PAUL DRIJVERS, PETER BOON, MARTIN VAN REEUWIJK

8. ALGEBRA AND TECHNOLOGY

> An algebra curriculum that serves its students well in the coming century may look very different from an ideal curriculum from some years ago. The increased availability of computers and calculators will change what mathematics is useful as well as changing how mathematics is done. At the same time as challenging the content of what is taught, the technological revolution is also providing rich prospects for teaching and is offering students new paths to understanding. (Stacey & Chick, 2000, p. 216)

INTRODUCTION

It is beyond any doubt that Information and Communication Technology (ICT) plays an increasingly important role in today's society and in the future professional practices of current students. This raises the question of whether technology might also play a similar role in algebra education and, if so, which role that would be.

In 2008, NCTM, the National Council for Teachers of Mathematics in the United States, formulated a position statement on the use of technology in mathematics education in general. A core paragraph in this document says:

> Technology is an essential tool for learning mathematics in the 21st century, and all schools must ensure that all their students have access to technology. Effective teachers maximize the potential of technology to develop students' understanding, stimulate their interest, and increase their proficiency in mathematics. When technology is used strategically, it can provide access to mathematics for all students.
> (NCTM, 2008, p. 1)

NCTM acknowledges the importance of technology and recognizes its potential, for example for enhancing students' understanding, for stimulating their interest, and for increasing proficiency. More specific for algebra education, a Study Group of the International Commission on Mathematical Instruction, a commission within the International Mathematical Union, focuses on the effects of technology on the teaching and learning of algebra (Stacey & Chick, 2000; Stacey, Chick & Kendal, 2004). The quotation at the top of this page points out the challenge educators have to face while developing contemporary technology-rich algebra education. How can the opportunities that technology offers to algebra education be exploited, without neglecting important aspects of algebraic skills? Which roles can new technologies play in algebra education, and in which way can the teaching and learning of algebra benefit?

P. Drijvers (ed.), Secondary Algebra Education, 179–202.

These are the main questions addressed in this chapter. Due to this focus on algebra education, some aspects of the integration of technology into mathematics education in general will remain unaddressed, such as the changing role of the teacher, changes in classroom arrangement and learning organization, and increasing opportunities for communication and collaborative learning in particular. For more information on these topics we refer to the recent work of Hoyles and Lagrange (2010).

AN EXAMPLE: TRIAL-AND-IMPROVE

Two students in grade 8, Annie and Michael, are working with the applet *Algebra Arrows*[i]. With the applet, they construct arrow chains, which in fact represent functions as input-output machines. The left screen in Figure 1 shows the work of this pair of students on the screen. The first task was to construct an arrow chain which gives 3, 3.2, 3.4, ... as an output table and apparently this worked out well.
The next task is to switch the order of the multiplication and the addition operations and still get the same table of output values (see Figure 1 on the right). The students start with some alternatives, such as 'plus 3 times 0.2' and 'plus 6 times 0.2'. Even if multiplying with 0.2 is correct, they change this factor into 1.2, so their chain is 'plus 3 times 1.2'. This results in an output table of 3, 4.2, 5.4, ..., which they realize is not correct. Next, the observer comes by their desk.

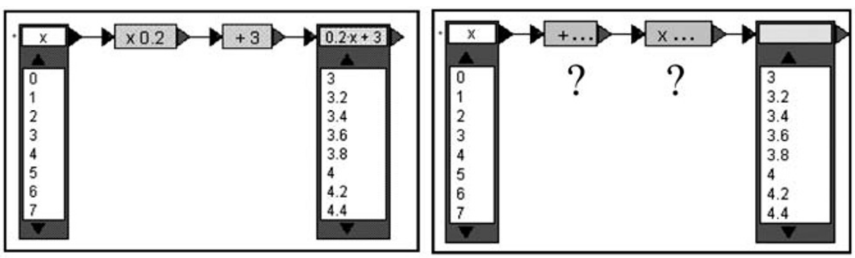

Figure 1. Student work (left) and next task (right)

Observer: Why isn't it [the factor of 1.2] correct?
Michael: Because we don't get the right numbers.
Annie: Oh, the integer numbers here ....
(She points at the integer parts of the numbers 3, 4.2, 5.4, ... in their output table.)
... each time they get one more, whereas here [the integer values of the numbers in the output table that is asked for in the task] constantly 3, 3, 3, 3.

Michael seems to be looking whether the output table shows the required values, whereas Annie pays attention to the increments in the table and notices that these have become 1.2 instead of 0.2, as is required in the task. The students change the factor of multiplication back to 0.2 and try chains such as 'plus 9 times 0.2' and 'plus 18 times 0.2'. In this way, after some trials and improvements, they get the correct chain: 'plus 15 times 0.2'. Then Michael notices the relation between the 'plus 3' in the original chain and the 'plus 15' in the current one: 15 times 0.2 equals 3!

This observation is typical for the learning of algebra using technology in more than one aspect. First, the two students are skilled and clever in using the buttons of the applet and in navigating through the menus. This facilitates their problem solving behaviour, which we could call 'trial-and-improve': they try several options at high speed, hoping to get closer to the solution. Michael's first reaction to the observer's question, "because we don't get the right numbers," suggests that sometimes this approach is (too much!) like haphazardly trying to get the correct answer. At first, the students do not notice that 'plus 15 times 0.2' comes down to the same as 'times 0.2 plus 3'. Meanwhile, the work with the applet at the end leads to the reasoning which shows a growing insight in the phenomenon. After the observer's intervention, the students think about the answer they got and find an explanation for it.

DIDACTICAL FUNCTIONS OF TECHNOLOGY IN ALGEBRA EDUCATION

The question we address now is which roles new technologies can play in algebra education. Before looking for specific answers to this question for each of the strands within school algebra, we first identify the following three global didactical functions for technology in algebra education: technology as a tool for doing algebra, as an environment for practicing skills, and as an environment for developing concepts. Let us consider each of these three didactical functions in more detail.

Technology as a tool for doing algebra

The first didactical function of technology in algebra education is the function of a tool for outsourcing algebraic procedures while doing algebra. Probably the student would be able to carry out the routine procedures by hand as well, but chooses not to spend his energy on that. Just like numerical calculations can be left to the calculator, tables of numerical values can be produced using spreadsheet software such as Excel, graphs can be drawn with graphical software or on a graphing calculator, and algebraic procedures can be left to a computer algebra system (CAS). In these cases, technology acts as a tool, as an 'algebra assistant', and offers a broad range of applications, not necessarily designed for educational purpose. To play this didactical role of tool for algebra, technology should fulfil several criteria, such as mathematical soundness and correctness, as well as flexible support of conventional algebraic notations, representations and operations.

A characteristic of the use of technology as a tool for carrying out algebraic procedures is that the initiative usually remains with the students; they decide whether or not to use the technology for this purpose. A second characteristic is that this didactical function of technology is 'didactics-free,' in the sense that this type of use does not involve a specific didactic approach to or view on the teaching and learning of algebra. The advantage of using technology as a tool for doing algebra is that it relieves the student from a lot of procedural work, and therefore allows for quick investigations of several examples or situations, which can lead to exploration, reflection, and theoretical proceedings.

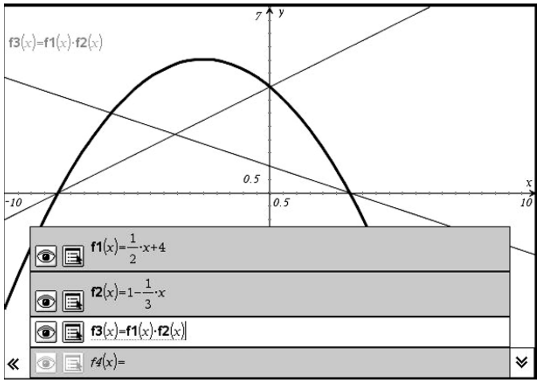

Figure 2. The task ‘multiplying lines’

An example of the latter approach is the ‘multiplying lines’ task (Figure 2). The graphs of two linear functions and their product function are drawn. The question is how specific properties of the product graph are related to those of the ‘building graphs’. Which relations exist between the zeros? What can you say about the vertex of the parabola? Which conditions do the linear functions need to fulfil in order for the parabola to touch the x-axis? In which cases does the vertex of the parabola coincide with the intersection point of the two lines? The technological environment – a graphing calculator in Figure 2, but it could just as easily be graphing software or Excel – takes over the drawing of the graphs and allows for exploration of the effects of changes in f_1 and f_2 on the product graph. The results of the exploration aim at inviting students to algebraic thinking.

Whether or not this works depends on the didactical setting. The danger of using technology as a generator of examples is that students stick to a superficial, phenomenological level of perception instead of entering into underlying fundamental reasoning. Through appropriate tasks and targeted questions, the teacher is in charge of focusing on this deeper thinking level.

Environment for practicing skills

A second didactical function of technology for the learning of algebra is the function of environment for practice. Technology offers several options for practicing algebraic skills. Through intelligent, diagnostic feedback, the technological environment can respond immediately to students' solutions and strategies. Randomization of task parameters allows for a huge variety of tasks, so that students can practice without straight repetition. The pace and length of the session is determined by the student himself. The technological tool is patient and consistent, and mistakes can remain invisible for peers and teacher. There is in fact no need for the teacher to correct mistakes, as this task is taken over by the tool; rather, the teacher can focus on the fundamental and conceptual difficulties that students encounter. The teacher does, however, determine the type of tasks; in that sense, the practice role of technology is often more teacher driven than is the case when ICT is used as just a tool for doing algebra. Also, a digital environment for practicing algebraic skills often implicitly contains didactical choices through the structure and sequence of algebra tasks. Therefore, the didactical function of an environment for practicing algebraic skills is not as didactics-free as the tool functionality described above. Criteria for appropriate tools for practicing algebra are good features for feedback on and registration of student work, and compatibility of problem solving strategies and procedures within the technological environment with those of paper-and-pencil algebra (Bokhove & Drijvers, 2010).

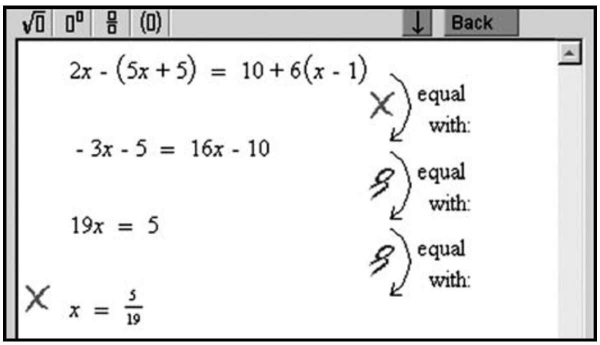

Figure 3. Practicing solving equations[i]

An example of a digital environment for solving equations is shown in Figure 3. The applet *Solving Equations* functions as an 'algebra-repetitor', which offers exercises, provides feedback, and motivates through its game-like reward structure. The applet consists of different versions or levels, that differ in the amount of support that is provided while solving the equations. At the basic level, the student just needs to indicate the operation that is needed, and the applet carries out the algebraic calculation. At the next level, the student has to carry out the algebraic operations himself, but he gets feedback on the correctness of the work. The third level is a self-assessment, which is corrected and graded by the applet. The fourth and final level is the test, which the teacher does not need to correct either.

Environment for developing concepts

A third didactical functionality of technology for the learning of algebra is its use for the development of concepts and mental models. The aim is to evoke specific thinking processes and to guide the development of the students' algebraic thinking. For example, ICT may help to visualize a concept, or present it in a dynamic way, which can lead to a more versatile and deeper conceptual understanding of the mathematical object or procedure. Also, the ICT environment can function as a generator of examples, which provoke the students' curiosity and invite generalization or investigation of relationships or properties.

This didactical functionality is the most complex of the three we distinguish. First, this type of use of technology requires a careful didactical analysis of the relationship between the use of the tool with its representations and techniques on the one hand, and the mathematical thinking and skills that the students are supposed to acquire on the other. This relationship is subtle and complex: a mismatch between the two may reduce the benefit of the work with technology to zero. In addition, more than the other two, this didactical functionality of technology is guided by the teacher and also embodies didactical choices and views.

Criteria for technology that supports concept development are a perfect match between the representations and techniques in the tool environment on the one hand, and the mental images and conceptual understanding on the other. Furthermore, some construction space is needed for students to develop their thinking.

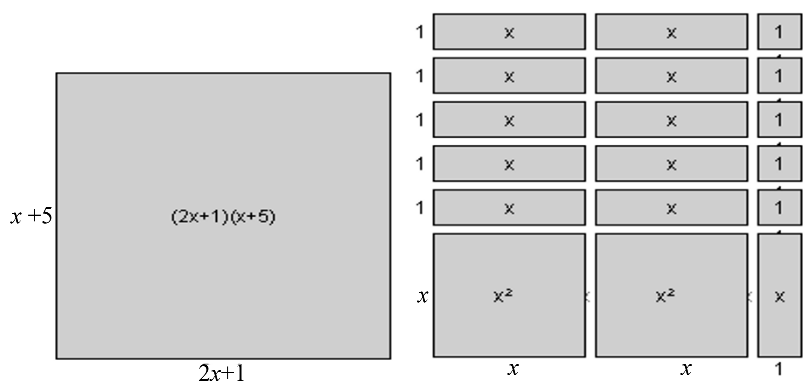

Figure 4. Multiplying two linear expressions in the applet Geometric Algebra 2D

An example of the use of technology to develop concepts is the work with the applet *Algebra Arrows*, described in Figure 1. It aims at the development of a mental image of the concept of function as an input-output machine, which transforms a number of input values into a strip of output values through a chain of operations. A second example is the applet *Geometric Algebra 2D*[i]. It represents an environment to use the area of tiles as a model to think about the multiplication of two algebraic factors (Figure 4). The applet offers opportunities for splitting up, moving and merging rectangular tiles, which represent algebraic expressions. This way, the area model becomes a meaningful model to the student, one they can fall back on in future, for example with expanding.

Such a conceptual model environment allows the student to investigate many different situations. By doing so, a distance emerges between the work in the digital environment and the concrete context that forms the motive for the task. The work within the technological environment will exceed the specific context; the reasoning with the model acquires a more general and more algebraic character. This invites abstraction and the development of a mental 'algebra world'. It is in this invitation that the power of technology as an environment for concept development lies; to exploit this power remains a task for the teacher.

Didactical functions intertwined

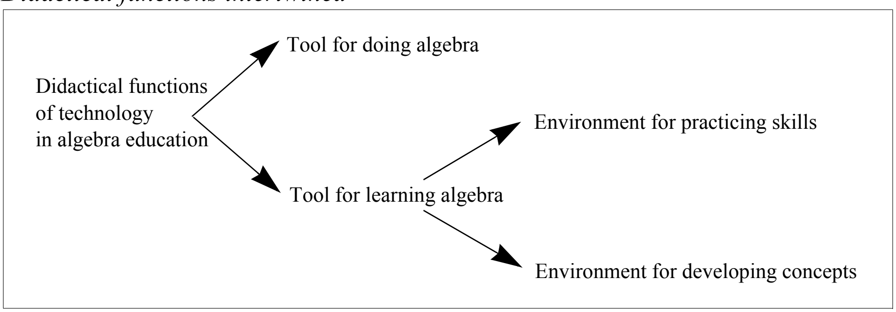

Figure 5. Schematic overview of didactical functions of technology in algebra education

Figure 5 shows a schematic overview of the three main didactical functions of technology in algebra education. It should be noted that these three functions are not properties of the technological tool, but of the way in which it is used in students' learning activities. This being said, some tools are more appropriate for specific didactical functions than others:

> Tools matter: they stand between the user and the phenomenon to be modelled, and shape activity structures. (Hoyles & Noss, 2003, p. 341)

The three didactical functions of digital tools are not mutually exclusive, but are intertwined. The insights that students develop need application in practice; practicing tasks and appropriate use of tools require conceptual understanding. As an example of the intertwinement of didactical functions, Figure 6 shows a sheaf of graphs for the set of functions $x \to x^4 + b \cdot x^2 + 1$. The didactical functionality of the technology is the tool function: the graphing, that the student could do by hand, is outsourced to the tool, because drawing a family of graphs is time consuming and not practical to do by hand. Meanwhile, through the visualization that the technology offers, and the opportunity to change for example window settings or parameter values, exploration becomes possible, and new questions arise. It seems that the curve through the vertices is a parabola, but is this really the case? How does the number of zeros depend on the value for the parameter *b*? This way, technology invites exploration, which leads to new insights and to the understanding of the concepts of parameter and families of functions.

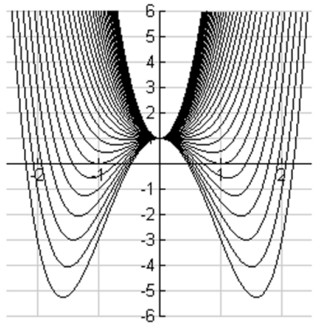

Figure 6. A sheaf of graphs: outsourcing the work to raise new questions

Nowadays, the different didactical functions of ICT for algebra education can be better exploited in educational practice than was the case in the past. Through the internet and increasing interoperability, students can continue their work at any time and in any place, and communicate with their peers and their teacher. The interactive whiteboard is a powerful means to make students engage in whole-class interactions, in which technology plays an important mediating role. Electronic learning environments such as Blackboard, Brainbox, and, more specific for mathematics, the Digital Mathematics Environment integrate many functionalities: they distribute and provide content, i.e. digital courseware, they host students' digital notebooks and portfolios, and supply virtual workspace in which collaborative work and communication are supported. The teacher can monitor the students' progress (see also Figure 14). In addition to this, he can arrange tools and, by using authoring tools, customize content and adapt it to mathematical or pedagogical goals. As a result, the teacher, in his role as designer of his course, acquires ownership of his teaching.

PATTERNS AND FORMULAS WITH TECHNOLOGY

In previous chapters of this book we distinguished three strands within algebra education: patterns and formulas, restrictions, and functions and graphs. How can the three didactical functions of technology be integrated in each of these strands? In this section, we answer this question for patterns and formulas; similar discussions of the two other strands follow in the subsequent sections.

Patterns and technology

As far as patterns are concerned, the main contribution of technology is that it can help to generate examples that invite sorting, pattern recognitions, generalization and investigation. Initially, technology functions as a tool for doing algebra. As the activities proceed, the use acquires the character of an environment for conceptual development. Figure 7 shows a first example of this, which concerns the reproduction with Excel of one of the arithmetic patterns described in Chapter 4. The regularity in the output begs for an algebraic proof. The technological environment, in this case spreadsheet software, supports the finding of similar arithmetic patterns. Research suggests, however, that young students (12-13 year old) may encounter difficulties while copying formulas in a spreadsheet (Haspekian, 2005).

A	B	C	D	E	F
n	**n^2**	**n-1**	**n+1**	**(n-1)*(n+1)**	**n^2-(n-1)*(n+1)**
1	1	0	2	0	1
2	4	1	3	3	1
3	9	2	4	8	1
4	16	3	5	15	1
5	25	4	6	24	1
6	36	5	7	35	1
7	49	6	8	48	1
8	64	7	9	63	1
9	81	8	10	80	1
10	100	9	11	99	1
11	121	10	12	120	1
12	144	11	13	143	1

Figure 7. Creating a pattern with Excel

An example of a higher level of pattern generation and recognition is shown in Figure 8. Computer algebra acts as a tool to factor expressions of the form $x^n - 1$ (Lagrange, 2000; Kieran & Drijvers, 2006). In itself, this is just a matter of pressing buttons. The results, however, raise several questions: in which cases does one get exactly 2 factors, and in which cases more than 2? Do we always get a factor $x + 1$ when n is even? How can you be sure of that? For $x^6 - 1$, we might have expected $(x - 1) \cdot (x^5 + x^4 + x^3 + x^2 + x + 1)$. Is the result shown on the screen equivalent with that? How does the software find its answers anyway, and how would we find the same results with paper and pencil? These and other reflective, mathematical questions enhance the development of algebraic meaning.

$$
\begin{array}{ll}
\text{factor } (x^2-1) & (x-1)\cdot(x+1)\\
\text{factor } (x^3-1) & (x-1)\cdot(x^2+x+1)\\
\text{factor } (x^4-1) & (x-1)\cdot(x+1)\cdot(x^2+1)\\
\text{factor } (x^5-1) & (x-1)\cdot(x^4+x^3+x^2+x+1)\\
\text{factor } (x^6-1) & (x-1)\cdot(x+1)\cdot(x^2+x+1)\cdot(x^2-x+1)\\
\text{factor } (x^7-1) & (x-1)\cdot(x^6+x^5+x^4+x^3+x^2+x+1)\\
\text{factor } (x^8-1) & (x-1)\cdot(x+1)\cdot(x^2+1)\cdot(x^4+1)\\
\text{factor } (x^9-1) & (x-1)\cdot(x^2+x+1)\cdot(x^6+x^3+1)
\end{array}
$$

Figure 8. Factoring expressions of the form $x^n - 1$ with a computer algebra tool

This example shows that computer algebra software can invite exploration and algebraic thinking. However, the use of computer algebra does require some prior time investment, as the algebraic flexibility such environments offer has as its price a syntactic rigidity in relation to entering expressions and commands. Because of this investment, computer algebra is also used as a backbone of front-end educational technology, that is put in action to check student results on algebraic equivalence with desired results, for example. As a result, computer algebra offers better error detection and therefore student feedback.

$$
\begin{aligned}
\frac{d}{dx}x^2 &= 2x\\
\frac{d}{dx}x^3 &= 3x^2\\
\frac{d}{dx}x^4 &= 4x^3\\
\frac{d}{dx}x^5 &= 5x^4\\
\frac{d}{dx}x^6 &= 6x^5\\
\frac{d}{dx}x^7 &= 7x^6
\end{aligned}
$$

Figure 9. A pattern of derivatives

Another example of the use of computer algebra for generating patterns is provided by Berry, Graham and Watkins (1994). The idea in the example is that students first use computer algebra to differentiate a number of functions, to investigate the pattern in the results, and finally reflect on the meaning of the differentiation (see Figure 9). This is an example of the so-called BlackBox-WhiteBox approach, in which students are first confronted with the results of working with technology, which are the motive for a subsequent investigation on what is really happening, what it means, and how one would find these results with paper and pencil. This approach is a reaction to the WhiteBox-BlackBox principle, where students carry out relevant algebraic operations by hand first, and only use ICT for outsourcing operations after skills and insights have been developed (Buchberger, 1990).

The above examples show how the didactical functions of tool for doing algebra and environment for concept development can be aligned: the algebraic power of the technology is used to generate examples that in a subsequent step are subject to algebraic reasoning.

Formulas and technology

In the second and third example of the previous section, formulas play a central role. The examples show that technology can generate formulas, and, in the case of computer algebra, transform them to other forms. It is interesting to notice that it is not always trivial to know which command leads to the form the user want to get, and, conversely, to recognize which 'story the algebraic form tells'. Figure 10, for example, shows how an algebraic function definition is rewritten by computer algebra through the commands factor and expand. An expert user, who is skilled in 'reading' formulas, recognizes the zeros and the vertical asymptote of the graph in the second form, and the equation of the other asymptote in the third form. The ability to interpret the computer algebra output requires a considerable amount of insight into the structure of algebraic expressions. That insight is part of the algebraic expertise which was labelled symbol sense in Chapter 1.

$$f(x) = \frac{4x^2 + 7x}{8x - 2} \quad \text{“Done”}$$

$$\text{factor}(f(x)) \qquad \frac{x \cdot (4 \cdot x + 7)}{2 \cdot (4 \cdot x - 1)}$$

$$\text{expand}(f(x)) \qquad \frac{1}{4x - 1} + \frac{x}{2} + 1$$

Figure 10. Rewriting a function definition

Several technological tools can help students to acquire that insight, and thus act as environment for the development of algebraic concepts. An example of this didactical functionality is the applet *AlgebraExpressions*, in which students create tree representations of algebraic expressions[i] (Figure 11). These expressions can have an increasing complexity, in which the structure of partial trees and the hierarchy of operations remain transparent. The tree representation is a model that can also back up students' paper-and-pencil work, when they encounter complex formulas.

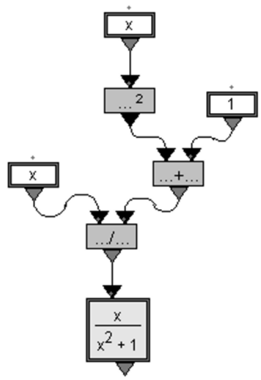

Figure 11. A tree representation of an expression made in the applet AlgebraExpressions

Entering formulas needs attention while working with formulas and expressions in technological environments. Some ICT-applications provide one-line formula entry, which means that using brackets is required. Figure 12, for example, shows how the expression

$$\frac{x}{x^2+1}$$

is entered in the graphing calculator TI-84 and in Excel. One can imagine a student forgetting to use brackets, and thereby accidentally entering

$$\frac{x}{x^2}+1$$

To avoid such mistakes, a two-dimensional 'pretty print' formula editor is preferable, and is getting more and more common.

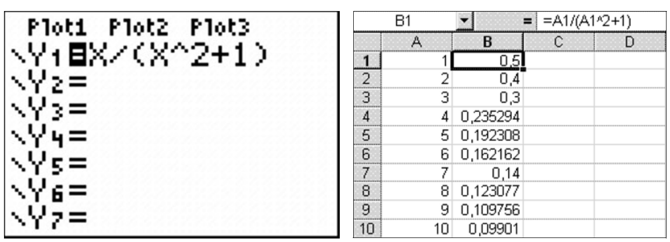

Figure 12. Entering expressions in the TI-84 and in Excel

RESTRICTIONS WITH TECHNOLOGY

For solving equations and dealing with restrictions, a range of technological tools is available, each with its own focus on mental models, practice or use. For practicing solving equations, applets can be used (see for example Figure 3). Figure 13 shows a variation of this applet, as well as how a student solves a similar equation with paper and pencil. The student's writing clearly reveals the transfer of strategy and notation from the applet environment to paper-and-pencil.

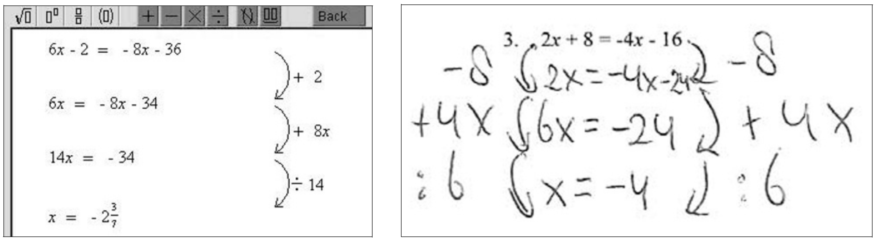

Figure 13. Solving equations: strategy transfer from technology to paper and pencil

There is a danger that work with technology has a fleeting character for the student. Screens appear, screens disappear and not much tangible remains after the session. To avoid this, student results can be saved in an individual digital workspace. This allows students to review and revise their work and in this way create their personal digital notebook; for the teacher, this type of registration offers means to monitor student progress, and to correct and eventually grade work. Figure 14 shows an example of the features of such a system for the teacher, in this case the Digital Mathematics Environment[ii]. In such systems, teachers can easily check the students' homework and, while preparing the next lesson, identify any difficulties they need to give more attention to.

The graphing calculator can be used as a tool for just solving equations graphically or numerically. One method consists of intersecting two graphs, corresponding to the left hand side and the right hand side of the equation, respectively. An advantage of this approach is that students develop a mental image of solving an equation as finding intersection points, which is an appropriate image for equations in one single variable. An alternative approach for solving equations with a graphing calculator is to use the solve module.

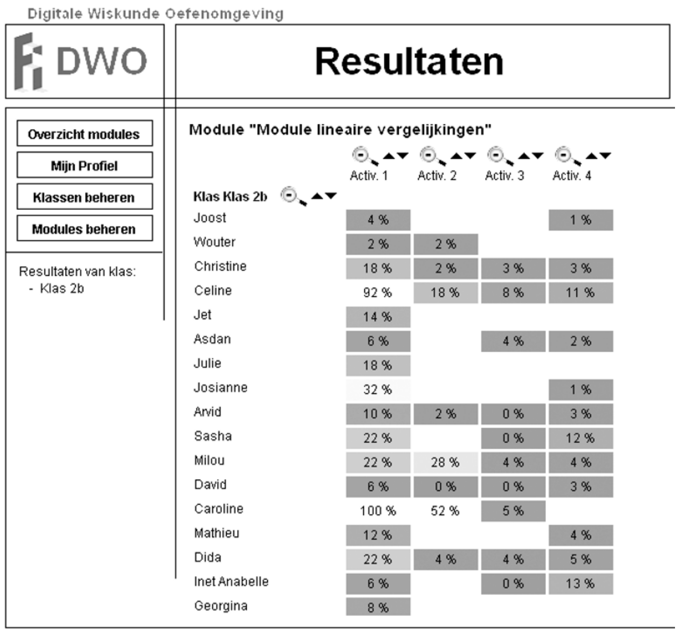

Figure 14. Screen shot of the Digital Mathematics Environment for student registration

If exact or symbolic solutions are required, a computer algebra tool is needed. Solving an equation with such CAS tools, which are available for both desktops and handheld devices, seems straightforward and easy. For novice users of this type of technological tool, however, this is not trivial, because the solve technique highlights aspects of solving equations that often remain underexposed in work with paper and pencil (Drijvers & Gravemeijer, 2004). For example, students often are unaware of the differences between algebraic expressions and equations, which leads to trying to solve for example $x^2 + b \cdot x + 1$ instead of $x^2 + b \cdot x + 1 = 0$. Also, in cases of equations with more than one variable, students often do not realize that an equation is always solved *with respect to an unknown*, and we can not expect the technology to know which variable plays that role in the equation at stake. The unknown, therefore, must be specified, something which often remains implicit while solving with paper and pencil. If the solution of a parametric equation turns out to be an expression instead of a numerical value, students may feel that 'nothing is really solved', as their interpretation of a solution is restricted to numerical outcomes. As a final issue while solving equations with a computer algebra tool, the upper part of Figure 15 shows that the solution of the general quadratic equation in some cases is represented differently from the form in which it usually appears in text books. The solution of the second equation, $x^2 + b \cdot x + 1 = 0$, is not copied correctly by one of the students in her notebook (Figure 15 bottom part).

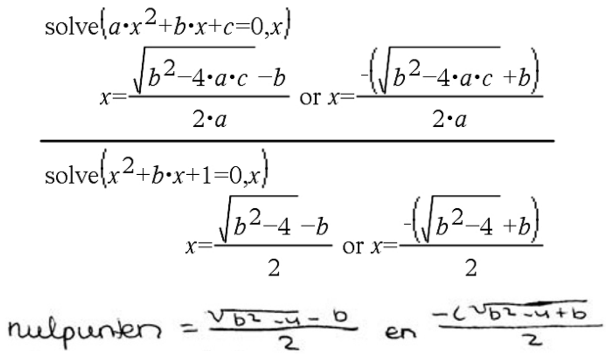

Figure 15. Difficult representations in computer algebra software

Solving an equation algebraically using computer algebra requires intertwined technical and conceptual insight. This is expressed graphically in Figure 16, taken from Drijvers & Gravemeijer, 2004. In general, the execution of a problem solving procedure in a computer algebra environment highlights different insights than the paper-and-pencil method: there is a certain distance to the executive work, but the work must be formulated at a more abstract level. One has to 'make the work be done' instead of doing it oneself. This requires a much deeper awareness of underlying conceptual aspects.

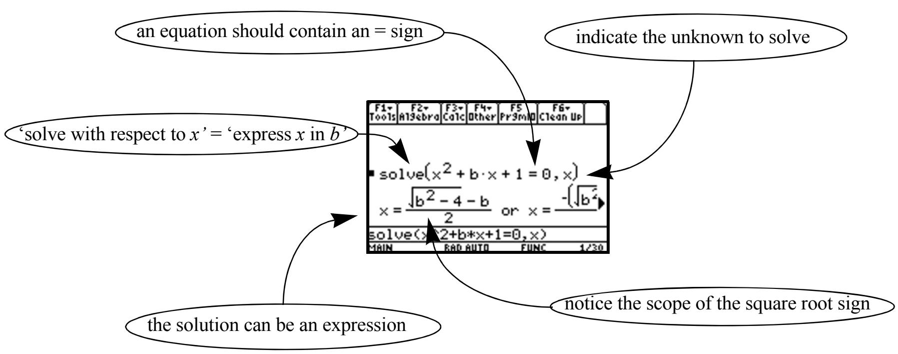

Figure 16. Conceptual and technical aspects of solving equations with CAS

FUNCTIONS AND GRAPHS WITH TECHNOLOGY

The concept of function and its representations

In this section we address the part technology can play in the teaching and learning of functions and graphs. In which ways can technology support the acquisition of the function concept?

The introductory example of this chapter concerns the applet *Algebra Arrows* (see Figure 1). This applet provides options for building arrow chains of operations, an activity which is intended to support students' concept image of functions as input-output machines (see also Chapter 6). It emerged that students also used the arrow chains as representations of functions on paper (Figure 17).

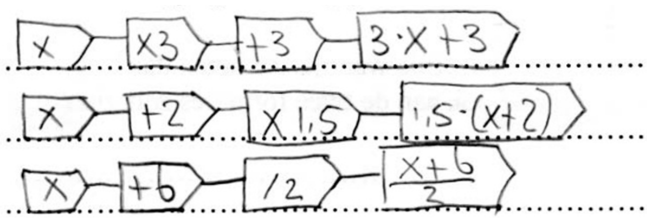

Figure 17. Transfer of the arrow chain notation to paper-and-pencil work

The same applet can be used to evoke the concept image of a function as a mathematical object with different, interrelated representations (Figure 18). In this case, the input is variable, and this variation causes the output to vary as well. With the function as input-output machine as a point of departure, the different function representations appear in one window: the arrow chain, the table, the graph and the formula (Doorman et al., in press; Drijvers et al., 2007). These representations are connected to each other. For instance, when scrolling through the values of x in the input table, the output value in the other table changes accordingly, as well as the point in the graph. This allows students to experience the different representations as different views on the same mathematical object. This way, thee applet provides an environment that supports the development of an integrated function concept. As such, it is not unique; many technological environments offer means to view different function representations simultaneously, and to study the effects of changes in one representation to the others. Technology has a lot to offer here.

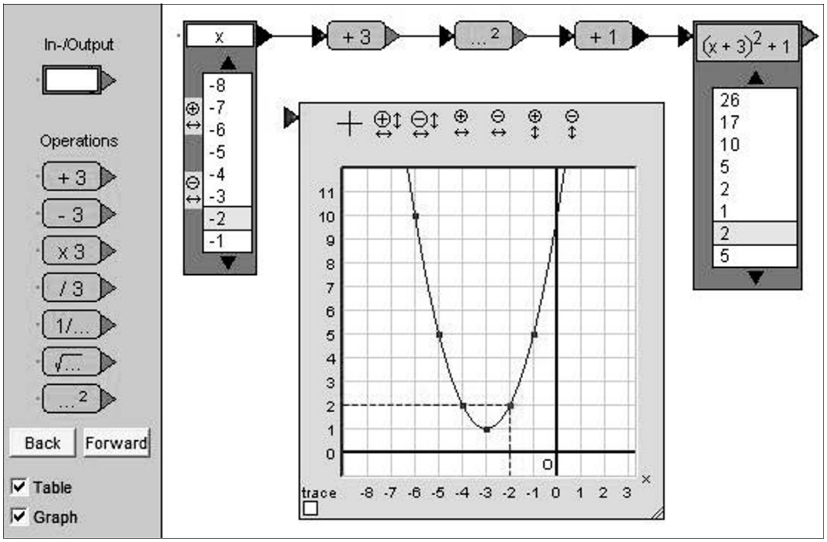

Figure 18. Function with different representations in the applet Algebra Arrows[i]

Graphs as particular function representations

With formulas, graphs can be seen as the most important function representations. ICT tools such as graphing calculators, graphical software and spreadsheet software can draw graphs quickly and accurately. By changing the window settings, students can easily get different views on a graph and zoom in on relevant details. By tracing a graph, students can investigate the co-variation of dependent and independent variable (Figure 19, left screen). The independent variable is no longer a placeholder or a generalized number, but a changing quantity which runs through the horizontal axis, causing the dependent variable to change on the vertical axis.

Probably the most powerful image of a variable as a changing quantity is generated through a slider bar, as available in Excel and many other function graphing tools. By dragging the pointer along the slider bar, the student can dynamically vary the value of a variable, for instance a parameter, in a seemingly continuous way. The right screen of Figure 19 provides an example, which is unfortunately static on paper.

In short, there are many technological tools that generate tables and graphs. Students can use them to explore change and to experience the dynamic character of a variable. These ICT-applications are considered meaningful, as they enrich the students' concept image of function. Whereas in the past, graphs used to be the end point of a laborious algebraic function investigation, now they form accessible starting points for further exploration (Kindt, 1992ab). In this way, the use of technology gradually affects the content and pedagogy of algebra education.

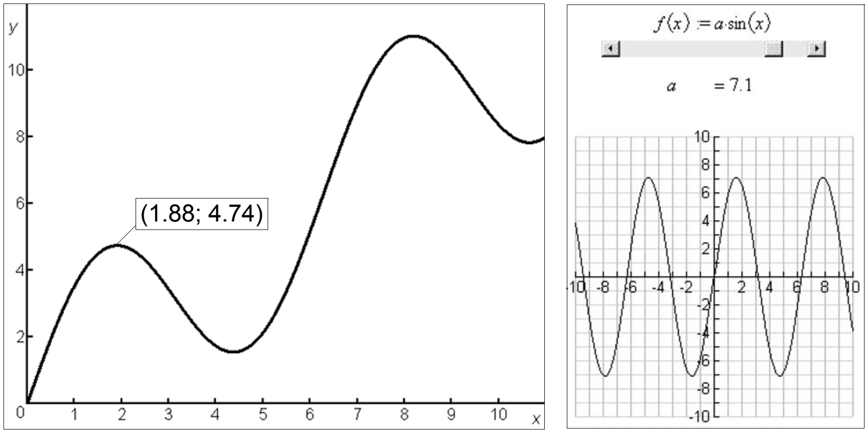

Figure 19. Tracing a graph and using a slider bar

Even if technology can be very useful for graphing functions, we need to be aware of the difference between the graph as a set of pixels that students see on a screen, and the mathematical object of a graph, which in fact comes down to the function definition as a set of ordered pairs. Particularly when screen resolution is low, as is the case on graphing calculators, the difference can be striking and students are not always able to bridge the gap between the two.

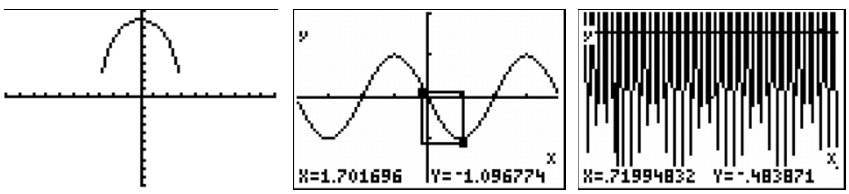

Figure 20. Misleading graphs on the screen of a graphing calculator

As an example, Figure 20 shows two misleading graphs. The left one is the graph of $x \to 3\sqrt{9-x^2}$ on the domain [-10, 10]. The half circle that the graph in fact is, does not touch the x-axis on the screen of the – first generation! – graphing calculator. The graph in the middle is the graph of $x \to \sin(95 \cdot x)$ on the interval [-2π, 2π]. However, zooming in on the box shown in the middle screen provides the graph in the right screen. Apparently, the graph in the middle screen is too smooth and hides much of the function's variation! Student will have to learn to deal with graphical limitations such as the ones shown here. Classroom discussions are a way to make explicit the differences between discrete graphs consisting of approximated screen pixels, and smooth, continuous graphs as they exist in mathematical theory. Another teaching strategy is to exploit technology's limitations by challenging students to create misleading graphs on their screens. This somewhat surprising task may fascinate students and can invite deeper understanding of graphs.

The above example shows that the graphing options of technological tools can be used to work on unusual tasks. Another example of this is shown in Figure 6. In line with this is the example in Figure 21. The task is to find an equation of the curve that 'touches' each of the line segments of this pattern, or, as an easier variant, to show that the graph of the function $f(x) = x + 10 - 2\sqrt{10x}$ has this property.

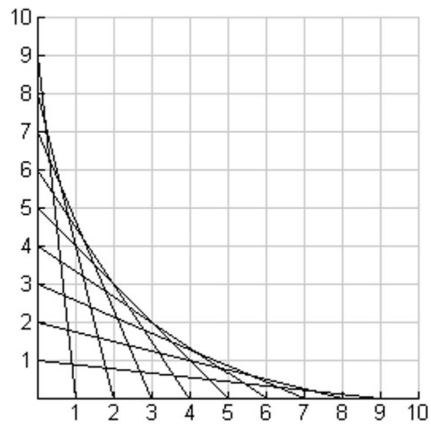

Figure 21. A sheaf of segments motivating algebraic questions

To summarize this section, we conclude that technology offers opportunities to work with formulas, to draw graphs and tables, and to combine and integrate different function representations. The technology plays the didactical roles of tool to carry out the work, and of environment for concept development.

CONCLUSION AND REFLECTION

Conclusion

The central question in this chapter is which roles new technologies can play in algebra education, and how the teaching and learning of algebra can benefit from these roles. In answer to the first part of this question, three didactical functions of technology in algebra education are distinguished: the function of tool for carrying out the algebraic work, the function of environment for practicing skills, and the function of environment for concept development. These three didactical functions, which differ in their degree of guidance by the teacher, are not mutually exclusive and may merge. However, each function does put specific demands on the technology. For the tool functionality, it is important that conventional mathematical notations can be used, that a standard repertoire of algebraic procedures is available and that it is carried out correctly. For the functionality of environment for practice, it is important that the techniques supported by the technology match with the paper-and-pencil strategies that students need to master. Furthermore, adequate feedback is an important feature. For technology as an environment for developing concepts, a requirement is that the activities, techniques and representations in this environment will indeed evoke the concept images and insights as intended. Even if these criteria may sound trivial, it is not always easy in practice to foresee the subtleties of the use of technology in each of the three roles in detail while preparing a lesson. This brings us to the second part of the question: how can the teaching and learning of algebra benefit optimally from these roles?

For each of the three algebra strands, patterns and formulas, restrictions, and functions and graphs, the chapter provides examples of meaningful ICT applications, which aim at capitalizing on the opportunities technology offers for algebra education. These opportunities can be labelled as variation and dynamics (e.g. see Figure 19), as generation of examples that invite pattern recognitions and generalization (e.g. see Figure 8 and Figure 9), as visualization (e.g. see Figure 1 and Figure 11) and finally also concern exploration and investigation (e.g. see Figure 2). This set of opportunities, exemplified in these concrete tasks, forms the answer to the question of how to use technology in algebra education. These opportunities have in common that they can help in changing the student from a passive 'consumer' of algebra into an active investigator, which may improve students' motivation as well as the efficiency of their learning.

Reflection

In this reflection we first focus on the *role of the teacher*. In spite of the positive description of the opportunities technology offers for algebra education, ICT is not a panacea that will make all old didactical difficulties of algebra education disappear. Exploiting the opportunities identified above requires a profound didactical consideration and preparation of the way in which technology plays a role in the learning process of the algebra topic under consideration, and how this is made concrete in the mathematics lesson. Research suggests that this so crucial didactical consideration and preparation is not an easy job for the teacher (Drijvers & Trouche, 2008; Drijvers et al., in press; Lagrange & Ozdemir Erdogan, 2009; Ruthven, Deaney, & Hennessy, 2009). Even if the feedback functionality of an ICT environment, for example, can relieve the teacher from providing feedback, other aspects of teacher-student interaction cannot be taken over by the technology. There remains much to do for the teacher: raising reflective questions, summarizing, enhancing convergence by means of whole-class discussions, sketching lines of thought, inviting exploration of and reasoning about the results found through the use of technology, relating the work with technology and the work with paper and pencil, monitoring student achievements, diagnosing difficulties students encounter in working with technology, etcetera. All these issues require thoughtful attention, and may require the development of new teaching techniques and didactical skills. For technological tools that will be used for a longer period, such as for example Excel or graphing calculators, the teacher may want to orchestrate the development of shared machine skills, so that a set of standard techniques emerges in the class. Also, the teacher may need to take care of the changing didactical contract. For example, students need to develop a critical attitude towards the limited power of technology in mathematical proofs and will need some guidelines on the paper-and-pencil skills they are supposed to master. Responding to all the needs and questions the use of technology brings to the fore, the teacher may extend and adapt the didactic repertoire of teaching techniques en orchestrations used in teaching.

A prerequisite for this process to happen, of course, is a good infrastructure. This comes down to good and accessible ICT facilities, adequate technical support, the possibility to access work from home as well as at school, and the availability of appropriate means for communication with students and evaluation of their work. Technological developments such as wireless networks, netbooks, handheld computers and interactive whiteboards contribute to such an infrastructure.

As a second reflection, we want to address an important issue in the discussion on the role of ICT in mathematics education in general, and algebra education in particular: the tension that is often assumed to exist between the use of technology for algebra and *the acquisition of procedural algebraic skills with paper and pencil*. Do students learn how to carry out algebraic work with paper and pencil, if they can outsource all the work to a technological device? What is the relationship between the use of technology for algebra and paper-and-pencil basic skills? As a first remark, we claim that the use of ICT can contribute to the development of algebraic insight and the mastery of algebraic skills, as they play a role in paper-and-pencil work. Several studies (e.g. Heid, 1988) suggest that techniques carried out in a technological environment prepare for the algebraic by-hand skills. A prerequisite for this transfer to take place is that the techniques used with the technology are to a certain extent similar to the paper and pencil ones, and that students are able to reconcile the results of their paper and pencil work with the output technology provides (Kieran & Drijvers, 2006). In addition to this, the use of technology can complement the work with paper and pencil. The example shown in Figure 16 suggests that solving an equation in a computer algebra environment stresses other aspects than solving by hand, such as the notion that one of the variables plays the role of the unknown. Similarly, solving an equation graphically with a graphing calculator with an intersect technique highlights the idea that solving an equation can be considered as finding an intersection point of graphs. As applying ICT-techniques stresses different aspects compared to the paper-and-pencil work, it can complement the traditional methods.

Meanwhile, algebraic work with technology often has a different character in comparison to work with paper and pencil, as the student takes on more the position of a supervisor than that of labourer. Because one cannot be a good supervisor without experience as labourer, some paper-and-pencil skill remains indispensable. Paper-and-pencil skills need to be acquired, practiced and maintained in order to remain operational. If the use of technology means that skills are not maintained, it can only be expected that mastery decreases. Our concern, therefore, is to find a balanced combination of algebra 'with the mind, on paper and on a screen'. Even if we do acknowledge the additional value of technology in mathematics education, it will not render paper and pencil redundant, but rather support and complement it.

As a final reflection, let us briefly consider the *future of technology in algebra education*. The development of ICT tools, which mainly takes place outside the educational community, is expected to continue at a high pace. Think of mobile technology, netbooks and handheld computers, of serious gaming. Learning can happen any time, any place, on interoperable platforms; communication facilities guarantee that the learning process does not need to be a solitary one. Therefore, connectivity in more than one sense is a key idea for future developments (Drijvers, Kieran & Mariotti, 2010). The opportunities for the teacher to monitor, support and evaluate student work will further increase. Digital portfolios are ways to avoid the fleeting character that sometimes characterises the use of technology. Assessment can take place digitally as well, and can be flexible in time and in content. Teacher and students communicate through the digital learning environment and during digital meeting hours. These developments have pedagogical consequences. We already mentioned the need to find an equilibrium between paper-and-pencil skills and the skills that a technological environment requires to become a meaningful algebra tool in the hands of a student. The exact position of this equilibrium depends on the goals of algebra education, which are subject to reconsideration due to the current technological developments. As a tentative outcome, one may expect a shift towards processes such as mathematizing and modelling, at the cost of basic procedural skills. The ability to translate a problem situation into algebraic terms and into machine techniques, for example, is likely to become more important than it already is, as is the case for the ability to relate graphical and algebraic properties. Flexible problem solving behaviour is required, as the affordances and constraints of the technology will appeal to creative and inventive problem solving behaviour. Assessment of these types of higher order skills is not easy, but it seems logical that technology will play a role there as well. Meanwhile, assessment will also include paper-and-pencil tests for basic algebraic skills.

Conclusive for the success of the use of technology in algebra education will be the way in which teachers and the mathematics educational community as a whole manage to integrate the new media into teaching in a natural and meaningful way. To make the somewhat optimistic scenario sketched above come true, it is crucial that teachers' professional expertise concerning the use of technology in mathematics education will be further developed and that the design of good practice teaching examples and courses for professional development will be facilitated.

NOTES

i Available at http://www.fi.uu.nl/wisweb/en/

ii Available at http://www.fi.uu.nl/dwo/en/

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