Chapter 12
Renormalization Theory

So far we have shown only the lowest order calculations, but when computing higher order corrections, divergences must show up. A method for deducing finite consequences by a suitable interpretation is called a *renormalization theory*. As mentioned once or twice before, the basic idea of such a formalism is to specify a way of separating the Lagrangian density into the free part and the interaction part. This grouping is related to the definition of the interaction picture. We call the interaction picture defined by the correct grouping a *renormalized interaction picture*. Several properties of the Green's functions discussed in the previous chapter hold true in the renormalized interaction picture, while they may not hold true in other pictures.

In this chapter, in order to show that such a separation of the Lagrangian or the Hamiltonian is not necessarily trivial, we first review the scattering theory in non-relativistic quantum mechanics. The formal system discussed here has many similarities with the S-matrix theory based on the reduction formula given in the last chapter.

12.1 Lippmann–Schwinger Equation

In this section, we introduce the formal logic for the standard quantum mechanical system. This theory has a lot in common with the theory of Green's functions. What is important in the scattering problem is the way we formulate the boundary conditions. This issue is discussed in detail in my book, "Relativistic Quantum Mechanics" [78], but here we shall present it in a slightly different order.

In this section, we formulate the scattering problem using the notion of Stueckelberg causality:

> Assuming that the potential $V(t)$ is a function of time, if $V(t) = 0$ in the region $t < T$, then there exists no scattered wave for $t < T$.

K. Nishijima, *Quantum Field Theory*,
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The principle above has already been used for the quantization of free fields in Sect. 4.3 and for the derivation of the Yang–Feldman equation in Sect. 6.3. Here we apply it to the non-relativistic formulation. We consider the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi(t) = [H_0 + V(t)]\psi(t) \ , \tag{12.1}$$

where we have omitted the spatial coordinate for simplicity, and $V(t)$ is defined by

$$V(t) = \begin{cases} V \ , & t > T \ , \\ 0 \ , & t < T \ . \end{cases} \tag{12.2}$$

For $t < T$,

$$\mathrm{i}\frac{\partial}{\partial t}\psi(t) = H_0\psi(t) \ . \tag{12.3}$$

Since this is the equation for the incident wave, we write its solution as

$$\psi(t) = \psi_{\mathrm{in}}(t) \ , \quad H_0\psi_{\mathrm{in}}(t) = E\psi_{\mathrm{in}}(t) \ . \tag{12.4}$$

Therefore, the scattered wave appears at a generic time t and its equation is

$$\psi(t) = \psi_{\mathrm{in}}(t) + \psi_{\mathrm{scatt}}(t) \ , \tag{12.5}$$

$$\left(\mathrm{i}\frac{\partial}{\partial t} - H_0\right)\psi_{\mathrm{scatt}}(t) = V(t)\psi(t) \ . \tag{12.6}$$

To solve this equation, we introduce the Green's function, which is the solution of the equation

$$\left(\mathrm{i}\frac{\partial}{\partial t} - H_0\right)K_{\mathrm{ret}}(t, \boldsymbol{x} : t', x') = \delta(t - t')\delta^3(\boldsymbol{x} - \boldsymbol{x}') \ , \tag{12.7}$$

$$K_{\mathrm{ret}}(t, \boldsymbol{x} : t', \boldsymbol{x}') = 0 \ , \quad t < t' \ . \tag{12.8}$$

Therefore, the solution of (12.6) satisfying the causality condition is

$$\psi_{\mathrm{scatt}}(t, \boldsymbol{x}) = \int \mathrm{d}t'\mathrm{d}^3x' K_{\mathrm{ret}}(t, \boldsymbol{x} : t', \boldsymbol{x}')V(t', \boldsymbol{x}')\psi(t', x') \ . \tag{12.9}$$

Here, taking the limit $T \to -\infty$, the t-dependence of the potential disappears, whence

$$\psi_{\mathrm{scatt}}(t, \boldsymbol{x}) = \int \mathrm{d}t'\mathrm{d}^3x' K_{\mathrm{ret}}(t, \boldsymbol{x} : t', \boldsymbol{x}')V(\boldsymbol{x}')\psi(t', x') \ . \tag{12.10}$$

Then, if H_0 does not depend on the space-time coordinate,

$$K_{\text{ret}}(t, \boldsymbol{x} : t', \boldsymbol{x}') = K_{\text{ret}}(t - t', \boldsymbol{x} - \boldsymbol{x}') \ . \tag{12.11}$$

Thus,

$$\psi(t, \boldsymbol{x}) = \psi_{\text{in}}(t, \boldsymbol{x}) + \int \mathrm{d}t' \mathrm{d}^3 x' K_{\text{ret}}(t - t', \boldsymbol{x} - \boldsymbol{x}') V(\boldsymbol{x}') \psi(t', \boldsymbol{x}') \ . \tag{12.12}$$

This is the integral equation governing the scattering, and corresponds to the Yang–Feldman equation. To remove the time variable, we take

$$\psi_{\text{in}}(t, \boldsymbol{x}) = \mathrm{e}^{-\mathrm{i}Et} \psi_{\text{in}}(\boldsymbol{x}) \ , \quad \psi(t, \boldsymbol{x}) = \mathrm{e}^{-\mathrm{i}Et} \psi(\boldsymbol{x}) \ , \tag{12.13}$$

leading to

$$\psi(\boldsymbol{x}) = \psi_{\text{in}}(\boldsymbol{x}) + \int \mathrm{d}^3 x' G(\boldsymbol{x} - \boldsymbol{x}' : E) V(\boldsymbol{x}') \psi(\boldsymbol{x}') \ , \tag{12.14}$$

where

$$G(\boldsymbol{x} - \boldsymbol{x}' : E) = \int \mathrm{d}t' \mathrm{e}^{\mathrm{i}E(t - t')} K_{\text{ret}}(t - t', \boldsymbol{x} - \boldsymbol{x}') \ . \tag{12.15}$$

We now introduce the Fourier representation of the retarded Green's function:

$$K_{\text{ret}}(t, \boldsymbol{x}) = \frac{1}{(2\pi)^4} \int \mathrm{d}E \, \mathrm{d}^3 p \, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{x} - \mathrm{i}Et} K(E, \boldsymbol{p}) \ . \tag{12.16}$$

Furthermore, we restrict H_0 to the form

$$H_0 = \frac{\boldsymbol{p}^2}{2m} \ . \tag{12.17}$$

Thus, the equation for $K(E, p)$ obtained from (12.7) can be written in the form

$$\left(E - \frac{\boldsymbol{p}^2}{2m} \right) K(E, \boldsymbol{p}) = 1 \ . \tag{12.18}$$

As will be shown later, the solution of this equation satisfying the boundary condition (12.8) is

$$K(E - \boldsymbol{p}) = \left(E - \frac{\boldsymbol{p}^2}{2m} + \mathrm{i}\epsilon \right)^{-1} \ , \tag{12.19}$$

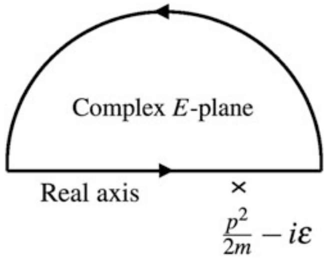

Fig. 12.1 Path of E-integration in (12.22)

where ϵ is a positive infinitesimal. Taking $E = k^2/2m$, Eqs. (12.15) and (12.16) imply

$$G(\boldsymbol{x} - \boldsymbol{x}' : E) = -\frac{m}{2\pi} \frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x} - \boldsymbol{x}'|} . \tag{12.20}$$

Thus, (12.14) can be written in the form

$$\psi(\boldsymbol{x}) = \psi_{\mathrm{in}}(\boldsymbol{x}) - \frac{m}{2\pi} \int \mathrm{d}^3x' \frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x} - \boldsymbol{x}'|} V(\boldsymbol{x}')\psi(\boldsymbol{x}') . \tag{12.21}$$

This equation indicates that the scattered wave becomes an outward-directed spherical wave as a consequence of causality. When $t - t' < 0$,

$$K_{\mathrm{ret}}(t - t', \boldsymbol{x} - \boldsymbol{x}') = \frac{1}{(2\pi)^4} \int \mathrm{d}^3p \, \mathrm{e}^{\mathrm{i}\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')} \int \mathrm{d}E \frac{\mathrm{e}^{-\mathrm{i}E(t-t')}}{E - \frac{p^2}{2m} + \mathrm{i}\epsilon} . \tag{12.22}$$

Then, since $-\mathrm{i}E(t - t') = \mathrm{i}E|t - t'|$, if $\mathrm{Im}\, E > 0$, this exponential decreases exponentially with $\mathrm{Im}E$. Hence, selecting the integration path as in Fig. 12.1 and taking into account the fact that the pole is located outside the semicircle, it vanishes by Cauchy's theorem. Thus, if $t - t' < 0$, then $K_{\mathrm{ret}} = 0$ and we see that (12.8) is satisfied.

Clearly, using (12.22) and taking $\Psi = \psi$ and $\Phi = \psi_{\mathrm{in}}$, Eq. (12.21) can be written formally as

$$\Psi = \Phi + \frac{1}{E - H_0 + \mathrm{i}\epsilon} V\Psi \quad (E > 0) , \tag{12.23}$$

In fact, looking at the Fourier representation of the retarded Green's function (12.22), we see that this corresponds to the operator which gives the energy denominator in the equation above. From (12.23) and (12.3),

$$(E - H_0)\Psi = V\Psi , \quad (E - H_0)\Phi = 0 . \tag{12.24}$$

Moreover, for the bound state ($E < 0$), the inverse of $(E - H_0)$ is uniquely defined, so we obtain the homogenous equation

$$\Psi = \frac{1}{E - H_0} V \Phi \quad (E < 0) \; . \tag{12.25}$$

The pair of Eqs. (12.23) and (12.25) is known collectively as the *Lippmann–Schwinger equation* [114]. The solution of (12.23) consists of the incident wave and the outward-going spherical wave, and it can be written as

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E_a - H_0 + \mathrm{i}\epsilon} V \Psi_a^{(+)} \; . \tag{12.26}$$

Mathematically, it is useful to consider the solution which consists of the incident wave and the inward-going spherical wave. The equation for such a state can be obtained by replacing $\mathrm{i}\epsilon$ with $-\mathrm{i}\epsilon$:

$$\Psi_a^{(-)} = \Phi_a + \frac{1}{E_a - H_0 - \mathrm{i}\epsilon} V \Psi_a^{(-)} \; . \tag{12.27}$$

The two cases are solutions of the same Schrödinger equation under different boundary conditions:

$$(E_a - H_0)\Psi_a^{(\pm)} = V \Psi_a^{(\pm)} \; . \tag{12.28}$$

Thus, $\left\{\Psi_a^{(+)}\right\}$ and $\left\{\Psi_a^{(-)}\right\}$ form complete systems separately. Solving (12.26) sequentially,

$$\begin{aligned} \Psi_a^{(+)} &= \Phi_a + \frac{1}{E_a - H_0 + \mathrm{i}\epsilon} \left(1 + V \frac{1}{E_a - H_0 + \mathrm{i}\epsilon} + \ldots \right) V \Phi_a \\ &= \Phi_a + \frac{1}{E_a - H_0 + \mathrm{i}\epsilon} \left(1 - V \frac{1}{E_a - H_0 + \mathrm{i}\epsilon} \right)^{-1} V \Phi_a \; . \end{aligned} \tag{12.29}$$

Then using $A^{-1}B^{-1} = (BA)^{-1}$,

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E_a - H + \mathrm{i}\epsilon} V \Phi_a \quad (H = H_0 + V) \; . \tag{12.30}$$

We call this the *Chew–Goldberger formal solution* [115]. Another solution is

$$\Psi_a^{(-)} = \Phi_a + \frac{1}{E_a - H - \mathrm{i}\epsilon} V \Phi_a \; . \tag{12.31}$$

Although this formal solution is in fact not useful for solving the problem, it is very useful for deriving the general properties of the S-matrix, and we will show this application below.

We assume that the state of the incident wave is suitably normalized, i.e.,

$$(\Phi_b, \Phi_a) = \delta_{ba} \ . \tag{12.32}$$

Therefore, combining the LS equation with the CG formal solution,

$$\begin{aligned}
\left(\Psi_b^{(+)}, \Psi_a^{(+)}\right) &= \left(\Phi_b + \frac{1}{E_b - H + \mathrm{i}\epsilon} V\Phi_b, \Psi_a^{(+)}\right) \\
&= \left(\Phi_b, \Psi_a^{(+)}\right) + \left(\Phi_b, V\frac{1}{E_b - H - \mathrm{i}\epsilon}\Psi_a^{(+)}\right) \\
&= \left(\Phi_b, \Phi_a + \frac{1}{E_a - H_0 + \mathrm{i}\epsilon} V\Psi_a^{(+)}\right) + \left(\Phi_b, V\frac{1}{E_b - H - \mathrm{i}\epsilon}\Psi_a^{(+)}\right) \\
&= (\Phi_b, \Phi_a) + \left(\frac{1}{E_a - E_b + \mathrm{i}\epsilon} + \frac{1}{E_b - E_a - \mathrm{i}\epsilon}\right)(\Phi_b, V\Psi_a^{(+)}) \\
&= (\Phi_b, \Phi_a) \ ,
\end{aligned} \tag{12.33}$$

so we see that $\left\{\Psi^{(+)}\right\}$ forms an orthonormal system just as $\{\Phi\}$ does. The same is true of $\left\{\Psi^{(-)}\right\}$. Thus, the transformation matrix between these two pairs of complete orthonormal systems, viz.,

$$S_{ab} = \left(\Psi_b^{(-)}, \Psi_a^{(+)}\right) \ , \tag{12.34}$$

is unitary. Comparing with (11.136), $\Psi^{(+)}$ and Ψ^{-} correspond to Φ^{in} and Φ^{out}, respectively. The unitarity condition is

$$S^\dagger S = SS^\dagger = 1 \ . \tag{12.35}$$

Starting with the definition and modifying it suitably, the S-matrix above can be written as

$$\begin{aligned}
S_{ba} &= \left(\Phi_b + \frac{1}{E_b - H - \mathrm{i}\epsilon} V\Phi_b, \Psi^{(+)}a\right) \\
&= \left(\Phi_b, \Psi_a^{(+)}\right) + \left(\Phi_b, V\frac{1}{E_b - H + \mathrm{i}\epsilon}\Psi_a^{(+)}\right) \\
&= (\Phi_b, \Phi_a) + \left(\Phi_b, \frac{1}{E_a - H_0 + \mathrm{i}\epsilon} V\Psi_a^{(+)}\right) + \left(\Phi_b, V\frac{1}{E_b - H + \mathrm{i}\epsilon}\Psi_a^{(+)}\right) \\
&= \delta_{ba} + \left(\frac{1}{E_a - E_b + \mathrm{i}\epsilon} + \frac{1}{E_b - E_a + \mathrm{i}\epsilon}\right)\left(\Phi_b, V\Psi_a^{(+)}\right) \\
&= \delta_{ba} - 2\pi\mathrm{i}\delta(E_b - E_a)\left(\Phi_b, V\Psi_a^{(+)}\right) \ .
\end{aligned} \tag{12.36}$$

Although in the transformation above we have expressed $\Psi^{(-)}$ in terms of Φ, expressing $\Psi^{(+)}$ in terms of Φ, we obtain

$$S_{ba} = \delta_{ba} - 2\pi \mathrm{i}\delta(E_b - E_a)\big(\Psi_b^{(-)}, V\Phi_a\big) \ . \tag{12.37}$$

Therefore, if $E_b = E_a$, the transition amplitude T_{ba} can be written in a symmetric form:

$$T_{ba} = \big(\Phi_b, V\Psi_a^{(+)}\big) = \big(\Psi_b^{(-)}, V\Phi_a\big) \ . \tag{12.38}$$

We now express the unitarity of the S-matrix in terms of T. Using T as above,

$$T_{ba}^\dagger = T_{ab}^* = \big(\Phi_a, V\Psi_b^{(+)}\big)^* = \big(\Psi_b^{(+)}, V\Phi_a\big) \ ,$$

$$\begin{aligned}
T_{ba}^\dagger - T_{ba} &= (\Psi_b^{(+)}, V\Phi_a) - (\Phi_b, V\Psi_a^{(+)}) \\
&= (\Phi_b, V\Phi_a) + \left(\frac{1}{E_b - H + \mathrm{i}\epsilon} V\Phi_b, V\Phi_a\right) \\
&\quad -(\Phi_b, V\Phi_a) - \left(\Phi_b, V\frac{1}{E_a - H + \mathrm{i}\epsilon} V\Phi_a\right) \\
&= \left(V\Phi_b, \left(\frac{1}{E_b - H - \mathrm{i}\epsilon} - \frac{1}{E_b - H + \mathrm{i}\epsilon}\right) V\Phi_a\right) \\
&= 2\pi\mathrm{i}(V\Phi_b, \delta(E_b - H)V\Phi_a) \ ,
\end{aligned} \tag{12.39}$$

then inserting the complete system $\{\Psi^{(-)}\}$, we obtain

$$\begin{aligned}
T_{ba}^\dagger - T_{ba} &= 2\pi\mathrm{i}\sum_n (V\Phi_b, \Psi_n^{(-)})\delta(E_b - E_n)\big(\Psi_n^{(-)}, V\Phi_a\big) \\
&= 2\pi\mathrm{i}\sum_n T_{bn}^\dagger \delta(E_b - E_n) T_{na} \ ,
\end{aligned} \tag{12.40}$$

where

$$T_{bn}^\dagger = T_{nb}^* = \big(\Psi_n^{(-)}, V\Phi_b\big)^* = \big(V\Phi_b, \Psi_n^{(-)}\big) \ .$$

If we inserted the other complete system $\{\Psi^{(+)}\}$ instead of $\{\Psi^{(-)}\}$, then instead of (12.40) we would get

$$2\pi\mathrm{i}\sum_n T_{bn}\delta(E_b - E_n)T_{na}^\dagger \ . \tag{12.41}$$

Thus, the unitarity condition can be written as

$$\begin{aligned} T^{\dagger}_{ba} - T_{ba} &= 2\pi \mathrm{i} \sum_n T^{\dagger}_{bn} \delta(E_b - E_n) T_{na} \\ &= 2\pi \mathrm{i} \sum_n T_{bn} \delta(E_b - E_n) T^{\dagger}_{na} \ . \end{aligned} \tag{12.42}$$

Using this unitarity condition, we can reproduce the optical theorem already discussed in Sect. 9.4. Although the content is exactly the same, let us express it in terms of the notation used in this section.

The probability per unit time for the transition $a \to b$ is

$$w_{ba} = 2\pi \delta(E_b - E_a) |T_{ba}|^2 \ . \tag{12.43}$$

If we take the sum over all probable final states, then from (12.42),

$$w_a = \sum_b w_{ba} = 2\pi \sum_b \delta(E_b - E_a) |T_{ba}|^2 = \frac{1}{\mathrm{i}} (T^{\dagger}_{aa} - T_{aa}) = -2 \mathrm{Im} T_{aa} \ . \tag{12.44}$$

Starting with the two-particle state a, the total cross-section can be obtained as

$$\sigma_a = \frac{\Omega}{v_{\mathrm{rel}}} w_a \ , \tag{12.45}$$

where Ω is the volume of quantization and v_{rel} is the relative speed of the two particles. Therefore, from (12.44),

$$\sigma_a = -\frac{2\Omega}{v_{\mathrm{rel}}} \mathrm{Im} T_{aa} \ . \tag{12.46}$$

Then, since the final result does not depend on Ω, we take $\Omega = 1$. The scattering amplitude $f(\theta)$ in the scattering potential is given by

$$\begin{aligned} f(\theta) &= -\frac{m}{2\pi} \int \mathrm{d}^3 x \, \mathrm{e}^{-\mathrm{i} \boldsymbol{k}_f \boldsymbol{x}} V(\boldsymbol{x}) \psi(\boldsymbol{x}) \\ &= -\frac{m}{2\pi} \big(\Phi_f, V \Psi_i^{(+)} \big) \ . \end{aligned} \tag{12.47}$$

Computing w_{fi}/v_{rel} for scattering into a constant solid angle, the cross-section becomes

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{2\pi}{v}\frac{1}{(2\pi)^3}\int k_f^2 \mathrm{d}k_f \delta\left(\frac{k_f^2}{2m} - \frac{k_i^2}{2m}\right)|T_{fi}|^2 \quad \left(v = \frac{k}{m}\right)\\ &= \left(\frac{m}{2\pi}\right)^2 |T_{fi}|^2\\ &= |f(\theta)|^2 \ ,\end{aligned} \tag{12.48}$$

which reproduces the well known result. In this regard, however, $T_{fi} = \left(\Phi_f, V\Psi_i^{(+)}\right)$. In addition, from (12.46), the total cross-section becomes

$$\sigma = -\frac{2}{v}\mathrm{Im}T_{aa} = -\frac{2}{v}\left(-\frac{2\pi}{m}\right)\mathrm{Im}f(0) = \frac{4\pi}{k}\mathrm{Im}f(0) \ , \tag{12.49}$$

which also reproduces the optical theorem in its well-known form.

Although in the scattering potential the asymptotic form of the wave function has been obtained easily, it turns out that the asymptotic form of the abstract Lippmann–Schwinger state vector $\Psi^{(+)}$ is given by

$$\Psi_a^{(+)} \sim S\Phi_a = \Phi_a - 2\pi i\delta(E_a - H_0)V\Psi_a^{(+)} \ . \tag{12.50}$$

Let us compare this asymptotic form with $\Psi_a^{(+)}$ itself:

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E_a - H_0 + \mathrm{i}\epsilon}V\Psi_a^{(+)} \ .$$

From this we understand that the asymptotic form can be derived if we make the following replacement for the scattered wave:

$$\frac{1}{E_a - H_0 + \mathrm{i}\epsilon} \to -2\pi\mathrm{i}\delta(E_a - H_0) \ , \tag{12.51}$$

or

$$\Psi_a^{(+)} \sim \Phi_a - 2\pi\mathrm{i}\delta(E_a - H_0)\left[(E_a - H_0)\Psi_a^{(+)}\right] \ , \tag{12.52}$$

where we multiply by $\delta(E_a - H_0)$ after multiplying by $(E_a - H_0)$. This operation reminds us of the Lehmann–Symanzik–Zimmermann (LSZ) asymptotic condition in Sect. 11.7. In (11.159), K_y corresponds to $(E_a - H_0)$, and $\Delta(y - x)$ corresponds to $\delta(E_a - H_0)$. Moreover, the asymptotic form satisfies the equation for the free particle, viz.,

$$(E_a - H_0)S\Phi_a = 0 \ . \tag{12.53}$$

This corresponds to (11.124).

Equation (12.52) is the reduction formula. It plays an important role in the discussion about recombination reactions. So far we have assumed that the separation of the Hamiltonian into the free part and the interaction part is unique, but in general, if a bound state appears, this uniqueness is lost. It is due to recombination reactions that this grouping in the initial state differs from that in the final state. For instance, consider a reaction such as

$$\mathrm{n} + \mathrm{d} \to \mathrm{n} + \mathrm{n}' + \mathrm{p} \ , \tag{12.54}$$

where d stands for the deuteron and n′ has been labeled by the prime in order to distinguish it from the other meson n. The total Hamiltonian is

$$H = T_{\mathrm{p}} + T_{\mathrm{n}} + T_{\mathrm{n}'} + V_{\mathrm{np}} + V_{\mathrm{n'p}} + V_{\mathrm{nn'}} \ , \tag{12.55}$$

where T and V denote the kinetic energies and the potentials for the two-body forces, respectively. The decomposition of the Hamiltonian corresponding to the initial state is

$$H_0 = T_{\mathrm{p}} + T_{\mathrm{n}} + T_{\mathrm{n}'} + V_{\mathrm{n'p}} \ , \quad V = V_{\mathrm{np}} + V_{\mathrm{nn'}} \ . \tag{12.56}$$

This is because, if we do not insert $V_{\mathrm{n'p}}$ into H_0, there is no way to make d. On the other hand, the decomposition in the final state is

$$H_0' = T_{\mathrm{p}} + T_{\mathrm{n}} + T_{\mathrm{n}'} \ , \quad V' = V_{\mathrm{np}} + V_{\mathrm{n'p}} + V_{\mathrm{nn'}} \ . \tag{12.57}$$

In general, we introduce two decompositions, one for the initial state and one for the final state:

$$H = H_a + V_a = H_b + V_b \ . \tag{12.58}$$

The free state vectors corresponding to each decomposition are Φ_a and Φ_b satisfying

$$(E_a - H_a)\Phi_a = 0 \ , \quad (E_b - H_b)\Phi_b = 0 \ . \tag{12.59}$$

Of course, in order for the transition $a \to b$ to occur, we must have $E_b = E_a$. Let us derive the transition amplitude T_{ba} in this case. Corresponding to the initial state, we have

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E_a - H + \mathrm{i}\epsilon} V_a \Phi_a \ . \tag{12.60}$$

Then, to construct the asymptotic form corresponding to the final state, we use the formula

$$\frac{1}{A} - \frac{1}{B} = \frac{1}{B}(B - A)\frac{1}{A} \ . \tag{12.61}$$

Therefore,

$$\frac{1}{E_a - H + \mathrm{i}\epsilon} - \frac{1}{E_a - H_b + \mathrm{i}\epsilon} = \frac{1}{E_a - H_b + \mathrm{i}\epsilon} V_b \frac{1}{E_a - H + \mathrm{i}\epsilon} \ , \tag{12.62}$$

which yields

$$\Psi_a^{(+)} = \Phi_a + \frac{1}{E_a - H_b + \mathrm{i}\epsilon}\left(1 + V_b \frac{1}{E_a - H + \mathrm{i}\epsilon}\right) V_a \Phi_a \ . \tag{12.63}$$

In the following, we write $E_a = E_b = E$ and make the replacement

$$\frac{1}{E - H_b + \mathrm{i}\epsilon} \ \longrightarrow \ -2\pi \mathrm{i}\delta(E - H_b) \ , \tag{12.64}$$

corresponding to (12.51). Then the asymptotic form corresponding to the final state is

$$\Psi_a^{(+)} \sim -2\pi \mathrm{i}\left(1 + V_b \frac{1}{E - H + \mathrm{i}\epsilon}\right) V_a \Phi_a \ . \tag{12.65}$$

Thus, T_{ba} can be given by the inner product of this asymptotic form and Φ_b :

$$\begin{aligned} T_{ba} &= \left(\Phi_b, \left(1 + V_b \frac{1}{E - H + \mathrm{i}\epsilon}\right) V_a \Phi_a\right) \\ &= \left(\left(1 + \frac{1}{E - H - \mathrm{i}\epsilon} V_b\right) \Phi_b, V_a \Phi_a\right) \\ &= \left(\Psi_b^{(-)}, V_a \Phi_a\right) \ . \end{aligned} \tag{12.66}$$

Note also that, corresponding to (12.38), there are also two ways to represent T_{ba}. Using $\Psi_a^{(+)}$ instead of $\Psi_b^{(-)}$,

$$T_{ba} = \left(\Psi_b^{(-)}, V_a \Phi_a\right) = \left(\Phi_b, V_b \Psi_a^{(+)}\right) \ . \tag{12.67}$$

This is because, subtracting one from the other,

$$\begin{aligned}\left(\Psi_b^{(-)}, V_a\Phi_a\right) - \left(\Phi_b, V_b\Psi_a^{(+)}\right) &= (\Phi_b, V_a\Phi_a) - (\Phi_b, V_b\Phi_a) \\ &\quad + (\Phi_b, V_a\Phi_a) - \left(\Phi_b, V_b\frac{1}{E - H + \mathrm{i}\epsilon}V_a\Phi_a\right) \\ &= (\Phi_b, (V_a - V_b)\Phi_a) \\ &= (\Phi_b, (H_b - H_a)\Phi_a) \\ &= (E_b - E_a)(\Phi_b, \Phi_a) = 0 \,.\end{aligned}$$

This shows that the two expressions in (12.67) are equal.

12.2 Renormalized Interaction Picture

In Chap. 8, we described the computational method for obtaining the S-matrix in the interaction picture based on a covariant perturbation theory. To define the interaction picture, we have to decompose the Hamiltonian or the Lagrangian into the free part and the interaction part. We took this decomposition to be trivial, but it is clear from the discussion about the recombination reaction in the previous section that this decomposition is not unique. Although at the lowest level of the perturbation it has not posed serious problems, it will turn out that this difference between decomposition methods has an important implication when computing higher order corrections.

We thus set down several conditions to determine the decomposition method. These conditions are called *renormalization conditions*. The interaction picture defined by the decomposition satisfying these conditions is called the *renormalized interaction picture*. In fact, it is in the renormalized interaction picture that the Gell-Mann–Low relation derived in Chap. 11, the related asymptotic conditions, and so on, all hold true, although we have not stated this clearly up to now. Another aspect of renormalization, and in general only this aspect is emphasized, is that it can remove the divergences appearing in higher order corrections. Indeed, it was through this that, in the period after World War II, there was a major development of QED. A theory in which divergences can be removed in this way is said to be *renormalizable*, and renormalizablity has been promoted as one of the guiding principles. It was also an important motivation for the more recent development of gauge theories.

Let us now go back to the problem of the decomposition of the Hamiltonian:

1. We have two complete orthonormal systems of eigenstates of the Hamiltonian:

$$\left\{\Psi_a^{(+)}\right\} \quad \text{and} \quad \left\{\Psi_a^{(-)}\right\} \,. \tag{12.68}$$

Furthermore, in relativistic quantum mechanics, these are eigenstates of the four-momentum:

$$P_\mu \Psi_a^{(\pm)} = (p_\mu)_a \Psi_a^{(\pm)} \ . \tag{12.69}$$

The S-matrix element is given by

$$S_{ab} = \left(\Psi_b^{(-)}, \Psi_a^{(+)}\right) \ . \tag{12.70}$$

It is clear from this expression for the S-matrix element that these two complete systems can be considered to be the same as

$$\{\Phi_a^{\text{in}}\} \quad \text{and} \quad \{\Phi_a^{\text{out}}\} \ . \tag{12.71}$$

2. In field theory, the vacuum state Φ_0 and the stable one-particle state Φ_α satisfy the conditions

$$\Phi_0^{\text{in}} = \Phi_0^{\text{out}} \ , \quad \Phi_\alpha^{\text{in}} = \Phi_\alpha^{\text{out}} \ . \tag{12.72}$$

If we express these conditions in terms of the S-matrix, then

$$S\Phi_0 = \Phi_0 \ , \quad S\Phi_\alpha = \Phi_\alpha \ , \tag{12.73}$$

where (12.72) implies that Φ_0 and Φ_α have been written without distinguishing between the in-state and the out-state. Equations (12.72) and (12.73) are called *renormalization conditions*.

Then, since the condition on the vacuum is satisfied by (8.70), this means that the S-matrix is defined by dropping all bubble diagrams. It turns out that we define the interaction picture in such a way as to satisfy these conditions. This requires as a consequence reintroducing several kinds of physical observable using the following two kinds of renormalization

- **Mass Renormalization.** In field theory, no elementary particles are in the bare state, because they have self-interactions. For example, even within the framework of classical theory, a charged particle carries the Coulomb field. Thus, considering an electron, the conditions (12.73) should hold true for the electron carrying its own field and not for the bare electron. Due to the existence of the self-interaction, the mass of an electron increases by what we call the *self-energy* δm. Thus, the observable mass of an electron is not just the originally given mass m, but changes to $m + \delta m$. We call m and $m + \delta m = m_{\text{obs}}$ the *bare mass* and the *observed mass*, respectively. We interpret this by saying that the one-electron state is the state with mass m_{obs}. We determine the self-energy δm from (12.73), i.e., we consider m_{obs} as the given mass rather than m. In the sense of absorbing δm into the mass, we call this reinterpretation a *renormalization of the mass*.

- **Charge Renormalization.** As discussed in Chap. 8, due to the phenomena of vacuum polarization, the vacuum behaves like a dielectric medium in field theory. Thus, taking the permittivity of the vacuum as ϵ, when the distance r between two charges e_1 and e_2 is large enough, the Coulomb potential between them is

$$V = \frac{e_1 e_2}{\epsilon} \frac{1}{4\pi r} . \tag{12.74}$$

However, since the permittivity of the vacuum is normalized to unity,

$$V = (e_1)_{\text{obs}} (e_2)_{\text{obs}} \frac{1}{4\pi r} . \tag{12.75}$$

Comparing (12.74) and (12.75),

$$(e_1)_{\text{obs}} = \frac{e_1}{\sqrt{\epsilon}} , \quad (e_2)_{\text{obs}} = \frac{e_2}{\sqrt{\epsilon}} , \tag{12.76}$$

where $(e)_{\text{obs}}$ is the experimental value. Thus, we must reinterpret $(e)_{\text{obs}}$ as the given charge, and not the bare charge e. Since the permittivity of the vacuum ϵ is absorbed into the definition of the electric charge, this is called a *renormalization of the electric charge*.

Renormalization can thus be additive, as for the mass, or multiplicative, as for the electric charge. Moreover, when we compute the self-energy δm or the permittivity ϵ using QED, we find that they diverge.

What we have discussed above is the renormalization of specific quantities due to the self-energy and the permittivity. There exists a procedure called *renormalization of the field operators*, which is a slightly abstract renormalization condition. This has already been required in Chap. 11 and expressed in (11.145). There exists a condition whereby the wave function of a given body is not varied by introducing interactions, i.e., the normalization is unchanged. What this condition means will be explained in detail later.

12.3 Mass Renormalization

We call a Feynman diagram which starts with and ends in a one-electron state a *self-energy diagram*. Any diagram which cannot be separated into two disconnected diagrams by cutting a single electron line is called an *irreducible self-energy diagram* (see Fig. 12.2 left). Others are said to be *reducible* (see Fig. 12.2 right).

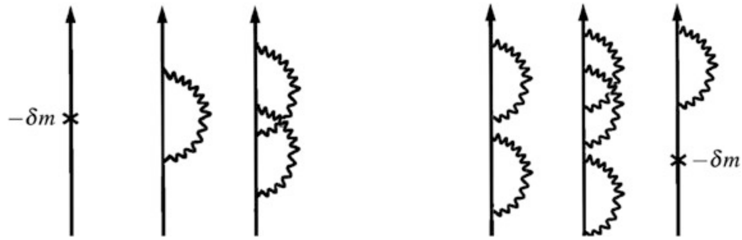

Fig. 12.2 *Left*: Irreducible self-energy diagram. *Right*: Reducible self-energy diagram

Considering the S-matrix element for the case where an electron with the four-momentum p enters and electron in the same state comes out, we have

$$\langle p|S|p\rangle = \langle p|p\rangle + \sum_{n=1}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_n \langle p|T[\mathscr{H}_{\mathrm{int}}(x_1)\ldots\mathscr{H}_{\mathrm{int}}(x_n)]|p\rangle_{\mathrm{conn}}. \tag{12.77}$$

Here, in order to drop contributions from bubble diagrams, after the second term we keep only contributions from the connected parts in which bubble diagrams are omitted. The subscript 'conn' stands for dropping bubble diagrams which are not connected to any points in $x_1, \ldots, x_n$. The condition (12.73) requires the terms after the second term on the right-hand side of (12.77) to vanish. Taking m_{obs} as the mass in the free part, the difference from m, viz., δm, is included in the interaction part, so that in a first approximation where we consider only the renormalization of the mass, we can write

$$\mathscr{H}_{\mathrm{int}} = -\mathrm{i}e\bar{\psi}\gamma_\mu\psi A_\mu - \delta m\bar{\psi}\psi \ . \tag{12.78}$$

Computing the S-matrix element to order e^2, if we consider δm to be of order e^2, then from the Feynman–Dyson rule,

$$\langle p'|S^{(2)}|p\rangle = \int \mathrm{d}^4 x \langle p'|\bar{\psi}(x)|0\rangle \left[\mathrm{i}\delta m^{(2)} + e^2 \int \mathrm{d}^4 y\, \gamma_\mu S_{\mathrm{F}}(y)\gamma_\mu D_{\mathrm{F}}(y)\mathrm{e}^{-\mathrm{i}p\cdot y}\right] \langle 0|\psi(x)|p\rangle \ , \tag{12.79}$$

where we have taken into account only the first two irreducible self-energy diagrams shown above. This is because the third is of order e^4. We now introduce

$$\Sigma^{*(2)}(p) = \mathrm{i}e^2 \int \mathrm{d}^4 y\, \gamma_\mu S_{\mathrm{F}}(y)\gamma_\mu D_{\mathrm{F}}(y)\mathrm{e}^{-\mathrm{i}p\cdot y} \ , \tag{12.80}$$

where the computation above is carried out in the Fermi–Feynman gauge corresponding to $\alpha = 1$. In general, the sum over contributions from all irreducible self-energy diagrams is called a *proper self-energy operator* or a *mass operator*.

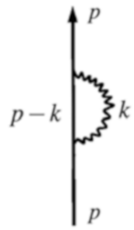

Fig. 12.3 Self-energy diagram in momentum space

Hence,

$$\langle p'|S^{(2)}|p\rangle = \mathrm{i}\int \mathrm{d}^4x \langle p'|\bar{\psi}(x)|0\rangle\big[\delta m^{(2)} - \Sigma^{*(2)}(p)\big]\langle 0|\bar{\psi}(x)|p\rangle \ . \tag{12.81}$$

The renormalization condition requires the expression above to vanish. Writing m_{obs} simply as m, we now have (Fig. 12.3)

$$\begin{aligned}
\Sigma^{*(2)}(p) &= \frac{\mathrm{i}e^2}{(2\pi)^4}\int \mathrm{d}^4k\gamma_\mu \frac{\mathrm{i}(p-k)\cdot\gamma - m}{(p-k)^2+m^2-\mathrm{i}\epsilon}\gamma_\mu \frac{1}{k^2-\mathrm{i}\epsilon} \\
&= \frac{\mathrm{i}e^2}{(2\pi)^4}\int \mathrm{d}^4k \frac{-2\mathrm{i}(p-k)\cdot\gamma - 4m}{(p-k)^2+m^2-\mathrm{i}\epsilon}\frac{1}{k^2-\mathrm{i}\epsilon} \\
&= \frac{\mathrm{i}e^2}{(2\pi)^4}\int \mathrm{d}^4k \int \mathrm{d}x \frac{-2\mathrm{i}(p-k)\cdot\gamma - 4m}{(k^2+xp^2-2xp\cdot k+xm^2-\mathrm{i}\epsilon)^2},
\end{aligned} \tag{12.82}$$

where we have used the fact that $\gamma_\mu\gamma_\mu = 4$ and $\gamma_\mu\gamma_\lambda\gamma_\mu = -2\gamma_\lambda$. Making the change of variable

$$k \ \rightarrow \ k' = k - xp \ , \tag{12.83}$$

rewriting (12.82), and dropping odd-order terms in k', we obtain

$$\Sigma^{*(2)}(p) = \frac{-\mathrm{i}e^2}{(2\pi)^4}\int_0^1 \mathrm{d}x \int \mathrm{d}^4k' \frac{2\mathrm{i}(1-x)p\cdot\gamma + 4m}{\big[k'^2 + x(1-x)p^2 + xm^2 - \mathrm{i}\epsilon\big]^2} \ . \tag{12.84}$$

Since $p^2 = -(\mathrm{i}p\cdot\gamma)^2$, $\Sigma^*(p)$ can be identified with a function of $\mathrm{i}p\cdot\gamma$, and expanding this as a power series in $(\mathrm{i}p\cdot\gamma + m)$, we have

$$\Sigma^{*(2)}(p) = A + B(\mathrm{i}p\cdot\gamma + m) + C(p) \ , \tag{12.85}$$

where $C(p)$ is the sum over all terms higher than the second order in $(\mathrm{i}p\cdot\gamma + m)$. Therefore, A can be obtained if we set $\mathrm{i}p\cdot\gamma = -m$ in $\Sigma^{*(2)}(p)$. This yields

$$A = \frac{-\mathrm{i}e^2}{(2\pi)^4}\int_0^1 \mathrm{d}x\, 2m(1+x)\int \mathrm{d}^4k' \frac{1}{(k'^2 + x^2m^2 - \mathrm{i}\epsilon)^2} \ . \tag{12.86}$$

To carry out this k'-integral, we use

$$\frac{1}{(k'^2 + x^2 m^2 - \mathrm{i}\epsilon)^2} = 2 \lim_{\Lambda\to\infty} \int_{m^2}^{\Lambda^2} \frac{x^2 \mathrm{d}M^2}{(k'^2 + x^2 M^2 - \mathrm{i}\epsilon)^3} . \tag{12.87}$$

Plugging this into (12.86) and carrying out the k'-integral, we obtain

$$\begin{aligned} A &= \frac{2\pi^2 e^2 m}{(2\pi)^4} \int_0^1 \mathrm{d}x (1+x) \int_{m^2}^{\Lambda^2} \frac{\mathrm{d}M^2}{M^2} \\ &= \frac{3}{2\pi} \alpha m \ln \frac{\Lambda}{m} , \quad \text{where } \alpha = \frac{e^2}{4\pi} . \end{aligned} \tag{12.88}$$

Going back to (12.81) and using the wave functions (3.191a), (3.191b), (3.192a) and (3.192b),

$$\langle p' | S^{(2)} | p \rangle = \frac{\mathrm{i}(2\pi)^4}{V} \delta^4(p' - p) \bar{u}(p') \big[\delta m^{(2)} - \Sigma^{*(2)}(p) \big] u(p) . \tag{12.89}$$

Using Dirac's equation for $u(p)$, we can replace $-p \cdot \gamma$ by $-m$ in the equation above. Thus, the condition that (12.89) should vanish can be written as

$$\delta m^{(2)} = A . \tag{12.90}$$

This stands for the self-energy of the electron up to order e^2. It is clear from (12.88) that this diverges logarithmically in the limit $\Lambda \to \infty$. However, what we should emphasize here is that the renormalization condition (12.73) requires a choice of interaction part of the form (12.78).

12.4 Renormalization of Field Operators

We have used (12.90) to understand the meaning of A in the expansion (12.85), but what about B? To answer this, we write down the condition for the normalization of the one-body wave function to remain unchanged when interactions are introduced:

$$\big\langle 0 \big| T[\psi(x) U(\infty, -\infty)]_{\mathrm{conn}} \big| p \big\rangle = \langle 0 | \psi(x) | p \rangle . \tag{12.91}$$

The Feynman diagram corresponding to the left-hand side is shown in Fig. 12.4.

Here, Σ^* is the contribution from all the irreducible self-energy diagrams except for δm. It should be clear from the diagram above that the contribution from the

Fig. 12.4 Feynman diagram corresponding to the left-hand side of (12.91)

left-hand side of (12.91) is

$$\frac{\mathrm{i}p\cdot\gamma+m}{\mathrm{i}p\cdot\gamma+m+\Sigma^*(p)-\delta m}u(p) = \Big\{1-\big[\Sigma^*(p)-\delta m\big]\frac{1}{\mathrm{i}p\cdot\gamma+m} + \big[\Sigma^*(p)-\delta m\big]\frac{1}{\mathrm{i}p\cdot\gamma+m}\big[\Sigma^*(p)-\delta m\big]\frac{1}{\mathrm{i}p\cdot\gamma+m}+\cdots\Big\}u(p)\,. \tag{12.92}$$

The discussion so far is based on the assumption that the interaction part is given by (12.78). Expanding Σ^* as suggested by (12.78), we can take $\mathrm{i}p\cdot\gamma = -m$ because $u(p)$ appears in (12.92). This yields

$$\lim_{\mathrm{i}p\cdot\gamma+m\to 0}\frac{\mathrm{i}p\cdot\gamma+m}{\mathrm{i}p\cdot\gamma+m+\Sigma^*(p)-\delta m} = \frac{1}{1+B}\,. \tag{12.93}$$

This is not equal to unity unless $B = 0$, so (12.91) cannot be satisfied. Recall the origin of the condition that the normalization is unchanged. In fact, it originally arose from the asymptotic condition (11.126) for the operator in the Heisenberg picture. It was not assumed that the operator used in this case was the same as the original one. Hence, we may consider that the Heisenberg operator satisfying the asymptotic condition has a different normalization from the original Heisenberg operator appearing in the Lagrangian. We call operators satisfying the asymptotic condition *renormalized field operators*. They carry the subscript r and we assume the following multiplicative renormalization:

$$\boldsymbol{\psi}(x) = Z_2^{1/2}\boldsymbol{\psi}(x)_{\mathrm{r}}\,,\quad \bar{\boldsymbol{\psi}}(x) = Z_2^{1/2}\bar{\boldsymbol{\psi}}(x)_{\mathrm{r}}\,,\quad \boldsymbol{A}_\mu(x) = Z_3^{1/2}\boldsymbol{A}_\mu(x)_{\mathrm{r}}\,. \tag{12.94}$$

The multiplicative renormalization has already appeared in the renormalization of the electric charge, and we shall see that this is in fact closely related to the renormalization of operators mentioned above.

Going back to the general gauge, let us express the whole Lagrangian in terms of renormalized operators. Since in this case the operators are originally the Heisenberg operators, we should use bold face for them, but there should be no confusion if the usual type face is used:

$$\mathscr{L} = -Z_2\bar{\psi}_\mathrm{r}\Big[\gamma_\mu\big(\partial_\mu - \mathrm{i}eZ_3^{1/2}A_{\mu\mathrm{r}}\big) + m - \delta m\Big]\psi_\mathrm{r} - \frac{1}{4}Z_3F_{\mu\nu\mathrm{r}}F_{\mu\nu\mathrm{r}} - \frac{1}{2\alpha}Z_3(\partial_\mu A_{\mu\mathrm{r}})^2 \,. \tag{12.95}$$

Although $m - \delta m$ should be written as $m_\mathrm{obs} - \delta m$, as already mentioned, for simplicity we have written m_obs as m. We decompose (12.95) into the free part and the interaction part as follows:

$$\mathscr{L}_\mathrm{f} = -\bar{\psi}_\mathrm{r}(\gamma_\mu\partial_\mu + m)\psi_\mathrm{r} - \frac{1}{4}F_{\mu\nu\mathrm{r}}F_{\mu\nu\mathrm{r}} - \frac{1}{2\alpha_\mathrm{r}}(\partial_\mu A_{\mu\mathrm{r}})^2 \,, \tag{12.96}$$

$$\mathscr{L}_\mathrm{int} = (1 - Z_2)\bar{\psi}_\mathrm{r}(\gamma_\mu\partial_\mu + m)\psi_\mathrm{r} + (1 - Z_3)\frac{1}{4}F_{\mu\nu\mathrm{r}}F_{\mu\nu\mathrm{r}} + \mathrm{i}eZ_2Z_3^{1/2}A_{\mu\mathrm{r}}\bar{\psi}_\mathrm{r}\gamma_\mu\psi_\mathrm{r} + Z_2\delta m\bar{\psi}_\mathrm{r}\psi_\mathrm{r} \,, \tag{12.97}$$

where α_r is the renormalized gauge parameter. Hence, the gauge parameter changes under renormalization. The interaction picture corresponding to the partition above is called the *renormalized interaction picture*. The renormalized gauge parameter α_r is defined by

$$\alpha_\mathrm{r} = \alpha Z_3^{-1} \,. \tag{12.98}$$

Since we will only use the renormalized interaction picture in the following discussions, for simplicity we will drop the subscript r. In perturbation theory, several kinds of renormalization constant can be expanded as power series in e^2. We thus assume the expansions

$$\delta m = \delta m^{(2)} + \delta m^{(4)} + \cdots \,, \tag{12.99}$$

$$Z_2 = 1 + Z_2^{(2)} + \cdots \,, \tag{12.100}$$

$$Z_3 = 1 + Z_3^{(2)} + \cdots \,. \tag{12.101}$$

In the interaction picture, the interaction includes derivatives of field operators, so we use Matthew's theorem to express the S-matrix in the form

$$S = 1 + \sum_{n=1}^{\infty}\frac{\mathrm{i}^n}{n!}\int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n T^*\big[\mathscr{L}_\mathrm{int}(x_1)\ldots\mathscr{L}_\mathrm{int}(x_n)\big]_\mathrm{conn} \,. \tag{12.102}$$

We now repeat the discussion in the last section in this new interaction picture:

$$\langle p'|S^{(2)}|p\rangle = \int \mathrm{d}^4x \langle p'|\bar{\psi}(x)|0\rangle\mathscr{S}(p)\langle 0|\psi(x)|p\rangle \,. \tag{12.103}$$

The new quantity $\mathscr{S}(p)$ is given by the following equation, corresponding to $\mathrm{i}[\delta m^{(2)} - \Sigma^{*(2)}(p)]$ in (12.81):

$$\mathscr{S}(p) = \mathrm{i}Z_2\delta m + \mathrm{i}(1 - Z_2)(\mathrm{i}p \cdot \gamma + m) - \mathrm{i}\Sigma^*(p) \; . \tag{12.104}$$

In the lowest order approximation, Σ^* is given by

$$\Sigma^*(p) = \mathrm{i}e^2 Z_2^2 Z_3 \int \mathrm{d}^4 y \, \gamma_\mu S_\mathrm{F}(y)\gamma_\mu D_\mathrm{F}(y)\mathrm{e}^{-\mathrm{i}p\cdot y} \; . \tag{12.105}$$

Replacing Z_2 and Z_3 by 1, this coincides with the expression in the last section. Then in general, expanding as

$$\Sigma^*(p) = A + B(\mathrm{i}p \cdot \gamma + m) + C(p) \; , \tag{12.106}$$

the expression (12.103) vanishes. Using the fact that the normalization of the one-particle wave function is unchanged, we have

$$Z_2\delta m = A \; , \quad 1 - Z_2 = B \; . \tag{12.107}$$

It turns out that δm and Z_2 can be determined from this renormalization condition. Hence,

$$\mathscr{S}(p) = -\mathrm{i}C(p) \equiv -\mathrm{i}\Sigma^*_\mathrm{ren}(p) \; . \tag{12.108}$$

This equation defines the renormalized mass operator Σ^*_ren. Although $A^{(2)}$ and $B^{(2)}$ diverge logarithmically, $C^{(2)}(p)$ is finite.

12.5 Renormalized Propagators

In the renormalized interaction picture, the electron propagator is defined by

$$S_\mathrm{F}(x-y) = \langle 0 | T[\psi(x)\bar{\psi}(y)] | 0 \rangle = \frac{-\mathrm{i}}{(2\pi)^4} \int \mathrm{d}^4 p \, \mathrm{e}^{\mathrm{i}p\cdot(x-y)} S_\mathrm{F}(p) \; ,$$

with

$$S_\mathrm{F}(p) = \frac{1}{\mathrm{i}p \cdot \gamma + m} \; . \tag{12.109}$$

A propagator including higher order corrections can be expressed in the Heisenberg picture by

$$S'_F = \langle \mathbf{0} | T[\boldsymbol{\psi}(x)\bar{\boldsymbol{\psi}}(y)] | \mathbf{0} \rangle = \frac{\langle 0 | T^*[\psi(x), \bar{\psi}(y), U(\infty, -\infty)] | 0 \rangle}{\langle 0 | U(\infty, -\infty) | 0 \rangle} . \tag{12.110}$$

In in the renormalized interaction picture,

$$U(\infty, -\infty) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n T^*\left[\mathscr{L}_{int}(x_1) \dots \mathscr{L}_{int}(x_n)\right] . \tag{12.111}$$

From the discussion about the Feynman diagram, the Fourier transform of S'_F is

$$\begin{aligned} S'_F(p) &= S_F(p) - S_F(p)\Sigma^*_{ren}(p)S_F(p) + \cdots \\ &= S_F(p) - S_F(p)\Sigma^*_{ren}(p)S'_F(p) . \end{aligned} \tag{12.112}$$

This is called *Dyson's equation*. Its solution is

$$S'_F(p) = S_F(p)\left[1 + \Sigma^*_{ren}(p)S_F(p)\right]^{-1} = \left[ip \cdot \gamma + m + \Sigma^*_{ren}(p)\right]^{-1} . \tag{12.113}$$

Since $\Sigma^*_{ren}(p)$ is a sum over terms higher than second order in $ip \cdot \gamma + m$,

$$\lim_{ip\cdot\gamma+m \to 0} (ip \cdot \gamma + m)S'_F(p) = 1 . \tag{12.114}$$

This is an important property of the renormalized propagator. Then in the computation in Sect. 12.3, and in particular in (12.84), we make the change of variables from the Feynman parameter x to M, where

$$m^2 = (1 - x)M^2 , \tag{12.115}$$

whence $S'_F(p)$ can be written to order e^2 as

$$\begin{aligned} S'_F(p) &= S_F(p) - S_F(p)\Sigma^{*(2)}_{ren}(p)S_F(p) \\ &= \frac{1}{ip \cdot \gamma + m} + \frac{e^2}{16\pi^2} \int_m^{\infty} \frac{dM}{M^3(M^2 - m^2)} \left[\frac{(M + m)^2(M^2 + m^2 - 4mM)}{ip \cdot \gamma + M - i\epsilon} \right. \\ &\qquad \left. + \frac{(M - m)^2(M^2 + m^2 + 4mM)}{ip \cdot \gamma - M + i\epsilon} \right] . \end{aligned} \tag{12.116}$$

This integral diverges at $M = m$. In this case, we need to improve the approximation near the mass shell using some suitable method. We will return to this when discussing the renormalization group method in Chap. 20. However, what we call the ultraviolet divergence disappears completely.

Next, let us study the propagator of the electromagnetic field. Here we consider the renormalized Fermi–Feynman gauge $\alpha_{\mathrm{r}} = 1$. In the interaction picture,

$$\delta_{\mu\nu} D_{\mathrm{F}}(x-y) = \langle 0 | T[A_\mu(x) A_\nu(y)] | 0 \rangle = \frac{-\mathrm{i}}{(2\pi)^4} \delta_{\mu\nu} \int \mathrm{d}^4 k \mathrm{e}^{\mathrm{i}k\cdot(x-y)} D_{\mathrm{F}}(k) . \tag{12.117}$$

In the renormalized Heisenberg picture including higher order corrections, this is not proportional to $\delta_{\mu\nu}$:

$$D'_{F_{\mu\nu}}(x-y) = \langle \mathbf{0} | T[\boldsymbol{A}_\mu(x) \boldsymbol{A}_\nu(y)] | \mathbf{0} \rangle = \frac{\langle 0 | T^*[A_\mu(x) A_\nu(y) U(\infty, -\infty)] | 0 \rangle}{\langle 0 | U(\infty, -\infty) | 0 \rangle} . \tag{12.118}$$

Let us compute corrections up to order e^2. In this case, the diagrams which should be taken into account are shown in Fig. 12.5.

The terms in the interaction Lagrangian needed for this calculation are

$$\frac{1}{4}(1 - Z_3) F_{\mu\nu} F_{\mu\nu} + \mathrm{i}e Z_2 Z_3^{1/2} A_\mu \bar{\psi} \gamma_\mu \psi .$$

The first term and iterations of the second term correspond to the first diagram and the second diagram in Fig. 12.5, respectively. However, to order e^2, Z_2 and Z_3 in

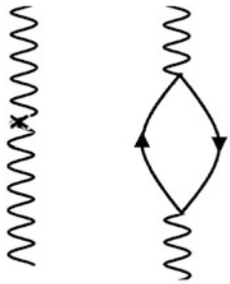

Fig. 12.5 Feynman diagrams for the electromagnetic field propagator up to order e^2

the second term can be set equal to 1:

$$
\begin{aligned}
D'_{F_{\mu\nu}}(x-y) &= \delta_{\mu\nu} D_{\mathrm{F}}(x-y) \\
&\quad - e^2 \int \mathrm{d}^4x' \mathrm{d}^4x'' D_{\mathrm{F}}(x-x') D_{\mathrm{F}}(x''-y) \mathrm{Tr}\left[\gamma_\mu S_{\mathrm{F}}(x'-x'')\gamma_\nu S_{\mathrm{F}}(x''-x')\right] \\
&\quad + \frac{\mathrm{i}}{2}(1-Z_3) \int \mathrm{d}^4x' \left(\delta_{\mu\nu}\frac{\partial}{\partial x'_\rho} - \delta_{\mu\rho}\frac{\partial}{\partial x'_\sigma}\right) D_{\mathrm{F}}(x-x') \,. \qquad (12.119)
\end{aligned}
$$

Taking the Fourier transform of this equation yields

$$
\begin{aligned}
D'_{F_{\mu\nu}}(k) &= \frac{\delta_{\mu\nu}}{k^2 - \mathrm{i}\epsilon} \\
&\quad - \frac{\mathrm{i}e^2}{(2\pi)^4} \frac{1}{(k^2-\mathrm{i}\epsilon)^2} \int \mathrm{d}^4 p \mathrm{Tr}\left[\gamma_\mu \frac{1}{\mathrm{i}p\cdot\gamma + m - \mathrm{i}\epsilon} \gamma_\nu \frac{1}{\mathrm{i}(p-k)\cdot\gamma + m - \mathrm{i}\epsilon}\right] \\
&\quad + (1-Z_3)\frac{1}{k^2-\mathrm{i}\epsilon}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2-\mathrm{i}\epsilon}\right). \qquad (12.120)
\end{aligned}
$$

The second term is the Fourier transform of the expression

$$\left\langle 0 \middle| T^*[j_\mu(x), j_\nu(y)] \middle| 0 \right\rangle .$$

And formally, this satisfies the condition

$$\frac{\partial}{\partial x_\mu}\left\langle 0 \middle| T^*[j_\mu(x), j_\nu(y)] \middle| 0 \right\rangle = 0 \,. \qquad (12.121)$$

In momentum space, this condition becomes

$$k_\mu \int \mathrm{d}^4 p \mathrm{Tr}[\ldots] = 0 \,. \qquad (12.122)$$

The integral in the equation above, denoted by $f_{\mu\nu}(k)$, has the general form

$$f_{\mu\nu}(k) = \delta_{\mu\nu} f(k^2) - k_\mu k_\nu g(k^2) \,. \qquad (12.123)$$

From the condition (12.122),

$$k_\mu f_{\mu\nu}(k) = k_\nu\left[f(k^2) - k^2 g(k^2)\right] = 0 \,. \qquad (12.124)$$

Thus,

$$f_{\mu\nu}(k) = (k^2\delta_{\mu\nu} - k_\mu k_\nu) g(k^2) \,. \qquad (12.125)$$

Computing the trace in the integral in (12.122),

$$\mathrm{Tr} = \frac{4\left[\delta_{\mu\nu}(p^2 - p\cdot k + m^2) - 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu\right]}{(p^2 + m^2 - \mathrm{i}\epsilon)\left[(p-k)^2 + m^2 - \mathrm{i}\epsilon\right]} \ .$$

Therefore,

$$\begin{aligned} f_{\mu\nu}(k) &= 4\int \mathrm{d}^4 p \frac{\delta_{\mu\nu}(p^2 - p\cdot k + m^2) - 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu}{(p^2 + m^2 - \mathrm{i}\epsilon)\left[(p-k)^2 + m^2 - \mathrm{i}\epsilon\right]} \\ &= 4\int_0^1 \mathrm{d}x \int \mathrm{d}^4 p \frac{\delta_{\mu\nu}(p^2 - p\cdot k + m^2) - 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu}{\left[p^2 + m^2 + x(k^2 - 2p\cdot k) - \mathrm{i}\epsilon\right]^2} \ . \end{aligned}$$

We make the change of variables

$$p \ \to \ p' = p - xk \ . \tag{12.126}$$

Thus,

$$f_{\mu\nu}(k) = 4\int_0^1 \mathrm{d}x \int \mathrm{d}^4 p' \frac{N}{\left[p'^2 + m^2 + x(1-x)k^2 - \mathrm{i}\epsilon\right]^2} \ , \tag{12.127}$$

where

$$N = \delta_{\mu\nu} p'^2 - 2p'_\mu p'_\nu + \delta_{\mu\nu}\left[m^2 + x(1-x)k^2\right] - 2x(1-x)(\delta_{\mu\nu}k^2 - k_\mu k_\nu) \ . \tag{12.128}$$

All terms except for the last produce terms proportional to $\delta_{\mu\nu}$. Hence, for this to coincide with (12.125), only the last term can survive. Thus,

$$f_{\mu\nu}(k) = -8(\delta_{\mu\nu}k^2 - k_\mu k_\nu)\int_0^1 \mathrm{d}x\, x(1-x) \int \frac{\mathrm{d}^4 p}{\left[p^2 + m^2 + x(1-x)k^2 - \mathrm{i}\epsilon\right]^2} \ , \tag{12.129}$$

and we can write

$$\begin{aligned} \frac{1}{\left[p^2 + m^2 + x(1-x)k^2 - \mathrm{i}\epsilon\right]^2} &= \frac{1}{(p^2 + m^2 - \mathrm{i}\epsilon)^2} \\ &\quad + \left\{ \frac{1}{\left[p^2 + m^2 + x(1-x)k^2 - \mathrm{i}\epsilon\right]^2} - \frac{1}{(p^2 + m^2 - \mathrm{i}\epsilon)^2} \right\} \ . \end{aligned}$$

Therefore, the expression for $D'_{F_{\mu\nu}}$ becomes

$$D'_{F_{\mu\nu}}(k) = \frac{1}{k^2 - \mathrm{i}\epsilon} + \frac{8\mathrm{i}e^2}{(2\pi)^4}\frac{k^2\delta_{\mu\nu} - k_\mu k_\nu}{(k^2 - \mathrm{i}\epsilon)^2}\int_0^1 x(1-x)\mathrm{d}x$$
$$\times \int \mathrm{d}^4 p \left\{\frac{1}{[p^2 + m^2 + x(1-x)k^2 - \mathrm{i}\epsilon]^2} - \frac{1}{(p^2 + m^2 - \mathrm{i}\epsilon)^2}\right\}$$
$$+ \frac{k^2\delta_{\mu\nu} - k_\mu k_\nu}{(k^2 - \mathrm{i}\epsilon)^2}\left[1 - Z_3 + \frac{8\mathrm{i}e^2}{(2\pi)^4}\int_0^1 x(1-x)\mathrm{d}x \int \frac{\mathrm{d}^4 p}{(p^2 + m^2 - \mathrm{i}\epsilon)^2}\right]. \tag{12.130}$$

Then to determine Z_3, corresponding to (12.114) in the case of the electron, or the equation

$$S_\mathrm{F}(p)\Sigma^*_\mathrm{ren}(p)u(p) = 0\ , \tag{12.131}$$

we adopt the condition

$$D_\mathrm{F}(k)\Pi^*_\mathrm{ren}(k)_{\mu\nu}e_\nu = 0\ , \tag{12.132}$$

where e_ν is the polarization vector of transverse photons and Π^* is called the *proper self-energy operator of the photon* or the *polarization operator*, which corresponds to Σ^* in the electron case. Thus, similarly to Dyson's equation (12.112) for Σ^* in the electron case, Π^* is defined by

$$\begin{aligned} D'_{F_{\mu\nu}} &= \delta_{\mu\nu}D_\mathrm{F}(k) - D_\mathrm{F}(k)\Pi^*_\mathrm{ren}(k)_{\mu\nu}D_\mathrm{F}(k) + \cdots \\ &= \delta_{\mu\nu}D_\mathrm{F}(k) - D_\mathrm{F}(k)\Pi^*_\mathrm{ren}(k)_{\mu\lambda}D'_{F_{\lambda\nu}}(k)\ . \end{aligned} \tag{12.133}$$

To order e^2,

$$\Pi^*_\mathrm{ren}(k)_{\mu\nu} = -\frac{8\mathrm{i}e^2}{(2\pi)^4}(k^2\delta_{\mu\nu} - k_\mu k_\nu)\int_0^1 x(1-x)\mathrm{d}x$$
$$\times \int \mathrm{d}^4 p\left[\frac{1}{\left[p^2 + m^2 + x(1-x)k^2 - \mathrm{i}\epsilon\right]^2} - \frac{1}{(p^2 + m^2 - \mathrm{i}\epsilon)^2}\right]$$
$$+ \frac{8\mathrm{i}e^2}{(2\pi)^4}(\delta_{\mu\nu}k^2 - k_\mu k_\nu)\left[1 - Z_3 + \frac{8\mathrm{i}e^2}{(2\pi)^4}\int_0^1 x(1-x)\mathrm{d}x\int\frac{\mathrm{d}^4 p}{(p^2 + m^2 - \mathrm{i}\epsilon)^2}\right]. \tag{12.134}$$

Then, in the one-photon state, taking into account $k^2 = 0$ and $k \cdot e = 0$, the condition (12.132) requires the coefficient of $\delta_{\mu\nu}k^2 - k_\mu k_\nu$ to vanish when $k^2 = 0$. Since the first term in (12.134) satisfies this condition, it implies that the second term vanishes, i.e.,

$$Z_3 = 1 + \frac{8ie^2}{(2\pi)^4} \int_0^1 x(1-x)\mathrm{d}x \int \frac{\mathrm{d}^4 p}{(p^2 + m^2 - i\epsilon)^2} . \quad (12.135)$$

Therefore, carrying out the Feynman integral,

$$\Pi^*_{\text{ren}}(k)_{\mu\nu} = \frac{e^2}{2\pi^2}(k^2\delta_{\mu\nu} - k_\mu k_\nu) \int_0^1 x(1-x) \ln \frac{m^2}{m^2 + x(1-x)k^2 - i\epsilon} \mathrm{d}x . \quad (12.136)$$

In particular, when $|k^2| \ll m^2$,

$$\Pi^*_{\text{ren}}(k)_{\mu\nu} \approx \frac{e^2}{60\pi^2}(\delta_{\mu\nu}k^2 - k_\mu k_\nu)\left(-\frac{k^2}{m^2}\right) . \quad (12.137)$$

We see that, from the renormalization condition (12.132), Z_3 is uniquely determined by (12.135), and it turns out that D'_{F} is divergenceless.

12.6 Renormalization of Vertex Functions

We have seen that the propagator becomes finite in the renormalized interaction picture. However, there is one thing that does not become finite without multiplicative renormalization, namely the vertex function. We now consider its renormalization. Up to now, we have investigated the proper self-energy diagrams, but these are all related to two-point functions or propagators. We now consider the corrections to the vertex function γ_μ shown in Fig. 12.6.

These are diagrams in which the propagator, including the self-energy diagram, is removed from the Feynman diagrams of the three-point function. They give the correction to the vertex operator. With this correction, the vertex function γ_μ is replaced by the vertex function $\Gamma^{(0)}_\mu$. This is the same as replacing S_{F} and D_{F} by S'_{F} and $D'_{F_{\mu\nu}}$.

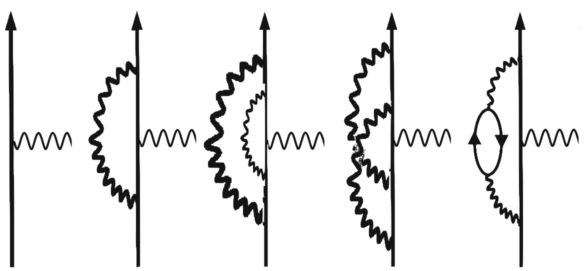

Fig. 12.6 Corrections to the vertex function

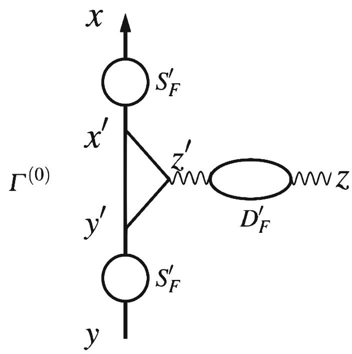

Fig. 12.7 Feynman diagram for the complete three-point function

Thus, the Feynman diagram corresponding to the complete three-point function (Fig. 12.7) can be written like the one above, i.e., the general three-point function can be expressed as a product of three propagators and one vertex function. Corresponding to this diagram, we have the expression

$$\begin{aligned}\left\langle \mathbf{0} \middle| T^*[\boldsymbol{\psi}(x), \bar{\boldsymbol{\psi}}(y), \boldsymbol{A}_\nu(z)] \middle| \mathbf{0} \right\rangle &= \frac{\left\langle 0 \middle| T^*[\psi(x), \bar{\psi}(y), A_\nu(z), U(\infty, -\infty)] \middle| 0 \right\rangle}{\langle 0|U(\infty, -\infty)|0\rangle} \\ &= -eZ_2 Z_3^{1/2} \int \mathrm{d}^4x' \mathrm{d}^4y' \mathrm{d}^4z' S'_{\mathrm{F}}(x-x') \Gamma_\mu^{(0)}(x', y'; z') \\ &\quad \times S'_{\mathrm{F}}(y'-y) D'_{F_{\mu\nu}}(z'-z) \,. \end{aligned} \tag{12.138}$$

Here, the Heisenberg operators and the interaction picture are renormalized. In the renormalized interaction picture, a propagator is automatically renormalized, but the vertex function is not. To the lowest order,

$$\Gamma_\mu^{(0)}(x, y, z) = \gamma_\mu \delta^4(x-z)\delta^4(y-z) \,. \tag{12.139}$$

Since in general $\Gamma_\mu^{(0)}$ becomes a function of $(x-z)$ and $(y-z)$, we define its Fourier representation by

$$\Gamma_\mu^{(0)}(x, y, z) = \frac{1}{(2\pi)^8} \int \mathrm{d}^4p \, \mathrm{d}^4q \, \mathrm{e}^{\mathrm{i}p\cdot(x-z)+\mathrm{i}q\cdot(z-y)} \Gamma_\mu^{(0)}(p, q) \,. \tag{12.140}$$

Therefore, according to (12.139), to lowest order,

$$\Gamma_\mu^{(0)}(p, q) = \gamma_\mu \,. \tag{12.141}$$

We renormalize the vertex function in which higher-order corrections are included by the equations

$$\Gamma_\mu(p,q) = Z_1 \Gamma_\mu^{(0)}(p,q) \ , \tag{12.142}$$

$$\bar{u}(p)\Gamma_\mu(p,p)u(p) = \bar{u}(p)\gamma_\mu u(p) \ . \tag{12.143}$$

However, in (12.143), we have assumed that p is on the electron mass shell. Replacing $\Gamma_\mu^{(0)}$ on the right-hand side of (12.138) by Γ, it turns out that the coefficient in front of the integral on the right-hand side of that equation is given by

$$e_{\text{obs}} = e Z_1^{-1} Z_2 Z_3^{1/2} \ . \tag{12.144}$$

In fact, this combination corresponds to the electric charge observed in experiments. This is manifested by several properties called *low-energy theorems*. Comparing (12.144) with (12.76), the permittivity of the vacuum is

$$\sqrt{\epsilon} = Z_1 Z_2^{-1} Z_3^{-1/2} \ . \tag{12.145}$$

Although the definition of the renormalized interaction picture is unchanged, it is more useful to rewrite e as e_{obs}. Then we write

$$\begin{aligned} e Z_2 Z_3^{1/2} &= e Z_1^{-1} Z_2 Z_3^{1/2} - (1 - Z_1) e Z_1^{-1} Z_2 Z_3^{1/2} \\ &= e_{\text{obs}} - (1 - Z_1) e_{\text{obs}} \\ &\equiv e_{\text{obs}} - \delta e \ . \end{aligned} \tag{12.146}$$

Thus, in the interaction part of the Lagrangian density, we rewrite as follows:

$$\mathrm{i} e Z_2 Z_3^{1/2} \bar{\psi}\gamma_\mu \psi A_\mu = \mathrm{i}(e_{\text{obs}} - \delta e)\bar{\psi}\gamma_\mu \psi A_\mu \ . \tag{12.147}$$

So from now on, we use powers of e_{obs} rather than powers of e in the perturbation theory. We have

$$\delta e = (1 - Z_1) e_{\text{obs}} = \mathscr{O}(e_{\text{obs}}^3) \ . \tag{12.148}$$

The relation between Γ_μ and $\Gamma_\mu^{(0)}$ is

$$(e_{\text{obs}} - \delta e)\Gamma_\mu^{(0)}(x,y,z) = e_{\text{obs}} \Gamma_\mu(x,y,z) \ . \tag{12.149}$$

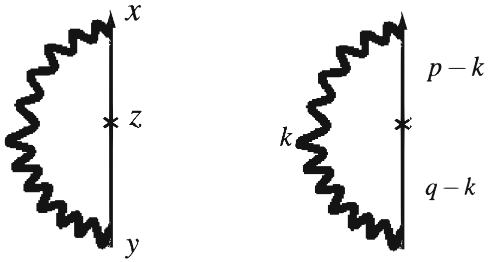

Fig. 12.8 Feynman diagram corresponding to the second term on the right-hand side of (12.150) in the x-space (*left*) and the p-space (*right*)

Computing the left-hand side up to order e_{obs}^3,

$$e_{\text{obs}}\Gamma_\mu(x,y,z) = (e_{\text{obs}} - \delta e)\gamma_\mu \delta^4(x-z)\delta^4(y-z) \\ + e_{\text{obs}}^2 \gamma_\lambda S_{\text{F}}(x-z)\gamma_\mu S_{\text{F}}(z-y)\gamma_\lambda D_{\text{F}}(x-y)\ . \tag{12.150}$$

The Feynman diagram corresponding to the second term on the right-hand side is shown in the x-space and the p-space in Fig. 12.8.

Taking the Fourier transform of (12.150), we obtain

$$e_{\text{obs}}\Gamma_\mu(p,q) = (e_{\text{obs}} - \delta e)\gamma_\mu \\ + \frac{(-\text{i})^3}{(2\pi)^4} e_{\text{obs}}^3 \int \text{d}^4k \gamma_\lambda \frac{1}{\text{i}(p-k)\cdot\gamma + m - \text{i}\epsilon}\gamma_\mu \frac{1}{\text{i}(q-k)\cdot\gamma + m - \text{i}\epsilon}\gamma_\lambda \frac{1}{k^2 - \text{i}\epsilon}\ . \tag{12.151}$$

To determine δe, we use (12.143). Rationalizing the denominator of (12.151), the numerator becomes

$$N = \gamma_\lambda \big[\text{i}(p-k)\cdot\gamma + m\big]\gamma_\mu \big[\text{i}(q-k)\cdot\gamma + m\big]\gamma_\lambda\ . \tag{12.152}$$

Summing over λ, we can use the following formulas to calculate products of the γ-matrices:

(1) $\gamma_\lambda\gamma_a\gamma_\lambda = -2\gamma_a$,

(2) $\gamma_\lambda\gamma_a\gamma_b\gamma_\lambda = 4\delta_{ab}$,

(3) $\gamma_\lambda\gamma_a\gamma_b\gamma_c\gamma_\lambda = -2\gamma_c\gamma_b\gamma_a$.

The numerator assumes the form

$$N = -2\text{i}(q-k)\cdot\gamma\gamma_\mu \text{i}(p-k)\cdot\gamma - 4\text{i}m\big[(p-k)_\mu + (q-k)_\mu\big] - 2m^2\gamma_\mu\ . \tag{12.153}$$

Equation (12.151) then becomes

$$e_{\text{obs}}\Gamma_\mu(p,q) = (e_{\text{obs}} - \delta e)\gamma_\mu + \frac{(-\mathrm{i}e_{\text{obs}})^3}{(2\pi)^4}\int \mathrm{d}^4k \frac{N}{\left[(p-k)^2+m^2-\mathrm{i}\epsilon\right]\left[(p-k)^2+m^2-\mathrm{i}\epsilon\right](k^2-\mathrm{i}\epsilon)} \ . \tag{12.154}$$

In order to carry out the k-integral, we use the following formula and change of variables:

$$\frac{1}{abc} = 2\int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \delta(1-\Sigma x_i)\frac{1}{(x_1a+x_2b+x_3c)^3} \ , \tag{12.155}$$

$$P = \frac{1}{2}(p+q) \ , \quad \Delta = p-q \ . \tag{12.156}$$

Up to a factor of 2, the denominator of the integral in (12.154) can be written

$$\int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \delta\big(1-\sum x_i\big) f(x_1,x_2) = \int_D \mathrm{d}x_1\mathrm{d}x_2 f(x_1,x_2) \ , \tag{12.157}$$

where

$$f(x_1,x_2) = \left[\left(P^2+\frac{\Delta^2}{4}+m^2\right)(x_1+x_2) + (P-k)\cdot\Delta(x_1-x_2) - 2k\cdot P(x_1+x_2) + k^2 - \mathrm{i}\epsilon\right]^{-3} \ . \tag{12.158}$$

The domain of integration D is shown in Fig. 12.9 (left). Changing the variables in (12.157) according to

$$x_1+x_2 = u \ , \quad x_1-x_2 = 2v \ , \tag{12.159}$$

the domain of integration for u and v is given by D', depicted in Fig. 12.9 (right). We now change the integration variable from k to k':

$$k' = k - uP - v\Delta \ . \tag{12.160}$$

The part corresponding to (12.155) becomes

$$2\int_{D'} \mathrm{d}u\,\mathrm{d}v\left[k'^2 + \left(P^2+\frac{\Delta^2}{4}+m^2\right)u + 2P\cdot\Delta v - (uP+v\Delta)^2 - \mathrm{i}\epsilon\right]^{-3} \ . \tag{12.161}$$

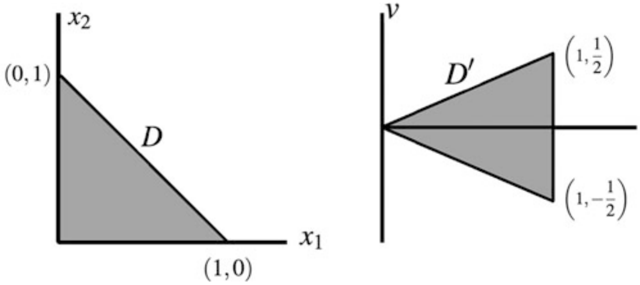

Fig. 12.9 Domains of integration for evaluating (12.154)

We now express the numerator in terms of k'. We can drop terms linear in k' because they give zero when we carry out the k'-integral. Since the integral is still complicated, we assume that p and q are on the mass shell, sandwich it between $\bar{u}(p)$ and $u(q)$, and use the relations

$$\bar{u}(p)(\mathrm{i}p \cdot \gamma + m) = \bar{u}(p)\left(\mathrm{i}P \cdot \gamma + \frac{1}{2}\mathrm{i}\Delta \cdot \gamma + m\right) = 0 ,$$

$$(\mathrm{i}q \cdot \gamma + m)u(p) = \left(\mathrm{i}P \cdot \gamma - \frac{1}{2}\mathrm{i}\Delta \cdot \gamma + m\right)u(q) = 0 .$$

Moreover, when $\bar{u}(p)Au(q) = \bar{u}(p)Bu(q)$, we write $A \sim B$. From the mass-shell condition,

$$p^2 + m^2 = q^2 + m^2 = P^2 + \frac{\Delta^2}{4} + m^2 = 0 , \quad P \cdot \Delta = 0 . \tag{12.162}$$

Therefore, the term in square brackets in (12.161) simplifies as follows:

$$k'^2 + m^2u^2 + \Delta^2\left(\frac{u^2}{4} - v^2\right) - \mathrm{i}\epsilon . \tag{12.163}$$

Furthermore, assuming that Δ is small, we only keep terms linear in Δ in the denominator and the numerator, and drop those in Δ^2. Using the formula

$$2\mathrm{i}P_\mu \sim -\sigma_{\mu\nu}\Delta_\nu - 2m\gamma_\mu , \tag{12.164}$$

we obtain

$$e_{\mathrm{obs}}\Gamma_\mu(p,q) \sim (e_{\mathrm{obs}} - \delta e)\gamma_\mu - 4\frac{\mathrm{i}e_{\mathrm{obs}}^3}{(2\pi)^4}\int_{D'} \mathrm{d}u\, \mathrm{d}v \int \mathrm{d}^4k' \frac{N'}{(k'^2 + m^2u^2 - \mathrm{i}\epsilon)^3} + \mathscr{O}(\Delta^2) , \tag{12.165}$$

where

$$N' = -u(1-u)m\sigma_{\mu\nu}\Delta_\nu + m^2\gamma_\mu\big[(1-u)^2 - 4(1-u) + 1\big] + \frac{1}{2}\gamma_\mu k'^2 \ . \tag{12.166}$$

In the limit $\Delta \to 0$, the right-hand side should be equal to $e_{\text{obs}}\gamma_\mu$, so

$$\delta e = -\frac{4\mathrm{i}e_{\text{obs}}^3}{(2\pi)^4}\int_{D'} \mathrm{d}u\,\mathrm{d}v \int \mathrm{d}^4k' \frac{m^2(u^2+2u-2) + k'^2/2}{(k'^2 + m^2u^2 - \mathrm{i}\epsilon)^3} \ . \tag{12.167}$$

The right-hand side turns out to diverge logarithmically. Inserting (12.167) into (12.165),

$$\begin{aligned} e_{\text{obs}}\Gamma_\mu(p,q) &\sim e_{\text{obs}}\gamma_\mu + 4\frac{\mathrm{i}e_{\text{obs}}^3}{(2\pi)^4}\int_{D'} \mathrm{d}u\,\mathrm{d}v \int \mathrm{d}^4k' \frac{u(1-u)m\sigma_{\mu\nu}\Delta_\nu}{(k'^2 + m^2u^2 - \mathrm{i}\epsilon)^3} + \mathscr{O}(\Delta^2) \\ &= e_{\text{obs}}\gamma_\mu - \frac{\alpha}{2\pi}\frac{e_{\text{obs}}}{2m}\sigma_{\mu\nu}\Delta_\nu + \mathscr{O}(\Delta^2) \ . \end{aligned} \tag{12.168}$$

Here, α is, of course, equal to $e_{\text{obs}}^2/4\pi$. This equation is the one derived as the third-order perturbation. The effective Hamiltonian which yields the same result up to the first order in Δ is

$$\mathscr{H}_{\text{eff}} = -\frac{\alpha}{2\pi}\frac{e_{\text{obs}}}{4m}\bar{\psi}\sigma_{\mu\nu}\psi F_{\mu\nu} \ . \tag{12.169}$$

This term, which is gauge invariant, is called the *Pauli term*. When there is only a magnetic field, applying the non-relativistic approximation, we have

$$\mathscr{H}_{\text{eff}} = -\frac{\alpha}{2\pi}\frac{e_{\text{obs}}}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{H} \ . \tag{12.170}$$

This tells us that the electron acquires a supplementary magnetic moment in addition to $e_{\text{obs}}/2m$ in the Dirac theory. This increase is called the *anomalous magnetic moment*. As a result, the magnetic moment of the electron up to this order is given by

$$\frac{e_{\text{obs}}}{2m}\left(1 + \frac{\alpha}{2\pi}\right) \ . \tag{12.171}$$

This result was obtained by Schwinger and by Tomonaga et al. [116, 117], and it matches experimental values well. It is considered to be a great achievement of renormalization theory, which thus succeeded in explaining what we call the *Lamb shift* in the hydrogen atom, and has provided a foundation for the development of field theory.

So far, we have introduced Z_1, Z_2, and Z_3 as multiplicative renormalization constants. From (12.148) and (12.167), Z_1 is given by

$$Z_1 = 1 + \frac{4\mathrm{i}e_{\mathrm{obs}}^2}{(2\pi)^4} \int_{D'} \mathrm{d}u\, \mathrm{d}v \int \mathrm{d}^4 k' \frac{m^2(u^2+2u-2)+k'^2/2}{(k'^2+m^2u^2-\mathrm{i}\epsilon)^3} \,. \tag{12.172}$$

Although we have not provided an explicit computation of Z_2, it is given by (12.107), and computing to this order, it coincides with Z_1. We will discuss this equality in the next section.

12.7 Ward–Takahashi Identity

We now investigate in detail the relation (12.115) between the bare electric charge e and the observed electric charge e_{obs}. So far, we have only considered the electron. Let us consider the case where there is a wide variety of charged particles $a, b, \ldots$. Then,

$$(e_a)_{\mathrm{obs}} = Z_{1a}^{-1} Z_{2a} Z_3^{1/2} e_a \,, \quad (e_b)_{\mathrm{obs}} = Z_{1b}^{-1} Z_{2b} Z_3^{1/2} e_b \,, \ldots \,. \tag{12.173}$$

We consider the reaction

$$a + b \longrightarrow c + d \,. \tag{12.174}$$

In this case, it is the bare electric charge that is conserved by Noether's theorem:

$$e_a + e_b = e_c + e_d \,. \tag{12.175}$$

However, we know experimentally that charge conservation holds for the renormalized electric charges:

$$(e_a)_{\mathrm{obs}} + (e_b)_{\mathrm{obs}} = (e_c)_{\mathrm{obs}} + (e_d)_{\mathrm{obs}} \,. \tag{12.176}$$

In order for these two conservation laws to hold simultaneously, $Z_1^{-1} Z_2$ cannot depend on the type of charged particle, i.e.,

$$Z_{1a}^{-1} Z_{2a} = Z_{1b}^{-1} Z_{2b} = Z_{1c}^{-1} Z_{2c} = Z_{1d}^{-1} Z_{2d} \,. \tag{12.177}$$

Ward discovered that these equalities can be replaced by the following, which imply them [118]:

$$Z_{1a} = Z_{2a} \,, \ldots \,. \tag{12.178}$$

The equation $Z_1 = Z_2$ is referred to as the *Ward identity*. We shall now give its proof.

We take $\boldsymbol{\psi}^{(0)}$ and $\bar{\boldsymbol{\psi}}^{(0)}$ to be non-renormalized operators of the electric field. Setting

$$\boldsymbol{J}_\mu = \mathrm{i}\bar{\boldsymbol{\psi}}^{(0)}\gamma_\mu\boldsymbol{\psi}^{(0)} = \mathrm{i}Z_2\bar{\boldsymbol{\psi}}\gamma_\mu\boldsymbol{\psi} \ , \tag{12.179}$$

and using the fact that non-renormalized Heisenberg operators satisfy the canonical commutation relations for $x_0 = y_0$, we have

$$[\boldsymbol{J}_0(x), \boldsymbol{\psi}^{(0)}(y)] = -\boldsymbol{\psi}^{(0)}(y)\delta^3(x-y) \ , \quad [\boldsymbol{J}_0(x), \bar{\boldsymbol{\psi}}^{(0)}(y)] = \bar{\boldsymbol{\psi}}^{(0)}(y)\delta^3(x-y) \ . \tag{12.180}$$

Thus, if $\boldsymbol{J}_\mu$ is included in the T-product, then using $\partial_\mu\boldsymbol{J}_\mu = 0$, we obtain

$$\begin{aligned}\partial_\mu T\big[\boldsymbol{J}_\mu(z), \boldsymbol{\psi}^{(0)}(x), \bar{\boldsymbol{\psi}}^{(0)}(y)\big] &= T\big[[\boldsymbol{J}_0(z), \boldsymbol{\psi}^{(0)}(x)]\delta(z_0-x_0), \bar{\boldsymbol{\psi}}^{(0)}(y)\big] \\ &\quad + T\big[\boldsymbol{\psi}^{(0)}(x), [\boldsymbol{J}_0(z), \bar{\boldsymbol{\psi}}^{(0)}(y)]\delta(z_0-y_0)\big] \\ &= \big[\delta^4(z-y) - \delta^4(z-x)\big]T[\boldsymbol{\psi}^{(0)}(x), \bar{\boldsymbol{\psi}}^{(0)}(y)] \ .\end{aligned} \tag{12.181}$$

As we have seen before, at the point where the order of two time variables in the T-product are switched, a delta function in time shows up. Later, we will use a generalization of (12.181).

The renormalized interaction picture is defined by (12.96) and (12.97). On the other hand, expressing (12.181) in terms of renormalized operators,

$$\begin{aligned}\partial_\mu T\big[\boldsymbol{J}_\mu(z), \boldsymbol{\psi}(x), \bar{\boldsymbol{\psi}}(y)\big] &= \big[\delta^4(z-y) - \delta^4(z-x)\big]T[\boldsymbol{\psi}(x), \bar{\boldsymbol{\psi}}(y)] \\ &\quad - \left[\boldsymbol{\psi}(z)\frac{\delta}{\delta\boldsymbol{\psi}(z)} - \bar{\boldsymbol{\psi}}(z)\frac{\delta}{\delta\bar{\boldsymbol{\psi}}(z)}\right]T[\boldsymbol{\psi}(x), \bar{\boldsymbol{\psi}}(y)] \ .\end{aligned} \tag{12.182}$$

We will use this formula later by generalizing it to a certain extent.

In order to apply the reduction formula for the electromagnetic field, we introduce the differential operator

$$D_{\mu\nu}(\partial) = \delta_{\mu\nu}\Box - \partial_\mu\partial_\nu + \frac{1}{\alpha}\partial_\mu\partial_\nu \ . \tag{12.183}$$

Applying the reduction formula to Green's function in the renormalized interaction picture, we compute the quantity

$$\begin{aligned}D_{\mu\nu}(\partial)T^*\big[A_\nu(x)\ldots, U(\infty,-\infty)\big] &= T^*\left[\mathrm{i}\frac{\delta}{\delta A_\mu(x)}\ldots, U(\infty,-\infty)\right] \\ &\quad - T^*\big[\ldots[\mathscr{L}_{\mathrm{int}}(x)]_{A_\mu}, U(\infty,-\infty)\big] \ ,\end{aligned} \tag{12.184}$$

where an explicit expression for the Euler derivative is

$$[\mathscr{L}_{\text{int}}(x)]_{A_\mu} = eZ_3^{1/2} J_\mu - (1 - Z_3)(\delta_{\mu\sigma}\Box - \partial_\mu \partial_\sigma) A_\sigma(x) \ . \tag{12.185}$$

Inserting this into (12.184) and taking the derivative in the second term of (12.185) outside T^*, we have

$$\begin{aligned} D_{\mu\nu}(\partial) T^*[A_\nu(x) \ldots, U(\infty, -\infty)] = {} & T^*\left[\mathrm{i}\frac{\delta}{\delta A_\mu(x)} \ldots, U(\infty, -\infty)\right] \\ & - eZ_3^{1/2} T^*\big[J_\mu(x), \ldots, U(\infty, -\infty)\big] \\ & + (1 - Z_3)(\delta_{\mu\sigma} \Delta - \partial_\mu \partial_\sigma) T^*\big[A_\sigma(x), \ldots, U(\infty, -\infty)\big] \ . \end{aligned} \tag{12.186}$$

Here we use the relation

$$eZ_3^{1/2} = e(Z_1^{-1} Z_2 Z_3^{1/2}) Z_1 Z_2^{-1} = e_{\text{obs}} Z_1 Z_2^{-1}. \tag{12.187}$$

Next, we differentiate (12.186) with respect to x_μ and use (12.182). Since the last term vanishes, we obtain

$$\begin{aligned} \partial_\mu D_{\mu\nu}(\partial) T^*\big[A_\nu(x), \ldots, U(\infty, -\infty)\big] = {} & \partial_\mu T^*\left[\mathrm{i}\frac{\delta}{\delta A_\mu(x)} \ldots, U(\infty, -\infty)\right] \\ & + e_{\text{obs}} Z_1 Z_2^{-1} T^*\left[\left(\psi(x)\frac{\delta}{\delta\psi(x)} - \bar{\psi}(x)\frac{\delta}{\delta\bar{\psi}(x)}\right) \ldots, U(\infty, -\infty)\right] . \end{aligned} \tag{12.188}$$

Taking the vacuum expectation value of the above equation and using the Gell-Mann–Low formula, the relation between the Green's functions involving the renormalized Heisenberg operators is

$$\begin{aligned} \partial_\mu D_{\mu\nu}(\partial) \big\langle \mathbf{0} \big| T^*[\boldsymbol{A}_\nu(x) \ldots] \big| \mathbf{0} \big\rangle = {} & \partial_\mu \left\langle \mathbf{0} \middle| \mathrm{i}\frac{\delta}{\delta \boldsymbol{A}_\mu(x)} T^*[\ldots] \middle| \mathbf{0} \right\rangle \\ & + e_{\text{obs}} Z_1 Z_2^{-1} \left\langle \mathbf{0} \middle| T^*\left[\left(\boldsymbol{\psi}(x)\frac{\delta}{\delta\boldsymbol{\psi}(x)} - \bar{\boldsymbol{\psi}}(x)\frac{\delta}{\delta\bar{\boldsymbol{\psi}}(x)}\right) \ldots\right] \middle| \mathbf{0} \right\rangle . \end{aligned} \tag{12.189}$$

In the equation above, dots stand for a suitable product of Heisenberg operators.

One feature of this equation is that only renormalized operators show up. Since $Z_1 Z_2^{-1} = 1$, as will be shown later, we can now obtain the *Ward–Takahashi identity* [119]:

$$\begin{aligned} \partial_\mu D_{\mu\nu}(\partial) \big\langle \mathbf{0} \big| T^*[\boldsymbol{A}_\nu(x) \ldots] \big| \mathbf{0} \big\rangle = {} & \partial_\mu \left\langle \mathbf{0} \middle| \mathrm{i}\frac{\delta}{\delta A_\mu(x)} T^*[\ldots] \middle| \mathbf{0} \right\rangle \\ & + e_{\text{obs}} \left\langle \mathbf{0} \middle| T^*\left[\left(\boldsymbol{\psi}(x)\frac{\delta}{\delta\boldsymbol{\psi}(x)} - \bar{\boldsymbol{\psi}}(x)\frac{\delta}{\delta\bar{\boldsymbol{\psi}}(x)}\right) \ldots\right] \middle| \mathbf{0} \right\rangle . \end{aligned} \tag{12.190}$$

To understand the relevance of (12.189), we first insert $A_\sigma(y)$ in place of the dots to obtain

$$\partial_\mu D_{\mu\nu}(\partial_x)\langle\mathbf{0}|T^*[\boldsymbol{A}_\nu(x), \boldsymbol{A}_\sigma(y)]|\mathbf{0}\rangle = \mathrm{i}\partial_\sigma\delta^4(x-y)\ . \tag{12.191}$$

Expressing this in momentum space and taking into account the equation

$$\partial_\mu D_{\mu\nu}(\partial) = \frac{1}{\alpha}\Box\partial_\nu\ , \tag{12.192}$$

we find

$$\frac{1}{\alpha}k^2 k_\nu D'_{F_{\nu\sigma}}(k) = k_\sigma\ . \tag{12.193}$$

We now write Dyson's equation for D'_F in an arbitrary gauge, viz.,

$$D'_{F_{\nu\sigma}}(k) = D_{F_{\nu\sigma}}(k) - D_{F_{\nu\lambda}}(k)\varPi^*_{\lambda\mu}(k)D'_{F_{\mu\sigma}}(k)\ , \tag{12.194}$$

where

$$D_{F_{\mu\sigma}}(k) = \left(\delta_{\nu\sigma} - \frac{k_\nu k_\sigma}{k^2-\mathrm{i}\epsilon}\right)\frac{1}{k^2-\mathrm{i}\epsilon} + \alpha\frac{k_{\nu\sigma}}{(k^2-\mathrm{i}\epsilon)^2} \tag{12.195}$$

and

$$\varPi^*_{\lambda\mu}(k) = \varPi^*_{\mathrm{ren}}(k)_{\lambda\mu} = (k_\lambda k_\mu - \delta_{\lambda\mu}k^2)\varPi^*(-k^2)\ . \tag{12.196}$$

From this, we obtain the solution of (12.194) in the form

$$D'_{F_{\nu\sigma}}(k) = \left(\delta_{\nu\sigma} - \frac{k_{\nu\sigma}}{k^2-\mathrm{i}\epsilon}\right)\frac{1}{k^2-\mathrm{i}\epsilon}\frac{1}{1-\varPi^*(-k^2)} + \alpha\frac{k_\nu k_\sigma}{(k^2-\mathrm{i}\epsilon)^2}\ . \tag{12.197}$$

This certainly satisfies (12.193).

Another example is

$$\begin{aligned}\partial_\mu D_{\mu\nu}(\partial_z)&\langle 0|T^*[\boldsymbol{\psi}(x), \bar{\boldsymbol{\psi}}(y), \boldsymbol{A}_\nu(z)]|\mathbf{0}\rangle \\ &= -e_{\mathrm{obs}}Z_1Z_2^{-1}\left[\delta^4(y-z)-\delta^4(x-z)\right]\langle\mathbf{0}|T^*[\boldsymbol{\psi}(x), \bar{\boldsymbol{\psi}}(y)]|\mathbf{0}\rangle.\end{aligned} \tag{12.198}$$

We combine (12.191) and this equation with the following equation:

$$\begin{aligned}&\langle\mathbf{0}|T^*[\boldsymbol{\psi}(x), \bar{\boldsymbol{\psi}}(y), \boldsymbol{A}_\nu(z)]|\mathbf{0}\rangle \\ &= -e_{\mathrm{obs}}\int \mathrm{d}^4x'\mathrm{d}^4y'\mathrm{d}^4z' S'_\mathrm{F}(x-x')\varGamma_\mu(x',y';z')S'_\mathrm{F}(y'-y)D'_{F_{\mu\nu}}(z'-z).\end{aligned} \tag{12.199}$$

Inserting the right-hand side of (12.199) into the left-hand side of (12.198) and using (12.191) to cancel D'_{F}, we obtain the following equation in momentum space:

$$-\,\mathrm{i}(p-q)_\nu S'_{\mathrm{F}}(p)\Gamma_\nu(p,q)S'_{\mathrm{F}}(q) = Z_1 Z_2^{-1}\big[S'_{\mathrm{F}}(p) - S'_{\mathrm{F}}(q)\big]\,. \tag{12.200}$$

Alternatively, differentiating both sides with respect to p_μ and setting $q = p$,

$$\Gamma_\mu(p,p) = -\mathrm{i}Z_1 Z_2^{-1}\frac{\partial}{\partial p_\mu}S'_{\mathrm{F}}(p)^{-1}\,. \tag{12.201}$$

We sandwiching this between $\bar{u}(p)$ and $u(p)$ and use the relations

$$S'_{\mathrm{F}}(p)^{-1} = \mathrm{i}p\cdot\gamma + m + \mathscr{O}\big((\mathrm{i}p\cdot\gamma+m)^2\big)\,,$$

$$\bar{u}(p)(\mathrm{i}p\cdot\gamma + m) = (\mathrm{i}p\cdot\gamma+m)u(p) = 0\,.$$

This leads finally to

$$\bar{u}(p)\gamma_\mu u(p) = Z_1 Z_2^{-1}\bar{u}(p)\gamma_\mu u(p)\,, \tag{12.202}$$

which yields Ward's identity

$$Z_1 = Z_2\,. \tag{12.203}$$

Thus, in the Ward–Takahashi identity (12.189), we can replace $Z_1 Z_2^{-1}$ by 1 to obtain the Ward–Takahashi identity given by (12.190). In general, an equation obtained by subtracting a divergence of the Green's function is called a Ward–Takahashi identity.

Equations (12.201) and (12.203) were found by Ward [118], while (12.200) was derived by Takahashi [119]. Equation (12.190) for the Green's function was proved by the author in the Fermi gauge [120], but it was eventually shown that it can be extended to an arbitrary gauge.

12.8 Integral Representation of the Propagator

The propagator with higher-order corrections can be expressed by a certain kind of integral representation as the superposition of free-field propagators with different masses. This was established by Umezawa et al. in the early 1950s [121–123].

12.8.1 Integral Representation

We consider a neutral scalar field and make the following general assumptions:

1. There exists a four-vector operator P_μ, which stands for the energy–momentum, satisfying

$$[P_\mu, P_\nu] = 0 , \tag{12.204}$$

$$[\boldsymbol{\varphi}(x), P_\mu] = \frac{1}{\mathrm{i}}\partial_\mu \boldsymbol{\varphi}(x) . \tag{12.205}$$

2. There exists a set $\{\Phi_{k,\alpha}\}$ of eigenstates of P_μ which form a complete system. Here, k_μ is an eigenvalue of P_μ and α is another quantity specifying the states:

$$P_\mu \Phi_{k,\alpha} = k_\mu \Phi_{k,\alpha} . \tag{12.206}$$

Of course, as $\{\Phi\}$, we can choose either $\{\Phi^{\text{in}}\}$ or $\{\Phi^{\text{out}}\}$. In what follows, we assume the existence of the vacuum and write it as $|\mathbf{0}\rangle$. Affixing a prime to the invariant function in the presence of interactions,

$$\begin{aligned}\langle \mathbf{0}|\boldsymbol{\varphi}(x)\boldsymbol{\varphi}(y)|\mathbf{0}\rangle &= \mathrm{i}\Delta^{(+)\prime}(x-y) ,\\ \langle \mathbf{0}|[\boldsymbol{\varphi}(x), \boldsymbol{\varphi}(y)]|\mathbf{0}\rangle &= \mathrm{i}\Delta'(x-y) ,\\ \langle \mathbf{0}|T[\boldsymbol{\varphi}(x)\boldsymbol{\varphi}(y)]|\mathbf{0}\rangle &= \Delta'_{\mathrm{F}}(x-y) .\end{aligned} \tag{12.207}$$

Then, to carry the Fourier expansion, we introduce the matrix element

$$\langle \mathbf{0}|\boldsymbol{\varphi}(x)|\boldsymbol{k}, \boldsymbol{\alpha}\rangle = a_{k,\alpha}\mathrm{e}^{\mathrm{i}k\cdot x} . \tag{12.208}$$

Hence,

$$\begin{aligned}\langle \mathbf{0}|\boldsymbol{\varphi}(x)\boldsymbol{\varphi}(y)|\mathbf{0}\rangle &= \sum_{k,\alpha}\langle \mathbf{0}|\boldsymbol{\varphi}(x)|\boldsymbol{k}, \boldsymbol{\alpha}\rangle\langle \boldsymbol{k}, \boldsymbol{\alpha}|\boldsymbol{\varphi}(y)|\mathbf{0}\rangle\\ &= \sum_{k,\alpha}|a_{k,\alpha}|^2\mathrm{e}^{\mathrm{i}k\cdot(x-y)} .\end{aligned} \tag{12.209}$$

We now introduce the Lorentz invariant function

$$\rho(-p^2) = (2\pi)^3\sum_{k,\alpha}|a_{k,\alpha}|^2\delta^4(p-k) \geq 0 . \tag{12.210}$$

Inserting this into (12.209), for the time-like four-momentum p, we have

$$\begin{aligned} \mathrm{i}\Delta^{(+)\prime}(x-y) &= \frac{1}{(2\pi)^3}\int \mathrm{d}^4 p\theta(p_0)\rho(-p^2)\mathrm{e}^{\mathrm{i}p\cdot(x-y)} \qquad (-p^2 \geq 0) \\ &= \int_0^\infty \mathrm{d}\kappa^2\rho(\kappa^2)\frac{1}{(2\pi)^3}\int \mathrm{d}^4 p\theta(p_0)\delta(p^2+\kappa^2)\mathrm{e}^{\mathrm{i}p\cdot(x-y)} \\ &= \int_0^\infty \mathrm{d}\kappa^2\rho(\kappa^2)\mathrm{i}\Delta^{(+)}(x-y;\kappa^2) \ . \end{aligned} \tag{12.211}$$

Generalizing this to an arbitrary function, we obtain

$$\Delta^{(+)\prime}(x) = \int_0^\infty \mathrm{d}\kappa^2\rho(\kappa^2)\Delta^{(+)}(x;\kappa^2) \ , \tag{12.212}$$

so in the presence of interactions, the invariant two-point function can be expressed by the integral representation as a superposition of invariant two-point functions for free fields with different masses.

The spectral function $\rho(\kappa^2)$ for a free field with the mass m is

$$\rho(\kappa^2) = \delta(\kappa^2 - m^2) \ . \tag{12.213}$$

If interactions are introduced, we can decompose into contributions from the one-particle intermediate states and the multi-particle intermediate states:

$$\rho(\kappa^2) = c\delta(\kappa^2 - m^2) + \sigma(\kappa^2) \ . \tag{12.214}$$

We note, however, that σ vanishes below the lowest invariant mass $2m$ in the two-particle system, i.e.,

$$\sigma(\kappa^2) = \theta(\kappa^2 - 4m^2)\sigma(\kappa^2) \ . \tag{12.215}$$

If we apply the renormalization condition, namely that the normalization of the wave function becomes the same as that of the free field in the one-particle state, then c in (12.214) becomes unity, i.e.,

$$\rho(\kappa^2) = \delta(\kappa^2 - m^2) + \sigma(\kappa^2) \ . \tag{12.216}$$

We now introduce the Fourier representation of the propagator:

$$\Delta'_{\mathrm{F}}(x) = \frac{-\mathrm{i}}{(2\pi)^4}\int \mathrm{d}^4 k\, \mathrm{e}^{\mathrm{i}k\cdot x}\Delta'_{\mathrm{F}}(-k^2) \ . \tag{12.217}$$

Hence,

$$\Delta'_{\mathrm{F}}(-k^2) = \frac{1}{k^2+m^2-\mathrm{i}\epsilon} + \int_{4m^2}^{\infty} \mathrm{d}\kappa^2 \frac{\sigma(\kappa^2)}{k^2+\kappa^2-\mathrm{i}\epsilon} \ . \tag{12.218}$$

We introduce the renormalization constant Z_φ, noting that, for the electron field $Z_\psi = Z_2$ and for the electromagnetic field $Z_A = Z_3$:

$$\boldsymbol{\varphi}^{(0)}(x) = Z_\varphi^{1/2}\boldsymbol{\varphi}(x) \ , \tag{12.219}$$

where the superscript (0) denotes unrenormalized quantities. Therefore,

$$\langle \mathbf{0}|[\boldsymbol{\varphi}^{(0)}(x), \boldsymbol{\varphi}^{(0)}(y)]|\mathbf{0}\rangle = Z_\varphi \int \mathrm{d}\kappa^2 \rho(\kappa^2)\mathrm{i}\Delta(x-y;\kappa^2) \ . \tag{12.220}$$

Differentiating both sides of this equation with respect to x_0 and then setting $x_0 = y_0$, the left-hand side becomes $-\mathrm{i}\delta^3(x-y)$ from the canonical commutation relation. In addition, on the right-hand side, using (4.13),

$$\left.\frac{\partial}{\partial x_0}\Delta(x-y;\kappa^2)\right|_{x_0=y_0} = -\delta^3(x-y) \ , \tag{12.221}$$

whence

$$-\mathrm{i}\delta^3(x-y) = -\mathrm{i}\delta^3(x-y) Z_\varphi \int \mathrm{d}\kappa^2 \rho(\kappa^2) \ . \tag{12.222}$$

The integral representation of the renormalization constant can now be obtained immediately as

$$Z_\varphi^{-1} = \int \mathrm{d}\kappa^2 \rho(\kappa^2) = 1 + \int \mathrm{d}\kappa^2 \sigma(\kappa^2) \geq 1 \ . \tag{12.223}$$

This inequality originates from the positive-definite metric assumed for the scalar field. Thus,

$$1 \geq Z_\varphi \geq 0 \ . \tag{12.224}$$

12.8.2 Self-Energy

As shown above, the renormalization constant is written in terms of the integral representation via the spectral function. But what about the self-energy? To discuss this problem, we consider the following simple model:

$$\mathscr{L} = -\left(\partial_\lambda \Phi^\dagger \partial_\lambda \Phi + M_0^2 \Phi^\dagger \Phi\right) - \frac{1}{2}\left[(\partial_\lambda \varphi)^2 + m_0^2 \varphi^2\right] - g_0 \Phi^\dagger \Phi \varphi \ . \qquad (12.225)$$

Here φ stands for a neutral scalar field, Φ and $\Phi^\dagger$ for charged scalars, and m_0 and M_0 for the bare masses associated with each field, which are related to the observed masses m and M via

$$M^2 = M_0^2 + \delta M^2 \ , \quad m^2 = m_0^2 + \delta m^2 \ . \qquad (12.226)$$

We introduce multiplicative renormalizations via the relations

$$\boldsymbol{\Phi}^{(0)} = Z_2^{1/2} \boldsymbol{\Phi} \ , \quad \boldsymbol{\varphi}^{(0)} = Z_3^{1/2} \boldsymbol{\varphi} \ , \quad g_0 = Z_1 Z_2^{-1} Z_3^{-1/2} g \ . \qquad (12.227)$$

The renormalized field equations are

$$(\Box - m^2)\boldsymbol{\varphi} = Z_1 Z_3^{-1} g \boldsymbol{\Phi}^\dagger \boldsymbol{\Phi} - \delta m^2 \boldsymbol{\varphi} \ , \qquad (12.228)$$

$$(\Box - M^2)\boldsymbol{\Phi} = Z_1 Z_2^{-1} g \boldsymbol{\Phi} \boldsymbol{\varphi} - \delta M^2 \boldsymbol{\Phi} \ . \qquad (12.229)$$

Using (12.228), we can derive an integral representation of δm^2. Likewise, we can use (12.229) to derive an integral representation of δM^2. Expressing (12.228) in terms of m_0,

$$(\Box_x - m_0^2)\langle \mathbf{0} | [\boldsymbol{\varphi}(x), \boldsymbol{\varphi}(y)] | \mathbf{0} \rangle = Z_1 Z_3^{-1} g \langle \mathbf{0} | [\boldsymbol{\Phi}^\dagger(x) \boldsymbol{\Phi}(x), \boldsymbol{\varphi}(y)] | \mathbf{0} \rangle \ . \qquad (12.230)$$

Differentiating this equation with respect to y_0 and setting $y_0 = x_0$, the right-hand side becomes

$$\langle \mathbf{0} | [\boldsymbol{\Phi}^\dagger(x) \boldsymbol{\Phi}(x), \dot{\boldsymbol{\varphi}}(y)] | \mathbf{0} \rangle = 0 \qquad (x_0 = y_0) \ . \qquad (12.231)$$

It thus turns out that the left-hand side also vanishes. The left-hand side should still vanish if we carry out the operation discussed above in the integral representation:

$$(\Box^2 - m_0^2)\langle \mathbf{0} | [\boldsymbol{\varphi}(x), \boldsymbol{\varphi}(y)] | \mathbf{0} \rangle = \mathrm{i} \int \mathrm{d}\kappa^2 \rho_\varphi(\kappa^2)(\kappa^2 - m_0^2) \Delta(x - y; \kappa^2) \ .$$

Differentiating this with respect to y_0 and setting $y_0 = x_0 = 0$, Eq. (12.221) implies

$$\mathrm{i}\delta^3(x - y) \int \mathrm{d}\kappa^2 (\kappa^2 - m_0^2) \rho_\varphi(\kappa^2) = 0 \ ,$$

and hence,

$$\int \mathrm{d}\kappa^2(\kappa^2 - m^2)\rho_\varphi(\kappa^2) = -\delta m^2 \int \mathrm{d}\kappa^2 \rho_\varphi(\kappa^2) = -Z_3^{-1}\delta m^2 \ . \tag{12.232}$$

Thus,

$$\delta m^2 = -Z_3 \int \mathrm{d}\kappa^2(\kappa^2 - m^2)\rho_\varphi(\kappa^2) = -Z_3 \int \mathrm{d}\kappa^2(\kappa^2 - m^2)\sigma(\kappa^2) < 0 \ . \tag{12.233}$$

This equation implies that δm^2 is always negative. This is, of course, a predictable result. If we assume that, in (12.226), m^2 is positive and finite and m_0^2 is positive and infinite, then δm^2 can only be negative and infinite.

12.8.3 Integral Representation of the Electromagnetic Field Propagator

We now apply the previous discussion for the scalar field to the electromagnetic field. We first introduce a propagator for the electromagnetic field:

$$\langle \mathbf{0}|T^*[\boldsymbol{A}_\mu(x)\boldsymbol{A}_\nu(y)]|\mathbf{0}\rangle = \frac{-\mathrm{i}}{(2\pi)^4}\int \mathrm{d}^4k\, \mathrm{e}^{\mathrm{i}k\cdot(x-y)} D'_{F_{\mu\nu}}(k) \ . \tag{12.234}$$

For the free electromagnetic field, $D_{F_{\mu\nu}}$ is given by (12.195). Furthermore, it is clear from (12.197) that the effects of the interaction appear only in the transverse part of the wave. Recalling that

$$D_{\mu\nu}(\partial)\boldsymbol{A}_\nu = -\boldsymbol{j}_\mu \ , \tag{12.235}$$

$$\boldsymbol{j}_\mu = -\mathrm{i}e_{\mathrm{obs}} Z_2 \bar{\boldsymbol{\psi}}\gamma_\mu \boldsymbol{\psi} - (1 - Z_3)(\delta_{\mu\sigma}\Box - \partial_\mu\partial_\sigma)\boldsymbol{A}_\sigma \ , \tag{12.236}$$

we introduce the current $\boldsymbol{j}_\mu$ and define the function

$$\langle \mathbf{0}|T^*[\boldsymbol{j}_\mu(x)\boldsymbol{j}_\nu(y)]|\mathbf{0}\rangle = \frac{-\mathrm{i}}{(2\pi)^4}\int \mathrm{d}^4k\, \mathrm{e}^{\mathrm{i}k\cdot(x-y)} \Pi_{\mu\nu}(k) \ . \tag{12.237}$$

Therefore,

$$k^4 \mathrm{Im} D'_{F_{\mu\nu}}(k) = \mathrm{Im}\Pi_{\mu\nu}(k) \ . \tag{12.238}$$

We now seek an integral representation of $\Pi_{\mu\nu}$. From the Lorentz covariance,

$$\langle \mathbf{0}|\boldsymbol{j}_\mu(x)\boldsymbol{j}_\nu(y)|\mathbf{0}\rangle = \mathrm{i}\delta_{\mu\nu}\int \mathrm{d}\kappa^2\sigma_1(\kappa^2)\Delta^{(+)}(x-y;\kappa^2) \qquad (12.239)$$
$$+\mathrm{i}\partial_\mu\partial_\nu\int \mathrm{d}\kappa^2\sigma_2(\kappa^2)\Delta^{(+)}(x-y;\kappa^2)\ .$$

Note also that, since $\partial_\mu \boldsymbol{j}_\mu = 0$,

$$\sigma_1(\kappa^2)+\kappa^2\sigma_2(\kappa^2)=0\ . \qquad (12.240)$$

We thus have

$$\sigma_1(\kappa^2)=\kappa^2\sigma(\kappa^2)\ ,\quad \sigma_2(\kappa^2)=-\sigma(\kappa^2)\ , \qquad (12.241)$$

$$\big\langle \mathbf{0}\big|\boldsymbol{j}_\mu(x)\boldsymbol{j}_\nu(y)\big|\mathbf{0}\big\rangle = \mathrm{i}\int \mathrm{d}\kappa^2\sigma(\kappa^2)(\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\Delta^{(+)}(x-y;\kappa^2)\ . \qquad (12.242)$$

Taking $\mu=\nu=4$,

$$-\big\langle \mathbf{0}\big|\boldsymbol{j}_0(x)\boldsymbol{j}_0(y)\big|\mathbf{0}\big\rangle = \mathrm{i}\int \mathrm{d}\kappa^2\sigma(\kappa^2)\nabla^2\Delta^{(+)}(x-y;\kappa^2)\ . \qquad (12.243)$$

Because $\boldsymbol{j}_\mu$ is gauge invariant, the indefinite metric cannot appear on the left-hand side. Hence,

$$(2\pi)^3\sum_n\big|\langle \boldsymbol{n}|\boldsymbol{j}_0(0)|\mathbf{0}\rangle\big|^2\delta^4(p_n-k)=\boldsymbol{k}^2\sigma(-k^2)\geq 0\ , \qquad (12.244)$$

i.e., $\sigma(\kappa^2)$ is positive-definite as in the case of the scalar field:

$$\sigma(\kappa^2)\geq 0\ . \qquad (12.245)$$

Using this result, for the integral representation of $\Pi_{\mu\nu}$, we have

$$\Pi_{\mu\nu}(k)=(k_\mu k_\nu-\delta_{\mu\nu}k^2)\Pi(-k^2)\ , \qquad (12.246)$$

$$\Pi(-k^2)=\int \mathrm{d}\kappa^2\frac{\sigma(\kappa^2)}{k^2+\kappa^2-\mathrm{i}\epsilon}+\mathrm{const.} \qquad (12.247)$$

Since the renormalization condition requires $\Pi(0)$ to vanish,

$$\Pi(-k^2)=\Pi(-k^2)-\Pi(0)=-k^2\int\frac{\mathrm{d}\kappa^2}{\kappa^2}\frac{\sigma(\kappa^2)}{k^2+\kappa^2-\mathrm{i}\epsilon}\ . \qquad (12.248)$$

At the lowest order in the perturbation theory,

$$\sigma(\kappa^2) = \frac{e_{\mathrm{obs}}^2}{12\pi^2}\left(1 + \frac{2m^2}{\kappa^2}\right)\sqrt{1 - \frac{4m^2}{\kappa^2}}\theta(\kappa^2 - 4m^2) \; . \tag{12.249}$$

The integral representation of the electromagnetic field propagator is

$$D'_{F_{\mu\nu}}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 - \mathrm{i}\epsilon}\right)\left[\frac{1}{k^2 - \mathrm{i}\epsilon} + \int \frac{\mathrm{d}\kappa^2}{\kappa^2}\frac{\sigma(\kappa^2)}{k^2 + \kappa^2 - \mathrm{i}\epsilon}\right] + \alpha\frac{k_\mu k_\nu}{(k^2 - \mathrm{i}\epsilon)^2} \; . \tag{12.250}$$

12.8.4 Goto–Imamura–Schwinger Term

The integral representation of the vacuum expectation value of the commutator of $\boldsymbol{j}_\mu$ and $\boldsymbol{j}_\nu$ is

$$\langle\mathbf{0}|[\boldsymbol{j}_\mu(x), \boldsymbol{j}_\nu(y)]|\mathbf{0}\rangle = \mathrm{i}\int \mathrm{d}\kappa^2\sigma(\kappa^2)(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\Delta(x - y; \kappa^2) \; . \tag{12.251}$$

To obtain this result, we have only used the gauge invariance of $\partial_\mu\boldsymbol{j}_\mu = 0$ and $\boldsymbol{j}_0$.

Assuming that $\mu = 0$ and $\nu = k$ (1, 2, 3) and setting $x_0 = y_0$,

$$\begin{aligned}\mathrm{i}\langle\mathbf{0}|[\boldsymbol{j}_0(x), \boldsymbol{j}_k(y)]|\mathbf{0}\rangle &= -\int \mathrm{d}\kappa^2\sigma(\kappa^2)\big[\partial_0\partial_k\Delta(x - y; \kappa^2)\big]_{y_0=x_0} \\ &= \frac{\partial}{\partial x_k}\delta^3(x - y)\int \mathrm{d}\kappa^2\sigma(\kappa^2) \; .\end{aligned} \tag{12.252}$$

Since $\sigma(\kappa^2) \geq 0$, the right-hand side does not vanish. Since $j_\mu = -\mathrm{i}e\bar{\psi}\gamma_\mu\psi$ at the lowest order, this contradicts the result obtained from the canonical commutation relation, i.e., using the commutation relation,

$$[j_0(x), j_k(y)] = 0 \quad (x_0 = y_0) \; . \tag{12.253}$$

The term which survives on the right-hand side of (12.252) is called the *Goto–Imamura–Schwinger term* [124, 125]. We carry out the following computation in the interaction picture using $\partial_\mu j_\mu = 0$:

$$\partial_\mu T[j_\mu(x), j_\nu(y)] = \delta(x_0 - y_0)[j_0(x), j_\nu(y)] \neq 0 \; . \tag{12.254}$$

The reason why this vanished in (12.121) is that we used the T*-product instead of the T-product, i.e.,

$$\langle \mathbf{0} | T^*[\boldsymbol{j}_\mu(x), \boldsymbol{j}_\nu(y)] | \mathbf{0} \rangle = \int \mathrm{d}\kappa^2 \sigma(\kappa^2)(\delta_{\mu\nu}\Box - \partial_\mu \partial_\nu) \Delta_\mathrm{F}(x-y;\kappa^2). \tag{12.255}$$

In this case, it turns out that (12.121) obviously holds. As shown above, the representation which includes a product of field operators at the same point often has a singularity. A different result is often obtained from the one derived by a simple computation.

As a similar example, if we consider a neutral vector field and assume that the mass is zero, then since $\partial_\mu \varphi_\mu = 0$,

$$\begin{aligned}[][\varphi_\mu(x), \varphi_\nu(y)] &= \mathrm{i}\left(\delta_{\mu\nu} - \frac{1}{m^2}\frac{\partial^2}{\partial x_\mu \partial x_\nu}\right)\Delta(x-y;m^2) \\ &= \mathrm{i}\int \mathrm{d}\kappa^2 \frac{\delta(\kappa^2 - m^2)}{m^2}(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\Delta(x-y;\kappa^2)\,. \end{aligned} \tag{12.256}$$

Therefore,

$$\sigma(\kappa^2) = \frac{1}{m^2}\delta(\kappa^2 - m^2)\,, \tag{12.257}$$

$$\mathrm{i}\langle 0 | [\varphi_0(x), \varphi_k(y)] | 0 \rangle = \frac{\partial}{\partial x_k}\delta^3(x-y)\frac{1}{m^2}\,. \tag{12.258}$$

The derivation here is exactly the same as for the Goto–Imamura–Schwinger term.

In this chapter, we have discussed the concept of renormalization by asking how we can separate the Lagrangian into the free part and the interaction part. We have shown that, at the next to lowest order, the divergence disappears, or rather that the renormalization constant can be chosen so as to make the divergence vanish. In the renormalization theory, the following stance is taken: the observed parameters can be replaced by finite experimental values, while the bare parameters which are not observed directly are allowed to be divergent. But if we now compute higher order corrections, can we be sure there will still be no divergence? Although this issue has been discussed by many people, we shall not pursue it here. We will discuss this aspect from a slightly different point of view in the last chapter.