physical braids – the ethnocomputing approach allows us to model aspects that were previously considered irrelevant.

OBSERVATIONS OF STUDENTS USING PCSDTS

The programmable version of Cornrow Curves is currently under investigation. Although we do not have a complete study we present here some initial observations. In the following descriptions we use the pronoun "I" since there was a singular observer.

Observer 1

The school I was assigned was an Alternative School serving students who were expelled from the regular system due to chronic disciplinary problems. The group I worked with was 100% African American, and included only one female student. The variation in attendance was so extreme – essentially a different group of students each time – that it precluded any comprehensive analysis, but it was still possible to make some general observations. First, the software's strength is in its ability to engage students by getting to apply their own experiences to mathematics in a way that showed math that underlies their own understanding. Their enthusiasm was great: they were jumping in to answer rather than needing me to call on them, as I had seen in other lessons. They were talking over each other trying to get my attention; at one point one of the students was on her feet.

One conversation was particularly striking to me: in the course of explaining the concept of ethnomathematics, I posed the following statement to the students: "One plus one equals two, except where it equals three." Initially they pointed out that they didn't think it was true, but as we reviewed possible counter examples (such as an added fee for performing a transaction) the students gained a greater appreciation for how you can think of math as a symbol system that was invented for modeling the world, and that the symbols might be developed differently as long as they were used consistently. One of the students then came up with his own example: he pointed out that you could think of $1 + 1 = 3$ as a model for buying "loosies," or individual cigarettes (where it was common to offer three for the price of two as a bargain). The discussion about the use of math continued after the bell; more than anything, it showed that engagement with the material can be enhanced when entering an area where they can contribute their own knowledge and creativity.

The CSDTs are not by any means a panacea; but even in one of the worst possible educational circumstances, they can still be an opportunity for students to begin a mathematical or computational conversation about the world they actually share: in this case, one of cornrows, loose cigarettes, and alternate interpretations of the dominate discourse.

Observer 2

Work at the second school is, at the time of this writing, an on-going study. This middle school is classified as a "high needs" public school and is located in an urban area, with about 650 students in grades 6–8. The student population is about 55% African American, 17% Hispanic, and 9% multiracial, with 72% eligible for free lunch. In addition, the school also faces the daily challenges of educating students that range from cooperative to completely disruptive. As a first trial for the software, I chose a subset of students that would be considered co-operative and willing but who also ranged in academic ability.

The software was used by members of the seventh grade science club, composed of students that had chosen to participate in science enrichment activities out of an interest in science. These students worked with the program for about an hour, during which I recorded observations concerning their use of the program. I paid particular attention to how many objects (in the form of curves) they created and the complexity of the pattern they were able to produce in that period of time. In addition, I recorded my perception of their reactions as they made scripts with the program building blocks, called codelets, to accomplish drawing tasks that interested them.

At the beginning of the work session, I provided them with a brief demo of how to simulate a braid using a script, followed by an overview of how the software functioned. I started with how objects were created (in this pCSDT, there is only one type of object, the plait, but the user can create multiple instantiations of that object) and how the scripting panel worked, and the expectation that each script should start with an "On Begin" event. I explained that once an object is created, the codelet panel fills with all the available codelets for the object.

> First place an "On Begin" Event codelet in the scripting panel, then click on the Methods panel and add method codelets to define the plait pattern you want to create. For example, to create a new curve, click on "Create New Object," choose "Plait" and then begin selecting method codelets to complete the curve definition.

I then demonstrated by adding some codelets and clicking "Begin," so the students would know how to run their scripts and see the results they produced. Finally, I demonstrated deleting an object from the Object Panel by right clicking the object to be deleted and choosing "Delete." After this brief summary on object creation, deletion, and script building I encouraged the students to give it a try.

The students worked individually on netbook computers and each began working with the software. For the remainder of the trial time, I did my absolute best to not interfere unless a student had forgotten to use an "On Begin" Event codelet at the top of their script, and only if they seemed unable to resolve an issue themselves. In wandering from student to student, I did occasionally ask "What were you trying to do?" if something apparently unexpected had occurred, and otherwise just simply praised them on the work that they were doing as general encouragement.

During the course of my observations, I observed that student ability in working with the software ranged from having great difficulty with the programming process to working with relative ease. I will focus on two students at opposite ends of this spectrum; their pseudonyms are Tomas (male Latino) and Zahira (female African American). Tomas demonstrated a fair amount of proficiency in working with the programming aspects of the software, and later mentioned some previous experience with programming. Zahira was at the other end of the spectrum.

Tomas quickly created the scripts necessary to generate a curve on the screen. He had little difficulty in navigating the interface to find the event, control, and method codelets that were necessary to accomplish the task and only once, when I happened to be near him did he ask a question concerning loop creation. Once I reviewed with him how to insert the variables in the control structure for a "do while" loop, he proceeded to complete his script and clicked "Begin." I heard an audible gasp from Tomas, and upon returning to him I found that the results he was expecting was not what was displayed on the screen. I asked him "What were you trying to do?" and he explained how he wanted the loop to function and how he wanted the plait to be drawn across the screen. Upon closer inspection, I realized he had placed the “Duplicate” codelet before the Repeat-While loop, but I did not give him the solution. Rather I suggested that he go back through his script step by step and see if he could figure out how to fix it.

Zahira had taken a different approach to drawing a curve of plaits on the screen, as she was creating new objects for each plait in the curve. Although Zahira was not getting the program results that had originally been demonstrated at the beginning of the session, she was still very engaged in creating her curve on the screen in the manner in which she was able to do so. In addition to creating new plait objects and placing them on the screen using the initial (*x,y*) value in the properties panel, she was also making use of the rotate and dilate codelets resulting in an approximation of the results of the initial program demonstration. While I was observing her working she looked up and asked “how do you make it do the curve on its own?” There is nothing more gratifying in a high needs school than to have a student say “help me.”

I took Zahira back through the original example and demonstrated the “Do While” codelet. I also reviewed the different panels containing the Controls, Methods, and Events. Having completed the review, I did not offer any more suggestions unless Zahira asked additional questions. I paid closer attention to Zahira as she worked for the rest of the session because I really wanted to know if she succeeded at climbing the learning curve. She continued to work steadily and did succeed at assembling a loop before the session ended.

At some point, I heard what I thought was an “Ah-hah!” from Tomas which immediately drew me back over to where he was working. He had successfully debugged his script and discovered that he had put the “Duplicate” codelet in the wrong place. Having moved “Duplicate” to the correct place the script functioned according to his expectations, which resulted in a very happy Tomas.

In addition to my observations of Tomas debugging a script and Zahira grappling with the beginnings of programming, there were other interesting indications of learning taking place. After hearing a groan from one student, I noticed a hand go up to the screen and trace along the Cartesian coordinate lines of the grid, followed by an “Oh!” and what seemed to be an adjustment of the starting (*x,y*) values, terminating in a “Yay!” Another student spent a significant amount of time experimenting with the starting angle of the plait – it seemed as though every time I passed by where this student was working, the plait was being rotated yet again, quite probably through most of the 360 degrees that are available for rotation!

ANALYSIS OF PRELIMINARY RESULTS.

Inquiry-based learning, in which students either invent a question themselves, or have their inquiry assigned to them, is increasingly supported by innovations in pedagogy (cf. Minstrell & van Zee, 2000). A crucial component of the education theory supporting inquiry learning is that of scaffolding, in which some temporary conceptual aid allows a student to advance their understanding, such that with a firmer grasp on new concepts, they can then climb to higher levels. Brush and Saye

(2002) introduce the terminology of "hard" and "soft" scaffolding. They refer to teachers as providing "soft" scaffolding, by which they mean it is contingent and adapted to circumstances. In contrast, they suggest that multimedia systems, of the type they introduce (which consists primarily of hyperlinked media to support high school social studies inquiry), can be labeled "hard scaffolding" because the designer must pre-plan whatever learning aids will be available.

In our case, neither category fits well: the scaffolding is contingent, not pre-planned. We never anticipated that a student would generate a braid simply by creating each plait as a separate object. But it is not a contingency generated by a teacher; rather it is a contingency generated by the interaction between a student and a digital medium that is sufficiently flexible and powerful to allow creative explorations. Rather than call this hard or soft inquiry, a better category might be "mangled inquiry." Both Tomas and Zahira's struggle with writing a script can be described using the model of "The Mangle" (Pickering, (1995) in which he detailed how scientific discovery occurs as a "dance of agency." Pickering describes the failure to accomplish a particular goal as "resistance" (in the language of Pickering, nature resisting a "capture" of its agency by some model or machine). The scientist then responds by seeking a new strategy to overcome that failure – changing models or machines or procedures until she finds one that works ("accommodation"). In the case of Tomas, the resistance came in the form of a logic error that placed the "Duplicate" codelet outside the control loop in his script. The resulting struggle to find a solution eventually led to accommodation when he moved the codelet inside the loop. However it is critical to understand that in Pickering's view, there is not simply one "correct" model or machine. Multiple different accommodations are possible, including a change of goal. And in fact, there are multiple locations within the script that would have allowed Tomas to successfully generate a braid, although the behavior might have varied slightly (e.g., there would be one less plait if you duplicated after the variable is incremented). Zahira's case consisted of two stages: she attempted an initial strategy that was temporarily successful, creating the braid with individual objects, which allowed her to proceed to the point where she was able to set a higher goal for herself (from the goal of merely making a braid by any means necessary, to the goal of having the script automatically generate the entire braid sequence).

Indeed, we can view our own attempts through this same lens of "mangled inquiry". Our planned evaluation system at the Alternative School met with initial resistance; there was no way to use pre/post evaluations given the enormous variation in student attendance. But we accommodated that resistance by focusing on the discussions that followed the software experience, and thus gained some insights into the elements that increased students' engagement in math and computing conversations.

Inquiry learning works best when it is open-ended. Students need to be able to ask questions, pose answers, and explore the implications of those answers – not necessarily "the one right answer" but rather discovering what new patterns emerge when those answers are used. In that exploration, new questions can then be developed for further consideration. The new pCSDTs offer exactly that scenario. As drawings are created they can be changed by adding additional coding elements. This added complexity will result in scripts that could benefit from rewriting, and the results offer new horizons for further exploration. As Resnick et al. (2009) note about MIT's *Scratch*, it is critical to offer a "low floor" (easy to get started) and "high ceiling" (enormous room for expansion).

CONCLUSION

In closing, we quote from one of the second observer's notes:

> The working session was nearing an end and I requested that students begin to clean up by shutting down their netbooks. There were no students that seemed happy that the session was at an end, in fact all of them seemed genuinely disappointed and expressed an interest in working with the software again. Several wanted to save the script that they had been working on, and I quickly walked them through how to do that. Yesterday, Zahira's friend who was helping with the pattern was practically giddy at the prospect that the tool was designed around the cornrow hair style. She just brightened right up as I was explaining it.

As our study shifts to quantitative data, comparing pre/post tests between control and experimental groups, we will need to leave these qualitative observations behind. But it is certainly these experiences that provide our motivation. If ethnomathematics is the scaffolding that allows ethnocomputing to emerge – scaffolding based as much on issues of social justice and opposition to racism as it is on mathematical modeling – then we do indeed stand on the shoulders of giants, however mangled our inquiry.

ACKNOWLEDGEMENT

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PART III

LEARNING TO SEE MATHEMATICALLY

The relation between what we see and what we know is never settled. (Berger, 1972, p. 7)

In the title of this section, the word "see" refers to both the physical act of visual perception and the mental act of understanding.[1] The metaphor of *understanding as seeing* is so pervasive that it takes an effort to become aware of it: "I *see* what you are saying. It *looks* different from my *point of view*. What is your *outlook* on that?" (Lakoff & Johnson, 1980, p. 48). In this section very different aspects of what it means to *see* when doing mathematics, communicating about mathematics, and teaching mathematics are discussed. In fact, this whole book is about *viewing* mathematical practices from different *perspectives* than customary. And, as indicated by the title of the section, the three chapters deal not just with seeing, in the dual sense, but with *learning to see,* that is, becoming able to see something that one couldn't previously see, and with ways of facilitating this learning. (If you think about it, so much of the development of mathematics may be characterized as driven by new ways of seeing/representing, such as the liberation of conceptualization of number from its geometric interpretation.) The objects of learning to see discussed in this section are as diverse as geometrical figures, proofs, data, students as people, a teacher's teaching, the nature and purposes of mathematics, alternative forms of representation, relationships between teacher and students.

In chapter 10, Wolff-Michael Roth considers examples of seeing mathematical objects, both in terms of the work done with the eyes (processes) and how that results (in a context of previous culturally shaped experience) in mental objects (products). Abstraction is "both a process (the action of drawing out from a situation) and an object (the product, the concept)" (Maheux, Thom, & Roth, 2009, p. 73). The very specific example that Roth examines in detail in his chapter serves to illustrate strong philosophical and culturally-based critiques of traditional views of epistemology that characterize mathematics as transcendental, external, and objective. The notion that the body and the mind occupy different realms is rejected and the sources of mathematics are recognized as the individual's body and collective, culturally-framed, lived experience. Such critiques also rip open naïve conceptions of representation and the relationship between believing and seeing.[2] Somewhat underemphasized in Roth's chapter – but clear in other of his writings – is the dependence of the perception of, say, the line drawing of a cube on individual past experience which, itself, is embedded in a cultural milieu.[3] Thus, consider, for example, how a Yup'ik conceptualizes a square as a perceptual/kinesthetic construction, beginning in the center (Lipka, Wong, Andrew-Ihrke, & Yanez, chapter 7, this volume).

From his phenomenological analysis of the perception of a line drawing as a cube, the proof that the measures of the internal angles of a triangle add up to 180 degrees, and some other examples, Roth draws educational implications about the importance of active learning, the need to distinguish between the account of a proof, for example, and its re-enactment by the student (for which the account can provide a guide – a parallel would be between a recipe on paper and the actually cooking of a dish). These insights are aligned with those of many mathematics educators, reached by reflecting on experience of interacting with children. For example, “didactical phenomenology” (Freudenthal, 1983) is firmly based in children’s physical and social experience and, in particular, “the best palpable material you can give the child is its own body” (Freudenthal, 1991, p. 76). This is precisely shown in a recent analysis of second-grade children’s learning of three-dimensional geometry, which arises at the intersection of their incarnate experiences, the embedding culture, and the societal relations that they produce together with their teachers (Roth, 2011). Moreover, modes of relations that are inaccessible to deliberate consciousness – such as the rhythm of speech and pitch (intonation) – play constitutive roles in the coordination of the relations and, with these, in the constitution of sense. There are definite parallels to chapter 12 by Swetz, which presents a multitude of examples from many cultures of the intimate relationship between the body and measurement.

In chapter 11, Chris Jordan, in conversation with Swapna Mukhopadhyay, talks about his attempts to facilitate another form of mathematical seeing – looking at data about collective social life and seeing the implications. In a quotation attributed to George Bernard Shaw (cited at the start of the chapter), the ability to be moved by statistics is taken as the mark of the highest intelligence. We may term this ability “statistical empathy” (Mukhopadhyay & Greer, 2007, p. 117).

It is well recognized that people, in general, find difficulty in making sense of very large numbers. Rarely in people’s lives do they look at an aggregation of, say, a million objects, knowing that it is a million. There are some examples, however. Frank Swetz, when teaching in the Philippines, organized students in a school to collect a million bottle caps. In other cases, an attempt to give an intuitive feel for numbers is done by expressing the numbers in equivalent, but more accessible form (see Frankenstein, chapter 13, this volume, for many examples). In Jordan‘s work. as it appears on a computer, he can transcend the limitations of static photographs. In particular, the zooming facility afforded by software allows the viewer to move between a close-up one-to-one representation of a very large number and a view of the aggregate. Alternatively, as in the gallery version of his representation of 2,300,000 folded prison uniforms (representing the approximate number of Americans incarcerated in 2005), the sheer size of the installation (23ft. x 10ft.) in relation to the smallness of each photographic element showing a single uniform is also very powerful. A point forcefully made in the conversation between Jordan and Mukhopadhyay is the relationship of the individual to the collective, and a possible elaboration of Jordan’s work, as they discuss, would be to be able to click on a single element and see individual information about that case.

In chapter 12, Frank Swetz greatly enlarges the scope of learning to see in mathematics, drawing on numerous and varied examples from his wide experience in many cultures. He cites many examples of how re-enactments by students can help them to see the essence of mathematical relationships[4]. These examples include cases where the mathematics is anchored to a physical/perceptual series of actions, such as having students who have learned some trigonometry going out and applying it to measure the height of a distant building (which can be independently verified). This experience allows them to understand the trigonometry in a very different way.

Swetz also talks about seeing in the context of interpersonal relationships with students – having insight into your own teaching, understanding the students in relation to their lived experience, and the interpersonal relationships with them (Gay, 2009). Such aspects are very clearly described in an account of an anthropological study of an American classroom (Spindler & Spindler, 2000). George Spindler was given an assignment to observe a teacher who was regarded extremely positively by all, including himself. At first, Spindler was nonplussed about what to observe – everything was "normal." Then

> eventually I began to see the teacher and the pupils as "natives," engaging in rituals, interaction, roleplaying, selective perception, cultural conflict, sociometric networks, defensive strategies, and so on. (Spindler & Spindler, 2000, p. 202).

Following his observations, Spindler undertook to make the teacher more able to see how he actually behaved towards the students.

NOTES

1 The logo of Sense Publishers that you can see on the spine of this book is a Native American symbol "the eye of the medicine man," representing wisdom. Likewise, in Hinduism and Buddhism, the third eye, or inner eye, refers to enlightenment (and note the metaphor within that word!).

2 Historically, the drive for rigor in academic mathematics led to a distrust of visual arguments. In the extreme, this was manifested by the Bourbaki group banning figures from textbooks on geometry. The computer (which, according to Mandelbrot, "put the eye back into mathematics," has helped to overcome this self-imposed restriction. Further, the role of intuition in mathematics, which often has a strong visual component, remains undertheorized. Another big idea that is related is that of mathematics as an empirical science.

3 The cultural influence on perception of 2-dimensional conventions for representing 3-dimensional objects is perhaps clearest in the long development of perspective paintings in many cultures.

4 A very fine teacher and writer of accessible books explaining mathematics, W. W. Sawyer, provisionally defined mathematics as "the classification and study of all possible patterns" (Sawyer, 1955, p. 12) and went on to state that "for every pattern that appears, a mathematician feels he ought to know why it appears." Another of his books has the title *Vision in Elementary Mathematics* (Sawyer, 1964).

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WOLFF-MICHAEL ROTH

10. THE WORK OF SEEING MATHEMATICALLY[1]

The reigning epistemologies in mathematics education take (visual) perception as an unproblematic phenomenon, assuming that students see (i.e., perceive and understand) the curriculum materials presented in the way that a knowing mathematics teacher/educator intends them to be seen (i.e., perceived and understood). Moreover, all current epistemologies – including not only all forms of constructivism but also all embodiment and enactivist theories – miss a fundamental contradiction: Students cannot see and therefore intend the object of learning precisely because they do not yet know it and therefore are asked to learn it. There is therefore an essentially passive dimension in knowing and learning that current epistemologies do not theorize: If I cannot intend the mathematical learning object, it somehow has to be given to me, or reveal itself to me, so that I come to see and understand (Roth & Radford, 2011). This leads to inherent contradictions that the existing epistemologies cannot overcome; attempts on the part of (radical) constructivists to overcome this phenomenon, which has come to be known as the *learning paradox* simply reiterate the position of the individual as the source of all knowing.[2]

In this chapter, I present an investigation of visual perception concerning real-world and ideal-mathematical objects. I show that new, previously unknown objects are not simply seen (intentionally) but that they are given to the subject of mathematical activity as the movements of the eyes are shaped by structures in the world. I show how mathematical seeing (perceiving and understanding) is grounded in immanent – i.e., immediate and unmediated – processes, which constitute initially immanent forms of knowing that are not subject to sign mediation. Moreover, the objects of experience that the children of today encounter, much as the ancient Greek encountered them, *are not* the mathematical objects that they subsequently learn (learned) about. These objects will be geometrical and ideal, whereas their sensuous experiences that children encounter initially through the senses of touch and vision inherently are real objects that only in the limit (of engineering) approximate the ideal objects that they denote. I offer a radical re-theorizing of mathematics along the lines of my recent work on mathematical cognition.[3]

Throughout this chapter, I insist on the difference between the lived experience of mathematically seeing and the *accounts* of experience of seeing in mathematics that societal actors – children, teachers, or lay and professional mathematicians – provide when asked about what they see. Almost all research, both quantitative and

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 227–245.

qualitative, is concerned with *accounts* of experiences of mathematical seeing rather than with the living/lived work of mathematical seeing.[4] I articulate the difference between the two and provide some guidance with respect to the ways of going about researching the lived work rather than accounts thereof. In this, I counter the false belief that our perceptual experiences are "constructed," and I insist that the real work (doing, seeing) that makes mathematics an objective science is actually lived and the result of our living/lived, sensuous bodies rather than that of the constructivist mind.[5] In this manner, I articulate and elaborate an approach that is an incommensurable, asymmetrical alternate approach to *formal (including constructivist) analyses* of living/lived experiences of mathematical seeing.

THE LIVING/LIVED WORK OF SEEING MATHEMATICALLY

I begin this investigation with two practical inquiries, which, when readers engage with these, allow them to *live* the experiences of mathematically *seeing* a geometrical object as a specific object (cube) and *doing/seeing* the proof of the angle sum of a triangle. The *immanent* aspects of these living and lived experiences are radically different from accounts of experiences that are articulated in one or the other way for someone else or for the person himself/herself.

Case 1: What Makes a Cube a Cube?

To start our inquiry into the difference between the lived work of seeing mathematically – the living/lived, sensuous experience of mathematical seeing – and an account of (the experience of) mathematical seeing, consider the drawing in Figure 10.1. What do you see? Take a moment to look at the figure and find an answer before you proceed reading.

The figure is known in psychological research as the Necker Cube. Although there are but a few black lines on a two-dimensional sheet of paper of white color, most research participants report something like "I see a (three-dimensional) cube," "I see a cube from below that extends from front right to back left," or "I see a cube from the top that extends from the front left to the back right." When asked further, participants may outline – by moving their fingers along certain lines salient in their perception – where they see the different surfaces of the particular cube they see. In their statements – which may be provided verbally alone or communicated using a range of semiotic resources – they provide *accounts* or reports of experience. What they have not provided access to is the actual, lived work of seeing that is obliquely referred to in their accounts/reports.

Qualitative researchers, including researchers employing phenomenography, tend to be interested in reporting all the different things that research participants have reported seeing, which, in addition to a cube, may simply be a set of lines, or an assembly of several flat geometrical figures, and so on. Constructivist mathematics educators may be tempted to say that these participants "constructed" the particular cube or cubes that they see. In fact, should it not be strange that

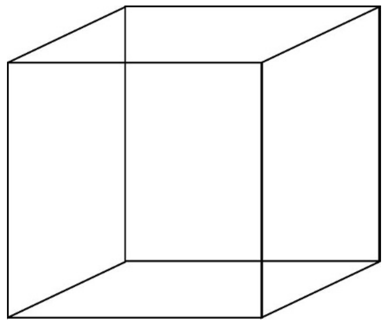

Fig. 10.1 This diagram has been used in psychological research on perception and is known as the Necker Cube

participants report seeing this or that cube given that there are only lines on a flat page? How is it possible to see something three-dimensional when there are only two dimensions? Both sets of research reports are limited, as they do not get us any closer to the real question of the lived work (experience) that is denoted in the reports/accounts that provide us with the structures that people exhibit to one another. So what more is there? Related to this question we may distinguish between *formal analysis* and *ethnomethodology* (Garfinkel, 1996). The former approaches to research report structures, here perceptual structures, whereas the latter is concerned with the living work that brings the structures about, here the perceptual structures. Ethnomethodology, as its descriptive name suggests, is concerned with the methods by means of which people (Gr. *ethnos*) produce and exhibit to each other the structures of social action, whereas formal analysis, generally having to specify particular research methods, is concerned with the identification of the structures. Phenomenological studies, too, are concerned with the conditions that produce this or that sensual experience rather than with the phenomena as they are given to us in our senses.

So what is the lived perceptual work underlying the report of seeing this or that cube? The drawing (Fig. 10.1) allows us to investigate perception and how we come to see what we see. Upon first sight, you may see a cube, if you see a cube at all, from slightly above extending from the front left to the back and right (Appendix, Fig. A1a).[6] But, if you see a cube, you might actually see one from below and extending from front right to the left back (Fig. A1b). These two perceptions are the two spatial configurations that participants report seeing in psychological experiments, where these perceptions are categorized as "cognitive illusions." Rather than wondering about illusions, let us engage in the analysis of the living/lived work of perception to find out what is at the origin of the perception of the cube in one or the other way (i.e., from below or from above). We may do so, for example, by exploring how to quickly switch back and forth from the cube seen slightly from above to the other one seen from below.

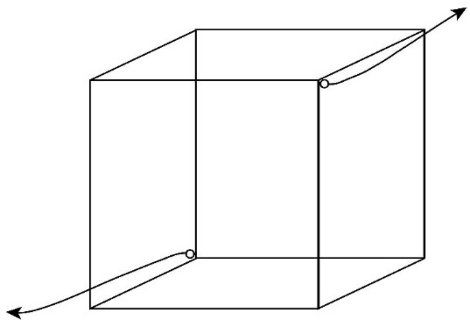

Fig. 10.2 Placing the gaze at one of the two vertices and following the trajectory toward a vanishing point *gives* rise to one or the other cube in sensuous experience

To begin with, look at the figure (Fig. 10.1) and allow the first cube to appear, for example, the one that you see from below and extending into the back toward the left, and then intend seeing the other one until you see it. Move back to see the first; return to the second. You might also do this: look at the first cube, the one seen from the bottom and extending toward the back and left. Close your eyes – but intend to see the other cube upon opening the eyes again. Practice until you can switch between the two in the rapid flicker of the eyelids. Once you achieve this, observe what is happing with your eyes during the flicker. That is, how can you generate *this* or *that* experience voluntarily and intentionally?

You may notice that if you place your eyes to the lower left corner that appears inside the set of lines and then move toward a non-present vanishing point to the left ("along the surface") – this may be along the edge leading from the "front" vertex toward the back left – then the cube-seen-from-below becomes instantly apparent (Fig. 10.2). Similarly, focusing on the equivalent vertex further up and right and then moving along the edge "backward" to a non-existing vanishing point allows you to see a cube-from-above (Fig. 10.2). That is, unbeknownst to your intellectual consciousness, the *movement* of the eye from one of the two vertices toward a non-existing vanishing point in the back to the left or right of the diagram creates one or the other perceptual experience. This, therefore, is a statement about how the work of seeing produces the cube even if we do not attend to it. If the eyes do not make these movements, then the cubes do not appear and the lines remain on a flat surface. Most importantly, therefore, this experiment shows us that the cube is not (intentionally) constructed because when you looked at the figure for the first time, the cubes appeared, you did not intentionally construct it. And for the very first time you looked at the figure, you might have not seen any cube at all or only one and not the other. That this is so can be accentuated by taking a common form of puzzle, where a person initially sees nothing but splotches (Fig. 10.3). Most people have to gaze at the image for a while and then see some figure; turning the page around, the same figure can be seen again.[7] Why do we see these figures? They arise from the movements of the eyes, which we initially *cannot*

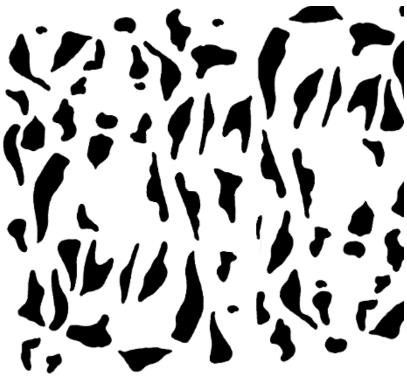

Fig. 10.3 The forms that come to be seen are not "constructions" but arise from the invisible via the ground of the image

intend because the figures are invisible. The initially arbitrary movements are shaped by the material image, making the image eventually appear. But once you have seen what there is to be seen, you can find it again, initially perhaps with some difficulty, but eventually at will. At this time, the required eye movements know themselves, they constitute "kinetic melodies" that occur again such that you may again see what there is to see. We do not know where the originary movements come from, much as we do not know where mathematical insights come from.[8] They are more or less haphazard but come to be honed in repeated execution.

How do the eyes know to move like this to make the cube appear? The answer extends the possibility of this text, but I have worked out a possible response based on the phenomenology of the flesh. Briefly stated, this knowing emerges from first uncoordinated movements during which the corporeal movement (of the eye) auto-affects itself such that it develops the capacity to move and develops an immanent memory of this capacity. In other words, during first arbitrary, random movements, corporeal-kinetic movement forms (archetypes) emerge that would be more ancient, more basic than any "image schemas" or "sensorimotor schemas," if they exist at all. Nothing is constructed at that point because there are no tools available for the construction; in fact, this capacity, the self-knowledge producing the movement precedes any intentional movement, any intention to act, and any intentional thought. Before I can *intentionally* move the eyes, these have to *immanently* know that they can move.

It is clear in the preceding account of the perceptual work that different movements of the eyes underlie the different visuo-sensual experiences; that is, if there is a different movement, a different experience is produced. If our living flesh does not produce some movement, this form of experience is not available to the person. The source of the movements underlying our disciplinary visions and *di*visions can be located in the *habitus*, sets of structured structuring dispositions that we cannot ever access but the results of which express themselves in praxis. There is a dialectical process at work, because habitus is shaped by the social and material field that it inhabits, but habitus itself allows the social and material field to appear in specific ways. Ultimately, this mutual dependence leads to the fact that habitus and field are homologous.

We can push this analysis further – but this is difficult and requires considerable practice. The question we attempt to answer is this: How do we see one and the same cube over an extended time? Or, equivalently, is the eye movement from the vertex to the corresponding vanishing point necessary for us to see a cube? To reach an answer, fixate, for example, the lower vertex. Or, equivalently, attempt to have both cubes appear at the same time. You may not be able to achieve this feat on your first few attempts – psychologists generally use equipment that allows them to fix an image to a specific location on the retina. But as soon as you achieve this feat – that is, as soon as your eye is fixed so that the parts of the image continuously fall onto the same equivalent spots on the retina – you notice that the figure dissolves completely and you do not experience anything except a dark grey perceptual field. This graying tends to start at the periphery of your perceptual field and move inward. You no longer see lines. That is, as soon as the eye no longer moves, you cannot see the lines and even less a cube. To see a line or cube, the eye needs to move back and forth between the cube and some other place that constitutes the ground against which the cube appears as the figure. The eye does work to produce the sensuous experience that you have. Once the movements exist, we may speak of the construction of the cube, as the eye now knows how to move to bring forth the cube. *But initially*, the eye was not in the position to construct anything because it did not know how to move or that some movement would produce anything. In one sense, the cube is a cube because the eye finds it again upon moving away, and to generate the cube, my eye has to move from the vertex to its corresponding vanishing point.

The upshot of this investigation is this: We do not just see or recognize a cube because of some mirror image that is produced on the retina. Rather, our eyes have to do work, and associated with this work there are changes on the retina. Based on the changing images, and based on prior experience, we have learned to see cubes. This is not the outcome of a conscious construction, but rather, it is the result of a shaping of our eyes' movements as they follow the structures that they find in the world. The visible *is given to* the person, a recognition that is widely shared among artists and philosophers. Artists *find* what there is to see after having finished the painting rather than communicating what is already visible to them in and through the painting. We can see cubes because our eyes know what they have to do to make a cube appear. It is in the non-perceived movement of the eye that the distension and dehiscence between the cubical figure and the ground occurs and that the former comes to detach itself from the latter. But we should not think of the image as something standing before the ground, as if projected against a screen; rather, in the image the ground is rising to us. It is not merely, as enactivist theorists would say, that the organism is bringing forth a world – *initially* the world gives itself to the organism, which learns how to make any figure reappear, at

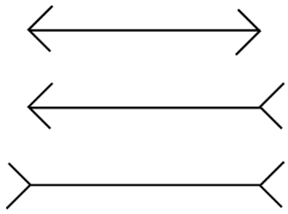

Fig. 10.4 The Müller-Lyer "illusion" makes line segments of equal length to be of different length

which point we might describe the process as a bringing forth. That is, the movements of the eyes are not random, not constructed, but they are entrained by the structures of the material world in which the organism is embedded – movement patterns and the structures of the world are homologous, as I note above. The eyes *follow* lines and thereby are entrained into certain movement patterns that are not their own but that arise from the structures in the world. This then leads to the fact that "it is in reference to my flesh that I apprehend the objects in the world" so that "in my desiring perception I discover something like a flesh of objects" (Sartre, 1943, p. 432). It is in reference to my flesh that I apprehend the objects of the world, which means "that I make myself passive in relation to them and that they are revealed to me from the point of view of this passivity, in it and through it" (p. 432). There is therefore a fundamentally passive component to perception that tends to be obliterated in the (social, radical) constructivist literature but that is essential to understand the dual, subjective/objective nature of mathematics that has become the point of unresolved contention between formal and constructivist accounts of mathematics.

We can enact further phenomenological investigations relevant to geometry by, for example, investigating the conditions for seeing an angle or seeing two lines as equal or unequal. Thus, in *Geometry as Objective Science in Elementary Classrooms* (Roth, 2011a), I exhibit how the movements of the eyes make us see two line segments of demonstrably equal length appear to have different lengths (Fig. 10.4). The Müller-Lyer illusion is produced as the eye follows the inward and outward pointing arrows at their ends in a different way. At its very heart the phenomenon is based on the same movement processes that allow us to see a drawing as a cube. Thus, such a perception of equality of lengths important to perception in geometry is explained by the movements of the eyes in the context of particular configurations of lines. This illusion is sustained even when we have measured the two lines and therefore know that the two lines are of equal length. That is, we are passive with respect to our perception even when "we know better." There is therefore nothing constructive about the originary experience, it is happening to us. We come to see what we see because of the movement of the eyes, movements that our eyes, as an aspect of our living/lived bodily selves, are given as originary, archetypal corporeal-kinetic forms.

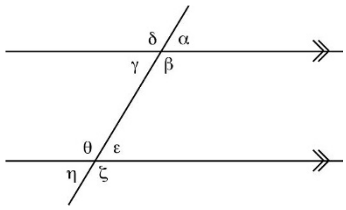

Fig. 10.5 The angles produced when a line crosses two parallel (») lines

We can sum up this first part of our investigation by saying that there are two parts to perception: (a) the account or gloss of what is mathematically seen and (b) the living/lived work of mathematical seeing that underlies the account. Qualitative research generally and phenomenographically oriented qualitative research specifically investigate and report on these accounts; this kind of research presents us with the structures that either the participants or the researchers report. It is our phenomenological analysis that actually leads us to an understanding the living/lived work that produces the different experiences that people report.

Case 2: Mathematical Seeing while Proving the Angle Sum of a Triangle

In the following description of mathematical practices, using proving as an example, I follow the kind of studies produced in the field of ethnomethodology of mathematics. This work is concerned with the irreducible relation of living/lived work and accounts of this work. These descriptions are consistent with the phenomenological studies of the foundation of mathematics (geometry), which recognize the co-presence of lived (subjective) and formal (objective) dimensions of mathematics. Accordingly, there are records and accounts of mathematical proofs, on the one hand, and the living/lived labor of doing a proof, on the other hand.

The Proof Account The proof that the internal angle sum of a triangle is 180° involves a drawing (Fig. 10.5) and the following. In a first step, we note the relationships between angles that are produced when a line crosses two parallel lines (marked by the sign "»").

- The pairs (α, ε), (β, ζ), (η, γ), and (θ, δ) are known as corresponding angles; corresponding angles are equal (i.e., $\alpha = \varepsilon$, etc.).
- The pairs (α, γ), (β, δ), (ε, η), and (ζ, θ) are known as vertically opposite angles; vertically opposite angles are equal (i.e., $\alpha = \gamma$, etc.).
- The pairs (ε, γ) and (θ, β) are alternate angles. Alternate angles are equal (i.e., $\varepsilon = \gamma$) – because of (a) and (b).

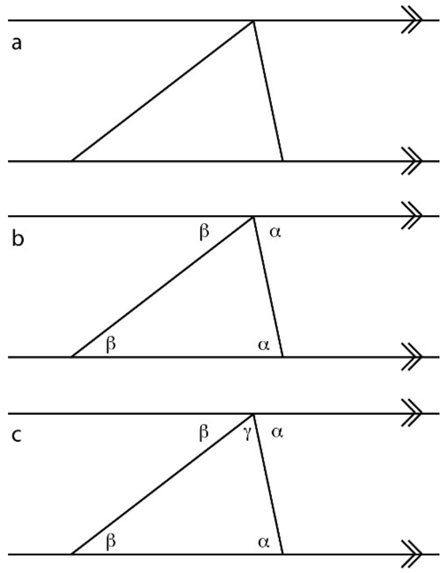

Fig. 10.6 Steps in and part of the account for the proof that the interior angle sum of a triangle is 180°

With these identities in place, we can prove that in the Euclidean plane, the angle sum in a triangle is 180° – if the total angle around a point is defined as 360°. This proof includes the following steps together with three diagrams (Fig. 10.6).

- Any triangle can be drawn such that the base lies on one of two parallel lines and the opposing vertex on the other (Fig. 10.6a).
- We know that alternate angles are equal, as marked (Fig. 10.6b).
- Hence, because of configuration of lines at the upper parallel, that α, β and γ add up to 180°, that is, $\alpha + \beta + \gamma = 180°$. Therefore three angles in a triangle add up to 180°.

The preceding steps and figures do not constitute the entirety of the proof; rather, they constitute what we know to be the proof account. These are the parts that one might find in a textbook on geometry, on a website, or, in the case of new mathematical discoveries, in relevant journals. This is the part, therefore, that allows us to re-do the proof over and over again, which certainly has been done so since antiquity, when the proof was done for a first time. For example, the reviewers of an article take the submitted proof as instructions for doing the proof, checking whether there are "no holes" in the proof procedure. When they get the same result, their own subjective work has reproduced the objective account. The proof becomes a fact. In written form, this account suffices to be able to hand the proof procedure down – initially, to share it with others in the prover's community.

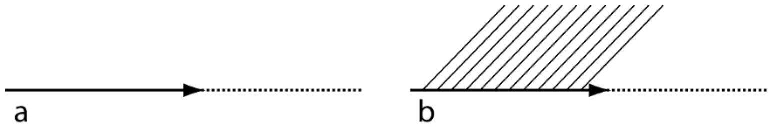

Fig. 10.7 In the dynamic of drawing a line, the plane becomes bisected, here denoted by a hatched and an unhatched part

Ordinarily, newcomers to a discipline learn these practices in face-to-face work with others who monitor and give feedback to correct actions; but the written accounts are such that they allow others to re-discover the proof in their own praxis.[9] That is, as initially arbitrary and tentative actions are marked as subject to correction, the student tries again. Once such actions receive approval, then the immanent generating mechanism, the self-affected movement, can now or after some trials reproduce the action intentionally. This possibility for the rediscovery of the proof in fact constitutes the objective and tradable nature of geometry as objective science. Thus, "the important function of *writing* is to enable the *continual objectivity* of ideal sense entities in the curious form of virtuality" (Husserl, 1939, p. 212). The ideal (subjective) objects exist virtually in the world in written form, and they therefore can be actually produced at any time. The lived praxis (labor) within which this written account *counts* as the proof, however, is not contained in the written account. *It is precisely this lived work that we are interested in here and in ways of capturing it*. We already see some of what is involved in the preceding inquiry concerning what makes a cube a cube. To bring this proof to life we actually need to do it in and as of living/lived labor for which the written record has to provide sufficient resources.

The Living/Lived Work of Mathematical Seeing in Proving I am interested in the living/lived work within which such accounts constitute the resources that allows us to count what is happening as a proof. Part of the kind of work involved is articulated in the first subsection, that is, the lived work of seeing. In the present instance, for example, this living/lived work includes the re/cognition that pairs of corresponding, opposite, or alternate angles are equal. That these pairs of angles are equal presupposes the seeing of each angle – where the work of *seeing* is described above. Such *seeing* is related to the living/lived work of drawing multiple lines, each of which bisects the plane (Fig. 10.7). This work involves particular movements, kinesthetic structures or kinetic trajectories, which are inscribed in the living/lived body (the flesh) where it constitutes an immanent form of knowing. From the perspective of the living/lived work, the writing gesture produces the divisions of our pre-geometrical perceptual experience of left/right, up/down, and so on. Even if the movements initially are arbitrary and random, they constitute traces that mark differences in space, and thereby shape the perceptual experiences that follow.

When, after the completion of the first line (involving a complete bisection of the plane), a second line is added, it, too, bisects the plane. Four sectors are thereby

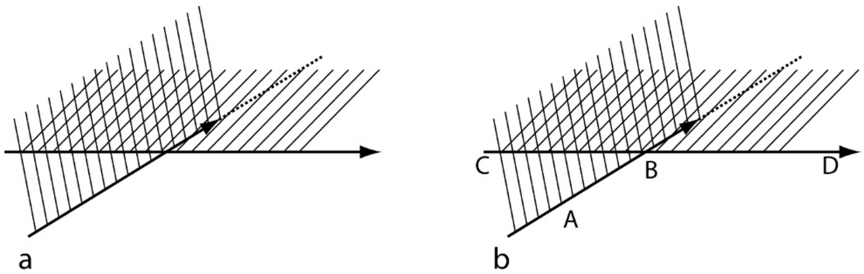

Fig. 10.8 Two intersecting lines produce four sectors

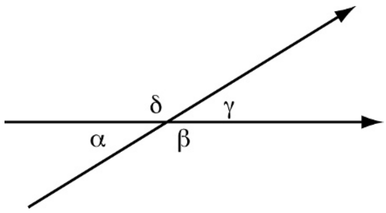

Fig. 10.9 The placement of the labels a and b is apparently disengaged from the temporal practice of drawing the figure

produced, which appear in three different hatchings: not hatched, once-hatched, and twice-hatched (Fig. 10.8a).

I could have also drawn the second line in the reverse and produced the same account. For this reason, the angles enclosing the single-hatched areas are the same. What is in the first drawing the angle forming first to the left and then to the right will be, upon beginning the diagram from the other side, again first to the left and then to the right. In this very act of drawing, we also produce an order that goes with the naming of locations (Fig. 10.8b). In this way, the unfolding from drawing the AB line with respect to CD forms angles ABC and ABD, which we may also name, following the tradition, by the Greek letters α and β (as well as the equivalent angles γ and δ) (Fig. 10.9). Here, the order in the actual making constitutes a conceptual order: "The temporally placed label of an angle or its apparently disengaged placement in a finished figure exhibits this seen relationship as a proof-specific relevance" (Livingston, 1987, p. 96). The conceptual order is *in* and *arises from* the movement rather than from the constructive mind, if there indeed should exist something of that kind. Mind and sensorimotor schema are *postkinetic*, as are all accounts of mathematical experience.

The relationships between the lines, angles, bisectors, and sectors have to be seen; this seeing, as shown above, is based on the movements of the eyes, movements that we are not in conscious control of. Not surprisingly, phenomenological philosophers have recognized the fundamental passivity that is associated with a first cognition that such seeing involves. Any first formation of sense therefore has two passive moments: the first existing in the first cognition and the second in the fact of the retention of this first cognition. Thus, "the passivity of the initially darkly awakened (insight) and the eventually increasing clarity of that which appears is accompanied by the possibility of a change in the activity of a *remembrance*, in which the past experience is lived again actively and quasi anew" (Husserl, 1939, p. 211). The memory is awakened passively but can be transformed back into corresponding activity. The *re*cognized relationship may therefore be maintained throughout the proof procedure, which leaves as its end result a sequence of diagrams (Figs. 10.5, 10.6). In the drawing, we do not specify a particular angle to be produced. Any work that produces two, non-parallel lines suffices to get us to this point. This fact produces the generality of the proof procedure.

This immemorial, subjective memory is important in the constitution of geometry as an *objective* science in and through the subjective, living/lived, sensuous work of the geometer. A sense-forming act that came about spontaneously can be actively/passively remembered, and therefore reproduced not only by the original individual but by any other individual as well. It is in the reproduction of the living/lived work that the evidence of the identity between original and subsequent act arises: "That which now is originally reconstituted is the same as what was evident before" (Husserl, 1939, p. 211). That is, together with the original sense formation comes the possibility of an arbitrary number of repetitions that are identical in the chain of repetitions. That is, the very subjective, living/lived work of doing and seeing geometry that allows me to recognize relationships again make for the social nature of geometry and its historicity as objective science.

Interestingly, the very generality of the proof derives from the way in which the sensuous work generally and the sensuous work of seeing specifically unfolds. For example, in the drawing of a line that crosses two parallel lines and labeling alternate angles using the same letter, the proof makes available that any such line could have been drawn, which in fact occurs when the second line between the two parallels is drawn such as to form a triangle. The very possibility to have one line between parallel lines with alternate angles enables all other lines. The relations between the angles in configurations of parallel lines crossed by a third thereby imply the angle sum of the triangle to be 180°. From the way in which living/lived work draws parallel lines and sees the equivalent angles that follow from (the idea of) parallelism simultaneously constitute the angle sum to be 180°. That this is so can be discovered over and over again because (necessarily written) proof-accounts describe, like a recipe, their own work. It is precisely "in this particularistic way, the generality of our proof-account's description was evinced in and as the lived, seen, material details of the proof" (Livingston, 1987, p. 108). The very nature of geometry as objectivity science arises from the demonstrability and visibility of its procedures in the living/lived (subjective) work of proving, including the living/lived work of mathematical seeing. Anyone may reproduce the living/lived work anywhere. In sum, therefore, we realize that the "generality of our proof both is in and not in the proof-account; it is in that proof-account through the pairing of that account with its lived-work" (p. 108).

In this brief description, we can see how the living/lived work of producing, seeing, and labeling the angles is actually accomplished. This drawing, seeing, and labeling is available to those present; this drawing, seeing, and labeling makes the work objectively available to those present. But this sensuous work does not (and cannot) appear in the proof account proper, where the lines and labels appear disengaged from the actual movements of drawing, seeing, and labeling. All of these involve our living/lived, sensuous body in the manner described in the first section above for the eyes' work that makes a cube from a set of lines. Seeing an angle involves fewer lines, but nevertheless requires the movement of the eye that puts into relation the two unfolding lines, the half planes, and the seeing of the intersecting planes against the background (generally white). Even imagining an angle or a line in our minds or recognizing someone else drawing an angle or a line *requires* the activation of the same immanent movements in us that operate when we actually see or draw a line. This fact has been recognized over 200 years ago through phenomenological analysis and has been recently substantiated by neuroscientific studies on the function of mirror neurons. The account, as we might find in textbooks, is disengaged from this living/lived work, but it may serve as a resource on the part of the learner, as an instruction for reliving the sensuous work of proving in and through his/her own living praxis of drawing, seeing, and labeling. The relation between accounts and the lived work can be stated in this way (Husserl, 1939): In textbooks the actual production of the primal geometrical idealities is surreptitiously substituted by means of drawn figures that render concepts visual-sensibly intuitable. It is up to the students to find in their own subjective sensuous work the practical relevance of the instruction, which in the present example would be the proof-specific relevance of the lines, markings, naming, and so forth.

We can see that in this pairing of proof account and lived, sensuous work of proving there is the possibility of a pedagogy. In fact it has been said that the proof account is "completely and hopelessly a pedagogic object – it teaches the lived-work that it itself described" (Livingston, 1987, p. 104). This is so because we can see in it a *formulation* of the work that is described, much like an instruction that presents both what is to be done and what will be found as an outcome of the actions. However, this condition still does not solve the ultimate problem of the difference between the account and the lived work: the students have to find in their own living/lived corporeal actions the relevance of this or that definition, this or that instruction, this or that description of an outcome. There is a surplus in the transitivity of the living/lived action over its ideation that constitutes the difference between living/lived work and any account thereof.

In this section, I articulate but the beginning of an analysis that indicates the nature of the lived work as distinct from the objective accounts produced and handed down for millennia from the ancient Greek to the present day. The *accounts* of mathematical seeing and doing – though not the subjective work of perceiving – are objectively available to all the generations; the lived (subjective) work of mathematical seeing and doing has to be enacted each and every time by the person actually doing or following (observing) the proof. In this way, the subjective enactment of geometrical seeing and the objectively available account have to be intertwined to make geometry the objective historical science that it is. The sensuous work has nothing to do with a mental construction, as the movements underlying the (intentional) drawing of a line emerge from experiences that have nothing at all to do with intentions. These are originary movements that have nothing do with the "(embodied) image schematas" of cognitive science and embodiment/enactivist accounts but may be thought of as *archetypal corporeal-kinetic* forms or as kinetic melodies that would enable any such schemata, if they were to exist at all.

OF PERCEPTUAL WORK AND ACCOUNTS OF PERCEPTION

In a text on the formal structures of practical action, Garfinkel and Sacks (1986) propose a way of theorizing the ways in which accounts of structures and the generally invisible work that brings these structures about are related (Fig. 10.10). Thus, the expression "doing [proving the sum of the internal angles of a triangle is 180°]" consists of two parts. The text between brackets "[]" topicalizes a particular practice that social scientists and educational researchers might be interested in; the text is a gloss of what a researcher or lay participant might say that is happening. For example, observing a student, a teacher might explain to the researcher visiting the classroom that the former is "proving the sum of the internal angles of a triangle is 180°." This text is the *account* for what is currently happening. Similarly, if asked by the researcher what she has been doing, the student might gloss, "I was proving that the sum of the internal angles of a triangle is 180°." Almost all research in the social sciences and education is of this kind; ethnomethodologists refer to this kind of research as *formal analysis*. Research methods are provided in articles to articulate how the researchers arrived at identifying the structures that appear between the gloss marks (i.e., between "[" and "]"). But formal analysis does not capture the first part of the expression: it misses the "doing." This moment of the expression allows us to ask a pertinent research question, paraphrasing Garfinkel and Sacks: "What is the work for which 'proving the sum of the internal angles of a triangle is 180°' is that work's accountable text?" or "What is the work for which 'proving the sum of the internal angles of a triangle is 180°' is that work's proper gloss?"

In contrast to constructive formal analysis, ethnomethodology is interested in specifying *the work* by means of which the structures are produced that are accounted for and glossed by the bracketed texts. In other words, the question ethnomethodology pursues is that in the living/lived work, for example, of proving that the internal sum of a triangle (on the Euclidean plane) is 180°. Once we know the organization of the living/lived work, we are able to predict the kinds of results people produce in the same manner as we can predict what kind of entities people will see when looking at the diagram known as the Necker Cube. However, from knowing the accounts, we cannot infer the nature of the lived work. For this

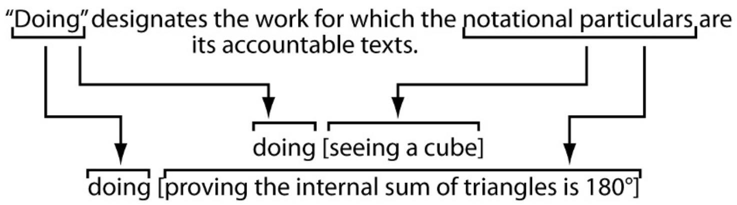

Fig. 10.10 Conceptualization of the difference between the *work* ('doing') that produces a phenomenon and the description of the experience (seeing a cube, proving the internal sum of a triangle

reason, phenomenological and ethnomethodological accounts of mathematics are related to formal analyses – whether quantitative or qualitative – in asymmetrically alternate ways. This is not to say that ethnomethodology disputes the accounts provided by formal analysis; those achievements can be demonstrated and are demonstrated in and as the outcomes of the living/lived work of doing mathematics. This asymmetry is radical and incommensurable, but nevertheless obtains to related aspects of mathematics. Ethnomethodology (as phenomenology) is not in the business of "interpreting" signs that people produce. Rather, its "fundamental phenomenon and its standing technical preoccupation in its studies is to find, collect, specify, and make instructably observable the endogenous production and natural accountability of immortal familiar society's most ordinary organizational things in the world, *and to provide for them both and simultaneously as objects and procedurally, as alternate methodologies*" (Garfinkel, 1996, p. 6). The two examples I use here constitute such materials that allow readers, in and through engaging the work specified, to experience the living/lived, worksite-specific (inherent lived) praxis of *doing* and *seeing* mathematically.

IMPLICATIONS FOR PEDAGOGY

The upshot of this approach is that no account can get us closer to the actual living/lived experience of seeing and doing mathematically, even when, and precisely because, persons retrospectively *talk about* their living/lived mathematical experiences. Therefore, no textbook paragraph or professor utterance can *tell* us to see mathematically. This is so because these accounts inherently involve *re*presentations of the sensuous experience, that is, means of making some past experience present again. We do not get in this way at the sensuous experiences themselves. In any instance imaginable, these representations – the means of making a past presence present again – are different from the sensuous work in the living present. Only metaphysics will make a claim to the contrary, because it has not recognized that ever since the Greek antiquity, scholars have attempted to access living/lived *Being* in and through externalities, that is, beings (representations). Being (capital B) and beings are not the same thing, though in metaphysical accounts of knowing and learning (which includes all forms of constructivism from Kant to the present day), the latter are freely substituted for the former. Therefore, the dehiscence of Being and beings is never recognized – but this is precisely the divide that I see between all forms of formal analysis and ethnomethodology, the former being concerned with beings (identifiable, identified structures) and the latter with Being, the never-ending living/lived labor of producing the structures identified in the asymmetrically alternate way in formal analyses. By their very *re*presentational nature, therefore, *pedagogical instructions* are radically different from what they intend to instruct: seeing, doing mathematically.

It should be clear, therefore, that mathematical seeing cannot be taught explicitly, because the forces and movements underlying seeing are invisible and inaccessible to consciousness. I can notice what my eyes do, but only because they already have developed the competency. I can voluntarily move my eyes from one vertex to another, from a vertex to a vanishing point (Fig. 10.2) because my eyes already master these movements and therefore give me the capacity to intend particular movements. Before that – e.g., during the initial look at an image (e.g. Fig. 10.3) – my movements are inherently arbitrary. That is, mathematical seeing requires particular forms of movement capacities that emerge from initial arbitrary and random movements. If my flesh does not yet know them, I cannot intend these movements precisely because I do not know how to enact the movements required for seeing mathematically. In a praxis framework, this impossibility to teach disciplinary perception may be attributed to the invisibility of habitus:

> Given that what is to be communicated consists essentially of a *modus operandi*, a mode of scientific production which presupposes a definite mode of perception, a set of principles of vision and di-vision, there is no way to acquire it other than to make people see it in practical operation or to observe how this *scientific habitus*) we might as well call it by its name) "reacts" in the face of practical choices – a type of sampling, a questionnaire, a coding dilemma, etc. – without necessarily explicating them in the form of formal precepts. (Bourdieu, 1992, p. 222)

That is, one can observe only the *effect* of habitus, which is a particular form of vision and *di*vision, never habitus itself, because it is immanent in the movements. The only way of support that can be offered is by having the student participate in the actual praxis where disciplinary seeing, the forms of vision and *di*vision are in action. This allows students, as I suggest above, to identify in their own, initially unintended movements those that yield results similar to those that they can see brought about by the more experienced person.

DIRECTIONS FOR RESEARCH

Readers will notice that in my approach to lived experience of mathematical seeing, I am not interested in asking people what they have seen while engaging in this or that mathematical task. Any response I might receive is only a *re*presentation of the sensuous work of seeing filtered through the particular perspectives of the person. It has been noticed that what a practitioner has to say retrospectively about what s/he has done does not get us any closer to the lived praxis than what a theoretician says. Accounts of experience are as far from experience as any other description including the accounts a theoretician might provide; they constitute but *rationalizations* of an originary event given everything else that we have experienced and learned since then. We know very well – as the popular adage goes – that hindsight always has 20/20 vision. Retrospective accounts always and continuously are subject to change; what I get from people when I ask for accounts of experience, therefore, depends on when and under what conditions I ask. What I am interested in instead is this: (a) the enabling of a situation whereby the interested reader experiences the living/lived work of seeing mathematically that is the focus of my research (e.g., while doing the entire proof, including the drawing, seeing, concluding) and (b) an understanding of the fundamental living/lived processes that enable this or that sensuous experience (e.g., how we come to see a cube as a cube, a line as a line, etc.).

The kind of distinctions I make in the preceding sections allow us to move from accounts of doing and seeing mathematically to the actual sensuous labor (work) of mathematical doing and seeing. The two stand in an incommensurably and asymmetrically alternate relation. I am not interested in the interpretation of signs people produce but in the sensuous labor of doing mathematics. That is, I am not interested in local practices as texts that are interpreted for their "meaning." Rather, I am interested in accessing the sensuous labor of mathematics as events that are "in detail identical with themselves, and not representative of something else" (Garfinkel, 1996, p. 8). This requires attention to the "witnessably recurrent details of ordinary everyday practices," which literally "constitute their own reality" (p. 8). We see above that knowing the work allows us to specify the structures that formal analytic procedures identify. This means, that "you can use ethnomethodology to recover in phenomenal ordered details – in a phenomenal field of ordered details the work that makes up, at the worksite, the design, administration, and carrying off of investigations with the use of formal analytic practices. You can't do it the other way around" (p. 10).

Much of the living/lived work goes unnoticed – not in the least discoverable in the disattention that formal analysts pay to the living/lived work of doing mathematics. In fact, phenomenological analyses that focus on *Life* show that it remains invisible, especially to the so-called sciences of life, *bio*logy. However, under special circumstances, parts of this work are to be exhibited: in situations of trouble, for example, when experienced scientists struggle with the classification of a specimen or when scientists struggle with providing an expert reading of a graph even though it was taken from an introductory course of their own domain. With respect to research method, what really matters in, and to, praxis is made available and perceivable only in the actual living/lived work of doing research – one has to experience it to be able to see it. To allow readers to re/live the work in and through their own living/lived bodies, reading/seeing or hearing accounts are insufficient. What research of the living/lived mathematics experience can do is:

- to provide for situations that make the phenomenon instructably observable such that in doing what the instructions say, the reader experiences in and through his/her living/lived labor the relevant mathematics; the phenomenological investigations of seeing a cube and proving the angle sum (the work is only partially detailed) would be of that kind.
- to provide something like a musical score, which, when readers actually "play the tune," allows them to live the mathematical conversations presented in the same way as musicians live the music written by some classical composer who, in most cases, no longer lives (e.g., Roth & Bautista, 2011).

In summary, therefore, to get at the living/lived work, we need research to go about differently than what formal analysis allows us to do. There is no difference whether formal analysis denotes itself as qualitative or as quantitative. Distinctly different are phenomenological and ethnomethodological approaches, because they are concerned with the living/lived work of doing mathematics. No retrospective account can get at this because of the inherent, unavoidable dehiscence between Being and beings, presence and the making present of the present (representation). But we have to inquire into the living/lived work, because this is the only way accessible to the "inner-historical," nature of mathematics, the very problem of its objectivity continually re/produced living/lived (subjective) sense-building and sense-producing work of everyone in the culture doing/seeing mathematics. We cannot understand mathematical seeing as a living/lived form of life unless we gain access to the very engine that keeps it alive, produces and transforms it across generations: the sensuous work of doing/seeing mathematics.

NOTES

1 The work described in this chapter was made possible by several grants from the Social Sciences and Humanities Research Council of Canada. Aspects of this work were presented as part of the lecture series "Alterative Forms of Knowing (in) Mathematics" at Portland State University and at the WISDOM[e] conference at the University of Wyoming. Some parts of the text were initially published in *Forum Qualitative Social Research*.

2 In a chapter devoted to the *learning paradox*, von Glasersfeld (2001) states that "far from being *given*, what is called 'data' can be seen as the result of the experiencer's own construction" (p. 143). In this, he simply reiterates the (neo-)Kantian position of conceptions preceding sensuous experience. This flies in the face of many 20th-century philosophical analyses, affirmed by the neurosciences, according to which an essential aspect of knowing *is* given to the subject.

3 See, for example, Roth, 2010, 2011a.

4 Marx/Engels (1962) frequently use the adjective "lebendig [living]" and the corresponding noun "Lebendigkeit [vivacity]" of things that are *alive* and changing in contrast to things that are dead and unchanging. In his focus on the living person, Marx explicitly situates all economic phenomena in the phenomenological life of the individual. I pair the "living" with "lived" in the expression living/lived, because at a number of very different levels, human beings are not only alive but also live (experience) this state of being alive.

5 During the WISDOM[e] conference, Pat Thompson and Les Steffe suggested that I did not understand (radical) constructivism. But all they were doing was reiterate the subjectivist idealist position that von Glasersfeld has laid out, a position that many philosophers have shown to be untenable in the face of real data. I deconstruct this position in several recent works (Roth, 2011a, 2011b).

6 The two cubes that participants in psychological studies tend to report seeing in Figure 10.1

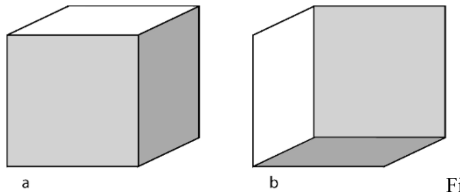

Fig. A1

7 There is a small Dalmatian dog on the left. When the page is rotated through 180˚, there is the same Dalmatian dog, again on the left.

8 Arguing that these structures come from the unconscious only gets us deeper into Western metaphysics, as the unknown and unknowable now are explained in terms of cognitive structures currently not available to consciousness. Thus, "[t]he non-presence always has been thought in the form of the present . . . or as a modalisation of the present" (Derrida, 1972, p. 36–37). This is also the fundamental point and problem both in de Saussurian semiology and Freudian analysis.

9 *Praxis* denotes the real situation where the living/lived work occurs; it generally is not characterized by thematization and "metacognition." *Practices* refer to the patterned action and therefore denote something apparent to a theoretical gaze rather than to the regard of the practitioner.

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CHRIS JORDAN

11. RUNNING THE NUMBERS

A Conversation

> It seems to me that the best art is political and you ought to be able to make it unquestionably political and irrevocably beautiful at the same time. (Toni Morrison)[1]

> It is the mark of a truly intelligent person to be moved by statistics. (George Bernard Shaw)[2]

One of Chris Jordan's goal as an artist is to enable people to be affected by numbers that tell important stories about collective responsibility. He does this by digitally assembling pictorial arrays showing large numbers of objects reflective of contemporary society. While these objects are familiar to the point of being unnoticed, we lack a sense of their aggregated numbers, which are normally conveyed only through the abstraction of numbers, a representation low in impact. Accordingly, he set himself the task of devising a way to show these numbers in a more directly graspable form, and the solution he came up with was to exploit the capacity of computers to manipulate images.

By way of example, take the bare statistic that in the year 2007, 2,000,000 plastic bottles were used and discarded in the USA *every five minutes* (Fig. 11.1). We have some sense that this is a big number, but how big? As individuals, we rarely see more than a few plastic bottles. We could estimate how many we use in a year, but what would that number mean to us? Jordan's computer image shows 2,000,000 plastic bottles. In this representation, there is a one-to-one mapping between elements in the image and the objects being represented, as opposed to the compact representation of two million as a mere 7 digits. When we first look at the image, it looks like an abstract painting. However, using the capability of the computer to zoom in, we can look more and more closely, and we begin to see that the image is made up of bottles, and finally we can make out individual bottles with familiar Coke and Pepsi labels (Fig. 11.2).

Another composition, Cap Seurat (Jordan, 2011), is a take on Georges Seurat's painting *A Sunday on La Grande Jatte* (Fig. 11.3). It represents "400,000 plastic bottle caps, equal to the average number of plastic bottles consumed in the United States every minute." The very large number – the magnitude of which could be hard to grasp – is embedded in an initially aesthetically pleasing, familiar, and non-threatening image. However, zooming in, Seurat's illusion of an image created by

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 247–259.

Fig. 11.1 *Plastic Bottles 60" x 120"* (Jordan, 2007) depicts two million plastic beverage bottles, the number used in the US every five minutes

pointillism morphs into a mass of familiar bottle caps – blue, green, black – that are all around us (Fig. 11.4).

Although we may be vaguely aware that a large number of plastic bottles are piling up daily somewhere as waste, we can hardly comprehend the impact of its bulk. In his own words, "Statistics can feel abstract and anesthetizing, making it difficult to connect with and make meaning of 3.6 million SUV sales in one year, for example, or 2.3 million Americans in prison, or 32,000 breast augmentation surgeries in the US every month." By zooming in and out, we can start to make a connection between the individual bottle discarded, a single act of throwing away a bottle, and the cumulative effect of this individual action being replicated across the nation, and often around the world. Work in this style is collected in his Internet presentations and book under the title *Running the Numbers: An American Self-portrait* (Jordan, 2009). Story-telling numbers featured in this work include:

Number of cell phones retired in the US every day in 2007: 426,000
Number of gun-related deaths in the US in 2004: 29,569
Number of Americans incarcerated in 2005: 2,300,000
Number of plastic beverage bottles used in every five minutes in US in 2007: 2,000,000
Number of plastic beverage cups used on airlines flights in the US every six hours: 1,000,000
Number of sheets of office paper used in the US every five minutes in 2007: 15,000,000
Estimated number of plastic bags consumed around the world every ten seconds: 240,000
Amount of money spent in dollars on the war in Iraq every hour in 2007: 12,500,000
Number of Americans who die from smoking cigarettes every six months: 200,000
Number of items of junk mail that are printed, shipped, delivered, and disposed of in the US every three seconds: 9,960
Number of gallons of oil consumed around the world every second: 48,000

Fig. 11.2 Zoomed-in image of *Plastic Bottles* (Jordan, 2007)

Number of Indian farmers who have committed suicide since 1997 when Monsanto introduced its genetically modified cotton seeds containing terminator technology into that region: 200,000
Estimated number of pounds of plastic pollutant that enter the world's oceans every hour: 2,400,000

Recently, he has been photographing and filming on Midway Atoll, a cluster of Pacific islands 2,000 miles away from any landmass, where ocean currents bring massive quantities of the plastic detritus of today's world. The effects are tragically seen in the carcasses of albatrosses scattered across the islands, in which a few bones and feathers surround piles of plastic, which the birds have ingested from the ocean (Fig. 11.5).

Chris Jordan does not aspire to tell people how to behave, but he does want them to have a sense of the implications of their individual actions in relation to the collective actions that are destroying our world.

On May 5, 2011, Chris Jordan and Swapna Mukhopadhyay had an informal conversation at Jordan's Seattle studio where three magnificent gongs hang from the ceiling near a work table covered with large prints. Various photos are displayed on the walls; books, musical instruments, art-related stuff are all around.

Fig. 11.3 Cap Seurat 60” x 90" in one panel, and 88" x 132" in 3 panels (Jordan, 2011). The image depicts 400,000 plastic bottle caps, equal to the average number of plastic bottles consumed in the US every minute

Chris and Swapna sat facing each other, with a bank of computers behind Chris. He lit a candle and sage smudge sticks, struck a Tibetan singing bowl, and the conversation began.

RESONANCE

Chris: Resonance. I am a big fan of resonance. I love the metaphor of things that resonate. In a way maybe that’s what we all are: resonators.

Swapna: Do you want to talk about that and how it relates to your work?

Chris: Well, one jumping-in place is simply to recognize how important it is to face the gravity of the times we live in. That’s where my work originates, at least in part. I envy people who lived back when there wasn’t such a need for global collective change. Not so long ago, artists could devote all their time to exploring beauty, or the nature of perception, and be satisfied with that as their life’s work. These times feel different.

Swapna: In what way?

Chris: Well, leaders in lots of different fields right now are telling us that we’ve got only a few years to move into a radically new way of living. And if we fail, the consequence will be a shift in our biosphere over several centuries that will bring human civilization as we know it to an end. Every day our actions are

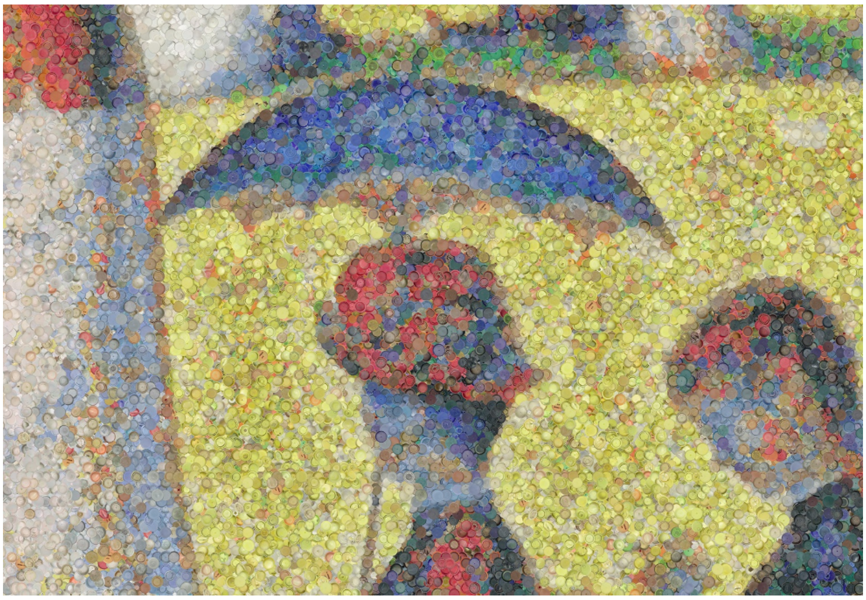

Fig. 11.4 Zoomed-in image of Cap Seurat (Jordan, 2001)

inadvertently deciding the fate of humankind, and to know that and not focus my work on it in some direct way would feel to me that I was living a useless life. And I wonder how beauty, humor, and joy fit into that picture, and what realistically can be done by one dude with a camera?

Swapna: There seems to be some humility in this approach.

Chris: I'm not sure I would characterize it that way. I think I still have a lot to learn about humility. For me, it's a position of urgency, because there's so much at stake. How does one pose these uncomfortable questions with grace and respect, and deliver them in a way that reaches others meaningfully? It's like trying to solve a Rubik's cube.

Swapna: So one always has to choose a conduit. How do you feel about that, your medium as a conduit?

Chris: I like that idea, and it gets back to resonance, oscillating a vibe out into the field. But what's the deeper message that's being resonated? I don't think art's role is to give out answers, in an activism kind of way. My work aims more toward self-inquiry, looking into the unseen processes that got us where we are. I am interested in framing these issues without judgment, in a way that attempts to honor their complexity, facing our individual roles too, including my own.

But I'm also not totally hands-off in terms of having an opinion. The way I choose my subjects, there's a point of view there. I care about the state of our culture, and believe in our ability to change. I'm interested in what motivates us below the level of awareness, and about the relationship of grief and beauty in

Fig. 11.5 Midway: Message from the Gyre

our spirits. I don't want to stand back at a detached distance with a viewpoint that is fundamentally cynical or ironic. There's plenty of that out there already, and it doesn't advance the ball.

Swapna: One of the things that appealed to many people at your lecture at Portland State University was the story you told about leaving the legal profession.

Chris: I used to tell that story a lot, but now it's sounding kind of old to me. At its heart, the lawyer thing for me was about my fear of living. I come from an artistic family, and have been photographing and studying photography for more than twenty years. But I couldn't find the courage to take the risk of failing, so instead I went to law school and copped out of the game for a while there. I'm glad to be back in the fray. In terms of my Portland lecture[3], it's so inspiring to have a student come up and say she made a decision in her life based on something I did. It makes me want to work harder than ever.

Swapna: You are a teacher in a way.

Chris: Hum, well I don't think of myself like that, at least not at this point in my life. What would I have to teach? But, maybe it's helpful to say here's the path I'm choosing, and here are some mistakes I have made along the way, for what's that worth.

Swapna: Do you look at the work of political artists? Does that interest you?

Chris: You know I have to admit that I don't feel connected with a lot of what's going on in the visual art world right now. Fine art tends to live a strange parallel existence with the rest of the world, sometimes without much contact between the two. It's disappointing because art has so much potential as a means of affecting culture, but in my view a lot of that is being wasted right now, especially at the big institutional level. There are plenty of exceptions, and I do see some exciting new threads emerging lately, artists and curators who are reaching out directly to the public, engaging the issues of our time head-on. There are pockets of brilliant creativity and courage out there, especially at the fringes.

Swapna: I think maybe there is something your work has in common with Banksy's[4], in that it tries to present uncomfortable information in a direct way. Anyone with basic intelligence will be able to understand what is being represented because of the way you've represented it.

Chris: Banksy! A real ass-kicker – I love what that dude's up to. And yeah, I do want my work to be as transparent as possible. I'm trying to say my thing in a way that a fourth grader can get, but embracing complexity and subtlety at the same time, without oversimplifying. One influence that way is Mark Twain, who I studied extensively in college. I have huge reverence for his form of storytelling: he respects his audience, but tweaks their minds too, using sleight of hand to gently nudge them into some gnarly psychic territory. That's a powerful approach, something I aspire to. For me, part of that process is also about getting myself out of the way, making my work about the subject, not about me or about the art itself.

Swapna: Tell us about the mathematics part of your *Running The Numbers* series. It seems to me you are interested in helping us comprehend big numbers. Can you talk about that?

Chris: That's a part of it, but it isn't only large numbers that I'm interested in – it's specifically numbers that relate to our unconscious mass culture. The issues I focus on are collective behaviors that tend to go unacknowledged, unspoken, in the shadow. Millions of people can join in a trend, like buying and driving increasingly dangerous and impractical off-road vehicles for vague reasons we can't even articulate; or objectifying women's bodies to the point where hundreds of thousands of American women come to believe they need various kinds of surgery to attract love into their lives. These kinds of phenomena can be motivated by feelings below the level of awareness, with cumulative effects that no one wants or intends.

And these behind-the-scenes feelings can reside in powerful forms, like anger and rage, building up collectively and manifesting as cultural horrors that seem to come out of nowhere. Right now American culture is permeated with hostility, and we aren't talking about it openly. It seeps into our political process as vitriol and hatred, splitting people into smaller and smaller groups and leading to a kind of public nastiness that was unheard of even a couple of decades ago. The underlying feelings are legitimate, and deserve to be talked about and worked through; but instead they remain mostly underground, morphing into behaviors like violence. It's frightening to watch it seethe, ready to turn into an atrocity if the wrong conditions all combine.

As an artist, I'm interested in shining a light into those dark corners, illuminating what's lurking below the level of awareness. Because the moment we become aware of an unconscious process, then we have choices that we didn't have previously. So, I'm not just interested in showing what 384,000 of some random thing looks like; I want to illustrate 384,000 breast augmentations surgeries, so that as a culture we might begin to wonder into this tragedy on a deeper level. Just as we speak, I saw this horrendous news of a Barbie mom giving her seven-year-old daughter a voucher for breast augmentation when she is sixteen.[5]

These mass phenomena are everywhere we look, right in the details of our own lives. Think about the strange fact that every year Americans voluntarily buy, drink, and dispose of more than 200 billion plastic bottles of processed sugar and chemical ingredients that are toxic to our bodies. I want to hold up a mirror and reflect these things back to us. What we each do with that information is not for me to say, but let's start by facing it together.

Swapna: Where do your ideas come from?

Chris: Mostly they just appear out of nowhere. Like the idea for my latest piece came when I was standing naked out in my garden at about 2:00 am, looking up at the full moon and thinking about all the Americans who are losing their homes right now. I was reflecting on the old cliché that whatever good or bad things are happening in our various lives, we all look at the same moon. And right then an image hit me: the moon made of thousands of credit cards, representing all the people in the US who are being forced into bankruptcy. It felt like a strong idea because it contained a dark humor, with layers of horror and sadness, plus visual interest and some potential for beauty, all wrapped up together so you can't tell them apart anymore. Those are the ingredients I look for.

It's a bizarre experience, that moment of creation. Creation isn't even the right word, because it's something that comes *to* me, rather than *from* me. A lot of artists describe their ideas like that, where you can't really claim credit for having thought it up, it just arrives, with a zapping sound. Then from that point, it's just a matter of executing the work, which for that image meant collecting hundreds of credit cards, photographing each one individually, blurring the names and numbers one by one in Photoshop, and then building 29,000 of them into a giant image. A hundredth of a second of creative spark, followed by hundreds of hours of pure tedium. I like the final result of that one though – from a distance it looks pretty convincingly like the moon.

Swapna: The relationship between the individual and the collective. What would it be like if you could click on one of those credit cards and get the story of one of the individuals?

Chris: Hey, nice one, I love that idea, it would really bring the issue home. I'm always searching for ways to make these global issues more personal. It's a challenge, and as you suggest, at bottom it's about the individual's place in the collective. I'm interested in how we each fit in. Like, do I matter? Can one person's behavior really make a difference? When you look behind the cliché that every vote counts, our insignificance is hard to face. How can we feel empowered as individuals when we are each just one of 6.8 billion, with the world's population increasing by 200,000 new people every day? Our collective behavior is causing a multi-leveled catastrophe, but each one of us is so small that our own impact is abstract and infinitesimal. How do we make sense of that? One thing that's clear is that humanity's collective power is immense, whether in a negative or positive way. What is the key to triggering a new kind of collective consciousness that could change everything? I think art could play a role.

Swapna: How do we take that next step?

Chris: Well that's a big question, isn't it? I think a key ingredient could be our feelings, and perhaps that's where art comes in. As individuals, our feelings are our connection with the world. If we can't find the part of ourselves that feels something about these issues – our fear and anger about the extinctions and wars, our love and joy for the tremendous beauty of our world, our grief about what is being lost – then it's no wonder we aren't acting more decisively. I think as Americans we've become frighteningly disconnected from how we feel about things. It's not that we're unfeeling people; I think we've just gotten lost, and are overwhelmed with abstract information and surface noise. Our news comes to us as vast numbers that have no meaning: millions of that, billions of that, and whether it is deaths or crimes or amounts of money or whatever, it becomes hard to feel any of it.

Running The Numbers tries to point in the direction of comprehending some of these issues so we can relate to them more consciously, and perhaps feel something more directly and openly. I think if we could actually stand in front of a pile of the dead bodies of all the people we have killed in the Iraq war, or a mountain of all the money we have spent on it – an actual pile of a trillion dollars, or stand in front of all the schools that could have been built with that money, or all the food that could have been given to people who need it, I think we would feel something quite profound. It might not be a happy feeling, but it sure would wake us up. Otherwise it's just a big abstract concept characterized by numbers we can't comprehend, that we read on our computer screens before clicking on to the next thing.

Swapna: Abstraction can take away from feeling. Your pieces have an interesting play that way, abstract, and yet pointing toward something that's not abstract. How do you think about it?

Chris: That part of *Running The Numbers* is a source of dissonance for me. To illustrate global issues, the images are forced into some degree of conceptual abstraction, but I'm trying to shape them so they have a personal impact. They attempt to point in the general direction toward comprehension of these issues, but I'm constantly aware of how limited they are in their ability to really convey what I want to say.

For example, take the image of 11,000 jet trails that depicts eight hours of our commercial flights. If you were really to look at the resources being expended by our jet travel, you would have to consider all the metals extracted from the earth in mining operations all over the planet to create the machines and factories that build the jets and all of their components; the operation of the airports, including all the vehicles and buildings, and the mining and manufacturing behind that; and all of the power and fuel being consumed at all those stages, and so on. Air travel is an incomprehensibly massive and impactful global process, and my *Jet Trails* piece gestures toward all that only in a vague way. When I look at it like that, I realize how much further there is to go to in facing these issues.

Swapna: Your work is not just about large numbers, it's also about how things are scaled up in a reality that we otherwise don't see.

Chris: Yea, that's another piece of the puzzle – these issues are invisible. The scale of our mass culture is never collected in one place where we can go and see it, and behold it with our senses. There is no way to photograph the millions of tons of plastic in our oceans, or all the Americans who are losing their health insurance coverage, and so on. To try to grasp these phenomena on their full scale, the only information we have to work with is statistics – giant numbers with lots of zeros that our minds and hearts cannot relate to in any meaningful way. And yet the phenomena themselves are profoundly important for us to connect with, so how do we do it?

Swapna: Can we act and think both locally and globally at the same time, is it becoming impossible at this point?

Chris: That's an interesting question that I was talking with some Chinese artists about recently. They come from a two-thousand-year tradition that subverts the importance of the individual in favor of the smooth functioning of the collective. Western culture, especially America, is all about empowering the individual with as few limits as possible – for better or worse. And yet, despite coming from these two diametrically opposite philosophical directions, our two cultures face the same problem right now: We have hundreds of millions of individuals acting on their own, and collectively we are inadvertently contributing to a disaster that threatens humanity's very existence.

So how do we foster what futurists call "global collective intelligence?" We are still operating under top-down management, shaped like a pyramid, with so-called leaders at the top, the power elites, who in most cases lack integrity or wisdom, and who control massive amounts of resources and people. That paradigm clearly is not working anymore, if it ever did in the first place.

Swapna: Power structure – we live in a hierarchical world. If that's gone, it could get us to a place where we're on the same page. Do you feel hopeful about that?

Chris: Lately, not so much. For a while there, I felt like my work was part of a bigger movement that was on the cusp of a big, spreading transformation. I heard lots of names for it and my favorite was the Great Turning. People around the globe were tuning in to the fragility of our world, our interconnectedness with it, our reliance on the balance of nature for our own continued existence, and new paradigms of ethical stewardship and compassionate social justice. These ideas seemed to be catching on and propagating fast, like new grass growing up through an old cracked parking lot. It looked like in a decade or two we would see a deep change in human culture, an evolutionary step that would transform society within our lifetimes.

What I and a lot of other people hadn't anticipated is the incredibly powerful pushback from the boardrooms of corporate America: big oil, big chemical, big coal, military-industrial, factory farming, the finance and insurance industries, Fox News, and so on. They have effectively bullied the American psyche into a paralyzed state of fear and shame, and meanwhile the big deadline is passing. It's such a tragedy. And what for, after all? I suspect all those guys are secretly leading alienated and joyless lives.

Swapna: But what about young people, and the power of education, both formal and informal? Informal learning – it's what your work is about, no? How is it, what responsibilities do we have?

Chris: Well, you could say that education is the big hope, particularly mass communication like television and films. Nothing else can reach so many people so fast. But look at the appalling effect the right wing is having on the media and education right now. They've got us back to arguing creationism and denying the validity of science, while we emit another nine gigatons of carbon into our atmosphere this year.

One thing that's clear is the people who are alive now aren't the ones who are going to bear the brunt of what we're doing. Scientists say it's going to take a couple of hundred years for the disaster to fully take hold. I've sat with the world's leading climate scientist Pieter Tans of NOAA in Boulder and heard the straight scoop about global climate change. He said there is currently enough oil in the known reserves (and we're still drilling) to bring the carbon in our atmosphere up to 2,000 parts per million, assuming we burn it all. And that will alter the chemistry of the atmosphere and the oceans, and change the climate enough to set off an already cascading extinction that is likely to kill off most of life on Earth except for algae and fungus and those scary fish at the bottom of the sea. How do we tell that to our children? It is stunning to think about the destruction we're causing.

And yet, maybe it actually is us who are bearing the brunt, on a spiritual level. Knowing the harm we are doing to our world, and not addressing it, in my view is causing us an extremely deep psychic pain. We can all pretend we're having fun, laugh loud and clink our beer mugs, but our collective failure of integrity is always there, like the proverbial bloody rhino head in the corner of the room. Coming out of our denial looks from a distance like a bad experience, but I think it would be an ecstatic one, if we could find the courage to take the risk together.

And on another plane, when you think more broadly about the beauty and miracle of our existence, about what's actually happening – that right now you and I are embodied conscious beings sitting on the surface of a round planet in outer space, with hundreds of trillions of miles of cold emptiness in every direction, then we can remember that this truly is a wondrous place. Our biosphere is the only known sanctuary of its kind in the Universe, and we're damaging it irreparably for reasons that are very difficult to justify, or even articulate.

From that perspective, the damage we are doing is so far beyond tragic that there aren't even words. When I think about that, the only response I can come up with is to focus on living locally and in the present, and channel my feelings into my work. Do the work and resonate it out there into the field. It feels impotent, but what else is there to do?

Swapna: Do you find it depressing?

Chris: Yeah, maybe. It's hard to admit, I guess. There's a lot of talk about the need to stay hopeful these days, and I have been taking a close look at the nature of my own hope as part of my work on Midway Island. I have found that – for me – hope pretty much turns out to be an empty puff of smoke. It's a weak and disempowering feeling, a kind of passive optimism or faith that things will get better someday, independently of anything I do. It's a moving target, rooted in the future instead of the present; vague and abstract, like happiness – with a not-yet quality to it, always just beyond reach. So with some relief, I'm learning to give up on hope lately, not in favor of negativity or hopelessness, but just trying to live more with whatever I'm experiencing from moment to moment, as uncomfortable as that can be at times. Of course that's all easy to say, but depression is a risk when you make a practice of looking honestly at the state of our world.

Swapna: How do you deal with the sadness of knowing what is coming and the anticipation of all of that? And our roles?

Chris: On a personal level, it's difficult to balance the layers of horror and grief – along with the ironic fact of my own complicity – against the privilege and love and fulfillment I feel in my own life. I love my family, my friends, my work, and the sacred gift of being alive. I am trying to learn what it means to live ethically, as messy and hypocritical as that frequently turns out to be. I want to allow myself to feel joy and celebrate close relationships; and yet a part of me never stops thinking about the plastic in our oceans, those dolphins on the beaches with their ears bleeding, the five million children who are dying right now from malnutrition, the thousands of women raped in the Congo, and so on. And despite everything that's happening, the downward trend, life remains mysterious and wondrous and more beautiful than there are words for. I want my heart to grow big enough to bear it all, but I don't know how. Not even a clue.

Swapna: Have you thought about making books for children?

Chris: I would love to do that. Children really get this stuff. I think they have evolved one step beyond us, and when I get to hang out with kids, I see their minds are pre-wired for the new paradigm in a way that constantly amazes me. There are lots of cool projects I'd love to do if I had more resources to work with. Right now I'm directing a film about my experiences with the plastic-filled birds on Midway Island. What a stunning place that is, a multi-layered mirror for humanity, an acupuncture point for our globe. Stay tuned. . . .

Swapna: Thank you so much for an engaging conversation.

Chris: And you. Cheers.

NOTES

1 Morrison, T. (1984). Rootedness: The ancestors as foundation. In M. Evans (Ed.), *Black women writers (1950-1980): A critical evaluation*, p. 345. New York: Anchor.

2 http://thinkexist.com/quotation/it_is_the_mark_of_a_truly_intelligent_person_to/207497.html.

3 Jordan, C. (April 30, 2009). *Running the Numbers*. Lecture at Portland State University. http://www.media.pdx.edu/dlcmedia/events/AFK/.

4 This British street artist with an international reputation shows and explains some of his best work at www.banksy.co.uk

5 http://www.phillyburbs.com/lifestyle/moms/mom-gives-the-gift-of-breast-augmentation-to-her-year/article_32ab5b5c-92af-11e0-bcdf-001a4bcf6878.html and http://www.huffingtonpost.com/2011/06/09/human-barbie-boob-job-voucher_n_873705.html?ncid=edlinkusaolp00000008.

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FRANK SWETZ

12. TO KNOW HOW TO SEE

The Realities of Learning and Teaching Mathematics

A doting admirer, fascinated by dynamic representations of swirling waters and lifelike depictions of the human form, asked Leonardo da Vinci how he could so vividly capture the essence of his subjects. The master responded, “One must know how to see.” For da Vinci “to see” meant to understand his subjects, that is, to study every aspect of their being, what made them what there were, so he could capture that essence, that dynamic or spirit, in his painting. So thoroughly did he wish to relate to and capture the human form that he undertook anatomical dissections, sketching the relationships of muscle to bone, attempting to understand the mechanics of posture and movement. In da Vinci’s case, his seeing resulted in exquisite detail of his subjects, whether a wheat field swept by the wind, flowing water or the tantalizing gaze of a noble lady (Nathan & Zöllner, 2007). The results of his vision, both physical and accomplished through the power of his imagination, added a transcendental dynamic to his paintings and sketches, eliciting emotional responses from their viewers as they also began “to see”, to feel and to understand the subject.

In daily conversation, colloquial use of the terms “to see” or “seeing” often conveys more than just a visual experience; the word designates the arriving at a level of understanding of a concept. You explain something to a colleague, and eventually they might say “I see”; a curtain has been lifted, a window has been opened for their mental gaze. I have a personal, rather embarrassing, but I believe illustrative instance when I experienced a moment of understanding, an epiphany. As a child, I enjoyed listening to classical music on the radio. In elementary school, I was exposed to music appreciation and learned the various instruments of an orchestra and their functions. My later years found me attending concerts and musical presentations including those rendered by string ensembles. I always enjoy the music, but in viewing stringed instruments, I also retained a nagging question. I wondered why the sound boxes of violins and cellos contained those peculiar concave indentations on their sides? Did it help project the sound, or was it just a creative whim of the instrument’s designer? Even queries to musicians did not produce a satisfactory answer. Finally, well into my 40s, I “saw” the purpose of this design: to facilitate the bow’s contact with the strings. The function of this feature was really so obvious, but remained elusive to me for years, until I saw it in a different light, an operational light.

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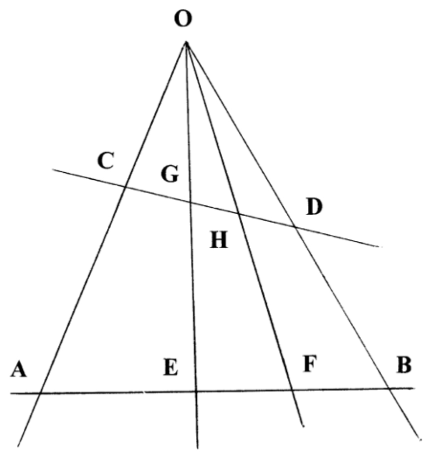

Fig. 12.1 Seeing at several levels is required

A chess master, in entering a competition, must see the game before him or her, that is, visualize, in their mind, the opponent's possible moves and countermoves and devise an appropriate opening and strategy to proceed. So too, it is with mathematics teachers in working with their subject matter and their students. Teachers must "know how to see." Even experienced veterans of the classroom should occasionally stop and relook at both the mathematics they teach and the way in which they teach it and the learning needs of their students. In particular, as they encounter diverse student differences in backgrounds and preparation, the tasks of teaching will become more challenging. These new viewings can take place through windows of pedagogical advancements, cultural understanding, historical enrichment and/or social economic realizations and expose the new landscape of learning. Consider how these new insights might take place.

A mathematical exercise that involves seeing at several levels is shown in Figure 12.1. Here a pencil of lines extends from point O. They are intersected by the straight lines CD and AB. In a usual learning situation, a student may be asked to list all the triangles she sees in this configuration. This task indeed involves the physical act of seeing and the comprehension and recognition of the properties of a triangle. At a more challenging level of seeing, the student could be required to list all obtuse triangles. Even more demanding would be to request the designation of all quadrilaterals present in the figure. Finally, the seeing task could become open-ended, requiring the listing of all polygons contained in the illustration. Thus the teacher and student can see at many levels of understanding in the same diagram.

I am a strong advocate of concrete learning aids. A task I always assign to my mathematics teacher trainees is to bring a small carton to class. Cartons such as toothpaste containers and soap and snack food boxes usually appear. Then, I ask them to tell the other members of the class what mathematical properties they

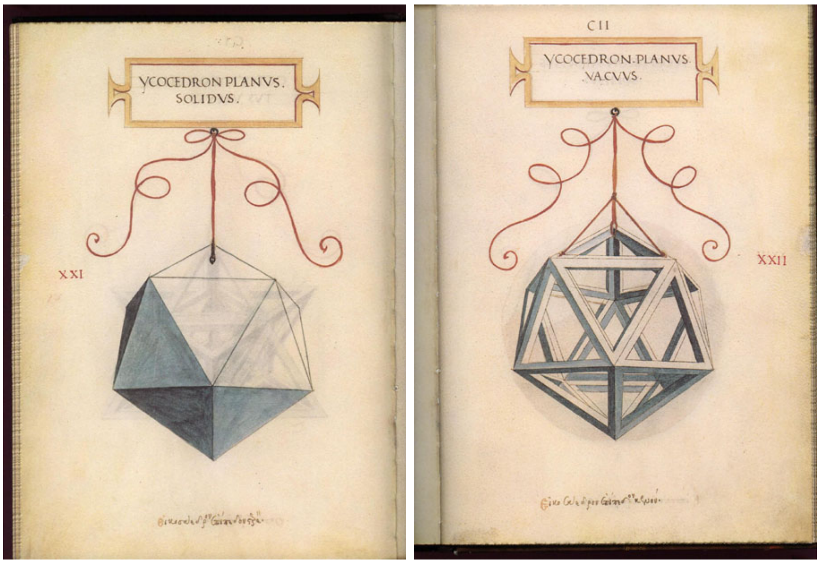

Fig. 12.2 Da Vinci's icosahedron (left) and the net or graph of the icosahedron (right)

could teach using their container. Of course, discussions concerning shapes and volumes are given. An imaginative student may even suggest computing the price/volume ratio of the item conveyed by the box. All is well and good. When the class is convinced that they have exhausted all possible teaching examples evident among their items, I suggest disassembling the containers along their seams. The flat shapes are now laid on a surface and I ask them to describe the patterns they see. Many new shapes emerge. These geometric shapes facilitate the making of the container, as does the placing of the creases, and yet they usually remain unseen. Da Vinci, in illustrating a book for his friend, the mathematician Lucia Pacioli (ca. 1445–1509) sketched a set of the Platonic solids. These particular solids have fascinated mathematicians and philosophers for thousands of years. Da Vinci not only drew the solids but he also drew a net consisting of vertices and edges for each body (Swetz, 2008). He saw beyond the usual properties of solid geometry to view these solids in a topological perspective. See an example of his technique in Figure 12. 2.

One of my favorite "seeing" exhibits is a souvenir of my travels, a small basket, shown below in Figure 12.3. I bring it into a class and let the students pass it around and look at it. Then I ask if they have any questions about the basket. Several inquiries might come forward: Where did it come from? What is it made of? What was it used for? In each instance, the students have "seen" this basket in a different light: origin, structure, purpose. But other viewings are also possible. I wish to use this object to enhance mathematics learning. Do you see any mathematics in it? It is a cylinder with a circular top. The top serves as a lid and

Fig. 12.3 A basket

contains a decorative tiling. When one looks at the tiling, one can see parallelograms, octagons or cubes in perspective embedded in it. As a basket, it has a volume. What is its volume? (The diameter is 5 in. and the height 3 in.) Does this volume indicate anything? The basket is from Northern Thailand and was made by the Mao Hill Tribe. Probably, the person who wove this basket had no formal schooling or training in mathematics. And yet there is much mathematics obvious in its structure and decoration. This is a basket to contain sticky rice; a person's individual serving at a meal. How does this volume of rice compare with the amount of food eaten at a meal in a western home? Insights are provided into the uses of mathematics. People use mathematics to decorate common objects. Students can even duplicate some of the patterns shown in this basket through paper-folding exercises. An exposure to ethnomathematics, the natural mathematics of a people, is achieved (Nunes, 1992). Issues of environmental adaptation are exposed such as the use of a basket as a food container, rattan weaving, and the consumption of rice as a main dietary staple. Through this basket, students can obtain a global perspective on both mathematics and the lifestyle and needs of a distant people.

SEEING THE MATHEMATICS

Too often, we teach the textbook, assign the conveniently listed problems and exercises, and move on. But in a learning sense, is this meaningful or really busy work? The perennial student questions: "Why should I study this?" and "What is this stuff good for?" remain. Can we give deeper meaning to the mathematics by relating it to current affairs, local developments, historical background or cultural relevance? African American students may show more interest in the learning of geometry if they are exposed to the patterns of Kinta cloth, batik prints from West

Africa, or network designs devised by the Bantu (Gerdes, 1999). Hispanic students with Latin American backgrounds may take pride in the mathematical accomplishments of their ancestors as revealed in: Aztec calendars; Mayan number glyphs (Closs, 1986), including a concept for zero and the Inca number storage device the quipu. Environmentally conscious students may build upon the knowledge that people in Southeast Asia measure trees by holding hands around them. A sepemelukis is the arc of circumference grasped by a human hug (Swetz, 1985). The techniques of these original "tree huggers" can be employed to measure local trees. Origins for common units of measurement taught in the school can be investigated and discussed. Who would have thought that the units of measure "the pint" and "the quart" are based on "the mouthful" (Klein, 1974). A system of rice measure employed in Southeast Asia originates in the jemput, a pinch of rice taken by the fingers of one hand, and advances to the cupak, the capacity of half of a coconut shell. Such associations make complete sense when recognized. Archaic units of measure that possess historical or cultural significance can be sought out and explored. The hogshead was a common unit of volume measure in Colonial America. It was used to import rum from the Caribbean; in turn, the rum trade was closely associated with the slave trade. How big is a hogshead? Why is our unit of measure for weight, the pound, abbreviated by "lb" rather than the letter "p" which would seem appropriate? Answers to such questions are telling, they reinforce an understanding of the uses of mathematics and its implications.

Mathematics is just not invented. It does not drop down from the sky in complete form but has had a long and involved human history. Much effort has gone into its refinement and adoption. Perhaps it would be encouraging for students who are experiencing difficulty with mathematics to learn that such great men as Johannes Kepler (1571-1630) struggled with the problem of determining planetary orbits for fifteen years before formulating his Laws of Planetary Motion, and the scientist Albert Einstein (1879-1955) once failed a school mathematics examination. In 1993, at a conference in Princeton, New Jersey, British mathematician Andrew Wiles revealed a solution to Fermat's Last Theorem. This problem had been posed by the French scholar Pierre de Fermat (1601-1665) in about 1637. It appears deceptively simple: given the equation $X^n + Y^n = Z^n$, (where X, Y, Z, and n are positive integers) show that no solution is possible for $n > 2$. In the intervening centuries, the world's greatest mathematicians could not solve this problem (Singh, 1997). Wiles worked on the problem for over seven years before arriving at his solution. Mathematics poses a challenge and requires persistence. Even simple concepts, the ones which we readily accept and use today, may have had a long and difficult period of gestation. The "Hindu-Arabic" numerals, after being introduced into Europe, were considered suspect for over 500 years. They were foreign symbols from India and pagan, that is, used by the Muslims and Hindus. Historically, many schemes or algorithms were developed to accomplish longhand multiplication by the use of pencil and paper. Students would be amused to view some of these schemes, learn their fanciful names and even to try their hand at multiplying two three digit numbers using "the grating" or "by the

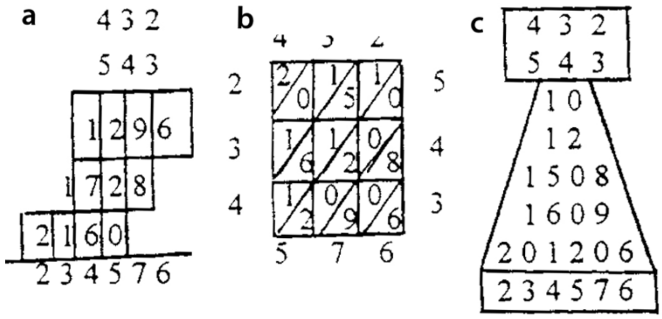

Fig. 12.4 Multiplication algorithms of the 13th and 14th century. a. By the chessboard. b. By the grating. c. By the bell

bell" (Fig. 12.4). Viewing mathematics in a cultural or historical light can make it more meaningful both for the teacher and the students.

Suppose a new student from Cambodia arrives in middle school in the USA. She performs division of common fractions differently from that which is known and practiced by the class. Given the problem, 2/3 ÷ 1/4, she finds a least common denominator for the two fractions, 12, converts both fractions to twelfth's and performs a division with the numerators, obtaining an answer of 2 2/3. That is: 2/3 ÷ 1/4 = 8/12 ÷ 3/12 = 8 ÷ 3 = 2 2/3. Is she correct? How should the teacher respond? Should the teacher enforce "the correct way" to perform this operation, one used by the class and accepted as a standard method, invert the divisor and multiply, or accept this new method? Which method has more mathematical merit? Should this new method be shown to the rest of the class? These are all issues that the teacher must confront in seeing this new situation.

As a teacher, do I see mathematics, and thus plan my lessons – through the textbook, the required examinations, or the level of student understanding and performance? Certainly, I would choose the last and reconcile it with the other standards that must be met. This might require employing a different teaching strategy than that followed in the textbook. An activity might supplant a reading. For example, a trigonometry class, after being exposed to the concept of the tangent function, could be required to obtain remote outside heights. By taking actual measurements, they could determine the height of the school building, available flagpole, or telephone pole. Trigonometry evolved through measurement of distance, what is more natural than to teach it by replicating the activities that spawned it? Where possible, students should experience applications and see the usefulness of what they are learning. To see mathematics is also to see the mistakes one can make while working problems. In any problem-performing situation where students make mistakes, in order to correct those mistakes, they must first see that they are wrong and must be able to see how to obtain the correct answer. Similarly, a teacher grading student exercises must see the mistakes made and decide on the seriousness of the mistake. Is it due to carelessness or confusion of concepts? Is this mistake prevalent among other class members? If so, what is wrong? Seeing such a situation as a teacher I would review how I taught the material and how the particular examination exercises were phrased. Is the mistake due to something I did or omitted to do? If so, how may I remedy the situation? Consider the following examples of student mistakes. In elementary algebra class, a student is required to reduce the expression $(x^2 - 49) / (x - 7)$ and show the necessary calculations. The student factors x into x^2 and 7 into 49 to obtain the answer x + 7.

Correct method:

$$x^2 - 49 = (x - 7)(x + 7)$$

so

$$(x^2 - 49) / (x - 7) = x + 7$$

Student's method:

$$\overset{x+7}{\frac{\cancel{x}^2 - \cancel{49}}{\cancel{x} - \cancel{7}}} = x + 7$$

This is the correct answer, but in analyzing the procedure we can see that the student does not understand the factoring of binomials and the reduction of algebraic fractions, the very concept that is being tested by this task. In another instance, a student must demonstrate knowledge of basic trigonometric functions. Given triangle ABC where angle C is a right angle and the measure of angle A equals 32°, side BC of the triangle equals six units. Find the lengths of side AC and AB. The student's work is as follows:

M (Θ) = 32^0 , sin*32* = 6/AB AB= 6/.5299 = 11.32
[Diagram] Here*,* tan*32* = 6/AC AC = 6/.848 =7.075
[Mistake! The value of tan 32 should be .62487 and AC = 9.548]

In analyzing the situation, it appears that the student looked at the problem and realized that to obtain the solution requires use of tangent and sine functions. It appears that in carrying out this task, she confused the reading of tangent 32°. This seems to be a mistake of carelessness in confusing values and should be considered as such.

SEEING THE STUDENT

How do we see our students? Do we see them superficially, by their appearance, rather than more deeply through their words and actions that better reveal their backgrounds and interests? How does their life experiences, socioeconomic status or cultural background impact on their interest in mathematics in school learning

(Lamb & Fullarton, 2002)? Here is another personal anecdote to illustrate the easy pitfalls a teacher can encounter in working with students. While supervising one of my students in a middle school mathematics class, my attention was drawn to a particular student. He looked rather rough and stood out from his peers. This visual impression led me to believe that he might be a troublemaker. I shared my impression with the student teacher as a warning. I developed a bias and transmitted it. While driving home that afternoon, I thought about this incident, and suddenly I saw my mistake. I did not look deep enough. This boy looked shaggy; he badly needed a haircut and wore a faded flannel shirt, threadbare at the collar. He was simply a poor boy among more affluent peers, and, because of this, he stood out. When, as a young boy, I often went to school needing a haircut, awaiting my father's attention, the barbershop being too expensive. I also wore faded or "hand-me-down" clothes. In that classroom, I was seeing myself as a youth. That night, I telephoned the student, explained my ill-conceived evaluation, and confessed my shame for the suggestions I made concerning that young boy.

Another socioeconomic impact on a teaching situation occurred when I was supervising student teachers in New York City's Harlem, in the 1960's. One teacher shared her frustration with me concerning a lesson on fractions. She intended to associate the topic with daily life activities by discussing a pound of butter being divided into quarters. She asked the class, "When you purchase butter in a grocery store, how many pieces are there in the package?" To her amazement, they responded "five" and the conclusion emerged that a pound contained five quarters. Since so many students in the class held the same opinion, she was baffled. They were strong in their conviction. This error occurred because of the particular choice for the example, butter, and the timing of this discussion. First of all, most students in this neighborhood did not eat butter because it was too expensive. They used a cheaper spread, a form of margarine, but referred to it as "butter." At this time, a major margarine manufacturer conducted a promotion by selling 5/4 pounts of their spread in a package. The buyer received a bonus of a quarter pound stick. Students, believing this package contained a pound, concluded that there were 5 quarters in the pound. This situation recalled for me an incident where I made a similar socioeconomic faux pas. For an eighth-grade class, I devised a lesson involving consumer activities: the purchasing of fuel oil for home heating. My class consisted of inner-city students from poor homes. They did not heat their houses with oil, but, at this time, used coal. This incident highlighted for me the need to know my students better, to attempt to see them in their home settings and neighborhood environments.

A large university that adopted an open enrollment program soon encountered the issues of accepting students with inadequate or minimal academic preparation. In particular, basic mathematical skills were lacking among a majority of the new student population. The question of mathematical remediation arose. Many of these students were older, and had enjoyed careers or partial careers in the work force or military service. All of them had already been taught mathematics in their school careers and had probably undergone remedial teaching procedures, perhaps several times, without experiencing success. The usual type of remediation, call it "special tutoring," or attendance at a "mathematics laboratory," would be recognized by the students as being unhelpful and even demeaning: "Yet again treating us like kids." But still they had to be trained in mathematics as many of them now sought college careers in business or engineering fields. Recognizing this situation, those students known to have a weak mathematical background were required to take a special statistics course. Now, intellectually, they felt they were not stepping down but stepping up. For them, statistics was a higher mathematics, if you will, more glamorous in its appeal, and since it was frequently referenced in news media, possessed a contemporary relevance. In the introduction of the basic statistical concepts, mathematical skills would be revised, taught once again and reinforced. In learning about means, medians, modes, and standard deviations, these students would be required to organize and process data and be forced to do computations. Perhaps under a deception, remediation was undertaken by "better seeing" the students' learning positions. They were accommodated. Their mathematical skills were strengthened and the program was a success.

A similar academic strategy can be employed with middle school students by introducing them to a mathematical activity outside the usual school environment. Let them play the stock market. A local financial management organization will gladly supply a speaker to explain the workings of the stock market to a class: the process of investing, buying stocks, and the necessity for diversification. Then each student in the class will be given a virtual amount of money to invest, $10,000, which will be used to purchase no more than five stocks. They must build a portfolio and manage it through the school year: buying and selling as desired. The paying of commissions and taxes must be provided for from the given finances. They will be encouraged to read about the stock market in investment magazines, and to consult daily stock listings in newspapers. These items are available in the school's library. Periodically, each student will report on their financial standing. Their reports should employ graphs and include percentage gains or losses. Within the class a natural competition will arise as to see who can earn the most money and perhaps a winner can be officially recognized with a small prize at the end of the school year. For the usual urban student from a working class background, the stock market is for rich people and their ability, even if theoretical, to participate in the stock market and begin to understand it is very empowering knowledge. This empowerment will drive a spirit of personal and class competition that will promote mathematical activity.

Do we underrate our students? I think, in many instances, we expect less of them than what they are able to do. I have found that by issuing small challenges, problem-solving situations that I have left fairly open-ended, students have responded. For example, after discussing the value of pi, and using it in some simple computations, I have required a class working in small groups to measure a series of round objects obtaining their diameters and circumferences and arrive at a numerical estimation for pi (Swetz, 1994). The values for each student group are displayed and discussed as to the possible reasons for differences. Finally, the varied results of the class efforts are averaged and compared to the known value for pi (to some level of precision). In a similar manner, simple modeling exercises can be assigned for group activity. Modeling situations may involve working with a set of data, organizing it and drawing mathematical solutions (Swetz & Hartzler, 1991). For example, one easy exercise I have assigned several times is for the class to predict rising sea levels for a future period using a collection of measurements made over past years. After completing this task and understanding the mathematical implications revealed, the students begin to appreciate the concept of global warming. They had heard about it, and now they began to realize its effects. If they employed graphing to solve this problem, they can see the future results of sea level rise. The crux of a modeling situation is that the students must decide what to do in order to obtain the desired result. They make the decisions, decide on the necessary calculations, draw conclusions, and present and justify those conclusions to others. I have found the results of such teaching very satisfying both from a teaching standpoint and a learning standpoint. Usually, the students appreciate the opportunity for exploration and the aura of self-responsibility. After working through such a modeling exercise, one student commented, "Now that's doing real mathematics!" Can a teacher want more?

Cultural conditioning influences how and what people learn. In gender studies involving mathematical performance and problem solving, mathematics is often described as an aggressive discipline: the proof is found, a correct answer is arrived at, and this is the way it is. No discussion. But this sort of activity often does not fit the life patterns and thought processes of many people. While teaching in Asia, I was frustrated by the lack of student classroom participation, and the reluctance of students to discuss or demonstrate their work. In meeting with students privately, they would not look me in the eye and rather stared at the floor during our conversations. Finally, I discussed my concern over behavior with a local colleague. He said, "They have shown you respect. By looking downwards, they are humbling themselves before you. To look you in the eyes would be very brazen." As for this reluctance of responding in class, this is face-saving where a wrong answer may bring personal shame and reflect on one's family. Self-image is very important, not only in Asian populations but to all learners. The concept of self-image affects the way students learn; but it also may affect the way students are taught. A self-supporting cycle can be established. In my foreign travels, I have often been dismayed by observing university lecturers teaching concepts of higher mathematics to rather unsophisticated students at inappropriate levels. The teaching was complex and obscure, ensuring the status of the "learned professor" by erecting an unnecessary wall of difficulty between the students and the material they were supposed to learn. When a question did arise from the student audience it was rebuffed or dismissed as an expression of ignorance or poor attention, the fault of the student. I would think that this is what teaching is all about: answering questions of ignorance, but to these lecturers such questions represented a threat to their status of authority.

Another facet of problem solving that I have observed is that some people first consider the implications of the problem before they attack the problem. They work from the outside in, basically saying we have these conditions, and it is traced to this problem. To better understand, remove or alleviate these conditions, we must solve the problem. Others, including Americans, start with the problem, and then consider its implications. They work from the inside out. It seemed to me in working with students that have ambivalent attitudes towards mathematics it will be more meaningful to work from the outside in. We should justify the mathematics before we do it, not after it has been done. In many instances, affective teaching is more productive than cognitive teaching. Students must be convinced that what they are learning is important and will benefit them personally. Authors of early mathematics textbooks were always careful to clarify the usefulness of the mathematics before they began their formal expositions of it. For example, the author of the Rhind Papyrus (1650 BCE), one of the oldest known mathematics textbooks, entices his reader with the promises that his examples will provide “insights into all things” and “a knowledge of obscure secrets” (Burton, 1997, p. 35). The Chinese scholar Sun Zi, in the preface to his fourth century CE Mathematical Classic, admonishes his students that “mathematics has prevailed for thousands of years and has been used extensively without limitations. If one neglects its study, one will not be able to achieve excellence and thoroughness” (Lam & Ang, 1992, p. 151). Similar testimonies of importance and usefulness were contained in American textbooks up through the 19th century. Unfortunately, this encouragement to study mathematics is no longer so forcefully supplied in modern textbooks: it must be promoted by the teacher. Sometimes I feel we teach too much mathematics and not enough about it.

In general teaching, the conflicts of self-image and limitations of cultural conditionings can be alleviated by the use of small group learning and reporting strategies. Individuals are protected and supported within a small group or team. A small group of peers supplies a community of support and encouragement and provides non-threatening means for mutual learning. Of course, individual assessment must take place, but the impact of this requirement can be lessened by the prevailing group mentality. A group will strive to outperform the other groups. Individual test scores of members can be averaged to represent group or team performance. In this way outcomes could be compared with that of other groups. Such a strategy encourages peer tutoring and support in this scheme: the teacher becomes an interactive, nonthreatening guide. The group dynamics sets standards and encourages performance.

Working with a particular ethnic or racial group of students, teachers should have some feelings for this group’s cultural backgrounds and priorities (Cuevas & Driscoll, 1993). For example, in an early educational experience, a teacher may be frustrated with difficulty in establishing the principles of serial ordering necessary for the introduction of ordinal counting. Lining up a collection of objects and requiring a child to designate the first, second, and so on in this sequence may result, as far as the observer is concerned, in a wrong answer. The student subject exhibits an inverse ordering, choosing the first object closer to the observer and further from him or herself. Again, this phenomenon may be culturally ingrained as a ritual of respect. The question should then be asked, “First in relation to what or whom?” A reference system must be established that both the teacher and the learner agree upon. Many cultures have different traditional reference systems:

Chinese will say there are five directions; Zuni people of North America use a reference of seven directions. In a truly spatial and mathematical sense, these people are more correct than their Western or technological savvy peers. The Chinese rely on the four compass directions South, North, East, and West and "here" the location in which they stand; the Zuni follow the same principle, only they include the directions of "up" and "down." There are even traditional tribes in the Amazon that reference locations and directions in terms of a circle – their houses are circular, and their living plots are circular. For these people, for purposes of orientation, a system of polar coordinates would be more meaningful than that supplied by the standard rectangular system (Lizarralde & Beckerman, 1982). Every person views reality within their own world view built upon cultural and environmental conditioning. Students from "non-carpentered societies," that is, where the existence of square corners on manmade structures are exceptional, will have difficulties adjusting to straight line perspectives as required in isometric illustrations (Mitchelmore, 1980). Some cultures are more exact in their mathematical considerations, for example, when estimating time or distance.

Cultures develop and refine those systems of mathematical comprehension that are important to them in their living circumstances. Thus, a so-called "primitive person," a jungle dweller, may in some respects be more mathematically sophisticated than a civilized counterpart, for example, regarding the estimation of travel time by foot over rugged terrain. In this respect, an informative and worthwhile class exercise would be to occasionally examine how other people do their mathematics (Ascher, 2002). Considerations of the mathematics of measurement, the use of geometric patterns in decoration, or the estimation of distance and time, would be quite revealing. Inhabitants of the island of Borneo speak of small distance as "monkey paces, setapak" and for very minute measurements refer to the thickness of their fingernails.

Mathematical practices vary among peoples, and even within generations. Recalling and examining some of these practices is a worthwhile mathematical exercise. For example, the class of students may be asked to interview their grandparents to learn some of the mathematical techniques they used in their lives that are no longer practiced. I remember, as a child, my mother sizing socks for me by wrapping them around my clenched fist. If the sock circumscribed my fist, it fit my foot. A similar situation takes place today in a different regional setting. Filipino mothers, in the absence of dressing rooms in stores, measure their child's waist by using their neck. In sizing pants for their sons, these mothers will place the waistband around the child's neck. If the waistband circumscribes the neck, theoretically, the pants will fit. I remember a physical task required of a child to determine if they were developed enough to enter kindergarten. The child would be required to reach over their head with one arm and touch their ear on the far side of the head. Are such estimation techniques accurate? Simple classroom exercises could verify these practices. In introducing or refining the concept of measurement to students, cultural-historical examples can be built upon, such as the inch evolving from the width of a man's thumb. The extensive concept of hand/body measurements: foot, yard, hand span, cubit, and fathom can be discussed and demonstrated. A simple exercise would be to construct a ruler on the basis of the thumb width of the student or her/his parent; class comparisons should lead to the necessity of standards for measurement. Whose thumb do we use? In the Middle Ages when a king ruled people, whose thumb set the standard? A class would readily, and correctly, suggest "the king." King Edward II of England set an inch as the "length of three barley corns, round and dry, placed end to end." In a few occasions a more democratic approach to setting a standard was followed: in 16th century Germany, the measure of a "foot" was determined by measuring the lengths of 16 male, Sunday churchgoers, and taking an average. Setting standards in mathematical and scientific fields requires an authority. Through such discussions, students begin to understand the worldwide applications and need for mathematics. Its utility becomes more appreciated as its mysteries diminish.

SEEING ONESELF

Most teachers are constantly re-evaluating their performance. Unless one is satisfied to teach the textbook, they are always trying to improve their understanding of both the mathematics they teach and the learning needs of their students (Hill & Lubienski, 2007). Granted, in today's teaching/learning environments there are many nagging distractions that diminish this effort. But still, occasionally try to be objective and look into your relationship with teaching and your students. Are you an easy prey to stereotyping, accepting some students as "naturally lazy" or hostile to school learning rather than evaluating that student as an individual in your classroom and under your directions? A student may relate to one teacher differently than to another. If you primarily teach students of one ethnic and/or cultural background, it would behoove you to learn something about their backgrounds and interests. Get to meet their parents if possible. Visit their neighborhoods, learn the concerns and particular interests of the students in question, and occasionally incorporate these concerns and interests in class lessons. Working with migrant farm workers, I would be aware of the fluctuating prices of the produce they work with: these prices will determine their family earnings. If the picking price for tomatoes decreases by two cents a box, how much less will a family of tomato pickers earn in a week? If the price of gasoline increases by $0.10 a gallon, how would this price change impact on a family of migrant workers who depend on their old pickup truck for transportation? The teacher is part of many communities; at least one of which he or she shares with her students. The better a teacher knows that community, the better he can teach to its needs and its children. Occasionally, ethnic groups conduct special festivals or block parties. Your presence at such an event will be noticed and appreciated as an effort in reaching out to understand that community.

Perhaps you are bored with the mathematics that you teach. You need some new perspectives. Mathematics has a story to tell. It has a long history of people and achievements. To know that history provides both an appreciation of the mathematics itself in the knowledge of the diverse people and conditions that contributed to its development. This diversity can be brought to the classroom through simple discussions and the use of anecdotes: for example, by reviewing the saga of how our use of common fractions that originated in China was adopted by the Arabs and introduced into Europe by Fibonacci, an Italian merchant. Or, discuss the sacred concept of the circular orbits of the planets established by the Greek astronomer Ptolemy (ca. 100–178); and modified, challenged, and corrected by such people as the Damascene astronomer Ibn al-Shatir, the Catholic monk Copernicus, and the Renaissance scientists Johannes Kepler and Galilei Galileo. The conflict of science and religion that sometimes arises is illustrated by Galileo's trial by the Catholic Church. Actual problems from different historical periods can be assigned and discussed as to what they tell us about how the people lived. In different places and at different times, who was allowed to do mathematics? Was mathematics a privileged activity limited for use by priests or philosophers? Why during the 15th and 16th centuries did the European attitudes towards mathematics change? What caused this change? Why, through much of history, were women not expected/permitted to do mathematics? Why did the magical concept of numbers arise with such designations as: lucky numbers, unlucky numbers, perfect numbers, friendly numbers and female and male numbers? To understand and stress the continual human involvement with mathematics is most important for teachers and their students.

CONCLUSIONS

Sometimes "seeing" and really understanding is very difficult, as illustrated by my personal violin story. Frequently, even the most obvious features of a subject can remain unrecognized. Sometimes we just don't want to see it. Mathematics teachers should talk to their colleagues who have different insights into teaching mathematics. They should also talk to their students on their reactions to certain lessons or the teaching of particular concepts. Which did they like or dislike? Workshops and professional meetings provide opportunities for the exchange of ideas. Attending presentations and interacting with a larger population of teachers can often open a new window of understanding. Much teaching enrichment comes through the interchanges of experience and opinions. In the 1970s, an educational claim was that the learning and teaching of mathematics was "culture free" (Davison, 2007). As a result, all children could learn mathematics regardless of their backgrounds. All could succeed in this subject. Then it was felt that students overcoming failure in one school subject [mathematics] would transfer renewed efforts to succeed in other subjects. Finally, success in mathematics would encourage such populations as inner-city children, recent immigrants, and others to experience achievement and learning satisfaction. This ill-conceived notion ended in failure. Mathematics and all school subjects are affected by the ethnic, socioeconomic, and cultural backgrounds of the children in question. Students will, at best, only learn what they want to learn, what they feel is relevant to them, and what resonates with their backgrounds. Thus as teachers, we must discover and build upon that relevance. Unless we are willing to look, to see, to understand and to act, the fruits of our teaching efforts will remain limited. As da Vinci also said, "There are three classes of people: those who see, those who see when they are shown, those who do not see." Let us not be in the last group.

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PART IV

MATHEMATICS EDUCATION FOR SOCIAL JUSTICE

> This is a great discovery, education is politics! After that, when a teacher discovers that he or she is a politician, too, the teacher has to ask, What kind of politics am I doing in the classroom? That is, in favor of whom am I being a teacher? By asking in favor of whom am I educating, the teacher must also ask against whom am I educating. Of course, the teacher who asks in favor of whom I am educating and against whom, must also be teaching in favor of something and against something. This "something" is just the political project, the political profile of society, the political "dream." (Freire, 1987, p. 46)

A seminal, and pioneering, study reports the re-invention of a Freirean pedagogy in the context of teaching mathematics to working-class adults in Boston (Frankenstein, 1983). In reading this account, it seems that fundamental ideas in Freire lent themselves very naturally to this recontextualization – to the point that it seems strange that Freire never explicitly wrote about mathematics education (D'Ambrosio, 1997). In his approach to teaching literacy through raising political consciousness around generative themes from the lived experiences of his students, Freire surely must have been aware of quantitative aspects of those experiences. An explanation of this apparent paradox may lie in Freire's view of mathematics shared in an interview that opened the 8th International Congress of Mathematics Education in 1996 in Sevilla:

> In my generation of Brazilians from the North-East, when we referred to mathematicians, we were referring to something suited for gods or for geniuses. There was a concession for the genius individual who might do mathematics without being a god. (Freire, D'Ambrosio, & Mendonça, 1997, p. 8)

In any case, in the spirit of Freire's insistence that his pedagogy be not copied, but *reinvented in context*, the relevance of his work and philosophy to mathematics education has been recognized, most notably by the two contributors to this part of the book. Frankenstein has worked on mathematical literacy, in particular in relation to the critical analysis of data and quantitative arguments. One central aspect of her work is to make clear that, while much of mathematical modeling uses complex technical mathematics and is not easy to understand for non-specialists, there are plenty of cases where the mathematics per se is within the grasp of an intelligent adult, and the key requirements are a critical disposition and a sense of agency. The current situation in the USA, at the time of writing, affords a clear example. The mathematics of gross inequality in the distribution of wealth and earnings, and particularly the acceleration of this inequality in recent decades, is simple to communicate and understand, yet only since the start of the Occupy Wall Street protests in September 2011 have the brute facts become part of the general discourse. As amply illustrated in chapter 13, Frankenstein puts a lot of work and ingenuity into finding graphical, pictorial, photographic, auditory, narrative, and other ways to communicate quantitative information that can be used to better understand a given situation. And, as she makes clear, the form in which we put numerical information fundamentally affects how easily it can be grasped, and hence it is interpreted.

The influence of Freire is also explicitly apparent in the work of Eric (Rico) Gutstein, for example in the title of his book *Reading and Writing the World with Mathematics* (Gutstein, 2006) which echoes Freire's themes of "reading the world," that is, understanding situations, and "writing the world" i.e., acting upon it so as to change it, encapsulating Freire's belief in the essential dialectic between reflection and action. Gutstein's description of the context within which he works exemplifies growing awareness that much of the literature on mathematics education ignores the reality of the worlds in which the students live, a reality that includes poverty, violence, and social injustice – and resilience and resistance – in many parts of the world, by no means limited to the "undeveloped" world.

In terms of curriculum development, Gutstein aims at a balance among three forms of mathematical knowledge, that he labels Classical, Community, and Critical. Classical mathematics is the familiar academic mathematics that, to a great extent, dominates mathematics education in most classrooms. Community mathematics draws on the knowledge of the community. Critical mathematical knowledge refers to the application of mathematics in critiquing, and acting upon, social and political issues. The development of curriculum within these guidelines is complex. A central organizational framework is provided by the Freirean notion of "generative themes" that are chosen after discussion, and often suggested by the students themselves. For example, one year Gutstein taught a twelfth-grade class that, through dialogue, agreed on five themes: analysis of election data from the 2008 U.S. presidential election, displacement of population in the course of gentrification within the students' communities, HIV/AIDS, criminalization, and sexism (Gutstein, 2012). A leitmotif running through these themes is the importance of interrelations among race, class, and gender.

The complexity of what Gutstein is attempting is illustrated by attempts to model the spread of HIV/AIDS (Gutstein, 2012). He openly describes how it proved very difficult to mathematically model such phenomena with the students, since the most appropriate methods involving differential equations are beyond the students' range. In this untypical case, the class, for a few days, resembled a social studies class without mathematics. On other rare occasions, the students did mathematics without direct connection to sociopolitical issues. Mostly, however, the work achieved a balance in "the use of real world activities that promote students' learning about their social world *while* they are learning mathematics and, at the same time, learn about mathematics *while* they are engaging with real world activities" (Atweh, 2012). In the work of both Frankenstein and Gutstein, the political stance is overt – in contrast to the not uncommon claim that mathematics and mathematics education are politically neutral, reflecting the "ideology of no ideology." Likewise, much of the writing in this book is explicitly political (and the rest implicitly so).

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MARILYN FRANKENSTEIN

13. QUANTITATIVE FORM IN ARGUMENTS[1]

Many people argue that too much data about the injustices in our world make us numb to the realities of those situations in people's lives. Wasserman's editorial cartoon about the Israeli occupation of Palestine makes this point[2]:

However, we do need to know the meaning of the numbers describing our realities in order to deepen our understandings of our world. One of my proudest academic moments was when *The Nation* published an edited version of the following letter I wrote responding to an article by Howard Zinn, in which he argues that the numerical descriptions of the deaths from the US war on Afghanistan can obscure those horrors.

> February 11, 2002
> To the Editor:
>
> Howard Zinn's article ("The Others," February 11) is a powerful reminder of the horrors that are perpetrated in the world on all the days in addition to 9/11. I, too, cried as I saw the portraits of the 9/11 victims. I, too, was crying not only for them, and not only for the victims of the wars the USA and other powers create, but also for the millions who die every year because of

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 283–297.

> economic terrorism – from unsafe working conditions that kill "by accident," to unjust working conditions that result in death from preventable causes such as hunger.
>
> Binu Mathew reports in Z magazine (January, 2002) that at this point in time the original death toll of 8,000 caused by the Bhopal gas leak at Union Carbide's factory has increased to 20,000, growing every month by 10-15 people succumbing to exposure-related diseases. Union Carbide management delayed sounding the public siren for 15 hours, and continues to obstruct full revelations which would have helped decrease some of this horrific toll. I, too, wondered, if detailed, in-depth TV and newspaper portraits of these victims, and of, say, the 12 million children who die from hunger every year (Moore, 1998), would wake up our collective consciousness.
>
> Zinn makes another important point that I stress with my Quantitative Reasoning classes at the College of Public and Community Service (University of Massachusetts/Boston): statistical data can distance us from a deep empathy and understanding of the conditions of people's lives. Of course, the data are important because they reveal the institutional structure of those conditions. But, also, quantitatively confident and knowledgeable people can use those data to deepen their connections to humanity. Those 12 million children are dying faster than we can speak their names.

In essence, the quantitative point of my letter is about the form in which we put numerical information.[3] Changing the form can help us make sense of quantities whose significance we cannot grasp. Changing the form through basic calculations can allow us to feel the impact of those quantities through better understandings. Further, knowing the most effective form in which to present those quantities in arguing for creating a just world is an important skill in a critical mathematical[4] literacy curriculum. I suggest that knowing the most meaningful quantitative form in which to express information is necessary to understand what is going on.

CHANGING THE FORM TO UNDERSTAND LARGE QUANTITIES

In 2010, the American Friends Service Committee and the National Priorities Project sponsored a contest for youth to make a short video with the theme "If I had a trillion dollars." The goal was to have young people who "are affected by the ongoing costs of the wars and our federal budget priorities … but are often not part of the conversation or the anti-war, pro-peace movement," present their ideas for spending what at that time had been the estimate of the costs of the wars on Iraq and Afghanistan.[5]

One of the two national contest winners was *AmplifyMe*, a non-profit group that uses "[P]op culture to inspire people to think differently about their power to create change in their lives and the lives of others through civic engagement." The kinds of projects that the youth wanted to spend this money on included: "a new medical center for people who don't have access to health insurance or people who just can't afford it," "a center for the homeless and underprivileged to reconnect with

Table 13.1. Key domestic programs vs. major military programs fiscal 1998 budget

Domestic Program	*Military Program*
Home-heating assistance for low-income families ($1 billion)	Cost of 1 Arleigh Burke destroyer ($1 billion)
Raise Pell grants to $3000 ($1.7 billion)	Cost of 1 B-2 bomber ($2.1 billion)
Head Start for young children ($4.3 billion)	Cost for 1 Seawolf attack submarine ($4.3 billion)
Drug prevention programs ($2.2 billion)	Request for F-22 fighter program ($2.2 billion)

(Center for Defense Information, quoted in Herman, 1997, p. 43)

society and also to get food and appropriate clothes," and "a different type of community center to keep the young teenagers out of the street and prevent gang violence." Only one youth suggested a project that would use up more than a miniscule fraction of the $1 trillion: "I would donate the trillion dollars to better our economy to have more jobs available." This lack of perception is because most of us have no idea of how large $1 trillion really is; and, although most of us will never personally deal with these sums of money, collectively we are spending these sums of money on our wars, with no concrete idea of how much that war spending is stealing from our human services.

– But, by changing the form of numbers, we can get some idea of how large they are. Ask people to guess, without calculating, the answers to the first two questions to get a sense of how little sense we all have about the meaning of large numbers. About how long, at the non-stop rate of one number per second, would it take to count to one billion?[6]
– About how long, at the non-stop rate of $1,000 per hour, would it take to spend $1,000,000,000?[7]

And, then, of course, to get a sense of the size of 1 trillion, you need to multiply by 1,000.[8]

In terms of what human services $1 trillion can buy, one of the largest and well-equipped community health centers in the country is currently building an expansion at the cost of $24 million. So, building one such health center would use up 24 ten-thousandths of one percent (0.0024%) of the $1 trillion dollars. Or, for $1 trillion we could build over 40,000 such facilities. To put this number in a more realistic context, as of 2007, there were about 19,500 municipal governments in the country.

The political and real meaning of the dollars we spend on wars is also revealed in various other specific comparisons (Table 13.1)[9]: in 1997, for example, was "a B-2 bomber worth more than twice the $800 million currently being saved by cutting 150,000 disabled children with 'insufficiently severe' disabilities? Is $248 billion for the military and $31 billion for education a proper balance in the use of federal funds?" (Herman, 1997, p. 43).

Further, other seemingly large numbers are not so large when placed in context, in a different quantitative form. Without proper context, people's lack of quantitative understanding results in their voting against their own interests (Baker,

Table 13.2. What does he mean?

WHAT DOES HE MEAN?

"Our troops … will not be asked to fight with one hand tied behind their back."
– President Bush, national address, January 16 [, 1991]

Tons of bombs dropped on Vietnam by the U.S.: 4,600,000
Tonnage dropped on Cambodia and Laos: 2,000,000
Tonnage dropped by the Allies in World War II: 3,000,000
Gallons of Agent Orange sprayed: 11,200,000
Gallons of other herbicides: 8,000,000
Tons of napalm dropped: 400,000
Bomb craters: 25,000,000
South Vietnamese hamlets destroyed by the war: 9,000 (out of 15,000)
Acres of farmland destroyed: 25,000,000
Acres of forest destroyed: 12,000,000
Vietnamese killed: 1,921,000
Cambodians killed (1969-1975): 200,000
Laotians killed (1964–1973): 100,000
Vietnamese, Cambodians & Laotians wounded: 3,200,000
Total refugees by 1975: 14,305,000
American troops who served in Vietnam: 2,150,000
American troops killed: 57,900

Note: From *The Nation* (February 18, 1991)

2005). For example, in the mid-90's a Kaiser poll found that 40% of people ranked welfare for those with low incomes as one of the two largest items in the federal budget – the overall number was under 4%, even according to the government statistics that mislead in a way that underestimates human services (as compared with 22% for Social Security and 18% for military spending). Baker blames the misperception on the way the numbers for welfare are reported – $16 billion for a core welfare program sounds gigantic, but in context it is only 0.6% of total federal spending. He feels that when people have such an exaggerated view of current welfare spending, they are unlikely to support increases in programs for those with low incomes.

QUANTITATIVE FORM IN ARGUMENTS

One consideration in understanding, evaluating, and constructing arguments whose claims are supported by quantitative evidence is the form in which this evidence is presented. Is it clear, or is it misleading? Is it powerful, or is it likely to be ignored? In this section of the chapter, I focus on examples in which the latter question is explored.[10] The examples illustrate various ways I ask students to reflect upon how the forms of quantitative data affect the meanings we take from information.

We discuss a table from *The Nation* (Table 13.2), which presents a counter to the first Bush's claim about our former wars, in particular our war on Vietnam. Is this numerical formulation powerful? What might be more powerful forms in which to express the numbers supporting this counter-argument?

I also use a letter that Helen Keller wrote in 1911 to a suffragist in England: “You ask for votes for women. What good can votes do when ten-elevenths of the land of Great Britain belongs to 200,000 people and only one-eleventh of the land belongs to the other 40,000,000 people? Have your men with their millions of votes freed themselves from this injustice?” (quoted in Zinn, 1995, p. 337). As part of understanding Helen Keller’s argument, students are asked to discuss how she uses numbers to support her claim, to evaluate whether that support makes her claim convincing, and to reflect on the form of her data – why she sometimes uses fractions, other times uses whole numbers, and whether there are alternative ways of presenting her quantitative evidence that would strengthen her argument.[11]

Another example of considerations about quantitative form comes from an argument we study about the globalization of workers in the garment trades. As part of that argument the Leslie Fay company’s Honduran workers’ earnings are compared with the company’s sales receipts (Gonzalez, 1995). Specifically, he expresses the quantities as a “grand total of $300” that all assembly line workers in Honduras cost Leslie Fay for one day (and he includes the information that this comes from 120 workers paid $2.50 per day), and $40,000 in retail sales that the company takes from the skirts those workers make in that one day. He does not say how many skirts they each make (another quantity which could reveal a different aspect of the exploitation.)[12] He does not calculate the workers’ yearly pay and compare it to the yearly retail sales of those Leslie Fay skirts.

One of the points brought out on our discussions is that if he had just focused on the average worker’s daily pay and the average amount the company made from selling the skirts she made, he would have compared $2.50 to $333. We speculate that he felt the $333 figure would not appear as outrageous to the reader as the $40,000, even though the numbers for one worker are in the same proportion as the numbers for 120 workers. Also, most readers can relate to $40,000 in terms of their yearly income – most would have a yearly income that is somewhere between half and double that figure. Since Leslie Fay is grossing that amount of money in one day, that adds to the outrageousness Gonzalez wants us to feel. If he had calculated the yearly sales – $14,600,000 – since most readers could not deeply grasp the meaning of such a giant number, its impact would have been less than when readers can think, “Leslie Fay makes more in one day than I make in a year!” And, even minimum-wage workers in the US could think, “In one day, they are paying their workers less than I make in half an hour – could it cost that much less to live decently in Honduras?”

The following is an argument supported by quantitative evidence in the form of a graph, and three forms of similar quantitative information that could have been used to support the claim.[13] Students are asked to identify the claim, the reasoning and the evidence Sklar (1999) provides in an editorial graph (Fig. 13.1). Then we compare the different forms of the quantitative evidence given in the related arguments. Students reflect on the questions: Which quantitative form provides the most powerful support for the claim? What other kinds of numerical data would

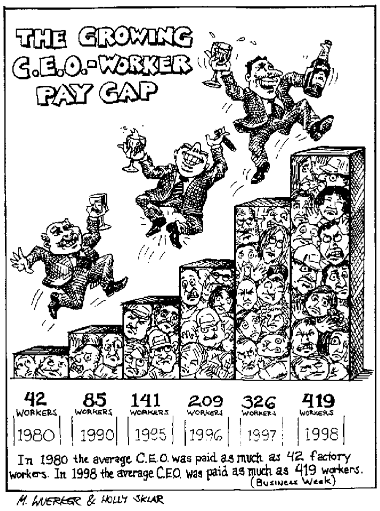

Fig. 13.1 The growing C.E.O.-worker pay gap

further strengthen the claim? What data or reasoning would a counter-argument present?

- The cartoon graph "The growing C.E.O-worker pay gap" (Fig. 13.1).[14]
- Similar quantitative evidence from Jackson (2001): If the minimum wage had risen at the same pace as executive pay since 1990, it would be $25.50 an hour, not $5.15; if average pay for production workers had risen at the same level as CEO pay since 1990, the annual salary would be $120,491, not $24,668.
- Analysis from *United for a Fair Economy* (Sklar, 1999. p. 63): "If the real 555-foot Washington Monument reflects average 1998 CEO pay, then a scaled-down replica representing average worker pay would be just 16 inches tall – 5 inches shorter than in 1997. Back in 1980, the Workers Monument was over 13 feet tall – reflecting a CEO-worker wage gap of 42 to 1."[15]
- The cartoon "How long does it take to earn $8,840?" (Fig. 13.2).[16]

QUANTITATIVE FORM IN AURAL ARGUMENTS

In addition to various written and graphical forms of quantitative information, sound can also be used to help people grasp data. Business as usual was disrupted as part of a traveling exhibit, *Witness to Violence,* to dramatize the toll of domestic violence. A bell was rung in the Massachusetts State House's Great Hall every 15 seconds on February 14, 1995. Each ring represented that, on average, another woman had been beaten somewhere in the US during that time interval. I ask my quantitative reasoning students to write a commentary that uses quantitative data to highlight the emotional and political impact of domestic violence. They are given

Fig. 13.2 How long does it take to earn $8,840? (Wasserman, 1996)

hints about how to elaborate on the numerical data in such a way that it would make sense to as many people as possible and would capture the attention and imagination of most people. We also discuss media coverage of the issue. The results of the studies are what decide how they will be considered (Pozner, 1999). “A small number of faulty studies showing women and men as ‘equal’ batterers get a lot of press coverage and respect. On the other hand, many more carefully researched studies, including those conducted by the U.S. Department of Justice, showing most victims of domestic violence are women, and most perpetrators of this violence, against women and men, are men, are ignored and/or attributed to “feminists” promoting “half-truths based on ideological dogma.”

The Sound of Inequality is another example, a fascinating way to experience, aurally, the concentration of wealth in the US. It is part of *The Working Group on Extreme Inequality*, formed in 2007, connected to the *Institute for Policy Studies*, and believes that the fight against poverty and economic insecurity “has to both ‘raise the floor’ and challenge the concentrated wealth and power that increasingly sit at the top of our economic ladder.”

QUANTITATIVE FORM IN ARTISTS’ ARGUMENTS

In addition to various written and graphical forms of quantitative information, the arts present different kinds of opportunities for people to understand and use quantities in arguments. As Toni Morrison states: “Data is not wisdom, is not knowledge” (quoted in Caiani, 1996, p. 3). Caiani goes on to add:

> In contrast to stories told in a living language filled with images taken from the human world, facts, statistics, data and bits of information, valuable as they are, slide in and out of memory without fully engaging sustained, powerful connections to the whole being. Data are important, are necessary, but not all by themselves, not alone. Analysis and facts are not able to give a face, eyes, a body to the suffering, joy, love, anguish of the people … Art, I am convinced (as indeed have been many scientists like Pascal and Einstein), is at least as necessary as the sciences in grasping reality if we are ever to effect the change we seek in our long struggle to be human. … Our efforts to make a better world through a narrow, reductive, isolated, scientific method which relies on the accumulation of data and its business-like interpretation, will fail. (p. 3)

The following examples of art encode quantitative information in ways that make us understand – what does this amount mean? The numbers are the data of our world – our wars. The art allows us to understand the quantities in ways we could not understand from the numbers alone.

- "The other Vietnam Memorial" refers to the original famous Vietnam Memorial in Washington, D.C. by artist Maya Lin, which lists the names of 57,939 Americans killed during the Vietnam War. In the "other" Vietnam Memorial, Chris Burden etched 3,000,000 names onto a monumental structure that resembles a Rolodex standing on its end. These names represent the approximate number of Vietnamese people killed during the U.S. involvement in the Vietnam War, many of whom are unknown. Burden reconstructed a symbolic record of their deaths by generating variations of 4000 names taken from Vietnamese telephone books. By using the form of a common desktop object used to organize professional and social contacts, Burden makes a pointed statement about the unrecognized loss of Vietnamese lives. (Notes from the Museum of Contemporary Art in Chicago, IL).
- The work of Chris Jordan gives us a powerful way of visualizing the impact humans are having on our environment. He describes his "Running the Numbers" series as viewing "contemporary American culture through the austere lens of statistics. Each image portrays a specific quantity of something: fifteen million sheets of office paper (five minutes of paper use); 106,000 aluminum cans (thirty seconds of can consumption) and so on. My hope is that images representing these quantities might have a different effect than the raw numbers alone, such as we find daily in articles and books. Statistics can feel abstract and anesthetizing, making it difficult to connect with and make meaning of 3.6 million SUV sales in one year, for example, or 2.3 million Americans in prison, or 32,000 breast augmentation surgeries in the US every month."
- Artist Alfredo Jaar (born in Chile, works in New York City) went to Rwanda in 1994 to try to understand and represent the slaughter of "possibly a million Tutsis and moderate Hutus" during three months of Prime Minister Jean Kambanda's term. "Even after 3000 [photographic] images, Jaar considered the

tragedy to be unrepresentable. He found it necessary to speak with the people, recording their feelings, words and ideas. … In Jaar's Galerie Lelong installation, a table containing a million slides is the repetition of a single image, The Eyes of Gutete Emerita." The text about her reads: "Gutete Emerita, 30 years old, is standing in front of the church. Dressed in modest, worn clothing, her hair is hidden in a faded pink cotton kerchief. She was attending mass in the church when the massacre began. Killed with machetes in front of her eyes were her husband Tito Kahinamura (40), and her two sons Muhoza (10) and Matriigari (7). Somehow, she managed to escape with her daughter Marie-Louise Unumararunga (12), and hid in the swamp for 3 weeks, only coming out at night for food. When she speaks about her lost family, she gestures to corpses on the ground, rotting in the African sun." The art review ends with a comment about the numbers: "The statistical remoteness of the number 1,000,000 acquires an objective presence, and through the eyes of Gutete Emerita, we witness the deaths, one by one, as single personal occurrences" (Rockwell, 1998; for images go to http://www.alfredojaar.net/ and, under "projects", view Rwanda).

ADDITIONAL CONSIDERATIONS ABOUT QUANTITIES IN ARGUMENTS

There are other kinds of considerations about quantities that are important in understanding, evaluating, and making powerful arguments that challenge the global imperialism that has trickled down into every corner of our world. In addition to the more typical ways of classifying various misleading statistical accounts (like misinterpreting correlation as implying causation), two overarching questions are important to consider: What are the political, as opposed to scientific/mathematical, aspects involved in the data presented? Is the measure chosen the most accurate way of describing or analyzing the situation, or, in other words, is the correct answer being given to the wrong question?

It is important to understand which aspect of quantitative evidence is mathematical fact and which is political, and therefore, subject to debate. Much data are presented as if they were neutral descriptions of reality, masking the political choices that produced the data. For example, once the government determines which categories of workers count as part of the labor force, and which categories of workers count as unemployed, rewriting that information in percent form is a mathematical algorithm for which there is only one answer and about which it does not make sense to argue. The politics about which we can, and should, argue comes in decisions made by the government such as to count part-time workers who want full-time work as fully employed, and to not count workers who have looked for over a year and not found a job as unemployed.[17]

It is also important to determine which quantitative measure gives the most accurate picture of a particular issue. For example, the ultra-conservative Heritage Foundation argues that our progressive federal income tax is terribly unfair because of facts such as in 1997 the top 1% income group paid 33.6% of all federal income taxes, while the share of all taxes paid by the bottom 60 percent was only

5.5%. But a left (and even a liberal) perspective would argue that is the wrong measure by which to judge tax equity. It makes much more sense to look at what happens to the share of total pre-tax and total after-tax income. That picture reveals only a small progressivity: a slight upward shift for the bottom four quintiles, and a slight downward shift for the top income quintile. Ellen Frank (2002) then argues:

> If one believes that Ken Lay deserved no less than the $100 million he collected from Enron last year, while the burger-flippers and office cleaners of America deserve no more than the $6.50 an hour they collect, then a progressive tax would seem immoral. But if one believes that incomes are determined by race, gender, connections, power, luck and (occasionally) fraud, then redistribution through the tax system is a moral imperative. (p. 44)

Frank goes on to discuss the impact of other kinds of federal taxes, such as Social Security taxes (which are capped at $90,000), excise taxes and so on which are regressive, as well as state and local regressive levies like sales taxes. She hypothesizes that adding all these taxes together would "almost certainly find that the U.S. tax system, as a whole, is not progressive at all" (p. 44).[18]

CONCLUSION

In "Scenes from the Inferno," Alexander Cockburn (1989) wrote about some of the realities behind the so-called worldwide triumph of capitalism. One of his illustrations is particularly relevant to understanding how the wrong quantitative measure has real consequences in people's lives. He relates how in some neighborhoods of Santiago, Chile, "the diet of 77 to 80% of the people does not have sufficient calories and proteins, by internationally established standards, to sustain life." Under Pinochet, the dictator of Chile during that country's period of "triumphant capitalism," malnutrition was measured in relation to a person's weight and height, in contrast to the usual comparison of weight and age. "So a stunted child is not counted as malnourished, and thus is not eligible for food supplements, because her weight falls within an acceptable range for her height" (p. 510). The overarching goal underlying a critical-mathematical literacy curriculum precisely is to explore the connections between understanding the outrageousness of collecting such statistics, and struggling to change the outrageousness of such conditions.

In this chapter, however, I have focused on the importance of the form in which the measures are actually expressed. I argue for the importance of critical-mathematical literacy curricula also consisting of reflections on questions such as whether it is more powerful to state that "The wealthiest 1 percent of Americans control about 38 percent of America's wealth." Part of struggling to change our world in the direction of more justice is knowing how to clearly and powerfully communicate the outrageousness.

NOTES

1 A revised version of Frankenstein, 2007.

2 Of course, there is more here than just that interpretation. Professor Arthur B. Powell, of Rutgers University, provided many helpful suggestions about this manuscript, reading the cartoon as saying that as time goes on, people tend to loose interest in protesting injustices and as a consequence perpetrators of injustices eventually can act with impunity.

3 A similar publication of which I am very proud is an edited version of the following letter responding to an editorial in the *Boston Globe*. To my delight, they titled it "Secretary of Death" and included a drawing of Kissinger, which made the letter the central one on the editorial page. In this case, the focus is on how a particular quantitative form can be correct, but highly misleading.

> November 30, 2002
> Dear Editor:
> I commend the *Globe* for writing an editorial questioning the appointment of Henry Kissinger to lead what is supposed to be an independent commission investigating intelligence failures in the September 11. 2001 attacks.
>
> However, as a teacher of Quantitative Reasoning, I was dismayed to see your misleading presentation of numerical data. You mention Kissinger's responsibility for "thousands" of civilian deaths in Indochina, Bangladesh, and East Timor. In facts, very clearly documented by Christopher Hitchens' review of USA government declassified documents detailing Kissinger's legal crimes (*The Trial of Henry Kissinger* , London and New York: Verso, 2001), Pentagon figures for deaths in Vietnam during the period under which Kissinger's deceits and policies prolonged that war, are: 31,205 of our citizens, 86,101 South Vietnamese, and 475,609 Vietnamese "enemy." The US Senate Subcommittee on Refugees estimated that during that time more than 3 million civilians were killed, injured, or made homeless (p. 41). In Bangladesh, "the eventual civilian death toll has never been placed at less than half a million and has been put as high as 3 million" (p. 46). In East Timor, 200,000 people, approximately one-third of the population, were slaughtered as a direct result of the "green light" then United States Secretary of State Kissinger gave to Indonesian dictator General Suharto. (p.93)
>
> Although your statement that Kissinger's policies resulted in "thousands" of civilian deaths is technically accurate, clearly it presents a misleading picture. In a conservative tallying from just these three areas of the world, Kissinger is responsible for *more than one million* deaths, *hundreds of thousands* of them civilian deaths.

4 Broadly speaking, criticalmathematics [one word] defines a pedagogical perspective that connects education in mathematics with critical analyses of social, economic, cultural, and political issues and with movements for social change. Criticalmathematics educators teach for understanding rather than memorization, start with the mathematical knowledge of students, and engage students to reflect critically on both the substance and process of their learning. In agreement with ethnomathematics, this perspective counters Eurocentric historiography of mathematics and considers the interaction of culture and mathematical knowledge. Moreover, criticalmathematics attends to the dynamics of power in society to understand the effects of racism, sexism, ageism, heterosexism, monopoly capitalism, imperialism, and other alienating, totalitarian institutional structures and attitudes. Finally, criticalmathematics educators attempt to develop commitment to build more just structures and attitudes and personal and collective empowerment needed to engage these tasks. For an elaboration on these ideas and the history of the Criticalmathematics Educators Group, see Frankenstein (1983, 1998), and Powell (1995).

5 At the time, the costs of the wars had been estimated at $1 trillion. More recent estimates, from Brown University's Watson Institute for International Studies http://costsofwar.org/ accessed on July 2, 2011, that include future costs for items such as veterans care (as of Fall, 2010 there were over half a million disability claims submitted to the Veterans Administration) and debt payments

connected directly to borrowing for the wars, are between $3.2 and $4 trillion. The report also emphasizes the direct human costs – conservatively estimating 225,000 dead and 7.8 million war refugees, loss of civil liberties in the USA, and human rights violations abroad.

6 Answer: about 32 years – most people guess way too little or way too big.

7 Answer: over one century – 114 years rounded to the nearest year.

8 There are also some really interesting ways of getting a sense of the size of finite numbers larger than one billion. For trillions, one of my favorites is: Our blood contains tiny red corpuscles, each like a circular disc somewhat flattened in the center and about 0.007 mm in diameter and 0.002 mm thick. In a drop of blood (1 cubic mm) there are about 5 million of these corpuscles. An average adult has about 3,500,000 cubic mm of blood. If all these corpuscles were strung out in a single chain, about how many times could this chain be wound around the earth's 25,000-mile equator? (Note: 1 mile = 5280 feet; 1000 mm = 1 meter; 1 meter is approximately 3 feet.) (Answer: about 3 times)

A science fiction story whose theme is making sense of super-gigantic finite numbers is *The Universal Library* (Lasswitz,195]) and " Postscript to 'The Universal Library' " (Ley, 1958).

9 Thad Williamson, in *Dollars and Sense* (July/August 2004, p.12) provided a number of different ways of looking at what the money spent by the USA in the war on Iraq – $151 billion by the end of 2004 – could have been used for. This money, in his calculations, could have provided in the developing world for two years: food for half of the malnourished people in the world, a comprehensive, global HIV/AIDS treatment/prevention program, clean water and sanitation to all who lack them, and immunizations for all children.

The General Accounting Office found that this bomber deteriorates in rain, heat, and humidity and "must be sheltered or exposed only to the most benign environments." These special shelters are not available at the very military bases in which the plane was proposed to be located. The plane also needs as much as 124 hours of maintenance time for each hour in the air. In 74% of the trial missions its radar "could not tell a rain cloud from a mountainside." Further, since the early 1980's its theoretical mission to evade Soviet radar to penetrate to targets in the former Soviet Union, has ended with the end of the "cold war." Finally, the then-claim of Soviet military superiority, used to justify requests for 21 of these bombers, has been challenged (Herman, 1997).

Herman (1997) points out that the cost per F-22 plane is "five times that of the still quite serviceable F-15, which the F-22 is designed to replace." Even at the initial unveiling, the media quoted some critics saying "we are in an arms race with ourselves" (p. 42).

10 Some references to investigate this further are Frankenstein (1994) and Griffiths, Irvine and Miles (1979).

11 When considering this example, I also include information about the politics of knowledge. When Helen Keller became an active socialist, the Brooklyn *Eagle*, a newspaper that had previously treated her as a hero, wrote "her mistakes spring out of the manifest limitations of her development." Howard Zinn relates that the *Eagle* did not print her reply about the unfairness of attacking her physical challenges and not her ideas: "Oh, ridiculous Brooklyn *Eagle*! What an ungallant bird it is! Socially blind and deaf, it defends an intolerable system a system that is the cause of much of the physical blindness and deafness which we are trying to prevent … If I ever contribute to the Socialist movement the book that I sometimes dream of, I know what I shall name it: Industrial Blindness and Social Deafness." (Zinn, 1995, pp. 337–338).

12 In Las Vegas, for example, in a unionized hotel laundry shop (with much better pay and other conditions than the non-unionized shops), workers count and bundle 1800 washcloths an hour; other workers separate stained and ripped sheets from the total and feed the good sheets, "310 an hour, into a machine that irons and folds them. The pillowcase feeder has to complete 900 an hour; the napkin feeder, 1300" (Wypijewski, 2002). The workers in the "soil sort" need to sort 700 to 1050 pounds of dirty linen an hour. For this they get paid $9.09 to $10.45 an hour plus benefits.

[13] It is particularly challenging to analyze visual arguments, which is one reason I include them in my curriculum. The next section on artists' arguments carries this kind of critical visual thinking even further.

[14] Reprinted with permission from Matt Wuerker.

[15] Recently, I came across a similar presentation in Johnston (2003), an important book revealing the details of how the USA current tax structures transfer money not only from the poor to the wealthy, but even from the upper middle classes to the top 0.1 percent wealthiest. To visualize what he calls the *enormous chasm* in incomes (“just 28,000 men, women and children had as much income in 2000 as the poorest 96 million Americans”), he suggests we “imagine these two groups in geographic terms. The super-rich would occupy just one-third of the seats at Yankee Stadium, while those at the bottom as the equivalent of every American who lives west of Iowa – plus everyone in Iowa” (p. 41).

[16] Reprinted with permission from Dan Wasserman.

[17] And, of course, there is the Wal-Mart example, where many of their counted-as-employed-in-the-unemployment-statistics full-time workers earn so little that, even under the current stinginess of our government, they are eligible for various kinds of public assistance. The Democratic Staff of the House Committee on Education and the Workforce (2004) estimated that “one 200-person Wal-Mart store may result in a cost to federal taxpayers of $420,750 per year – about $2,103 per employee. Specifically, the low wages result in the following additional public costs being passed along to taxpayers: $36,000 a year for free and reduced lunches for just 50 qualifying Wal-Mart families; $42,000 a year for Section 8 housing assistance, assuming 3 percent of the store employees qualify for such assistance, at $6,700 per family; $125,000 a year for federal tax credits and deductions for low-income families, assuming 50 employees are heads of households with a child and 50 are married with two children; $100,000 a year for the additional Title 1 expenses, assuming 50 Wal-Mart families qualify with an average of two children; $108,000 a year for the additional federal health care costs of moving into state children's health insurance programs (S-CHIP), assuming 30 employees with an average of two children qualify; and $9,750 a year for the additional costs for low income energy assistance” (p. 9). Further kinds of costs that Wal-Mart extracts form taxpayers are detailed in Robert Greenwald's documentary “Wal-Mart: The High Cost of Low Price” (2005). To get current information about the hidden costs of Walmart, see the Walmart Watch website: http://walmartwatch.org/

[18] Another example that illustrates how to shift the argument by using more appropriate measures involves what Paul Krugman (2005) called the right-wing's “little black lies” about Social Security for African-Americans. Of course, the whole premise of the argument that Social Security privatization would be fairer to Blacks since their life expectancy is shorter than Whites, so they collect fewer benefits, speak volumes about racism in our society! Imagine stating this outrageous fact and then remaining silent about the obvious health-care and other programs which would eliminate that fact! But, in addition to this moral consideration, as Spriggs (2004) argues, there are problems with using the life-expectancy measure to calculate the benefits collected. One key concern he discusses is that since Social Security “also serves families of workers who become disabled or die, a correct measure would take into account all three risk factors – old age, disability, and death. Both survivor benefits and disability benefits, in fact, go disproportionately to African Americans” (p. 18).

Another example involves the actual creation of a measure to counter the argument that heavily populated countries with many poor people are ruining our environment. Holtzman (1999) relates that Mathis Wackernagel and William Rees, community planners at the university of British Columbia, developed a measure of the impact of each country on the world's resources. “The ecological footprint measures the resources consumed by a community or a nation, whether they come from the community's backyard or from around the globe. … Wackernagel and Rees ask how many hectares … are needed per person to support a nation's consumption of food, housing, transportation, consumer goods, and services. They calculate how much fossil energy use, land degradation, and garden, crop, pasture, and forest space it takes to produce all that consumers buy." Using this measure, each person in the USA consumes 5.1 hectares, while each person in India only consumes 0.4 hectares. So, even with a population 4 times that of the USA, India has only one-third the impact on our global environment that the USA does. To get current information see the website of the Global Footprint Network: http://www.footprintnetwork.org/en/index.php/GFN/page/at_a_glance/

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ERIC GUTSTEIN

14. CONNECTING COMMUNITY, CRITICAL, AND CLASSICAL KNOWLEDGE IN TEACHING MATHEMATICS FOR SOCIAL JUSTICE[1]

In this chapter, I describe conceptually, and give an example of, an aspect of teaching mathematics for social justice – teachers' attempts to connect three forms of knowledge: community, critical, and classical. The setting is a Chicago public high school, oriented toward social justice, whose students are all low-income African Americans and Latinas/os. Drawing from the experience of creating and teaching a mathematics project that emerged from a central disruption in the life of the school community, I discuss complexities and challenges of creating, from students' lived experiences, curriculum that simultaneously develops their critical sociopolitical consciousness and mathematical proficiencies.

INTRODUCTION

Teaching and learning mathematics for social justice has its roots in the mathematics education work of Skovsmose (1994, 2004) and Frankenstein (1987, 1998), among others. It builds on work in critical pedagogy, in particular that of Freire (1970/1998) and others such as Giroux (1983) and McLaren (2007) and also draws upon culturally relevant pedagogy (Ladson-Billings, 1994, 1995b; Tate, 1995). Though proponents and researchers describe it in different ways (e.g., some refer to it as "critical mathematics"), there are certain common pedagogical aims. Two of the most central are that students develop both critical consciousness and mathematical competencies, and there is also the view that these two areas of learning need to be dialectically interwoven by both teachers and students in a conscious manner. That is, mathematics should be a vehicle for students to deepen their grasp of the sociopolitical contexts of their lives, and through the process of studying their realities – using mathematics – they should strengthen their conceptual understanding and procedural proficiencies in mathematics. One of the principal ways for teachers to support students in moving toward these interconnected goals is for the students to engage in mathematical investigations in the classroom of specific aspects of their social and physical world (see Gutstein & Peterson, 2005 for reports by K–12 teachers on efforts to do so). There are indeed, few extended studies of teaching and learning mathematics for social justice in K–

S. Mukhopadhyay & W.-M. Roth (eds.), Alternative Forms of Knowing (in) Mathematics, 299–311.

12 urban classrooms (Brantlinger, 2006; Gutstein, 2006; Turner, 2003). These reports shed light on the complexities of enacting critical mathematics pedagogy and certainly point out some of the difficulties in what is mostly uncharted territory. In this chapter, I highlight one particularly challenging quandary and illustrate it with a short vignette. There is much work to do in theorizing and practicing social justice mathematics, and my purpose here is to point out some issues that I believe currently face those of us who want students to learn mathematics as a vehicle for social change. The matter I discuss is the complexity of building on students' and communities' knowledge while simultaneously supporting the development of their mathematical competencies and critical awareness. I examine it from the perspective of my own work in Chicago (and its public schools) where I have lived, worked, and taught for the past 12½ years, first teaching my own middle-school mathematics class for several years, and for the past few years, working with a new social justice high school in mathematics classes.

CONNECTING COMMUNITY, CRITICAL, AND CLASSICAL KNOWLEDGE

We[2] have adopted a framework in the school's mathematics team of trying to synthesize what we call *community*, *critical*, and *classical knowledge* (Gutstein, 2006), or the "three C's." These concepts are not new, but their interrelations have been under-elaborated with respect to mathematics education. We recognize that these may be contested definitions, and we consider the categories (and our thinking) to be provisional and fluid. By *community knowledge*, we mean several different, but related, components of knowledge and culture. It refers to what people already know and bring to school with them. This includes the knowledge that resides in individuals and in communities that usually has been learned out of school (e.g. their *funds of knowledge*; see Moll, Amanti, & González, 2005). It involves how people understand their lives, their communities, power relationships, and their society. We also mean the cultural knowledge people have, including their languages and the ways in which they make sense of their experiences. Some refer to this as "indigenous knowledge," "traditional knowledge," "popular knowledge," or "informal knowledge" (including with respect to mathematics, e.g., Knijnik, 1997; Mack, 1990). Two examples serve to illustrate our meaning. In *Rethinking Columbus*, Tajitsu Nash and Ireland (1998) describe the knowledge of a typical Amazonian elder, who

> has memorized hundreds of sacred songs and stories; plays several musical instruments; and knows the habit and habitat of hundreds of forest animals, birds, and insects, as well as the medicinal uses of local plants. He can guide his sons in building a two-story tall house using only axes, machetes, and materials from the forest. He is an expert agronomist. He speaks several languages fluently; knows precisely how he is related to several hundred of his closest kin; and has acquired sufficient wisdom to share his home peacefully with in-laws, cousins, children, and grandchildren. Female elders

> are comparably learned and accomplished. (Tajitsu Nash & Ireland, 1998, p. 112)

The other example is from *Pedagogy of Hope* (Freire, 1994, pp. 44–49). In it, Freire recounts a conversation with a group of Chilean farmers. They were having a rousing discussion when the farmers suddenly silenced themselves and asked the "professor" (i.e., Freire) to tell them what he knew. Freire wrote that he was unsurprised by this, having experienced it before, and proceeded to challenge the farmers to a game. They were to stump each other with questions that the other could not answer. Freire went first and asked, "What is the Socratic maieutic?" The farmers laughed, could not answer, then baffled Freire with the question, "What's a contour curve?" The game continued, each stumping the other, until finally the score was 10:10. The point was clear – Freire's knowledge and the farmers' knowledge were both valid and valuable. Each knew things that the other did not; each had to respect the others' – and their own – knowledge. What the farmers knew, from years of shared lived experience, is what we term community knowledge. *Critical knowledge* is knowledge about the sociopolitical conditions of one's immediate and broader existence. It includes knowledge about why things are the ways that they are and about the historical, economical, political, and cultural roots of various social phenomena. Various authors (e.g., Giroux, 1983; Macedo, 1994) have described *critical literacies*, and we mean essentially the same idea, Freire referred to as "reading the world" (Freire & Macedo, 1987) In his earlier work on literacy campaigns, he discussed *culture circles* in which groups of workers, peasants, and farmers studied *codifications* (representations of daily life, usually pictorial) and reflected on their meanings (Freire, 1970, 1973). Those sessions allowed the culture circle members to examine their lives from different perspectives, and the process of collectively decoding the representations led the individuals to deepen their understanding of the phenomena. Freire's pedagogy thus provided the opportunities for people to transform their community knowledge about the everyday world that they had often normalized into critical knowledge about the same situations.

It is often the case that community knowledge already is critical, but context matters. For example, relatively young adolescents (e.g., middle-school students) may have knowledge about their life situations, but it is not often critical. Whether it is critical depends on several things, including their experiences, those of their families and communities, the level of political consciousness at the time, and the strength of existing social movements. In contrast, adults who are engaged in various struggles may have community knowledge that is quite critical. As an example, a battle is currently taking place in Chicago to stop the displacement of low-income people of color (in particular, African Americans) through gentrification (Lipman & Haines, 2007). Many adults in the affected communities have a clear and critical understanding of the political forces allied against them, including their geneses and various forms of subterfuge. I have heard parents in communities where public housing has been demolished (and not replaced) and schools closed (and reopened for "new" residents) eloquently elaborate who and what forces are responsible for their removal, and why. So the lines between community and critical knowledge are not always clear. A major thesis of Freire's work is that *problem-posing* pedagogies can present life situations back to people (whether in or out of school) so that they may pose questions themselves and transform their community knowledge into a more critical state, and consequently be drawn into action to challenge unequal, oppressive relations of power.

The lines between classical and the other forms of knowledge are not so clear either. *Classical knowledge* generally refers to formal, in-school, abstract knowledge. Our focus in terms of classical knowledge is that students have the competencies they need to pass all the gate-keeping tests they will face and to have full opportunities for life, education, and career choices. Classical mathematical knowledge clearly has high status in society, as many have commented (e.g. Apple, 2004), as well as a strong Eurocentric bias (Frankenstein & Powell, 1994; Joseph, 1997). Nonetheless, while we critique it, we recognize its power and cultural capital and argue that students need to develop it for several reasons. They need it for personal, family, and community survival, especially for students who come from economically marginalized spaces. But even more than that, we believe it is crucial that students appropriate, in this case, the "master's tools" with which to dismantle his house (cf. Lorde, 1984). We subscribe to Freire and Macedo's orientation toward what they referred to as "dominant" knowledge:

> To acquire the selected knowledge contained in the dominant curriculum should be a goal attained by subordinate students in the process of self and group empowerment. They can use the dominant knowledge effectively in their struggle to change the material and historical conditions that have enslaved them. (Freire & Macedo, 1987, p. 128)

To connect the three types of knowledge is no simple matter for many reasons. First, there is the question of how might teachers learn students' community knowledge. In Brazil, where Freire and others practice(d) these ideas, the process by which teachers investigate the *generative themes* of a community – key social contradictions in people's lives and the ways in which they understand them – is complicated. In Porto Alegre's *Citizen School Project*, there is a lengthy and involved ten-step process through which teachers, in collaboration with neighborhood adults, study community knowledge to develop school-wide, interdisciplinary curriculum based on the generative themes (Gandin, 2002). Freire (1970) elaborated his view of how researchers might investigate the themes within a specific community, and this also involved a detailed, multi-step process. There are still more issues, such as the question of how might teachers study community knowledge when they are outsiders to the community, language, and culture of their students (Delpit, 1988), or the fact that the generative themes identified by neighborhood adults may not coincide with those of the youth in schools.

Once educators begin to have a grasp of the community knowledge of their students and their families, then they can try to create curriculum based on those themes that will support both the development of critical and classical forms of knowledge. This also is quite complicated. First, there are the time constraints imposed on teachers and their working day (which also affects their capacity to investigate generative themes, although in Porto Alegre, teachers were paid for that work). When do teachers have the time to develop new innovative curriculum, let alone cope with all the other demands of teaching? For example, creating standards-based reform mathematics curricula in the US took massive amounts of time, money, and people. The reform curriculum with which I am most familiar, *Mathematics in Context* (MiC) (National Center for Research in Mathematical Sciences Education & Freudenthal Institute 1997-1998), required perhaps $8 million, 5 years, and close to 50 people working in two countries before it was fully operational. It is true that MiC was a connected, cohesive curriculum spanning four years (grades 5–8), and obviously developing curriculum for just one school community would require less time. But the time and people power alone needed to create quality curricula testify to the necessary resources required.

Second, to develop curriculum requires a different knowledge base than teaching, despite the interrelationship of the two. My personal knowledge of MiC's development and my professional judgment suggest that there are talented curriculum designers who would have difficulty teaching MiC in urban classrooms because, for example, they may not connect that well with the students or their communities. This is also probably true for other successful curriculum projects whose authors are primarily university-based mathematics educators. Conversely, there are successful mathematics teachers in urban schools who do not have the knowledge to create rich mathematics curriculum.

Third, successfully navigating the requirements of a standards-based mathematics curriculum is difficult enough, especially under the pressure of neoliberal accountability constraints like the *No Child Left Behind* legislation in the US that mandates repeated testing. But to do so while simultaneously providing opportunities for students to develop critical knowledge in mathematics classes is an added layer of complexity (Brantlinger, 2006; Gutstein, 2006). It is generally accepted that good (mathematics) teachers need to have content knowledge (Hill & Ball, 2004), pedagogical content knowledge (Shulman, 1986), and knowledge of students and their communities (Ladson-Billings, 1995a, 1995b); but in addition, to develop critical knowledge, teachers also need deep knowledge of social movements, history, culture, political economy, and local and global sociopolitical forces affecting students' lives, as well as particular dispositions toward social change and the politics of knowledge. Even when teachers do have these various knowledge bases, ensuring that the mathematics does not get lost when developing critical knowledge and supporting students' sociopolitical consciousness (in mathematics class) is no easy task – the dialectical interrelationships are complicated and more attention needs to be focused in this area, and more experience accumulated (Brantlinger, Buenrostro, & Gutstein, 2007).

In short, for many reasons, it is quite complex to create curriculum that starts from students' and their communities' lived experiences/knowledge and then simultaneously and with rich interconnections supports *both* mathematical power/ classical mathematical knowledge *and* a critical awareness of one's social context. No such mathematics curriculum currently exists that is broadly applicable partly because of the specificity of local situations, although there are several examples of projects and units of social justice mathematics that have been taught in urban schools (see, for example, Brantlinger, 2006; Frankenstein, 1998; Gutstein, 2006; Gutstein & Peterson, 2005; Osler, 2006; Turner, 2003). It will not be easy to create high-quality social justice mathematics curricula that teachers can adapt to their local settings, and even allowing for good curricula, the school change and professional development literature is clear that curriculum alone does not ensure effective and appropriate teaching – nor real learning ((Fennema & Scott Nelson, 1999). Efforts to work on connecting the "three C's," however we describe them, are needed, and how to do so is an open question with respect to both theory and practice.

An Example of Connecting the Three C's in Practice

I now turn to a short example of our work in a Chicago public high school for social justice in which we attempted to connect community, critical, and classical mathematical knowledge (see Gutstein, 2008b, for details). Briefly, a new school was built and opened in Fall 2005 after a group of residents in a Mexican immigrant community (*Little Village*) went on a 19-day hunger strike in 2001 (Russo, 2003). The residents struck for a new school for their community; the school board promised it, then reneged; and the hunger strike was the culmination of a multi-year struggle for a new school in the overcrowded neighborhood. The new school building houses four small schools and comprises a maximum of 350–400 students. Each has a different community-determined theme. The school I work with is the social justice high school (known to most as "Sojo").

Although Little Village is overwhelmingly Mexican, the Chicago Public School Board, under a 1980 federal desegregation mandate, racially integrated the open enrollment, neighborhood school by drawing the attendance lines into a bordering African American community, North Lawndale. Thus the schools are 30% African American and 70% Latina/o. However, by changing the attendance boundaries, the school board also limited Latina/o enrollment, causing friction for some Little Village residents who saw their children's spots in the new building "taken" by African Americans from North Lawndale. Furthermore, given Chicago's history of segregation, racist exclusion, and neighborhood and turf lines, there is an ambivalent relationship between the two communities. Students for the most part intermingle and work together in the school, although there are real tensions outside in the neighborhood.

In January 2006, during the first year when each school had about 100 ninth graders, a local Latino politician held a press conference and proposed a public referendum that the boundaries be redrawn to exclude North Lawndale African American students. Black students, understandably angry, hurt, and scared, immediately went to teachers to voice concerns about being removed from the school. Our mathematics team, on the initiative of one of the mathematics teachers, quickly developed a mathematics project (the "Boundaries Project") whose central question was this: What is a fair solution for both communities? While our assessment is that there were weaknesses in the project (e.g., we threw it together in two days because of the immediacy of the issue, and it was not clear how much mathematics students learned), our analysis also suggests that there were some considerable strengths. Most notable was that students were quite engaged, and we believe this is because the work students did was genuine. No one knew (or knows) the answer to the central question because, in fact, the solution to the problem has to be eventually determined by the two communities working together in concerted effort to ensure that there are enough spots in quality schools for all the students – something that is not the situation now, even with the new school. The project tied directly into students' lived experiences and generative themes – that is, it built on students' (and their families') community knowledge. The issues of interconnections between the two neighborhoods, their histories, and students' stereotypes toward each other all surfaced. Politically, the two main points with which we wanted students to grapple (i.e., as the development of critical knowledge) were that the differences between the communities were far outweighed by the commonalities, despite historical divide-and-conquer techniques used to pit communities of color against each other, and the above point that ultimately there were not enough quality schools for all the students. Mathematically, we asked students several questions about the numbers of Black and Brown students in the building at full enrollment given ratios different than the current 30:70, and the probability of a student from each community being accepted in a lottery (using different possible ratios). We also had them study census tract data and consider how to enlarge the boundaries in North Lawndale so that students from there would have the same chance to be accepted as the Little Village students. This entailed calculating acceptance probabilities for both communities, with various ratios of African American and Latina/o students – and this was further complicated mathematically because each neighborhood has different numbers of high-school aged students. Students also examined data for other nearby schools, as well as local area maps, and overall, they mathematized the central problem of having one new school building for too many students from two different communities. In our assessment, the complexity of the mathematics lay more in this requirement to draw out the mathematical components of the situation, than in any specific subpart or individual problem within the project.

While we know that a weeklong project can have only limited impact, we located the project within a four-year program of teaching and learning mathematics for social justice. We appreciate that the political aim of students using mathematics to develop an awareness of common issues for both communities is difficult to achieve (although we also note that the whole school is making its way toward social justice pedagogy and curriculum). First, the way the Chicago Public School Board altered the originally planned school boundaries was something we had to contend with – that is, the historical tensions were reignited and in the air. Second, the local politician exacerbated these by pitting the neighborhoods against each other and proposing that the schools serve only Little Village students. Third, the politics of the immigration rights movement and the huge immigration marches nationally and in Chicago (where close to a million people participated in two large demonstrations) interacted with the specific conditions in the school campus in which African American students reported (to African American staff) that they did not fully feel their place in the building.

The opportunity is there to work with students to deconstruct and politically explore this polarized context, but existing contradictions can impede the process. For example, only 5 of about 30 African American Sojo students attended the May 1, 2006 pro-immigrant rights rally in Chicago (the larger of the two). I ran into an African American friend at the march who felt uncomfortable with two of the ubiquitous, mass-produced signs at the rally: “We Are All Immigrants” and “Immigrants Built America,” neither of which is historically accurate and both of which negate the presence, contributions, experiences, and exploitation of both African Americans and Native Americans. There is a racially coded subtext here that is visible in the school and larger society both, with respect to “good” and “not-so-good” “minorities.” Chicago employers report that they prefer hiring Mexican workers to African American ones because they were supposedly more “compliant.” When asked about popular perceptions in their community about African Americans, Latina/o students report the stereotype that “Black people are lazy,” while some African American students suggest that Mexican workers are “taking our jobs.” A recent New York Times article conveyed these misconceptions well (Swarns, 2006). In a town in the state of Georgia, where Africans and African Americans created most of the wealth and toiled mightily for centuries either as slaves or low-paid, exploited workers, a 51-year-old Mexican worker was quoted as saying:

> They don’t like to work, and they’re always in jail. If there’s hard work to be done, the blacks, they leave and they don’t come back. That’s why the bosses prefer Mexicans and why there are so many Mexicans working in the factories here.

The point here is that community knowledge is affected by popular misconceptions and myths. Although this project had its limitations (Gutstein, 2008b), a strength was that we were able to tap into, and build on, students’ community knowledge, and students were able to develop some critical and classical mathematical knowledge. The experience gives us (and others) some insight into the challenges and possibilities of teaching mathematics for social justice, although this was not a case in which we consciously investigated students’ community knowledge. Rather, the generative theme emerged because of the dynamics of the situation. We might have ignored students’ realities and kept to the already planned curriculum. Our analysis is that to have done so would have been a mistake and a missed opportunity to engage students and provide them a chance in school to examine their own lived experiences, deepen their sociopolitical awareness, and learn mathematics. One positive outcome we point to is that involving students in this particular project played a role in enculturating students to social justice pedagogy and reshaping their views of mathematics; their journaling after the project provided evidence for this assertion.

CONCLUSION

In the school year 2006–07, our mathematics team began planning a more in-depth, extended unit centered around *displacement* in an attempt to build on a generative theme salient for both communities. The specific local and broader national contexts shape our understanding of displacement. First, gentrification is a major issue in Chicago. While it affects many urban areas in the US, it is particularly severe here because the city power structure (i.e., Mayor Daley and his administration, major finance capitalists, and the real estate/ development machine) is in the throes of attempting to reshape Chicago as a global city (Lipman, 2004). The mayor and the school board are currently in the process of closing 60–70 neighborhood schools and creating 100 "new" ones, most of which are in the same school buildings but with large infusions of resources historically denied in the past (Lipman & Haines, 2007). Many of the communities experiencing school closings are being rapidly gentrified. North Lawndale is very much on the list of affected neighborhoods, and has been referred to as "ground zero" by activists battling the redevelopment although the amount of new construction (e.g., condos) is still relatively small as of this writing. Thus displacement in the North Lawndale context refers to the oncoming gentrification in the community. Second, in Little Village, displacement refers to the removal of people out of the country altogether, back to Mexico. The U.S. House of Representatives passed a bill in September 2006 to build a 700-mile fence along the Mexican-U.S. border, and shortly afterwards, the Senate began considering the fence as well. In a small town of 37,000 located about 40 miles from Chicago, in early October 2006, town officials proposed an ordinance to penalize landlords who rented to undocumented immigrants and employers who hired them. Three thousand people showed up at the Town Hall in protest. Many residents of Little Village are undocumented, and the threat of expulsion from the community and country altogether is quite real. Thus both communities are faced with issues of displacement.

An appropriate challenge that we pose to ourselves is how do we know that this matters to students and community members, that this is really a generative theme when we have not done (for example) the thorough investigation conducted by Brazilian teachers to uncover community knowledge? In October 2006, we conducted focus group discussions and in-class discussions with small groups of students to explore this issue. In our conversations with close to 60% of the sophomore class, students overwhelmingly expressed support and interest in the proposed unit. We also know, by the strength of the social movements for immigrant rights and against gentrification, that these issues matter profoundly to people (both adults and youth) in the affected communities. The tremendous number of people in the streets in support of immigrants and their rights is powerful evidence of this, and while the struggle against gentrification involves far fewer people, the level of consciousness and determination in impacted neighborhoods is quite high (Lipman & Haines, 2007). We can *read the world* and understand clearly that the issue of displacement has deep meaning in Chicago.

Whereas we have sketched out a political framework for this project, and have some clarity on how the community and critical knowledge fit in, there are certainly multiple challenges ahead of us. A key one is the connection of classical knowledge. The mathematics of change is central in understanding displacement in North Lawndale and Little Village. Specific issues we plan to have students investigate include the changing demographics of the communities, the change in the cost and availability of properties, and the issues of affordability for people in the area. We want students to analyze the trends and the possibilities, as well as to think about possible actions to take, in conjunction with activists in their communities. We know from other gentrifying Chicago neighborhoods that the battle to stay in the area is an extremely difficult one, but there are community development corporations that are building or rehabilitating housing that is fairly affordable to many existing residents. This also entails mathematical analysis. Finally, we plan on having students investigate the mathematics of home ownership, loans, mortgages, and development schemes so that they begin to understand how capitalism works, and how real estate developers and banks profit while communities such as theirs experience extreme economic poverty and dis- and under-investment in basic human needs. All this will equip them with knowledge they will need as they become adults and have to fight to maintain their place in the neighborhood, city, and country. This, ultimately, is the goal of teaching (mathematics) for social justice – that students become agents of social change and join in, and eventually lead, the struggles to remake our world for peace and justice.

POSTSCRIPT

I wrote this text originally in 2007. Its key premise is that teaching critical mathematics (or *reading and writing the world with mathematics*) involves teachers in developing (and teaching) curriculum based on students' lives and knowledge while simultaneously supporting the development of their critical sociopolitical consciousness and mathematical competencies. This follows Freire's (1970/1998) view that the starting (but not ending) point for a liberatory education is the reality of the learners themselves. In the chapter, I give a brief example from nineth-grade mathematics classes in a neighborhood Chicago public high school where I work. I also pointed out some conceptual and pragmatic difficulties. At the time, no research existed on any long-term efforts to do this in high school mathematics classes.

Since that time, the ninth graders grew up, and when they were 12th graders in the 2008–09 school year, I taught a regular-track mathematics class at the school in which *all* the contexts we studied came from my students' lives. The content was an eclectic mix of algebra, pre-calculus, discrete mathematics, probability and statistics, and number. But the contexts were chosen by the 21 students and me (before the year started) and included (a) the mathematics of the 2004 US presidential election (was it "stolen?") and implications for the 2008 election; (b) neighborhood displacement (gentrification/foreclosure, immigration/deportation);

(c) spread of HIV-AIDS; (d) criminalization of youth/people of color; and (e) sexism (Gutstein, 2012). A team of graduate students and I developed curriculum frameworks, and I created most of the curriculum before the school year and on the fly, borrowing liberally from various sources.

This later work (see Gutstein, 2012) was an attempt to put into practice, and study, a full-year, Freirean approach to developing liberatory mathematics from students' lives – that is, to connect *community*, *critical*, and *classical* knowledge. Briefly, we learned that: (a) urban high school students of color can study their social reality with mathematics while developing both mathematics competencies and sociopolitical consciousness; (b) students choosing the contexts supported their engagement and mathematics/sociopolitical learning; (c) students and I co-constructed the classroom environment that supported their collaborative work and learning; (d) teaching critical mathematics required both "up-front" and ongoing, on-the-fly curriculum creation; and (e) teachers' own sociopolitical consciousness is part of interweaving mathematics and social justice.

NOTES

1 This is a revised version of Gutstein (2007). Although this text is single-authored, the teaching, planning, assessment, and analysis of the boundaries project in this story was collectively done with three other people besides the author: Joyce Sia (teacher), Phi Pham (teacher), and Patricia Buenrostro (mathematics support staff). The research described here was partially supported by a grant from the National Science Foundation to the Center for the Mathematics Education of Latinos (No. ESI-0424983). The findings and opinions expressed here are those of the author and do not necessarily reflect the views of the funding agency.

2 "We" refers to the school's two mathematics teachers (Phi Pham and Joyce Sia) and the other mathematics support staff-person (Patricia Buenrostro), and me. Together, we constituted the school "mathematics team."

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SWAPNA MUKHOPADHYAY, WOLFF-MICHAEL ROTH, &
BRIAN GREER

EPILOGUE

Why Bother about Diversity of Mathematical Practices?

The term epilogue derives from the Greek *epilogos*, the concluding or perorating part (Gr. *epi-*, in addition) of speech (Gr. *logos*). In more recent times, the term also denotes the concluding part of a literary work, a summary. An interesting form of an epilogue consists in the metalogue, a conversation that takes previous conversations or text to take it to another level (Bateson, 1980). Metalogue literally is talk about talk, meta-talk. The interesting aspect about authentic talk is its unpredictable, dialogical nature, so that even an author of a truly literary work, such as Feodor Dostoevsky, cannot anticipate where a dialogue between protagonists or the internal dialogue of a single protagonist will lead (Bakhtin, 1984). It is a genre that allows new forms of understanding to emerge even within a team of authors and editors, because their differences come to exhibited rather than disappearing in the authorial voice – e.g., that of the introductions to the different parts of this book for which we take collective responsibility (Tobin & Roth, 2006). After completing and repeatedly editing the remainder of the book, we began the following conversation with some comments or fragmentary statements, and then allowed our conversation to emerge and evolve in the very dialogical spirit that we bring to our individual work more generally. We encourage readers to read this text as a starting point for further conversations that they have with others in their respective contexts.

* * *

Brian: As a framework within which to embed the discussion, we can consider four distinct families of mathematical practice: (a) mathematics-as-discipline, (b) mathematics in action, (c) non-academic mathematical practices, and (d) mathematics-as-school-subject. The last is our main concern, and we will focus on how it relates to the other three.

Swapna: To make a strong contrast to the pervasive simplism in which these phenomena are typically framed, we need some term like “activity system” that reflects the complexity of the phenomena under discussion. Illustrative of such simplism, we may point to, for example, the reduction of mathematics to structured content to be transmitted and regurgitated, the intellectually impoverished efforts of curriculum designers, textbook authors, and test constructors, the dehumanizing effects of much experimental psychology as

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applied to mathematics teaching/learning, the acritical assumptions underlying mathematical modeling of social as well as physical phenomena, particularly as such modeling is typically taught in schools, and so on.

Michael: This is precisely what I believe today Martin Heidegger critiqued in his analysis of technology. He was not concerned with the fact that we use cars or machines, airplanes or kitchen devices. Rather, he was concerned with the underlying ideology of simple cause-effect relations underlying not only physical phenomena but also social-psychological phenomena. Nearly everything that we can see in mathematical curriculum development can be characterized by the idea that there is a quick fix; and if we just spend enough money, we can get the right curriculum that "fixes" all kids, all teachers, all problems. Thus, in psychological theories or large-scale curriculum implementation, human beings are but pawns determined psychologically and sociologically. There is little place for *why* they might use mathematics in their everyday lives in this or that way, that is, there is no place for the *reasoned* ways in which we go about our mundane everyday affairs, including shopping, calculating sports statistics, adjusting the amounts of ingredients in a recipe to adapt it for the particular purpose, and so on.

Swapna: This is why at the birth of the lecture series on which this book is based I was motivated to give an airing to the ideas of ethnomathematics. My aim was to invite a diverse audience of educators, mathematicians, and "just plain folks" to contemplate a conception of mathematics connected with people's lived experience, and thereby gain a sense of mathematics as relevant to their personal lives, and hence a feeling of identity as a user of mathematics with agency, of which their experience of mathematical schooling may well have deprived them.

Michael: Reading through the chapters of this book, it can be said without exaggeration that you brought together a very interesting group of people that is contributing to a rethinking of mathematics education.

Brian: You talked about the positivistic ideology that has infected psychology and, in particular, educational psychology. While cognitive science was certainly an advance on Behaviorism, it was characterized, as Gardner (1985, p. 41) put it, by a "de-emphasis on affect, context, culture and history", a comment that still applies to much of the work in mathematics education. However, there have been powerful counter-developments within psychology, notably situated cognition and cultural psychology, and the dominance of mathematics education as a research field by the disciplines of psychology and mathematics itself has been considerably ameliorated by the infusion of more human-oriented disciplines and methodologies.

Swapna: As is clear, also, the political nature of mathematics and mathematics education has inevitably emerged. In the last section of the book, in particular, we see arguments for the potential of mathematics education as a weapon for social/political change. It seems to me that the family of theories termed activity theory affords another theoretical resource for this attempt to reconceptualize towards a humanistic paradigm.

Michael: You are right, there is a range of theories that have been developed from the effort of Lev S. Vygotsky and Alexei N. Leont'ev to create a humanist Marxist theory. They are concerned with understanding the material and ideal aspects of everyday human activity: the concerns people bring to their work, their affect and impulses, their consciousness, and their personalities. In the two main versions that I am familiar with – the more structuralist Finnish model and the more dynamic model that was developed in Berlin in an attempt to develop a critical psychology – there is a fundamental concern for action and changing the world. Thus, both branches of activity theory emerged as attempts to build theories that follow Marx's eleventh thesis on Feuerbach: whereas philosophers attempted to understand the world, *the real point is to change it*. Human beings are not cultural dopes, are not determined by their environment. Human beings are human beings because they have the capacity to change the natural and social environment.

Swapna: I do not think that activity theory inherently gives me a framework for action, though it certainly provides a much richer foundation for deciding on action in relation to one's ethical and political positions. I personally find some of the applications of the theory to bring about societal change a little simplistic in the sense that they treat the manifestations of the system – subject, object, tools, rules, division of labor, and community – as entities in their own right, which they are precisely not. They are parts of a whole, and without this whole, they make no sense. That is, we need to begin thinking in terms of systems rather than in terms of its manifestations.

Michael: In my view, this is so because many scholars simply take pieces of activity theory and fit them into the technical-technological ideology that I note above. There appears to be an inherent push to identify bits and pieces and to build theories in the way engineers build cars: Take independent pieces and then put them together like we built things with Meccano sets. There are no systems out there but forms of discourse that are designed to model life as a system. When you think about life in the way process philosophers encourage us to do – Heraclitus being (one of) the first, Georg W. F. Hegel, Friedrich Nietzsche, and "post-structuralist" philosophers some more recent examples – then you notice that they do not think about life and the world in terms of systems, that is, structures, but in terms of flux, continuous change.

Swapna: That's so true – these systems emerge as a part of life, over extended periods of time. They evolve, within traditions using oral and symbolic forms of communication across time and space, and are embedded in larger activity systems. We know that educational policy, as a top-down imposition, often does not intermesh with existing practices. I am reminded of a case of formal schooling in rural India where the locals never had an opportunity to participate in the literacy movement. This particular group of people, although very poor, has been functioning on their own terms over the ages. (There is no romanticizing the poor here, they also definitely should be given an option to participate in developmental projects and need to be formally educated.) Formal education, in terms of efforts to teach how to read and write, typically follows the banking model whereby the teacher, generally from a different community, deposits the desired knowledge in the learners. This particular group of people has been the lowest of the low in terms of their economic status for very many years. The school as a mode of development was initially exciting for them, but they soon realized that the schools were not valuing what they traditionally do; their children, now formally educated, were progressively becoming misfits in their own communities. Rural poor usually do not speak up, but they started showing their resentment by asking what good is education if it disrupts the community and takes the young away? I consider this as an example of a mismatch in which the larger activity system attempted to assimilate a subsystem without accommodation.

Michael: This embeddedness in larger activity systems, or, perhaps more appropriately, the interconnectedness of activity systems to constitute society is often not addressed by those who subscribe to activity theory. Thus, for example, there are many who use activity theory in mathematics education. But the level of analysis is the individual or the classroom while students engage in some task (e.g., factoring polynomials). But, in my view, this is not an "activity." The cultural and historical activity that students realize is "schooling," and the products of schooling are grades. Thus, for example, I am not at all surprised that students and teachers focus on grades rather than on mathematical understanding – it is the grades that constitute the object/motive and students and teachers act accordingly. There are other settings where we observe precisely the same thing – for example, one of our studies showed that in the upgrading and licensing processes of mariners, college instructors and students collude to assure high levels of success on examinations well-knowing that what students learn is not useful on the vessels where they will subsequently work.

Brian: Alexandre Pais (2012), in penetratingly analyzing the very extensive literature on equity and mathematics education, makes strong points relating to what you've just said. In particular, he argues that most of the work on equity, while initially recognizing that the problems are societal, proposes solutions within mathematics education – better curricula, better teacher education, and so on. He also argues that much of mathematics education research resembles the drunk looking for the key where the light is good rather than where he lost it, and repeating over and over what has hitherto produced no success. He also analyzes the processes of grading, credentialing, constructing failure/success, within a framework of dialectical materialism.

Swapna: A starting position is that all forms of mathematical activity are human activities, involving the body and the emotions as well as the mind – more accurately, involving the whole person as a system that integrates all these aspects of personhood and more – hence irreducibly social/cultural, involving tool use, communication, performance, ritual . . . I am constantly reminded of Alan Bishop's formulation of six pancultural activities – counting, measuring, locating, designing, playing, and explaining – that are essentially mathematical.

Michael: Again, many analyses that I have seen only apparently use activity theory – which in my sense simply has been absorbed into constructivist theories, individual-radical and social alike, where the emphasis is on knowing separate from life as a whole. To me, all forms of constructivism are but the latest instantiation of metaphysics, giving primacy to mind (discourse) and forgetting that we are whole persons, who are anxious, fearful, excited, panic-ridden, high-flying, eating, drinking, hungry, satiated beings in flesh and blood. Thus, how is it possible to consider what students do in class independent of all the other object/motives that they pursue, that is, all the other activity systems that they participate in during their daily lives? And this is where I see a big difference with ethnomathematical analyses, which tend to be concerned with the manner in which various mathematical representations are involved in the daily lives of the people, focusing on their concerns and needs rather than on mental process.

Swapna: Once you move towards this wider perspective, it becomes clear that mathematics and mathematics education are deeply political, in both overt ways and ways that need archaeological explorations to unearth.

Michael: It is interesting that you mention the political, which I believe we cannot ever eliminate using cultural-historical activity theory as it was intended, a framework that allows us to analyze *societal structures* and their effects. I frequently point out the problems with the translations of activity theoretical work from Russian and German, where there is a focus on society, inherently political. Thus, the pair of adjectives *gesellschaftlich/obshestvenii*, which should translate into English as "societal," tends to be rendered instead by the term "social." This latter adjective, however, is completely apolitical whereas the former implies the political. If, in Leont'ev's category *personality,* the individual is a singular "knot-work" of all objective/motives that a person engages in, then personality is a political category. *We are political beings, shaped through and through by the societal relations that we have and continue to participate in*. We are, Vygotsky pointed out with reference to Karl Marx, the totality of societal relations that we are part of in the course of our lives. We are political beings because the realization of object/motives of activities meets generalized societal needs. That is, Leont'ev defines his category in terms of the ensemble of generalized needs that a person participates to satisfy or that it satisfies itself. All these parts that would allow us to better understand mathematical-representations-in-use (as objects or tools) are evacuated from most analyses that profess to use activity theory.

Swapna: The implicit assumption in typical mathematics education is that mathematics-as-school-subject can be unproblematically derived from mathematics-as-discipline (for example, the Common Core State Standards recently developed and being implemented in the US essentially consist of a specification of content, formally described). From many perspectives, including ethnomathematics, and the general position that (mathematics) education should relate to children's lived experience, acknowledging cultural diversity, non-academic mathematical practices should also shape mathematics-as-school-subject. For democratic citizenship, mathematics-as-school-subject should include preparation for agency in terms of having conceptual tools and a disposition to critique mathematics in action.

Michael: I think that we need to slowly unpack what you are saying here. Pertaining to your first proposition, the problematic issue appears to be that we take the object/motives of one activity, doing mathematics, and use it as a normative framework for what everyone else should do or know. But of course, historically and ontogenetically, the development or evolution of knowledge is in the reverse. We go from everyday ways of knowing mathematics and then formalize this knowledge until we get to the idealized mathematical objects. This was so, as Husserl showed, for the development of geometry by the early Greeks. For example, they dealt with rudimentary objects including the *kuklos* or *kubos*, and by continuous refinements of these objects in their environment, they also developed the circle and cube as ideal objects that had properties that none of their real materials could have: be ideally round, having lines without extension, having perfectly equal sides and angles. Moreover, even though they never saw a cube in its entirety – the six sides, eight vertices, twelve edges – they began to think of these as constitutive of the *cube*. The problem many mathematics educators make is that they do not keep separate the phenomenal objects of our lives – e.g., the many approximately cubical objects – and the ideal construct of a cube that geometers and mathematicians deal in.

Brian: The most complete argument for this point of view of which we are aware has come from Hans Freudenthal and his school. Starting to teach mathematics from the perspective of the end-point of historical development (which, indeed, is not finished) in the form of a formal structural architecture is a fundamental error characterized by Freudenthal as a "didactical inversion". Their work stands in stark contrast to the ahistorical view of mathematics as a set of formal structures that can be transmitted directly to students without any recapitulation, under instruction, of their development.

Michael: I agree. This shows you that we do not learn from scholarship. Thus, for example, Edmund Husserl provided us with a really good analysis of the historical development of mathematics, one moment of which is the formal aspect that is only the outcome of the second aspect, the subjective productive activity. It is only in the dialectical structure–agency relation that mathematics comes alive and therefore continues to develop. Another important issue that we ought to theorize better is the difference between activities that realize mathematical object/motives and others where mathematical forms are used as tools in the pursuit of something. Thus, the fish culturists I studied were definitely *not* doing mathematics, even though there were many mathematical representations to be found in their workplace. When you ask them about it, they will say that they are not doing mathematics; their object/motive is the production of a healthy breed of fish. For this purpose they use mathematical representations and tools. When the Yupik built their tents or houses according to particular rules, they are not *doing mathematics* but they are *using* mathematical understandings. Their object/motives are very different and, according to cultural-historical activity theory, the attendant forms of consciousness are very different.

Swapna: The observation that the outsider is seeing mathematics in cultural practices where none exists from the perspective of the insider is a recurrent theme in relation to ethnomathematics, and also, in a more general form, within anthropology. It is the dilemma of outsiders attempting to interpret a different worldview in their own terms. (There is a historical aspect to this question also whereby a spurious claim of timelessness of mathematics is advanced through the imposition of "historically imposed continuity" (Rotman, 2006, p. 125).

Michael: The error of mathematics education is to think that one needs to learn mathematics to apply it. But this is an error in an epistemology that has separated knowing from application. This is the same metaphysical thinking that Immanuel Kant has formally described, a form of thinking inherent in constructivism. I do not therefore see how any hands-on activity or engagement in an ethnomathematical form *should* assist students to learn the formal mathematics that mathematicians take as their object. I have not yet seen analyses to show *why* a student after engaging in basket weaving, where she produces intricate designs, should be any better at taking tests or doing the kind of things that mathematicians do.

Brian: But what you've just said, Michael, could be turned around! Maybe her not doing better on tests reflects a problem with the testing (certainly as currently constituted). And you seem to be suggesting that being good at doing the kinds of things that mathematicians do is the predominant goal of school mathematics, something we have been seeking to problematize.

Swapna: I see the possibility of trying to exploit ethnomathematical expertise in the mathematics classroom as an open question. There certainly are people, some of whom have contributed to this book, who are trying to construct bridges from ethnomathematical practices to school mathematics curricula, especially at the elementary level. For example, Jerry Lipka and his team are currently investigating whether a rapprochement is possible between Yup'ik practices and ideas developed by Jere Confrey relating to the conceptual field of multiplicative structures. There are a number of mathematicians, such as Prigogine, who have suggested that alternative epistemologies (such as Navajo concepts of space and time, as studied by Pinxten) could be a fertile source of information for further developing mathematics-as-discipline.

Michael: I do not deny that we may see relations, but, as I point out above, the *concerns of the thinker* make the two endeavors different, and, according to Vygotsky and Leont'ev, the forms of consciousness that goes with them.

Swapna: Diversity is obvious in non-academic mathematical practices and mathematics in action. It is commonly denied in mathematics-as-discipline, which we can deconstruct. Likewise, most of the mathematics education literature relates to a "stereotypic math classroom." Skovsmose, Valero, and others have done useful work exposing this impoverished view.

Michael: We probably have to differentiate. I am concerned with the forms of consciousness that reflect the activity in which we are engaged. When you say that diversity is denied in mathematics, then I see this as the same as any other form of activity that enforces and stabilizes rules. Take any team sport, like soccer. There are certain rules to be played by, and the referee and linespersons are there to enforce the rules. You cannot play according to different rules, and if you do, then you receive a penalty – yellow or red card – or you may be excluded for several games, months, or for a lifetime from the sport. The case of mathematics is similar. As Husserl showed, it is precisely because of the subjectivity of doing mathematics that the objective forms, public records that result from mathematical activity, obtain normative function. Geometry is an objective science because as long as you play by the rules, you get the same results whether you do it in ancient Greece, in Europe, or somewhere in the Amazon forest.

Brian: It may not be so simple. A comment with the flavor "mathematics is the same for everyone" is typically followed by a reference to the counting numbers or the sum of the internal angles of a triangle. On the other hand, if you talk about probability or non-standard analysis, then it is by no means so clear that mathematics is the same for everyone. Ian Hacking (2001) wrote a paper in which he commented that (some) philosophers propose conclusions about the whole of mathematics based on analysis of a small subset. Raju (2007) argues that there are two streams of mathematics: "(i) from Greece and Egypt a mathematics that was spiritual, anti-empirical, proof-oriented, and explicitly religious, and (ii) from India via Arabs a mathematics that was pro-empirical, and calculation-oriented, with practical objectives" (p. 413). Raju traces the dominance of the former in the Eurocentric narrative of the history of mathematics as far back as the Christianization of the Roman Empire. And the emergence of the computer shifts the balance.

Michael: I think that you can find normative forms also in ethnomathematics: there are rules to follow. For example, the indigenous rules for tent or house design, the rules underlying the patterns that go into weaving blankets or baskets are "handed down." It is precisely with such processes of "handing something down" that Husserl's analysis was concerned. He suggested that there are certain material forms that become the referents for our own work: If our practices produce the material forms – baskets, tents, Euclidean geometry, you name it – to the extent that we recognize them as repetitions, then we also know to have played by the rules. I find that especially in "traditional" contexts, tradition plays a great normative function, perhaps even more so than in academic contexts where "innovation" is prized. So, from this perspective, there are also important regulatory functions in ethnomathematics, as in ethnobiology, traditional ecological knowledge, and other indigenous forms of knowing. The problem may be that mathematics educators do not capitalize on the diversity of activities in which mathematical forms appear but insist on instilling the object/motives of the one activity. And this is where there is a lack of diversity. But, in a way, you conflated diversity within a form of activity and diversity between forms of activity.

Swapna: It is true that cultural practices are largely carried on by tradition. However, I have noted in my studies of boat-builders in the Bay of Bengal that they are capable of adaptation to changes in available materials, availability of tools, and social needs. For example, the type of timber available has changed over the centuries, boat construction has been modified to accommodate motors, and they recently started using an electric drill. Fishermen are now using GPS devices, which they have assimilated to their navigational practices.

Michael: I was not intimating that change is impossible. I just find the normative function in traditional contexts is much stronger than in the kind of contexts where I work. My mother-in-law often comments on my cooking, "this is not the way to do this," my Italian and Portuguese tutors pointed out almost daily that I was gardening in a manner that "they never did it," and my beekeeper mentor said that I was "doing differently than everyone else." These people are from the generation that preceded mine, that is, from my parents' generation that was much more traditionalist than my own. All of these are comments grounded in, and reflective of, cultural practices that change slowly rather than reflective of the rapidity of current cultural change – just look at the changes related to computers, Internet use, or mobile electronic devices that have undergone changes at exponential rates.

Brian: To return to the question of whether mathematics can be used as a tool for social change, I agree with the point you made earlier, Michael, that the grounds for such attempts cannot be derived from activity theory as such (or from any other theory). More generally, as Skovsmose (2005) amongst others has argued, mathematics, as a tool, is neutral: it can be used for good or ill. The grounds for taking a political stance in relation to all forms of mathematical practice, accordingly, are external to mathematics-as-discipline, based on value systems and worldviews, such as the position that cultural diversity is a valuable human resource. In discussions of how to improve mathematics education, I find it absurd that the position is often taken that these are questions that can be settled by research. Research does have an important role but the most important questions, in my view, are questions of values, political questions.

Swapna: And, for me, the most central of these political questions, reflected in historical and contemporary manifestations of hegemony both between and within cultures, is the dominant status of formal knowledge over the knowledge that enables people to live meaningful lives.

Michael: I think that the statement of mathematics as a neutral tool needs to be revisited. Langdon Winner (1980) published a paper entitled by the rhetorical question "Do Artifacts have Politics?" In the paper, where he analyzes the height of bridges over the roads that went from New York City to the beaches, he definitely comes to this conclusion. These bridges are high enough for the cars of the rich people to pass but too low for the buses that the poor folks not owning a car had to take. So the rich could go to the beaches but the poor could not. Take another example of the political nature of mathematics in science. We teach graphing as if it were some neutral skill. But that very tool makes you think about nature in terms of factors and variables (Roth & Bowen, 1999b), which is an ideology contrasting ways of thinking characteristic of traditional ecological knowledge (Roth, 2001), taking us to Swapna's point about thinking systems in terms of their manifestations. We have done a lot of research that shows that even scientists have trouble understanding graphs, even those from introductory course to their own discipline. What struck me initially: even professors say inappropriate things when they attempt to translate a graph into everyday examples (e.g., Roth & Bowen, 1999a). All of this, for me, emphasizes the ideological, that is, political nature of *any* discourse, which is precisely the point that Mikhail Bakhtin and the members of his circle have made (e.g., Bakhtine [Volochinov], 1977). Saying that mathematical discourse is not ideological may constitute a form of idealism and certainly constitutes a form of exemptionalism.

Brian: Perhaps we can say that mathematics is not, in essence, good or bad, but once it is locked into the mechanistic/positivistic ideology you described earlier, and its representations become hardened, it readily affords applications that are dehumanizing.

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Swapna: To return to the question with which we started this wide-ranging trialogue, why do we celebrate diversity in mathematical practices and mathematics education? The main point for me in terms of celebrating diversity comes from the recognition that the world in which we live is diverse – we see biological and cultural diversity no matter where we look. People from different parts of the world, growing up in different environments with different physical demands have different approaches to survival, and to the intellectual, aesthetic, and spiritual activities that transcend survival. Despite this variation, there are cognitive universals at the level of intellectual functioning, including in mathematics. Given that the survival and development of humankind in the world depends on diversity, we need to sustain cognitive diversity, not least in terms of the theme of this book. To the extent that it is moving us towards linguistic and generally cultural homogeneity, globalization represents a threat to the vitality of humanity. Without diversity, the processes of selection have nothing to operate upon. Celebrating diversity is also about fairness in validating the historical intellectual achievements, in mathematics as elsewhere, that have been filtered out by the selective distortions of those holding power.

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