Chapter 5
Putting Davidson's Semantics to Work to Solve Frege's Paradox on Concept and Object

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Abstract What Frege's paradox on concept and object (FP) consists in and the manner in which Frege coped with it (the ladder strategy) are briefly reviewed (§ 5.1). An idea for solving FP inspired by Husserl's semantics is presented; it results in failure, for it leads to a version of Russell's paradox, the usual solution of which implies something like a resurgence of FP (§ 5.2). A generalized version of Frege's paradox (GFP) and an idea for solving it inspired by Davidson's semantics are presented; three theorems about recursive definability of truth are put forward and used to determine whether this idea can be successfully applied to certain putative forms of the Language of Science (§ 5.3). Proofs of these three theorems, in particular of the third, which answers a question that does not seem to have drawn logicians' attention, are then given (§ 5.4). Finally, it turns out that there is a tension between the proposed solution of GFP and the idea of Language of Science assumed so far in this paper, and a way of solving it is proposed (§ 5.5).

5.1 Frege's Paradox on Concept and Object (FP) and How Frege Put up with it

5.1.1. We all remember Frege's famous letter to Husserl dated May 24, 1891 (Frege 1976, Brief XIX/1; 1980, letter VII/1), in which the former objects to the latter's semantic analysis of concept-words and sums up the main points of his own new semantics in a chart, reproduced below.[1]

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[1] The capital letters that I have bestowed upon the translation of certain terms in this paper ("Proper Name" for "*Eigenname*", "Concept-word" for "*Begriffswort*", "Sense" for "*Sinn*", "Meaning" for "*Bedeutung*", "Truth-Value" for "*Warheitswert*", "Object" for "*Gegenstand*", "*Begriff*" for "Concept", etc.) are only there to remind readers that the terms are to be understood in the technical

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Sentence	Proper Name	Concept-word
↓	↓	↓
Sense of the Sentence (Thought)	Sense of the Proper Name	Sense of the Concept-word
↓	↓	↓
Meaning of the Sentence (Truth-value)	Meaning of the Proper Name (Object)	Meaning of the Concept-word (Concept)[a]

[a]At the bottom to the right of this box of the chart, Frege added: "→ Object that falls under the Concept", thus showing where he parted ways with Husserl with regard to Concept-words. For the former, the relation of Sense to Object was mediated by a Concept, while for the second, let it be said in Fregean terms, Senses referred directly to Object, which usurped the place of the Concept

The different columns may be understood as corresponding to different categories of entity. To give an idea of the difference between the different categories, Frege resorted, from that time on, to the metaphor of the completeness vs. the incompleteness, or the saturatedness vs. the unsaturatedness, of the entities under consideration, which paradoxically led him to consider Sentences as Proper Names, and the first column as a particular instance of the second. In the letter to Husserl, however, Frege did not feel the need to take that step expressly and, in my own presentation, I shall not do so either.

To each line corresponds one of the three levels—Expression, Sense, Meaning—of Fregean semantics. Frege defends the thesis of what I shall call the *categorial parallelism* of the *three* levels: Just as a Proper Name (a complete Expression) may complement a Concept-word (an incomplete Expression) to combine with it to make a Sentence (a complete Expression), so the Sense (complete) of a Proper Name may complement the Sense (incomplete) of a Concept-word to combine with it to make a Thought (a complete Sense), and so again an Object (a complete Meaning) may complement a Concept (an incomplete Meaning) to combine with it to make a Truth-Value (*sic,* a complete Meaning).

Thus, Concepts are not Objects, for example, the Concept *horse* is not an Object. In his article "Concept and Object" (Frege 1892b), Frege sought to refute Benno Kerry's objection. Kerry used the example: "The concept *horse*[2] is easily attained". In this sentence, he argued, the words "the concept *horse*" designate an object. Therefore, the concept *horse* is an object, some objects are concepts, and concepts are objects.

and more or less deviant, depending on the case, sense that Frege gave to the original. "*Eigenname*", in Frege's sense, is what is usually called a "singular term"; "*Bedeutung"* and "*Begriff*" are to be understood in a deviant sense, of which the chart provides an initial idea.

[2] Actually Kerry put quotation-marks around "horse" instead of italicizing it, as I do in Frege's wake to the same end.

Frege's reaction to Kerry's objection is extremely surprising. On the one hand, he gives in. Yes, he acknowledges, the words "the Concept *horse*" designate an Object, and the Concept *horse* is that Object. But on the other hand, he resists and persists to the point of paradox, what I call "Frege's paradox" (FP). If Concepts are not Objects and the Concept *horse* is an Object, then the Concept *horse* is not a Concept. This paradox is the price to be paid for holding on to the controversial thesis. Frege pays the price and holds on to the thesis that Concepts are not Objects.

Long ago, I proposed an in-depth analysis of Frege's paradox (Rouilhan 1988), but here (§§ 5.1–5.2) I shall restrict myself to showing readers the shortest path leading from FP to what I shall call the "generalized Frege's paradox" (GFP).

5.1.2. The situation, from a pragmatic point of view, is the following, according to Frege. By using the words "the Concept *horse*", one does not succeed in speaking about what they would like to speak, namely about a Concept, which is an incomplete entity; they only speak about a complete entity, more precisely, about an Object. Or, to speak from now on in a more *suggestive* manner than Frege did, when one uses the Proper Name "the Concept *horse*" to speak of the Concept *horse itself* (*in itself*, *as it is in itself*), they do not succeed in speaking about it, because they are trying to speak about it as they would speak of an Object, and they are indeed speaking only of an Object. In the following chart, which partially sums up the situation, the schematic Expression "Φ(ξ)" is replaceable by a Concept-word, and the letter "ξ" but marks the empty place of that Concept-word, liable to be occupied by a Proper Name in order to obtain a Sentence. The Expression "the Concept *horse*" must be construed as a variant of the Expression "the Concept *horse* (ξ)", of schema "the Concept Φ(ξ)".

Expression	"Φ(ξ)"	"The Concept Φ(ξ)"
Sense	Sense	Sense
Meaning	Φ(ξ) (it is the Concept Φ(ξ) *itself* and it is not an Object)	The Concept Φ(ξ) (it is *not* the Concept Φ(ξ) *itself*; it is an Object)

If one uses the Sentence "the Concept *horse* is not an Object" to illustrate the thesis that Concepts are not Objects, they do not say what they wanted to say, because what they are saying is literally false. And if one just states the succession of words "*horse* is not an Object" as if it were a Sentence, that does not work either, because the Concept-word "*horse*" is an incomplete Expression that cannot complete the incomplete Expression "is not an Object" so as to form a Sentence. The alleged Sentence "*horse* is not an Object" is the result of a category mistake and does not mean anything at all. In both cases, what one wanted to say was that the Concept *horse itself* is not an Object, but that, strictly speaking, cannot be said. It can only be suggested. And the same is so for Frege's thesis. If he uses the words "Concepts are not Objects" in the sense of "for every f, the Concept f is not an Object" (where "f" is a variable of the category of Concept-words), he is saying something that is literally false; and if it is in the sense of "for every f, f is not an Object", he is making a category mistake and is not saying anything at all. Of course, what he wanted to say was that the Concepts *themselves* are not Objects, but that, strictly speaking, cannot be said. It can only be suggested.

Thus, instead of seeing in the paradox of the Concept *horse* the symptom of an error to be spotted and corrected, Frege simply takes note of it and holds on obstinately to the thesis of the categorial difference between Concepts (*in themselves*) and Objects, which is at the origin of the paradox and implies its own ineffability. Towards the end of "Über Begriff und Gegenstand" (Frege 1892b), he lucidly makes the point:

> I admit that there is a quite peculiar obstacle in the way of an understanding with my reader. By a kind of necessity of language, my expressions, taken literally, sometimes miss my thought; I mention an Object, when what I intend is a Concept. I fully realize that in such cases I was relying upon a reader who would be ready to meet me half-way—who does not begrudge me a pinch of salt. (p. 196).

Frege's strategy for overcoming the obstacle is not essentially different from the one generally attributed, rightly or wrongly[3], to the Wittgenstein of the *Tractatus*, the *ladder strategy*. If one cannot *say* what they would like to *say* (for example that Concepts *themselves* are not Objects), at least they can *suggest* it (as I am used to saying) and count on the *good will* of the reader or the interlocutor (as Frege more or less said), at least they can *show* it (as Wittgenstein would say). One can do so by temporarily diverting language from its normal function as a means of expressing Sense in order to use it as a means for suggesting or showing what, strictly speaking, is an inexpressible non-Sense. When this unconventional usage of language has had its effect, when it has made it possible to see what needed to be seen, one will be able to go back to conventional usage and remain there. To say this in terms akin to those of the early Wittgenstein: what cannot be said, can be shown to those who have not seen it yet by setting up the ladder of non-Sense for them to climb. Once they have seen what needed to be seen, they will have to throw away the ladder and they will finally see the world aright.

5.2 An Idea for Solving FP Inspired by Husserl's Semantics and its Failure

5.2.1. A simple way of putting an end, at least temporarily, to the dispute with Kerry and of solving FP would have been to admit that Kerry was right and to acknowledge with him that Concepts (*themselves*) are definitely Objects. In fact, Frege's best

[3] In an article of 1991 (Conant 1991), James Conant argued that, despite appearances, the Wittgenstein of the *Tractatus* (1921) did *not* take up Frege's lesson. I shall retain only the following from Conant's long, subtle analysis: the ladder strategy in the *Tractatus* is not designed to make people see what cannot be said and can only be shown, for what someone who has climbed the ladder of non-sense is supposed to see is that *there is nothing to see*. The first colloquium mentioned above (n1), at which Conant was present, focused precisely on this article.

adversary in such a dispute would have been Husserl, Husserl of *Logische Untersuchungen* (Husserl 1900–1901)[4], so close to Frege in many respects. Like Frege, he distinguished between three levels: expression, meaning (*Bedeutung*) and object (*Gegenstand*) or objectivity (*Gegenstandlichkeit*)[5]. Like Frege, he recognized the categorial parallelism of the first two levels (those of expression and of meaning), which he explained in terms of dependence and independence. However, unlike Frege, he denied, with reasons to back it up, the existence of any parallelism between these first two levels and the third (that of objectivity, see n5). Evoking the idea that "categorematic expressions represent independent objects, and syncategorematic expressions dependent objects", Husserl objects that the expression of a *dependent moment* immediately provides a decisive counter-example (Investigation IV, § 8). For, Husserl thinks, as a common noun, this expression has an independent meaning, and that in no way keeps it from representing those dependent objects that are the said dependent moments. Admittedly, as his letter of May 24, 1891 precisely shows, Frege did not agree with Husserl about the semantics of concept-words. Husserl could, nonetheless, have made an analogous objection to the Fregean idea that complete Expressions (for example, Proper Names) Mean complete entities (that is Objects), and incomplete Expressions (for example, Concept-words), incomplete entities (in this case, Concepts *as they are in themselves*). He could have objected that an Expression of schema "the Concept $\Phi(\xi)$" immediately provides a decisive counter-example.

This, therefore, is what Frege should have acknowledged, he too, to his own advantage: that the categorial parallelism of the levels of Expression and of Sense did not extend to that of Meaning. One can definitely say that a Proper Name and a Concept-word are the constituents of a Sentence and explain that the Proper Name is precisely the sort of complement that the Concept-word needs to constitute with it the unity of the Sentence. One can definitely also say that the Sense of the Proper Name and the Sense (*in itself*) of the Concept-word are constituents of the Thought and explain that the Proper Sense is precisely the sort of complement that the Conceptual Sense (*in itself*, therefore incomplete) needs to constitute with it the unity of the Thought. But, unless possessed by the demon of analogy, one can certainly not say that the Object and the Concept (*in itself*) are constituents of the Truth-value and explain that the Object is precisely the sort of complement that the Concept (*in itself*, therefore incomplete) needs to constitute with it the unity of the Truth-value. One can do this no more than, more generally, they can say that an Object and a Function (*in itself*) are constituents of the Value of this Function (*in itself*) at this Object as argument and explain that the Object is precisely the sort of complement that the Function (*in itself*, therefore incomplete) needs to constitute with it the unity of the Value. This is, moreover, what Frege would end up understanding, as seen in his

[4] See more specifically Logical Investigation IV (in 1st ed., vol. II, 1901; 2[d] ed., vol. II.1, 1913).

[5] Within the context of his analysis, Husserl used "*Gegenstandlichkeit*" as a technical term having a certain extension greater than the ordinary term "*Gegenstand*".

1919 notes for Ludwig Darmstaedter (Frege 1919), where he would write that “[o]ne cannot say that Sweden is part of the Capital of Sweden”[6].

If he had done this, nothing, in the discussion with Kerry, would have kept him from saying that Concepts (*themselves*) *are* Objects, that moreover the same is so for entities of any category, including those whose incomplete nature would not have come into question, like for example the Sense (*itself*) of a Concept-word, and finally that one can say *everything*, that one can talk *about everything*. And in his famous letter to Husserl, he would have been able to draw up the following chart[7]:

Expression	“$\Phi(\xi)$”	“The Concept $\Phi(\xi)$”
Sense	Sense	Sense
Meaning	The Concept $\Phi(\xi)$ (it is Concept $\Phi(\xi)$ *itself* and it is an Object)	

The Fregean critique of the Husserlian semantics of concept-words would not for all that have lost its *raison d’être*. It would have only gained in simplicity and in credibility.

The identity of the Meanings of the two Expressions schematized by “$\Phi(\xi)$” and “the Concept $\Phi(\xi)$” would not have prevented neither of these Expressions from playing the role corresponding to its category in the formation of a Sentence or prevented its Sense from playing the role corresponding to its category in the formation of a Thought. Nothing would have changed in this regard with respect to Frege. There would just no longer have been a way back enabling one to find again the category of an Expression and that of its Sense—and thus the role of that Expression and its Sense in the formation of a Sentence and of a Thought—from the category of its Meaning.[8]

5.2.2. The solution to one paradox may hide another, and the solution of this other paradox may involve the return of the same.

[6] Frege was already aware of the difficulty when he wrote, as early as 1892: “One might also say that judgments (*Urteilen*) are distinctions of parts (*Teilen*) within Truth-values. [. . .] However, I have here used the word ‘part’ in a special sense. [. . .] This way of speaking can certainly be attacked [. . .]. A special term would need to be invented” (Frege 1892a, pp. 35–36, 1984, p. 165).

[7] Without neglecting to add to it at the bottom to the right of the chart: “→ Object falling under the Concept” (compare with the chart of Sect. 5.1.1).

[8] For this corrected version of Frege’s semantics under consideration, I am prepared to describe the role of a Proper Name, (schematized by) “Δ”, and that of a Concept-word, (schematized by) “$\Phi(\xi)$”, in the Sentence (schematized by) “$\Phi(\Delta)$” nearly as C. Wright did in 1998 (Wright 1998, cf. p. 260) (this is not a quotation): the Sense of “$\Phi(\xi)$” so relates it to the Concept $\Phi(\xi)$ that it may be used in concatenation with “Δ” in order to *ascribe* the Concept $\Phi(\xi)$ *to* Δ; and the Sense of “Δ” so relates it to Δ that it may be used in concatenation with “$\Phi(\xi)$” in order to *subsume* Δ *under* the Concept $\Phi(\xi)$. Indeed, the solution of FP inspired by Husserl outlined in Sect. 5.2.1 could be so presented as to be clearly, essentially equivalent to Wright’s solution. Unfortunately, as we shall see in Sect. 5.2.2, the theory of Concepts upon which these solutions are based falls prey to a certain version of Russell’s paradox. More will then be said about Wright.

For Frege, Concepts obeyed an extensional criterion of identity. Now, if all Concepts are Objects, then nothing any longer safeguards them from a certain version of Russell's paradox. It suffices to choose for "Φ(ξ)" the Concept-word:

$$\exists f(\xi = \text{the Concept } f(\zeta) \& \ \neg f(\xi))$$

(where "*f*" is a variable of the category of Concept-words) and, using "w(ξ)" as an abbreviation of this Concept-word, to ask the fateful question whether, yes or no,

$$\text{w(the Concept w}(\xi)).$$

If yes, then

$$\exists f(\text{the Concept w}(\xi) = \text{the Concept } f(\zeta) \& \ \neg f(\text{the Concept w}(\xi))),$$

whence, readily,

$$\neg\text{w(the Concept w}(\xi));$$

and if no, then

$$\forall f(\text{the Concept w}(\xi) = \text{the Concept } f(\zeta) \Rightarrow \ f(\text{the Concept w}(\xi))),$$

whence, readily,

$$\text{w(the Concept w}(\xi)).$$

From each answer follows the opposite answer.[9]

One is thus led to acknowledge that all Concepts *in themselves* are not Objects, that there are exceptions to the principle that all are. There are, perhaps, Concepts that *in themselves* are Objects, for example—let us admit it—, the Concept *horse*, but there are surely ones that are not, for example the Concept w(ξ). The Concept w(ξ) *itself* is not an Object. One cannot argue any longer, as Frege did, that the Concept *horse* is an Object and *thus* not a Concept, for the very same entity now is both an Object and a Concept. Nor can one argue that the Concept w(ξ) is an Object and *thus* not a Concept, for an Object *may* now be a Concept. Let me dwell on that point.

If, in spite of the version of Russell's paradox presented above, the Proper Name "the Concept w(ξ)" is to have a Meaning, as Frege required of all expressions of the

[9] This paradox was notably pointed out by T. Parsons in 1986 (Parsons 1986, pp. 454–455). Wright mentions it at the end of his article, but deals with it in a somewhat offhand manner: "This, like the recent resurgence of tuberculosis in the Western world, is a disappointment. But I do not think it is really an objection—too many of the family of paradoxes that exercised Russell survive the imposition of Frege's hierarchy to allow us to think that it gets to the root of that particular one" (Wright 1998, p. 263). Wright may be right in the second part of the last sentence, but not in the first one. The first lesson to be learnt from the paradox in question (see the next paragraph in the text) immediately gives rise to a sort of resurgence of FP itself. So the paradox in question *is* an objection.

Language of Science, this can only be an Object arbitrarily chosen to play this part, an Object *ad hoc*. As to whether the use of the Proper Name "the Concept w(ξ)" does or does not give rise to FP, I mean to the paradox according to which the Concept w(ξ) is not a Concept (*as it is in itself*), this depends on the Object chosen to play the part of the Meaning of the Proper Name "the Concept w(ξ)". If it is an Object that is not a Concept (*as it is in itself*), like the Moon—let us admit it—, for example, that is chosen to play this part, then we are entitled to claim that the Concept w(ξ) is not a Concept (*as it is in itself*), and thus FP is back. But if it is an Object like the Concept *horse*—which is nothing other than the Concept *horse itself*, as admitted at the beginning of the preceding paragraph—that is chosen for this part, then there is no reason to claim that the Concept w(ξ) is not a Concept (*as it is in itself*).

However, whether the use of the Proper Name "the Concept w(ξ)" gives rise or not to FP (in the sense specified above), the situation is still paradoxical. Since the Concept w(ξ) is an Object and the Concept w(ξ) *itself* is not one, the Concept w(ξ) is not the Concept w(ξ) *itself*, or, as Frege would have simply said, the Concept w(ξ) is not the Concept w(ξ). By using the Proper Name "the Concept w(ξ)", we do not therefore succeed in speaking of the Concept w(ξ) *itself*, we are speaking of an Object, and even of an Object that has nothing to do with the Concept w(ξ) *itself* at all. There are things that one would like to say, but cannot, etc.

The semantics of Concept-words (expressions schematized by "Φ(ξ)") and Proper Names obtained by prefixing them with the operator of nominalization "the Concept" (and thus schematized by "the Concept Φ(ξ)") is summed up below in terms of whether the Concept Φ(ξ) *itself* is or is not an Object.

1st case: the Concept Φ(ξ) *itself* is an Object (the case, for example—as we have admitted—, of the Concept *horse*)

<table>
<tr><td>Expression</td><td>"Φ(ξ)"</td><td>"The Concept Φ(ξ)"</td></tr>
<tr><td>Sense</td><td>Sense</td><td>Sense</td></tr>
<tr><td>Meaning</td><td colspan="2">The Concept Φ(ξ)
(this is the Concept Φ(ξ) itself and it is an Object)</td></tr>
</table>

2nd case: the Concept Φ(ξ) *itself* is not an Object (the case, for example, of the Concept w(ξ)

Expression	"Φ(ξ)"	"The Concept Φ(ξ)"
Sense	Sense	Sense
Meaning	Φ(ξ) (this is the Concept Φ(ξ) *itself* and it is not an Object)	the Concept Φ(ξ) (this is not the Concept Φ(ξ) *itself*, it is an *ad hoc* Object)

5.3 Generalized Frege's Paradox (GFP); an Idea for Solving it Inspired by Davidson's Semantics; Putting this Idea to the Test

5.3.1. What was at stake for Frege in his paradox was the possibility, for an author writing for readers, or a teacher speaking to students, of explicating the content of the Expressions of the Language of Science. The teacher was supposed to explain that there were different categories of Meaning, notably that of Concept and that of Object, that these categories were pairwise disjoint and that they were not to be confused, in particular, that no Concept was an Object any more than any Object was a Concept, etc. However, it transpired that, in saying this kind of thing, the teacher was ineluctably failing to say what he or she wanted to say, that what he or she wanted to say involved a category mistake and was therefore impossible to say.

The fact that the categories of Meaning were pairwise disjoint in character was not essential to FP. Let me here leave Frege and his terminology, but for the phrase "Language of Science". Generally, for a language taken to be the Language of Science to be open to a paradox of the same sort as FP, it suffices for this language to contain different categories of reference, or denotation, are not all included in a single category. In terms of variables: it suffices for this language to contain variables of different types whose domains of variation are not all included in a single domain of variation. The same reasons, *mutatis mutandis*, that hold in Frege's case lead to the same conclusion, namely, that it is impossible to explicate the content of the expressions of such a language without making a category mistake (relative to this supposed Language of Science; the resurgence of FP in Sect. 5.2.2 is a good example). This is what I call *generalized Frege's paradox* (GFP).

More precisely, the category mistake would have the form of surreptitious introduction of a new category of variable irreducible to those available in the supposed Language of Science. Let us call it *the Mistake*. If the impossibility in question were established, there would be no other solution for solving GFP than to require of any language taken to be the Language of Science that its variables range over domains that are all included in one of them (as it happens in particular and in the simplest way when all the variables range over one and the same domain). Then it would only remain to ascertain that complying with this requirement made it effectively possible to explicate, without making the Mistake, what expressions of such a language mean. But has the impossibility in question been established? Is it true that, when the requirement in question has not been met, a teacher wishing to explicate the content of the expressions of the language under consideration to a student is doomed to make the Mistake? I used to believe this, but I have not believed it for a long time (see Rouilhan 1988, pp. 186–187, and 2002, pp. 198–199). It is possible to solve GFP without having to shoulder the requirement in question.

My solution will be grounded on the basic idea of Davidson's semantics (Davidson 1984). Explicating what the expressions of a language, $\mathcal{L}$, mean is, as Davidson puts it, (not to *translate*, but) to *interpret* them. The interpretation of component expressions of statement (closed sentence) of $\mathcal{L}$ is determined by their contribution to the interpretation of the statements of $\mathcal{L}$ in which they occur. As for the statements of $\mathcal{L}$, according to an idea Frege himself had, which was taken up successively by Wittgenstein, Carnap and Davidson, their interpretation is determined by their *truth-conditions*. Davidson more specifically requires that those truth-conditions be stated in the form of what he calls a *recursive* ***theory*** *of truth à la Tarski* for $\mathcal{L}$. This is precisely the basic idea of Davidson's semantics, and the only one I want to exploit to solve GFP.

Whence the following idea of solution to GFP for a language, $\mathcal{L}$, taken to be the Language of Science: Either it is possible to construct a recursive theory of truth *à la* Tarski for $\mathcal{L}$ without making the Mistake, and GFP is solved; or this is impossible and GFP is an indirect proof that $\mathcal{L}$ cannot be the Language of Science, and again, at least indirectly, GFP is solved. In the latter case, the impression of paradox is liable to linger until further, direct reasons are found for not mistaking $\mathcal{L}$ for the Language of Science.

5.3.2. Now, let me speak about Tarski and truth. In his famous 1935 paper (Wb) on the concept of truth (Tarski 1935), Tarski reasoned within the framework, taken to be universal, of the extensional, simple theory of types, and examined the possibility of *explicitly defining* the concept of truth for object-languages *grounded on* this theory, that is to say, obtained from a segment (possibly the totality) of its language by adding finitely many constants of certain categories. We all remember the results obtained: (1) Tarski proposed a method for explicitly defining truth for languages of finite order through an explicit definition of the relation of satisfaction, itself obtained by a conversion of a recursive definition of this relation; (2) He demonstrated the impossibility of an explicit definition of truth for languages of infinite order; (3) He indicated that the nowadays so-called *minimal* axiomatic theory of truth for an infinite-order language, whose axioms are the so-called *T-sentences* for that language, is too weak for one to be able to prove the semantic version of the fundamental laws of logic there, and that the same is so of the extensions obtained from this theory by adding one or another of these laws as new axiom. [In his post-scriptum of 1936 (Tarski 1935), Tarski was to take into consideration languages other than those to which he had limited himself up to that point, in particular to languages grounded on some set theory or other of Zermelo and his successors.]

The path that led Tarski to an explicit definition of truth for a language, $\mathcal{L}$, when such a definition is possible, goes by way of the conversion of a recursive definition of satisfaction for $\mathcal{L}$ into an explicit definition. If one skips this step to go directly from a recursive definition of satisfaction to the explicit definition of truth in terms of satisfaction, the two definitions together constitute what I am calling here *a recursive* ***definition*** *of truth à la Tarski* for $\mathcal{L}$. In Wb, whenever Tarski constructed an *explicit* definition of truth for $\mathcal{L}$, a *recursive* definition of truth was available, but Tarski did not turn his attention to this point. If he did not do it, it is because he was seeking an explicit definition and that, if a recursive definition of truth for $\mathcal{L}$ is possible at all within the chosen framework, that of the extensional, simple theory of types, it can always be converted into an explicit definition. If $\mathcal{L}$ is of finite order, a recursive definition is quite possible and so is its conversion, why therefore would he have turned his attention to? And if $\mathcal{L}$ is of infinite order, a recursive definition is impossible. Otherwise, by conversion, an explicit definition would be possible as well, which is impossible [see above, result (2)]. Therefore, the question did not come up.

Yet, recursive definitions have their own advantages, an advantage over minimal theories, of course, whose essential weakness they do not share, but indeed an advantage also over explicit definitions. Tarski's *negative* theorem mentioned above is known to hold for many languages. Sometimes, the corresponding *positive* theorem for recursive definition of truth holds, but sometimes not.

In the following examples (theorems A-C), ZFC is Zermelo-Fraenkel set theory with axiom of choice [and without excluding individuals (in the sense of *Urelemente*)]. I note SSTT^{α} the initial (maybe total) segment of order $\alpha \leq \omega$ of the *monadic*, extensional, simple theory of types (with axiom of infinite and axiom of choice). $\mathrm{SSTT} = \mathrm{SSTT}^{\omega}$ is the simplest version of the simple theory of types, to which, as is well known, STT, the full, extensional, simple theory of types is reducible thanks to, e.g., Kuratowski's definition of ordered pairs. Let us say that a language, $\mathcal{L}$, is an *admissible* extension of the language of ZFC (SSTT^{α}, respectively) if, and only if, $\mathcal{L}$ is obtained from the latter language by adding finitely many constants each one of which is either a singular term or a predicate or functor of such a category that its addition is possible without adding new variables. If $\mathcal{L}$ is *such* an extension and the same is so of a certain extension, $\mathcal{M}$, of $\mathcal{L}$, let us say, naturally, that $\mathcal{M}$ is an *admissible* extension of $\mathcal{L}$. ZFC and SSTT^{α} for $\alpha \geq 4$ (α must be ≥ 4 for SSTT^{α} to contain Russell arithmetic) are of interest for us insofar as, *prima facie* (but see below), the Language of Science could plausibly be given the form of an admissible extension of any one of them. We know from Tarski that, *if a language, $\mathcal{L}$, is an admissible extension of the language of* ZFC (SSTT^{α} *with* $\alpha \geq 4$, *respectively*), *then an **explicit** definition of truth for $\mathcal{L}$ is **impossible** in any admissible extension of $\mathcal{L}$.* On the other hand, the corresponding results concerning *recursive* definitions of truth are the following.

Theorem A.—*Let $\mathcal{L}$ be an admissible extension of the language of* ZFC. *A **recursive** definition of truth for $\mathcal{L}$ is **possible** in some admissible extension of $\mathcal{L}$.*

This *positive* result is well known and very easy to prove, see Sect. 5.4.1.

Theorem B.—*Let $\mathcal{L}$ be an admissible extension of the language of* SSTT^{ω}. *A **recursive** definition of truth for $\mathcal{L}$ is **impossible** in any admissible extension of $\mathcal{L}$.*

This *negative* result is also well known, and hardly less easy to prove than theorem A, see Sect. 5.4.2.

Theorem C.—*Let $\mathcal{L}$ be an admissible extension of the language of* SSTT^{n} *with n natural number* ≥ 4. *A **recursive** definition of truth for $\mathcal{L}$ is **possible** in some admissible extension of $\mathcal{L}$.*

This result is the *positive* answer to a question that does not seem to have attracted logicians' attention. It is not that easy to prove, see the proof I propose in Sect. 5.4.3.

5.3.3 In a more general way, Tarski supposed a *translation* of an object-language, $\mathcal{L}$, into a metalanguage, $\mathcal{M}$, to be given, and sought the conditions of possibility of an *explicit definition* of truth for $\mathcal{L}$ in $\mathcal{M}$ relative to this translation—retrospectively, one can say that, in Wb, it went without saying that the translation was homographic[10]. He could just as well have taken interest in the less restrictive conditions of possibility of a *recursive definition* of truth for $\mathcal{L}$ in $\mathcal{M}$ relative to this translation (comp. above, § 5.3.2). Davidson starts, inversely, from a language, $\mathcal{L}$, whose meaning may be unknown to us, and asks for what form an *interpretation* of $\mathcal{L}$ in our used language, $\mathcal{M}$, supposed to give us this meaning should take. His answer is that such an interpretation should take the form of a *recursive* ***theory*** *of truth à la Tarski* for $\mathcal{L}$ in $\mathcal{M}$.

Actually, if such a recursive theory of truth is available, then it is possible recursively to define a (unique up to alphabetical change of bound variables) translation of $\mathcal{L}$ into $\mathcal{M}$ by following the clauses of the recursive theory of truth under consideration. Say that this translation *canonically* corresponds to that theory of truth, or that it is the *canonical* translation corresponding to that theory. The idea for a solution of GFP envisioned in the present paper can now be described in the following two ways. To solve this paradox for a language, $\mathcal{L}$, taken to be the Language of Science, it would suffice to show that it is possible, *without making the Mistake*, to construct a recursive *theory* of truth *à la* Tarski for $\mathcal{L}$ in some extension, $\mathcal{M}$, of $\mathcal{L}$, such that the corresponding canonical translation is homographic—or, equivalently, to construct a recursive *definition* of truth *à la* Tarski, corresponding to the homographic translation, for $\mathcal{L}$ in some extension, $\mathcal{M}$, of $\mathcal{L}$. If the latter construction is worked out for an *admissible* extension, $\mathcal{L}$, of ZFC ($SSTT^{\alpha}$, with $\alpha \geq 4$, respectively) in an *admissible* extension, $\mathcal{M}$, of $\mathcal{L}$, then the italicized condition above, relative to the Mistake, is obviously fulfilled.

It thus follows from theorems A-C that GFP *is* solvable in this way for any admissible extension of the language of ZFC (th. A) or $SSTT^{n}$ for $n \geq 4$ (th. C)—but *not* for any admissible extension of the language of $SSTT^{\omega}$ (th. B). I am not prepared here to enter into a discussion about the very notion of Language of Science, but I think that there are some *direct* reasons, independent of GFP, why such infinite-order languages as admissible extensions of the language of $SSTT^{\omega}$ could *not* play the part of the Language of Science (see above, § 5.3.1, last paragraph).

5.4 Proofs of Theorems About Recursive Definition of Truth Stated in the Preceding Section[11]

Let me rest content with giving proof of theorem A (B, C respectively) for a simple, exemplary, admissible extension of the language of ZFC ($SSTT^{\omega}$, $SSTT^{n}$ for $n = 6$ respectively), for it will become self-evident that the same method of proof could have been applied to any other explicitly given admissible extension of ZFC ($SSTT^{\omega}$, $SSTT^{n}$ for $n \geq 4$ respectively).

[10] It goes without saying that I am here borrowing this qualifier not from geometry, but from linguistics.

[11] The non-mathematically-minded reader may skip this section and go directly to Sect. 5.5.

5.4.1 Proof of Theorem A

The intended universe of ZFC is the class of what I shall call *objects*, viz. individuals (*Urelemente*) and sets. In one possible version, the signs of the language of ZFC are the variables, viz., the terms of a certain sequence (indexed by the set of non-null natural numbers) of objects, $\mathbf{v} = (\mathbf{v}_k)_{k \geq 1}$; the constants "¬", "∨", "∃", "=", "Set" (monadic predicate of sethood) and "∈" (dyadic predicate of membership); punctuation marks "(" and ")". Let $\mathcal{L}$ be the admissible extension obtained from that language by adding, for example, the constant "a" of the category of singular terms, and the constant "P" of the category of triadic predicates. The rules of formation are the usual ones. In the definitions below, "*x*", "*y*", and "*z*" are (primitive) variables of $\mathcal{L}$, and non-primitive symbols are contextually definable: "σ"and "τ" are sequence (of objects) variables; "$\mathbf{t}_1$", "$\mathbf{t}_2$ ", and "$\mathbf{t}_3$", term (of $\mathcal{L}$) variables; "**A**" and "**B**", (open or closed) sentence (of $\mathcal{L}$) variables; "*i*", "*j*" and "*k*", non-null natural number variables; "⌜" and "⌝", Quine's quasi-quotation marks; "$\Leftrightarrow_{ij}$", the operator of formal equivalence relative to variables "*i*" and "*j*"; etc.

We shall begin with an explicit definition of a predicate of relative denotation, "$\mathrm{Den}_{\mathcal{L}}$", for $\mathcal{L}$. Then we shall recursively define "$\mathrm{Sat}_{\mathcal{L}}$" in terms of the eliminable "$\mathrm{Den}_{\mathcal{L}}$". Finally, we shall explicitly define "$\mathrm{Tr}_{\mathcal{L}}$" in terms of "$\mathrm{Sat}_{\mathcal{L}}$". Thus, "$\mathrm{Tr}_{\mathcal{L}}$" will have been recursively defined in an admissible extension obtained from $\mathcal{L}$ by adding constants of the elementary syntax of $\mathcal{L}$ and the primitive predicate "$\mathrm{Sat}_{\mathcal{L}}$". There is no need to worry about coding the objects of this syntax. They are simply supposed to be themselves already there, somewhere in the intended universe of ZFC.

Denotation of a Term Relative to a Sequence The terms of a language are variables and constants of the same category as some variables; those of $\mathcal{L}$ are $\mathbf{v}_k$ for $k \geq 1$ and "a".

$x\ \mathrm{Den}_{\mathcal{L}}\ y, z \Leftrightarrow_{\mathrm{df}}$ *x* and *z* are a term, **t**, and a sequence, $\sigma = (\sigma_k)_{k \geq 1}$, respectively, such that $[(\mathbf{t}$ is of the form $\mathbf{v}_k\ \&\ \sigma_k = y) \vee (\mathbf{t} =$ "a" $\&\ \mathrm{a} = y)]$.

Thus, $\mathbf{v}_k\ \mathrm{Den}_{\mathcal{L}}\ y, \sigma \Leftrightarrow_k \sigma_k = y$ and "a" $\mathrm{Den}_{\mathcal{L}}\ y, \sigma \Leftrightarrow \mathrm{a} = y$.

Satisfaction of a Sentence by a Sequence A first clause insures that the dyadic relation $\mathrm{Sat}_{\mathcal{L}}$ can only hold between a sequence of objects and a sentence of $\mathcal{L}$. Four clauses then fix the conditions of satisfaction of an atomic sentence of $\mathcal{L}$ by a sequence:

- $\sigma\ \mathrm{Sat}_{\mathcal{L}}\ \ulcorner \mathbf{t}_1 = \mathbf{t}_2 \urcorner \Leftrightarrow \exists x \exists y (\mathbf{t}_1\ \mathrm{Den}_{\mathcal{L}}\ x, \sigma\ \ \&\ \ \mathbf{t}_2\ \mathrm{Den}_{\mathcal{L}}\ y, \sigma\ \ \&\ \ x = y)$;
- $\sigma\ \mathrm{Sat}_{\mathcal{L}}\ \ulcorner \mathrm{Set}\mathbf{t}_1 \urcorner \Leftrightarrow \exists x (\mathbf{t}_1\ \mathrm{Den}_{\mathcal{L}}\ x, \sigma\ \ \&\ \ \mathrm{Set}x)$;
- $\sigma\ \mathrm{Sat}_{\mathcal{L}}\ \ulcorner \mathbf{t}_1 \in \mathbf{t}_2 \urcorner \Leftrightarrow \exists x \exists y\ (\mathbf{t}_1\ \mathrm{Den}_{\mathcal{L}}\ x, \sigma\ \ \&\ \ \mathbf{t}_2\ \mathrm{Den}_{\mathcal{L}}\ y, \sigma\ \ \&\ \ x \in y)$;
- $\sigma\ \mathrm{Sat}_{\mathcal{L}}\ \ulcorner \mathrm{P}\mathbf{t}_1\mathbf{t}_2\mathbf{t}_3 \urcorner \Leftrightarrow \exists x \exists y \exists z (\mathbf{t}_1\ \mathrm{Den}_{\mathcal{L}}\ x, \sigma\ \&\ \mathbf{t}_2\ \mathrm{Den}_{\mathcal{L}}\ y, \sigma\ \&\ \mathbf{t}_3\ \mathrm{Den}_{\mathcal{L}}\ z, \sigma\ \&\ \mathrm{P}xyz)$.

Three clauses finally fix the conditions of satisfaction of a non-atomic sentence by σ according to the satisfaction of shorter sentences by this same sequence or by others, $\tau = (\tau_k)_{k \geq 1}$, connected to it:

- $\sigma\ \mathrm{Sat}_{\mathcal{L}}\ \ulcorner (\neg \mathrm{A}) \urcorner \Leftrightarrow \neg (\sigma\ \mathrm{Sat}_{\mathcal{L}}\ \mathbf{A})$;
- $\sigma\ \mathrm{Sat}_{\mathcal{L}}\ \ulcorner (\ \mathbf{A} \vee \mathbf{B}) \urcorner \Leftrightarrow (\sigma\ \mathrm{Sat}_{\mathcal{L}}\ \mathbf{A} \vee \sigma\ \mathrm{Sat}_{\mathcal{L}}\ \mathbf{B})$;
- $\sigma\ \mathrm{Sat}_{\mathcal{L}}\ \ulcorner \exists \mathbf{v}_i\ (\mathbf{A}) \urcorner \Leftrightarrow_i \exists \tau ((j \neq i \Rightarrow_j \tau_j = \sigma_j)\ \&\ \ \tau\ \mathrm{Sat}_{\mathcal{L}}\ \mathbf{A})$.

Truth of a Statement It is easily shown that a statement (closed sentence) of $\mathcal{L}$ is satisfied by any sequence or by none, whence the definition sought for the truth predicate, "$\mathrm{Tr}_{\mathcal{L}}$", for $\mathcal{L}$ in an admissible extension of $\mathcal{L}$:

$$\mathrm{Tr}_{\mathcal{L}}\mathbf{A} \Leftrightarrow_{\mathrm{df}} \mathbf{A} \text{ is a statement of } \mathcal{L} \;\&\; \forall\sigma(\sigma \text{ Sat}_{\mathcal{L}}\, \mathbf{A}).$$

5.4.2 Proof of Theorem B

The intended universe of SSTT^{ω} is composed of individuals and classes corresponding to a simply infinite hierarchy of types (or orders), viz., the type (or order) 1 for individuals, 2 for classes of individuals, 3 for classes of classes of individuals, etc. In one possible version, the signs of the language of SSTT^{ω} are, but for differences to be presently explained, the same as those of the language of ZFC. Variables are typed: for any explicitly given $i \geq 1$, the variables of order i, ranging over the domain of the entities of order i, are the terms of a certain sequence, or, more precisely, K-sequence, noted $\mathbf{v}_{\mathrm{K}}{}^{(i)} = (\mathbf{v}_{\mathrm{K}}{}^{(i)}{}_{k})_{k\geq 1}$, where a K-sequence is a class of K-ordered pairs of a certain sort (see below), and a K-ordered pair (of entities of the same order) is an ordered pair as coded, or defined, by Kuratowski. "Set" and "$\in$" have been eliminated, and "=" maintained as corresponding to identity between individuals.[12] Natural numbers are assumed to be defined *à la* Russell and, except for any duly marked exception, to be of the lowest possible order, viz., 3, their definition is of order 4, and so is Russell arithmetic. Now, let $\mathcal{L}^{\omega}$ be the admissible extension of the language of SSTT^{ω} obtained by adding, for example, the constants "a" of the category of singular terms, denoting the individual a, and "P" of the category of dyadic predicates whose first argument values are entities of order 3 and second argument values entities of order 5. Further notions and notations are progressively introduced when needed.

It is impossible to construct a recursive definition of truth for $\mathcal{L}^{\omega}$ in any admissible extension of $\mathcal{L}^{\omega}$, for such a definition would be convertible in this extension into an explicit definition of truth for $\mathcal{L}^{\omega}$, which since Tarski we know is impossible. However it is easy to find a *method* for recursively defining truth for any explicitly given, *finite*-order, initial segment of $\mathcal{L}^{\omega}$, in some admissible extension of $\mathcal{L}^{\omega}$.

Let us present this method by means of an example, by constructing a recursive definition of truth for the initial fragment, $\mathcal{L}^{6}$, of $\mathcal{L}^{\omega}$ obtained by eliminating all the

[12] It is well known that "=" is definable in the language of SSTT^{ω} in terms of other primitives, but the same is not so for the language of any explicitly given, finite, initial segment, SSTT^{n}, of SSTT^{ω}. Whence my maintaining of "=" in the language of SSTT^{ω}, for the sake of the overall simplicity of the present Sect. 5.4. I maintain "=" as primitive only for individuals, because identity for two entities of any explicitly given order > 1 is definable in terms of that primitive.

variables of order >6. (This exercise will also prove useful in the next Sect. 5.4.3.) But for a few differences, this definition resembles that of Sect. 5.4.1.

Denotation of a Term Relative to six K-sequences of Entities of Order 1, . . . , 6 *Respectively* The terms of $\mathcal{L}^6$ are, for each explicitly given i, $\mathbf{v}_{K}{}^{(i)}{}_k$ for $i \geq 1$, and "a". The relation in question, can only hold between a term of $\mathcal{L}^6$ of explicitly given order i such that $1 \leq i \leq 6$, an entity of order i, and six K-sequences, $\sigma_K{}^{(1)}, \ldots, \sigma_K{}^{(6)}$, of entities of order 1,. . ., 6 respectively. Indeed, there are six relations at stake here, which, by abuse of language, I shall uniformly note $\mathrm{Den}_{\mathcal{L}^6}$. Noting $\langle a, b\rangle_K$ the K-ordered pair whose terms are a and b (in this order),[13] for the relation $\mathrm{Den}_{\mathcal{L}^6}$ to hold between its eight arguments, they must more precisely be as follows:

1. a term, $\mathbf{t}^i$, of explicitly given order i such that $1 \leq i \leq 6$, and so of order 2 as an entity, if we hold with Tarski that an expression is a certain class of inscriptions and that inscriptions are individuals;[14]
2. an entity, y^i, of order i if the aforesaid term is of the form $\mathbf{v}_K{}^{(i)}{}_k$, and of order 1 if it is "a";
3. six sequences, $\sigma^{(1)} = (\sigma^{(1)}{}_k)_{k \geq 1}, \ldots, \sigma^{(6)} = (\sigma^{(6)}{}_k)_{k \geq 1}$, of entities of order 1, . . . , 6 respectively, which are classes of K-ordered pairs of certain form, from which the orders of the K-sequences are computable. See the chart below.

Sequence	$\sigma_K^{(1)}$	$\sigma_K^{(2)}$	$\sigma_K^{(3)}$	$\sigma_K^{(4)}$	$\sigma_K^{(5)}$	$\sigma_K^{(6)}$
Terms of the sequence	$\sigma_K^{(1)}{}_k$ for $k \geq 1$	$\sigma_K^{(2)}{}_k$ for $k \geq 1$	$\sigma_K^{(3)}{}_k$ for $k \geq 1$	$\sigma_K^{(4)}{}_k$ for $k \geq 1$	$\sigma_K^{(5)}{}_k$ for $k \geq 1$	$\sigma_K^{(6)}{}_k$ for $k \geq 1$
Order of these terms	1	2	3	4	5	6
Members of the sequence	$\langle k, \{\{\sigma_K^{(1)}{}_k\}\}\rangle_K$ for $k \geq 1$	$\langle k, \{\sigma_K^{(2)}{}_k\}\rangle_K$ for $k \geq 1$	$\langle k, \sigma_K^{(3)}{}_k\rangle_K$ for $k \geq 1$	$\langle \{k\}, \sigma_K^{(4)}{}_k\rangle_K$ for $k \geq 1$	$\langle \{\{k\}\}, \sigma_K^{(5)}{}_k\rangle_K$ for $k \geq 1$	$\langle \{\{\{k\}\}\}, \sigma_K^{(6)}{}_k\rangle_K$ for $k \geq 1$
Order of these members	5	5	5	6	7	8
Order of the sequence	6	6	6	7	8	9

[13] In Sect. 5.4.3 other ways of coding, or defining, ordered pairs will be put to work.

[14] Tarski's view in Wb was roughly as follows: *inscriptions* are concrete individuals with a form and a size; *expressions* of a language are equivalence classes of certain inscriptions with respect to the relation of having the same form and size; some of these expressions are *simple*, others are *complex*; complex ones result from simple ones through a finite process of *concatenation*, internal to order 2. For there to be, as needed, infinitely many expressions in the language under consideration, there should be infinitely many inscriptions. Tarski was perfectly lucid about the formidable problems surrounding such a requirement (see Tarski 1956 or 1983, p. 174).

If $i = 1$, then the relation $\mathrm{Den}_{\mathcal{L}^6}$ holds between such arguments if, and only if, $\mathbf{t}^1$ is of the form $\mathbf{v}_{\mathrm{K}}^{(1)}{}_k$ and $\sigma_{\mathrm{K}}^{(1)}{}_k = y^1$, or ($\mathbf{t}^1$ = a $\wedge$ a $= y^1$); and if $2 \leq i \leq 6$, then it holds if, and only if, $\mathbf{t}^i$ is of the form $\mathbf{v}_{\mathrm{K}}{}^{(i)}{}_k$ and $\sigma_{\mathrm{K}}{}^{(i)}{}_k = y^i$.

It would be easy, but space-consuming, to give to these considerations, for every explicitly given i such that $1 \leq i \leq 6$, the rigorous form of an explicit definition of "x^2 $\mathrm{Den}_{\mathcal{L}^6}$ y^i, $z^6{}_1$, $z^6{}_2$, $z^6{}_3$, $z^7{}_4$, $z^8{}_5$, $z^9{}_6$", and then to deduce that

$$\mathbf{v}_{\mathrm{K}}{}^{(i)}{}_k \ \mathrm{Den}_{\mathcal{L}^6} \ y^i, \sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \Leftrightarrow_k \sigma_{\mathrm{K}}{}^{(\mathrm{i})}{}_k = y^i;$$

$$\text{“}a\text{”} \ \mathrm{Den}_{\mathcal{L}^6} \ y^1, \sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \Leftrightarrow \mathrm{a} = y^1.$$

Satisfaction of a Sentence by six K-sequences of Entities of Order 1, ..., 6 *Respectively* A first clause insures that the heptadic relation $\mathrm{Sat}_{\mathcal{L}^6}$ can only hold between $\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)}$ and a sentence. Twelve (five plus six plus one) clauses then fix the conditions of satisfaction of an atomic sentence of $\mathcal{L}^6$ by $\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)}$:

- for any explicitly given i such that $1 \leq i \leq 5$:

$$\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ulcorner \mathbf{t}^{i+1}\mathbf{t}^i \urcorner \Leftrightarrow \exists x^i \exists x^{i+1}(\mathbf{t}^i \ \mathrm{Den}_{\mathcal{L}^6} \ x^i, \sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \ \& \ \ \mathbf{t}^{i+1} \ \mathrm{Den}_{\mathcal{L}^6} \ x^{i+1}, \sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \ \& \ \ x^{i+1}x^i);$$

- for any explicitly given i such that $1 \leq i \leq 6$:

$$\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ulcorner \mathbf{t}^i{}_1 = \mathbf{t}^i{}_2 \urcorner \Leftrightarrow \exists x^i \exists y^i(\mathbf{t}^i{}_1 \ \mathrm{Den}_{\mathcal{L}^6} \ x^i, \sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \ \& \ \ \mathbf{t}^i{}_2 \ \mathrm{Den}_{\mathcal{L}^6} \ y^i, \ \sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \ \& \ \ x^i = y^i);$$

- $\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ulcorner \mathbf{P}\mathbf{t}^3\mathbf{t}^5 \urcorner \Leftrightarrow \exists x^3 \exists x^5(\mathbf{t}^3 \ \mathrm{Den}_{\mathcal{L}^6} \ x^3, \sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \& \ \mathbf{t}^5 \ \mathrm{Den}_{\mathcal{L}^6} \ x^5, \sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \& \ \mathrm{P}x^3x^5)$.

Eight (one plus one plus six) clauses then fix the conditions of satisfaction of a non-atomic sentence by $\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)}$ according to the satisfaction of shorter sentences by these same sequences or by others connected with them; $\tau_{\mathrm{K}}^{(1)} = (\tau_{\mathrm{K}}^{(1)}k)_{k \geq 1}$, ..., $\tau_{\mathrm{K}}^{(6)} = (\tau_{\mathrm{K}}^{(6)}k)_{k \geq 1}$ are supposed to answer to the same constraints as $\sigma_{\mathrm{K}}^{(1)}$, ..., $\sigma_{\mathrm{K}}^{(6)}$ respectively.

- $\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ \ulcorner(\neg\mathbf{A})\urcorner \Leftrightarrow \neg(\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ \mathbf{A})$;
- $\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ \ulcorner(\mathbf{A} \vee \mathbf{B})\urcorner \Leftrightarrow (\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ \mathbf{A} \vee \sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ \mathbf{B})$;
- $\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ \ulcorner\exists\mathbf{v}_{\mathrm{K}}{}^{(1)}{}_k(\mathbf{A})\urcorner \Leftrightarrow_k \exists\tau_{\mathrm{K}}{}^{(1)}((j \neq k \Rightarrow_j \tau_{\mathrm{K}}{}^{(1)}{}_j = \sigma_{\mathrm{K}}{}^{(1)}{}_j) \ \& \ \tau_{\mathrm{K}}{}^{(1)}, \sigma_{\mathrm{K}}{}^{(2)}, \sigma_{\mathrm{K}}{}^{(3)}, \sigma_{\mathrm{K}}{}^{(4)}, \sigma_{\mathrm{K}}{}^{(5)}, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ \mathbf{A})$;
- $\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ulcorner\exists\mathbf{v}_{\mathrm{K}}{}^{(2)}{}_k(\mathbf{A})\urcorner \Leftrightarrow_k \exists\tau_{\mathrm{K}}{}^{(2)}((j \neq k \Rightarrow_j \tau_{\mathrm{K}}{}^{(2)}{}_j = \sigma_{\mathrm{K}}{}^{(2)}{}_j) \ \& \ \sigma_{\mathrm{K}}{}^{(1)}, \tau_{\mathrm{K}}{}^{(2)}, \sigma_{\mathrm{K}}{}^{(3)}, \sigma_{\mathrm{K}}{}^{(4)}, \sigma_{\mathrm{K}}{}^{(5)}, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ \mathbf{A})$;

....

- $\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ulcorner\exists\mathbf{v}_{\mathrm{K}}{}^{(6)}{}_k(\mathbf{A})\urcorner \Leftrightarrow_k \exists\tau_{\mathrm{K}}{}^{(6)}((j \neq k \Rightarrow_j \tau_{\mathrm{K}}{}^{(6)}{}_j = \sigma_{\mathrm{K}}{}^{(6)}{}_j) \ \& \ \sigma_{\mathrm{K}}{}^{(1)}, \sigma_{\mathrm{K}}{}^{(2)}, \sigma_{\mathrm{K}}{}^{(3)}, \sigma_{\mathrm{K}}{}^{(4)}, \sigma_{\mathrm{K}}{}^{(5)}, \tau_{\mathrm{K}}{}^{(6)} \ \mathrm{Sat}_{\mathcal{L}^6} \ \mathbf{A})$.

Truth of a Sentence Here is the definition of truth predicate in terms of satisfaction for $\mathcal{L}^6$ sought after in the admissible extension of $\mathcal{L}^6$ obtained by adding constants of the elementary syntax of $\mathcal{L}^6$ and the predicate "$\mathrm{Sat}_{\mathcal{L}^6}$":

$$\mathrm{Tr}_{\mathcal{L}^6}\mathbf{A} \Leftrightarrow_{\mathrm{df}} (\mathbf{A} \text{ is a statement of } \mathcal{L}^6 \;\&\; \forall\sigma_{\mathrm{K}}{}^{(1)} \ldots \forall\sigma_{\mathrm{K}}{}^{(6)}(\sigma_{\mathrm{K}}{}^{(1)}, \ldots, \sigma_{\mathrm{K}}{}^{(6)}\, Sat_{\mathcal{L}^6}\, \mathbf{A})).$$

The method illustrated is obviously applicable for any explicitly given, finite-order, initial fragment of $\mathcal{L}^\omega$ in order recursively to define a truth predicate for it in some admissible extension of $\mathcal{L}^\omega$. All these fragments form a naturally increasing sequence, whose limit, in a sense, is their union, and extensions of truth predicates for these fragments do so as well. The union of these fragments is $\mathcal{L}^\omega$, and the union of these extensions is, from some (if any) transcendent point of view, the class of true statements of $\mathcal{L}^\omega$, but this class cannot be the extension of any predicate of any admissible extension of $\mathcal{L}^\omega$. What is possible for the terms of a sequence need not be so for the limit. I repeat: A recursive definition of truth for $\mathcal{L}^\omega$ itself is impossible in any admissible extension of $\mathcal{L}^\omega$.

5.4.3 Proof of Theorem C

SSTT^n is the initial segment of SSTT^ω of order n for any explicitly given $n \geq 4$, and $\mathcal{L}^n$ is the corresponding initial segment of $\mathcal{L}^\omega$ as in Sect. 5.4.2. In Sect. 5.4.3.1, we prove that a recursive definition of truth is possible for $n \geq 4$ in the admissible extension obtained from $\mathcal{L}^n$ by adding primitive pairing functors, primitive constants of elementary syntax of $\mathcal{L}^n$, and primitive predicates of satisfaction for $\mathcal{L}^n$. The proof is based on a stratagem devised by Quine and Boolos (henceforth QB's trick). In Sect. 5.4.3.2, we try to prove that one can dispense with adding primitive pairing functors. It turns out that, with the best definitions of ordered pair available and QB's trick again, one obtains the result hoped for only for $n \geq 5$, not for $n = 4$. The latter case would merit further study, something remaining to be undertaken.

5.4.3.1. The K-sequences $\sigma_{\mathrm{K}}^{(1)}, \ldots, \sigma_{\mathrm{K}}^{(6)}$ (and the same holds for $\tau_{\mathrm{K}}^{(1)}, \ldots, \tau_{\mathrm{K}}^{(6)}$) involved in the recursive definition of truth for $\mathcal{L}^6$ given in Sect. 5.4.2 are of order 6, 6, 6, 7, 8, 9 respectively, so that the last three fall outside the intended universe of $\mathcal{L}^6$. Note that Kuratowski's definition of ordered pair, according to which an ordered pair is two orders higher than its terms, is to a large extent responsible for such an overflow. Let us abandon Kuratowski's definition and in place of it, for every explicitly given i such that $1 \leq i \leq 6$, use a primitive pairing functor,[15] say (by abuse of language, as if there were only one functor instead of six) "C", that can be attached

[15] Bourbaki did it, *mutatis mutandis*, in the first two editions, dated 1954 and 1960, of his fascicule containing the chapter on set theory, see Bourbaki 1954, chap. 2, § 1, n° 1 and § 2, n°1, but abandoned it in the third edition, dated 1966, and naturally in the one volume edition of book I, dated 1970. Curiously, the English translation of book I, dated 1968, goes back to the French, first edition of this fascicule instead of the second one.

to two terms, $\mathbf{t}^i{}_1, \mathbf{t}^i{}_2$, of order i to form a term of the same order, $\ulcorner \mathrm{C}\mathbf{t}^i{}_1\mathbf{t}^i{}_2 \urcorner$, or rather, more suggestively, $\ulcorner\langle \mathbf{t}^i{}_1, \mathbf{t}^i{}_2\rangle_{\mathrm{C}}\urcorner$, and such that

$$\langle x^i, y^i{}_2\rangle_{\mathrm{C}} = \langle u^i, v^i\rangle_{\mathrm{C}} \Rightarrow (x^i = u^i \;\;\&\;\; y^i = v^i).$$

The K-sequences $\sigma_{\mathrm{K}}^{(1)}, \ldots, \sigma_{\mathrm{K}}^{(6)}$ can now be replaced by C-sequences, say $\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(6)}$, of order 4, 4, 4, 5, 6, 7 respectively, and only the last one falls outside the intended universe of $\mathcal{L}^6$.

It is possible to rid ourselves of this last C-sequence $\sigma_{\mathrm{C}}{}^{(6)}$ by the means of QB's trick.[16] Generally speaking, and using an outdated terminology dating back to Euler, the gist of QB's trick consists in coding a *single-valued function*, *f*, whose value at every argument, *x*, is a class, *fx*, by the *multiple-valued function* whose values at *x* are the members of *fx*. More specifically, here, the C-sequence $\sigma_{\mathrm{C}}{}^{(6)}$ can be coded by the relation, R, holding exactly between any natural number $k \geq 1$ and each element of $\sigma_{\mathrm{C}}{}^{(6)}{}_k$, this relation being itself coded by the class, Σ^6, of ordered pairs of the form $\langle\{\{k\}\}, x^5\rangle_{\mathrm{C}}$ such that $\mathrm{R}kx^5$. The sequence $\sigma_{\mathrm{C}}{}^{(6)}$, of order 7, is thus coded by the class Σ^6, of order 6, of the $\langle\{\{k\}\}, x^5\rangle_{\mathrm{C}}$ such that $\sigma_{\mathrm{C}}{}^{(6)}{}_k x^5$. So, for any j and any x^5, $\sigma_{\mathrm{C}}{}^{(6)}{}_k x^5$ if, and only if, $\Sigma^6\ \langle\{\{k\}\}, x^5\rangle_{\mathrm{C}}$; whence the two lemmas that will be used below (in the second one, T^6 is supposed to code $\tau_{\mathrm{C}}{}^{(6)}$ as Σ^6 does $\sigma_{\mathrm{C}}{}^{(6)}$):

$$\text{Lemma 1. } \sigma_{\mathrm{C}}{}^{(6)}{}_k = x^6 \Leftrightarrow_k \left(x^6 x^5 \Leftrightarrow_{x^5} \Sigma^6\langle\{\{k\}\}, x^5\rangle_{\mathrm{C}}\right);$$

$$\text{Lemma 2. } \tau_{\mathrm{C}}{}^{(6)}{}_k = \sigma_{\mathrm{C}}{}^{(6)}{}_k \Leftrightarrow_k \left(\mathrm{T}^6\langle\{\{k\}\}, x^5\rangle_{\mathrm{C}} \Leftrightarrow_{x^5} \Sigma^6\langle\{\{k\}\}, x^5\rangle_{\mathrm{C}}\right).$$

But what about the K-sequence $\mathbf{v}_{\mathrm{K}}{}^{(i)}$ for any explicitly given i such that $1 \leq i \leq 6$? "$\mathbf{v}_{\mathrm{K}}{}^{(i)}$" is not a variable, but a (syntactic) constant, so that it is not the order of $\mathbf{v}_{\mathrm{K}}{}^{(i)}$, but that of its *members,* that matters. The *terms* of $\mathbf{v}_{\mathrm{K}}{}^{(i)}$, viz., variables of order i, are entities of order 2, thus its *members* are of order 5 (see the chart of Sect. 5.4.2), and there is no problem for them to be present in the intended universe of $\mathcal{L}^6$, nor would there be any problem for $\mathcal{L}^n$, for any explicitly given $n \geq 5$, with the K-sequences $\mathbf{v}_{\mathrm{K}}{}^{(i)}$, for any explicitly given i such that $1 \leq i \leq n$. However, there would *be* a problem for $\mathcal{L}^4$ with the K-sequences $\mathbf{v}_{\mathrm{K}}{}^{(i)}$, for any explicitly given i such that $1 \leq i \leq 4$. For the sake of uniformity, I shall replace the K-sequences $\mathbf{v}_{\mathrm{K}}{}^{(i)}$ of variables of $\mathcal{L}^6$ by the C-sequences of the same *terms* whose *members* are of order 3.

Now let me, as briefly as possible, present the recursive definition of truth sought for, which does not commit one to anything outside the intended universe of $\mathcal{L}^6$.

Relative Denotation The explicitly definable relation $\mathrm{Den}_{\mathcal{L}^6}$ can only hold between a term of $\mathcal{L}^6$, an entity of the same order as this term, five C-sequences, $\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}$, and a class, Σ^6, of ordered pairs of the form $\langle\{\{k\}\}, x^5\rangle_{\mathrm{C}}$; and it is such that, for any explicitly given i such that $1 \leq i \leq 5$,

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[16] Quine (1952) and Boolos (1985) used it to construct a recursive definition of truth for the languages of ML and ZF2 respectively, in the extension obtained from that language by adding a predicate of satisfaction.

$$\mathbf{v}_{\mathrm{C}}{}^{(i)}{}_k \text{ Den}_{\mathcal{L}^6}\ x^i, \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \Leftrightarrow_k \sigma_{\mathrm{C}}{}^{(i)}{}_k = x^i;$$

$$\mathbf{v}_{\mathrm{C}}{}^{(6)}{}_k \text{ Den}_{\mathcal{L}^6}\ x^6, \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \Leftrightarrow_k \left(x^6 x^5 \Leftrightarrow_{x^5} \Sigma^6\langle\{\{k\}\}, x^5\rangle_{\mathrm{C}}\right);$$[17]

$$\text{"a" Den}_{\mathcal{L}^6}\ x^1, \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \Leftrightarrow \mathrm{a} = x^1.$$

Satisfaction The first clause stipulates that the relation $\mathrm{Sat}_{\mathcal{L}^6}$ can only take place between five C-sequences, $\sigma^{(1)}, \ldots, \sigma^{(5)}$, a class, Σ^6, of ordered pairs of the form $\langle\{\{k\}\}, u\rangle_C$, and a sentence of $\mathcal{L}^6$. Clauses for atomic sentences:

- for any explicitly given i such that $1 \le i \le 5$:

$$\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \ulcorner \mathbf{t}^{i+1}\mathbf{t}^{i} \urcorner \Leftrightarrow \exists x^i \exists x^{i+1}(\mathbf{t}^i{}_1 \text{ Den}_{\mathcal{L}^6}\ x^i, \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \ \&\ \mathbf{t}^i{}_2 \text{ Den}_{\mathcal{L}^6}\ x^{i+1}, \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \ \&\ x^{i+1}x^i);$$

- for any explicitly given i such that $1 \le i \le 6$:

$$\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \ulcorner \mathbf{t}^i{}_1\mathbf{t}^i{}_2 \urcorner \Leftrightarrow \exists x^i \exists y^i(\mathbf{t}^i{}_1 \text{ Den}_{\mathcal{L}^6}\ x^i, \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \ \&\ \mathbf{t}^i{}_2 \text{Den}_{\mathcal{L}^6}\ y^i, \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \ \&\ x^i = y^i);$$

- $\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \ulcorner \mathrm{P}\mathbf{t}^3\mathbf{t}^5 \urcorner \Leftrightarrow \exists x^3 \exists x^5(\mathbf{t}^3 \text{ Den}_{\mathcal{L}^6}\ x^3, \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \ \&\ \mathbf{t}^5 \text{ Den}_{\mathcal{L}^6}\ x^5, \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \ \&\ \mathrm{P}x^3x^5).$

Clauses for non-atomic sentences:

- $\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \ulcorner(\neg\mathrm{A})\urcorner \Leftrightarrow \neg(\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \mathbf{A});$
- $\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \ulcorner(\mathbf{A} \vee \mathbf{B})\urcorner \Leftrightarrow (\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \mathbf{A}) \vee \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \mathbf{B});$
- $\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \ulcorner \exists \mathrm{v}_{\mathrm{C}}{}^{(1)}{}_k(\mathbf{A})\urcorner \Leftrightarrow_k \exists \tau_{\mathrm{C}}{}^{(1)}\left(\left(j \neq k \Rightarrow_j \tau_{\mathrm{C}}{}^{(1)}{}_j = \sigma_{\mathrm{C}}{}^{(1)}{}_j\right) \ \&\ \tau_{\mathrm{C}}{}^{(1)}, \sigma_{\mathrm{C}}{}^{(2)}, \sigma_{\mathrm{C}}{}^{(3)}, \sigma_{\mathrm{C}}{}^{(4)}, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \mathbf{A}\right);$
- $\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \ulcorner \exists \mathbf{v}_{\mathrm{C}}{}^{(2)}{}_k(\mathbf{A})\urcorner \Leftrightarrow_k \exists \tau_{\mathrm{C}}{}^{(2)}\left(\left(j \neq k \Rightarrow_j \tau_{\mathrm{C}}{}^{(2)}{}_j = \sigma_{\mathrm{C}}{}^{(2)}{}_j\right) \ \&\ \sigma_{\mathrm{C}}{}^{(1)}, \sigma_{\mathrm{C}}{}^{(2)}, \sigma_{\mathrm{C}}{}^{(3)}, \sigma_{\mathrm{C}}{}^{(4)}, \tau_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \mathbf{A}\right);$

....

- $\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \ulcorner \exists \mathbf{v}_{\mathrm{C}}{}^{(5)}{}_k(\mathbf{A})\urcorner \Leftrightarrow_k \exists \tau_{\mathrm{C}}{}^{(5)}\left(\left(j \neq k \Rightarrow_j \tau_{\mathrm{C}}{}^{(5)}{}_j = \sigma_{\mathrm{C}}{}^{(5)}{}_j\right) \ \&\ \sigma_{\mathrm{C}}{}^{(1)}, \sigma_{\mathrm{C}}{}^{(2)}, \sigma_{\mathrm{C}}{}^{(3)}, \sigma_{\mathrm{C}}{}^{(4)}, \tau_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \mathbf{A}\right);$
- $\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \ulcorner \exists \mathbf{v}_{\mathrm{C}}{}^{(6)}{}_k(\mathbf{A})\urcorner \Leftrightarrow_k \exists \mathrm{T}^6\left(\left(j \neq k \Rightarrow_j \left(\mathrm{T}^6\langle\{\{j\}\}, x^5\rangle_{\mathrm{C}} \Leftrightarrow_{x^5} \Sigma^6\langle\{\{j\}\}, x^5\rangle_{\mathrm{C}}\right)\right) \ \&\ \sigma_{\mathrm{C}}{}^{(1)}, \ldots, \sigma_{\mathrm{C}}{}^{(5)}, \mathrm{T}^6 \text{ Sat}_{\mathcal{L}^6} \mathbf{A}\right).$

Truth $\mathrm{Tr}_{\mathcal{L}}{}^6\mathrm{A} \Leftrightarrow_{\mathrm{df}} (\mathbf{A} \text{ is a statement of } \mathcal{L}^6 \ \&\ \forall\sigma_{\mathrm{C}}{}^{(1)} \ldots \forall\sigma_{\mathrm{C}}{}^{(5)} \forall\Sigma^6 (\sigma_{\mathrm{C}}{}^{(1)}, \ldots, \Sigma_{\mathrm{C}}{}^{(5)}, \Sigma^6 \text{ Sat}_{\mathcal{L}^6} \mathbf{A})).$[18]

[17] See lemma 1.
[18] See lemma 2.

The analysis of the construction of this definition of truth for $\mathcal{L}^6$ shows that it is easily transposable to $\mathcal{L}^n$ for any explicitly given $n \geq 4$.

5.4.3.2. One can do without primitive pairing functors, at least if $n \geq 5$.

To contain the overflow connected with the use of the notion of K-ordered pair, for which an ordered pair is two orders higher than its terms, one can, to begin with, replace this notion by another one that is more economical in terms of order. The first definition of the ordered pair such that an ordered pair is only one order higher than its terms is due to Quine and dates back to 1941.[19] Let us say that a Q_1-ordered pair, $\langle a, b\rangle_{Q_1}$, is, by definition, the class of singletons of member of a and of complements (with respect to the class of classes of the same order as a and b) of singletons of member of b. This definition only applies if a and b are classes, entities of order ≥ 2. It is adequate insofar as it implies that two Q_1-ordered pairs are identical only if their homologous terms are.

Subsequently (in his 1945), Quine had an idea for a second definition according to which an ordered pair is of the same order as its terms.[20] Let us say that a Q_2-ordered pair, $\langle a, b\rangle_{Q_2}$, with a and b of order m high enough for what follows to have a sense, is, by definition, $\#a \cup \flat b$, where $\#a$ results from a by simultaneously replacing in each of its members every natural number of order $m-2$ (if any) by its immediate successor (so that 0 of order $m-2$ does not belong to any of its members), and $\flat y$ results from $\#y$ by adding 0 of order $m-2$ to each of its members. $\#a$ and $\flat b$, and therefore also $\#a \cup \flat b$, are of order m. The notion of Q_2-ordered pair can only apply to classes, a, b, whose members can contain natural numbers, *i.e.* to classes of order ≥ 5. This definition is also adequate insofar as it implies that two Q_2-ordered pairs can be equal only if their homologous terms are.[21]

The following chart gives the easily computable order, whose knowledge is subsequently useful, of certain entities:

C-sequence	$\sigma_C^{(1)}$	$\sigma_C^{(2)}$	$\sigma_C^{(3)}$	$\sigma_C^{(4)}$	$\sigma_C^{(5)}$	$\sigma_C^{(6)}$
Order of a and b for any *member*, $\langle a, b\rangle_C$, of the C-sequence	3	3	3	4	5	6
Order of the K-sequence of the same entities as the C-sequence	6	6	6	7	8	9
Order of the Q_1-sequence of the same entities as the C-sequence	5	5	5	6	7	8
Order of the Q_2-sequence of the same entities as the C-sequence					6	7

[19] It was first related by Goodman (1941, p. 150, n5) and subsequently by Quine (1945).

[20] Quine's second definition introduces the notion of natural number of any order whatsoever ≥ 3, but for us this is an exception. Everywhere else in Sect. 5.4 of the present article, natural numbers are of order 3.

[21] The two Quinean definitions are mentioned in Scott and McCarty 2008, but not in Kanamori 2003, in spite of the latter's being historically much richer than former. Indeed, that is quite in order, given the respective theoretical aims of those papers.

It is possible recursively to define truth for $\mathcal{L}^6$ by exclusively using defined notions of ordered pair at our disposal. For example, we can use the sequences $\sigma_{Q_1}{}^{(1)}$, $\sigma_{Q_1}{}^{(2)}, \sigma_{Q_1}{}^{(3)}, \sigma_{Q_1}{}^{(4)}, \sigma_{Q_2}{}^{(5)}, \sigma_{Q_2}{}^{(6)}$, of orders 5, 5, 5, 6, 6, 7 respectively, and then code the one sequence of order > 6 by a class of order 6 thanks to QB's trick.[22] (And the same is so, *mutatis mutandis*, for $\mathcal{L}^n$ for any explicitly given $n \geq 6$.) In the case of $\mathcal{L}^5$, we can use the sequences $\sigma_{Q_1}{}^{(1)}$, $\sigma_{Q_1}{}^{(2)}$, $\sigma_{Q_1}{}^{(3)}$, $\sigma_{Q_1}{}^{(4)}$, $\sigma_{Q_2}{}^{(5)}$, of orders 5, 5, 5, 6, 6 respectively, and code the two sequences of order > 5 by classes of order 5 thanks to QB's trick.[23] On the other hand, the case of $\mathcal{L}^4$ is quite different. If we use the sequence $\sigma_{Q_1}{}^{(1)}$, $\sigma_{Q_1}{}^{(2)}$, $\sigma_{Q_1}{}^{(3)}$, $\sigma_{Q_1}{}^{(4)}$, of orders 5, 5, 5, 6 respectively, which is the best we can do with definitions of ordered pair at our disposal, then we can code these four sequences of order > 4 by classes of orders 4, 4, 4, 5 respectively, thanks to QB's trick, but then how could we get rid of the remaining class of order > 4? I do not know.

5.5 Tension Existing Between the Proposed Solution of GFP and the Idea of Language of Science, and How to Solve it

Among the admissible extensions of the language of ZFC and among those of the language of SSTT^n (for $n \geq 4$) respectable candidates may be found to take on the role of Language of Science. The idea is not a new one and it is what prompted me to take an interest in these extensions. I have accepted the basic idea of Davidson's semantics, and theorems A and C have provided me with a solution to GFP for all these extensions and therefore for the candidates in question. However, found among the admissible extensions of SSTT$^\omega$ are also candidates just as qualified, *prima facie*, to play the role of Language of Science, but theorem B has made my solution to GFP inapplicable to such languages. I have found this to be an indirect reason to deny those languages the right to play the role of Language of Science. As for direct reasons that could justify this prohibition, it would fall to a serious analysis of the very idea of Language of Science to produce them.

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[22] Noting Σ^6 the sixth-order class coding the seventh-order sequence $\sigma_{Q_2}{}^{(6)}$ (and likewise with T^6 and $\tau Q_2{}^{(6)}$), the two lemmas to be proved and applied can be obtained from lemmas 1 and 2 by replacing "C" by "Q_2".

[23] Noting $\Sigma^5{}_1$ and $\Sigma^5{}_2$ the fifth-order classes coding sixth-order sequences $\sigma_{Q_1}{}^{(4)}$ and $\sigma_{Q_2}{}^{(5)}$ respectively (and likewise with T^5_1 and T^5_2, and $\tau_{Q_1}{}^{(4)}$ and $\tau_{Q_2}{}^{(5)}$), there are now four lemmas to be proved and applied:

$$\sigma_{Q_1}{}^{(4)}{}_k = x^4 \Leftrightarrow_k (x^4 x^3 \Leftrightarrow_{x^3} \Sigma^5{}_1 \langle k,\ x^3 \rangle_{Q_1});$$

$$\sigma_{Q_2}{}^{(5)}{}_k = x^5 \Leftrightarrow_k (x^5 x^4 \Leftrightarrow_{x^4} \Sigma^5{}_2 \langle \{k\}, x^4 \rangle_{Q_2});$$

$$\tau_{Q_1}{}^{(4)}{}_k = \sigma_{Q_1}{}^{(4)}{}_k \Leftrightarrow_k (T^5{}_1 \langle k, x^3 \rangle_{Q_1} \Leftrightarrow_{x^3} \Sigma^5{}_1 \langle k, x^3 \rangle Q_1);$$

$$\tau_{Q_2}{}^{(5)}{}_k = \sigma_{Q_2}{}^{(5)}{}_k \Leftrightarrow_k (T^5{}_1 \langle \{k\}, x^4 \rangle_{Q_2} \Leftrightarrow_{x^4} \Sigma^6{}_1 \langle \{k\}, x^4 \rangle_{Q_2}).$$

But another, more profound, difficulty arises once again involving the idea of Language of Science. If a recursive definition of truth *à la* Tarski for an admissible extension of the language of ZFC or of $SSTT^n$ (for $n \geq 4$) taken to be the Language of Science, is possible in some admissible extension of this language, such a definition is nevertheless impossible *in* this language *itself*. Admittedly, this is but a manifestation of the Liar paradox and has nothing to do with one or another resurgence of GFP, but the fact of the matter is that the truth predicate for the so-called Language of Science under consideration is excluded from this language. Hence, one of two things. Either the recursive definition of truth *à la* Tarski—Tarski, the founder of semantics as science!—for the Language of Science is not a matter of Science, or it is a matter of Science, but then the language in question is not the Language of Science.

One will no more be able to solve this dilemma than one could solve GFP by considering the Language of Science, no longer in a static way, as I have done up to this point, although not without some ulterior motive, but in a dynamic way, as the moving, unforeseeable multiplicity of historically and geographically situated languages ever put to work in the enterprise of knowledge. For what is targeted as Language of Science in the dilemma and was so already in GFP, is obviously a language corresponding to a unified, stabilized, tame form of knowledge to which the enterprise of knowledge in general and as such ultimately aspires. It would now be required that the explication of the content of the expressions of such a language be possible, not only without making the Mistake relative to that language, but also without going beyond its limits. Which would not only imply a post-Tarskian solution to the Liar, but also… Also what? A corresponding post-Davidsonian semantics? There is no doubt that, technically and philosophically, greatest difficulties lay in store for the enterprise.

Awaiting better times, I am inclined to relax a bit the requirement of unity weighing upon the idea of Language of Science. In the best of cases, the unity of the so-called Language of Science would be not that of a single language, but that of an extensible, finite class of *suitable* (in a sense to be specified) extensions of a single language. The latter could be, for example, the language of ZFC or that of $SSTT^n$ for some $n \geq 4$ [24], with *suitability* then specified as admissibility. The dilemma could be solved in that way, at least in those exemplary cases, and the solution to GFP proposed in the present article would pass the test unscathed.

[24] If this common sub-language and the rules governing the use of its signs were called "*logical*", and and the signs and rules proper to its admissible extensions, "*extra-logical*", I could be said to be proposing to renounce the unity of science dear to the Vienna Circle and even its linguistic unity, for its logical unity alone. And then one would find back the version of the logical universalism that I have defended in a recent article (Rouilhan 2012).

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