Chapter 2
Review of Fundamental Equations

Abstract In this chapter we (re-)familiarize the reader with concepts from mathematics, thermodynamics, and aerodynamics that are fundamental to the methods presented in the remainder of the book. This includes a short review of vector algebra and partial-differential equations to provide the mathematical insight into the governing equations of fluid flow. Examples include the one-dimensional wave equation and the one-dimensional heat equation. A basic review of thermodynamics is given including the equation of state, the first law and the second law of thermodynamics. Also the isentropic relations between pressure, density and temperature are derived. In the aerodynamics section, the Navier-Stokes equations are derived in integral and derivative form. To simulate transonic flow, often simplifications of the Navier-Stokes equations are used. Therefore, it is shown how the Reynolds-Averaged Navier-Stokes (RANS) equations can be obtained and how the k-epsilon turbulence model can be used to close the RANS equations. For inviscid flow the Euler equations, and the full-potential equation are derived. Using examples from the literature, it is shown how well each of these models can predict the outcome of aerodynamic experiments in transonic conditions. This chapter contains 11 examples and concludes with 29 practice problems.

2.1 Introduction

Transonic flow conditions are encountered by the majority of today's jet aircraft. Almost all major airlines use high-subsonic jet transports on their medium to long haul flights. In addition, business jets, fighter aircraft, and military UAVs such as the Northrop X-47 are also confronted with transonic effects. To comprehend the physics of transonic flow, this chapter presents a review of physical and mathematical topics that form a basis for the subsequent chapters of the book. For further reading on these topics, the reader is referred to the reference list at the end of this chapter, that lists a number of text books that adequately explain the subject matter in more detail. After this short introductory section, we will present basic reviews on two mathematical topics: partial differential equations and vector algebra. Since mathematics is often referred to as "the language of physics," a good understanding is mandatory for the

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subsequent sections where we review the laws of thermodynamics and equations of motion in fluid flow.

2.2 Review of Partial Differential Equations

As the reader might remember from a course on elementary aerodynamics, the equations of motion of a particle in a fluid flow can be represented as a set of partial differential equations (PDEs) called the Navier-Stokes equations. Before we discuss these relatively complex equations we consider two more elementary equations: the one-dimensional heat and wave equations. These equations serve as examples of partial differential equations and how they can be solved. At the same time they are also part of a unique set of PDEs that actually have a closed-form solution. Unfortunately this is not true for most PDEs that describe physical processes such as the motion of a fluid. To solve these PDEs we need different solution techniques which often rely on a (numerical) approximation of the problem. However, to test such a numerical scheme we can always use the one-dimensional heat or wave equation because we know their exact solution. They can therefore serve as a test case for our numerical schemes to gain confidence in the accuracy of the approximation method.

In the subsequent two subsections, we do not attempt to review the broad subject of PDEs and their solution methods, but merely present a basic review of the topic within the context of the theory of aerodynamics and computational fluid dynamics.

2.2.1 One-Dimensional Wave Equation and Solution by D'Alembert

In this section we derive the governing equation that describes the transverse vibration of an elastic string, such as a guitar string. When solving this governing equation we should end up with an expression that describes each individual point on the string in both space (x, u) and time (t). First, consider a string of length L that we perturb and at time $t = 0$ we release the string such that it starts to vibrate. We want to evaluate its vertical displacement, $u(x, t)$ as shown in Fig. 2.1. To simplify this problem we assume that the mass per unit length (ρ) of the string is constant along the string and that it does not have any bending stiffness. In addition, we neglect the gravitational force and assume that the deflections remain small enough to justify the assumption that each point on the string only moves in the vertical direction.

To come up with the equation of motion for this string, we consider the force balance on an infinitesimal part within the string itself (P–Q in Fig. 2.1). We have α and β denoting the deviation angles from the horizontal axis, and T_1 and T_2 being the components of the internal forces at points P and Q, respectively. The horizontal components of the forces at points P and Q must be balanced:

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{constant} \tag{2.1}$$

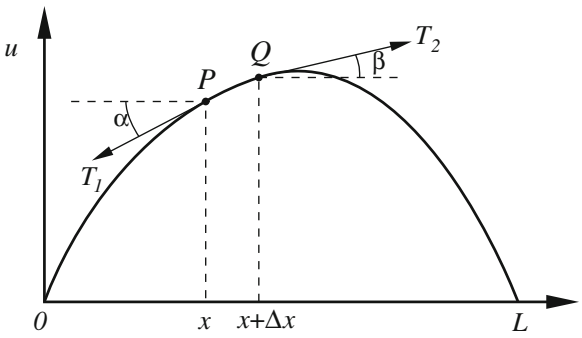

Fig. 2.1 Arbitrary transverse deflection, u of an elastic string at time t

Directing our attention to the vertical direction we know that the following force and acceleration balance should hold over the part Δx:

$$-T_1 \sin\alpha + T_2 \sin\beta = \rho \Delta x \frac{\partial^2 u}{\partial t^2} \tag{2.2}$$

Using (2.1) we can rewrite (2.2) as follows:

$$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} \tag{2.3}$$

Since $\tan\alpha = \partial u/\partial x|_x$ and $\tan\beta = \partial u/\partial x|_{x+\Delta x}$ we can now write a *partial* differential equation where both derivatives with respect to x and t appear:

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \tag{2.4}$$

If we subsequently take the limit as $\Delta x \to 0$ we obtain the well known one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad c^2 \equiv \frac{T}{\rho} \tag{2.5}$$

The wave equation is a second order (=the highest power of one of the differential terms), linear partial differential equation. Not only does this equation describe the transverse motion of a vibrating string it also describes the wave motion of a plane wave (e.g. acoustic wave) [16]. In that case we replace the excitation of the string by the sound pressure, $p'(x,t)$ and c becomes the speed of sound, $c = \sqrt{\gamma RT}$, where γ is the ratio of specific heats, R is the gas constant, and T is the static temperature in the calorically perfect gas.[1]

[1] The speed of sound will be derived in Sect. 2.5.

We can now proceed with a solution procedure of the wave equation. There are several approaches to derive a closed form solution of the problem, one of which is by using D'Alembert's approach. This approach relies on a transformation to the characteristic coordinates ξ and η according to:

$$\xi = x + ct, \qquad \eta = x - ct \tag{2.6}$$

Therefore, u now becomes a function of ξ and η. The partial derivatives should therefore be rewritten with respect to these new independent variables. If we apply the chain rule to either side of the equation we transform (2.5) to:

$$u_{\xi\eta} = \frac{\partial^2 u}{\partial\eta\partial\xi} = 0 \tag{2.7}$$

Note that we shift to the notation of denoting a partial derivative as a subscript to the dependent variable. In the subscript notation the subscripts are written in the order in which we differentiate, whereas in the "∂" notation the order is opposite. As the reader can readily observe, the substitution of the new independent variables have transformed the problem such that it can be (easily) solved by two successive integration steps:

$$\frac{\partial u}{\partial \xi} = h(\xi),$$

where $h(\xi)$ is an arbitrary function of ξ. A second step of integration yields the following:

$$u = \int h(\xi)\mathrm{d}\xi + \psi(\eta),$$

where $\psi(\eta)$ is an arbitrary function of η. Since the integral of an arbitrary function in ξ is another arbitrary function in ξ, say $\phi(\xi)$, the solution is of the form $u(\xi, \eta) = \phi(\xi) + \psi(\eta)$. Substituting the original values for ξ and η in the previous equation results in the following closed form solution:

$$u(x, t) = \phi(x + ct) + \psi(x - ct) \tag{2.8}$$

If we assume that two arbitrary functions, $f(x)$ and $g(x)$ describe the initial position and velocity, respectively, we can express the functions $\phi(x)$ and $\psi(x)$ in terms of the these initial conditions. We have:

$$u(x, 0) = f(x) = \phi(x) + \psi(x) \tag{2.9}$$

$$u_t(x, 0) = g(x) = \phi_\xi \xi_t + \psi_\eta \eta_t \tag{2.10}$$

Using (2.6), we can evaluate that $\xi_t = c$ and $\eta_t = -c$. Since at $t = 0$ we have $\xi = \eta = x$, (2.10) reduces to:

$$u_t(x, 0) = g(x) = c\phi_x(x) - c\psi_x(x) \tag{2.11}$$

If we integrate both sides of (2.11) between x_0 and x and divide both sides by c we obtain:

$$\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\mathrm{d}s \quad \text{with} \quad k(x_0) = \phi(x_0) - \psi(x_0). \tag{2.12}$$

Now, to get $\phi(x)$ we simply add (2.12) to (2.9) and divide both sides by two. Similarly, to obtain $\psi(x)$ we subtract (2.12) from (2.9) and divide both sides by two. The following results are found:

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\mathrm{d}s + \frac{1}{2}k(x_0) \tag{2.13}$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\mathrm{d}s - \frac{1}{2}k(x_0) \tag{2.14}$$

Substituting $\phi(x + ct)$ and $\psi(x - ct)$ in the above expressions and inserting everything back into (2.8) results in the following solution to this initial value problem:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s. \tag{2.15}$$

For a brief moment, let us return to interpretation of this result with respect to the original problem of the vibrating string and assume that the initial velocity, $u_t(x,0) = g(x) = 0$. To come up with a valid solution that satisfies the boundary conditions ($u(0) = u(L) = 0$), the following should hold:

$$u(0,t) = \frac{1}{2}\left[f(ct) + f(-ct)\right] = 0 \tag{2.16}$$

$$u(L,t) = \frac{1}{2}\left[f(L+ct) + f(L-ct)\right] = 0 \tag{2.17}$$

From (2.16) we learn that the initial shape function $f(x)$ is odd. In other words it is anti-symmetric with respect to the u-axis. Combining this with (2.17) we obtain that $f(L + ct) = f(-L + ct)$, which shows that the initial function should have a period of $2L$.

Example 2.1 Calculate the solution to the homogeneous wave equation (2.5) for $u(x,0) = f(x) = \sin(2\pi x)$, $u_t(x,0) = g(x) = 0$, and $c = 1$.

Solution:

Using D'Alembert's solution we can directly substitute $f(x) = \sin(\pi x)$ and $g(s) = 0$ in to (2.15). We obtain:

$$u(x,t) = \sin 2\pi(x+t) + \sin 2\pi(x-t)$$

A graphical representation of this string is sketched over the interval $[0,\ 1]$ between time $t = 0$ and $t = 0.25$ with time increments $\Delta t = 0.05$ (Fig. 2.2).

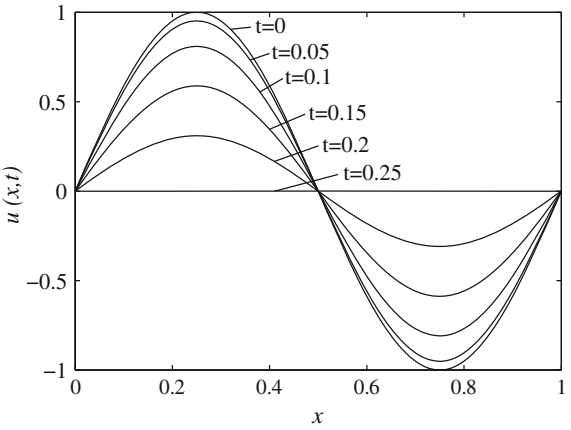

Fig. 2.2 Solution to the wave equation corresponding to the initial conditions of Example 2.1

To investigate the effect of a disturbance function on the excitation of the vibrating string we consider the inhomogeneous wave equation:

$$u_{tt} - c^2 u_{xx} = f(x,t), \tag{2.18}$$

where $f(x,t)$ is the disturbance function. We assume that the string is initially undisturbed, i.e.:

$$u(x,0) = u_t(x,0) = 0 \tag{2.19}$$

Using the characteristic coordinates as in (2.6), we can rewrite $u(x,t)$ as follows:

$$u(x,t) = u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right) = v(\xi,\eta) \tag{2.20}$$

Similarly, we can transform the forcing function: $f(x,t) = G(\xi,\eta)$. Note that when $t = 0$ we have $\xi = \eta$ and $\xi = x$. Therefore, we can write the first initial condition:

$$u(x,0) = u(\xi,0) = v(\xi,\xi) = 0 \tag{2.21}$$

To transform the second initial condition we write the first partial derivative of v with respect to ξ in terms of u_x and u_t by employing the chain rule (see Problem 2.5):

$$v_\xi = \frac{1}{2}u_x + \frac{1}{2c}u_t \tag{2.22}$$

Substituting the equalities that hold when $t = 0$ (same as above) we have:

$$v_\xi(\xi,\xi) = \frac{1}{2}u_x(\xi,0) + \frac{1}{2c}u_t(\xi,0) \tag{2.23}$$

Now, we know that the second part of the right-hand side of the equation is zero due to the initial condition $u_t(x, 0) = 0$. And, since $u(x, 0) = 0$ we know that $1/2u(x, 0)|_{x=\xi}$ must also be zero. Therefore, we now have the second initial condition in transformed coordinates:

$$v_\xi(\xi, \xi) = 0 \tag{2.24}$$

Finally, we write the PDE in characteristic coordinates according to:

$$v_{\xi\eta} = -\frac{1}{4c^2} G(\xi, \eta) \tag{2.25}$$

We are interested in the excitation at a particular time and location, $u(x_0, t_0)$, which corresponds to $v(\xi_0, \eta_0)$. Therefore, we need to integrate the left-hand side of (2.25) twice with appropriate boundary conditions:

$$\int_{\eta_0}^{\xi_0} \int_{\eta_0}^{\xi} v_{\xi\eta}(\xi, \eta) \mathrm{d}\eta \mathrm{d}\xi$$

Solving the inner integral by using the fundamental theorem of calculus (FTC)[2] results in:

$$\int_{\eta_0}^{\xi} v_{\xi\eta}(\xi, \eta) \mathrm{d}\eta = v_\xi(\xi, \xi) - v_\xi(\xi, \eta_0)$$

We note that the first term on the right-hand side must equal zero, due to the initial condition (2.24). The second integration step thus results in:

$$-\int_{\eta_0}^{\xi_0} v_\xi(\xi, \eta_0) \mathrm{d}\xi = v(\eta_0, \eta_0) - v(\xi_0, \eta_0)$$

Again, due to the initial condition (2.21) for $\xi = \eta_0$, the first term on the right-hand side equals zero, which leaves us with the negative of what we were initially looking for: $-v(\xi_0, \eta_0)$. Now, by performing the same integration on the right-hand side of (2.25) and bringing the minus sign to the other side of the equality sign (which cancels) we have a solution in characteristic coordinates:

$$v(\xi_0, \eta_0) = \frac{1}{4c^2} \int_{\eta_0}^{\xi_0} \int_{\eta_0}^{\xi} G(\xi, \eta) \mathrm{d}\eta \mathrm{d}\xi \tag{2.26}$$

To change back to the original coordinate system, the determinant (for explanation of the determinant see Sect. 2.3) of the Jacobian, $\det(J)$, needs to be calculated, since $\mathrm{d}\eta\mathrm{d}\xi = \det(J)\mathrm{d}x\mathrm{d}t$. It is left to the reader (Problem 2.6) to show that this equals $2c$.To

[2] The fundamental theorem of calculus specifies the relationship between the two central operations of calculus: differentiation and integration [11].

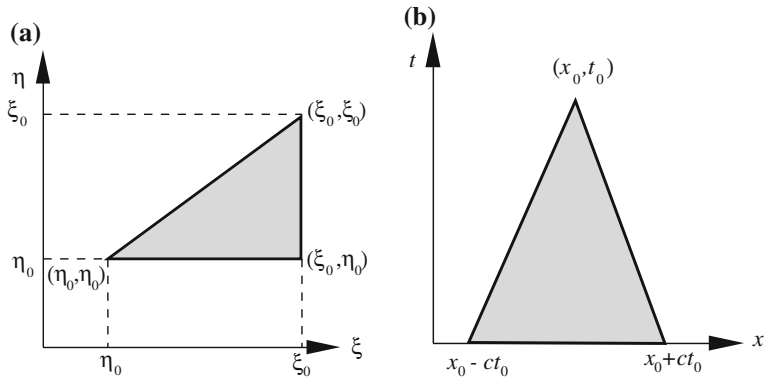

Fig. 2.3 Transformation of the area of integration between the characteristic and the original coordinate system. **a** Region of integration in the (ξ, η)-plane. **b** Region of integration in the (x, t)-plane

transfer the limits of the integrals it is convenient to look at the region over which we are integrating $G(\xi, \eta)$. This region is displayed in Fig. 2.3a. If we look at the vertices of this triangle, and what they represent in the physical (x, t)-plane, we note that:

$$\begin{aligned}(\xi_0,\ \eta_0) &\rightarrow (x_0,\ t_0)\\ (\xi_0,\ \xi_0) &\rightarrow (x_0 + ct_0,\ 0)\\ (\eta_0,\ \eta_0) &\rightarrow (x_0 - ct_0,\ 0)\end{aligned}$$

The region of integration in the (x, t)-plane that corresponds to these vertices is shown in Fig. 2.3b. We therefore have:

$$u(x_0, t_0) = \frac{1}{2c} \int_0^{t_0} \int_{x_0 - ct_0}^{x_0 + ct_0} f(x, t) \mathrm{d}x \mathrm{d}t \tag{2.27}$$

The region of integration in the physical domain can be used to see whether a point in the domain is influenced by a particular disturbance. When the area under the graph of the disturbance function and the area of the integration region (partially) overlap, then the point in the domain is influenced by the disturbance. However, if there is no overlap at all, the disturbance function does not affect that particular point in the domain. The characteristic lines ($\xi = x + ct$ and $\eta = x - ct$) therefore mark the boundaries of the so-called *domain of dependence*. The solution, therefore, only depends on a disturbance that happens within this domain of dependence. Any disturbance outside this region does not have any effect on the excitation of this point. In Example 2.2 this is demonstrated.

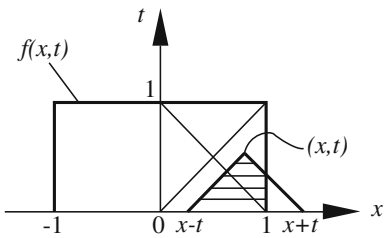

Fig. 2.4 Region of integration (striped) for Example 2.2

Example 2.2 Consider the following problem:

$$u_{tt} - c^2 u_{xx} = f(x,t) = \begin{cases} 1 & \text{for } 0 < t < 1, \quad -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$u(x,0) = 0$$

$$u_t(x,0) = 0$$

Calculate $u(x,t)$ explicitly for (x,t) satisfying: $x > t$, $x < 1$, and $x + t > 1$.

Solution:

To solve this problem we use (2.27) with $c = 1$. Due to the simple forcing function, $f(x,t)$, we can directly see that the solution $u(x,t) = 1/2 \int \int_{\Delta(x,t)} f(x,t) \mathrm{d}x\mathrm{d}t$, where $\Delta(x,t)$ represents the region of integration. This region is schematically shown in Fig. 2.4. Using simple geometry we calculate the area of this region $\Delta(x,t) = \frac{t^2}{2} - \frac{x^2}{2} - xt - x - t - \frac{1}{2}$. The solution then becomes:

$$u(x,t) = \frac{1}{4}(t^2 - x^2 - 2xt + 2x + 2t - 1)$$

2.2.2 One-Dimensional Heat Equation and Solution by Fourier Series

In this section we look at the other famous PDE, the heat equation. In the one-dimensional case we consider a bar with length L (Fig. 2.5). We are interested in the temperature, T, and its distribution over the bar with time, depending on the boundary and initial conditions. We assume that the bar is of constant cross section

Fig. 2.5 Model problem for the one-dimensional heat equation: a bar of length L

and homogeneous material and that it is perfectly insulated, such that heat flows only along the x-direction.

The derivation of the heat equation starts with Fourier's law of heat conduction that states that the heat flux, q, is negatively proportional to the temperature gradient, $\partial T/\partial x$, according to (in one-dimensional space):

$$q = -k\partial T/\partial x, \tag{2.28}$$

where k is the thermal conductivity of the material. The negative sign indicates the direction of heat flow is from hot to cold, i.e. in the opposite direction to the temperature gradient. In the absence of work done, a change in internal energy per unit volume in the material, ΔE, is proportional to the change in temperature, ΔT. That is,

$$\Delta E = c\rho\Delta T \tag{2.29}$$

where c is the specific heat capacity and ρ is the mass density of the material. Choosing zero energy at the absolute zero temperature, this can be rewritten as

$$E = c\rho T. \tag{2.30}$$

The increase in internal energy in a small spatial region of the material

$$x - \Delta x \leq \xi \leq x + \Delta x \tag{2.31}$$

over the time period

$$t - \Delta t \leq \tau \leq t + \Delta t \tag{2.32}$$

is given by

$$c\rho \int_{x-\Delta x}^{x+\Delta x} [T(\xi, t+\Delta t) - T(\xi, t-\Delta t)]\, d\xi = c\rho \int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \frac{\partial T}{\partial \tau}\, d\xi d\tau, \tag{2.33}$$

where the FTC was used. Additionally, with no work done and the absence of any heat sources or sinks, the change in internal energy in the interval $[x-\Delta x, \quad x+\Delta x]$ is accounted for entirely by the flux of heat across the boundaries. By Fourier's law, this is

$$k \int_{t-\Delta t}^{t+\Delta t} \left[\frac{\partial T}{\partial x}(x+\Delta x, \tau) - \frac{\partial T}{\partial x}(x-\Delta x, \tau) \right] d\tau = k \int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \frac{\partial^2 T}{\partial \xi^2}\, d\xi d\tau \tag{2.34}$$

again by the fundamental theorem of calculus. In higher dimensions, the divergence theorem (Sect. 2.3.3) is used instead. By conservation of energy (Sect. 2.5.3),

$$\int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} [c\rho T_\tau - kT_{\xi\xi}]\, d\xi d\tau = 0. \tag{2.35}$$

This is true for any rectangle $[t-\Delta t, \quad t+\Delta t]\times[x-\Delta x, \quad x+\Delta x]$. Consequently, the integrand must vanish identically:

$$c\rho T_t - kT_{xx} = 0. \tag{2.36}$$

Which can be rewritten as:

$$T_t = \frac{k}{c\rho}T_{xx}, \tag{2.37}$$

or:

$$\frac{\partial T}{\partial t} = \frac{k}{c\rho}\left(\frac{\partial^2 T}{\partial x^2}\right) \tag{2.38}$$

which is the heat equation. The coefficient $k/(c\rho)$ is called thermal diffusivity and is denoted by α for convenience.

To solve the heat equation (2.38), we use the 'separation-of-variables' technique [18] where we substitute

$$T(x,t) = F(x)G(t) \tag{2.39}$$

Denoting $\partial/\partial t$ with $(\dot{\ldots})$ and $\partial/\partial x$ with $(\ldots)'$, we can write (2.38) as:

$$\frac{\dot{G}}{\alpha G} = \frac{F''}{F} = -\beta^2. \tag{2.40}$$

Here we introduce a constant, β, that is conveniently supplied with a minus sign to aid in the solution procedure. We can see that each fraction in (2.40) can be written as a separate ordinary differential equation (ODE) according to:

$$F'' + \beta^2 F = 0 \tag{2.41}$$

$$\dot{G} + \alpha\beta^2 G = 0 \tag{2.42}$$

Equation (2.41) is second order in space and Eq. (2.42) is first order in time.

To find the eigenfunctions of the heat equation we need a set of boundary equations that give us some details about the heat transfer conditions of the rod at the ends. In the present case we choose to keep the ends of the rods at zero temperature: $T(0,t) = T(L,t) = 0$. A general solution to (2.41) is:

$$F(x) = A\cos(\beta x) + B\sin(\beta x) \tag{2.43}$$

Using the specified boundary conditions, we know that $A = 0$, and $B\sin\beta L = 0$. To avoid the trivial solution ($B = 0$) we know that $\sin\beta L = 0$ and thus, $\beta_n = \frac{n\pi}{L}$ for $n = 1, 2, \ldots$. We have found the following solution to the boundary-value problem:

$$F_n(x) = B_n(x)\sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad n = 1, 2, \ldots \tag{2.44}$$

Let us now return to (2.42) and substitute the eigenvalue $\lambda_n = \beta_n\sqrt{\alpha}$:

$$\dot{G} + \lambda_n^2 G = 0 \tag{2.45}$$

This has the general solution

$$G_n(t) = C_n e^{-\lambda_n^2 t} \quad \text{for} \quad n = 1, 2, \ldots, \tag{2.46}$$

where C_n are arbitrary constants. This results in the following final solution of this problem:

$$T_n(x,t) = F_n(x)G_n(t) = D_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t} \quad \text{for} \quad n = 1, 2, \ldots \tag{2.47}$$

where D_n are arbitrary constants. These are the eigenfunctions of the problem and $\lambda_n = \frac{cn\pi}{L}$ are the eigenvalues.

In order to solve the heat equation we need an initial condition, $f(x)$. This initial condition is prescribing the temperature distribution along the bar at time $t = 0$ must comply with the boundary conditions. We know that the sum of each of the solutions in (2.47) is also a solution to (2.38) and we consider the following series of eigenfunctions:

$$T(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t} \tag{2.48}$$

Since at $t = 0$ we need to satisfy the initial condition the following relation must be satisfied:

$$T(x,0) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) \hat{=} f(x) \tag{2.49}$$

Multiplying both sides by $\sin\frac{n\pi x}{L}$ and integrating both sides over the interval $[0, L]$ results in the following coefficients of the Fourier sine series:

$$D_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \tag{2.50}$$

Example 2.3 Solve the heat equation, $T_t = T_{xx}$ on the interval $0 < x < 1$ with boundary conditions $T(0,t) = T_x(1,t) = 0$ and initial condition $T(x,0) = 1$.

Solution:

Using the same methodology as above we separate the temperature function in the product of $F(x)$ and $G(t)$. We have

$$F(x) = A\cos(\beta x) + B\sin(\beta x)$$

Again, we state that due to the first boundary condition $A = 0$. The second boundary condition implies: $B\beta\cos(\beta x) = 0$. This equation is satisfied when

$$\beta = \pi\left(n + \frac{1}{2}\right) \quad \text{for} \quad n = 0, 1, 2, \ldots$$

This results in the following general solution:

$$F_n(x,t) = B_n \sin\left[\pi x\left(n + \frac{1}{2}\right)\right]$$

Substituting this in (2.39), with $G_n(t)$ being the same as in (2.46) and $\lambda_n = \pi\left(n + \frac{1}{2}\right)$, we now have the following solution:

$$T(x,t) = \sum_{n=0}^{\infty} D_n \sin\left[\pi x\left(n + \frac{1}{2}\right)\right] e^{-\pi^2\left(n+\frac{1}{2}\right)^2 t} \tag{2.51}$$

Setting $T(x,0) = 1$ and knowing that $\int_0^1 \sin\left[\pi\left(n + \frac{1}{2}\right)x\right]\sin\left[\pi\left(m + \frac{1}{2}\right)x\right]\mathrm{d}x = 0$ for $n \neq m$ we multiply both sides of (2.51) with $\sin\pi\left(m + \frac{1}{2}\right)x$ and integrate between 0 and 1:

$$\int_0^1 \sin\left[\pi x\left(m + \frac{1}{2}\right)\right]\mathrm{d}x = \frac{1}{2}D_m \quad \rightarrow \quad D_m = \frac{2}{\pi\left(m + \frac{1}{2}\right)}$$

This gives the following final solution (switching back from m to n for convenience):

$$T(x,t) = \sum_{n=0}^{\infty} \frac{2}{\pi\left(n + \frac{1}{2}\right)} \sin\left[\pi x\left(n + \frac{1}{2}\right)\right] e^{-\pi^2\left(n+\frac{1}{2}\right)^2 t} \tag{2.52}$$

In Fig. 2.6, we see a graphical interpretation of the solution. Note how the initial condition is satisfied by showing almost a complete horizontal line at $t = 0$. The boundary condition at $x = 0$ is also satisfied, while we can see that the slope of the line at $x = 1$ is zero, indicating that we have also satisfied this boundary condition correctly.

2.2.3 Conservation Form of PDEs

The equations of motion of fluid flow that are presented in this chapter (and throughout the text) are in conservation form (also referred to as conservative form). This implies that the partial differential equations (PDEs) that describe the physics of the flow have coefficients that are either constant or, if variable, their derivatives do not appear in the equation. We use the following example to illustrate this form.

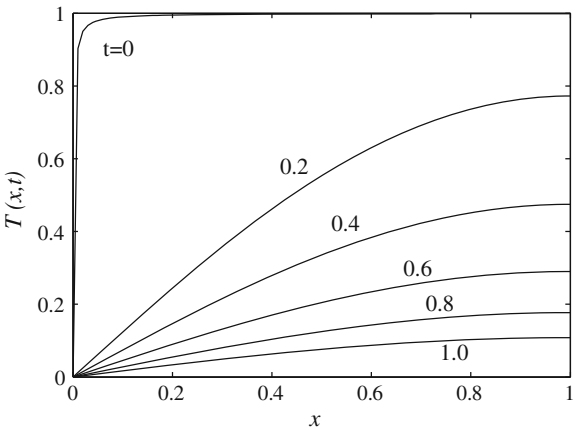

Fig. 2.6 Graphical representation of (2.52) for the first 200 components of the Fourier series expansion

Example 2.4 To appreciate the difference between the conservative and nonconservative form of a partial differential equation we consider the one-dimensional heat equation (2.38) without the assumption that ρ, c, or k are constant with x. In other words, the thermal diffusivity, α is a function of x: $\alpha = \alpha(x)$. First this equation is formulated in conservative form:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\alpha \frac{\partial T}{\partial x}\right) \tag{2.53}$$

The non-conservative form of the same equation reads:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{\partial \alpha}{\partial x}\frac{\partial T}{\partial x} \tag{2.54}$$

Note how in (2.53) the coefficient, α, can vary with position (x) but its derivative does not appear in the equation. Therefore, (2.53) is in conservation form. In (2.54) the derivative on the RHS of (2.53) has been expanded and now contains the derivative term $\partial\alpha/\partial x$, which is a non-conservative term in the equation. Therefore we deem (2.54) to be in non-conservative form.

The same logic that is used for the one-dimensional heat equation can be expanded to the equations of motion that describe fluid flows (Sect. 2.5). These governing equations must hold at any distinct point in the flow. To formulate these equations we take a *Eulerian approach* where we consider a fixed control volume through which the fluid passes. The conservation form for each PDE at such a point also allows us to formulate finite-difference representations that provide a good approximation to the PDEs. These, in turn, can be used in an iterative numerical code to obtain the global properties of a flow field. The conservative formulation is therefore often used to solve partial differential equations by numerical methods.

2.2.4 Classification of Partial Differential Equations

A second order partial differential equation (PDE) can be classified based on the coefficient of its second-order derivative terms. For an arbitrary function, ϕ, a second-order differential equation in two dimensions reads:

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = H \tag{2.55}$$

where a, b, and c are coefficients and $H = H(\phi, \phi_x, \phi_y, x, y)$. Along the so-called *characteristic curves* of the PDE the following should hold:

$$a(\mathrm{d}y)^2 - b(\mathrm{d}x\mathrm{d}y) + c(\mathrm{d}x)^2 = 0 \tag{2.56}$$

Solving for $\mathrm{d}y/\mathrm{d}x$ results in:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.57}$$

Depending on the term under the radical in (2.57), the family of characteristic curves displays different behavior. When $b^2 - 4ac > 0$ there are two distinct families of real characteristic curves. In this case the PDE is termed *hyperbolic*. When $b^2 - 4ac = 0$ a single family of real characteristic curves is found and the PDE is called *parabolic*. Finally, when $b^2 - 4ac < 0$ the RHS of (2.57) is complex and no real characteristic curves exist. The PDE is termed *elliptic* in this case.

Example 2.5 Consider the wave equation (2.5) (using χ for wave speed to avoid confusion in notation):

$$u_{tt} - \chi^2 u_{xx} = 0$$

Determine whether this equation is hyperbolic, parabolic, or elliptic, and determine, if possible, its characteristic curves.

Solution:

The coefficients of the wave equation are: $a = 1$, $b = 0$, and $c = -\chi^2$. We calculate the discriminant to be $4\chi^2$ and since χ is real we know that $4\chi^2 > 0$ and that the wave equation is therefore hyperbolic. Substituting the coefficients in (2.57) gives the two characteristic curves: $\mathrm{d}t/\mathrm{d}x|_{1,2} = \pm\frac{1}{\chi}$.

Every PDE that is represented in an orthogonal coordinate system can be transformed to its *canonical* form. A coordinate transformation takes place and every point in the original coordinate system is mapped onto the arbitrary coordinate system. In a two-dimensional space, this would imply mapping the x and y coordinates onto the ξ–η plane: $(x, y) \rightarrow (\xi, \eta)$, which we also used for the wave equation. How do we find these characteristic coordinate transformation? Well, that is quite simple: we employ (2.57) to find the slope of the characteristic lines and integrating this results in an expression for the characteristic curves.

Example 2.6 Find the characteristic curves for the wave equation of Example 2.5.

Solution:

For (2.5) we have $\mathrm{d}t/\mathrm{d}x|_{1,2} = \pm\frac{1}{\chi}$. Integrating this gives:

$$t_1 = \frac{1}{\chi}x + \eta$$
$$t_2 = -\frac{1}{\chi}x + \xi,$$

where, η and ξ are integration constants. Rearranging these equations results in the relations of (2.6).

In this arbitrary (orthogonal) coordinate system, the hyperbolic, parabolic, and elliptic PDEs assume a particular canonical (natural) form. Let us examine these canonical forms for each class of second order PDEs. We assume that the solution $u(x, t)$ is written in the transformed coordinates as $\phi(\xi, \eta)$.

Hyperbolic PDE Two characteristic forms exist:

$$\phi_{\xi\xi} - \phi_{\eta\eta} = h_1(\phi_\xi, \phi_\eta, \phi, \xi, \eta)$$
$$\phi_{\xi\eta} = h_2(\phi_\xi, \phi_\eta, \phi, \xi, \eta)$$

Similar to the wave equation, for a two-dimensional problem, two characteristic lines (or *characteristics*) can be drawn that intersect at the point of interest (x_0, t_0) (see Fig. 2.7a). The region between the characteristic lines that is present in a flow that is described by hyperbolic PDEs is called the *region of influence* [1]. A disturbance in the flow only propagates in the region of influence. Conversely, any point in the flow is influenced by the initial data that falls between the two characteristics. As we saw before, this region is termed the *domain of dependence*. Any disturbance that occurs in the domain of dependence influences the state of the point where the characteristic lines cross. Hyperbolic PDEs describe the steady inviscid supersonic flow, as well as the unsteady inviscid flow (Euler equation).

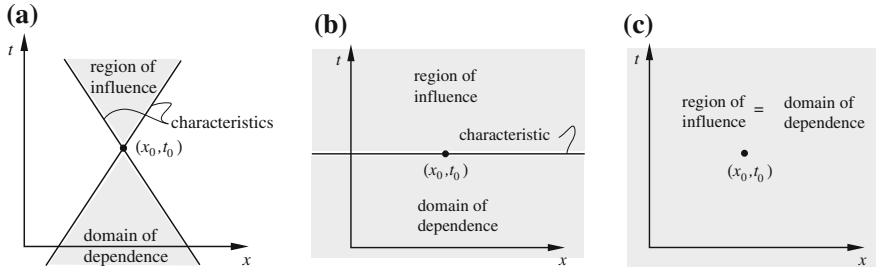

Fig. 2.7 Domain of dependence and region of influence for the three different classes of second order PDEs. **a** Hyperbolic. **b** Parabolic. **c** Elliptic

Parabolic PDE The characteristic equation in canonical form can be written as either:

$$\phi_{\xi\xi} = h_3(\phi_\xi, \phi_\eta, \phi, \xi, \eta) \quad \text{or}$$
$$\phi_{\eta\eta} = h_4(\phi_\xi, \phi_\eta, \phi, \xi, \eta)$$

Similar to hyperbolic PDEs, the parabolic PDEs have a region of influence. However, this region is now unbounded by characteristics and therefore spans the entire space (can also be time) beyond the point where the initial disturbance takes place (see Fig. 2.7b). In addition, the entire region before this point influences this point in the flow. Parabolic PDEs describe steady boundary layer equations along with "parabolized" viscous flows.

Elliptic PDE The canonical form of an elliptic PDE can be written in its characteristic coordinates as follows:

$$\phi_{\xi\xi} + \phi_{\eta\eta} = h_5(\phi_\xi, \phi_\eta, \phi, \xi, \eta)$$

Any disturbance in a flow field that is described by elliptic PDEs is felt throughout the entire flow. In other words, the entire flow domain forms the region of influence. Conversely, every point in the flow domain influences any other point in the flow domain. Therefore, the entire flow can also be viewed as the domain of dependence (see Fig. 2.7c). Elliptic PDEs describe the steady subsonic, inviscid flow along with an incompressible inviscid flow field.

Consider the two-dimensional potential equation for steady, irrotational, inviscid flow (we will derive this equation in Sect. 2.7.2):

$$\left[1 - \frac{\Phi_x^2}{c_0^2}\right]\Phi_{xx} + \left[1 - \frac{\Phi_y^2}{c_0^2}\right]\Phi_{yy} - 2\frac{\Phi_x \Phi_y}{c_0^2} = 0 \tag{2.58}$$

The velocity potential function is denoted with $\Phi(x, y)$, with the velocities, $\Phi_x(x, y)$ and $\Phi_y(x, y)$, as well as the speed of sound, c_0. The three coefficients of interest can be identified as follows:

$$a = 1 - \frac{\Phi_x^2}{c_0^2} \qquad b = -\frac{2\Phi_x \Phi_y}{c_0^2} \qquad c = 1 - \frac{\Phi_y^2}{c_0^2} \tag{2.59}$$

The discriminant for this second order PDE becomes:

$$b^2 - 4ac = \frac{\Phi_x^2 + \Phi_y^2 - c_0^2}{c_0^2} = M^2 - 1, \tag{2.60}$$

where M is the Mach number of the flow. For subsonic flows $M < 1$ and the potential equation is elliptic. When the flow is supersonic $M > 1$ and the equation

becomes hyperbolic. At sonic condition the equation is parabolic. In transonic flow the potential equation therefore changes type between the different flow domains.

It is important to distinguish the class of the PDE because different solution techniques should be used to solve them. The "problem" in transonic flows is that the most convenient equations in both supersonic and subsonic flow are not of the same type. Any type of numerical scheme that attempts to approximate one of these equations is therefore doomed to fail. Steady supersonic and steady subsonic flows can therefore not be modeled with the same numerical approach. However, *unsteady* flows can be described by a hyperbolic scheme whether the flow is subsonic or supersonic. This is the reason that in transonic aerodynamics the unsteady equations of motion form the basis for the solution of the flow field. The solution is found by marching in time until the solution converges to a steady-state value.

2.3 Review of Vector Algebra

As demonstrated in Sect. 2.5 the equations of motion of a fluid flow can be conveniently represented in vector form. The vector notation is often used to represent the governing flow equations and their (numerical) approximations. Vector notation allows us to represent a set of equations in a single line, which make it easier to perform mathematical operations in order to derive certain numerical approximation schemes. The following sections give a review of basic vector algebra including a review of the most common operations and their notation used in this text.

2.3.1 Vectors, Vector Fields, and Scalar Fields

Some properties of a fluid flow are represented as scalar values (such as p, T, and ρ). Scalars have a magnitude but no direction. Velocity, however, is represented by a vector, because it has both a magnitude and a direction. In this text a vector is denoted with a bold font. For example, the velocity vector is denoted with $\boldsymbol{V}$, while its magnitude, $V = |\boldsymbol{V}|$. In two dimensions, the velocity vector is denoted with $\boldsymbol{V} = \begin{pmatrix} u \\ v \end{pmatrix}$, where u and v are the magnitude of the velocity vector components in the two orthogonal directions.

Addition and subtraction of vectors (and also matrices) is the same as for scalars. Each entry of the first vector (matrix) is added or subtracted to the corresponding entry in the second vector (matrix). This does require the two vectors (matrices) to be of identical dimension. We distinguish different forms of vector multiplication. For two vectors, **a** and **b** the inner product (also called dot product or scalar product) is defined as:

$$\mathbf{a} \cdot \mathbf{b} = (a_1 \cdots a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i b_i \tag{2.61}$$

Here $(a_1 \cdots a_n) = \mathbf{a}^T$, with T denoting the transposition of vector $\mathbf{a}$. When two vectors are orthogonal to each other, the inner product of these vectors is zero.

The outer product refers to the tensor product of two vectors. Given a vector $\mathbf{a} = (a_1, \ldots, a_m)$ with m elements and a vector $\mathbf{b} = (b_1, \ldots, b_n)$ with n elements, their outer product $\mathbf{a} \otimes \mathbf{b}$ is defined as the $m \times n$ tensor $\mathbf{A}$ obtained by multiplying each element of $\mathbf{a}$ by each element of $\mathbf{b}$. Thus, the outer product defines every entry of $\mathbf{A}$ according to $A_{mn} = a_m b_n$. Notation of the outer product often neglects the "$\otimes$" symbol. For example, the outer product of the vector $\boldsymbol{V}$ with itself can be denoted as $\boldsymbol{V}\boldsymbol{V}$, as we will see in Sect. 2.5.2.

The third form of multiplication is the so called the cross product (also called vector product). The cross product of two vectors is a vector:

$$\mathbf{a} \times \mathbf{b} = ab \sin\theta \mathbf{n} = \mathbf{v} \tag{2.62}$$

where θ is the measure of the smaller angle between $\mathbf{a}$ and $\mathbf{b}$ ($0° \leq \theta \leq 180°$), a and b are the vector magnitudes of vectors $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{n}$ is a unit vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ in the direction given by the right-hand rule. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel (i.e., the angle θ between them is either $0°$ or $180°$), by the above formula, the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the zero vector $\mathbf{0}$. In the present text the cross product in the three-dimensional Euclidian space is considered and the vectors $\mathbf{a}$ and $\mathbf{b}$ therefore each have three entries. This allows us to calculate the entries for $\mathbf{v}$ using the third-order determinant:

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}, \tag{2.63}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors in the respective directions x, y, and z.

Example 2.7 For the vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (4, 5, 6)$, calculate the inner product, the outer product and the vector product of $\mathbf{a}$ with $\mathbf{b}$ and $\mathbf{b}$ with $\mathbf{a}$.

Solution:

The inner product is calculated according to (2.61):

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

The outer product can be calculated according to:

$$(1,2,3)^T \otimes (4,5,6) = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

To calculate the vector product we use (2.63):

$$\mathbf{a} \times \mathbf{b} = (2 \cdot 6 - 3 \cdot 5, 3 \cdot 4 - 1 \cdot 6, 1 \cdot 5 - 2 \cdot 1) = (-3, 6, -3)$$
$$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) = (3, -6, 3)$$

In vector calculus we have two kinds of functions: scalar functions and vector functions. The output of a scalar function at a particular point P is a scalar:

$$f = f(P)$$

Vector function, $\mathbf{g}$, is based on the vectorial function evaluation at a particular point P:

$$\mathbf{g} = \mathbf{g}(P) = (g_1(P), g_2(P), g_3(P))$$

The output of the vector function is a vector. The domain where a vector function is defined is a region of space (can also be a surface or a curve in space). Within this region (or on that surface or line) we say that this vector function defines a vector field. In Cartesian coordinates a vector field can be denoted by

$$\mathbf{g}(x, y, z) = (g_1(x, y, z), g_2(x, y, z), g_3(x, y, z)),$$

while a scalar function in Cartesian coordinates can be written as $f(x, y, z)$.

2.3.2 Gradient of a Scalar Field

The following sections present various operations that can be performed on vector and scalar fields. One of these operations is the gradient of a scalar field. The gradient of a scalar field is a vector field which points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is the greatest rate of change. The gradient (grad) of a scalar field is defined as (in three-dimensional Cartesian coordinates):

$$\text{grad}\, f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \tag{2.64}$$

The ∇ (read: nabla or del) operator is defined as the following differential vector operator:

$$\nabla \equiv \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} \tag{2.65}$$

We can define a scalar function whose gradient forms the three components of the velocity field for irrotational, inviscid flow. This so-called potential-flow equation reduces the system of equations to a single equation, thereby simplifying the problem of finding the velocity field considerably. For example, the velocity distribution in a fluid flow can be represented by the vector field $\mathbf{V}$. At any point, P, throughout the fluid this vector field can be related to the gradient of a scalar field, Φ, at this point: $\mathbf{V}(P) = \nabla\Phi(P)$. Since this is true throughout the entire physical domain we define the velocity potential function as:

$$\mathbf{V} \equiv \nabla\Phi \tag{2.66}$$

Example 2.8 Consider a fluid flow where the velocity vector field is given by $\boldsymbol{V} = (2x, yz^2, zy^2)$. Determine the velocity potential function that describes this flow, if it exists.

Solution:

To find $\Phi(x, y, z)$ find the primitive function of each of the components of the vector field:

$$\Phi = \int 2x\mathrm{d}x = x^2 + C_1(y, z)$$
$$\Phi = \int yz^2\mathrm{d}y = \frac{y^2z^2}{2} + C_2(x, z)$$
$$\Phi = \int zy^2\mathrm{d}z = \frac{z^2y^2}{2} + C_3(x, y)$$

It is possible to combine the above equations into a single expression for the following potential flow function: $\Phi = x^2 + \frac{1}{2}y^2z^2 + C_4$, where C_4 is a constant. When we employ (2.66) to Φ, we find $\boldsymbol{V}$. This confirms that Φ exists and is indeed the potential function of $\boldsymbol{V}$.

2.3.3 Divergence of a Vector Field

To assess the magnitude of 'change' of a vector field we use the divergence (div) of a vector field defined as:

$$\mathrm{div}\,\boldsymbol{g} = \nabla \cdot \boldsymbol{g} = \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z} \tag{2.67}$$

where $\boldsymbol{g} = (g_1, g_2, g_3)$.

Example 2.9 For the vector field of Example 2.8 calculate the divergence.

Solution:

Employing (2.67) we obtain the following:

$$\text{div}\, \boldsymbol{V} = \nabla \cdot \boldsymbol{V} = 2 + z^2 + y^2$$

2.3.4 Curl of a Vector Field

To study the rotation of a vector field we employ the curl. The curl is defined as:

$$\text{curl}\, \boldsymbol{g} = \nabla \times \boldsymbol{g} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ g_1 & g_2 & g_3 \end{vmatrix} \tag{2.68}$$

$$= \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right)\boldsymbol{k}$$

In a velocity field, the curl is called the vorticity vector and is related to the rotation of the flow. Rotation of the flow is introduced due to the viscosity of the gas and leads to turbulent conditions. We will see later that the rotation of a flow field is directly tied to the formation of curved shock waves in supersonic flow via Crocco's theorem.

Example 2.10 Determine the curl of the vector field of Example 2.8.

Solution:

We can directly apply (2.68):

$$\text{curl}\, \boldsymbol{V} = \nabla \times \boldsymbol{V} = (2zy - 2zy)\mathbf{i} + (0 - 0)\mathbf{j} + (0 - 0)\mathbf{k} = \mathbf{0}$$

In Example 2.10 we see that the specified vector field is irrotational. This agrees with the fact that we *can* define a potential equation. This confirms that we can define a potential function only when the vector field is irrotational.

2.3.5 Relation Between Volume, Surface, and Line Integrals

In this section we briefly review the relation between volume, surface, and line integrals. First, we consider an enclosed volume, $\mathscr{V}$, with an outer surface $\boldsymbol{S}$. The *divergence theorem* (usually attributed to Gauss) states that the flux of a vector field

through the surface equals the divergence of that vector field inside the enclosed volume. For a velocity vector field, $\boldsymbol{V}$, this theorem can be written as follows[3]:

$$\iint_S \boldsymbol{V} \cdot \mathbf{d}\boldsymbol{S} = \iiint_{\mathscr{V}} \nabla \cdot \boldsymbol{V} \mathrm{d}\mathscr{V} \tag{2.69}$$

The *gradient theorem* states that the gradient of a scalar field, p, integrated over the control volume, equals the scalar field integrated over the surface vector of the control volume:

$$\iint_S p\mathbf{d}\boldsymbol{S} = \iiint_{\mathscr{V}} \nabla p \mathrm{d}\mathscr{V} \tag{2.70}$$

where $\mathbf{d}\boldsymbol{S}$ is the element of the surface vector in the normal direction to the surface pointing outward. Finally, we can relate the curl of a vector field over a control surface, S, to the closed line integral of the vector field in a counter-clockwise direction over its boundary, C. This is the *Kelvin-Stokes theorem* (often referred to as Stokes' theorem):

$$\oint_C \boldsymbol{V} \cdot \mathbf{d}\boldsymbol{C} = \iint_S (\nabla \times \boldsymbol{V}) \cdot \mathbf{d}\boldsymbol{S} \tag{2.71}$$

2.4 Review of Thermodynamics

Thermodynamics is the science that studies the energy conversion between mechanical work and heat. In fluid flow, thermodynamics plays an important role in transferring energy from one state to the other. A simple example is a stagnation point at the leading edge of a wing, where the flow is brought to rest and its kinetic energy is transferred to heat. Similar effects arise in the boundary layer over the wing where the friction between the air and the structure also converts kinetic energy into heat. As we will see in Sect. 2.5 the first law of thermodynamics forms the basis of the energy equation in fluid flow. This section presents the thermodynamic parameters that we find in a fluid flow and their interrelation via the state law, the first law of thermodynamics, and the second law of thermodynamics. It is emphasized that this section is merely a review of the thermodynamic relations that we will use in subsequent sections and chapters. For a more complete and in-depth discussion of the topic the reader may consult Sonntag and Van Wylen [21]. For more information on the application of thermodynamics to aircraft propulsion, the reader is referred to Farokhi [9].

[3] This theorem applies to any vector field, not only $\boldsymbol{V}$ but also mass flow or force fields.

2.4.1 Perfect Gas Relations

In fluid flow, the thermodynamic properties of fluid are characterized by scalar fields. The variables of interest for the present discussion are pressure (p in Pa or psi), density (ρ in kg/m^3 or lb/ft^3), temperature (T in K or °F), and the specific internal energy (e in J/kg or BTU/lb). According to the *state principle* of thermodynamics, the local thermodynamic state is determined by any two independent state variables. If e and ρ are chosen as those variables, both pressure, p, and temperature, T, are dependent on these two variables. The subsequent paragraphs present the equations that relate the state variables to each other via a set of laws.

For a calorically perfect gas (intermolecular forces are assumed negligible and the volume of individual molecules is infinitesimally small) the equation of state reads:

$$p = \rho RT, \tag{2.72}$$

where R is the gas constant for a specific gas. In terms of specific volume, $v = 1/\rho$, (2.72) can also be expressed as $pv = RT$. A new thermodynamic state variable is defined as the sum of internal energy, e, and the product of pressure and specific volume and is called specific enthalpy:

$$h = e + pv = e + RT \tag{2.73}$$

Even though specific enthalpy is introduced here without any physical motivation, we will see later that it is indeed a fundamental state variable for aero-thermodynamic analysis. Both specific energy and specific enthalpy (which also has unit J/kg) are both related to the temperature for a perfect gas:

$$e = e(T) \tag{2.74}$$

$$h = h(T) \tag{2.75}$$

In transonic flow we assume that the gas is perfect and we can consequently find

$$\mathrm{d}e = c_v \mathrm{d}T \tag{2.76}$$

$$\mathrm{d}h = c_p \mathrm{d}T \tag{2.77}$$

where c_v is the specific heat at constant volume and c_p the specific heat at constant pressure. These coefficients can be assumed to be constant for air up to temperatures of 1,000 K, which is the basis for the calorically perfect gas assumption. Based on this assumption we can now define the following relations between the specific heat coefficients, c_v and c_p, the specific internal energy and enthalpy, e and h, the gas constant, R, and the ratio of specific heats, γ:

$$e = c_v T, \quad h = c_p T, \quad \gamma = \frac{c_p}{c_v}, \quad c_v = \frac{R}{\gamma - 1}, \quad c_p = \frac{\gamma R}{\gamma - 1}, \quad R = c_p - c_v \tag{2.78}$$

Using the above relations, the temperature and pressure can now be defined in terms of the independent state variables e and ρ:

$$p = (\gamma - 1)\rho e, \quad T = \frac{(\gamma - 1)e}{R} \tag{2.79}$$

For dry air at temperatures that are normally encountered during flight, the gas constant equals $R = 287.04\,\text{J/kg/K}$.

The viscosity (μ) and thermal conductivity (k) are properties of a gas that are both dependent on the temperature. The viscosity of the fluid is responsible for the momentum transport on molecular level. The dynamic viscosity of an ideal gas is:

$$\mu = \frac{1}{3}\rho\lambda\overline{c} \tag{2.80}$$

Here λ is the mean free path between molecules and $\overline{c}$ is the average velocity of the molecules. Examining this equation we can see that the viscosity increases with the average velocity and therefore with temperature. Sutherland's equation states that the viscosity of air is dependent on its temperature according to the following relation:

$$\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + S}{T + S} \tag{2.81}$$

where μ_0 and T_0 are a reference viscosity and temperature, respectively, and S is the Sutherland temperature which is $S = 110\,\text{K}$ for air. For ISA (international standard atmosphere) conditions these are $\mu_0 = 1.7894 \times 10^{-5}\,\text{kg/m/s}$ and $T_0 = 288.16\,\text{K}$. The thermal heat conductivity was already introduced in Fourier's law (2.28) and is a property of the gas. For air $k = 0.024\,\text{W/m/K}$ at a temperature of 273 K. The Prandtl number relates the viscosity and the thermal conductivity of a fluid according to:

$$\text{Pr} = \frac{c_p}{k}\mu \tag{2.82}$$

Since k, μ, and c_p are dependent on the temperature, the Prandtl number is dependent on temperature as well. However, for air over a substantial temperature range (up to 600 K) we can assume the Prandtl number remains constant at 0.71 [3]. In Fig. 2.8 the viscosity and conductivity of air are presented as a function of temperature, along with the assumed values for the constants that have been used. We see that there is an appreciable change in conductivity and viscosity with temperature, which can be important if we model high-speed fluid flows.

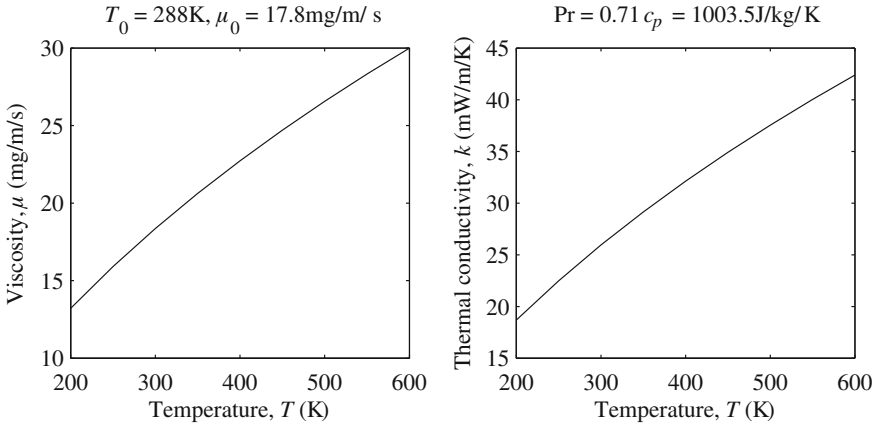

Fig. 2.8 Graphical representation of (2.81) and (2.82) for dry air

2.4.2 First Law of Thermodynamics

The first law of thermodynamics states that energy can only be transformed from one form into another. In other words, the summation of all energies in a closed system remains constant. The system has hypothetical boundaries to the surroundings that we can define based on the problem under consideration. The change in specific internal energy of the system, de, can be related to the increment in specific work done by the system, δw, and the specific heat added to the system, δQ:

$$\mathrm{d}e + \delta w = \delta Q \tag{2.83}$$

As we noted in Sect. 2.4.1, e is a state variable. That means that de is an exact differential which only depends on the difference between the initial (e_1) and final state (e_2) of the process:

$$\int_1^2 \mathrm{d}e = e_2 - e_1 \tag{2.84}$$

Contrary to that, the heat and work depend on the path between the two states. Therefore, their changes are represented with a δ in (2.83). To demonstrate this process dependency Fig. 2.9 shows how the total amount of work changes with the path taken from point 1 to point 2. Note that this figure shows expansion processes where $p\mathscr{V}^n = \text{constant}$. This is a polytropic process, with p and $\mathscr{V}$ representing the pressure and volume of the gas, respectively. The exponent, n, depends on the process. For example if $n = 0$ we have an isobaric process, when $n = 1$ we have an isothermal process (using the perfect gas law). Of course, we can define an infinite number of ways to get from state 1 to state 2, each described by a different value of n.

If work is done by a system under adiabatic conditions (meaning no heat addition or extraction, hence $\delta Q = 0$) the internal energy (read: temperature) of the system

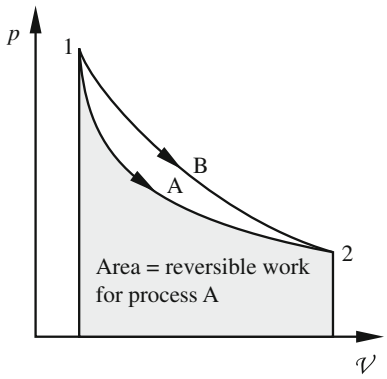

Fig. 2.9 Work depends on the path (process) between state 1 and state 2

changes according to:

$$\mathrm{d}e = -\delta w \tag{2.85}$$

For a process that is reversible (no dissipative phenomena occur), we can calculate the specific work (w) according to:

$$w = \int_1^2 p\mathrm{d}v = \begin{cases} \frac{p_2 v_2 - p_1 v_1}{1-n} & \text{for} \quad n \neq 1 \quad \text{(non-isothermal condition)} \\ p_1 v_1 \ln \frac{v_2}{v_1} & \text{for} \quad n = 1 \quad \text{(isothermal condition)} \end{cases} \tag{2.86}$$

Substituting the perfect gas law for pv in (2.86), we get the following expression for non-isothermal processes:

$$w = -\frac{R(T_2 - T_1)}{n-1} \tag{2.87}$$

Since we assume that the change in energy equals the work done on the system and using (2.76), we can write:

$$\Delta e = c_v(T_2 - T_1) = -w \tag{2.88}$$

Combining (2.87) and (2.88) leads, after simplification, to the following statement:

$$n = \frac{c_p}{c_v} = \gamma \tag{2.89}$$

which demonstrates that for an adiabatic and reversible process the following is true:

$$pv^\gamma = \text{constant} \tag{2.90}$$

Let us recap the result of this process. We have made two assumptions in the previous derivation: (1) no heat addition or extraction; (2) no dissipative phenomena occur. Finally, we implicitly assumed that the total amount of mass within the system

remained the same, which is inherent in the definition of a thermodynamic system. Under these assumptions, the process is adiabatic and reversible. As the reader might remember from a course on thermodynamics, this process is called isentropic. In an aerodynamic context this means that the dissipative phenomena such as thermal conductivity, viscosity and mass diffusion are assumed to be absent from the flow. In addition, no heat is added or extracted from the flow by means of heating or cooling the flow, respectively. Flows that exhibit the formation of shock waves are therefore per definition anisentropic as this process is irreversible and dominated by viscous properties in the flow.

2.4.3 Second Law of Thermodynamics

The first law of thermodynamics only states that energy is conserved during a thermodynamic process. However, it does not tell us anything about the direction in which the energy flow is taking place. This is governed by the second law of thermodynamics that states that over time, differences in temperature, pressure, and chemical potential tend to even out in a physical system that is isolated from the outside world. To quantify this 'evening-out' process, a property called entropy, S (denoted by s for specific entropy), is introduced. Entropy is related to the change in heat in a reversible system:

$$\mathrm{d}s = \frac{\delta Q_{\mathrm{rev}}}{T}, \tag{2.91}$$

where T is the temperature of the system. Entropy is a state property meaning that the change in entropy of a system going from one state to another is the same for all processes. Note that for an irreversible process where dissipative phenomena occur, we could always assign an effective δQ_{rev} that relates the initial state and the end state to each other. However, it is more revealing to say that these dissipative processes produce their own entropy, $\mathrm{d}s_{\mathrm{irrev}}$, i.e.:

$$\mathrm{d}s = \mathrm{d}s_{\mathrm{rev}} + \mathrm{d}s_{\mathrm{irrev}} = \frac{\delta Q_{\mathrm{rev}}}{T} + \mathrm{d}s_{\mathrm{irrev}} \tag{2.92}$$

This statement says that $\mathrm{d}s$ has two parts, one is reversible and the other is irreversible due to dissipation phenomena. The dissipative phenomena within the system always increase the entropy of the system:

$$\mathrm{d}s_{\mathrm{irrev}} \geq 0 \tag{2.93}$$

If $\mathrm{d}s_{\mathrm{irrev}} = 0$ we have a truly reversible process. If $\mathrm{d}s$ exceeds the reversible limit, the process is irreversible and entropy is produced. Combining (2.92) and (2.93) we express the second law of thermodynamics as:

$$\mathrm{d}s \geq \frac{\delta Q}{T} \tag{2.94}$$

$$\mathrm{d}s \geq 0 \quad \text{for adiabatic processes} \tag{2.95}$$

When a process is isentropic (literally meaning "equal disorder") the entropy of the system and its surroundings remains constant from state 1 to state 2.

Combining the first and second law of thermodynamics, (2.83) and (2.91), respectively, produces Gibbs' equation:

$$T\mathrm{d}s = \mathrm{d}e + p\mathrm{d}v \tag{2.96}$$

If we use the chain rule we can find the change in enthalpy (2.73) to be $\mathrm{d}h = \mathrm{d}e + p\mathrm{d}v + v\mathrm{d}p$. Combining this with (2.96) we can now write two alternative forms for the first law of thermodynamics, one in terms of specific energy and one in terms of specific enthalpy:

$$T\mathrm{d}s = \mathrm{d}e + p\mathrm{d}v \tag{2.97}$$

$$T\mathrm{d}s = \mathrm{d}h - v\mathrm{d}p \tag{2.98}$$

By inserting the relations between temperature and specific energy (2.76) and specific enthalpy (2.77), respectively, and by utilizing the state law (2.72) we can find two expressions for $\mathrm{d}s$: one in terms of specific volume and one in terms of specific pressure (see Problem 2.22). Integrating both relations between the state properties at state 1 and state 2 results in the following expressions for the change in entropy of a calorically perfect gas:

$$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1} \tag{2.99}$$

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{2.100}$$

These forms of Gibbs' equation are useful in aero-thermodynamic calculations. For an isentropic flow, i.e. $s_1 = s_2$, we can establish the isentropic relations between temperature, density, and pressure (see Sect. 2.4.4).

Example 2.11 We consider a one-dimensional flow field of air with a discontinuity in state properties due to the presence of a normal shock wave. In front of the shock, the static pressure is 50 kPa, while right behind the shock it is 75.6 kPa. The static temperature over the shock increases with a factor of 1.128. Calculate the corresponding change in entropy over the shock and the corresponding density ratio between state 2 and 1.

Solution:

We calculate the pressure ratio to be $p_2/p_1 = 1.512$ and enter this in (2.100) to find $s_2 - s_1 = 0.746\,\text{J/kg/K}$. Rewriting (2.99) we get the following:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \exp\left[\frac{(s_1 - s_2) + c_v \ln\left(\frac{T_2}{T_1}\right)}{R}\right]$$

Substitution of the appropriate values and constants gives $\rho_2/\rho_1 = 1.34$.

The example above is actually a demonstration of how the state properties of the flow change when it goes through a normal shock wave. Pressure, density, temperature, and entropy all increase depending on the Mach number of the flow in front of the shock wave. In the above example this Mach number is 1.2 and the entropy change is still relatively small. Notice, though, that even at this relatively low Mach number the shock wave creates a relatively large pressure, temperature, and density rise.

2.4.4 Isentropic Relations

Based on the relations (2.99) and (2.100) we can investigate the change in state properties for an isentropic (adiabatic and reversible) process. In this case $\mathrm{d}s = 0$ and we can rewrite (2.99) and (2.100) to show the relation between the pressure, density and temperature ratios in an isentropic process. We set the LHS of (2.99) and (2.100) to zero and perform some manipulations to find the following:

$$\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{-c_v/R} \tag{2.101}$$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{c_p/R} \tag{2.102}$$

By using the relations of (2.78), we can make the following substitutions:

$$\frac{c_v}{R} = \frac{1}{\gamma - 1} \quad \text{and} \quad \frac{c_p}{R} = \frac{\gamma}{\gamma - 1}$$

With these substitutions and subsequent manipulations (see Problem 2.23) we can now state the isentropic relations:

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \tag{2.103}$$

Now that we have the isentropic relations we ask ourselves: why are they so important? We know that in reality a gas flow is viscous and thermally conducting and therefore dissipative phenomena always occur. However, we also know that these phenomena are only dominant in small regions in the flow: the boundary layer and inside the shock waves. Outside of these regions the dissipative phenomena are so small that they can often be neglected when we want to calculate the local state properties. However, within these regions the isentropic relations do not hold and we cannot relate the state properties in this simple way. This becomes evident in the subsequent section and chapters of this text.

2.5 Equations of Fluid Motion

The following sections give an overview of the fundamental equations of motion in (transonic) fluid dynamics. For a more comprehensive treatment of this subject matter we refer to introductory texts on aerodynamics such as the work by Anderson [3]. Most of what is presented in this chapter follows the text of Tannehill et al. [19].

Before we start discussing the equations of motion we introduce two non-dimensional numbers that are often used to characterize a fluid flow: the Reynolds number and the Mach number. The Reynolds number is a measure for the ratio of inertial forces to viscous forces in a moving fluid. Its definition is as follows:

$$\mathrm{Re}_l = \frac{\rho V l}{\mu} \tag{2.104}$$

where l is a characteristic length, for example the chord length of an airfoil or the diameter of a pipe. A second non-dimensional number to characterize a fluid flow is its Mach number. The Mach number is defined as the ratio of the local flow velocity to the local speed of sound:

$$M = \frac{V}{a} \tag{2.105}$$

The local speed of sound, a, is a function of the state properties of the fluid. For a perfect gas the speed of sound is solely dependent on the temperature of the gas:

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)} = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma R T} \tag{2.106}$$

2.5.1 Conservation of Mass

We consider a fluid flow through a fixed control volume $\mathscr{V}$. We consider the balance between the mass flowing in and out of the control volume and the time rate of change

of the mass inside the control volume. We use the empirical law that mass can neither be destroyed nor created during this process. Mass can only exit the control volume through its enclosed surface, **S**. The resulting mass flowing out of the control volume can be written as follows:

$$\oiint_S \rho \boldsymbol{V} \cdot \mathbf{d}\boldsymbol{S} \tag{2.107}$$

where $\boldsymbol{V}$ is the velocity vector. Note that the vector $\mathbf{d}\boldsymbol{S}$ has a magnitude of dS and a direction **n**, which is perpendicular to the surface of dS. We can express the time rate of increase in mass inside $\mathscr{V}$ using the following volume integral:

$$\frac{\partial}{\partial t} \iiint_{\mathscr{V}} \rho \mathrm{d}\mathscr{V} \tag{2.108}$$

The change in mass within the control volume should be in balance with the net mass flux through the control surface. This conservation principle results in the following continuity equation in integral form:

$$\oiint_S \rho \boldsymbol{V} \cdot \mathbf{d}\boldsymbol{S} + \frac{\partial}{\partial t} \iiint_{\mathscr{V}} \rho \mathrm{d}\mathscr{V} = 0 \tag{2.109}$$

Equation (2.109) is the conservation of mass in integral form. Applying the divergence theorem to the surface integral combines both terms within one volume integral (see Problem 2.24). From this integral the conservation form of the continuity equation can be found. The conservation of mass law in differential form is:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{V}) = 0 \tag{2.110}$$

Let us briefly analyze this equation. It is a first order partial differential equation. At a particular point in space it describes the change in density with time and the divergence of the product of density and velocity. We can simplify this equation even further by introducing the substantial derivative operator, D/Dt. The substantial derivative combines the local derivative ($\partial/\partial t$) and the so-called convective derivative ($\boldsymbol{V} \cdot \boldsymbol{\nabla}$), to represent the "total" derivative:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \boldsymbol{\nabla} \tag{2.111}$$

In the case of the density in two Cartesian dimensions it reads:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\rho = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} \tag{2.112}$$

This equation physically states that the change in density is due to temporal and spatial variations. Using the definition of the substantial derivative (2.111) we can rewrite the continuity equation (2.110) in substantial-derivative form:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\nabla \cdot \boldsymbol{V}) = 0 \tag{2.113}$$

When homogeneous, incompressible flow is considered, the first term on the LHS can be dropped and only have $\nabla \cdot \boldsymbol{V} = 0$. However, generally speaking this is only a valid approximation at Mach numbers below $M = 0.3$ where density variations are less than 5 %. In transonic flows with typical Mach numbers in the range of 0.8–1.3 we cannot use the incompressible flow assumption.

2.5.2 Conservation of Linear Momentum

The momentum theorem in fluid mechanics is the counterpart of Newton's second law of motion in solid mechanics which states that the time rate of change of linear momentum of a body of mass, m, must be equal to the net forces that act on that body:

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{V}) \tag{2.114}$$

Let us again consider a fluid passing through a finite control volume. The forces on the fluid in this control volume can be divided into forces that are acting on the fluid (such as gravity) and forces that are acting on the control surface, S (pressure and shear forces). In Fig. 2.10 it is schematically shown how the body force, pressure and stress vectors act on a control volume in Cartesian coordinates. Note that τ_{ij} is the stress on the surface of the control volume with normal vector i in

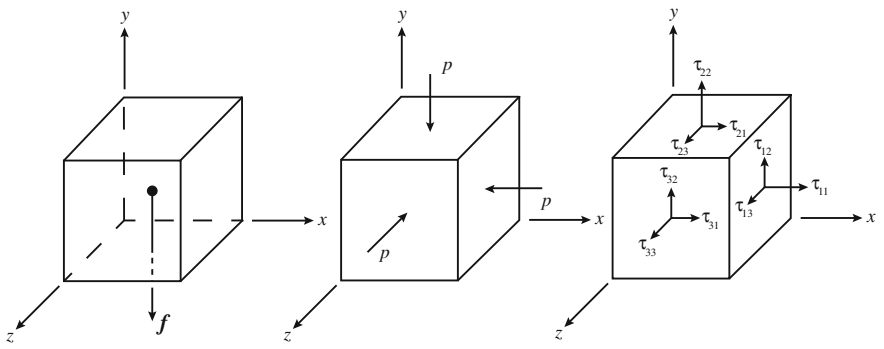

Fig. 2.10 Schematic control volume with body force (*left*), surface pressure (*center*) and shear stresses (*right*)

the direction j. The stress components for which $i \neq j$ are shear stress components, while the stress components normal to the surface ($i = j$) are associated with the *thermodynamic pressure*. The thermodynamic pressure can be perceived as the force exerted on the control volume walls as fluid molecules coincide with it during their random movement. The latter becomes important only for such effects where fluid compressibility is essential. Examples would include shock waves and sound propagation. For incompressible flows the thermodynamic pressure is zero.

If we denote the body forces per unit volume by $\rho \boldsymbol{f}$ we can represent the total body force by:

$$\text{body force} = \iiint_{\mathscr{V}} \rho \boldsymbol{f} \, \mathrm{d}\mathscr{V} \tag{2.115}$$

The most common body force per unit volume is the gravitational force for which $\rho \boldsymbol{f} = \rho \boldsymbol{g}$. The surface force due to pressure always acts perpendicular to the surface of the control volume:

$$\text{surface force due to pressure} = -\oiint_{S} p \mathbf{d}\boldsymbol{S} \tag{2.116}$$

Note that the pressure force is negative, since the pressure acts in the opposite direction to the surface normal vector, $\boldsymbol{n}$. Finally, the surface force due to friction can be written as follows:

$$\text{viscous surface force} = \iint_{S} \boldsymbol{\tau}_{ij} \cdot \mathbf{d}\boldsymbol{S} \tag{2.117}$$

where $\boldsymbol{\tau}_{ij}$ is the viscous shear stress tensor. In three dimensions this tensor is expressed as follows:

$$\boldsymbol{\tau}_{ij} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \tag{2.118}$$

A tensor can be perceived as a multi-dimensional vector. The inner product of a tensor with a vector therefore results in a vector, rather than a scalar as we noted in Sect. 2.3. In three dimensions, this tensor consists of three vectors, which form its columns. The inner product should be applied to each of these vectors. The entries of the resulting vector correspond to the scalar result of each of these operations. We can combine each of the force components and form the LHS of (2.114):

$$\boldsymbol{F} = \iiint_{\mathscr{V}} \rho \boldsymbol{f} \, \mathrm{d}\mathscr{V} - \oiint_{S} p \mathbf{d}\boldsymbol{S} + \oiint_{S} \boldsymbol{\tau}_{ij} \cdot \mathbf{d}\boldsymbol{S} \tag{2.119}$$

Now, let us turn our attention to the RHS of (2.114) and look at the time rate of change of linear momentum. Similar to the components in the continuity equation we distinguish two contributions to the time rate of change of momentum: first due to the momentum change with time of the fluid inside the control volume and second

due to momentum entering and leaving the control volume. These components can be defined as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{V}) = \frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho \boldsymbol{V}\mathrm{d}\mathscr{V} + \oiint_{S}(\rho \boldsymbol{V}\cdot\mathbf{d}\boldsymbol{S})\boldsymbol{V} \tag{2.120}$$

We can now combine (2.119) and (2.120) to obtain a first version of the fluid-flow momentum balance in integral form:

$$\frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho \boldsymbol{V}\mathrm{d}\mathscr{V} + \oiint_{S}(\rho \boldsymbol{V}\cdot\mathbf{d}\boldsymbol{S})\boldsymbol{V} = \iiint_{\mathscr{V}} \rho \boldsymbol{f}\mathrm{d}\mathscr{V} - \oiint_{S} p\mathbf{d}\boldsymbol{S} + \oiint_{S} \boldsymbol{\tau}_{ij}\cdot\mathbf{d}\boldsymbol{S} \tag{2.121}$$

We can expand the shear stress terms in this equation in terms of state variables and fluid constants, but let us first evaluate the differential form of the momentum equation.

By applying the gradient and divergence theorems (Sects. 2.3.2 and 2.3.3, respectively) appropriately to each of the individual terms in (2.121) we can rewrite this equation only in terms of volume integrals, which allows us to evaluate only the integrand. This results in the following differential form of the momentum equation:

$$\frac{\partial}{\partial t}(\rho \boldsymbol{V}) + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{V}\boldsymbol{V}) = \rho \boldsymbol{f} - \boldsymbol{\nabla} p + \boldsymbol{\nabla}\cdot\boldsymbol{\tau}_{ij} \tag{2.122}$$

Note that $\boldsymbol{V}\boldsymbol{V}$ is the outer product of $\boldsymbol{V}$ with itself, leading to the second-order velocity tensor. For Newtonian fluids (stress-rate of strain relation is linear) the relation between the shear stress tensor and velocity the components can be written as follows:

$$\boldsymbol{\tau}_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\mu'\frac{\partial u_k}{\partial x_k} \quad i, j, k = 1, 2, 3 \tag{2.123}$$

where μ is the dynamic viscosity and μ' is second (or bulk) coefficient of viscosity of the fluid. Furthermore, u_i is the velocity component of $\boldsymbol{V}$ in the direction of i and δ_{ij} is the Kronecker delta.[4] Following Stokes' hypothesis:

$$\mu' = -\frac{2}{3}\mu \tag{2.124}$$

Therefore, the viscous stress tensor, $\boldsymbol{\tau}_{ij}$, can be written as:

$$\boldsymbol{\tau}_{ij} = \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] \tag{2.125}$$

[4] The Kronecker delta is defined as follows: $\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$.

By expanding the LHS of (2.122) using the chain rule and subsequent substitution of the continuity equation (see Problem 2.25), we can write the momentum equation in substantial-derivative notation according to:

$$\rho \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} = \rho \boldsymbol{f} - \nabla p + \nabla \cdot \boldsymbol{\tau}_{ij} \tag{2.126}$$

Subsequent substitution of (2.123) yields the following set of momentum equations:

$$\rho \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} = \rho \boldsymbol{f} - \nabla p + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\mu\frac{\partial u_k}{\partial x_k}\right] \tag{2.127}$$

We emphasize that (2.127) is a set of three second-order partial-differential equations (if three-dimensions are considered). These equations are often referred to as the Navier-Stokes equation named after Claude-Louis Navier and George Gabriel Stokes.

2.5.3 Conservation of Energy

The first law of thermodynamics (2.83) forms the basis for the energy balance between a fluid going through a fixed control volume and its surroundings:

$$\mathrm{d}e + \delta w = \delta Q \tag{2.83}$$

Note that (2.83) is the balance between the rate of change in internal (or specific) energy with the sum of the rate of specific heat added to the fluid and the rate of specific work done by the fluid. In order to express the first law in energy per unit volume we need to multiply (2.83) by the density:

$$\rho \mathrm{d}e + \rho \delta w = \rho \delta Q \tag{2.128}$$

Let us expand each of those terms in terms of state variables of the fluid. We start with the LHS of (2.128), which is the change of internal energy. Remember, that this law is written for a gas in stationary condition. Since we are considering a moving fluid, the kinetic energy per unit mass is to be added. If we denote the total energy per unit volume with E_t we have:

$$E_t = \rho\left(e + \frac{V^2}{2}\right) \tag{2.129}$$

Since the subject of this text is aerodynamics, we have intentionally omitted a potential energy term in (2.129). The time rate of change of total energy in the control volume can now be written as:

$$\text{time rate of change of total energy inside control volume} = \frac{\partial}{\partial t} \iiint_{\mathscr{V}} E_t \mathrm{d}\mathscr{V} \tag{2.130}$$

Since fluid is entering and leaving the control volume, the kinetic and internal energy changes. Each particle that leaves the control volume has a volume-specific energy density of E_t. The total rate of volume-specific total energy leaving the control volume therefore equals the volumetric flux over the boundary $\mathbf{d}\boldsymbol{S}$ times E_t: $E_t \boldsymbol{V} \cdot \mathbf{d}\boldsymbol{S}$. Integrating over the entire control surface yields the following:

$$\text{time rate of energy transfer across control surface} = \iint_S E_t \boldsymbol{V} \cdot \mathbf{d}\boldsymbol{S} \tag{2.131}$$

Now, let us turn our attention to the heat term in (2.128). First, we consider the time rate of change of heat inside the control volume. Similar to the energy flux, we define two components: the time rate of change of heat inside the control volume and the heat flow going in and out of the volume via the control surface. We switch in nomenclature from specific heat (δQ, unit: J/kg) to heat per unit volume ($\rho \delta Q$, unit: J/m^3). The time rate of change of heat inside the control surface is given by:

$$\text{time rate of change of heat inside control volume} = \frac{\partial}{\partial t} \iiint_{\mathscr{V}} \rho Q \mathrm{d}\mathscr{V} \tag{2.132}$$

By using Fourier's law of heat conduction (see also (2.28) in Sect. 2.2.2), the heat flow ($\boldsymbol{q}$, unit: J/s/m^2) is linearly related to the temperature gradient:

$$\boldsymbol{q} = -k \nabla T \tag{2.133}$$

where k is the coefficient of thermal conductivity and T is the temperature of the gas. The heat flow into the control volume is in opposite direction to the control surface vector $\mathbf{d}\boldsymbol{S}$. Therefore, the rate of heat addition is negatively related to the heat flux over the control surface $\mathbf{d}\boldsymbol{S}$:

$$\text{heat flow across control surface} = -\iint_S \boldsymbol{q} \cdot \mathbf{d}\boldsymbol{S} \tag{2.134}$$

Having examined the energy and heat flux in the control volume, the last part of (2.128) is the rate of work being done on the fluid in the control volume. We consider three different contributions that are related to the three different forces we defined in Sect. 2.5.2. First we have the time rate of change in work due to the body forces. We know that work can be defined as the force times a displacement. Hence, the time rate of change of work can be expressed as the force multiplied by the velocity. Therefore, we have the following:

$$\text{time rate of change of work due to body forces} = \iiint_{\mathscr{V}} \rho \boldsymbol{f} \cdot \boldsymbol{V} \mathrm{d}\mathscr{V} \tag{2.135}$$

Secondly, we have the pressure on the control surface that does work on the fluid inside the control surface. We take the inner product of the RHS of (2.116) with $\boldsymbol{V}$ to obtain the work flux due to this component:

$$\text{time rate of work done due to pressure on the control surface} = -\iint_S p\mathbf{d}\boldsymbol{S}\cdot\boldsymbol{V} \tag{2.136}$$

Finally, we look at the rate of work done due to the shear stress on the control surface. Similar to the pressure-induced work flux, the work flux due to the shear stress can be found by taking the inner product of the RHS of (2.117) with $\boldsymbol{V}$:

$$\text{time rate of work done due to shear stress on the control surface} = \iint_S (\boldsymbol{\tau}_{ij}\cdot\mathbf{d}\boldsymbol{S})\cdot\boldsymbol{V} \tag{2.137}$$

Remember that $\boldsymbol{\tau}_{ij}$ is the stress tensor as defined in (2.118).

We have now defined the time derivative of each of the components of (2.128) and we can formulate the first version of the energy balance between the control volume and its surroundings:

$$\frac{\partial}{\partial t}\iiint_{\mathscr{V}} E_t \mathrm{d}\mathscr{V} + \iint_S E_t\boldsymbol{V}\cdot\mathbf{d}\boldsymbol{S} = \frac{\partial}{\partial t}\iiint_{\mathscr{V}}\rho Q\mathrm{d}\mathscr{V} - \iint_S \boldsymbol{q}\cdot\mathbf{d}\boldsymbol{S} + \iiint_{\mathscr{V}}\rho\boldsymbol{f}\cdot\boldsymbol{V}\mathrm{d}\mathscr{V} - \iint_S p\mathbf{d}\boldsymbol{S}\cdot\boldsymbol{V} + \iint_S(\boldsymbol{\tau}_{ij}\cdot\mathbf{d}\boldsymbol{S})\cdot\boldsymbol{V} \tag{2.138}$$

We can cast (2.138) in differential form by applying the gradient and divergence theorems to the appropriate terms and changing everything to volumetric integral formulation. The integrand of that equation yields the following energy balance:

$$\frac{\partial E_t}{\partial t} + \nabla\cdot E_t\boldsymbol{V} = \frac{\partial(\rho Q)}{\partial t} - \nabla\cdot\boldsymbol{q} + \rho\boldsymbol{f}\cdot\boldsymbol{V} - \nabla(p\cdot\boldsymbol{V}) + \nabla\cdot(\boldsymbol{\tau}_{ij}\cdot\boldsymbol{V}) \tag{2.139}$$

The first term on the LHS of (2.139) represents the rate of increase of E_t in the control volume, while the second term represents the total energy lost due to convection through the control volume. The first term on the RHS is the amount of heat produced by external factors. The second term is the rate of heat lost by means of conduction. The third term represents the work done by body forces while the last two terms represent the work done by normal and shear stresses on the surface, respectively.

By subsequently employing the continuity and momentum equation (see Problem 2.26) we can write the energy equation in a convenient substantial-derivative form:

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} + p(\nabla\cdot\boldsymbol{V}) = \frac{\partial(\rho Q)}{\partial t} - \nabla\cdot\boldsymbol{q} + \nabla\cdot(\boldsymbol{\tau}_{ij}\cdot\boldsymbol{V}) - (\nabla\cdot\boldsymbol{\tau}_{ij})\cdot\boldsymbol{V} \tag{2.140}$$

The sum of last two terms in this equation is termed the *dissipation function*, Φ, and represent the rate at which mechanical energy is expended due to viscosity when

the fluid is deformed. The LHS of (2.140) can be rewritten in terms of enthalpy by employing (2.73) and the continuity equation (see Problem 2.27). The following energy balance results:

$$\rho \frac{\mathrm{D}h}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} + \frac{\partial(\rho Q)}{\partial t} - \nabla \cdot \boldsymbol{q} + \Phi \tag{2.141}$$

where the dissipation function is:

$$\Phi = \nabla \cdot (\boldsymbol{\tau}_{ij} \cdot \boldsymbol{V}) - (\nabla \cdot \boldsymbol{\tau}_{ij}) \cdot \boldsymbol{V} \tag{2.142}$$

From a mathematical point of view we classify (2.141) as a second order partial differential equation. Note that the second order terms appear only in the dissipation function (see Problem 2.27).

If we start from (2.139) and assume isentropic conditions (inviscid: $\mu = 0$ and adiabatic: $\mathrm{d}(\rho Q)/\mathrm{d}t = 0$ and $k = 0$), we can write the energy equation as follows:

$$\rho \frac{\mathrm{D}(e + V^2/2)}{\mathrm{D}t} = -\nabla(p \cdot \boldsymbol{V}) \tag{2.143}$$

Here, we have assumed the body forces, $\boldsymbol{f}$, to be negligible. Using the continuity equation (2.113), we can rewrite this equation as follows:

$$\rho \frac{\mathrm{D}(h + V^2/2)}{\mathrm{D}t} = -\frac{\partial p}{\partial t} \tag{2.144}$$

In steady conditions, we can therefore derive that:

$$h + \frac{V^2}{2} = H = \text{constant} \tag{2.145}$$

where H is the definition of the total enthalpy. If we use the relation between enthalpy and static temperature (2.77), we can write:

$$c_p T + \frac{V^2}{2} = c_p T_t = \text{constant} \tag{2.146}$$

where T_t is the total temperature. For a flow where all the streamlines emanate from the same uniform freestream, the temperature and velocity are therefore uniquely correlated through (2.146). We can manipulate (2.146) and include the definition of the speed of sound (2.106) to obtain:

$$\frac{T_t}{T} = 1 + \frac{\gamma - 1}{2} M^2 \tag{2.147}$$

Equation (2.147) states that there exists a unique relationship between the Mach number and the static temperature in the flow. For a given total temperature, T_t, in the freestream, Eq. (2.147) shows that the static temperature decreases with Mach number. If we combine (2.147) with the isentropic relations (2.103) we can express the temperature, density and pressure as a function of the Mach number. We have tabulated this relation for Mach numbers ranging from 0 through 10 in Appendix A.

2.5.4 Conservation Form of the Navier-Stokes Equations

We now make two assumptions about the nature of the flow we consider. The first assumption is that the body forces (such as gravity) are so small compared to the surface forces that it is unnecessary to account for them during calculations. In addition, we assume that there is no change in heat per unit volume due to volumetric heating. Accordingly, we declare that:

$$\boldsymbol{f} = 0 \qquad \frac{\mathrm{d}(\rho Q)}{\mathrm{d}t} = 0 \tag{2.148}$$

Let us rewrite the three equations that represent the conservation of mass (2.110), momentum (2.122) and energy (2.139), respectively, and apply the assumptions of (2.148):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0 \tag{2.110}$$

$$\frac{\partial}{\partial t}(\rho \boldsymbol{V}) + \nabla \cdot (\rho \boldsymbol{V}\boldsymbol{V}) + \nabla p - \nabla \cdot \boldsymbol{\tau}_{ij} = 0 \tag{2.149}$$

$$\frac{\partial E_t}{\partial t} + \nabla \cdot E_t \boldsymbol{V} + \nabla \cdot \boldsymbol{q} + \nabla p \cdot \boldsymbol{V} - \nabla \cdot (\boldsymbol{\tau}_{ij} \cdot \boldsymbol{V}) = 0 \tag{2.150}$$

This shortened form permits us to explicitly rewrite all five equations in their conservation form:

$$\frac{\partial \rho}{\partial t} + \frac{(\partial \rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{2.151a}$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} + \frac{\partial (\rho uw)}{\partial z} + \frac{\partial p}{\partial x} - \left(\frac{\tau_{xx}}{\partial x} + \frac{\tau_{xy}}{\partial y} + \frac{\tau_{xz}}{\partial z} \right) = 0 \tag{2.151b}$$

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho uv)}{\partial x} + \frac{\partial (\rho v^2)}{\partial y} + \frac{\partial (\rho vw)}{\partial z} + \frac{\partial p}{\partial y} - \left(\frac{\tau_{xy}}{\partial x} + \frac{\tau_{yy}}{\partial y} + \frac{\tau_{yz}}{\partial z} \right) = 0 \tag{2.151c}$$

$$\frac{\partial (\rho w)}{\partial t} + \frac{\partial (\rho uw)}{\partial x} + \frac{\partial (\rho vw)}{\partial y} + \frac{\partial (\rho w^2)}{\partial z} + \frac{\partial p}{\partial z} - \left(\frac{\tau_{xz}}{\partial x} + \frac{\tau_{yz}}{\partial y} + \frac{\tau_{zz}}{\partial z} \right) = 0 \tag{2.151d}$$

$$\frac{\partial E_t}{\partial t}+\frac{\partial(uE_t)}{\partial x}+\frac{\partial(vE_t)}{\partial y}+\frac{\partial(wE_t)}{\partial z}+\frac{\partial(up)}{\partial x}+\frac{\partial(vp)}{\partial y}+\frac{\partial(wp)}{\partial z}$$
$$+\left[\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right]-\left[\frac{\partial}{\partial x}(u\tau_{xx}+v\tau_{xy}+w\tau_{xz})\right.$$
$$\left.+\frac{\partial}{\partial y}(u\tau_{xy}+v\tau_{yy}+w\tau_{yz})+\frac{\partial}{\partial z}(u\tau_{xz}+v\tau_{yz}+w\tau_{zz})\right]=0 \tag{2.151e}$$

The Navier-Stokes equations consists of a time-dependent continuity equation for conservation of mass, three time-dependent conservation of momentum equations and a time-dependent conservation of energy equation. There are four independent variables in the problem: x, y, z and time t. There are six dependent variables; the pressure p, density ρ, and temperature T (which is contained in the energy equation through the total energy E_t) and three components of the velocity vector (u, v, w). All of the dependent variables are functions of all four independent variables. The differential equations are therefore a set of partial differential equations. We can reduce the number of dependent variables to five by adding two algebraic equations to close this system, namely (2.72) and (2.76) that together relate pressure, p, to the state variables ρ and e (see Problem 2.18). We are then left with five partial differential equations and five unknowns: u, v, w, ρ, and e.

It can be convenient to combine these equations in vector form such that they become more compact. By doing so, we can formulate this set of equations in the following vector form (see Problem 2.28):

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}+\frac{\partial \mathbf{F}}{\partial y}+\frac{\partial \mathbf{G}}{\partial z}=0 \tag{2.152}$$

where:

$$\mathbf{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ E_t\end{bmatrix} \qquad \mathbf{E}=\begin{bmatrix}\rho u\\ \rho u^2+p-\tau_{xx}\\ \rho uv-\tau_{xy}\\ \rho uw-\tau_{xz}\\ (E_t+p)u-u\tau_{xx}-v\tau_{xy}-w\tau_{xz}+q_x\end{bmatrix}$$

$$\mathbf{F}=\begin{bmatrix}\rho v\\ \rho uv+p-\tau_{xy}\\ \rho v^2+p-\tau_{yy}\\ \rho vw-\tau_{yz}\\ (E_t+p)v-u\tau_{xy}-v\tau_{yy}-w\tau_{yz}+q_y\end{bmatrix} \qquad \mathbf{G}=\begin{bmatrix}\rho w\\ \rho uw-\tau_{xz}\\ \rho vw-\tau_{yz}\\ \rho w^2+p-\tau_{yz}\\ (E_t+p)w-u\tau_{xz}-v\tau_{yz}-w\tau_{zz}+q_z\end{bmatrix} \tag{2.153}$$

The first and last row in these vectors represent the continuity and energy equation, respectively. The three middle rows represent the three components of the momentum equation. The form above is often used because it is easier to code in numerical form. It still represents the full equations of motion. Note that this set in the subsequent text is referred to as the Navier-Stokes (NS) equations.

2.6 Reynolds-Averaged Navier-Stokes Equations

The Navier-Stokes equations can be solved numerically by using direct numerical simulation (DNS). This means that the whole range of spatial and temporal scales of the turbulence must be resolved. All the spatial scales of the turbulence must be resolved in the computational mesh, from the smallest dissipative scales, up to the integral scale L, associated with the motions containing most of the kinetic energy. This requires to have a high spatial density of the volumetric mesh in combination with a small time step. It can be shown that the number of floating-point operations required to complete this simulation is proportional to the number of mesh points and the number of time steps. The number of operations grows with the third power of the Reynolds number: Re_L^3 [4]. For practical engineering problems, the number of required operations would exceed the maximum number of the most powerful computers that are currently available. To increase the minimum spatial and temporal scales and hence reduce the number of floating point operations, the Reynolds-averaged Navier-Stokes equations (RANS) can be used. In this section we show how the Reynolds-averaged Navier-Stokes equations can be obtained. Because RANS equations relate the turbulence in the flow to the mean flow properties , we introduce new dependent variables. Therefore, additional equations are necessary to "close" the RANS equations. Many turbulence models have been developed to close the RANS equations. As an example, we present the so-called k-epsilon model, which is often used in engineering practice.

2.6.1 Incompressible Reynolds-Averaged Equations of Motion

Conventional Reynolds-averaging is done by splitting the flow parameters into a time-averaged part and a fluctuating part. This is called *Reynolds decomposition* [15]:

$$u = \bar{u} + u' \qquad v = \bar{v} + v' \qquad w = \bar{w} + w' \qquad \rho = \bar{\rho} + \rho' \tag{2.154}$$

$$p = \bar{p} + p' \qquad h = \bar{h} + h' \qquad T = \bar{T} + T' \qquad H = \bar{H} + H' \tag{2.155}$$

Here, the total enthalpy is defined by $H = h + u_i u_i/2$ and the barred parameters represent averages. The fluctuation terms that appear in the equations above become zero when they are time-averaged. For example:

$$\overline{u'} = \frac{1}{\Delta t} \int_0^{\Delta t} u' dt = 0 \tag{2.156}$$

The equations of motion of Sect. 2.5 are modified by substitution of the flow parameters that we have defined in (2.155). Subsequently, each of the equations is time averaged. The result yields a set of equations that have averaged fluctuating terms. Each of the terms in the equations of motion that have averaged fluctuating terms

are subsequently set to zero. This yields a more compact version of these equations, including the fluctuating terms. We briefly discuss each of the governing equations below.

When substituting the relations of (2.154) in the continuity equation (2.110) the following averaged equation can be obtained:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u} + \overline{\rho' u'_j}\right) = 0 \tag{2.157}$$

For incompressible flow, the fluctuating and unsteady density terms can be dropped from this equation. Furthermore, $\bar{\rho} = \rho =$ constant. What we are left with is a much reduced continuity equation:

$$\frac{\partial \bar{u}_j}{\partial x_j} = 0 \tag{2.158}$$

Starting from the Navier-Stokes equation (2.127) and neglecting body forces, the Reynolds-averaged momentum equation can be obtained by substitution of the relations (2.154) and applying subsequent time averaging:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\bar{\rho}\bar{u}_i + \overline{\rho' u'_j}\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_i\bar{u}_j + \bar{u}_i\overline{\rho' u'_j}\right) \\ &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \bar{u}_j\overline{\rho' u'_i} - \bar{\rho}_j\overline{u'_i u'_j} - \overline{\rho' u'_i u'_j}\right) \end{aligned} \tag{2.159}$$

where

$$\bar{\tau}_{ij} = \mu\left[\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \bar{u}_k}{\partial x_k}\right] \tag{2.160}$$

For incompressible flow, this can be simplified to:

$$\frac{\partial}{\partial t}\left(\rho\bar{u}_i\right) + \frac{\partial}{\partial x_j}\left(\rho\bar{u}_i\bar{u}_j\right) = -\frac{\partial \bar{p}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \rho\overline{u'_i u'_j}\right) \tag{2.161}$$

As can be seen from (2.161) the Reynolds-Averaged Navier-Stokes (RANS) equation exhibits additional terms with respect to the original NS equations. These fluctuating terms need to be related to the average flow parameters in order to close the RANS equations. Since the fluctuating terms are in the same bracket as the averaged stress terms they are often termed Reynolds stresses. Closure of the Reynolds-averaged equations via turbulence models is the topic of Sect. 2.6.3. To interpret the equations properly, consider Eq. (2.161) in substantial-derivative form:

$$\rho\frac{D\bar{u}_i}{Dt} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial\left(\bar{\tau}_{ij}\right)_{\text{lam}}}{\partial x_j} + \frac{\partial\left(\bar{\tau}_{ij}\right)_{\text{turb}}}{\partial x_j} \tag{2.162}$$

Particle acceleration of mean motion | Mean pressure gradients | Laminar-like stress gradient for the mean motion | Apparent stress gradients due to transport of momentum by turbulent fluctuations

This shows how the Reynolds stresses appear as an addition to the mean flow parameters that are very much comparable to the original NS equations. The Reynolds-averaged stress is related to the fluctuating velocities according to:

$$\left(\bar{\tau}_{ij}\right)_{\text{turb}} = -\rho\overline{u_i' u_j'} \tag{2.163}$$

Since $\overline{u_i' u_j'} = \overline{u_j' u_i'}$, the Reynolds stress tensor, $\bar{\tau}_{ij}$ is symmetric. Considering three dimensions, it consists of six new variables. Unfortunately, we do not have six additional equations. We therefore require additional equations to close this system.

The Reynolds-averaged energy equation can be found by substitution of the relations (2.154) in (2.140):

$$\begin{aligned}\frac{\partial}{\partial t}\left(\bar{\rho}\bar{H}\overline{\rho' H'}\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_j\bar{H} + \bar{\rho}\overline{u_j' H'} + \overline{\rho' u_j'}\bar{H} + \overline{\rho' u_j' H'} + \bar{u}_j'\overline{\rho' H'} - k\frac{\partial \bar{T}}{\partial x_j}\right) \\ = \frac{\partial \bar{p}}{\partial t} + \frac{\partial}{\partial t}\left[\bar{u}_i\left(-\frac{2}{3}\mu\delta_{ij}\frac{\partial \bar{u}_k}{\partial x_k}\right) + \mu\bar{u}_i\left(\frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j}\right)\right] \\ -\frac{\partial}{\partial t}\left[\frac{2}{3}\mu\delta_{ij}\overline{u_i'\frac{\partial \bar{u}_k'}{\partial x_k}} + \mu\left(\overline{u_i'\frac{\partial u_j'}{\partial x_i}} + \overline{u_i'\frac{\partial u_i'}{\partial x_j}}\right)\right]\end{aligned} \tag{2.164}$$

For incompressible flows the energy equation can be written as follows:

$$\begin{aligned}\frac{\partial \rho\bar{H}}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j\bar{H} + \rho\overline{u_j' H'} - k\frac{\partial \bar{T}}{\partial x_j}\right) \\ = \frac{\partial \bar{p}}{\partial t} + \frac{\partial}{\partial x_j}\left[\mu\bar{u}_i\left(\frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j}\right) + \mu\left(\overline{u_i'\frac{\partial u_j'}{\partial x_i}} + \overline{u_i'\frac{\partial u_i'}{\partial x_j}}\right)\right]\end{aligned} \tag{2.165}$$

Since the last bracketed term within the square brackets on the RHS of (2.165) is often small compared to the other terms within the square brackets, it is usually neglected. What remains is an energy equation comparable to the original energy equation (2.140) except for the fluctuating term $\rho\overline{u_j' H'}$. Since this term shares its brackets with heat flux terms it is often termed the Reynolds heat flux. A similar analysis can be done for the energy equation where the Reynolds heat flux component is:

$$-\left(\nabla\cdot\boldsymbol{q}\right)_{\text{turb}} = \frac{\partial}{\partial x_j}\left(-\bar{\rho}c_p\overline{T' u_j'} - c_p\overline{\rho' T' u_j'} - \bar{u}_j c_p\overline{\rho' T'}\right) \tag{2.166}$$

The Reynolds-averaged equations form a set of five partial differential equations with more than five unknowns. As was briefly explained, additional fluctuating terms need to be related to mean flow properties via appropriate turbulence models. We discuss a typical turbulence model in Sect. 2.6.3.

2.6.2 Compressible Reynolds-Averaged Equations of Motion

The Reynolds-averaged equations in the previous section assume a constant density of the fluid. In Mach numbers beyond 0.3 it can be shown that compressibility effects become important in determining the correct flow parameters [3]. This section shows how time-averaging along with mass-weighting of the equations of motion results in an equivalent set of equations that describe an equivalent compressible flow.

In a Reynolds averaged compressible flow, it is convenient to apply a mass-weighted averaging in addition to the time averaging.Mass-averaging the flow parameters can be done by multiplying the parameters by the density, averaging this product and dividing by the average density:

$$\tilde{u} = \frac{\overline{\rho u}}{\bar{\rho}} \quad \tilde{v} = \frac{\overline{\rho v}}{\bar{\rho}} \quad \tilde{w} = \frac{\overline{\rho w}}{\bar{\rho}} \quad \tilde{h} = \frac{\overline{\rho h}}{\bar{\rho}} \quad \tilde{T} = \frac{\overline{\rho T}}{\bar{\rho}} \quad \tilde{H} = \frac{\overline{\rho H}}{\bar{\rho}} \tag{2.167}$$

The following fluctuating quantities are defined:

$$\begin{aligned} u &= \tilde{u} + u'' \quad & v &= \tilde{v} + v'' \quad & w &= \tilde{w} + w'' \quad & \rho &= \bar{\rho} + \rho' \\ p &= \bar{p} + p' \quad & h &= \tilde{h} + h'' \quad & T &= \tilde{T} + T'' \quad & H &= \tilde{H} + H'' \end{aligned} \tag{2.168}$$

where the primed parameters stand for time dependent deviations from the average which are zero when integrated over time (i.e. $\overline{u'} = \frac{1}{\Delta t}\int_0^{\Delta t} u' dt = 0$). The averages of the doubly primed fluctuating quantities are not zero. Instead the time average of the doubly primed fluctuations multiplied by the density equals zero (i.e. $\overline{u''} = \frac{1}{\Delta t}\int_0^{\Delta t} \rho u'' dt = 0$).

We substitute the averaged flow parameters (2.168) in the equations of motion of Sect. 2.5. Substituting the averaged flow parameters into the continuity equation (Eq. 2.110) and subsequently time averaging the equation yields:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_j\right) = 0 \quad j = 1, 2, 3 \tag{2.169}$$

To develop the Reynolds momentum equation in mass-weighed variables, the flow parameters defined in (2.168) are substituted in Eq. (2.127). After averaging and elimination of terms that are zero, the complete Reynolds momentum equation in mass-weighted variables becomes:

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{u}_j\right)+\frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_i\tilde{u}_j\right)=-\frac{\partial \bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij}-\overline{\rho u_i'' u_j''}\right)\quad i,j=1,2,3 \tag{2.170}$$

where, neglecting viscosity fluctuations, $\bar{\tau}_{ij}$ is as follows:

$$\bar{\tau}_{ij}=\mu\left[\left(\frac{\partial \tilde{u}_i}{\partial x_i}+\frac{\partial \tilde{u}_j}{\partial x_j}\right)-\frac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}\right]+\mu\left[\left(\frac{\partial \overline{u''}_i}{\partial x_i}+\frac{\partial \overline{u''}_j}{\partial x_j}\right)-\frac{2}{3}\delta_{ij}\frac{\partial \overline{u''}_k}{\partial x_k}\right] \tag{2.171}$$

A brief look at the expression reveals a more complex expression than the original, non-averaged expression of Eq. (2.125). In practice, however, the viscous terms with doubly primed fluctuations are likely candidates for being neglected based on their magnitude compared to the mass-averaged variables. By substituting the Reynolds-averaged continuity equation (2.169) in the Reynolds-averaged momentum equation (2.170) and employing the substantial derivative yields the following [19]:

$$\underbrace{\bar{\rho}\frac{D\tilde{u}_i}{Dt}}_{\text{Particle acceleration of mean motion}}=\underbrace{-\frac{\partial \bar{p}}{\partial x_i}}_{\text{Mean pressure gradients}}+\underbrace{\frac{\partial\left(\bar{\tau}_{ij}\right)_{\text{lam}}}{\partial x_j}}_{\substack{\text{Laminar-like}\\ \text{stress gradient}\\ \text{for the mean motion}}}+\underbrace{\frac{\partial\left(\bar{\tau}_{ij}\right)_{\text{turb}}}{\partial x_j}}_{\substack{\text{Apparent stress gradients due to transport of momentum by}\\ \text{turbulent fluctuations and}\\ \text{deformations attributed to fluctuations}}} \tag{2.172}$$

Note that the Reynolds-averaged momentum equation (above) has the same form as the original Navier-Stokes equation, Eq. (2.127), with the addition of a turbulent stress term. Explicitly, the laminar and turbulent stress terms are given by:

$$\left(\bar{\tau}_{ij}\right)_{\text{lam}}=\mu\left[\left(\frac{\partial \tilde{u}_i}{\partial x_i}+\frac{\partial \tilde{u}_j}{\partial x_j}\right)-\frac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}\right] \tag{2.173}$$

$$\left(\bar{\tau}_{ij}\right)_{\text{turb}}=-\overline{\rho u_i'' u_j''}+\mu\left[\left(\frac{\partial \overline{u''}_i}{\partial x_i}+\frac{\partial \overline{u''}_j}{\partial x_j}\right)-\frac{2}{3}\delta_{ij}\frac{\partial \overline{u''}_k}{\partial x_k}\right] \tag{2.174}$$

To arrive at the Reynolds form of the energy equation, the mass-weighted variables (2.168) are substituted in (2.140). Subsequent elimination of terms that go to zero yields the following Reynolds energy equation in mass-weighed variables:

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{H}\right)+\frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_j\tilde{H}+\overline{\rho u_j'' H_j''}-k\frac{\partial \bar{T}}{\partial x_j}\right)=\frac{\partial \bar{p}}{\partial t}+\frac{\partial}{\partial x_j}\left(\tilde{u}_i\tilde{\tau}_{ij}+\overline{u_i''\tau_{ij}}\right) \tag{2.175}$$

A similar analysis can be performed on the energy equation. Apart from the laminar and turbulent stress terms a laminar and turbulent heat flux term can be defined. The new apparent turbulent stresses and flux terms that appear in both the Reynolds-averaged momentum and energy equation should be treated as new variables. Therefore, additional equations are required that make assumptions regarding the apparent turbulent quantities and the mean flow variables.

2.6.3 Turbulence Modeling: The k-Epsilon Model

The fluctuation terms that are introduced in the Reynolds-averaged equations of motion need to be related to the average flow values. Several models have been introduced that couple those two parameters. Ranging from simple algebraic models to more rigorous models, each of these approaches attempts to describe the equivalent behavior of an essentially random flow. The only way to verify whether a turbulence model is effective, is by experimental verification. This section presents one of the many turbulence models that are used in practice, the k-epsilon model.

The Reynolds stress in incompressible flow amounts to $-\overline{\rho u_i' u_j'}$, see (2.163). For compressible flow this is $-\overline{\rho u_i'' u_j''}$ when the molecular viscosity (the second term of (2.174)) is neglected. The *Boussinesq assumption* relates the Reynolds stress to the mean-flow parameters according to:

$$-\overline{\rho u_i' u_j'} = 2\mu_T S_{ij} - \frac{2}{3}\delta_{ij}\left(\mu_T \frac{\partial \bar{u}_k}{\partial x_k} + \rho \bar{k}\right) \tag{2.176}$$

where μ_T is the turbulent viscosity, $\bar{k}$ is the kinetic energy of turbulence: $\bar{k} = \overline{u_i' u_i'}/2$, and the rate of the mean strain tensor S_{ij} given by:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{2.177}$$

To predict the value of the turbulent kinetic energy, $\bar{k}$, a transport equation is developed from the Reynolds-averaged Navier-Stokes equations. Using Boussinesq's assumption for eddy viscosity (2.176) this transport equation reads (in substantial-derivative form):

$$\rho \frac{D\bar{k}}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_T}{\mathrm{Pr}_k}\right)\frac{\partial \bar{k}}{\partial x_j}\right] + \left(2\mu_T S_{ij} - \frac{2}{3}\rho \bar{k}\delta_{ij}\right)\frac{\partial u_i}{\partial x_j} - C_D \rho \frac{\bar{k}^{3/2}}{l} \tag{2.178}$$

A derivation of this equation is beyond the scope of this text but can be found in texts on turbulence modeling (i.e. [22] or [6]). In (2.178) the Prandtl number for turbulent kinetic energy (Pr_k) appears as a closure constant ($\mathrm{Pr}_k = 1.0$). C_D is the dissipation coefficient and has been experimentally shown to be $C_D = 0.164$. The term on the LHS of this equation represents the particle rate of increase of $\bar{k}$. The terms on the RHS of Eq. (2.178) represent the diffusion rate, the generation rate and the dissipation rate of $\bar{k}$, respectively. Here l is a characteristic length, referred to as the *mixing length*. The mixing length can be interpreted as follows: a fluid parcel will conserve its properties for a characteristic length, l, before mixingwith the

surrounding fluid. Prandtl [14] described that the mixing length "may be considered as the diameter of the masses of fluid moving as a whole in each individual case; or again, as the distance traversed by a mass of this type before it becomes blended in with neighboring masses…"

The dissipation rate of $\bar{k}$ which is embedded in the last term in (2.178) is often represented with ε: $\varepsilon = C_D \frac{\bar{k}^{3/2}}{l}$. Similar to the analysis of $\bar{k}$, a transport equation developed from the Reynolds-averaged Navier-Stokes equations can be established for ε:

$$\rho \frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_T}{\mathrm{Pr}_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + C_{\varepsilon 1}\frac{\varepsilon}{\bar{k}}\left(2\mu_T S_{ij} - \frac{2}{3}\rho\bar{k}\delta_{ij}\right)\frac{\partial u_i}{\partial x_j} - C_{\varepsilon 2}\rho\frac{\varepsilon^2}{\bar{k}} \tag{2.179}$$

The term on the LHS of Eq. (2.179) represents the particle rate of increase in dissipation, while the terms on the RHS represent the diffusion, generation and dissipation rates of ε. In terms of $\bar{k}$ and ε the terms in Eqs. (2.178) and (2.179) are as follows:

$$l = C_D\frac{\bar{k}^{3/2}}{\varepsilon} \qquad \mu_T = C_\mu \rho \frac{\bar{k}^2}{\varepsilon} \qquad C_\mu = C_D^{4/3} \tag{2.180}$$

By appropriate substitution, the $\bar{k}$-ε model becomes a set of two coupled partial differential equations with two unknowns: $\bar{k}$ and ε.

Complementary to closure of the Reynolds-averaged momentum equations, the $\bar{k}$-ε closure for the heat flux terms in the energy equation assumes the following apparent turbulent conductivity:

$$\rho\overline{u_j' H'} = \rho c_p \overline{x_j' T'} = -k_T \frac{\partial T}{\partial x_j} \tag{2.181}$$

Analogous to the laminar heat conduction coefficient, k, the turbulent heat conductive heat coefficient is defined according to:

$$k_T = \frac{c_p \mu_T}{\mathrm{Pr}_T} \tag{2.182}$$

where Pr_T is the turbulent Prandtl number that is most commonly takes on the value of $\mathrm{Pr}_T = 0.9$. The combination of (2.181) and (2.182) form the closure for the energy equation. That is, if the last bracketed term in (2.175) is neglected because it is small compared to the other terms within the square brackets. Table 2.1 displays the constants that are to be used in the $\bar{k}$-ε model.

Table 2.1 Model constants for $\bar{k}$-ε model [19]

C_μ	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	Pr_k	Pr_ε	Pr_T
0.09	1.44	1.92	1.0	1.3	0.9

The k-epsilon method that has been presented is tailored towards closure of the incompressible Reynolds-averaged equations of motion. Similar models have been developed for the closure of the Reynolds-averaged, mass-weighted equations of motions that were developed in Sect. 2.6.2. The closure of the compressible Reynolds equations is beyond the scope of this text. Even though in transonic aerodynamics the flow is compressible, the incompressible turbulence models have been proven to give good predictions up to Mach 5 [17]. The $\bar{k}$-ε method for compressible flow is treated extensively by Launder and Spalding [12] and Mohammadi and Prionneau [13].

2.7 Equations of Motion for Inviscid Flows

The Reynolds-averaged Navier-Stokes equations can be further simplified if heat transfer and viscous effects are neglected. This is often done to reduce the number of independent variables. If cast into a numerical scheme to solve these equations, the equations of motion for inviscid flow can be solved must faster. This can be advantageous if the computational domain is large (i.e. the problem is solved on a grid with a large number of nodes) and/or when the analysis has to be frequently repeated (e.g. as part of a design optimization loop). In this section we discuss two reduced forms of the equations of motion: the Euler equations (Sect. 2.7.1) and the potential flow equation (Sect. 2.7.2).

2.7.1 Euler Equations

In this section a reduced model is presented of the complete Navier-Stoke equations. By neglecting the heat-transfer terms as well as the viscous terms, these equations describe the flow of an inviscid, non-conducting fluid. The resulting set of equations is referred to as the Euler equations, although strictly speaking, the name of Euler should only be attached to the inviscid momentum equation. In addition to the aforementioned assumptions, it is also assumed that there is no external heat transfer, such that the $\partial(\rho Q)/\partial t$ term in the energy equation can be dropped.

The continuity equation is neither dependent on the viscosity nor on the heat-transfer coefficient. The continuity equation (2.113) remains therefore unchanged in the inviscid case:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\boldsymbol{\nabla} \cdot \boldsymbol{V}) = 0 \tag{2.113}$$

When the viscous terms in the momentum equation (2.127) are neglected, the following equations result:

$$\rho \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} = -\boldsymbol{\nabla} p \tag{2.183}$$

Neglecting the viscous and heat-transfer terms in the energy equation (2.140) results in the following:

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} + p(\nabla \cdot \boldsymbol{V}) = 0 \tag{2.184}$$

Alternatively, this equation can be written in terms of enthalpy, h, by modifying (2.141):

$$\rho \frac{\mathrm{D}h}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} \tag{2.185}$$

Equations (2.113), (2.183), and (2.184) are generally known as the Euler equations. The compressible Euler equations can be written in conservation form according to (2.152). However, the vector representation is now simpler than for the full NS equations:

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E_t \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (E_t + p)u \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} \rho v \\ \rho uv + p \\ \rho v^2 + p \\ \rho vw \\ (E_t + p)v \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ (E_t + p)w \end{bmatrix} \tag{2.186}$$

It is of interest to take a closer look at the momentum equations (2.183). Because of the cleaner representation of the momentum equations we can say something about the application of the Euler equations in transonic flow. From kinematics, the acceleration of a fluid particle, $D\boldsymbol{V}/Dt$, can be expressed as:

$$\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} = \frac{\partial \boldsymbol{V}}{\partial t} + \nabla \left(\frac{V^2}{2} \right) - \boldsymbol{V} \times \boldsymbol{\zeta} \tag{2.187}$$

where $\boldsymbol{\zeta} = \nabla \times \boldsymbol{V}$ is the vorticity of the flow. Equation (2.187) is referred to as Lagrange's acceleration formula. It can be substituted in (2.183) to give an alternate form of the Euler equations:

$$\frac{\partial \boldsymbol{V}}{\partial t} + \nabla \left(\frac{V^2}{2} \right) - \boldsymbol{V} \times \boldsymbol{\zeta} = -\frac{1}{\rho} \nabla p \tag{2.188}$$

The vorticity can be related to the specific entropy, s, according to Crocco's equation:

$$\frac{\partial \boldsymbol{V}}{\partial t} - \boldsymbol{V} \times \boldsymbol{\zeta} = T \nabla s - \nabla \left(\frac{V^2}{2} \right) \tag{2.189}$$

This equation can be derived from the first and second laws of thermodynamics, (2.83) and (2.94), respectively. Note that the LHS of (2.189) is also present in the Lagrange form of the momentum equation (2.188). A physical interpretation of

Crocco's theorem shows that whenever an enthalpy or entropy gradient is present in the flow, it must be rotational. This is particularly important for transonic flow, since a shockwave often terminates a supersonic portion. From the second law of thermodynamics it is known that there is an entropy discontinuity across the shock-wave, which means the flow cannot be assumed irrotational [2]. This is an important finding because it shows that the Euler equations are the simplest formulation of the equations of motion which still capture the effect of the shock wave on the flow.

In the 1980s the airfoil analysis code ISES was developed by MIT (currently further developed into the multi-element airfoil code MSES). This program solves the two-dimensional Euler equations interacting with the boundary layer equations. As is shown in Fig. 2.11 and from [8], the prediction of the pressure distribution $[C_p = (p - p_\infty)/\frac{1}{2}\rho_\infty V_\infty^2]$ and the two-dimensional drag polar is quite accurate. The sharp pressure gradient on the upper surface at approximately 60 % of the chord length, indicates the presence of a shock wave. If we focus on the airfoil below the pressure distribution, we see that the streamline that separates the inviscid flow from the viscous boundary layer has been drawn. Up to the shock wave this line virtually coincides with the airfoil contour, indicating a very thin boundary layer. However, due to the sharp adverse pressure gradient, the boundary layer thickness grows and the streamline is displaced upwards. The flow outside the boundary layer therefore 'sees' a different body than the airfoil. This added thickness (known as *displacement thickness*) is important to accurately determine the experimentally obtained pressure distribution. Note that in this figure the pressure distributions are shown for the same lift coefficient ($c_l = 0.743$), while the angles of attack are slightly different. In predicting the pressure distribution over an airfoil it has been shown that this often gives a better correlation to experimental results. Producing these results can be done in under a minute of computation time, making this a suitable tool for airfoil design.

To stress the importance of including a boundary layer in the calculations we also show results for flow about a transonic airfoil without the inclusion of the boundary layer. In Fig. 2.12 we see that the pressure distribution has been predicted using

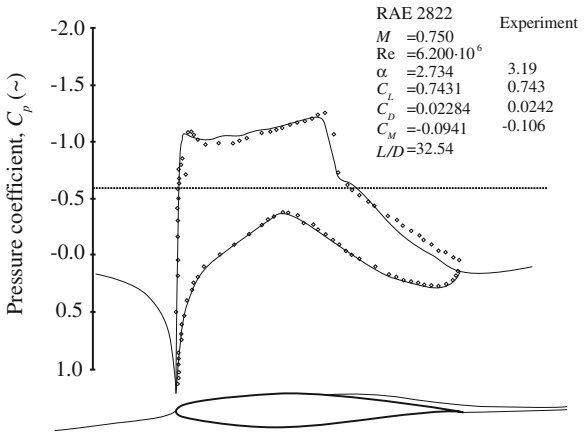

Fig. 2.11 Comparison between predicted and measured pressure distribution about the RAE2822 airfoil (from [7, 8], respectively)

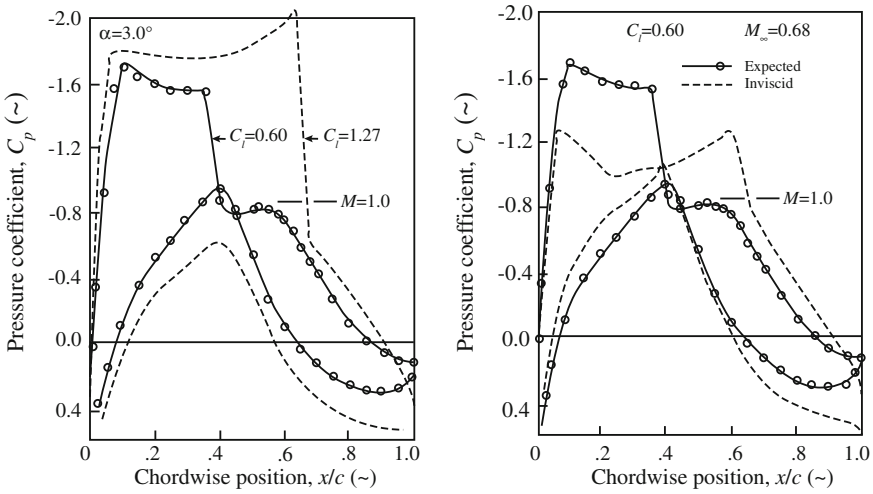

Fig. 2.12 Comparison between predicted and measured pressure distribution at constant angle of attack (*left*) and constant lift coefficient (*right*) (after Ref. [5])

the same angle of attack as in the experiment (left) and at the same lift coefficient as in the experiment (right). None of these predictions gives an accurate result. This shows that we must be extremely careful when assessing the results from a purely inviscid solver. Given the short computation time of viscous-inviscid solvers such as MSES, it is therefore advised to include the presence of the boundary layer when predicting the pressure distribution over a body. In particular at low-Reynolds numbers and transonic Mach numbers this is an important prerequisite for obtaining reliable results.

2.7.2 Potential Flow Equation

Even though the previous analysis showed that the vorticity in transonic flow cannot be neglected, there is a body of literature that is based on steady, inviscid, irrotational flow with no body forces [10]. It can be shown that for weak normal shocks ($M_1 \approx 1$) the entropy change is relatively small. Based on this finding the approximation can be made that transonic flow is irrotational and that that therefore the curl of the velocity field (see Sect. 2.3.4) is zero:

$$\nabla \times \boldsymbol{V} = 0 \tag{2.190}$$

In potential flow we pose that there exists a scalar potential function, Φ, which' partial derivatives equal the velocity components:

$$\nabla \Phi = \boldsymbol{V} \tag{2.191}$$

In Cartesian coordinates, we align the x-axis with the freestream velocity, $\boldsymbol{V}_\infty$, and represent the local velocity components, u, v and w by

$$u = \frac{\partial \Phi}{\partial x} = \Phi_x \tag{2.192a}$$

$$v = \frac{\partial \Phi}{\partial y} = \Phi_y \tag{2.192b}$$

$$w = \frac{\partial \Phi}{\partial z} = \Phi_z \tag{2.192c}$$

If we assume the flow to be steady, the continuity equation (2.113) reduces to:

$$\nabla \cdot (\rho \boldsymbol{V}) = 0 \tag{2.193}$$

In Cartesian coordinates this results in the following expression:

$$\frac{\partial}{\partial x}(\rho \Phi_x) + \frac{\partial}{\partial y}(\rho \Phi_y) + \frac{\partial}{\partial z}(\rho \Phi_z) = 0 \tag{2.194}$$

Note in this equation that the partials ρ_x, ρ_y, and ρ_z appear. We will find an expression for each of these density derivatives by considering the potential form of the steady, inviscid, irrotational momentum equation.

If we assume that the flow is steady (i.e. $\partial/\partial t = 0$) the Lagrange form of the momentum equation (2.188) can be reduced to:

$$\nabla \left(\frac{V^2}{2} \right) = -\frac{1}{\rho} \nabla p \tag{2.195}$$

In differential form we can rewrite (2.195) as follows:

$$\mathrm{d}p = -\rho \mathrm{d} \left(\frac{V^2}{2} \right) = -\rho \mathrm{d} \left(\frac{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}{2} \right) \tag{2.196}$$

In isentropic flow the speed of sound (a) is given by (2.106). We can substitute the expression for the speed of sound (2.106) in (2.196) and obtain:

$$\mathrm{d}\rho = -\frac{\rho}{a^2} \mathrm{d} \left(\frac{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}{2} \right) \tag{2.197}$$

If we return to $\nabla \rho$ and consider each of the partials ρ_x, ρ_y, and ρ_z we obtain:

$$\rho_x = -\frac{\rho}{a^2} \frac{\partial}{\partial x} \left(\frac{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}{2} \right) \tag{2.198a}$$

$$\rho_y = -\frac{\rho}{a^2}\frac{\partial}{\partial y}\left(\frac{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}{2}\right) \tag{2.198b}$$

$$\rho_z = -\frac{\rho}{a^2}\frac{\partial}{\partial z}\left(\frac{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}{2}\right) \tag{2.198c}$$

If we substitute (2.198a–2.198c) in (2.194) we obtain the *full potential equation* for steady, inviscid, irrotational and isentropic flow:

$$\left[1 - \frac{\Phi_x^2}{a^2}\right]\Phi_{xx} + \left[1 - \frac{\Phi_y^2}{a^2}\right]\Phi_{yy} + \left[1 - \frac{\Phi_z^2}{a^2}\right]\Phi_{zz} - 2\frac{\Phi_x\Phi_y\Phi_{xy}}{a^2} - 2\frac{\Phi_x\Phi_z\Phi_{xz}}{a^2} - 2\frac{\Phi_y\Phi_z\Phi_{yz}}{a^2} = 0 \tag{2.199}$$

In the late 1980s and early 1990s the analysis code Matrics-V was developed by the National Aerospace Laboratory of the Netherlands to estimate the aerodynamic forces on wing-fuselage combinations (three-dimensional) in high-subsonic conditions. This code relies on a viscous-inviscid formulation, where the full potential equation (2.199) is numerically solved in the inviscid domain outside the boundary layer. In the boundary layer the boundary layer equations are numerically solved. Subsequently, an interaction algorithm is used to match the flow properties at the

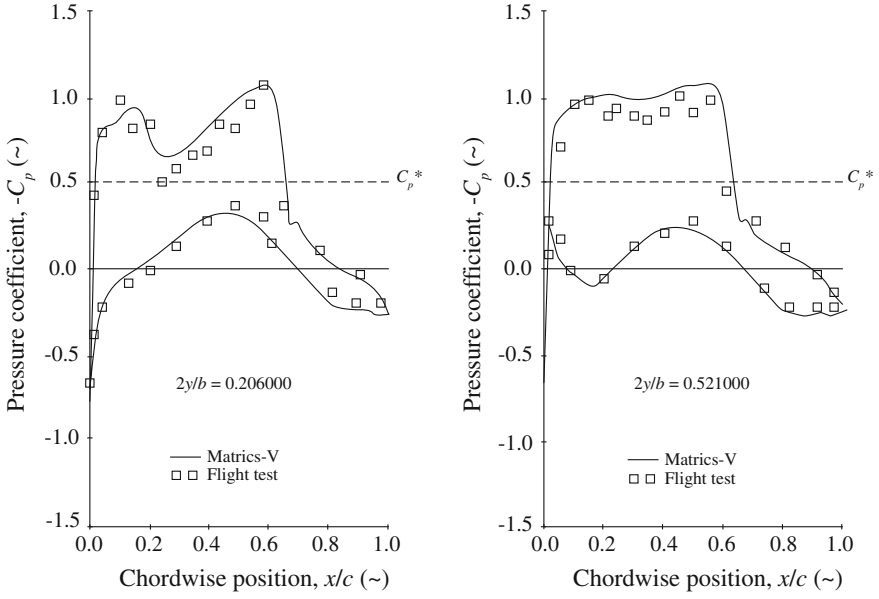

Fig. 2.13 Comparison of predicted and measured pressure distributions about two sections of the Fokker 100 wing (from [20]). Note that this is a two-dimensional representation of results that were obtained from three-dimensional calculations (Matrics-V)

edge of the boundary layer to the inviscid flow outside the boundary layer [20]. This analysis code was specifically developed to investigate the flow about wing-body combinations. The objective was to develop a code that could produce results in less than week of computation time. Due to the increase in computational power, this has now (2015) reduced to less than five minutes on a personal computer, including the generation of the mesh. In Fig. 2.13 a comparison between the predicted and measured pressure distribution at two spanwise wing stations of the Fokker 100 can be seen. We see that the prediction of the pressure distribution is in excellent agreement with the measured data at the same angle of attack. The accurate prediction in combination with the short computation time make this type of code a viable candidate in the early stages of preliminary wing design.

2.8 Summary

In the preceding sections we have briefly reviewed the fundamental tools that the reader needs to be familiar with in order to properly understand the material that is presented in the subsequent chapters. We have done a very basic review of partial differential equations, their solution methods, and their classification. We have also re-acquainted ourselves with the mathematical operations in vector algebra. We have explained the fundamental laws of thermodynamics: the state law that relates the thermodynamic state variables to each other, the first law that balances work, heat and internal energy, and the second law that tells us the direction in which a thermodynamic process takes place. Finally, we have applied all this knowledge in defining the equations of motion of fluid flow. When expanded we have shown that the equations of motion are a coupled system of five partial differential equations [(2.110), (2.127) (3 components), and (2.139)] with five unknowns: ρ, e, and the three velocity components, u, v, w. These equations are often referred to as the Navier-Stokes equations and they can be cast in different forms: integral form, substantial-derivative form, and conservative-derivative form. Because solving these equations requires vast computational resources we have presented derivatives of these equations. In the Reynolds-averaged Navier-Stokes equations the flow is decomposed into a mean flow with fluctuating terms. A turbulence model can then be used to relate those fluctuating terms to the mean flow parameters. A further simplification of the Navier-Stokes equations is obtained through the assumption of having an inviscid and non-conducting flow. The resulting equations are often referred to as the Euler equations and they can provide an accurate prediction of the flow about a body, in particular when they are used in parallel with the boundary-layer equations. If also the assumption is made that the flow is irrotational, the Euler equations reduce to the full-potential equation. Solving the three-dimensional full-potential equation in combination with the boundary-layer equations can still produce accurate predictions of transonic flow about bodies, as long as the shock strength is relatively weak.

Problems

Review of Partial Differential Equations

2.1 Show that (2.7) results from (2.5) by using the chain rule and substituting $\xi = x + ct$ and $\eta = x - ct$.

2.2 Demonstrate that $u(x,t) = \sin\left(\frac{\pi(x+ct)}{L}\right) + \sin\left(\frac{\pi(x-ct)}{L}\right)$ is a solution to the following problem:

$$u_{tt} = c^2 u_{xx}$$
$$u(x,0) = 2\sin\left(\frac{\pi x}{L}\right)$$
$$u(0,t) = u(L,t) = 0$$

2.3 Consider a string of steel wire, measuring 1 m in length and weighing 0.5 N. It is stretched by a force of 100 N. What is the corresponding speed c of the transverse waves?

2.4 Consider the wave equation (2.5) with the following boundary conditions: $u(0,t) = u_x(L,t) = 0$.

(a) Use separation of variables technique to calculate the eigenvalues, eigenfunctions and general solution.
(b) Now, assume $L = \pi$ and $c = 1$. With initial conditions $u(x,0) = 0$ and $u_t(x,0) = 1$, calculate the solution for $u(x,t)$.
(c) With initial conditions $u(x,0) = \sin(x/2)$ and $u_t(x,0) = 2\sin(x/2) - 3\sin(5x/2)$ calculate the solution for $u(x,t)$.

2.5 Demonstrate that (2.22) can be derived from $u_t = 0$ by using the chain rule and the transformation (2.6).

2.6 Show that the Jacobian for the change in characteristic coordinates (2.6) equals $2c$.

2.7 For the same problem as in Example 2.2, calculate the solution for:

(a) $x < t, t > 0, x + t < 0$, and check $u_{tt} - u_{xx} = 1$.
(b) $x > 1, t < x < t + 1$, and check $u_{tt} - u_{xx} = 0$.

2.8 Show that (2.43) is indeed a solution of (2.41).

2.9 Demonstrate that from (2.49) we can obtain (2.50) by using the method outlined in the text.

2.10 Using the separation-of-variables technique find a solution to the following problem:

$$u_t = k u_{xx} \qquad (0 \leqq x \leqq L)$$

$$\begin{aligned} u(0,t) &= 0 && (t > 0) \\ u_x(L,t) &= 0 && (t > 0) \\ u(x,0) &= f(x) && (0 < x < L) \end{aligned}$$

2.11 For Problem 2.10, provide calculate D_n in closed form for $L = \pi$ and $f(x) = x$.

2.12 Consider the homogenous heat equation, $c^2 u_{xx} - u_t = 0$. Determine whether this equation is hyperbolic, parabolic, or elliptic.

2.13 Consider the potential equation on p. 74 (2.199). When $M > 1$, determine the characteristic coordinates and write the equation in its canonical form.

Review of Vector Algebra

2.14 Let $\mathbf{a} = (-2,\ 5,\ -9)$ and $\mathbf{b} = (4,\ 2,\ 7)$. Calculate the following:

(a) $\mathbf{a} + \mathbf{b}$
(b) $\mathbf{a} \cdot \mathbf{b}$
(c) $\mathbf{a} \times \mathbf{b}$
(d) $\mathbf{b} \times \mathbf{a}$
(e) $\mathbf{a} \otimes \mathbf{b}$

2.15 For the following velocity vector fields, find the potential function if it exists. Check that (2.66) holds in each case. $\phi(x, y, z)$: (a) $(x,\ 2y,\ 3z)$, (b) $(yz,\ xz,\ xy)$, (c) $(ye^x,\ e^x,\ 1)$, (d) $(2y,\ 5x,\ 0)$.

2.16 For the vector fields of Problem 2.15, calculate the divergence.

2.17 For the vector fields of Problem 2.15, calculate the curl. Compare your results to those of Problem 2.15. What do you notice?

Review of Thermodynamics

2.18 With c_p as given in Fig. 2.8 and R as given in the text, calculate c_v, γ, e, and h for air at an altitude of 10 km under ISA conditions.

2.19 For the values given in Problem 2.18 also calculate the viscosity of the air, μ and the thermal conductivity, k.

2.20 Calculate the change in specific internal energy, Δe, for air that is being compressed isentropically. Assume that the initial temperature is 288 K and the final temperature, due to compression, 340 K.

2.21 Show that (2.97) and (2.98) can be derived by combining the first and second law of thermodynamics, (2.83) and (2.94), respectively.

2.22 Derive the following relations by starting from (2.97) and (2.98), respectively.

$$\mathrm{d}s = c_v \frac{\mathrm{d}T}{T} + R\frac{\mathrm{d}v}{v} \tag{2.200}$$

$$\mathrm{d}s = c_p \frac{\mathrm{d}T}{T} - R\frac{\mathrm{d}p}{p} \tag{2.201}$$

2.23 Demonstrate that (2.103) can be derived by combining (2.101) and (2.102) and applying the substitutions as described in the text.

Equations of Fluid Motion

2.24 Use the divergence theorem (2.69) to prove that (2.109) and (2.110) are mathematically identical.

2.25 Consider the LHS of (2.122). Show that by using (2.110) this can be simplified to $\rho\frac{\mathrm{D}V}{\mathrm{D}t}$.

2.26 Consider the energy equation (2.139).

(a) By employing the continuity equation (2.110) show that the following is true:

$$\rho\frac{\mathrm{D}(E_t/\rho)}{\mathrm{D}t} = \frac{\partial E_t}{\partial t} + \nabla\cdot E_t\boldsymbol{V}$$

(b) Demonstrate that also the following relation is true:

$$\rho\frac{\mathrm{D}(E_t/\rho)}{\mathrm{D}t} = \rho\frac{\mathrm{D}e}{\mathrm{D}t} + \rho\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}\boldsymbol{V}$$

(c) Use (2.122) to show that the second term in the equation above can be written as:

$$\rho\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}\boldsymbol{V} = \rho\boldsymbol{f}\cdot\boldsymbol{V} - \nabla p\cdot\boldsymbol{V} + (\nabla\cdot\boldsymbol{\tau}_{ij})\cdot\boldsymbol{V}$$

(d) Using the three equations above, demonstrate that (2.140) is identical to (2.139).

2.27 Consider the energy equation in substantial-derivative form (2.140).

(a) Show, by employing the continuity equation, that the following identity is true:

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} + p(\nabla\cdot\boldsymbol{V}) = \rho\frac{\mathrm{D}h}{\mathrm{D}t} - \frac{\mathrm{D}p}{\mathrm{D}t}$$

(b) Explicitly write out the dissipation function, Φ, in terms of the velocity components in three-dimensional Cartesian coordinates. Use the assumption that $\mu' = \frac{2}{3}\mu$.

2.28 Write out (2.110), (2.149), and (2.150) in three-dimensional Cartesian coordinates and demonstrate that you can write this as (2.152).

2.29 Consider the system of Eq. (2.152).

(a) Demonstrate that pressure, p, can be expressed as a function of the the state variables ρ and e and the ratio of specific heats, γ by using (2.72) and (2.76).
(b) Show that the heat flow, $\boldsymbol{q}$, can expressed in the state variable e. Assume that k is constant.
(c) Repeat the previous exercise, but now assume that k is variable and related to temperature according to (2.81) and (2.82). Is this a conservative differential equation?

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