Chapter 22
Validity and Truth-Preservation

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Abstract The revisionary approach to semantic paradox is commonly thought to have a somewhat uncomfortable corollary, viz. that, on pain of triviality, we cannot affirm that all valid arguments preserve truth (Beall 2007, 2009; Field 2008, 2009b). We show that the standard arguments for this conclusion all break down once (i) the *structural rule of contraction* is restricted and (ii) how the premises can be aggregated—so that they can be said to *jointly* entail a given conclusion—is appropriately understood. In addition, we briefly rehearse some reasons for restricting structural contraction.

Logical orthodoxy has it that valid arguments preserve truth (see e.g. Etchemendy 1990; Harman 1986, 2009):

(VTP) If an argument is valid, then, if all its premises are true, then its conclusion is also true.

Intuitive as it may seem, this claim, on natural enough interpretations of 'if' and 'true', turns out to be highly problematic. Hartry Field has argued that its most immediate justification requires all the logical and semantic resources that yield the standard semantic version of Curry's Paradox. Worse yet, both Field and Jc Beall have observed that the claim that valid arguments preserve truth almost immediately

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yields absurdity via Curry-like reasoning in most logics (Field 2008; Beall 2007, 2009). Moreover, Field has argued that, by Gödel's Second Incompleteness Theorem, any semantic theory that declares all valid arguments truth-preserving must be inconsistent (Field 2006, 2008, 2009b, 2009a). We can't coherently require that valid arguments preserve truth, or so the thought goes.[1]

Two main ingredients are required for this conclusion: that the conditional occurring in VTP detaches, i.e. satisfies *Modus Ponens*, and the *naïve view of truth*, viz. that (at the very least) the truth predicate must satisfy the (unrestricted) T-Scheme

$$(\textit{T-Scheme})\ Tr(\ulcorner\alpha\urcorner) \leftrightarrow \alpha,$$

where $Tr(...)$ expresses truth, and $\ulcorner\alpha\urcorner$ is a name of α. Both assumptions lie at the heart of the leading contemporary *revisionary approaches* to semantic paradox. These include recent implementations (see e.g. Brady 2006; Field 2003, 2007, 2008; Horsten 2009) of the *paracomplete* approach inspired by Martin and Woodruff (1975) and Kripke (1975), as well as *paraconsistent* approaches (see e.g. Asenjo 1966; Asenjo and Tamburino 1975; Priest 1979, 2006a, 2006b; Beall 2009). Paracomplete approaches solve paradoxes such as the Liar by assigning the Liar sentence a value in between truth and falsity, thus invalidating the Law of Excluded Middle. Paraconsistent approaches solve the Liar by taking the Liar sentence to be both true and false, avoiding absurdity by invalidating the classically and intuitionistically valid principle of *Ex Contradictione Quodlibet*. Both approaches have sought to preserve room for a detaching conditional that underwrites the T-Scheme. And when such a conditional threatens to reintroduce absurdity through Curry's Paradox, both approaches have offered a common diagnosis: they take it to show that this conditional cannot satisfy the law of contraction:

$$(\textit{Contraction})\ (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta).$$

More generally, they require that a theory of truth be *robustly contraction free* ('rcf', for short); free, essentially, of a conditional satisfying Contraction and other natural principles such as *Modus Ponens* (Restall 1993).

In this paper, we assume for argument's sake the naïve view of truth, and argue that this view doesn't in fact require rejecting VTP. However, maintaining VTP requires more than revising logic so as to ensure that Contraction is no longer a theorem. Rather, it involves adopting a logic that lacks one or more of the rules usually thought to correspond to basic features of reasoning in the context of assumptions. We will focus on the *structural* rule of contraction

$$(\mathsf{SContr})\frac{\Gamma, \alpha, \alpha \vdash \beta}{\Gamma, \alpha \vdash \beta}$$

Once SContr is rejected, we will see, the standard objections against VTP all break down. The standard arguments against VTP at best support the weaker conclusion

[1] Shapiro (2011) refers to the the claim that VTP and the naïve view of truth we introduce in the next paragraph yield triviality as the 'Field-Beall thesis'.

that, given the naïve view of truth, *either* VTP *or* SContr (or perhaps some other structural feature of the consequence relation) should be rejected.

To be sure, rcf theorists, especially Field, are aware of the existence of substructural revisionary approaches. Field dismisses them, though, as "radical" (Field 2008, p. 10), and as "very desperate measures" that are, ultimately, not needed (Field 2009a, p. 350). He writes:

> I haven't seen sufficient reason to explore this kind of approach (which I find very hard to get my head around), since I believe we can do quite well without it. ... [Hence] I will take the standard structural rules for granted. (Field 2008, pp. 10–11; also 283n)

However, while we agree with Field that more work needs to be done to make sense of a failure of SContr, we'd like to stress that giving up VTP is *also* a radical move. What is more, revisionary theorists have at least one powerful reason to reject SContr. Let us assume, as is often done, that the "valid" arguments include those whose goodness depends on rules governing the truth and validity predicates (McGee 1991; Whittle 2004; Priest 2006a, 2006b; Field 2007, 2008; Zardini 2011). Then there exist validity-involving versions of Curry's Paradox which cannot be solved by revising the logic's *operational* rules (those governing the behavior of logical vocabulary) to ensure that the theory is robustly contraction free. This is because the only operational rules these versions of Curry's Paradox employ are a pair of rules governing a validity predicate, rules that are arguably essential to that predicate's expressing validity (Shapiro 2011; Beall and Murzi 2013).

It has long been known that Curry-paradoxical reasoning can be blocked by adopting a "substructural" logic lacking SContr.[2] Yet we're not aware of any detailed examinations of how the various challenges to VTP are affected by adopting such logics.[3] What makes matters delicate is that all the challenges to VTP involve arguments with *multiple premises*. Hence how we may respond to the challenges depends crucially on how we understand what it means for a conclusion to follow validly from all of the premises *taken jointly*. Even stating what truth-preservation amounts to requires us to represent such joint consequence using some logical connective in place of the above informal 'all' or (in the case of arguments with finitely many premises) in place of the corresponding 'and.' Once SContr is rejected, various possibilities open up for the logical behavior of such an 'and', with different choices having different implication for the challenges to VTP. Moreover, the possibility arises that there are *two* suitable connectives, corresponding to different modes in which premises may be understood as taken jointly. Our chief aim is to clarify this poorly understood complex of issues and challenge the received wisdom that VTP is incompatible with revisionary approaches to paradox.

Two final qualifications. The structural feature of validity encapsulated in SContr isn't the only standardly accepted structural feature whose rejection would block the validity-involving versions of Curry's Paradox and allow a defense of VTP against the standard objections. An alternative "substructural" strategy, proposed by Ripley

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[2] See Slaney (1990), Restall (1994) and Field (2008, p. 283n).

[3] There is some relevant discussion in Shapiro (2011) and Zardini (2011).

(2013), involves restricting the *transitivity* of validity as reflected in the structural rule of Cut.[4] While we will occasionally remark on this approach, we do not have space to compare it with the strategy of giving up SContr.[5] In what follows, we will assume (as rcf theorists typically do) that validity is transitive. Likewise, we won't here be able to discuss the various ways in which one might try to make sense of and motivate the failure of SContr.[6]

The remainder of this paper is structured thus. §1 introduces the standard arguments in favor of rejecting VTP. §2 observes that VTP follows from what we call the *naïve view of validity*, viz. that the validity predicate satisfies (generalisations of) the Rule of Necessitation and the T axiom. It then rehearses some reasons for thinking that the naïve view of validity is in tension with SContr, and considers a couple of possible objections to this claim. §3 examines various possible interpretations of VTP, interpretations that become available once SContr is rejected. Specifically, it considers different ways of understanding the claim that an argument's premises are *all* true, as one finds in linear logic and what we call dual-bunching logics. It then argues that, once SContr is rejected, the standard objections to VTP are all blocked. §4 offers some concluding remarks.

22.1 Three Challenges to VTP

We focus on *three* challenges to VTP: that the most obvious argument in defense of this principle rests on inconsistent premises, that VTP yields triviality via Curry-like reasoning, and that Gödel-like reasoning shows that no consistent recursively axiomatizable semantic theory can endorse VTP.

22.1.1 The Validity Argument and Curry's Paradox

Field (2008, §2.1, §19.2) considers an argument, which he calls the Validity Argument, to the effect that "an inference is valid if and only if it is logically necessary

[4] Weir (2005) also addresses semantic paradox by restricting the transitivity of validity, though this shows up in his natural deduction system as a structure-based restriction on the use of *operational* rules.

[5] Both of these "substructural" approaches to semantic paradox have an advantage worth mentioning: they allow for a *unified* approach to the paradoxes of self-reference (Weir 2005; Zardini 2011; Ripley 2013), as opposed to the piecemeal approach proposed by current rcf theories, where similar paradoxes, e.g. the Liar and Curry, are treated in radically different ways. In recent unpublished work, Beall uses the desideratum of uniformity as one motivation for a new approach to paradox—one that retains the standardly accepted structural rules but gives up on a detaching conditional altogether. For a sketch of that approach, see Beall (2011).

[6] For discussion of this important topic, see Shapiro (2011), Zardini (2011), Beall and Murzi (2013), Mares and Paoli (2014). [Note added in proof: see also Shapiro (2015).]

that it preserves truth" (Field 2008, p. 284). If sound, the argument for this biconditional's 'only if' direction would seem to establish VTP. However, Field argues, it can't be sound. Let's use $\alpha_1, ..., \alpha_n \vdash \beta$ to mean that "the argument from the premises $\alpha_1, ..., \alpha_n$ to the conclusion β is logically valid" (Field 2008, p. 42). And let *Tr*-I and *Tr*-E, respectively, be the rules that one may infer $Tr(\ulcorner\alpha\urcorner)$ from α in any context of assumptions, and vice versa. Then Field reasons thus (we have adapted his terminology):

> *'Only if' direction*: Suppose $\alpha_1, ..., \alpha_n \vdash \beta$. Then by *Tr*-E, $Tr(\ulcorner\alpha_1\urcorner), ..., Tr(\ulcorner\alpha_n\urcorner) \vdash \beta$; and by *Tr*-I, $Tr(\ulcorner\alpha_1\urcorner), ..., Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$. By $\wedge$-E, $Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$. So by $\to$-I, $\vdash Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \to Tr(\ulcorner\beta\urcorner)$. That is, the claim that if the premises $\alpha_1, ..., \alpha_n$ are true, so is the conclusion, is valid, i.e. holds of logical necessity.
> *'If' direction*: Suppose $\vdash Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \to Tr(\ulcorner\beta\urcorner)$. By *Modus Ponens*, $Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$. So by $\wedge$-I, $Tr(\ulcorner\alpha_1\urcorner), ..., Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$. So by *Tr*-I, $\alpha_1, ..., \alpha_n \vdash Tr(\ulcorner\beta\urcorner)$; and by *Tr*-E, $\alpha_1, ..., \alpha_n \vdash \beta$. (Field 2008, p. 284).[7]

Two features of this Validity Argument call for comment. First, notice that it is conducted in a metalanguage containing a validity predicate (the turnstile), but no truth predicate. In taking the argument to establish VTP, then, Field is assuming that the object-language sentence $Tr(\ulcorner\alpha_1\urcorner) \wedge ... \wedge Tr(\ulcorner\alpha_n\urcorner) \to Tr(\ulcorner\beta\urcorner)$ expresses the claim *that if* $\alpha_1, ..., \alpha_n$ *are all true, so is* β. In §3, we will see that once giving up structural contraction is an option, it becomes controversial whether the claim that "all premises are true" should be expressed using a connective for which the inferences Field justifies using $\wedge$-I and $\wedge$-E are valid. Second, one might worry that the Validity Argument presupposes its own conclusion. The argument establishes that if an argument is valid, then the conditional claiming that the argument preserves truth will likewise be *valid*. But we couldn't take this as establishing VTP itself unless we took for granted that *valid sentences are true*—a claim that is a special case of VTP. Still, even if the Validity Argument doesn't suffice to establish VTP, it does undermine the objections that have been offered against VTP. That is because these objections (which all involve multi-premise arguments) don't purport to challenge the claim that *valid sentences are true*. Thus the Validity Argument should count as a defense of VTP.[8]

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[7] It may help to make Field's reasoning for the 'only if' direction explicit in natural deduction format, for the special case where we are considering an argument from the single premise α to the conclusion β. Complications raised by the multiple-premise case will be discussed in §3.

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \beta \qquad \dfrac{Tr(\ulcorner\alpha\urcorner) \vdash Tr(\ulcorner\alpha\urcorner)}{Tr(\ulcorner\alpha\urcorner) \vdash \alpha}\,Tr\text{-E}}{Tr(\ulcorner\alpha\urcorner) \vdash \beta}\,\mathsf{Cut}}{Tr(\ulcorner\alpha\urcorner) \vdash Tr(\ulcorner\beta\urcorner)}\,Tr\text{-I}}{\vdash Tr(\ulcorner\alpha\urcorner) \to Tr(\ulcorner\beta\urcorner)}\,\to\text{-I}$$

[8] In §2.1, we will see that if our object-language contains a validity predicate, it is also possible to derive VTP using an intuitively compelling elimination rule for that predicate. While we will discuss only a predicate that applies to single-premise arguments, a generalized version of that derivation would be subject to all our conclusions about the Validity Argument.

Field suggests that the Validity Argument, though it "looks thoroughly convincing at first sight," can't be accepted, since it relies on Tr-I, Tr-E, $\to$-I, and $\to$-E, "which the Curry Paradox shows to be jointly inconsistent" (Field 2008, pp. 43, 284). Let us unpack this a little. The Diagonal Lemma allows us to construct a sentence κ which, up to equivalence, intuitively says that, if it's true, then (say) you will win the lottery. Assuming that our theory of truth T is strong enough to prove the Diagonal Lemma, this means that

$$\vdash_T \kappa \leftrightarrow (Tr(\ulcorner\kappa\urcorner) \to \bot).$$

Let Π now be the following derivation of the further theorem $Tr(\ulcorner\kappa\urcorner) \to \bot$:

$$\dfrac{\dfrac{\dfrac{\vdash_T \kappa \leftrightarrow Tr(\ulcorner\kappa\urcorner) \to \bot \qquad \dfrac{Tr(\ulcorner\kappa\urcorner) \vdash_T Tr(\ulcorner\kappa\urcorner)}{Tr(\ulcorner\kappa\urcorner) \vdash_T \kappa}\,Tr\text{-E}}{Tr(\ulcorner\kappa\urcorner) \vdash_T Tr\ulcorner\kappa\urcorner \to \bot}\,{\to}\text{-E} \qquad Tr(\ulcorner\kappa\urcorner) \vdash_T Tr(\ulcorner\kappa\urcorner)}{\dfrac{Tr(\ulcorner\kappa\urcorner), Tr(\ulcorner\kappa\urcorner) \vdash_T \bot}{Tr(\ulcorner\kappa\urcorner) \vdash_T \bot}\,\mathsf{SContr}}\,{\to}\text{-E}}{\vdash_T Tr(\ulcorner\kappa\urcorner) \to \bot}\,{\to}\text{-I}$$

Using Π, we can then 'prove' that you will win the lottery:

$$\dfrac{\begin{matrix}\Pi \\ \vdash_T Tr(\ulcorner\kappa\urcorner) \to \bot\end{matrix} \qquad \dfrac{\dfrac{\vdash_T \kappa \leftrightarrow (Tr(\ulcorner\kappa\urcorner) \leftrightarrow \bot) \qquad \begin{matrix}\Pi \\ \vdash_T Tr(\ulcorner\kappa\urcorner) \to \bot\end{matrix}}{\vdash_T \kappa}\,{\to}\text{-E}}{\vdash_T Tr(\ulcorner\kappa\urcorner)}\,Tr\text{-I}}{\vdash_T \bot}\,{\to}\text{-E}$$

This is the (standard) conditional-involving version of Curry's Paradox, or c-Curry, as we'll call it.[9] The derivation makes use of Tr-I, Tr-E, $\to$-I and $\to$-E, just like the Validity Argument. Hence, Field argues, one can't accept the latter without thereby validating the former. Rcf theorists invalidate c-Curry by rejecting $\to$-I, thus resisting Π's final step (Priest 2006b; Field 2008; Beall 2009; Beall and Murzi 2013). Therefore, Field suggests, they must reject the 'only if' direction of the Validity Argument, too.

However, as Field notes, the above derivation makes use of the rule SContr. Hence if SContr is rejected—as proposed in this context by Brady (2006), Zardini (2011), Shapiro (2011), and Beall and Murzi (2013)—Curry's paradox no longer stands in the way of our embracing the principles used in the Validity Argument for VTP. One complication: we should note in advance that it isn't clear that all types of contraction-free logics we will be considering support theories of arithmetic that prove a Diagonal Lemma. Where this isn't the case, the reader should suppose that some other means of self-reference built into our semantic theory is responsible for the Curry paradoxes

[9] This terminology was introduced in Beall and Murzi (2013).

we will be considering. In what follows, we will ignore this complication, and assume that T has the resources for at least simulating self-reference.

Will rejecting SContr allow us to endorse the Validity Argument, then? As we will see below, matters are not this simple. Field's argument makes crucial use of rules governing the conjunction symbolized by $\wedge$. Once we no longer accept the standard structural rules, however, the rules for conjunction can take non-equivalent forms, and the soundness of the Validity Argument now depends on which of the available rules for $\wedge$ we accept. In §3, we will examine which of the contraction-free logics that have been proposed in response to semantic paradox underwrite the Validity Argument.[10]

22.1.2 From VTP to Absurdity via the Modus Ponens Axiom

In addition to criticizing the most obvious *defense* of VTP, Field offers two arguments according to which VTP can't be embraced without absurdity. In the remainder of this section, then, let us examine whether we can at least *affirm* that valid arguments preserve truth. For simplicity's sake, we focus for now on arguments with only one premise. (Issues raised by multiple-premise arguments will be considered in detail in §3 below.) We will try to affirm VTP in the object-language itself, by introducing a predicate *Val*(x, y) which intuitively expresses that the argument from x to y is valid. VTP may now be naturally represented thus (see Beall 2009):

$$(\mathsf{V0})\ Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \rightarrow (Tr(\ulcorner\alpha\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)).^{11}$$

As Field and Beall point out, V0 entails absurdity, based on principles accepted by rcf theorists (Field 2006, 2008; Beall 2007, 2009).

[10] Let us briefly consider how the Validity Argument fares on the alternative substructural approach that restricts transitivity. In the version of c-Curry given above, in natural deduction format, SContr is the only structural rule used. By contrast, the parallel Curry derivation in sequent calculus format will conclude with the following use of the structural rule of Cut

$$\frac{\vdash_T Tr(\ulcorner\kappa\urcorner) \quad Tr(\ulcorner\kappa\urcorner) \vdash_T \bot}{\vdash_T \bot}$$

Ripley (2013) proposes a semantic theory that blocks c-Curry reasoning by invalidating Cut. His theory adds rules for Tr to a sequent calculus with entirely classical operational rules and structural rules except for Cut, which is no longer admissible in the presence of the truth rules. We would like to make two observations about Ripley's proposal. On the one hand, since it retains the rule $\rightarrow$-I, it allows a defense of the Validity Argument's "only if" direction (his truth rules replace Cut in note 7 above), and thus of VTP. On the other hand, though Ripley's theory also endorses the *conclusion* of every instance of the Validity Argument's "if" direction, it won't allow the above intuitive *argument*, since it renders the rule $\rightarrow$-E inadmissible. See note 46 below.

[11] Strictly speaking, this should be expressed by a universal generalisation on codes of sentences, but, for the sake of simplicity, we won't bother.

Since rcf theorists do not accept the rule $\rightarrow$-I, we will need two additional ingredients to obtain paradox from $\mathsf{V0}$. First, the rules Tr-I and Tr-E no longer suffice; our semantic theory T needs to underwrite all instances of the T-Scheme. Second, we will use the principle that if $\vdash_T \alpha \leftrightarrow \beta$, then α and β are intersubstitutable within conditionals.[12] Given these presuppositions, $\mathsf{V0}$ entails

$$(\mathsf{V1})\ Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \rightarrow (\alpha \rightarrow \beta).$$

Now let us assume, as rcf theorists do, that our theory T implies the validity of a single-premise version of the *Modus Ponens rule*:

$$(\mathsf{VMP})\ Val(\ulcorner(\alpha \rightarrow \beta) \wedge \alpha\urcorner, \ulcorner\beta\urcorner).$$

Hence $\mathsf{V1}$ in turn entails the *Modus Ponens axiom*:

$$(\mathsf{MPA})\ (\alpha \rightarrow \beta) \wedge \alpha \rightarrow \beta.$$ [13]

However, Meyer et al. (1979) show that MPA generates Curry's Paradox. The only additional ingredient we need is the claim that it is a theorem that "conjunction is idempotent," i.e. that $\vdash \alpha \leftrightarrow \alpha \wedge \alpha$.

To see why this is so, recall that we have assumed T is strong enough to ensure $\vdash_T \kappa \leftrightarrow (Tr(\ulcorner\kappa\urcorner) \rightarrow \bot)$. Hence, given the T-Scheme and the above substitutivity principle, $\vdash_T \kappa \leftrightarrow (\kappa \rightarrow \bot)$. We can now derive absurdity starting with the relevant instance of MPA:

$$(\kappa \rightarrow \bot) \wedge \kappa \rightarrow \bot.$$

Substituting κ for the equivalent $\kappa \rightarrow \bot$ gives us $\kappa \wedge \kappa \rightarrow \bot$. In view of our assumption that $\vdash_T \kappa \leftrightarrow \kappa \wedge \kappa$, another substitution of equivalents yields $\kappa \rightarrow \bot$. By substituting κ for $\kappa \rightarrow \bot$ once again, we get κ. Finally, we use $\rightarrow$-E to derive $\bot$ from $\kappa \rightarrow \bot$ together with κ.

Since VTP and VMP jointly entail the paradox-generating MPA, it would thus appear that rcf theorists can't consistently assert that valid arguments preserve truth.[14] Field (2008, p. 377) and Beall (2009, p. 35) accept the foregoing argument, and consequently reject the claim that valid arguments are guaranteed to preserve truth (assuming, again, that truth-preservation is expressed using a detaching conditional that underwrites the T-Scheme). The need to reject VTP is a perhaps surprising, although ultimately unavoidable, corollary of the revisionary approach to paradox, or so they argue.[15]

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[12] This principle is endorsed by Field (2008, p. 253) and Beall (2009, pp. 28, 35).
[13] Following Restall (1994), this is sometimes referred to as *pseudo Modus Ponens*. See also Priest (1980), where it is described as the "counterfeit" *Modus Ponens* axiom.
[14] See Beall (2007); Beall (2009, pp. 34–41], Shapiro (2011, p. 341) and Beall and Murzi (2013).
[15] For Field, who rejects excluded middle, rejecting an instance of VTP doesn't mean accepting its negation. Beall, by contrast, accepts that there are valid arguments, e.g. the argument from κ and $\kappa \rightarrow \bot$ to $\bot$, that fail to preserve truth. However, as Field and Beall both note, Beall's position *doesn't* require accepting that there are valid arguments whose premises are all true and whose conclusion is false. See Field (2006, p. 597) and Beall (2009, p. 36).

22.1.3 *From* VTP *to Inconsistency via the Consistency Argument*

A second argument for rejecting VTP (Field 2006, 2008, 2009b) proceeds via Gödel's Second Incompleteness Theorem, which states that no consistent recursively axiomatisable theory containing a modicum of arithmetic can prove its own consistency. Field first argues that *if* an otherwise suitable semantic theory could prove that all its rules of inference preserve truth, it could prove its own consistency. Hence, by Gödel's theorem, no semantic theory that qualifies as a "remotely adequate mathematical theory" can prove that its rules of inference preserve truth. Yet insofar as we endorse the orthodox semantic principle VTP, Field says, we should be able to consistently add to our semantic theory an axiom stating that its rules of inference preserve truth (see Field 2009a, p. 351n10). Hence, he concludes, we should reject VTP.

To establish the first step in this argument against VTP, Field considers what he calls the Consistency Argument (Field 2006, pp. 567–568). This is an argument which, one might think, one should be able to run *within* any theory T containing a truth predicate satisfying the unrestricted T-Scheme. The argument proceeds by "(i) inductively proving within T that all its theorems are true, and (ii) inferring from the truth of all theorems of T that T is consistent." Though intuitively sound, the Consistency Argument must fail if T is to be consistent.

Field's claim is that the failure of the Consistency Argument must be blamed on an illicit appeal to VTP. He observes that (ii) can't be problematic for those who hold that "inconsistencies imply everything." The target theories "certainly imply $\neg Tr(\ulcorner 0 = 1 \urcorner)$, so the soundness of T would imply that '$0 = 1$' isn't a theorem of T; and this implies that T is consistent" (Field 2008, p. 286–287). However, (ii) will be equally unproblematic for any *paraconsistent* theorist who holds that an adequate semantic theory must imply the universal generalization over instances of the schema $\neg Tr(\ulcorner \alpha \wedge \neg\alpha \urcorner)$. In this case as well, if T could prove that all its theorems are true, it would thereby prove that no contradiction is a theorem (Field 2006, pp. 593–595). Field therefore concludes that the problem with the Consistency Argument must lie with (i). The argument by induction alluded to in (i) proceeds as follows: "(1) Each axiom of T is true, (2) Each rule of inference of T preserves truth [in the sense of VTP, whence] (3) All theorems of T are true." Field argues persuasively that "[t]he only place that the argument can conceivably go wrong is ... in (2)" (Field 2008, p. 287). This conclusion is endorsed by Beall (2009, pp. 115–116).

In sum, not only does the seemingly obvious Validity Argument in favor of VTP fail, but there are at least two arguments against accepting VTP—or so contemporary revisionary wisdom goes. As Beall writes: "such a claim ... needs to be rejected, and I reject it" (Beall 2009, p. 35).

22.2 Naïve Validity and Validity Curry

What role, then, if any, is left for the notion of validity, if we can no longer affirm that valid arguments preserve truth? Field (2008, 2009b, 2015) suggests that validity normatively constrains belief: very roughly, one shouldn't fully believe the premises of a valid argument without fully believing its conclusion. We take no position here on whether the role of the notion of validity can be explained without recourse to truth-preservation.[16] Instead, we'll suggest in the remainder of this paper that revisionary theorists *need not* and *should not* reject VTP. Provided they accept certain basic principles that would appear to govern the notion of validity, revisionary theorists are required on pain of paradox to adopt the very kind of logic that allows them to embrace VTP.

22.2.1 Naïve Validity

Still restricting our attention to single-premise arguments, consider the following two principles for the use of the validity predicate: that, if one can derive ψ from ϕ, one can derive on no assumptions that the argument from ϕ to ψ is valid, and that, from ϕ and the claim that the argument from ϕ to ψ is valid, one can infer ψ.[17]

Both rules are highly intuitive. If $Val(x, y)$ expresses *validity*, it seems natural to assume that an adequate semantic theory T must include the following introduction rule for $Val(x, y)$, which, by analogy with $\rightarrow$-I or Conditional Proof, we'll call *Validity Proof*:

$$(\mathsf{VP})\frac{\alpha \vdash_T \beta}{\vdash_T Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner)}$$

If T's rules are valid, and we can derive β from α in T, then T must be able to assert the sentence $Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner)$, expressing that the argument from α to β is valid. But it also seems natural to assume that T contains an elimination rule for $Val(x, y)$, which we'll call *Validity Detachment*:

$$(\mathsf{VD})\frac{\Gamma \vdash_T Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \quad \Delta \vdash_T \alpha}{\Gamma, \Delta \vdash_T \beta}$$

[16] For the record, we think that even if VTP holds, an explanation of the role of the notion of validity will have to involve normative considerations such as those Field advances.

[17] To the best of our knowledge, these rules are first discussed in Priest (2010). For further discussion, see Beall and Murzi (2013) and Murzi (2014). Shapiro (2011) proposes introducing a validity predicate governed by the equivalences $Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \dashv\vdash_T \alpha \Rightarrow \beta$, where $\Rightarrow$ is an *entailment connective* whose introduction and elimination rules in turn render VP and VD derivable. Such a connective is common in the tradition of relevant and paraconsistent logic: see e.g. Anderson and Belnap (1975, p. 7) and Priest and Routley (1982).

If, from a given context of assumptions, we can derive in T the sentence α and from another context we can derive that the argument from α to β is valid, then it must be possible (from the assumptions taken together) to derive β.[18]

The rules VP and VD can also be viewed as *generalizations* of natural rules for a predicate that expresses logical truth: namely, analogues of the rule of Necessitation and of a rule corresponding to the T axiom. To see this, it is sufficient to instantiate VP and VD using a constant $\top$ expressing logical truth. Instantiating VP yields a notational variant of Necessitation, rewritten using our two place predicate $Val(x, y)$ in place of a necessity operator:

$$(\mathsf{NEC}^*)\frac{\top \vdash_T \beta}{\vdash_T Val(\ulcorner \top \urcorner, \ulcorner \beta \urcorner)}$$

Likewise, instantiating VD thus

$$\frac{\Gamma \vdash_T Val(\ulcorner \top \urcorner, \ulcorner \beta \urcorner) \quad \top \vdash_T \top}{\Gamma, \top \vdash_T \beta}$$

yields a notational variant of a rule corresponding to the T axiom for a necessity operator:

$$(\mathsf{T}^*)\ \ Val(\ulcorner \top \urcorner, \ulcorner \beta \urcorner), \top \vdash_T \beta$$

The intuitiveness of our rules VP and VD is thus underscored by the close connection they underwrite between the behavior of a predicate expressing logical truth and the behavior of an operator expressing logical necessity.

We will therefore call the view that 'valid' satisfies VP and VD the *naïve view of validity* (Murzi 2014). One first point that deserves emphasis is that, on the naïve view of truth we've assumed at the beginning of this paper, such a view entails $\mathsf{V0}$, our object-language statement of VTP for single-premise arguments. This can be shown using what is essentially a version of Field's Validity Argument, except that the validity of the argument from α to β is now expressed using an object-language predicate rather than using a turnstile in the metalanguage:[19]

[18] We have written the rule VP without side assumptions. That is because the acceptability of a version including side assumptions

$$(\mathsf{VP}^*)\frac{\Gamma, \alpha \vdash_T \beta}{\Gamma \vdash_T Val(\ulcorner \alpha \urcorner, \ulcorner \beta \urcorner)}$$

depends on the properties of the structural comma. For example, if the comma obeys *weakening* and we get $\beta, \alpha \vdash_T \beta$, then VP^* allows us to derive $\beta \vdash_T Val(\ulcorner \alpha \urcorner, \ulcorner \beta \urcorner)$. But where β is contingent, it shouldn't follow from β that it is entailed by any sentence. A similar problem arises if the comma obeys *exchange*. From VD and Cut we get $Val(\ulcorner \alpha \urcorner, \ulcorner \alpha \urcorner), \alpha \vdash_T \alpha$, whence exchange yields $\alpha, Val(\ulcorner \alpha \urcorner, \ulcorner \alpha \urcorner) \vdash_T \alpha$ and VP^* allows us to derive $\alpha \vdash_T Val(\ulcorner Val(\ulcorner \alpha \urcorner, \ulcorner \alpha \urcorner) \urcorner, \ulcorner \alpha \urcorner)$. But if α is contingent, it shouldn't follow from α that it is entailed by a logical truth. Zardini (2013), whose comma obeys both weakening and exchange, avoids these problems by restricting the side assumptions in VP^* to logical compounds of *validity claims*. See also Priest and Routley (1982).

[19] Ripley (2013) offers a similar defense of VTP, using VP and the sequent $\alpha, Val(\ulcorner \alpha \urcorner, \ulcorner \beta \urcorner) \vdash_T \beta$. Shapiro (2011) explains that on the version of the naïve view presented there (see note 17 above), $Val(\ulcorner \alpha \urcorner, \ulcorner \beta \urcorner)$ implies $Tr(\ulcorner \alpha \urcorner) \Rightarrow Tr(\ulcorner \beta \urcorner)$.

$$
\begin{prooftree}
\AxiomC{$Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner) \vdash_T Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner)$}
\AxiomC{$Tr(\ulcorner\alpha\urcorner) \vdash_T Tr(\ulcorner\alpha\urcorner)$}
\RightLabel{Tr-E}
\UnaryInfC{$Tr(\ulcorner\alpha\urcorner) \vdash_T \alpha$}
\RightLabel{VD}
\BinaryInfC{$Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner), Tr(\ulcorner\alpha\urcorner) \vdash_T \beta$}
\RightLabel{Tr-I}
\UnaryInfC{$Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner), Tr(\ulcorner\alpha\urcorner) \vdash_T Tr(\ulcorner\beta\urcorner)$}
\RightLabel{$\to$-I}
\UnaryInfC{$Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner) \vdash_T Tr(\ulcorner\alpha\urcorner) \to Tr(\ulcorner\beta\urcorner)$}
\RightLabel{$\to$-I}
\UnaryInfC{$\vdash_T Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner) \to (Tr(\ulcorner\alpha\urcorner) \to Tr(\ulcorner\beta\urcorner))$}
\end{prooftree}
$$

A second point to notice is that, natural though they may seem, VP and VD lead us into trouble—which should of course be expected, since NEC* and T* are nothing but the key ingredients of the Myhill–Kaplan-Montague Paradox, or Paradox of the Knower (Myhill 1960; Kaplan and Montague 1960; Murzi 2014).[20]

22.2.2 Validity Curry

The Diagonal Lemma allows us to construct a sentence π, which intuitively says of itself, up to equivalence, that it validly entails that you will win the lottery:

$$\vdash_T \pi \leftrightarrow Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)$$

Let Σ now be the following derivation of the further theorem $Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)$:

$$
\begin{prooftree}
\AxiomC{$\pi \vdash_T \pi$}
\AxiomC{$\vdash_T \pi \leftrightarrow Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)$}
\RightLabel{$\to$-E}
\BinaryInfC{$\pi \vdash_T Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)$}
\AxiomC{$\pi \vdash_T \pi$}
\RightLabel{VD}
\BinaryInfC{$\pi, \pi \vdash_T \bot$}
\RightLabel{SContr}
\UnaryInfC{$\pi \vdash_T \bot$}
\RightLabel{VP}
\UnaryInfC{$\vdash_T Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)$}
\end{prooftree}
$$

Using Σ, we can then ‘prove’ that you will win the lottery

[20] Shapiro (2011) identifies two challenges to the naïve view: a “direct argument” that it leads straight to paradox, and an “indirect argument” that it entails a version of the paradox-producing VTP.

$$
\dfrac{\begin{array}{c}\Sigma\\ \vdash_T Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)\end{array} \qquad \dfrac{\vdash_T \pi \leftrightarrow Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner) \qquad \begin{array}{c}\Sigma\\ \vdash_T Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)\end{array}}{\vdash_T \pi}\to\text{-E}}{\vdash_T \bot}\mathsf{VD}
$$

Our revisionary theory of truth and validity, *T*, proves on no assumptions that you will win the lottery.[21] Call this the Validity Curry, or v-Curry, for short, to contrast it with the standard conditional-involving version of Curry's Paradox, or c-Curry.[22]

As we explained above, rcf theorists invalidate c-Curry by rejecting →-I. Unlike c-Curry, however, the v-Curry Paradox makes no use of →-I, and hence it cannot be invalidated by rejecting such a rule. On the other hand, the above derivation of v-Curry presupposes SContr (Beall and Murzi 2013). Hence if VP and VD hold, there is only one revisionary way out of the v-Curry Paradox, viz. rejecting SContr, thus adopting a *substructural* logic—a logic where some of the standardly accepted structural rules fail (Shapiro 2011; Beall and Murzi 2013; Murzi 2014; Zardini 2011).[23]

Before examining in §3 how rejecting SContr affects VTP and the Validity Argument, we'd first like to offer a partial defence of our claim that v-Curry Paradox is a reason for revisionary logician to adopt a substructural logic. To this end, we'll consider in the next section two natural responses to the claim that the Validity Curry is a genuine paradox of validity, and offer replies on the substructural logician's behalf.

22.2.3 A Genuine Paradox of Validity

If the v-Curry Paradox isn't a genuine paradox of validity, one of VP and VD must not unrestrictedly hold. As it turns out, there are *prima facie* compelling reasons

[21] To the best of our knowledge, the first known occurrence of the Validity Curry is in the 16th-century author Jean de Celaya. See Read (2001, fn. 11–12) and references therein. Albert of Saxony includes a contrapositive version of the paradox among his "insolubles" (Read 2010, p. 165). A more recent version can be found in Priest and Routley (1982), and surfaces again in Whittle (2004, fn. 3), Clark (2007, pp. 234–235) and Shapiro (2011, fn. 29). For a first comprehensive discussion of the Validity Curry, see Beall and Murzi (2013). For a defence of the claim that Validity Curry is a genuine paradox of validity, see §2.3 below and Murzi (2014).

[22] This terminology was first introduced in Beall and Murzi (2013). Ultimately, however, the distinction in terms of predicate versus connective may not be the essential one. Whittle (2004) and Shapiro (2011) discuss a version of Curry's Paradox, involving a "consequence connective" or "entailment connective," which poses much the same challenge to rcf theorists as does v-Curry. [See also Shapiro (2015, p. 82).]

[23] For an early anticipation of the argument from naïve validity to the rejection of SContr (in the form of multiple discharge of assumptions), see Priest and Routley (1982). Priest and Routley, whose entailment connective obeys analogues of VP and VD, discuss several resulting paradoxes which they blame on the "suppression of innocent premises." By contrast, Ripley (2013) blocks v-Curry at the final step using VD, which is inadmissible in his nontransitive theory for the same reason that →-E is inadmissible. See note 46 below.

for restricting both.[24] One argument against VP runs thus. In order to establish $Val(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner)$ as a theorem using "Validity Proof," it is said, we should be required to produce a logically valid argument from α to β. Yet subderivation Σ above doesn't establish the argument from π to $\bot$ as *logically* valid, for two reasons. First, this subderivation relies on a substitution instance of the *logically invalid* biconditional proved by the Diagonal Lemma, viz. $\pi \leftrightarrow Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)$. Second, it uses VD, and, it might be objected, surely such a rule isn't logical. More precisely, Roy Cook (2012) has argued that the T-Scheme isn't logically valid, if by logical validity one means truth under all uniform interpretations of the non-logical vocabulary. Using the same reasoning, we could conclude that VD doesn't preserve logical validity.[25]

These objections have an important virtue: they help us understand what the v-Curry Paradox really is a paradox of. More precisely, they show that the v-Curry Paradox is not paradox of *purely logical*, or *interpretational*, in John Etchemendy's term, validity (Etchemendy 1990).[26] Indeed, a recent result by Jeff Ketland shows that purely logical validity *cannot* be paradoxical. Ketland (2012) proves that Peano Arithmetic (PA) can be conservatively extended by means of a predicate expressing logical validity, governed by intuitive principles that are themselves derivable in PA. It follows that purely logical validity is a consistent notion if PA is consistent, which should be enough to warrant belief that purely logical validity simply *is* consistent.

However, it seems to us that there are broader notions of validity than purely logical validity.[27] Thus, neither of the above objections applies to versions of the v-Curry Paradox in which 'valid' expresses *representational* validity, whereby (roughly) validity is equated with preservation of truth in all possible circumstances (Read 1988; Etchemendy 1990; McGee 1991). In this sense, at least intuitively, the arithmetic required to prove the Diagonal Lemma is valid and VD is validity-preserving.[28]

[24] Thanks to Roy Cook and Jeff Ketland for raising these potential concerns.

[25] Field (2008, §20.4) himself advances versions of this line of argument, while discussing what is in effect a validity-involving version of the Knower Paradox resting on NEC^* and T^*. See especially Field (2008, p. 304 and p. 306). On the question whether his conception of the extension of the validity predicate consistently allows him to do so, see note 27 below.

[26] Here we take the logical vocabulary to be the standard vocabulary of some first-order, perhaps non-classical, logic.

[27] Several semantic theorists, including rcf theorists such as Field and Priest, resort to notions of validity that are not purely logical. For instance, Field (2007, 2008) extensionally identifies validity with, essentially, preservation of truth in all ZFC models of a certain kind, thus taking validity to (wildly) exceed purely logical validity. (Incidentally, it seems to us that this use of 'valid' is in tension with the purely logical sense Field (2008) appeals to at p. 304 and especially p. 306.) Likewise, McGee (1991, p. 43–49) takes logical necessity to extend to arithmetic and truth-theoretic principles.

[28] It might be objected that such a notion of validity presupposes VTP, and hence cannot be appealed to in the present context, where the question whether VTP can be consistently upheld is the very point at issue. Our modest aim here, however, is simply to suggest that someone who *already thinks*, following perhaps logical orthodoxy, that valid arguments preserve truth and that, accordingly, consequence is to be explicated in terms of truth-preservation has a reason—the v-Curry Paradox—to reject SContr. Once SContr is rejected, the standard challenges to VTP no longer stand, as we'll see in §3 below. But, it seems to us, no illicit or question-begging appeal to

Nor does the objection that VP cannot be legitimately applied to non-purely-logical subderivations apply to conceptions of validity which take 'valid' to express the consequence relation of one's semantic theory, provided that the naïve validity rules and enough arithmetic are part of that relation.[29] Insofar as VD preserves validity in one of these broader senses, and insofar as the VP and VD govern the use a predicate expressing validity in that sense, there is at least one—important—reading of 'valid' on which the use of VP in the v-Curry derivation is sound. The v-Curry Paradox is a paradox of *validity*, not *purely logical* validity.

To be sure, one might instead either reject VP on different grounds, or perhaps reject VD. One natural enough argument against the latter rule runs thus. Suppose validity is recursively enumerable. Then, one might argue, T^*, and hence VD, must fail. For, if validity is recursively enumerable, an argument is valid if and only if its conclusion can be derived from its premises in some recursively axiomatisable theory T. That is, the validity predicate $Val(x, y)$ is just a notational variant of $Prov_T(x, y)$, where this expresses that there is a T-derivation of y from x. Yet, the argument continues, we know from Löb's Theorem that, if T contains enough arithmetic (if it proves the so-called derivability conditions), T cannot contain, on pain of triviality, all instances of the provability-in-T analogue of T^*, $Prov_T(\ulcorner \top \urcorner, \ulcorner \alpha \urcorner) \rightarrow \alpha$. Hence, one might conclude, T may not contain all instances of T^* either, and hence of VD, *a fortiori*.

We find this conclusion problematic. It seems to us that rejecting VD, or VP, for that matter, isn't really a comfortable option for proponents of the naïve view of truth. In a nutshell, together with the naïve view of truth, the naïve view of validity is but an instance of the general thought underpinning the revisionary approach to paradox—what we may call the *naïve view of semantic properties*.[30] This is the view that one cannot revise naïve semantic principles without thereby also revising naïve semantic properties, and that, on pain of triviality, semantic properties should be held fixed, and *logic* must change. Arguably, the naïve view of semantic properties has it that validity *is* factive, and that we, and hence our semantic theory, must be able to say so, on pain of not being able to consistently assert what we know to be true. If T does indeed meet the conditions for Löb's Theorem, we would like to

VTP has been made in the course of the foregoing reasoning. We thank an anonymous referee for raising this potential concern.

[29] In fact, Cook (2014) shows how this response can be strengthened: it is possible to formulate a modified Validity Curry paradox in such a way that the arithmetic necessary to prove the Diagonal Lemma need not be included in the scope of the validity relation.

[30] This view is implicitly assumed in the work of contemporary revisionary theorists—see e.g. Priest (2006b); Field (2007, 2008), Beall (2009), Beall and Murzi (2013). In particular, it is implicit in their assumption that the paradoxes of validity are (in an interesting sense) of the same kind as the Liar and c-Curry. One defence of that possibly controversial assumption would involve arguing that the Paradox of the Knower is nothing but a weakened Liar, and that, as we've observed in §2.1, v-Curry is nothing but a generalised Knower, so that whatever the nature of the first paradox, it is inherited by the other two. See also Read (2001) and Beall and Murzi (2013). We should finally stress that in calling validity a semantic property, we merely intend to point to these parallels, without relying on any particular conception of what makes a property semantic.

suggest, then the correct reaction to the objection is instead to concede that $Val(x, y)$ can't be replaced with $Prov_T(x, y)$, and hence that naïve validity is not recursively enumerable.[31]

It might be objected that we could revise, or refine, our naïve conception of validity, which is after all naïve (McGee 1991, p. 45). But, then, a parallel argument would show that, when faced with the Liar Paradox, the c-Curry Paradox, and other paradoxes of truth, we should similarly revise our conception of *truth*, which is precisely what proponents of the naïve view of semantic properties take to be the *wrong* response to semantic paradox. For the time being, we'll assume that the Validity Curry is a genuine paradox of validity, and that giving up SContr, as suggested in Shapiro (2011) and Zardini (2011), is a legitimate revisionary response to it, and to semantic paradoxes more generally. We shall now argue that, on this admittedly controversial assumption, of which we've only offered a partial defence, all three arguments for rejecting VTP break down.

22.3 Validity and Truth-Preservation

All three challenges to VTP turn out to rest crucially on how our object-language expresses validity and truth-preservation for arguments with *multiple premises*. First, recall that Field argues that the most obvious defense of VTP, the Validity Argument, rests on principles that yield paradox. As we have pointed out, the Validity Argument presupposes that the truth-preservingness of an inference from $\alpha_1, ..., \alpha_n$ to β can be expressed using the object-language sentence $Tr(\ulcorner \alpha_1 \urcorner) \wedge ... \wedge Tr(\ulcorner \alpha_n \urcorner) \rightarrow Tr(\ulcorner \beta \urcorner)$. Second, the argument from VTP to PMP and absurdity used the simplifying assumption that the validity of the two-premise *Modus Ponens* rule can be expressed using a single-premise validity predicate as $Val(\ulcorner \alpha \wedge (\alpha \rightarrow \beta) \urcorner, \ulcorner \beta \urcorner)$. Finally, spelling out the Consistency Argument requires expressing in the object-language the claim that each of our semantic theory T's rules of inference preserves truth, where these will include multi-premise rules such as *Modus Ponens*.

[31] We don't have space to expand on this point here. Priest (2006b, §3.2) argues at length that the "naïve notion of proof" is recursive, whence naïve provability, a species of naïve validity, is recursively enumerable. Here we simply notice that his arguments are consistent with the view that naïve *validity* isn't. Finally, we'd like to point out that some SContr-free semantic theories extending contraction-free arithmetics may not be strong enough to satisfy Löb's Theorem's applicability conditions, in which case the objection from Löb's Theorem we are considering would not apply in the first place.

22.3.1 *Premise-Aggregating Connectives*

We will therefore assume that truth-preservation and validity for arguments with a finite number of premises can be expressed using some “premise-aggregating connective” $\odot$:[32]

(a) The claim that the argument from premises $\alpha_1, ..., \alpha_n$, taken together, to conclusion β preserves truth can be expressed in the object-language as $Tr(\ulcorner\alpha_1\urcorner) \odot ... \odot Tr(\ulcorner\alpha_n\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)$.

(b) The claim that the argument from premises $\alpha_1, ..., \alpha_n$, taken together, to conclusion β is valid can be expressed using the object-language's binary validity predicate as $Val(\ulcorner\alpha_1 \odot ... \odot \alpha_n\urcorner, \ulcorner\beta\urcorner)$.

Is there an understanding of the logical behavior of $\odot$ on which (a) and (b) are true, but each of our three challenges to VTP is blocked?

Before examining the three challenges in turn, we now consider the chief options for the rules governing $\odot$ in the context of a substructural natural deduction system. For the time being, we will work within a structural framework in which the “taking together” of assumptions—which we have indicated with commas to the left of the turnstile—can be represented using “multisets.” These are structures that behave like sets except for the fact that they keep track of the number of occurrences of each member (Meyer and McRobbie 1982a, 1982b). The philosophical significance of multiset structure in natural deduction has been explained in many ways, and the same is the case for the more complex structure we will consider later. This isn't the place to compare various interpretations or defend one of them.[33] Our aim, rather, is to explain how moving to a deduction system in which the structure referred to on the left of the turnstile is finer-grained than a set affects the standard objections to VTP.

Using multisets rather than (e.g.) sequences renders redundant Gentzen's structural rule of *exchange*:

$$(\mathsf{SExch})\frac{\Gamma, \alpha, \beta \vdash \gamma}{\Gamma, \beta, \alpha \vdash \gamma}$$

By contrast, SContr isn't redundant, nor is the structural rule of *weakening*:

$$(\mathsf{SWeak})\frac{\Gamma, \alpha \vdash \gamma}{\Gamma, \beta, \alpha \vdash \gamma}$$

Indeed, once one or more of SContr and SWeak is rejected, one can formulate operational rules for two different connectives, rules that become equivalent only in the presence of both SContr and SWeak. These are the rules that govern, respectively,

[32] For arguments with an infinite number of premises, we will need universal quantification to express truth-preservation. None of the objections to VTP we will consider, however, depend on consideration of infinite-premise arguments.

[33] We have each made different suggestions in previous work: Shapiro (2011) and Beall and Murzi (2013). [The interpretation of structure sketched in Shapiro (2011) is elaborated in Shapiro (2015).] See also Read (1988), Slaney (1990), Restall (2000), and Paoli (2002).

the "multiplicative" and "additive" conjunctions of linear logic, a multiset-based logic in which both SWeak and SContr are rejected (Girard 1987):[34]

$$(\otimes\text{-I})\frac{\Gamma \vdash \alpha \quad \Delta \vdash \beta}{\Gamma, \Delta \vdash \alpha \otimes \beta} \quad (\otimes\text{-E})\frac{\Gamma, \alpha, \beta \vdash \gamma \quad \Delta \vdash \alpha \otimes \beta}{\Gamma, \Delta \vdash \gamma}$$

$$(\&\text{-I})\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \,\&\, \beta} \quad (\&\text{-E1})\frac{\Gamma \vdash \alpha \,\&\, \beta}{\Gamma \vdash \alpha} \quad (\&\text{-E2})\frac{\Gamma \vdash \alpha \,\&\, \beta}{\Gamma \vdash \beta}$$

Since it will be important later, we note that the structural comma appears in the rules for the multiplicative $\otimes$, whereas it does not appear in the rules for the additive &. In the terminology of Belnap (1982, 1993), the additive rules are "structure-free" while the multiplicative rules are "structure-dependent." Finally, in this structural setting, our assumption of the transitivity of validity can be codified using the following version of the cut rule:

$$(\mathsf{Cut})\frac{\Gamma \vdash \alpha \quad \Delta, \alpha \vdash \beta}{\Delta, \Gamma \vdash \beta}$$

22.3.2 The Validity Argument

The first point we would like to make is that, in the absence of SContr, the 'only if' direction of the Validity Argument (the direction that would establish VTP) fails when the premise-aggregating connective $\odot$ is construed as the additive & in a multiset-based logic.

To see why, note that when rewritten using &, this direction of the Validity Argument requires deriving $Tr(\ulcorner \alpha_1 \urcorner) \,\&\, \ldots \,\&\, Tr(\ulcorner \alpha_n \urcorner) \vdash Tr(\ulcorner \beta \urcorner)$ from $Tr(\alpha_1), \ldots, Tr(\alpha_n) \vdash Tr(\beta)$. That in turn requires $n-1$ uses of the inference pattern

$$(\&\text{-L})\frac{\Gamma, \alpha_1, \alpha_2 \vdash \beta}{\Gamma, \alpha_1 \,\&\, \alpha_2 \vdash \beta}$$

Field himself justifies this inference by appeal to the rule &-E. Indeed, in the presence of SContr, either of our twin elimination rules &-E1 and &-E2 yields & -L. Here is a derivation using &-E2, SContr, Cut, and the reflexivity of validity:

$$\dfrac{\dfrac{\Gamma, \alpha_1, \alpha_2 \vdash \beta \qquad \dfrac{\alpha_1 \,\&\, \alpha_2 \vdash \alpha_1 \,\&\, \alpha_2}{\alpha_1 \,\&\, \alpha_2 \vdash \alpha_2}\,\&\text{-E2}}{\Gamma, \alpha_1, \alpha_1 \,\&\, \alpha_2 \vdash \beta}\,\mathsf{Cut} \qquad \alpha_1 \,\&\, \alpha_2 \vdash \alpha_1}{\dfrac{\Gamma, \alpha_1 \,\&\, \alpha_2, \alpha_1 \,\&\, \alpha_2 \vdash \beta}{\Gamma, \alpha_1 \,\&\, \alpha_2 \vdash \beta}\,\mathsf{SContr}}\,\mathsf{Cut}$$

[34] While linear logic is standardly presented in sequent calculus format, the above natural deduction rules appear in Avron (1988, p. 165), Troelstra (1992, p. 57) and O'Hearn and Pym (1999).

In a logic without SContr, on the other hand, & -L fails. Moreover, this remains the case if we accept SWeak, thus strengthening linear logic into what is known as an "affine" logic.[35]

Hence, insofar as we wish to preserve the Validity Argument while rejecting SCont (and thus avoiding c-Curry and v-Curry), we ought not interpret the premise-aggregating $\odot$ as the additive conjunction & of a multiset-based logic. On the other hand, both directions of the Validity Argument go through, even in the absence of SContr, provided that $\odot$ is construed as the multiplicative $\otimes$. Given $\alpha_1 \otimes \alpha_2 \vdash \alpha_1 \otimes \alpha_2$, the rule $\otimes$-E immediately yields the inference required for the argument's "only if" direction:[36]

$$(\otimes\text{-L})\,\frac{\Gamma, \alpha_1, \alpha_2 \vdash \beta}{\Gamma, \alpha_1 \otimes \alpha_2 \vdash \beta}$$

Indeed, with $\otimes$ as premise-aggregating connective, Elia Zardini (2011) has recently proved a generalization of the Validity Argument's "only if" conclusion.[37] And the "if" direction is no harder to establish.

Summarizing, we can say that Field's objection to the "only if" direction of the Validity Argument fails when our semantic theory is based on an underlying logic that lacks SContr, as long as this logic is multiset-based and we state the argument's conclusion using multiplicative conjunction. Admittedly, this method of vindicating the Validity Argument carries a cost. Multiset-based logics can contain no connectives that behave like the conjunction or disjunction of classical logic (see e.g. Belnap 1993). In the case of the additive connectives, for example, we lose *Distribution*: $\alpha \,\&\, (\beta \vee \gamma) \vdash (\alpha \,\&\, \beta) \vee (\alpha \,\&\, \gamma)$. On the multiplicative side, besides losing distribution of $\otimes$ over a corresponding multiplicative disjunction, we lose *Simplification*: $\alpha \otimes \beta \vdash \alpha$. Adding the rule SWeak, as Zardini proposes, restores the latter. But, as we will see below, we still lose *Square-increasingness*: $\alpha \vdash \alpha \otimes \alpha$.

However, adopting a multiset-based logic isn't the only way to vindicate the Validity Argument by rejecting a structural contraction rule. A second way is to use one of the many substructural logics in which assumptions are regarded as "taken together" in two different ways. In such "dual-bunching" logics, the structures referred to on the left of the turnstile are not multisets, but rather finer-grained "bunches" specified using two different punctuation marks (Read 1988; Slaney 1990; Restall 2000). This alternative is of interest for two reasons. First, unlike multiset-based logics, dual-bunching logics do feature connectives whose behavior is classical to the extent

[35] In that case, however, the "if" direction of the Validity Argument *will* go through for & as premise-aggregating connective. Deriving $Tr(\ulcorner\alpha_1\urcorner), ..., Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$ from $Tr(\ulcorner\alpha_1\urcorner) \,\&\, ... \,\&\, Tr(\ulcorner\alpha_n\urcorner) \vdash Tr(\ulcorner\beta\urcorner)$ requires the inverse of &-L, which obtains in the presence of SWeak.

[36] In single-conclusion sequent calculus formulations (which suffice for our purposes, as our derivations all involve the language's negation-free fragment), the connective $\otimes$ is governed by the twin rules $\otimes$-I and $\otimes$-L.

[37] Field's own reasoning, as sketched in §1.3, amounts to a special case of Zardini's proof: the case in which we are considering the truth-preservingness of a single-conclusion argument and employ no side assumptions. Zardini's proof does not depend on his acceptance of SWeak.

that they satisfy Distribution, Simplification, and Square-increasingness. Secondly, as we will see in §3.3, multiset-based and dual-bunching logics underwrite different interpretations of the way in which rejecting structural contraction blocks the argument against VTP via the *Modus Ponens* axiom.

The first kind of bunching is used to formulate all the *structure-dependent operational rules*. For this reason, it will be convenient to indicate this kind of bunching using the comma (though the semicolon is more standard). That way, we can retain our rules $\rightarrow$-I, $\rightarrow$-E, VD and $\otimes$-I, as long as Γ and Δ are now understood as bunches rather than multisets. On the other hand, we need a generalized version of $\otimes$-E, where $\Delta(\alpha, \beta)$ stands for any bunch of which α, β is a subbunch:[38]

$$(\otimes\text{-E}_{db})\frac{\Delta(\alpha,\beta)\vdash\gamma \quad \Gamma\vdash\alpha\otimes\beta}{\Delta(\Gamma)\vdash\gamma}$$

In dual-bunching logics, one or more of the standard structural rules SContr, SWeak or SExch is rejected for the comma.[39] Just as for multiset-based logics, rejecting SContr suffices to block the above derivations of c-Curry and v-Curry.

What is distinctive about dual-bunching logics is the introduction of a second kind of bunching of assumptions, which we will indicate using the colon. This "extensional" bunching *obeys all the standard structural rules*:

$$(\mathsf{eSContr})\frac{\Gamma(\Delta:\Delta)\vdash\beta}{\Gamma(\Delta)\vdash\beta} \quad (\mathsf{eSWeak})\frac{\Gamma(\Delta)\vdash\gamma}{\Gamma(\Delta':\Delta)\vdash\gamma} \quad (\mathsf{eSExch})\frac{\Gamma(\Delta:\Delta')\vdash\gamma}{\Gamma(\Delta':\Delta)\vdash\gamma}$$

Unlike the "intensional" comma, the colon need not get mentioned in operational rules for any connective.[40]

We are now ready to consider how the Validity Argument fares for dual-bunching logics. First, the reasoning challenged by Field goes through provided the conclusion is formulated using the structural comma together with the multiplicative $\otimes$ as the premise-aggregating connective. That is because we retain $\otimes$-L, now generalizable to

$$(\otimes-\text{L}_{db})\frac{\Gamma(\alpha_1,\alpha_2)\vdash\beta}{\Gamma(\alpha_1\otimes\alpha_2)\vdash\beta}$$

Construed this way, the Validity Argument's "only if" direction establishes that $\alpha_1, ..., \alpha_n \vdash_T \beta$ only if $\vdash_T Tr(\ulcorner\alpha_1\urcorner)\otimes ... \otimes Tr(\ulcorner\alpha_n\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)$. Moreover,

[38] For definitions, see Read (1988, §4.1) and Restall (2000, pp. 19–20). In sequent calculus formulations, $\otimes$-E$_{db}$ is replaced by $\otimes$-L$_{db}$ below. Sequent calculi of this type were developed independently for fragments of relevant logics by Minc (1976) and by Dunn, whose version appears in Anderson and Belnap (1975, §28.5). For natural deduction formulations, see Read (1988), Slaney (1990) and O'Hearn and Pym (1999), whose use of the comma we follow.

[39] Rather than rejecting the structural rule of *associativity*, we are avoiding the need for such a rule by allowing our comma to retain its variable polyadicity.

[40] Since assumptions can now be embedded in bunches specified using both comma and colon, we also need to generalize our statement of the cut rule:

$$(\mathsf{Cut}_{db})\frac{\Gamma\vdash\alpha \quad \Delta(\alpha)\vdash\beta}{\Delta(\Gamma)\vdash\beta}$$

a parallel result now holds for the connective &, known in this structural context as "extensional" conjunction.[41] This is because the fact that the colon obeys eSContr allows us to replicate the above derivation of &-L, yielding

$$(\text{\&-L}_{db})\frac{\Gamma(\alpha_1 : \alpha_2) \vdash \beta}{\Gamma(\alpha_1 \,\&\, \alpha_2) \vdash \beta}$$

Accordingly, the Validity Argument also goes through when the conclusion is formulated using the structural colon together with & as the premise-aggregating connective. Construed this way, it establishes that $\alpha_1 : ... : \alpha_n \vdash_T \beta$ only if $\vdash_T Tr(\ulcorner\alpha_1\urcorner) \,\&\, ... \,\&\, Tr(\ulcorner\alpha_n\urcorner) \rightarrow Tr(\ulcorner\beta\urcorner)$.[42] According to dual-bunching logics, then, there are *different kinds of multi-premise arguments*, represented using different antecedent structure, and the validity of each kind of argument entails a different kind of truth-preservation, expressed in the object-language using different premise-aggregating connectives.

There are thus at least two general ways to vindicate the Validity Argument by rejecting SContr: one can use a multiset-based logic with multiplicative conjunction as premise-aggregating connective, or a dual-bunching logic. Versions of both approaches are known to make possible a naïve theory of truth (either a consistent paracomplete theory or a nontrivial paraconsistent theory).[43] We will return to the difference between the two approaches in the next section. For now, we merely note that they yield logics that conflict for the fragment of the language whose only connectives are & and the corresponding disjunction $\vee$. Recall that the rules for these connectives *don't even mention* the nonstandard comma structure. It follows that on the dual-bunching approach, the *single-premise* validities of this fragment will be exactly those of the corresponding fragment of classical logic. As explained above, this stands in contrast to the conjunctive/disjunctive fragment of additive or

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[41] But see (Paoli 2007, pp. 569–571) for opposition to the standard claim that the extensional conjunction of such logics is "truth functional."

[42] The point extends naturally to cases in which the assumptions are aggregated using both kinds of structure. For instance, $\alpha_1 : (\alpha_2, \alpha_3) \vdash_T \beta$ only if $\vdash_T Tr(\ulcorner\alpha_1\urcorner) \,\&\, (Tr(\ulcorner\alpha_1\urcorner) \otimes Tr(\ulcorner\alpha_n\urcorner)) \rightarrow Tr(\ulcorner\beta\urcorner)$.

[43] Most work on this issue has concerned the closely parallel case of a naïve set theory featuring an unrestricted axiom of comprehension. For proofs of the consistency or nontriviality of unrestricted comprehension in some "weak relevant logics" that can be specified via dual-bunching natural deduction, see Brady (1983, 1989, 2006). For applications of Brady's techniques to naïve truth-theory, see Priest (1991) and Beall (2009), which do not however consider natural deduction systems. As for multiset-based logics, the consistency of unrestricted comprehension in an affine logic was shown by V. Grishin in 1974: see Došen (1993). For the consistency of a naïve truth theory based on an affine logic, see Zardini (2011).

multiplicative linear logic.[44] The philosophical interpretation of nonstandard antecedent structure—whether dual-bunching or multiset-based—remains a controversial and important issue. However, it isn't one we can address in this paper, which has the more limited aim of exploring how such logics allow a defense of VTP against the various challenges that have been raised against that thesis.[45]

22.3.3 From VTP to Absurdity via the Modus Ponens Axiom

We now turn to the objection that VTP entails the *Modus Ponens* axiom, and thus absurdity via c-Curry reasoning. Using a generic premise-aggregating connective, we can state, respectively, the validity of *Modus Ponens* and the *Modus Ponens* axiom as follows:

$$(\mathsf{VMP}_{\odot})\ Val(\ulcorner(\alpha \rightarrow \beta) \odot \alpha\urcorner, \ulcorner\beta\urcorner)$$

$$(\mathsf{MPA}_{\odot})\ (\alpha \rightarrow \beta) \odot \alpha \rightarrow \beta.$$

In §1.2 we saw that VTP, when expressed in the object-language, implies

$$(\mathsf{V1})\ Val(\ulcorner\alpha\urcorner, \ulcorner\beta\urcorner) \rightarrow (\alpha \rightarrow \beta).$$

It follows that if our naïve semantic theory implies $\mathsf{VMP}_{\odot}$, it also implies the absurdity-threatening $\mathsf{MPA}_{\odot}$. Thus, in order to evaluate the objection, we need to answer two questions:

(1) If we reject SContr, will our semantic theory still imply $\mathsf{VMP}_{\odot}$? Equivalently, in view of VP and VD, will our underlying contraction-free logic still give us $(\alpha \rightarrow \beta) \odot \alpha \vdash \beta$?
(2) If we reject SContr, will $\mathsf{MPA}_{\odot}$ still yield absurdity?

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[44] As Dave Ripley pointed out to us, a dual-bunching logic could also retain a connective $\&_A$ that behaves like the "additive" conjunction and disjunction of a multiset-based logic, for instance in failing to validate Distribution over the corresponding $\vee_A$. To achieve this, replace &-E1 and &-E2 with

$$(\&_A\text{-E1})\frac{\Gamma, \alpha \vdash \gamma \quad \Delta \vdash \alpha \&_A \beta}{\Gamma, \Delta \vdash \gamma} \quad (\&_A\text{-E2})\frac{\Gamma, \beta \vdash \gamma \quad \Delta \vdash \alpha \&_A \beta}{\Gamma, \Delta \vdash \gamma}$$

By contrast, in the presence of Cut_{db}, our original &-E1 and &-E2 have the same "extensional" effect as the rules

$$(\&\text{-E1}_{db})\frac{\Gamma(\alpha) \vdash \gamma \quad \Delta \vdash \alpha \& \beta}{\Gamma(\Delta) \vdash \gamma} \quad (\&\text{-E2}_{db})\frac{\Gamma(\beta) \vdash \gamma \quad \Delta \vdash \alpha \& \beta}{\Gamma(\Delta) \vdash \gamma}$$

[45] For relevant work on the interpretation of dual-bunching systems, see Read (1988) and Slaney (1990). For a recent and novel suggestion toward an interpretation of multiset-based systems, see Zardini (2011). For a sketch of a more deflationary approach to antecedent structure, see Shapiro (2011). [This sketch has since been elaborated in Shapiro (2015).]

A negative answer to (1) or (2) will show that the objection against VTP fails.[46]

The answers to these questions vary depending on which connective we employ as our $\odot$. For the additive & of a contraction-free logic, the answer to (1) is negative (Restall 1994, pp. 35–36). It should help to display how SContr is involved in the usual derivation:

$$\dfrac{\dfrac{\dfrac{(\alpha \to \beta) \mathbin{\&} \alpha \vdash (\alpha \to \beta) \mathbin{\&} \alpha}{(\alpha \to \beta) \mathbin{\&} \alpha \vdash \alpha \to \beta}\ \&\text{-E} \qquad \dfrac{(\alpha \to \beta) \mathbin{\&} \alpha \vdash (\alpha \to \beta) \mathbin{\&} \alpha}{(\alpha \to \beta) \mathbin{\&} \alpha \vdash \alpha}\ \&\text{-E}}{(\alpha \to \beta) \mathbin{\&} \alpha, (\alpha \to \beta) \mathbin{\&} \alpha \vdash \beta}\ {\to}\text{-E}}{(\alpha \to \beta) \mathbin{\&} \alpha \vdash \beta}\ \mathsf{SContr}$$

But the objection to VTP fails as well when we use the the multiplicative $\otimes$. This time, the answer to (1) is affirmative:

$$\dfrac{\dfrac{\alpha \to \beta \vdash \alpha \to \beta \qquad \alpha \vdash \alpha}{\alpha \to \alpha, \alpha \vdash \beta}\ {\to}\text{-E} \qquad (\alpha \to \beta) \otimes \alpha \vdash (\alpha \to \beta) \otimes \alpha}{(\alpha \to \beta) \otimes \alpha \vdash \beta}\ \otimes\text{-E}$$

However, now the answer to (2) is negative. That is because, as already noted in Meyer et al. (1979), the argument from $\mathsf{MPA}_{\odot}$ to absurdity depends essentially on the left-to-right direction of the *Idempotence* law $\vdash \alpha \leftrightarrow \alpha \odot \alpha$. But when we use multiplicative conjunction in a contraction-free logic, we lose this law (Zardini 2011). Again, notice how SContr is involved in its usual derivation:

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \alpha \vdash \alpha}{\alpha, \alpha \vdash \alpha \otimes \alpha}\ \otimes\text{-I}}{\alpha \vdash \alpha \otimes \alpha}\ \mathsf{SContr}}{\vdash \alpha \to \alpha \otimes \alpha}\ {\to}\text{-I}$$

In summary, to derive absurdity from VTP, the objector presupposes that there is some connective $\odot$ that meets two conditions:

[46] According to the theory proposed by Ripley (2013) based on Cobreros et al. (2012), which is "substructural" only in rejecting Cut, the objection to VTP we are considering in this section fails because MPA fails to yield absurdity. This is because the argument's final step from $\vdash_T \kappa \to \bot$ and $\vdash_T \kappa$ to $\vdash_T \bot$ fails. In Ripley's sequent calculus, the rule $\to$-E is inadmissible in the absence of Cut. Indeed, Ripley holds (p.c) that $\to$-E shouldn't be regarded as fundamental to the logic of a detaching conditional, as it covertly builds in extraneous transitivity in comparison with the sequent calculus rule

$$({\to}\text{-L})\dfrac{\Gamma \vdash \alpha \qquad \Delta, \beta \vdash \gamma}{\Delta, \alpha \to \beta, \Gamma \vdash \gamma}$$

To this, defenders of $\to$-E may reply that *each of* $\to$-E and $\to$-L builds in transitivity in comparison with $\alpha \to \beta, \alpha \vdash \beta$. It is true, as Ripley shows, that the transitivity built in by $\to$-E (which, given $\to$-I, yields Cut) can be blamed for paradox. But in view of the option of blaming paradox on SContr instead, this won't suffice to show that $\to$-L is a more fundamental rule than $\to$-E.

(a) it serves as premise-aggregator for the valid argument $\alpha \to \beta, \alpha \vdash \beta$, so that we have the single-premise rule $(\alpha \to \beta) \odot \alpha \vdash \beta$ and $\mathsf{VMP}_{\odot}$, and
(b) it satisfies the left-to-right direction of Idempotence, $\vdash \alpha \to \alpha \odot \alpha$.

Yet we have now seen that one or the other of these conditions fails for each of our candidate connectives. [47]

At this point, a critic of VTP might object that the response just given is at best incomplete. We have shown that the argument from VTP to absurdity fails, in the absence of SContr, when either & or $\otimes$ is used to state the premise $\mathsf{VMP}_{\odot}$. Still, the critic insists, our task remains that of explaining why the argument fails when $\odot$ expresses our *ordinary notion of conjunction*. After all, ordinary conjunction appears to satisfy both conditions (a) and (b): both single-premise *Modus Ponens* and Idempotence. If we are to avoid absurdity in the presence of a naïve theory of truth, we have argued, at least one of these appearances must be mistaken. The challenge is to explain which.

Zardini (2011, 2013) argues that condition (a) clearly holds for our "informal notion of conjunction." Accordingly, he maintains that ordinary conjunction is best captured by the multiplicative connective $\otimes$ of an affine logic—where the presence of SWeak guarantees such ordinary features as Simplification. Yet, as he recognizes, someone else might argue that condition (b) clearly holds for ordinary conjunction. More generally, we would add, one might maintain that the usual lattice properties are essential to our ordinary conjunction $\wedge$, whence from $\alpha \vdash \beta$ and $\alpha \vdash \gamma$ it must follow that $\alpha \vdash (\beta \wedge \gamma)$, even in the case where $\alpha = \beta = \gamma$.

We don't propose to settle this dispute about our informal notion of conjunction, or examine whether there is a univocal such notion.[48] Instead, we will now explain how the dispute is affected by the availability of dual-bunching logics. The chief reason Zardini insists that ordinary conjunction meets condition (a) is that he takes conjunction to be an all-purpose premise-aggregating connective. As he writes, conjunction is the connective we use to make explicit "how premises are combined in a multi-premise argument" (Zardini 2013). In order for $\odot$ to be conjunction, he holds, it is non-negotiable that it satisfy the rule

$$(\odot - L)\frac{\Gamma, \alpha_1, \alpha_2 \vdash \beta}{\Gamma, \alpha_1 \odot \alpha_2 \vdash \beta}$$

In a multiset-based logic without SContr, we have seen, the additive connective & violates $\odot$-L. We have a counterexample in the failure of $\alpha \to \beta, \alpha \vdash \beta$ to yield $(\alpha \to \beta)\ \&\ \alpha \vdash \beta$. This is the chief reason why he concludes that $\otimes$ has a stronger claim than & to represent our informal notion of conjunction.[49]

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[47] It makes no difference whether these connectives are those of a multiset-based or dual-branching logic. Nor, in the latter case, would it make a difference if we considered $\&_A$ of note 44 in place of &.
[48] For arguments to the contrary, see Paoli (2007), Mares and Paoli (2014).
[49] Hjortland (2012) has recently proposed using an affine logic with additive conjunction and disjunction in a revisionary approach to semantic paradox. We take no position here on whether the consideration just rehearsed poses a serious problem for that approach.

But once dual-bunching logics are an option, matters get more complicated. In such logics we have both &-L_{db} and $\otimes$-L_{db}. The additive & corresponds to one mode in which premises may be combined, marked by our colon, while the multiplicative $\otimes$ corresponds to another mode, marked by our comma (Read 1988). According to dual-bunching logics, & doesn't serve as premise-aggregating connective for *Modus Ponens*, since we don't have $\alpha \rightarrow \beta : \alpha \vdash \beta$. Yet & serves as premise-aggregating connective for other arguments, e.g. $\alpha : \beta \vee \gamma \vdash (\alpha \ \& \ \beta) \vee \gamma$. Hence it is no longer clear that Zardini's view, on which ordinary conjunction is multiplicative and obeys single-premise *Modus Ponens* but not Idempotence, holds an advantage over the alternative view on which ordinary conjunction is additive and satisfies Idempotence but not single-premise *Modus Ponens*. Giving up single-premise *Modus Ponens*, understood in terms of ordinary conjunction, needn't amount to giving up conjunction's role as a premise-aggregating connective in a natural deduction system. Of course, as we noted above, the philosophical significance of the twofold bunching of premises needs to be elucidated. But that is also the case for the simpler premise structure in multiset-based deduction systems.

In this section, we have shown that the standard argument from VTP to absurdity breaks down in substructural theories which do not validate SContr, and have explained how the details of *where* it breaks down depend on which connective of the contraction-free logic we use to represent the conjunction appealed to in the standard argument.

22.3.4 The Consistency Argument

Let us finally turn to the Consistency Argument, and the resulting challenge to VTP from Gödel's Second Incompleteness Theorem. There are two ways one might respond: argue that Gödel's limitative results don't obtain for theories of arithmetic based on contraction-free logics, or argue that the Consistency Argument fails for such logics. Since there are contraction-free theories of arithmetic for which the results hold, we won't rely exclusively on the former strategy.[50]

The Consistency Argument requires one to prove, within one's semantic theory T, the following induction step: if *all* conclusions of derivations of length $\leq n$ are true, then *all* conclusions of derivations of length $n + 1$ are true. To prove this, it suffices to prove, for each rule R, that

[50] Restall (1994, Chap. 11) shows that an arithmetic based on the dual-bunching contraction-free logic RWK (which he calls CK) is classical Peano arithmetic, but it isn't known whether RWK supports a nontrivial naïve semantic theory in which $Tr(\ulcorner \alpha \urcorner)$ is everywhere intersubstitutable with α (see Hjortland 2012).

(TP_R) If *all* the premises of an instance of R are true, then the corresponding instance of the conclusion will be true.[51]

Now consider a rule R such that the theory proves that R has precisely two premises. To establish TP_R we will then need to prove

($\mathsf{TP2}_R$) For all x, y, z such that x and y are the two premises of an instance of R and z its corresponding conclusion: if x is true *and* y is true, then z is true.

But how are we to understand the 'and' in $\mathsf{TP2}_R$?

If 'all' in TP_R is understood as the standard "lattice-theoretical" or additive quantifier (Paoli 2005), then $\mathsf{TP2}_R$ will only help establish TP_R provided 'and' is likewise construed as additive.[52] But when R is the two-premise *Modus Ponens*, we won't be able to prove TP_R on this construal. That is because we have already seen that we don't have any instance of $\vdash_T (\alpha \rightarrow \beta) \,\&\, \alpha \rightarrow \beta$. This should mean that we don't have any instance of $\vdash_T Tr(\ulcorner \alpha \rightarrow \beta \urcorner) \,\&\, Tr(\ulcorner \alpha \urcorner) \rightarrow Tr(\ulcorner \beta \urcorner)$ either, whence we can't prove the generalization TP_R. In fact, that is Field's own explanation of how the Consistency Argument breaks down for paracomplete and paraconsistent theories (Field 2008, pp. 377–378). Unlike Field, we don't attribute this breakdown to the argument's illicit appeal to VTP. In our view, rather, the breakdown of the Consistency Argument (on the standard interpretation of the quantifier) results from the argument's illicit use of & as premise-aggregator for the two-premise *Modus Ponens* rule.[53]

Perhaps, then, we could rescue the Consistency Argument by interpreting the 'all' in TP_R as some kind of multiplicative quantifier, one that stands to $\otimes$ the way the standard universal quantifier stands to &. Where R is *Modus Ponens*, we should indeed be able to prove $\mathsf{TP2}_R$ with 'and' interpreted as $\otimes$, since $\otimes$ does serve as premise-aggregator for *Modus Ponens*. If this is to help establish TP_R, however, we would need to know more about the envisioned multiplicative quantifier. Paoli (2005) and Mares and Paoli (2014) note that there is no accepted theory of how such a quantifier should behave. One option is presented by Zardini (2011) in the context of a multiset-based logic. But Zardini's multiplicative quantifier won't serve the purposes of anyone who wishes to use the Consistency Argument to criticize VTP.

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[51] Here we are no longer thinking of natural deduction rules, but rather of the rules of a Hilbert system, rules for generating theorems.

[52] Here is a rough explanation. In the course of deriving TP_R in our object-language, we will need to establish, under the assumption that three arbitrary sentences (denoted by a_1, a_2 and b) are the respective premises and conclusion of an instance of R, the claim $\forall x(x = a_1 \vee x = a_2 \rightarrow Tr(x)) \vdash Tr(b)$. Assuming $\forall$ is lattice-theoretical, this claim will follow from $Tr(a_1) \,\&\, Tr(a_2) \vdash Tr(b)$, whereas it won't follow from $Tr(a_1) \otimes Tr(a_2) \vdash Tr(b)$. For we have $\forall x \phi(x) \vdash \phi(a_1) \,\&\, \phi(a_2) ... \,\&\, ... \phi(a_n)$, but not $\forall x \phi(x) \vdash \phi(a_1) \otimes \phi(a_2) ... \otimes ... \phi(a_n)$. See Běhounek et al. (2007).

[53] Field himself claims that $\mathsf{TP2}_R$ will "obviously" fail to establish TP_R when the former is understood using what is, in effect, multiplicative conjunction. See Field (2006, p. 597) and Field (2008, p. 379). In his discussion, $Tr(\ulcorner \alpha \rightarrow \beta \urcorner) \rightarrow (Tr(\ulcorner \alpha \urcorner) \rightarrow Tr(\ulcorner \beta \urcorner))$ takes the place of $Tr(\ulcorner \alpha \rightarrow \beta \urcorner) \otimes Tr(\ulcorner \alpha \urcorner) \rightarrow Tr(\ulcorner \beta \urcorner)$, which is equivalent to the former in the logics we are considering. See also Priest (2010).

For he characterizes the behavior of the multiplicative quantifier using an ω-rule as (right-)introduction rule. Hence, the semantic theory based on this logic won't be recursively axiomatisable, and won't satisfy the conditions for Gödel's theorem.

22.4 Concluding Remarks

In this paper, we've argued for two main claims. First, the v-Curry Paradox shows that SContr is in tension with natural principles governing some (intuitive enough) notions of validity. Hence, if, as we've assumed, the validity relation is transitive, revisionary theorists have strong reason to give up SContr. Second, the standard challenges to VTP presented in §1 all break down once SContr is dropped. Rejecting SContr opens up non-classical ways of aggregating together premises—ways which no longer underwrite the objections to VTP. To be sure, it may be argued instead that the notion of validity that is shown to be paradoxical by the v-Curry Paradox should be rejected as incoherent. Validity, one might think, is interpretational, or purely logical, validity: truth on all uniform interpretations of the non-logical vocabulary. This, however, does not seem in line with the seemingly compelling thought, championed by rcf theorists such as Field (2007, 2008) and Priest (2006a, 2006b), that logical validity is a *species* of a more general notion of validity. Alternatively, it may be contended that paradox-prone notions of validity must be refined, and made less naïve (McGee 1991). But this, too, we've argued, doesn't seem like a viable option for proponents of the revisionary approach to paradox, who rather recommend revising our theory of logic, while preserving the naïve semantic principles. If neither of these foregoing options is viable, then SContr must be restricted on pain of triviality, and we can continue to maintain that valid arguments preserve truth.

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