Chapter 18
Quantization of Gauge Fields Using the Path-Integral Method

The quantization of the electromagnetic field, which is the simplest gauge field, was discussed in Chap. 5. The problem arising in that case was that, if we apply the usual quantization discussed in Chap. 4 using the gauge invariant Lagrangian density, then the operator $D(\partial)$ defined in (4.43) does not have an inverse. A related difficulty should thus arise in the path-integral method. In this chapter, we shall see how we can avoid this difficulty.

18.1 Quantization of Gauge Fields

Faddeev and Popov showed for the first time how to quantize gauge fields using the path-integral method [164]. Given a field A, in order to compute its Green's functions, we need to introduce the action integral $S[A]$:

$$S[A] = \int \mathrm{d}^4x \, \mathscr{L}(x) \, . \tag{18.1}$$

The vacuum expectation value of an arbitrary operator $F[A]$ containing the field A is given by

$$\langle F[A] \rangle = \frac{\displaystyle\int \mathscr{D}A \; F[A] \exp\left\{\mathrm{i}S[A]\right\}}{\displaystyle\int \mathscr{D}A \exp\left\{\mathrm{i}S[A]\right\}} \, . \tag{18.2}$$

However, the denominator and numerator are both divergent. The reason is that, if A is a gauge field, then since all configurations of the gauge field which can be obtained by gauge transformations from any given configuration correspond to exactly the same state physically, the same physical state will appear infinitely many

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times. This divergence corresponds to the fact that the differential operator $D(\partial)$ does not have an inverse. However, if $F[A]$ is a gauge invariant quantity, then such divergences will cancel between the denominator and the numerator.

Hence, if we introduce a gauge function Ω and write the corresponding transformation for the gauge field A as

$$A(x) \to A^{\Omega}(x) \, , \tag{18.3}$$

we need to divide each path integral in the denominator and in the numerator of (18.2) by

$$\int \mathscr{D}\Omega \, . \tag{18.4}$$

Faddeev and Popov provided a method to do this. We arrange for one configuration of the gauge field to correspond to one physical state. The condition for picking one configuration is called the *gauge condition*.

18.1.1 A Method to Specify the Gauge Condition

We can specify the gauge condition using a functional $f[A]$ of A and writing

$$f[A] = 0 \, . \tag{18.5}$$

Alternatively, for arbitrary A, we can choose a suitable Ω and require

$$f\left[A^{\Omega}\right] = 0 \, . \tag{18.6}$$

We can then define the gauge-invariant functional $\Delta_f[A]$ by

$$\Delta_f[A] \int \mathscr{D}\Omega \, \delta(f[A^{\Omega}]) = \text{const.} \tag{18.7}$$

Therefore,

$$\frac{\displaystyle\int \mathscr{D}A \exp\left\{\mathrm{i}S[A]\right\}}{\displaystyle\int \mathscr{D}\Omega} \sim \int \mathscr{D}\mathscr{A} \exp\left\{\mathrm{i}S[A]\right\} \Delta_f[A] \delta(f[A]) \, . \tag{18.8}$$

If we now introduce an external field, the generating functional of the Green's function is

$$\mathcal{T}_f[J] \sim \int \mathcal{D}A \exp\left\{\mathrm{i}\int \mathrm{d}^4x\left[\mathcal{L}(x) - J(x)A(x)\right]\right\} \Delta_f[A]\delta(f[A]) \,. \tag{18.9}$$

18.1.2 The Additional Term Method

Adding a term to S and setting

$$\varphi[A]\int \mathcal{D}\Omega \exp\left\{\mathrm{i}\Delta S[A^{\Omega}]\right\} = \text{const.}\,, \qquad \Delta S = \int \mathrm{d}^4x\,\Delta\mathcal{L}(x)\,, \tag{18.10}$$

as in the case above, the equation corresponding to (18.9) is

$$\mathcal{T}[J] \sim \int \mathcal{D}A \exp\left\{\mathrm{i}\int \mathrm{d}^4x\left[\mathcal{L}(x) + \Delta\mathcal{L}(x) - J(x)A(x)\right]\right\} \varphi[A]\,. \tag{18.11}$$

This is the customary way of quantizing gauge fields. We now turn to examples.

18.2 Quantization of the Electromagnetic Field

We apply the above method for quantizing gauge fields to the case of the electromagnetic field, which is the best known Abelian gauge field. The canonical quantization of the electromagnetic field is well understood. We shall now check whether the same result can be obtained using the path-integral method. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\bar{\psi}\left[\gamma_\mu(\partial_\mu - \mathrm{i}eA_\mu) + m\right]\psi - \frac{1}{4}F_{\mu\nu}F_{\mu\nu}\,,\\ &= \mathcal{L}_\mathrm{f} + \mathcal{L}_\mathrm{int}\,,\end{aligned} \tag{18.12}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and

$$\mathcal{L}_\mathrm{int} = \mathrm{i}e\bar{\psi}\gamma_\mu\psi A_\mu = j_\mu A_\mu\,. \tag{18.13}$$

This Lagrangian density is invariant under the gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x)\,, \quad \psi(x) \to \mathrm{e}^{\mathrm{i}e\lambda(x)}\psi(x)\,, \quad \bar{\psi}(x) \to \mathrm{e}^{-\mathrm{i}e\lambda(x)}\bar{\psi}(x)\,. \tag{18.14}$$

18.2.1 Specifying the Gauge Condition

Here we consider the Lorenz gauge and the Coulomb (or radiation) gauge:

$$f_{\mathrm{L}}[A] = \partial_\mu A_\mu = 0 \;, \quad f_{\mathrm{R}}[A] = \operatorname{div} \boldsymbol{A} = 0 \;. \tag{18.15}$$

In both cases, if we implement the gauge transformation with the gauge function $-\lambda$, then

$$\Delta_{\mathrm{L}}[A] \int \mathscr{D}\lambda \delta(\partial_\mu A_\mu - \Box\lambda) = \text{const.} \;, \tag{18.16}$$

$$\Delta_{\mathrm{R}}[A] \int \mathscr{D}\lambda \delta(\operatorname{div} \boldsymbol{A} - \varDelta\lambda) = \text{const.} \;, \tag{18.17}$$

noting that Δ does not depend on A in either case. For this reason, QED remains simple. We thus consider the generating functional in the Lorenz gauge:

$$\mathscr{T}[J, \eta, \bar{\eta}] \sim \int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi} \exp\left[\mathrm{i} \int \mathrm{d}^4 x (\mathscr{L} - \bar{\eta}\psi - \bar{\psi}\eta - J_\mu A_\mu)\right] \delta(\partial_\mu A_\mu) \;. \tag{18.18}$$

Setting $e = 0$, we carry out the path integral for the free field.

For the fermionic field, we take η and $\bar{\eta}$ to be anti-commuting c-numbers and consider

$$\int \mathscr{D}\psi \mathscr{D}\bar{\psi} \exp\left\{-\mathrm{i} \int \mathrm{d}^4 x \big[\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi + \bar{\eta}\psi + \bar{\psi}\eta\big]\right\} \;. \tag{18.19}$$

In order to evaluate this integral, we generalize the example in Sect. 17.2. We define an inner product by

$$(x, Ax) = \sum_{j,k} x_j^* A_{jk} x_k \;. \tag{18.20}$$

We then write a generalization of the integral (17.52):

$$I = \int \prod_j \mathrm{d}x_j \prod_k \mathrm{d}x_k^* \exp\Big\{ - \mathrm{i}\big[(x, Ax) + (x, y) + (y, x)\big]\Big\} \;. \tag{18.21}$$

Introducing the change of variables

$$x_j = -(A^{-1})_{jk} y_k + z_j \;, \tag{18.22}$$

we can carry out the integral, viz.,

$$I = \int \prod_j \mathrm{d}z_j \prod_k \mathrm{d}z_k^* \exp\left\{-\mathrm{i}\left[(z, Az) - (y, A^{-1}y)\right]\right\} \propto \exp\left[\mathrm{i}(y, A^{-1}y)\right] . \tag{18.23}$$

Making the same replacement as (17.56),

$$A^{-1}(x-y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p \frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{\mathrm{i}p\cdot\gamma + m - \mathrm{i}\epsilon} = \mathrm{i}S_{\mathrm{F}}(x-y) . \tag{18.24}$$

The integral (18.19) thus assumes the form

$$\exp\left[-\int \mathrm{d}^4 x \int \mathrm{d}^4 y \bar{\eta}(x) S_{\mathrm{F}}(x-y)\eta(y)\right] . \tag{18.25}$$

Although we should in fact take into account the anti-commutativity of the variables ψ, $\bar{\psi}$, η, and $\bar{\eta}$, here we have just given the result by analogy.

Now, for the electromagnetic field, using

$$\delta(\partial_\mu A_\mu) \sim \int \mathscr{D}B \exp\left[\mathrm{i}\int \mathrm{d}^4 x\, B(x)\partial_\mu A_\mu(x)\right] \tag{18.26}$$

and integrating by parts in the exponent above, we obtain

$$\int \mathscr{D}A_\mu \mathscr{D}B \exp\left[-\mathrm{i}\int \mathrm{d}^4 x \left\{\frac{1}{2}\left[A_\mu(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box)A_\nu + \partial_\mu B A_\mu - B\partial_\mu A_\mu\right] + J_\mu A_\mu\right\}\right] . \tag{18.27}$$

For a pair (A_μ, B), the operator corresponding to A_{ij} is then expressed by the matrix

$$\begin{pmatrix} \partial_\mu\partial_\nu - \delta_{\mu\nu} & \partial_\mu \\ -\partial_\nu & 0 \end{pmatrix} . \tag{18.28}$$

Its inverse matrix appears in the propagator. It can be shown to be

$$\begin{pmatrix} \dfrac{1}{\Box}\left(\dfrac{\partial_\mu\partial_\nu}{\Box} - \delta_{\mu\nu}\right) & -\dfrac{\partial_\mu}{\Box} \\ \dfrac{\partial_\nu}{\Box} & 0 \end{pmatrix} . \tag{18.29}$$

Inserting this, the path integral (18.27) becomes

$$\exp\left[-\frac{1}{2}\int \mathrm{d}^4 x \int \mathrm{d}^4 y J_\mu(x) D_{\mu\nu}(x-y) J_\nu(y)\right] , \tag{18.30}$$

where

$$D_{\mu\nu}(x) = \frac{-\mathrm{i}}{(2\pi)^4} \int \mathrm{d}^4k \frac{1}{k^2 - \mathrm{i}\epsilon} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 - \mathrm{i}\epsilon} \right) \mathrm{e}^{\mathrm{i}k\cdot x} \; . \tag{18.31}$$

The propagator appearing here, corresponding to the gauge condition (18.15), is written in the Landau gauge. The introduction of the auxiliary field B has already been discussed in Sect. 15.5.

18.2.2 The Additional Term Method

As an additional term, we choose

$$\Delta S = \int \mathrm{d}^4x \Delta\mathscr{L} \; , \quad \Delta\mathscr{L} = -\frac{1}{2\alpha}(\partial_\mu A_\mu)^2 \; . \tag{18.32}$$

Since $\varphi[A]$ does not depend on A, the generating functional has the simple form

$$\mathscr{T}[J, \eta, \bar{\eta}] \sim \int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi} \exp\left[\mathrm{i} \int \mathrm{d}^4x (\mathscr{L} + \Delta\mathscr{L} - J_\mu A_\mu - \bar{\eta}\psi - \bar{\psi}\eta) \right] \; . \tag{18.33}$$

The ψ-part is the same as above, but the propagator of the electromagnetic field is

$$D_{\mu\nu} = \frac{-\mathrm{i}}{(2\pi)^4} \int \mathrm{d}^4k \frac{1}{(k^2 - \mathrm{i}\epsilon)^2} \big(k^2\delta_{\mu\nu} - k_\mu k_\nu + \alpha k_\mu k_\nu \big) \mathrm{e}^{\mathrm{i}k\cdot x} \; , \tag{18.34}$$

where α is a gauge parameter. This form coincides with the integral expression already derived in (12.251) with $\sigma = 0$.

18.2.3 Ward–Takahashi Identity

The path-integral method gives the same result as the canonical quantization. We can use this method to derive other properties, such as the Ward–Takahashi identity [118, 119].

The propagator of the electron in the Landau gauge is

$$\big\langle \boldsymbol{\psi}(x)\bar{\boldsymbol{\psi}}(y) \big\rangle_{\mathrm{L}} = \frac{1}{N_{\mathrm{L}}} \int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi} \; \psi(x)\bar{\psi}(y) \mathrm{e}^{\mathrm{i}S[\psi,\bar{\psi},A]} \delta(\partial_\mu A_\mu) \; , \tag{18.35}$$

where N_L is a normalization factor given by

$$N_L = \int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi}\, e^{iS[\psi,\bar{\psi},A_\mu]}\delta(\partial_\mu A_\mu) \ . \tag{18.36}$$

We make the change of variables

$$\psi \to \psi' = e^{ie\lambda}\psi \ , \quad \bar{\psi} \to \bar{\psi}' = e^{-ie\lambda}\bar{\psi} \ . \tag{18.37}$$

Changing the integration variables to ψ' and $\bar{\psi}'$, and then rewriting them again as ψ and $\bar{\psi}$, the expression (18.35) takes the form

$$\begin{aligned}\langle \boldsymbol{\psi}(x)\bar{\boldsymbol{\psi}}(y)\rangle_L = &\frac{1}{N_L}\int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi}\ \psi(x)\bar{\psi}(y)e^{iS[\psi,\bar{\psi},A]}\delta(\partial_\mu A_\mu) \\ &\times \exp\left\{ie\big[\lambda(x)-\lambda(y)\big] - i\int d^4z j_\mu(z)\partial_\mu\lambda(z)\right\} \ ,\end{aligned} \tag{18.38}$$

where the last term is originated from the electron part of the Lagrangian density. Carrying out the functional differentiation of this equation with respect to $\lambda(x)$, and then setting that $\lambda = 0$,

$$ie\big[\delta^4(x-z)-\delta^4(y-z)\big]\langle \boldsymbol{\psi}(x), \bar{\boldsymbol{\psi}}(y)\rangle_L + i\partial_\mu\langle \boldsymbol{j}_\mu(z), \boldsymbol{\psi}(x), \bar{\boldsymbol{\psi}}(y)\rangle_L = 0 \ . \tag{18.39}$$

Taking the Fourier transform of this equation, we obtain the Ward–Takahashi identity (12.200):

$$-i(p-q)_\mu S_F'(p)\Gamma_\mu(p,q)S_F'(q) = S_F'(p) - S_F'(q) \ . \tag{18.40}$$

The discussion about the derivation above only refers to the fermionic (electron) part, and not the electromagnetic field, so it turns out that this result holds true for any gauge fields.

Next, we discuss the gauge transformations for Green's functions.

18.2.4 Gauge Transformations for Green's Functions

We ask ourselves what kind of relations exist among Green's functions in different gauges. As an example, we investigate the relation between the Landau gauge and the radiation gauge, viz.,

$$\langle \boldsymbol{\psi}(x)\bar{\boldsymbol{\psi}}(y)\rangle_R = \frac{1}{N_R}\int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi}\ \psi(x)\bar{\psi}(y)e^{iS[\psi,\bar{\psi},A]}\delta(\mathrm{div}\, \boldsymbol{A}) \ , \tag{18.41}$$

$$N_R = \int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi}\, e^{iS[\psi,\bar{\psi},A]}\delta(\mathrm{div}\, \boldsymbol{A}) \ . \tag{18.42}$$

We insert the following factor into the denominator and the numerator:

$$\int \mathscr{D}\lambda\delta(\Box\lambda - \partial_\mu A_\mu) = \text{const.} \tag{18.43}$$

If we now carry out a gauge transformation, then S and $\mathscr{D}A_\mu\mathscr{D}\psi\mathscr{D}\bar{\psi}$ are invariant. In the integral,

$$\delta(\Box\lambda - \partial_\mu A_\mu) \to \delta(\partial_\mu A_\mu) , \quad \delta(\text{div}\,\boldsymbol{A}) \to \delta(\text{div}\,\boldsymbol{A} + \Delta\lambda) , \tag{18.44}$$

and in the numerator, the following factor shows up:

$$\exp\left\{\mathrm{i}e\big[\lambda(x) - \lambda(y)\big]\right\} . \tag{18.45}$$

From (18.44),

$$\text{div}\,\boldsymbol{A} + \Delta\lambda = 0 , \tag{18.46}$$

so by solving this equation, we can find the constraint on λ :

$$\lambda(x) = \int \mathrm{d}^4 z \boldsymbol{\phi}(x-z)\cdot\boldsymbol{A}(z) , \tag{18.47}$$

$$\boldsymbol{\phi}(x) = -\delta(x_0)\nabla\left(\frac{1}{4\pi|\boldsymbol{x}|}\right) . \tag{18.48}$$

When we carry out the functional integration with respect to λ, $\delta(\text{div}\,\boldsymbol{A} + \Delta\lambda)$ disappears, whence

$$\langle\boldsymbol{\psi}(x)\bar{\boldsymbol{\psi}}(y)\rangle_{\mathrm{R}} = \left\langle\boldsymbol{\psi}(x)\bar{\boldsymbol{\psi}}(y)\exp\left\{\mathrm{i}e\int \mathrm{d}^4 z\big[\boldsymbol{\phi}(x-z) - \boldsymbol{\phi}(y-z)\big]\boldsymbol{A}(z)\right\}\right\rangle_{\mathrm{L}} . \tag{18.49}$$

Thus the propagator in the radiation gauge has been expressed in terms of the propagator in the Landau gauge.

18.3 Quantization of Non-Abelian Gauge Fields

Using the standard path-integral method for quantizing gauge fields, we consider the non-Abelian gauge fields.

18.3.1 A Method to Specify the Gauge Condition

We choose the gauge condition

$$f[A] = \partial_\mu A^a_\mu = 0 \,. \tag{18.50}$$

Applying an infinitesimal gauge transformation, from (14.23),

$$\delta A^a_\mu = \frac{1}{g}\partial_\mu \lambda^a + f_{abc} A^b_\mu \lambda^c \equiv \frac{1}{g}(\mathscr{D}_\mu \lambda)^a \,. \tag{18.51}$$

Thus, under an infinitesimal gauge transformation,

$$\left(\partial_\mu A^a_\mu\right)^\Omega = \partial_\mu\left(A^a_\mu + \delta A^a_\mu\right) = \partial_\mu \left[A^a_\mu + \frac{1}{g}(D_\mu \lambda)^a\right] . \tag{18.52}$$

In the usual way, we compute $\Delta_f[A]$. If we use (18.7), we have

$$\begin{aligned}
\int \mathscr{D}\Omega\,\delta\left[(\partial_\mu A^a_\mu)^\Omega\right] &= \int \mathscr{D}\lambda\,\delta\left[(\partial_\mu A^a_\mu)^\Omega\right] \\
&= \int \mathscr{D}\left[(\partial_\mu A^a_\mu)^\Omega\right] \left\{\frac{\mathscr{D}\left[(\partial_\mu A^a_\mu)^\Omega\right]}{\mathscr{D}\lambda}\right\}^{-1} \delta\left[(\partial_\mu A^a_\mu)^\Omega\right] \\
&= \left\{\frac{\mathscr{D}\left[(\partial_\mu A^a_\mu)^\Omega\right]}{\mathscr{D}\lambda}\right\}^{-1} .
\end{aligned} \tag{18.53}$$

Thus, $\Delta_f[A]$ is the functional Jacobian

$$\Delta_f[A] = \frac{\mathscr{D}\left[(\partial_\mu A^a_\mu)^\Omega\right]}{\mathscr{D}\lambda} = \det\left(-\frac{1}{g}\partial_\mu D_\mu\right) . \tag{18.54}$$

Normalizing this determinant to unity when $A^a_\mu = 0$,

$$\Delta_f[A] = \det\left(\frac{\partial_\mu D_\mu}{\Box}\right) . \tag{18.55}$$

To compute this expression, we use the method due to 't Hooft in 1971 [165]. The generating functional for the Green's functions is

$$\mathscr{T}[J] = \int \mathscr{D}A_\mu \exp\left\{\mathrm{i}S[A, J]\right\}\Delta_f[A]\delta(\partial_\mu A_\mu) \,, \tag{18.56}$$

where

$$S[A,J] = S[A] - \int \mathrm{d}^4x\, A_\mu^a(x) J_\mu^a(x) \,. \tag{18.57}$$

The δ-function in the integral is given by

$$\delta(\partial_\mu A_\mu) \sim \int \mathscr{D}B \exp\left[\mathrm{i}\int \mathrm{d}^4x\, B(x)\partial_\mu A_\mu(x)\right] , \tag{18.58}$$

but note that we do not discuss the normalization here. We use

$$\int \prod_j \mathrm{d}x_j \mathrm{d}y_j \exp\left[\mathrm{i}(x, Ay)\right] = (2\pi)^n (\det A)^{-1} \,. \tag{18.59}$$

And so we obtain

$$\det\left(\frac{\partial_\mu D_\mu}{\Box}\right)^{-1} \sim \int \mathscr{D}\varphi \mathscr{D}\bar{\varphi} \exp\left[\mathrm{i}\int \mathrm{d}^4x\, \bar{\varphi}(x)\partial_\mu D_\mu \varphi(x)\right] . \tag{18.60}$$

Note also that (18.60) is the inverse of (18.55). Let us therefore consider how to obtain the inverse.

We treat the expression (18.60) as a sum of loop contributions obtained by contractions among the scalar fields φ and $\bar{\varphi}$, while the gauge field A appears as an external line. According to the discussion in Sect. 11.2, this sum is the connected part, so in order to derive the inverse, we need to invert the sign of the connected part. However, the connected part consists of single loops obtained by contracting φ and $\bar{\varphi}$ before A is quantized. Thus, we must reverse the sign of each loop. As mentioned in Sect. 8.4, this reversal happens when φ and $\bar{\varphi}$ are anti-commutative, i.e., when they obey Fermi statistics. For a path integral involving these so-called *Grassmann numbers*, which anti-commute, we need an additional discussion, but for the moment we avoid getting further involved and just write down the result:

$$\Delta_f[A] \sim \int \mathscr{D}\varphi \mathscr{D}\bar{\varphi} \exp\left[\mathrm{i}\int \mathrm{d}^4x\, \bar{\varphi}(x)\partial_\mu D_\mu \varphi(x)\right] . \tag{18.61}$$

Although φ and $\bar{\varphi}$ are scalar fields, they obey Fermi statistics. It turns out that this introduces an indefinite metric. The effective Lagrangian density in this theory is

$$\mathscr{L}[A] + \bar{\varphi}\partial_\mu D_\mu \varphi + B\partial_\mu A_\mu \,. \tag{18.62}$$

Here we have summed indices standing for components, although this has not been written explicitly. This Lagrangian corresponds to the one in the Landau gauge in QED. The scalar fields φ and $\bar{\varphi}$ are called *Faddeev–Popov ghost fields*.

18.3.2 The Additional Term Method

We take $\Delta\mathscr{L}$ to be expressed in terms of $\partial_\mu A_\mu$ and define $\varphi[A]$ by

$$\varphi[A] \int \mathscr{D}\Omega \exp\left\{ \mathrm{i}\Delta S\big[(\partial_\mu A_\mu)^\Omega\big]\right\} = \text{const.} \tag{18.63}$$

In order to compute this, we use

$$\exp\left\{ \mathrm{i}\Delta S\big[(\partial_\mu A_\mu)^\Omega\big]\right\} \sim \int \mathscr{D}C \, \mathrm{e}^{\mathrm{i}\Delta S[C]} \delta\big[(\partial_\mu A_\mu)^\Omega - C\big] \, . \tag{18.64}$$

This computation is the same as the example above, so $\varphi[A]$ can be readily derived. As a consequence, the effective Lagrangian density, considering the first term as a gauge-invariant term, is

$$\mathscr{L}[A] + \bar{\varphi}\partial_\mu D_\mu \varphi + \Delta\mathscr{L} \, . \tag{18.65}$$

We choose the following form for $\Delta\mathscr{L}$:

$$\Delta\mathscr{L} = -\frac{1}{2\alpha}(\partial_\mu A_\mu)^2 \, . \tag{18.66}$$

Therefore, the total Lagrangian density is

$$\mathscr{L}[A] + \bar{\varphi}\partial_\mu D_\mu \varphi - \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 \, . \tag{18.67}$$

18.3.3 Hermitization of the Lagrangian Density

In the discussion so far, we used the effective Lagrangian density to compute the S-matrix and Green's functions. In the operator formalism, the Lagrangian density should be Hermitian. The Faddeev–Popov ghost term in (18.67) is not Hermitian. Integrating this term by parts,

$$\mathscr{L}_{\mathrm{FP}} \sim -\partial_\mu \bar{\varphi} D_\mu \varphi = -\partial_\mu \bar{\varphi}^a \left(\partial_\mu \varphi^a + g f_{abc} A_\mu^b \varphi^c\right) \, . \tag{18.68}$$

If φ and $\bar{\varphi}$ are Hermitian, then (18.67) is obviously not Hermitian. This is because φ and $\bar{\varphi}$ are anti-commutative scalar fields. We thus change the phase of this part:

$$\mathscr{L}_{\mathrm{FP}} \to \mathrm{e}^{\mathrm{i}\alpha} \mathscr{L}_{\mathrm{FP}} \, . \tag{18.69}$$

Consequently, the phases of the ghost propagator and the coupling constant for the ghost and the gauge field change according to

$$\langle \varphi(x)\bar{\varphi}(y)\rangle = D_{\rm F}(x-y) \to {\rm e}^{-{\rm i}\alpha} D_{\rm F}(x-y)\ , \tag{18.70}$$

$$g \to g{\rm e}^{{\rm i}\alpha}\ . \tag{18.71}$$

Note that, when φ and $\bar{\varphi}$ only appear in closed loops, the numbers of $D_{\rm F}$ and g are the same, so the contributions to the S-matrix or Green's functions are invariant under the above phase transformation. That is, it turns out that the phase α can be freely chosen. If we choose ${\rm e}^{{\rm i}\alpha} = -{\rm i}$, and write c and $\bar{c}$ instead of φ and $\bar{\varphi}$ [see (4.89)], we have

$$c^{\dagger} = c\ , \quad \bar{c}^{\dagger} = \bar{c}\ , \tag{18.72}$$

$$\mathscr{L}_{\rm FP} = {\rm i}\partial_{\mu}\bar{c}D_{\mu}c\ . \tag{18.73}$$

This Lagrangian density is then Hermitian.

The Lagrangian density in a general gauge (also called the α-gauge) is

$$\mathscr{L} = \mathscr{L}_{\rm int} + \mathscr{L}_{\rm GF} + \mathscr{L}_{\rm FP}\ , \tag{18.74}$$

where, dropping indices for the gauge field,

$$\mathscr{L}_{\rm inv} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu}, \tag{18.75}$$

$$\mathscr{L}_{\rm GF} = (\partial_{\mu}B)A_{\mu} + \frac{\alpha}{2}B\cdot B\ , \quad \text{or} \quad -\frac{1}{2\alpha}(\partial_{\mu}A_{\mu})^2, \tag{18.76}$$

$$\mathscr{L}_{\rm FP} = {\rm i}\partial_{\mu}\bar{c}D_{\mu}c. \tag{18.77}$$

The first term in (18.74) is gauge invariant, the second is a gauge-fixing term, and the third is a ghost term. This form was given by Kugo and Ojima in [166].

18.3.4 Gauge Transformations of Green's Functions

In the last section, we investigated the relations among Green's functions defined using different gauge conditions in QED. Here we discuss the different relations among Green's functions defined by including an additional term. The gauge-invariant term is the same, and we thus treat two theories which are physically equivalent in different gauges. Hence, we introduce two Lagrangian densities and two action integrals:

$$\mathscr{L}_{II} = \mathscr{L}_{I} + \Delta\mathscr{L}\ , \quad S_{II} = S_{I} + \Delta S\ . \tag{18.78}$$

Then considering the field operators A, B, C, ..., we introduce the Green's function in the second gauge:

$$\langle \boldsymbol{ABC\ldots} \rangle_{II} = \frac{1}{N_{II}} \int \mathscr{D}A_\mu \ldots ABC \ldots \exp(\mathrm{i}S_{II}) \,, \tag{18.79}$$

$$N_{II} = \int \mathscr{D}A_\mu \ldots \exp(\mathrm{i}S_{II}) \,. \tag{18.80}$$

Then from (18.78), we decompose S_{II}, considering ΔS as a perturbation and treating $\exp(\mathrm{i}\Delta S)$ like A, B, C, ... Therefore,

$$\begin{aligned} \langle \boldsymbol{ABC\ldots} \rangle_{II} &= \frac{1}{N_{II}} \int \mathscr{D}A_\mu \ldots ABC \ldots \exp(\mathrm{i}\Delta S) \exp(\mathrm{i}S_I) \\ &= \frac{N_I}{N_{II}} \frac{1}{N_I} \int \mathscr{D}A_\mu \ldots ABC \ldots \exp(\mathrm{i}\Delta S) \exp(\mathrm{i}S_I) \\ &= \frac{N_I}{N_{II}} \big\langle \boldsymbol{ABC\ldots} \exp(\mathrm{i}\Delta \boldsymbol{S}) \big\rangle_I \,, \end{aligned} \tag{18.81}$$

$$\frac{N_I}{N_{II}} = \frac{1}{N_I} \int \mathscr{D}A_\mu \ldots \exp(\mathrm{i}\Delta S) \exp(\mathrm{i}S_I) = \big\langle \exp(\mathrm{i}\Delta \boldsymbol{S}) \big\rangle_I \,. \tag{18.82}$$

We thus obtain

$$\big\langle \boldsymbol{ABC\ldots} \big\rangle_{II} = \frac{\big\langle \boldsymbol{ABC\ldots} \exp(\mathrm{i}\Delta \boldsymbol{S}) \big\rangle_I}{\big\langle \exp(\mathrm{i}\Delta \boldsymbol{S}) \big\rangle_I} \,. \tag{18.83}$$

This gives the relation among Green's functions in two different gauges. For example, considering (18.74), we choose the Landau gauge with $\alpha = 0$ and the gauge with $\alpha \neq 0$ for $\mathscr{L}_I$ and $\mathscr{L}_{II}$, respectively, and distinguish the Landau gauge by the index L. Then,

$$\langle \boldsymbol{ABC\ldots} \rangle_\alpha = \frac{\left\langle \boldsymbol{ABC\ldots} \exp\left[\frac{\mathrm{i}\alpha}{2} \int \mathrm{d}^4x\, \boldsymbol{B}(x) \cdot \boldsymbol{B}(x) \right] \right\rangle_{\mathrm{L}}}{\left\langle \exp\left[\frac{\mathrm{i}\alpha}{2} \int \mathrm{d}^4x\, \boldsymbol{B}(x) \cdot \boldsymbol{B}(x) \right] \right\rangle_{\mathrm{L}}} \,. \tag{18.84}$$

This equation shows the α-dependence of an arbitrary Green's function. We may also interpret operators appearing in the discussion above as being *unrenormalized*. The subscript L indicates that these Green's functions should be evaluated in Heisenberg's picture in the Landau gauge, while the subscript α indicates that they should be evaluated in the Heisenberg picture in the gauge $\alpha \neq 0$. This formula provides a basis for the discussion about the gauge invariance of various kinds of Green's functions.

18.4 Axial Gauge

In the last section, we introduced the effective Lagrangian density in the covariant gauge, and thereby understood the need for the Faddeev–Popov ghost. However, if we do not require manifest Lorentz covariance, there is a gauge in which we can quantize without ghosts. This is the axial gauge.

We replace the gauge condition (18.50) discussed in the last section by

$$n_\mu A_\mu^a = 0 \,, \tag{18.85}$$

where n_μ is a constant vector. For the infinitesimal gauge transformation (18.51),

$$\delta\big(n_\mu A_\mu^a\big) = \frac{1}{g} n_\mu \partial_\mu \lambda^a + f_{abc}(n_\mu A_\mu^b)\lambda^c \,. \tag{18.86}$$

Under the gauge condition (18.85),

$$\delta\big(n_\mu A_\mu^a\big) = \frac{1}{g} n_\mu \partial_\mu \lambda^a \,. \tag{18.87}$$

This is independent of A_μ. Hence,

$$\int \mathscr{D}\Omega\, \delta[(n_\mu A_\mu^a)^\Omega]$$

does not involve A_μ and $\Delta_f[A]$ is a constant. Therefore, it turns out that the Faddeev–Popov ghost term is not produced here.

Using the additional term method, if we choose

$$\Delta S[A] = -\frac{1}{2\alpha} \int \mathrm{d}^4 x \big(n_\mu A_\mu^a\big)^2 \,, \tag{18.88}$$

and set

$$\varphi[A] \int \mathscr{D}\Omega \exp\big\{\mathrm{i}S\big[A^\Omega\big]\big\} = \text{const.}\,, \tag{18.89}$$

then once again $\varphi[A]$ does not involve A_μ. Hence, we consider the effective Lagrangian density

$$\mathscr{L} = \mathscr{L}_{\text{inv}} - \frac{1}{2\alpha}\big(n_\mu A_\mu\big)^2 \,. \tag{18.90}$$

Now quantizing, leaving only terms in quadratic form and applying the variation principle,

$$D_{\mu\nu}(\partial)A_\nu^a = 0 \ , \tag{18.91}$$

where

$$D_{\mu\nu}(\partial) = (\delta_{\mu\nu} - \partial_\mu\partial_\nu) - \frac{1}{\alpha}n_\mu n_\nu \ . \tag{18.92}$$

Making the substitution $\partial_\mu\partial_\nu \to -k_\mu k_\nu$ in momentum space,

$$D_{\mu\nu}(k) = -k^2\delta_{\mu\nu} + k_\mu k_\nu - \frac{1}{\alpha}n_\mu n_\nu \ , \tag{18.93}$$

$$\left[\delta_{\mu\nu} - \frac{n_\mu k_\lambda + n_\lambda k_\mu}{n \cdot k} + \frac{n^2 + \alpha k^2}{(n \cdot k)^2}k_\mu k_\nu\right]D_{\lambda\nu}(k) = -k^2\delta_{\mu\nu} \ . \tag{18.94}$$

Thus, it turns out that the propagator is

$$\frac{1}{k^2 - \mathrm{i}\epsilon}\left[\delta_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{n \cdot k} + \frac{n^2 + \alpha k^2}{(n \cdot k)^2}k_\mu k_\nu\right] . \tag{18.95}$$

Problems with this gauge include the question of how to treat the pole $n \cdot k = 0$, and showing that computations of various physical quantities do not depend on the choice of n.

18.5 Feynman Rules in the α-Gauge

We now introduce the Feynman rules for the Lagrangian density (18.74). First, note that

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g(A_\mu \times A_\nu)^a \ , \tag{18.96}$$

$$(D_\mu c)^a = \partial_\mu c^a + g(A_\mu \times c)^a \ . \tag{18.97}$$

For the gauge group indices, we use the inner and the outer product symbols:

$$A \cdot B = \sum_a A^a B_a \ , \quad (A \times B)^a = \sum_{b,c} f_{abc}A^b B^c \ . \tag{18.98}$$

We split the Lagrangian density (18.74) into the free part $\mathscr{L}_\mathrm{f}$ and the interaction part $\mathscr{L}_\mathrm{int}$:

$$\mathscr{L}_\mathrm{f} = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2 + \mathrm{i}\partial_\mu \bar{c}\partial_\mu c \,, \tag{18.99}$$

$$\mathscr{L}_\mathrm{int} = -g\partial_\mu A_\nu (A_\mu \times A_\nu) - \frac{1}{4}g^2 (A_\mu \times A_\nu)(A_\mu \times A_\nu) + \mathrm{i}g\partial_\mu \bar{c}(A_\mu \times c) \,. \tag{18.100}$$

Expressing the Feynman rule in the Lagrangian formalism, it turns out that we assign the factors $\mathrm{i}(2\pi)^4$, $-\mathrm{i}/(2\pi)^4$, and (-1) to each vertex, propagator, and closed ghost loop, respectively. Considering the gauge particle as the gluon, the propagators are

$$\text{gluon} \qquad \frac{\delta_{ab}}{k^2 - \mathrm{i}\epsilon}\left[\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2 - \mathrm{i}\epsilon}\right] , \tag{18.101}$$

$$\text{ghost} \qquad \frac{\mathrm{i}\delta_{ab}}{k^2 - \mathrm{i}\epsilon} \,. \tag{18.102}$$

Moreover, we find the following three types of vertex function:

1. three-gluon vertex , 2. four-gluon vertex , 3. ghost–gluon vertex .

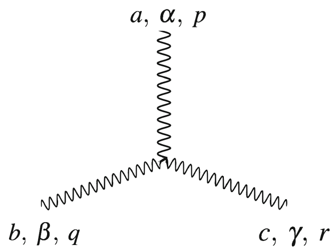
1. Three-gluon vertex (Fig. 18.1). Taking all the momenta of the incoming gluons, the vertex function is

$$-\mathrm{i}gf_{abc}\left[\delta_{\beta\gamma}(r-q)_\alpha + \delta_{\gamma\alpha}(p-r)_\beta + \delta_{\alpha\beta}(q-p)_\alpha\right] . \tag{18.103}$$

2. Four-gluon vertex (Fig. 18.2). In this case, the vertex function is

$$\begin{aligned} &- g^2 f_{gac} f_{gbd}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) - g^2 f_{gad} f_{gbc}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta}) \\ &- g^2 f_{gab} f_{gcd}(\delta_{\alpha\gamma}\delta_{\delta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) \,. \end{aligned} \tag{18.104}$$

Fig. 18.1 Three-gluon vertex



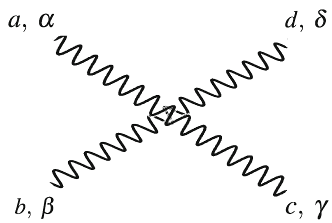


Fig. 18.2 Four-gluon vertex

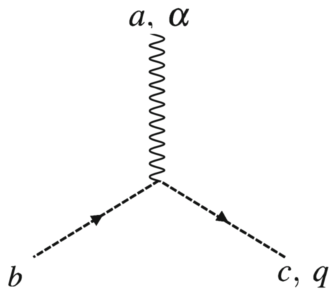


Fig. 18.3 Ghost–gluon vertex

3. Ghost–gluon vertex (Fig. 18.3). Considering the ghost lines to be directed from $\bar{c}$ to c and q to be the outgoing momentum, the vertex function is

$$-gf_{abc}q_\alpha \; . \tag{18.105}$$

Combining the above propagators and vertex functions, we can compute the S-matrix elements or Green's functions. Note that the four-momentum is conserved at each vertex. For the total amplitude, we then have conservation of four-momentum, viz., a factor

$$\delta^4(P_f - P_i) \; . \tag{18.106}$$

Moreover, we have to integrate over all the four-momenta k_i in closed loops, which are not affected by the overall conservation of four-momentum, i.e., we introduce the integrals

$$\int \mathrm{d}^4k_1 \ldots \int \mathrm{d}^4k_l \; , \tag{18.107}$$

where l is the number of closed loops.